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INTRODUCTION, CLASSIFICATION, SHORT HISTORY, AUXILIARY RESULTS, AND METHODS

Generally speaking, a functional equation is a relationship containing an unknown element, usually a function, which has to be determined, or at least partially identifiable by some of its properties. Solving a functional equation (FE) means finding a *solution*, that is, the unknown element in the relationship. Sometimes one finds several solutions (solutions set), while in other cases the equation may be deprived of a solution, particularly when one provides the class/space to which it should belong.

Since a relationship could mean the equality, or an inequality, or even the familiar "belongs to," designated by $=$, $\in$, $\subset$ or $\subseteq$, the description given earlier could also include the functional inequalities or the functional inclusions, rather often encountered in the literature. Actually, in many cases, their theory is based on the theory of corresponding equations with which they interact. For instance, the selection of a single solution from a solution set, especially in case of inclusions.

In this book we are mainly interested in FEs, in the proper/usual sense. We send the readers to adequate sources for cases of related categories, like inequalities or inclusions.

Functional Differential Equations: Advances and Applications, First Edition.
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1.1 CLASSICAL AND NEW TYPES OF FEs

The classical types of FEs include the *ordinary differential equations* (ODEs), the *integral equations (IEs) of Volterra* or Fredholm and the *integro-differential equations* (IDEs). These types, which have been thoroughly investigated since Newton's time, constitute the classical part of the vast field of FEs, or *functional differential equations* (FDEs).

The names Bernoulli, Newton, Riccati, Euler, Lagrange, Cauchy (analytic solutions), Dini, and Poincaré as well as many more well-known mathematicians, are usually related to the classical theory of ODE. This theory leads to a large number of applications in the fields of science, engineering, economics, in cases of the modeling of specific problems leading to ODE.

A large number of books/monographs are available in the classical field of ODE: our list of references containing at least those authored by Halanay [237], Hale [240], Hartman [248], Lefschetz [323], Petrovskii [449], Sansone and Conti [489], Rouche and Mawhin [475], Nemytskii and Stepanov [416], and Coddington and Levinson [106].

Another classical type of FEs, closely related to the ODEs, is the class of IEs, whose birth is related to Abel in the early nineteenth century. They reached an independent status by the end of nineteenth century and the early twentieth century, with Volterra and Fredholm. Hilbert is constituting his theory of linear IEs of Fredholm's type, with symmetric kernel, providing a successful start to the spectral theory of completely continuous operators and orthogonal function series.

Classical sources in regard to the basic theory of integral equations include books/monographs by Volterra [528], Lalesco [319], Hilbert [261], Lovitt [340], Tricomi [520], Vath [527]. More recent sources are Corduneanu [135], Gripenberg et al. [228], Burton [80, 84], and O'Regan and Precup [430].

A third category of FEs, somewhat encompassing the differential and the IEs, is the class of IDEs, for which Volterra [528] appears to be the originator. It is also true that E. Picard used the integral equivalent of the ODE $\dot{x}(t) = f(t,x(t))$, under initial condition $x(t_0) = x_0$, Cauchy's problem, namely

$$x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\,ds,$$

obtaining classical existence and uniqueness results by the method of successive approximations.

A recent reference, mostly based on classical analysis and theories of DEs and IEs, is Lakshmikantham and R. M. Rao [316], representing a rather comprehensive picture of this field, including some significant applications and indicating further sources.

The extended class of FDEs contains all preceding classes, as well equations involving operators instead of functions (usually from R into R). The classical categories are related to the use of the so-called Niemytskii operator, defined by the formula $(Fu)(t) = f(t, u(t))$, with $t \in R$ or in an interval of R, while in the case of FDE, the right-hand side of the equation

$$\dot{x}(t) = (Fx)(t),$$

implies a more general type of operator F. For instance, using Hale's notation, one can take $(Fx)(t) = f(t, x(t), x_t)$, where $x_t(s) = x(t+s)$, $-h \leq s \leq 0$ represents a restriction of the function $x(t)$, to the interval $[t-h, t]$. This is the finite delay case. Another choice is

$$(Fx)(t) = (Vx)(t),\ t \in [t_0, T],$$

where V represents an *abstract Volterra operator* (see definition in Chapter 2), also known as *causal operator*.

Many other choices are possible for the operator F, leading to various classes of FDE. Bibliography is very rich in this case, and exact references will be given in the forthcoming chapters, where we investigate various properties of equations with operators.

The first book entirely dedicated to FDE, in the category of delay type (finite or infinite) is the book by A. Myshkis [411], based on his thesis at Moscow State University (under I. G. Petrovskii). This book was preceded by a survey article in the Uspekhi Mat. Nauk, and one could also mention the joint paper by Myshkis and Eĺsgoĺtz [412], reviewing the progress achieved in this field, due to both authors and their followers. The book Myshkis [411] is the first dedicated entirely to the DEs with delay, marking the beginnings of the literature dealing with non-traditional FEs.

The next important step in this direction has been made by N. N. Krasovskii [299], English translation of 1959 Russian edition. In his doctoral thesis (under N. G. Chetayev), Krasovskii introduced the method of *Liapunov functionals* (not just functions!), which permitted a true advancement in the theory of FDEs, especially in the nonlinear case and stability problems. The research school in Ekaterinburg has substantially contributed to the progress of the theory of FDEs (including Control Theory), and names like Malkin, Barbashin, and Krasovskii are closely related to this progress.

The third remarkable step in the development of the theory of FDE has been made by Jack Hale, whose contribution should be emphasized, in respect to the constant use of the arsenal of Functional Analysis, both linear and nonlinear. A first contribution was published in 1963 (see Hale [239]), utilizing the theory of semigroups of linear operators on a Banach function space.

This approach allowed Hale to develop a theory of linear systems with finite delay, in the time-invariant framework, dealing with adequate concepts that naturally generalize those of ODE with constant coefficients (e.g., characteristic values of the system/equation). Furthermore, many problems of the theory of nonlinear ODE have been formulated and investigated for FDE (stability, bifurcation, and others (a.o.)). The classical book of Hale [240] appears to be the first in this field, with strong support of basic results, some of them of recent date, from functional analysis.

In the field of applications of FDE, the book by Kolmanovskii and Myshkis [292] illustrates a great number of applications to science (including biology), engineering, business/economics, environmental sciences, and medicine, including the stochastic factors. Also, the book displays a list of references with over 500 entries.

In concluding this introductory section, we shall mention the fact that the study of FDE, having in mind the nontraditional types, is the focus for a large number of researchers around the world: Japan, China, India, Russia, Ukraine, Finland, Poland, Romania, Greece, Bulgaria, Hungary, Austria, Germany, Great Britain, Italy, France, Morocco, Algeria, Israel, Australia and the Americas, and elsewhere.

The *Journal of Functional Differential Equations* is published at the College of Judea and Samaria, but its origin was at Perm Technical University (Russia), where N. V. Azbelev created a school in the field of FDE, whose former members are currently active in Russia, Ukraine, Israel, Norway, and Mozambique.

Many other journals are dedicated to the papers on FDE and their applications. We can enumerate titles like *Nonlinear Analysis (Theory, Methods & Applications)*, published by Elsevier; *Journal of Differential Equations; Journal of Mathematical Analysis and Applications*, published by Academic Press; *Differentsialuye Uravnenja* (Russian: English translation available); and *Funkcialaj Ekvacioj* (Japan). Also, there are some electronic journals publishing papers on FDE: *Electronic Journal of Qualitative Theory of Differential Equations*, published by Szeged University; EJQTDE, published by Texas State University, San Marcos.

1.2 MAIN DIRECTIONS IN THE STUDY OF FDE

This section is dedicated to the description of various types of problems arising in the investigation of FDE, at the mathematical side of the problem as well as the application of FDE in various fields, particularly in science and engineering.

A first problem occurring in relationship with an FDE is the *existence* or *absence* of a solution. The solution is usually sought in a certain class of functions (scalar, vector, or even Banach space valued) and "a priori" limitations/restrictions may be imposed on it.

In most cases, besides the "pure" existence, we need *estimates* for the solutions. Also, it may be necessary to use the numerical approach, usually approximating the real values of the solution. Such approximations may have a "local" character (i.e., valid in a neighborhood of the initial/starting value of the solution, assumed also unique), or they may be of "global" type, keeping their validity on the whole domain of definition of the solution.

Let us examine an example of a linear FDE, of the form

$$\dot{x}(t) = (Lx)(t) + f(t),\ t \in [0,T], \tag{1.1}$$

with $L : C([0,T],R^n) \to C([0,T],R^n)$ a linear, casual continuous map, while $f \in C([0,T],R^n)$. As shown in Corduneanu [149; p. 85], the unique solution of equation (1.1), such that $x(0) = x^0 \in R^n$, is representable by the formula

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)f(s)\,ds,\ t \in [0,T]. \tag{1.2}$$

In (1.2), the Cauchy matrix is given, on $0 \le s \le t \le T$, by the formula

$$X(t,s) = I + \int_s^t \tilde{k}(t,u)\,du, \tag{1.3}$$

where $\tilde{k}(t,s)$ stands for the conjugate kernel associated to the kernel $k(t,s)$, the latter being determined by the relationship

$$\int_0^t (Lx)(s)\,ds = \int_0^t k(t,s)x(s)\,ds,\ t \in [0,T]. \tag{1.4}$$

For details, see the reference indicated earlier in the text.

Formula (1.2) is helpful in finding various *estimates* for the solution $x(t)$ of the initial value problem considered previously.

Assume, for instance, that the Cauchy matrix $X(t,s)$ is bounded on $0 \le s \le t \le T$ by M, that is, $|X(t,s)| \le M$; hence $|X(t,0)| \le m \le M$, then (1.2) yields the following estimate for the solution $x(t)$:

$$|x(t)| \le m|x^0| + M\int_0^T |f(s)|\,ds, \tag{1.5}$$

with $T < \infty$ and f continuous on $[0,T]$. We derive from (1.5) the estimate

$$\sup_{0 \le t \le T} |x(t)| \le m|x^0| + M|f|_{L^1}, \tag{1.6}$$

which means an upper bound of the norm of the solutions, in terms of data.

We shall also notice that (1.6) keeps its validity in case $T = \infty$, that is, we consider the problem on the semiaxis R_+. This example shows how, assuming also $f \in L^1(R_+, R^n)$, all solutions of (1.1) remain bounded on the positive semiaxis.

Boundedness of all solutions of (1.1), on the positive semiaxis, is also assured by the conditions $|X(t,0)| \le m$, $t \in R_+$, and

$$\int_0^t |X(t,s)|\, ds \le M,\ |f(t)| \le A < \infty,\ t \in R_+.$$

The readers are invited to check the validity of the following estimate:

$$\sup_{t \ge 0} |x(t)| \le m|x^0| + AM,\ t \in R_+. \tag{1.7}$$

Estimates like (1.6) or (1.7), related to the concept of boundedness of solutions, are often encountered in the literature. Their significance stems from the fact that the motion/evolution of a man-made system takes place in a bounded region of the space. Without having estimates for the solutions of FDE, it is practically impossible to establish properties of these solutions.

One of the best examples in this regard is constituted by the property of *stability* of an equilibrium state of a system, described by the FDE under investigation. At least, theoretically, the problem of stability of a given motion of a system can be reduced to that of an equilibrium state. Historically, Lagrange has stated a result of stability for the equilibrium for a mechanical system, in terms of a variational property of its energy. This idea has been developed by A. M. Liapunov [332] (1857–1918), who introduced the method of an auxiliary function, later called *Liapunov function* method. Liapunov's approach to *stability* theory is known as one of the most spectacular developments in the theory of DE and then for larger classes of FDE, starting with N. N. Krasovskii [299].

The *comparison method*, on which we shall rely (in Chapter 3), has brought new impetus to the investigation of stability problems. The schools created by V. V. Rumiantsev in Moscow (including L. Hatvani and V. I. Vorotnikov), V. M. Matrosov in Kazan, then moved to Siberia and finally to Moscow, have developed a great deal of this method, concentrating mainly on the ODE case. Also, V. Lakshmikantham and S. Leela have included many contributions in their treaty [309]. They had many followers in the United States and India, publishing a conspicuous number of results and developments of this method. One of the last contributions to this topic [311], authored by Lakshmikantham, Leela, Drici, and McRae, contains the general theory of equations with causal operators, including stability problems.

The *comparison method* consists of the simultaneous use of Liapunov functions (functionals), and differential inequalities. Started in its general setting by R. Conti [110], it has been used to prove global existence criteria for ODE. In short time, the use has extended to deal with uniqueness problems for ODE by F. Brauer [75] and Corduneanu [114, 115] for stability problems. The method is still present in the literature, with contributions continuing those already included in classical references due to Sansone and Conti [489], Hahn [235], Rouche and Mawhin [475], Matrosov [376–378], Matrosov and Voronov [387], Lakshimikantham and Leela [309], and Vorotnikov [531].

A historical account on the development of the stability concept has been accurately given by Leine [325], covering the period from Lagrange to Liapunov. The mechanical/physical aspects are emphasized, showing the significance of the stability concept in modern science. The original work of Liapunov [332] marks a crossroad in the development of this concept, with so many connections in the theory of evolutionary systems occurring in the mathematical description in contemporary science.

In Chapter 3, we shall present stability theory for ODE and FDE, particularly for the equations with finite delay. The existing literature contains results related to the infinite delay equations, a theory that has been originated by Hale and Kato [241]. An account on the status of the theory, including stability, is to be found in Corduneanu and Lakshmikantham [167]. We notice the fact that a theory of stability, for general classes of FDE, has not yet been elaborated. As far as special classes of FDE are concerned, the book [84] by T. Burton presents the method of Liapunov functionals for integral equations, by using modern functional analytic methods. The book [43] by Barbashin, one of the first in this field, contains several examples of constructing Liapunov functions/functionals.

The converse theorems in stability theory, in the case of ODE, have been obtained, in a rather general framework, by Massera [373], Kurzweil [303], and Vrkoč [532]. Early contributions to stability theory of ODE were brought by followers of Liapunov, (see Chetayev [103] and Malkin [356]). In Chapter 3, the readers will find, besides some basic results on stability, more bibliographical indications pertaining to this rich category of problems.

As an example, often encountered in some books containing stability theory, we shall mention here the classical result (Poincaré and Liapunov) concerning the differential system $\dot{x}(t) = A(t)x(t)$, $t \in R_+$, $x : R_+ \to R^n$, and $A : R_+ \to \mathcal{L}(R^n, R^n)$ a continuous map. If we admit the commutativity condition

$$A(t)\int_0^t A(s)\,ds = \left(\int_0^t A(s)\,ds\right)A(t),\ t \in R_+, \tag{1.8}$$

then the solution, under initial condition $x(0) = x^0$, can be represented by

$$x(t,x^0) = x^0\,e^{\int_0^t A(s)\,ds},\ t \geq 0. \tag{1.9}$$

From this representation formula one derives, without difficulty, the following results:

Stability of the solution $x = \theta =$ the zero vector in R^n is equivalent to boundedness, on R_+, of the matrix function $\int_0^t A(s)\,ds$.

Asymptotic stability of the solution $x = \theta$ is equivalent to the condition

$$\lim_{t\to\infty} e^{\int_0^t A(s)\,ds} = O = \text{the zero matrix}. \tag{1.10}$$

Both statements are elementary consequences of formula (1.8). The definitions of various types of stability will be done in Chapter 3. We notice here that the already used terms, *stability* and *asymptotic stability*, suggest that the first stands for the property of the motion to remain in the neighborhood of the equilibrium point when small perturbations of the initial data are occurring, while the second term tells us that besides the property of stability (as intuitively described earlier), the motion is actually "tending" or approaching indefinitely the equilibrium state, when $t \to \infty$.

Remark 1.1 *The aforementioned considerations help us derive the celebrated stability result, known as Poincaré–Liapunov stability theorem for linear differential systems with constant coefficients.*

Indeed, if $A(t) \equiv A =$ constant is an $n \times n$ matrix, with real or complex coefficients, with characteristic equation $\det(\lambda I - A) = 0$, $I =$ the unit matrix of type $n \times n$, then we denote by $\lambda_1, \lambda_2, \ldots, \lambda_k$ its distinct roots ($k \leq n$). From the elementary theory of DEs with constant coefficients, we know that the entries of the matrix e^{At} are quasi-polynomials of the form

$$\sum_{j=1}^{k} e^{\lambda_j t} p_j(t), \tag{1.11}$$

with $p_j(t)$, $j = 1,2,\ldots,k$, some algebraic polynomials.

Since the commutativity condition (1.8) is valid when $A(t) \equiv A =$ constant, there results that (1.10) can hold if and only if the condition

$$Re\lambda_j < 0,\ j = 1,2,\ldots,k, \tag{1.12}$$

is satisfied. Condition (1.12) is frequently used in stability theory, particularly in the case of linear systems encountered in applications, but also in the case of nonlinear systems of the form

$$\dot{x}(t) = Ax(t) + f(t, x(t)), \tag{1.13}$$

when f—using an established odd term—is of "higher order" with respect to x (say, for instance, $f(t,x) = x^{\frac{3}{2}} \sin t$).

We will conclude this section with the discussion of another important property of motion, encountered in nature and man-made systems. This property is known as *oscillation* or *oscillatory motion*. Historically, the *periodic oscillations* (of a pendulum, for instance) have been investigated by mathematicians and physicists.

Gradually, more complicated oscillatory motions have been observed, leading to the apparition of *almost periodic* oscillations/vibrations. In the third decade of the twentieth century, Harald Bohr (1887–1951), from Copenhagen, constructed a wider class than the periodic one, called *almost periodic*.

In the last decade of the twentieth century, motivated by the needs of researchers in applied fields, even more complex *oscillatory motions* have emerged. In the books by Osipov [432] and Zhang [553, 554], new spaces of oscillatory functions/motions have been constructed and their applications illustrated.

In case of the *Bohr–Fresnel* almost periodic functions, a new space has been constructed, its functions being representable by generalized Fourier series of the form

$$\sum_{k=1}^{\infty} a_k e^{i(\alpha t^2 + \beta_k t)}, \; t \in R, \tag{1.14}$$

with $a_k \in \mathcal{C}$ and $\alpha, \beta_k \in R$, $k \geq 1$.

In the construction of Zhang, the attached generalized Fourier series has the form

$$\sum_{k=1}^{\infty} a_k e^{i q_k(t)}, \; t \in R, \tag{1.15}$$

with $a_k \in \mathcal{C}$, $k \geq 1$, and $q_k \in Q(R)$, $Q(R)$ denoting the algebra of polynomial functions of the form

$$q(t) = \sum_{j=1}^{m} \lambda_j t^{\alpha_j} \text{ for } t \geq 0, \tag{1.16}$$

and $q(t) = -q(-t)$ for $t < 0$; $\lambda_j \in R$, while $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$ denote arbitrary reals.

The functions (on R) obtained by uniform approximation with generalized trigonometric polynomials of the form

$$P_n(t) = \sum_{k=1}^{n} a_k \, e^{i\, q_k(t)} \tag{1.17}$$

are called *strong limit power functions* and their space is denoted by $\mathcal{SLP}(R,\mathcal{C})$.

A discussion of these generalizations of the classical trigonometric series and attached "sum" are presented in Appendix. The research work is getting more and more adepts, contributing to the development of this *third stage* in the history of oscillatory motions/functions.

In order to illustrate, including some applications to FDEs, the role of almost periodic oscillations/motions, we have chosen to present in Chapter 4 only the case of AP_r-almost periodic functions, $r \in [1,2]$, constituting a relatively new class of almost periodic functions, related to the theory of oscillatory motions. Their construction is given, in detail, in Chapter 4, as well as several examples from the theory of FDEs.

Concerning the first two stages in the development of the theory of oscillatory functions, the existing literature includes the treatises of Bary [47] and Zygmund [562]. These present the main achievements of the *first stage* of development (from Euler and Fourier, to contemporary researchers). With regard to the *second stage* in the theory of almost periodic motions/functions, there are many books/monographs dedicated to the development, following the fundamental contributions brought by Harald Bohr. We shall mention here the first books presenting the basic facts, Bohr [72] and Besicovitch [61], Favard [208], Fink [213], Corduneanu [129,156], Amerio and Prouse [21], and Levitan [326], Levitan and Zhikov [327]. These references contain many more indications to the work of authors dealing with the theory of almost periodic motions/functions. They will be mentioned in Section 4.9.

As an example of an almost periodic function, likely the first in the literature but without naming it by its name, seems to be due to Poincaré [454], who dealt with the representations of the form

$$f(t) = \sum_{k=1}^{\infty} a_k \sin \lambda_k t, \; t \in R. \tag{1.18}$$

Supposing that the series converges uniformly to f on R (which situation can occur, for instance, when $\sum_{k=1}^{\infty} |a_k| < \infty$), Poincaré found the formula for the coefficients a_k, introducing simultaneously the concept of mean value of a function on R:

$$M\{f\} = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)\,dt. \tag{1.19}$$

This concept was used 30 years later by H. Bohr, to build up the theory of almost periodic functions (complex-valued). The coefficients were given by the formula

$$a_k = \lim_{\ell\to\infty} (2\ell)^{-1} \int_{-\ell}^{\ell} f(t)\sin\lambda_k t\,dt. \tag{1.20}$$

1.3 METRIC SPACES AND RELATED CONCEPTS

One of the most frequent tools encountered in modern mathematical analysis is a *metric space*, introduced at the beginning of the twentieth century by Maurice Fréchet (in his Ph.D. thesis at Sorbonne). This concept came into being after G. Cantor laid the bases of the set theory, opening a new era in mathematics. The simple idea, exploited by Fréchet, was to consider a “distance” between the elements of an abstract set.

Definition 1.1 *A set S, associated with a map $d : S\times S \to R_+$, is called a metric space, if the following axioms are adopted:*

1) $d(x,y) \geq 0$, *with = only when* $x = y$;
2) $d(x,y) = d(y,x)$, $x,y \in S$;
3) $d(x,y) \leq d(x,z) + d(z,y)$, $x,y,z \in S$.

Several consequences can be drawn from Definition 1.1. Perhaps, the most important is contained in the following definition:

Definition 1.2 *Consider a sequence of elements/points $\{x_n; n \geq 1\} \subset S$. If*

$$\lim_{n\to\infty} d(x_n,x) = 0, \tag{1.21}$$

then one says that the sequence $\{x_n; n \geq 1\}$ converges to x in S.

Then x is called the limit of the sequence.

It is common knowledge that the limit of a convergent sequence in S is *unique*.

Since the concept of a metric space has gained wide acceptance in *Mathematics, Science and Engineering*, we will send the readers to the book of

Friedman [214] for further elementary properties of metric spaces and the concept of convergence.

It is important to mention the fact that the concept of convergence/limit helps to define other concepts, such as *compactness* of a subset $M \subset S$. Particularly, the concept of a *complete* metric space plays a significant role.

Definition 1.3 *The metric space (S,d) is called complete, if any sequence $\{x_n; n \geq 1\}$ satisfying the Cauchy condition, "for each $\epsilon > 0$, there exists an integer $N = N(\epsilon)$, such that $d(x_n, x_m) < \epsilon$ for $n, m \geq N(\epsilon)$, is convergent in (S,d)."*

Definition 1.4 *The metric space (S,d) is called compact, according to Fréchet, iff any sequence $\{x_n; n \geq 1\} \subset S$ contains a convergent subsequence $\{x_{n_k}; k \geq 1\}$, that is, such that $\lim_{k\to\infty} d(x_{n_k}, x) = 0$, for some $x \in S$.*

Definition 1.4 leads easily to other properties of a compact metric space. For instance, the diameter of a compact metric space S is finite: $\sup\{d(x,y); x,y \in S\} < \infty$. Also, every compact metric space is complete.

We rely on other properties of the metric spaces, sending the readers to the aforementioned book of Friedman [214], which contains, in a concise form, many useful results we shall use in subsequent sections of this book. Other references are available in the literature: see, for instance, Corduneanu [135], Zeidler [551], Kolmogorov and Fomin [295], Lusternik and Sobolev [343], and Deimling [190].

Almost all books mentioned already contain applications to the theory of FEs, particularly to differential equations and to integral equations. Other sources can be found in the titles referenced earlier in the text.

The metric spaces are a particular case of *topological spaces*. The latter represent a category of mathematical objects, allowing the use of the concept of *limit*, as well as many other concepts derived from that of limit (of a sequence of a function, limit point of a set, closure of a set, closed set, open set, a.o.)

If we take the definition of a topological space by means of the axioms for the family of *open* sets, then in case of metric spaces the open sets are those subsets A of the space S, defined by the property that any point x of A belongs to A, together with the "ball" of arbitrary small radius r, $\{y : d(x,y) < r\}$.

It is easy to check that the family of all open sets, of a metric space S, verifies the following axioms (for a topological space):

1. The *union* of a family of open sets is also an open set.
2. The *intersection* of a finite family of open sets is also an open set.
3. The space S and the empty set $\emptyset$ belong to the family of open sets.

Such a family, satisfying axioms 1, 2, and 3, induces a topology τ on S. Returning to the class of metric spaces, we shall notice that the couple (S,d) is inducing a topology on S and, therefore, any property of topological nature of this space is the product of the metric structure (S,d). The converse problem, to find conditions on a topological space to be the product of a metric structure, known as *metrizability*, has kept the attention of mathematicians for several decades of the past century, being finally solved. The result is known as the theorem of Nagata–Smirnov.

Substantial progress has been made, with regard to the enrichment of a metric structure, when Banach [39] introduced the new concept of *linear metric space*, known currently as *Banach space*.

Besides the metric structure/space (S,d), one assumes that S is a linear space (algebraically) over the field of reals R, or the field of complex numbers $\mathcal{C}$. Moreover, there must be some compatibility between the metric structure and the algebraic one. Accordingly, the following system of axioms is defining a *Banach space*, denoted $(S,\|\cdot\|)$, with $\|\cdot\|$ a map from S into R_+, $x \to \|x\|$, called a *norm*.

I. S is a linear space over R, in additive notation.

II. S is a *normed* space, that is, there is a map, from S into R_+, $x \to \|x\|$, satisfying the following conditions:

1) $\|x\| \geq 0$ for $x \in S$, $\|x\| = 0$ iff $x = \theta$;
2) $\|ax\| = |a|\ \|x\|$, for $a \in R$, $x \in S$;
3) $\|x+y\| \leq \|x\| + \|y\|$ for $x,y \in S$.

It is obvious that $d(x,y) = \|x-y\|$, $x,y \in S$, is a distance/metric on S.

III. (S,d), with d defined earlier, is a complete metric space.

Also, traditional notations for a Banach space, frequently encountered in literature, are $(B,\|\cdot\|)$ or $(X,\|\cdot\|)$, in the latter case, the generic element of X being denoted by x.

The most commonly encountered Banach space is the vector space R^n (or C^n), the norm being usually defined by

$$\|x\| = (|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$$

and called the Euclidean norm. Another norm is defined by $\|x\|_1 = \max(|x_1|, |x_2|, \ldots, |x_n|)$.

Both norms mentioned previously lead to the same kind of convergence in R^n, because $\|x\|_1 \leq \|x\| \leq \sqrt{n}\ \|x\|_1$, $x \in B$. This is the usual convergence on coordinates, that is, $\lim(x_1, x_2, \ldots, x_n) = (\lim x_1, \lim x_2, \ldots, \lim x_n)$.

A special type of Banach space is the *Hilbert space*. The prototype has been constructed by Hilbert, and it is known as $\ell^2(R)$, or $\ell^2(\mathcal{C})$, space. This fact occurred long before Banach introduced his concept of space in the 1920s. The ℓ^2-space appeared in connection with the theory of orthogonal function series, generated by the Fredholm–Hilbert theory of integral equations with symmetric kernel (in the complex case, the condition is $k(t,s) = \overline{k(s,t)}$). It is also worth mentioning that the first book on Hilbert spaces, authored by M. Stone [508], shortly preceded the first book on Banach space theory, Banach [39], 1932. The Banach spaces reduce to Hilbert spaces, in the real case, if and only if the rule of the parallelogram is valid:

$$\|x+y\|^2 + \|x-y\|^2 = 2\,(\|x\|^2 + \|y\|^2),\ x,y \in B. \tag{1.22}$$

Of course, the parallelogram involved is the one constructed on the vectors x and y as sides.

What is really specific for Hilbert spaces is the fact that the concept of *inner product* is defined for $x,y \in H$, as follows: it is a map from $H \times H$ into R (or $\mathcal{C}$), such that

1) $< x,y > = < y,x >$;
2) $< x,x > \geq 0$, the value 0 leading to $x = \theta$;
3) $< ax+by,z > = a < x,z > + b < y,z >$, with $a,b \in R$, $x,y,z \in H$.

In the complex case, one should change 1) to $< x,y > = \overline{< y,x >}$, 2) remains the same, and 3) must be changed accordingly.

If one starts with a Banach space satisfying condition (1.22), then the inner product is given by

$$< x,y > = \frac{1}{4}\,(\|x+y\|^2 - \|x-y\|^2),\ x,y \in B. \tag{1.23}$$

Conditions 1), 2), and 3), stated in the text, can be easily verified by the product $< x,y >$ given by formula (1.23).

A condition verified by the inner product is known as Cauchy inequality, and it looks

$$|< x,y >| \leq \|x\|\,\|y\|,\ x,y \in H. \tag{1.24}$$

It is easily obtained starting from the obvious inequality $\|x+ay\|^2 \geq 0$, which is equivalent to $\|x\|^2 + 2a < x,y > + a^2\|y\|^2 \geq 0$, which, regarded as a quadratic polynomial in a, must take only nonnegative values. This would be possible only in case the discriminant is nonpositive, that is, $|< x,y >|^2 \leq \|x\|^2\,\|y\|^2$, which implies (1.24). Using (1.24), prove that $\|x+y\| \leq \|x\| + \|y\|$, for any $x,y \in H$.

In concluding this section, we will define another special case of linear metric spaces, whose metric is invariant to translations. These spaces are known as linear Fréchet spaces. We will use them in Chapter 2.

If instead of a norm, satisfying conditions *II* and *III* in the definition of a Banach space, we shall limit the imposed properties to $\|x\| \geq 0$, accepting the possibility that there may be elements $x \neq \theta$, we obtain what is called a semi-norm. One can operate with a semi-norm in the same way we do with a norm, the difference appearing in the part that the limit of a convergent sequence is not necessarily unique.

Here precisely, the semi-norm is defined by the means of the following axioms related to a linear space E:

1) $\|x\| \geq 0$ for $x \in E$;
2) $\|ax\| = |a|\,\|x\|$, $a \in R$, $x \in E$;
3) $\|x+y\| \leq \|x\| + \|y\|$, $x, y \in E$.

In order to define a metric/distance on E, we need this concept: a family of semi-norms on E is called *sufficient*, if and only if from $|x|_k = 0$, $k \geq 1$, there results $x = \theta \in E$.

By means of a countable family/sequence of semi-norms, one can define on E the metric by

$$d(x,y) = \sum_{j=1}^{\infty} 2^{-j} \frac{|x-y|_j}{1+|x-y|_j},\ x,y \in E. \tag{1.25}$$

Indeed, $d(x,y) \geq 0$ for $x,y \in E$ and $d(x,y) = 0$ imply $|x-y|_j = 0$, $j \geq 1$. Hence, $x - y = \theta$, which means that the distance between two elements is zero, if and only if the elements coincide. The symmetry is obvious while the triangle inequality for $d(x,y)$ follows from the elementary inequality for nonnegative reals, $|a+b|(1+|a+b|)^{-1} \leq |a|(1+|a|)+|b|(1+|b|)$.

It is interesting to mention the fact that $d(x,y)$ is *bounded* on $E \times E$ by 1, regardless of the (possible) situation when each $|x|_j$, $j \geq 1$ is unbounded on E. This is related to the fact that a metric $d(x,y)$, on E, generates another *bounded* metric $d_1(x,y) = d(x,y)[1+d(x,y)]^{-1}$, with the same kind of convergence in E.

1.4 FUNCTIONS SPACES

Since our main preoccupation in this book is the study of solutions of various classes of FEs (existence, uniqueness, and local or global behavior), it is useful to give an account on the type of functions spaces we will encounter in subsequent chapters. As proceeded in the preceding sections, we will not provide all the details, but we will indicate adequate sources available in the existing literature.

We shall dwell on the spaces of *continuous functions* on R, or intervals in R, using the notations that are established in literature. Generally speaking, by $C(A,B)$ we mean the space of continuous maps from A into B, when continuity has a meaning. An index may be used for C, in case we have an extra property to be imposed. This is a list of spaces, consisting of continuous maps, we shall encounter in the book.

$C([a,b],R^n)$ will denote the Banach space of continuous maps from $[a,b]$ into R^n, with the norm

$$|x|_C = \sup\{|x(t)|;\, t \in [a,b]\}, \tag{1.26}$$

where $|\cdot|$ is the Euclidean norm in R^n. This space is frequently encountered in problems related to FEs, especially when we look for continuous solutions. But even in case of ODEs, the Cauchy problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, the proof is conducted by showing the existence of a continuous solution to the integral equation $x(t) = x_0 + \int_{t_0}^{t} f(s,x(s))\,ds$. Of course, the differentiability of $x(t)$ follows from the special form of the integral equation, equivalent to Cauchy problem, within the class of continuous functions. A basic property of the space $C([a,b],R^n)$, necessary in the sequel, is the famous criterion of *compactness*, for subsets $M \subset C$, known under the names of Ascoli-Arzelà criterion of compactness in $C([a,b],R^n)$: necessary and sufficient conditions, for the compactness of a set $M \subset C([a,b],R^n)$ are the boundedness of M and the equicontinuity of its elements on $[a,b]$.

The first property means that for the set, M, there exists a positive number, μ, such that $f \in M$ implies $|f(t)| \leq \mu$, $t \in [a,b]$.

The second property means the following: for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that $|f(t) - f(s)| < \epsilon$ for $|t-s| < \delta$, $t,s \in [a,b], f \in M$. It is also called equi-uniform continuity.

Proofs can be found in many textbooks, including Corduneanu [123]. The book by Kolmogorov and Fomin [295] contains the criterion but also an interesting proof of Peano's existence for Cauchy's problem, without transforming the problem into an integral equation, as a direct application of Ascoli-Arzela's result.

Let us notice that the compactness result of Ascoli–Arzelà is actually concerned with the concept of relative compactness.

What happens when the interval $[a,b]$ is replaced by $[a,b)$, or even $[a,\infty)$? The supremum norm used in (1.26) cannot be considered on the semi-open interval $[a,b)$ or on the half-axis $[a,\infty)$.

In this case, in order to obtain a distance between maps defined on $[a,\infty)$, the case $[a,b)$, $b<\infty$, being totally similar to the half-axis, we shall make recourse to the semi-norms

$$|x(t)|_k = \sup\{|x(t)|;\, t\in[a,a+k],\, k\geq 1\}, \tag{1.27}$$

which obviously form a sufficient family. On the linear space of continuous maps, from $[a,\infty)$ into R^n, we have the sufficient family of semi-norms defined by (1.27). Therefore, we can apply formula (1.25), which in this case becomes

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k}\left\{|x(t)-y(t)|_k\,[1+|x(t)-y(t)|_k]^{-1}\right\}, \tag{1.28}$$

with $|x(t)|_k$ defined by (1.27).

Therefore, the linear space of continuous maps, from $[a,\infty)$ into R^n, is a metric space. The property of completeness is the result of the fact that each $|x(t)|_k$ is a norm on the restricted space, $C([a,a+k],R^n)$, and this (plus continuity) implies the completeness of the space under discussion, which shall be denoted by $C([a,\infty),R^n)$ and sometimes by $C_c([a,\infty),R^n)$, the index denoting that the convergence induced by the metric is uniform on compact sets in $[a,\infty)$.

In summary, a metric structure as indicated earlier, is better—and natural—for $C([a,\infty),R^n)$, even though normed space (Banach structures are possible and useful for "parts" of $C([a,\infty),R^n)$.

We shall present now a class of Banach function spaces, denoted $C_g(R_+,R^n)$, where $g: R_+\to(0,\infty)$ is a continuous function, whose role is to serve as a *weight* for the concept of *boundedness*. Namely, $C_g(R_+,R^n)$ is defined by

$$\begin{aligned} C_g(R_+,R^n) = \{&x;\, R_+\to R^n \text{ continuous}\\ &\text{and such that } |x(t)|\leq A_x\, g(t),\, t\in R_+,\, A_x>0\}. \end{aligned} \tag{1.29}$$

The norm in $C_g(R_+,R^n)$ is given by

$$|x|_{C_g} = \sup\left\{\frac{|x(t)|}{g(t)};\, t\in R_+\right\}. \tag{1.30}$$

Obviously, $|x|_{C_g} = \inf A_x$, with A_x in (1.29). It is shown (see, for instance, Corduneanu [120]) that $C_g(R_+,R^n)$ is a Banach space, with the norm given by (1.30). The special case $g(t)\equiv 1$ on R_+ leads to the space of bounded functions on R_+, with values in R^n, the norm being the supremum norm. This space of bounded continuous functions on R_+ is denoted by $BC(R_+,R^n)$. It contains as subspaces several important Banach spaces, from R_+ into R^n, such as the space of functions with limit at ∞, $\lim_{t\to\infty} x(t) = x_\infty \in R^n$, which is encountered when we deal with the so-called *transient* solutions to FEs. This space is usually denoted by $C_\ell(R_+, R^n)$, and it is isomorphic and isometric to the space $C([0,1], R^n)$. Prove this statement! The subspace of $C_\ell(R_+, R^n)$, for which $\lim_{t\to\infty} x(t) = \theta$ is denoted $C_0(R_+, R^n)$, and it is the space of *asymptotic stability* (each motion, described by elements of $C_0(R_+, R^n)$, tends to the equilibrium point $x_\infty = \theta$). We will also deal with the subspace of space $BC(R, R^n)$, known as the space of almost periodic functions on R, with values in R^n (Bohr almost periodicity). It is denoted by $AP(R, R^n)$ and contains all continuous maps from R into R^n, such that they can be uniformly approximated on R by vector trigonometric polynomials: for each $\epsilon > 0$ and $x \in AP(R, R^n)$, there exists vectors $a_1, a_2, \ldots, a_n \in R^n$ and reals $\lambda_1, \lambda_2, \ldots, \lambda_m$, such that

$$\left| x(t) - \sum_{k=1}^{m} a_k e^{i\lambda_k t} \right| < \epsilon,\ t \in R. \tag{1.31}$$

Inequality (1.31) shows that $x \in AP(R, R^n)$ is as close as we want from oscillatory functions. For the classical types of almost periodic functions, see the books authored by Bohr [72], Besicovitch [61], Favard [208], Levitan [326], Corduneanu [129, 156], Fink [213], Amerio and Prouse [21], Levitan and Zhikov [327], Zaidman [547], and Malkin [355]. Appendix to this book will be dedicated to some new developments (not necessarily continuous functions).

We shall continue to enumerate function spaces, this time, having in mind the *measurable functions/elements*. These spaces of great importance in the development of modern analysis have appeared at the beginning of the past century, primarily due to Lebesgue’s discovery of measure theory. Actually, the first function spaces amply investigated in the literature are known as Lebesgue’s spaces or L^p-spaces.

The space $L^p(R, R^n)$, $p \geq 1$, is the linear space of all measurable maps from R into R^n, such that $\int_R |x(s)|^p\, ds < \infty$, the ds representing the Lebesgue measure on R. The norm of this space is

$$|x|_{L^p} = \left\{ \int_R |x(s)|^p\, ds \right\}^{\frac{1}{p}},\ 1 \leq p < \infty. \tag{1.32}$$

The case $p = \infty$ is characterized by

$$|x|_{L^\infty} = \text{ess-}\sup_{t\in R} |x(t)| < \infty. \tag{1.33}$$

The theory of these Banach spaces, whose elements are, in fact, equivalence classes of functions (i.e., two functions are equivalent, if and only if they coincide, except on set of points of Lebesgue measure zero) is largely diffused in many books/textbooks available. Let us indicate only Yosida [541], Lang [320], and Amann and Escher [20].

Let us consider now the problem of compactness (or relative compactness) in the space $L^p([a,b],R^n)$, providing a useful result (Riesz):

Let $M \subset L^p([a,b],R^n)$, $1 < p < \infty$, be a subset; necessary and sufficient conditions for the (relative) compactness of M are as follows:

(1) M is a bounded set in L^p, that is, $|x|_{L^p} \leq A < \infty$, for each $x \in L^p$;
(2) $\lim \int_a^b |x(t+h) - x(t)|^p\, dt = 0$ as $h \to 0$.

We notice the fact that $x(t+h)$ must be extended to be zero, outside $[a,b]$.

There are other criteria of compactness in L^p-spaces, due to Kolmogorov a.o. See, for instance, the items mentioned already, in this section, or Kantorovich and Akilov [274].

Some results concerning L^p-spaces, with a weight function, have been obtained by Milman [399], in connection with stability theory for integral equations.

Also, Kwapisz [305] introduced and applied to integral equations normed spaces of measurable functions, with a mixed norm, such as $x \to \sup\{g(t) \int_0^t |x(s)|\, ds\}$, on finite intervals or on the semiaxis R_+. Such spaces are useful when fixed-point theorems are used for existence of solutions to FEs. (See Kwapisz [304, 305]).

From the L^p-spaces theory, many other classes of measurable functions have been constructed, with important applications to the theory of FEs. An example, frequently appearing in literature, are the spaces L^p_{loc}. These spaces occur naturally when dealing with global existence of solutions.

For instance, the space $L^2_g(R,R^n)$ will consist of all measurable maps from R into R^n, such that $x \in L^2_g$ is determined by

$$\int_R g(t)\, |x(t)|^2\, dt < \infty, \tag{1.34}$$

where $g(t)$ is measurable from R to R_+. The norm adequate for this space is, obviously,

$$x \to \left\{ \int_R g(t)\, |x(t)|^2\, dt \right\}^{\frac{1}{2}}. \tag{1.35}$$

By using various weight functions g, one can achieve more generality in regard to the behavior/global properties of the solution.

Finally, we will mention the definition of $L^2_{\text{loc}}(R,R^n)$. This is a linear metric space (Fréchet), which belongs to the larger class of linear locally convex topological spaces.

In order to obtain the linearly invariant distance function for $L^2_{\text{loc}}(R, R^n)$, which consists of all locally integrable maps from R into R^n, that is, such that each integral

$$\int_{|t|\leq k} |x(s)|^2\, dt < +\infty,\ k \geq 1,$$

we will use a formula similar to (1.28).

Since

$$|x_k| = \left\{ \int_{|t|\leq k} |x(s)|^2\, ds \right\}^{\frac{1}{2}},\ k \geq 1,$$

is a semi-norm, the distance function on $L^2_{\text{loc}}(R, R^n)$ will be given (by definition)

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} |x-y|_k \,(1 + |x-y|_k)^{-1}. \tag{1.36}$$

The convergence in $L^2_{\text{loc}}(R, R^n)$ is, therefore, the L^2-convergence on each finite interval of R (or, on each compact set in R).

This definition for $L^2_{\text{loc}}(R, R^n)$ can be extended from R^n to Hilbert spaces, or even to Banach spaces.

The variety of function spaces encountered in investigating the solutions of functional differential equations is considerable, and we do not attempt to give a full list. We will mention one more category of function spaces, containing the L^p-spaces, for the sake of their frequent use in the theory of FDE. Apparently, these spaces have been first used by N. Wiener. Their definition and systematic use is given in Massera–Schäffer [374].

The space $M(R_+, R^n)$ consists of all locally measurable functions on R_+, such that

$$|x|_M = \sup_{t \in R_+} \left\{ \int_t^{t+1} |x(s)|\, ds \right\} < \infty.$$

The space $M_0(R_+, R^n) \subset M(R_+, R^n)$ with the same norm contains only those elements for which

$$\sup \left\{ \int_t^{t+1} |x(s)|\, ds \right\} \to 0 \; as\; t \to \infty.$$

Such functions are important in connection with the concept of *almost periodicity* and other applications in the theory of FDE.

1.5 SOME NONLINEAR AUXILIARY TOOLS

The development of Functional Analysis has brought to the investigation of various classes of FEs, such as integral equations, for instance, but later to the theory of partial differential equations, many tools and methods. Volterra, who started the systematic investigation of integral equations in the 1890s, considered the problem of existence for these equations as the inversion of an integral operator (generally nonlinear). At the beginning of the twentieth century, Fredholm created the theory of linear integral equations, with special impact on the spectral side. Hilbert went further in this regard, contributing substantially to the birth of the theory of orthogonal series, related to symmetric/hermitian kernels of the Fredholm type equations. Due to remarkable contributions from Fréchet, Riesz (F and M), and other mathematicians, the new field of functional analysis (i.e., dealing with spaces/classes whose elements are functions) made substantial progress, but it was centered around the linear problems.

Starting with the third decade of the twentieth century, essentially *nonlinear* results have appeared in the literature. One of the first items in nonlinear functional analysis is the theory (or method) of fixed points, generally speaking for *nonlinear* operators/maps. The history takes us back to Poincaré and Brouwer, when only finite dimensional (Euclidean) spaces were involved. The problem of fixed points was considered by mathematicians starting the third decade of the twentieth century, during which period both best known results have been obtained.

The *contraction mapping principle* was the first tool, due to Banach in the case of what we call now Banach spaces. The case of complete metric spaces, as it is usually encountered nowadays, is due to V. V. Nemytskii (*Uspekhi Mat. Nauk*, 1927). This principle, usually encompassing all the results that can be obtained by the iteration method (successive approximations), can be stated as follows:

Let (S,d) denote a complete metric space and $T : S \to S$ a map, such that

$$d(Tx,Ty) \leq a\,d(x,y),\ x,y \in S, \tag{1.37}$$

for fixed a, $0 \leq a < 1$. Then, there exists a unique $x^* \in S$, such that $Tx^* = x^*$.

The proof of this statement is based on the iterative process

$$x_{k+1} = Tx_k,\ k \geq 0,\ x_0 \in S, \tag{1.38}$$

the initial term x_0 being arbitrarily chosen in S. One easily shows that $\{x_k; k \geq 0\}$ is a Cauchy sequence in (S,d), whose limit is x^*. The uniqueness follows directly by application of (1.37).

Proofs are available in many books on Functional Analysis or FEs (see, for instance, Corduneanu [120, 135]).

Let us mention that an estimate for the "error" is easily obtained, namely

$$d(x^*, x_k) \leq a^k (1-a)^{-1} d(x_0, Tx_0),\ k \geq 1. \tag{1.39}$$

The fixed point theorem due to *Schauder* is formulated for Banach spaces and can be stated as follows:

Let B be a Banach space (over reals) and $T : B \to B$ a continuous map/operator such that

$$TM \subset M, \tag{1.40}$$

with $M \subset B$ a closed convex set, while TM is relatively compact. Then, there exists at least a fixed point of T, that is, $Tx^* = x^* \in M$.

The definition of a *convex* set, say M, is $x, y \in M$ implies $ax + (1-a)y \in M$, for each $a \in (0,1)$.

For a proof of the Schauder fixed-point result, see the proof of a more general result (the next statement) in Corduneanu [120]. *Schauder–Tychonoff* fixed point theorem. Let E be a locally convex Hausdorff space and $x \to Tx$ a continuous map, such that

$$TK \subset A \subset K \subset E, \tag{1.41}$$

where K denotes a convex set, A being compact. Then, there exists at least one $x^* \in A \subset K$, such that $Tx^* = x^*$.

Both fixed point results formulated already imply the relative compactness of the image $TK \subset A$, and we mention the fact that uniqueness does not, generally, hold.

The set of fixed points of a map, under appropriate conditions, satisfies certain interesting and useful properties, such as compactness, convexity, and others.

We invite the readers to check the property of compactness, under aforementioned conditions. Another fixed-point result, known as the *Leray–Schuader Principle* also involves the concept of compactness of the operator, but also the idea of an "a priori" estimate for searched solution. Namely, one considers the equation

$$x = Tx,\ x \in B, \tag{1.42}$$

with B a Banach space and $T : B \to B$ a compact operator (i.e., taking bounded sets in B, into relatively compact sets). One associates to (1.42) the parameterized equation

$$x = \lambda Tx,\ \lambda \in [0,1], \tag{1.43}$$

assuming that (1.43) is solvable in B for each $\lambda \in [0,1)$. Moreover, each solution x satisfies the "a priori" estimate

$$|x|_B \leq K < \infty,\ \lambda \in [0,1), \tag{1.44}$$

where K is a fixed number. Then, equation (1.42), which corresponds to $\lambda = 1$ in (1.43), possesses a solution in B.

The proof of the principle can be found in Brézis [77] and Zeidler [550,551], including also some applications.

Several other methods/principles in functional analysis are known and widely applied in the study (particularly, in existence results) of various classes of FEs. We will mention here the method based on *monotone operators* (see Barbu [45], Deimling [190], and Zeidler [551]).

The definition of a *strongly monotone* operator $A : H \to H$, with H a Hilbert space, is

$$< Ax - Ay, x - y >_H \geq m\,|x - y|_H^2,\ x, y \in H,$$

for some $m > 0$. A very useful result can be stated as follows:

Consider in H the equation

$$Ax = y, \tag{1.45}$$

with $A : H \to H$ strongly monotone. Further, assume A satisfies on H the Lipschitz condition

$$|Au - Av|_H \leq L\,|u - v|_H. \tag{1.46}$$

Then, for each $y \in H$, Equation (1.45) has a unique solution $x \in H$.

The proof of this result can be found in Zeidler [550], and it is done by means of Banach fixed point (contraction).

An alternate statement of the solvability property of (1.45), for any $y \in H$, is obviously the property of A to be onto H (or surjective).

A somewhat similar result, still working in a Hilbert space H, can be stated as follows: Equation (1.45) is solvable for each $y \in H$, when A is continuous, monotone, that is,

$$< Ax - Ay, x - y >_H \geq 0,\ x, y \in H, \tag{1.47}$$

and *coercive*

$$\lim < Ax, x >_H\ |x|_H^{-1} = \infty,\ as\ |x| \to \infty. \tag{1.48}$$

See the proof, under slightly more general conditions, in Deimling [190] or Barbu and Precupanu [46].

All aforementioned references, in regard to monotone operators, contain applications to various types of FEs.

Other methods/procedures leading to the existence of solutions of various classes of FEs are based on diverse form of the *implicit functions* theorem (in Banach spaces and Hilbert Spaces); see Zeidler [550].

When we deal with FDE with finite delay, which we will consider in Chapters 2 and 3, we use another constructive method called the *step method*. We briefly discuss this method, which is frequently used to construct solutions to FDE of the form

$$\dot{x}(t) = F(t, x(t), x(t-h)),\ x \in R^n, \tag{1.49}$$

with $h > 0$ the delay, or time delay. In order to make the first step in constructing the solution, it is necessary, assuming the initial moment is $t = 0$, to know $x(t)$ on the interval $[-h, 0]$. In other words, one has to associate to (1.49) the initial condition $x(t) = \phi(t)$, $t \in [-h, 0]$. Using the notation $x_t(s) = x(t+s)$, $s \in [-h, 0]$, this condition can be written in the form

$$x_0(t) = \phi(t),\ t \in [-h, 0]. \tag{1.50}$$

If we assign the initial function ϕ from a certain function space, say $C([-h, 0], R^n)$, then equation (1.49) becomes an ODE, on the interval $[0, h]$:

$$\dot{x}(t) = F(t, x(t), \phi(t-h)),\ t \in [0, h], \tag{1.51}$$

and the initial condition at $t = 0$ will be

$$x(0) = \phi(0). \tag{1.52}$$

The *second step*, after finding, from (1.51) to (1.52), $x(t)$ on $[0, h]$ will require to solve the ODE (1.49) on $[h, 2h]$, starting at $x(h)$, as found from $x(t)$, on $[0, h]$.

The process continues and, at each step, one finds $x(t)$, on an interval $[mh, (m+1)h]$, $m \geq 1$, solving the ODE (1.49),

$$\dot{x}(t) = F(t, x(t), x(t-h)),\ t \in [mh, (m+1)h], \tag{1.53}$$

under initial condition at mh, as determined from the preceding step. In this way, the solution $x(t)$ of (1.49), appears as a chain, say

$$x_1(t), x_2(t), \ldots, x_m(t), \ldots \tag{1.54}$$

each term in (1.54) being found from (1.53), for different values of m, taking as initial value for $x_m(t)$, the final value of $x_{m-1}(t)$, $m \geq 1$. In this way, one obtains a continuous solution on $[-h,T)$ for (1.49), where T denotes the largest value of T, such that the solution is defined ($T = \infty$, when the solution is global on R_+).

The method of integration by steps has several variants, and in order to make it more convenient for numerical purposes, one uses the construction of the chain by means of the recurrent equation

$$\dot{x}_{m+1}(t) = F(t, x_m(t), x_{m-1}(t-h)),\ m \geq 1,$$

which allows making each step by performing a single quadrature.

It is remarkable that the step method allows investigation of global problems, such as stability. See Kalmar-Nagy [273] for illustration.

1.6 FURTHER TYPES OF FEs

The mathematical literature has been enriched, by other types of FEs than those usually encountered in the classical period. We have in mind, particularly, the various classes of *discrete* equations, the *fractional-order differential* equations and the *difference equations*. Discrete equations have been largely investigated. Modern computers have been used to perform computational procedures on those equations, which have represented mathematical models with large number of variables and calculations.

It is true that some FEs, pertaining to the aforementioned types, have been involved in research (may be only sporadically) and we can mention here a cycle of papers due to Bochner [68–71] from 1929 to 1931, who investigated the almost periodicity of some classes of equations with differential, integral and difference operations. A very simple example is, for instance, the FE

$$\dot{x}(t) + ax(t-h) + \int_0^t A(t-s)x(s)\,ds = f(t),$$

for which an initial condition would be of the form $x_0(t) = \phi(t)$, $t \in [-h, 0]$, $h > 0$. See also Hale and Lunel [242].

In this section, we will only illustrate some types of equations belonging to those three categories mentioned above and state some results available in the literature.

Let us start by introducing some examples of *discrete* equations, by providing the necessary concepts leading to their solution belonging to various spaces of sequences.

One example is constructed by the space $s = s(N,R^n)$ or $S(Z,R^n)$ of all sequences $\xi = \{x_k; x_k \in R^n, k \geq 1\}$. These sequences form a Fréchet space, either real or complex, with the distance (compare with (1.36)),

$$d(\xi,\eta) = \sum_{k=1}^{\infty} 2^{-k} |\xi_k - \eta_k| \, (1 + |\xi_k - \eta_k|)^{-1}. \tag{1.55}$$

In case of sequences on Z, the sum must be considered from $-\infty$ to $+\infty$.

Within the space s, one can consider several sequence spaces, such as the space of all *bounded* sequences $s_b(N,R^m)$, respectively, $s_b(Z,R^m)$, with the supremum norm, its subspaces of *almost periodic* sequences or $s_\ell(N,R^m)$ which consists of all elements ξ of $s_\ell(N,R^m)$, such that $\lim \xi_k$ as $n \to \infty$ exists (finite!), (convergent sequences) or the subspace, traditionally denoted by $c_0(N,R^m)$, containing only those elements from $s_\ell(N,R^m)$ for which $\lim \xi_k = \theta \in R^m$, the null vector of R^m. Obviously, these concepts make sense when ξ belongs to a Banach space, in particular to Hilbert space $\ell^2(N,R^m)$.

Let E be a real Banach space and consider the operator equation, in discrete form

$$(Lx)(n) = (Gx)(n), \; n \in N, \tag{1.56}$$

where L denotes a linear operator on E, while G stands for a nonlinear operator on E. Equation (1.56) is a neutral one (it means, not solved in respect to the unknown element x). Under some conditions, we can reduce (1.56) to a normal form, namely

$$x(n) = (Fx)(n), \; n \in N, \tag{1.57}$$

which can be treated easier. Moreover, equations like (1.57) have caught the attention of researchers long time ago, the number of results being considerably higher than in case of (1.56).

If L has a bounded inverse operator on E, a situation warranted by the condition

$$|Lx|_E \geq m\,|x|_E, \; \forall x \in E, \; m > 0, \tag{1.58}$$

then equation (1.56) is equivalent to (1.57), with $F = L^{-1}G$. The term equivalent must be understood in the sense they have the same solutions (if any).

We assume that the "nonlinear" operator G satisfies the Lipschitz continuity:

$$|Gx - Gy|_E \leq K\,|x-y|_E, \; K > 0, \; x,y \in E. \tag{1.59}$$

Since $F = L^{-1}G$, we find easily

$$|Fx - Fy|_E = |L^{-1}(Gx - Gy)|_E \leq m^{-1}K|x - y|_E, \tag{1.60}$$

since $|L^{-1}|_E \leq m^{-1}$ (if in (1.58) we let $x = Ly$, $y \in E$, then we obtain $|Ly|_E \leq m^{-1}|y|_E$). From (1.60) one obtains for $x, y \in E$,

$$|Fx - Fy|_E \leq m^{-1}K|x - y|_E, \tag{1.61}$$

which implies the property of contraction (Banach) when

$$K < m. \tag{1.62}$$

Therefore, equation (1.57) has a unique solution in E. This property is then true for (1.56).

We will now apply this existence result, to obtain conditions for the existence of a solution $x = (x_1, x_2, \ldots, x_m, \ldots)$ in one of the spaces (of sequences) listed in the text, say $\ell^2(N, R^n)$. It is known that this is the space Hilbert used for constructing the infinite dimensional analysis (and geometry!). It is the prototype of separable Hilbert spaces (containing a countable subset, everywhere dense in it). Namely, we consider the infinite system of quasilinear equations

$$x_k = \sum_{j=1}^{\infty} a_{kj}x_j + f_k(x),\ k \geq 1, \tag{1.63}$$

where a_{kj} are real numbers and $x \to f_k(x)$, $k \geq 1$, are maps from E into itself. Obviously, (1.63) can be rewritten in concise form as

$$x = Ax + f(x), \tag{1.64}$$

with A the double infinite matrix

$$A = (a_{ij}),\ i,j = 1, 2, \ldots$$

and $f(x) = (f_1(x), f_2(x), \ldots, f_m(x), \ldots)$. It is quite obvious that (1.64) is of the same form as (1.57). Hence, we have to look for conditions assuring the contraction, on ℓ^2, of the operator on the right-hand side of (1.64). First, we shall assume an inequality of the form

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} |a_{kj}|^2 \leq \alpha^2 < \infty. \tag{1.65}$$

This guarantees that the operator A in (1.64) is bounded on $\ell^2(N,R^n)$. Concerning the nonlinear term $f(x)$, we assume that

$$\sum_{k=1}^{\infty} |f_k(x)|^2 < \infty,\ x \in \ell^2(N,R^n), \tag{1.66}$$

and the Lipschitz conditions hold

$$|f_k(x) - f_k(y)| \leq \beta_k\, |x-y|,\ x,y \in \ell^2,$$

with

$$\sum_{k=1}^{\infty} {\beta_k}^2 = \beta^2 < \infty. \tag{1.67}$$

The assumptions (1.65)–(1.67), allows us to write (with the ℓ^2-norm)

$$\begin{aligned} |Ax + f(x) - Ay - f(y)| &= |A(x-y) + f(x) - f(y)| \\ &\leq |A(x-y)| + |f(x) - f(y)| \\ &\leq (\alpha + \beta)|x-y|, \end{aligned}$$

which implies the contraction of the operator $x \to Ax + f(x)$ on $\ell^2(N,R^n)$, as soon as

$$\alpha + \beta < 1. \tag{1.68}$$

We can conclude that, under assumptions (1.65)–(1.68), the infinite system (1.63) (or (1.64)) has a unique solution in the space $\ell^2(N,R^n)$. This solution consists of a sequence of vectors $x^{(k)}$, $k \geq 1$, in the Hilbert space $\ell^2(N,R^n)$.

This discussion provides an elementary example of the kind of problems that may occur in applications. The discrete equations are adequate for treatment by means of the electronic computers. The literature in this field is very rich and various aspects and problems can be found in Collatz [108].

The following remark is related to condition (1.66). Namely, if (1.66) holds for an element $y \in \ell^2$, then it will be valid for any $x \in \ell^2$. This is a consequence of the obvious inequality

$$|f_j(x)|^2 \leq 2\,(|f_j(y)|^2 + \beta_j^2\, |x-y|^2),\ j \geq 1,$$

relying also on (1.67).

Another remark is that $x \in \ell^2(N,R^n)$ implies $x_m \to \theta \in R^n$, which implies the fact that only a finite number of coordinates of the solution can provide sufficient information about it (with regard to the numerical treatment of the problems).

We invite the readers to obtain the existence of a solution of the equation

$$\Delta f(x,m) = g(x,m),\ m \geq 0, \tag{1.69}$$

where $\Delta f(x,m) = f(x_{m+1}, m+1) - f(x_m, m)$, the sequence x_m, $m \geq 0$, being constructed recurrently with an arbitrary $c_0 \in s_b(N, R^n)$. In other words, we need conditions to obtain the existence of a bounded sequence $\{x_m; m \geq 0\} \subset s_b(N, R^n)$ for equation (1.69), which is of a neutral type. One uses the supremum norm.

The following conditions assure the existence of a unique solution:

1) $f : N \times R^n \to R^n$ is such that the equation

$$f(u,m) = v \in R^n$$

has a unique solution, say $u_m \in R^n$, for each v;

2) f satisfies the condition

$$|f(u,m)| \geq h(|u|),\ m \geq 1,$$

with $h(r)$, $0 \leq r < \infty$, monotonically increasing, while $h^{-1}(r)$ has sublinear growth,

$$h^{-1}(r) \leq \alpha r + \beta,\ r \geq 0,\ \alpha, \beta > 0;$$

3) $g : N \times R^n \to R^n$ satisfies

$$|g(u,m)| \leq c_m\, |u|,\ n \geq 0,\ u \in R^n,$$

where the series $\sum_{k=1}^{\infty} c_k$ is convergent.

Then, equation (1.69) has a unique solution in $s_b(N, R^n)$, for each initial choice $x_0 \in R^n$.

Details of the proof can be found in the paper by Corduneanu [146], together with other existence results.

Let us notice, with regard to the discrete FEs, that a rich literature is available. We mention some of the best known sources/books, in which a large variety of such equations (frequently including recurrent ones) is treated, with both classical or modern tools. Mickens [395], Kelley and Peterson [284], and Pinney [451].

The *difference equations* are adequate to be treated as discrete ones, as in above shown examples, or they can be considered in the class of equations with continuous argument. Let us look at the very elementary example of the equation $x(t+1) = 2x(t)$, on the real axis R. Then, looking for an exponential solution $x(t) = a^t$, $a > 0$, one finds that each function of the form $x(t) = c\,2^t$, with $c \in R$, is a solution depending on the continuous argument $t \in R$.

This second manner of looking at solutions depending on a continuous argument had enjoyed early attention in the literature, and we shall now dwell on these types of solutions. Following Shaikhet [493], we shall consider the scalar difference equation, somewhat simplified, namely

$$x(t+h_0) = F(t, x(t), x(t-h_1), \ldots, x(t-h_m)), \tag{1.70}$$

on the semiaxis $t > t_0 - h_0$, under initial conditions

$$x(s) = \phi(s),\ s \in S = [t_0 - h_0 - \max_{1\le k\le m}(h_k, t_0)]. \tag{1.71}$$

In (1.71), $x \in R$, $h_k > 0$, $1 \le k \le m$, $\phi : S \to R$ being the initial datum, usually continuous. The following growth condition is imposed on F:

$$|F(t, x_0, x_1, \ldots, x_m)| \le \sum_{j=0}^{m} a_j\,|x_j|, \tag{1.72}$$

with $a_j \in R_+ \setminus \{0\}, j = 0, 1, \ldots, m.$

We shall denote by $x(t; t_0, \phi)$ the solution of (1.70), (1.71). It is obtained by the step method, first on $[t_0, t_0 + h_0]$, by substituting ϕ to x on the right-hand side, then on $[t_0 + h_0, t_0 + 2h_0]$, substituting $x(t)$ already found on $[t_0, t_0 + h_0]$ on the right-hand side, and so on. This process uniquely determines $x(t)$, for $t > t_0 - h_0$.

Now we will anticipate on the concept of stability, which will be defined and developed in Chapter 3. More precisely, we shall formulate as follows the definitions of stability and asymptotic stability of the solution $x = 0$ of (1.70).

We notice first, on behalf of assumption (1.72), that $x = 0$ is a solution of (1.70).

Stability of $x = 0$ means that to each $\epsilon > 0$, $t_0 \ge 0$, there corresponds $\delta = \delta(\epsilon, t_0) > 0$, such that

$$|x(t; t_0, \phi)| < \epsilon \text{ for } t \ge t_0, \tag{1.73}$$

provided $|\phi|_c = \sup_{s\in S}|\phi(s)| < \delta(\epsilon, t_0)$.

Asymptotic stability means stability, and additionally,

$$\lim_{t\to\infty} x(t:t_0,\phi) = 0, \tag{1.74}$$

for all ϕ, such that $|\phi|_c < \eta(t_0)$.

We are now able to formulate a stability result for the zero solution of the difference equation (1.70), which is an example of an equation with difference and continuous argument.

Theorem 1.1 *Assume the existence of a functional (Liapunov type) $V(t) = V(t,x(t),x(t-t_1),\ldots,x(t-t_m))$, nonnegative and such that, for some positive c_1,c_2,p, one has*

$$V(t) \le c_1 \sup_{s\le t}|x(s)|^p,\ t\in[t_0,t_0+h_0), \tag{1.75}$$

$$\Delta V(t) = V(t+h_0) - V(t) \le c_2\,|x(t)|^p,\ t\ge t_0. \tag{1.76}$$

Then $x = 0$ is a stable solution for equation (1.70).

Proof. The conditions (1.75), (1.76) and the nonnegativity of $V(t)$ imply

$$V(t) \ge c_2\,|x(t)|^p,\ t\ge t_0, \tag{1.77}$$

$$V(t) \le V(t-h_0) \le V(t-2h_0)\ldots \le V(s),\ t\ge t_0, \tag{1.78}$$

with $s = t - \left[\frac{t-t_0}{h_0}\right] h_0 \in [t_0, t_0+h]$, which implies

$$\sup_{s\in[t_0,t_0+h_0]} V(s) \le c_1 \sup_{t\le t_0+h_0} |x(t)|^p. \tag{1.79}$$

After taking into account (1.70)–(1.72), we obtain for $t\le t_0+h_0$,

$$\begin{aligned}|x(t)| = |F(t,x(t),\ldots)| &\le a_0\,|\phi(t-t_0)| + \sum_{j=1}^m a_j\,|\phi(t-t_0-h_j)|\\ &\le \left(\sum_{j=0}^m a_j\right)\,|\phi|_c\\ &= A\,|\phi|_c,\end{aligned} \tag{1.80}$$

where $A = \sum_{j=0}^m |a_j| > 0$, which leads easily to the inequality

$$c_2\,|x(t)|^p \le c_1 A^p\,|\phi|_c^p,\ t\ge t_0, \tag{1.81}$$

relying also on (1.77)–(1.80).

The inequality (1.81), obtained for the solution $x(t) = x(t; t_0, \phi)$, shows that the stability is assured.

Moreover, imposing a rather restrictive condition, namely $A < 1$, the solution $x = 0$ of equation (1.70) is asymptotically stable. To obtain this conclusion, one should rely on (1.81) and $A < 1$, which leads to the inequality

$$|x(t)| \leq A^{\left[\frac{t-t_0}{h_0}\right]} |\phi|_c, \ t \geq t_0, \tag{1.82}$$

which assures (1.74) because $[\frac{t-t_0}{h_0}] \to \infty$ as $t \to \infty$. Of course $[\,.\,]$ denotes the integer part function. This inequality is obtained following the same steps as in case of deriving (1.80).

It is adequate to notice the fact that, in Shaikhet [493], one finds further similar results of stability, or integrability, as well as some procedures to construct Liapunov functions for difference equations. See also Kolmanovskii and Shaikhet [294], for related results.

In concluding this section, we will briefly illustrate some results and methods in the case of another class of FEs that came relatively recently under the scrutiny of researchers. We have in mind the class of *fractional differential equations*. The literature is rather vast, and we notice the apparition of a fractional calculus, for instance the one based on Caputo's concept of a fractional order derivative. This concept, similar to that known as Riemann–Liouville fractional-order derivative, implies the use of the operation of integration. Therefore, there is a rather close connection between the fractional differential equations and the (singular) integral or IDEs. It is worth mentioning that this new direction of research has been brought into actuality by its engineering applications (see, for instance, Caputo [88]).

An important distinction should be made when we deal with difference equations and look for solutions, which depend on a continuous variable or a discrete variable. Let us look again at the very simple equation, $x(t+1) = 2x(t)$, we considered earlier, in this section, for which $x(t) = c2^t$ is a family of continuous solutions on R, while regarded as a discrete one, the result is that each geometric progression $\{2c, 2^2c, \ldots, 2^mc, \ldots\}$ is a solution, defined on the positive integers. Apparently, the discrete variable is preferred in many papers, due to the fact that its results are most adequately handled by digital machines. And this aspect is stressed by the new tendency in mathematical modeling, with emphasis on discrete models.

Returning now to the class of fractional differential equations, based on the concept of fractional-order derivative, we will notice that its roots are in the work of the nineteenth century well known mathematicians Liouville and Riemann. Probably attracted by the beauty and usefulness of Euler's Γ (gamma) function, they discovered the fractional derivative of a function (also the fractional integral), which can be defined by means of Euler's function.

The Riemann–Liouville derivative, of (fractional) order q, of a function $f : R_+ \setminus \{0\} \to R$ is given by the formula

$$D^q_{0+}f(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)\,ds}{(t-s)^{q+1-n}},\ n = [q]+1, \tag{1.83}$$

where $[q]$ means the integer part of q.

The fractional derivative, introduced more recently by Caputo, is represented by the formula

$$D^\alpha_c f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s)\,ds, \tag{1.84}$$

with $n = [\alpha]+1$. It is assumed that $f(t)$ is n times differentiable on $R_+ \setminus \{0\}$.

There are other types of fractional derivatives, with frequent use, especially in applications. See the books by Miller and Ross [397] and Das [187], in which details and applications are provided. We shall illustrate the type of results obtained in the literature (both mathematical, i.e., theoretic and applied). The following result is taken from recent papers of Cernea [95, 96] stating the existence of continuous selections for initial value problems of the form

$$D^\alpha_c x(t) \in F(t, x(t)),\ a.e. \text{ on } [0,T], \tag{1.85}$$

under initial conditions

$$x(0) = x_0,\ x'(0) = x_1,$$

with $\alpha \in (1,2]$, $F : [0,T] \times R \to \mathcal{P}(R)$ a set valued map whose values are parts/subsets of R. Moreover, it is assumed that $x_0, x_1 \neq 0$. The following hypotheses are assumed:

H_1. The map $F : [0,T] \times R \to \mathcal{P}(R)$ has nonempty closed set-values and is measurable in $L([0,T]) \times B(R)$, where B denotes Borel measurability.

H_2. There exists $L(\cdot) \in L^1([0,T], R_+$, such that $a.e.$ on $t \in [0,T]$, the Lipschitz type condition

$$d_H(F(t,x), F(t,y)) \leq L(t)\,|x-y|, \tag{1.86}$$

for $(t,x), (t,y) \in [0,T] \times R$, where d_H denotes the Pompeiu–Hausdorff distance in $\mathcal{P}(R)$.

H_3. Let S be a separable metric space, and $a, b : S \to R$, $c : S \to (0, \infty)$ continuous maps. There exist continuous mappings $g, p : S \to L^1([0,T], R)$, and $y : S \to C([0,T], R)$, such that

$$(Dy(s))_c^\alpha(t) = g(s)(t),\, d(g(s)(t), F(t, y(s)(t)) \leq p(s), \tag{1.87}$$

a.e. for $t \in [0,T]$, $s \in S$.

In order to state the existence result, we need to introduce the following notation

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,\ \alpha > 0, \tag{1.88}$$

where $f : [0,\infty) \to R$ is locally Lebesgue integrable. This integral is known as the fractional integral of order $\alpha > 0$.

Let us now define the auxiliary function

$$\xi(s) = (1 - |I^\alpha L(t)|_c^{-1})\,[|a(s) - y(s)(0)| + T|b(s) - y'(s)(0)| \\ + c(s) + |I^\alpha p(s)|_c],\ s \in S. \tag{1.89}$$

The following result is an answer to the existence problem, formulated already, for (1.85):

Theorem 1.2 *Under assumptions H_1, H_2, and H_3, the existence of a solution to problem (1.85), under assigned initial conditions, is assured, in a sense to be specified below, if we impose the extra condition*

$$|I^\alpha L(t)|_c < 1. \tag{1.90}$$

The meaning of solution is as follows: There exists a mapping $x(\cdot) : S \to C([0,T], R)$, such that for any $s \in S$, $x(s)(\cdot)$ is a solution of the problem

$$D_c^\alpha z(t) \in F(t, z(t)),\ z(0) = a(s),\ z'(0) = b(s),$$

satisfying

$$|x(s)(t) - y(s)(t)| \leq \xi(s),\ (t,s) \in [0,T] \times S, \tag{1.91}$$

with y and ξ defined by (1.87) and (1.89), respectively.

The proof can be found in Cernea [95] and relies on a rather sophisticated construction. It is just a sample on how demanding this kind of problems are.

We are concluding this introductory Chapter, whose main purpose is to familiarize the reader with some concepts and methods we will use in the book, as well as summarily approaching some topics that will not be covered in the coming chapters (mainly, types of FEs, old or new, that are in the attention of many researchers).

At the same time, we want to emphasize the fact that other concepts, like equations on time scale, or equations with several independent variables, will not be represented in our presentation. The vastness of the mathematical and science and engineering production in this field is certainly one of the reasons we had to limit ourselves to a number of topics, namely existence, estimates, stability, and oscillations.

Some pertinent references for this section of the introductory chapter are Miller and Ross [397], Das [187], Kilbas, Srivastava and Trujillo [285], Pinney [451], Kelley and Peterson [284], Mickens [395], Abbas, Benchohra and N'Guérékata [2], and Diethelm [196].

More references/sources, concerning the topics summarily discussed in Section 1.6, can be found in those quoted the aforementioned references.