Chapter 1

The Mathematical Toolbox

1.1 INTRODUCTION

This chapter is essentially about applicable analytical tools. We cover items such as linear equations, simultaneous linear equations (involving two equations in two unknowns), the summation operator, sets (and set operations), general functions and graphs, and a brief exposure to differentiation and integration.

Our experience has indicated that some students might benefit from a review of the basic mathematical concepts typically studied in high school mathematics classes or in courses that might be offered in the first year of college. This is why we have included an optional appendix at the end of this chapter for those students in need of a refresher in topics such as exponents, factoring, fractions, and decimals and percents, among others. This review provides the student with a firm grounding in the fundamental mathematical tools needed for problem solving. Indeed, these basic foundation skills in elementary mathematics are necessary for developing and effecting business applications. For instance, derivatives of functions are not calculated for their own sake. One usually has to simplify a derivative in order for it to be useful in, say, determining the maximum of a profit function or the minimum of a cost function. While many students can readily apply the rules of differentiation, they often lack the algebraic skills required to simplify a derivative. (This is why some students describe the calculus as "too difficult.") The appendix to this chapter attempts to overcome this difficulty.

As to the approach followed throughout this text, and within this chapter proper, a few key notions have motivated our thinking. First, we want students to think in terms of the *modeling* of business activity. To this end, operating in terms of *functions* is critical. That is, any decision maker must identify those variables (called *independent* or *decision variables*) that are under his/her control. Once such variables are recognized and manipulated, other variables (called *response variables*) typically react in a deterministic and predictable fashion.

For instance, suppose a manager wants to increase the sales of some consumer product. What, if anything, can be done to facilitate an increase in sales revenue?

Introduction to Quantitative Methods in Business: With Applications Using Microsoft® Office Excel®, First Edition. Bharat Kolluri, Michael J. Panik, and Rao Singamsetti.

Companion website: www.wiley.com/go/Kolluri/QuantitativeMethods

What sort of decision variable can a manager manipulate? How about increasing the organizations advertising budget? Generally, firms advertise to increase sales (and/or to maintain market share). Thus, advertising expenditure is the decision or control variable and sales level is the response variable.

What might a revenue response look like? Possibly, the relationship between advertising (X) and sales revenue (Y) is one of direct proportionality (e.g., a dollar increase in advertising expenditure might always precipitate, say, 3 dollars in sales revenue), or maybe the advertising response increases at an increasing rate. Obviously, we are describing, in the first case, a linear relationship between advertising and revenue, and, in the second instance, a nonlinear one. Each is an example of a *functional relationship* between these two variables, which is of the general form $Y = f(X)$. For the linear case, structurally,

$$Y = f(X) = b_0 + b_1X.$$

And for the nonlinear (quadratic) case,

$$Y = f(X) = b_0 + b_1X + b_2X^2.$$

To reiterate, decision makers must think in terms of functions—in terms of relationships between decision variables and response variables.

Second, one might benefit from thinking in terms of *sets* of items. Here the use of set operations enables us to "create order" by organizing information in a very systematic fashion. This process will prove particularly useful when we introduce *uncertainty* into the decision-making picture and work with probability calculations.

To consider an additional decision-making theme/approach, let us return to the nonlinear advertising expenditure versus sales revenue relationship posited earlier. Specifically, how does one know if, at a particular advertising level, sales revenue is actually increasing at an increasing rate? To answer this question, we can employ a very useful and versatile analytical device, namely the requisite derivatives of a function at a given point—but more on this calculus approach later on.

1.2 LINEAR FUNCTIONS

An equation like $Y = b_0 + b_1X$ is known as a *linear function*. Here, b_1 is called the *slope* of the linear function; it measures the change in the value of the *dependent variable* Y as a result of a one unit change in the value of the *independent variable* X. It is also obtained as the ratio of the change in Y to the change in X. That is, $b_1 = \frac{\Delta Y}{\Delta X}$. b_1 is also called the coefficient of X. b_0 is known as the *Y-intercept*. It measures the value of Y when $X = 0$. Both b_0 and b_1 are constants.

Note the following:

- A horizontal line has zero slope.
- A vertical line has no slope or its slope is undefined.
- A line rising from left to right has positive slope.
- A line falling from left to right has negative slope.

1.3 SOLVING A SIMPLE LINEAR EQUATION FOR ONE UNKNOWN VARIABLE

Finding the particular value of the unknown variable in a linear equation is a useful algebraic exercise. When solving for the unknown variable in a single linear equation, all other values are assumed to be known. For example, let us solve the equation $6X + 42 = 0$. If we subtract 42 from both sides, the equation will be $6X = -42$. Dividing both sides by 6, X will be-7. It is a useful practice to substitute the answer into the original equation so as to check the solution. In our case, $6(-7) + 42 = 0$ is correct, so $X = -7$ is the solution. Let us say we have the equation, $4abX - 24 = 4$. To solve for X, all other values are assumed to be known. So, $4abX = 28$ and thus $X = 28/4ab = 7/ab$.

EXAMPLE 1.1

a. Solve the following equation:

$$4X - 24 = 0.$$

SOLUTION: Rewrite the given equation as

$$\begin{aligned} 4X &= 24, \\ X &= \frac{24}{4} = 6. \end{aligned}$$

b. Solve the following equation:

$$17X + 60 = 9.$$

SOLUTION: Rewrite the given equation as

$$\begin{aligned} 17X &= 9 - 60, \\ 17X &= -51, \\ X &= -\frac{51}{17} = -3. \end{aligned}$$

c. Solve the following equation:

$$\frac{1}{2}X - 10 = 2.$$

SOLUTION: Rewrite the given equation as

$$\frac{1}{2}X = 10 + 2 = 12,$$

$$X = 24.$$

d. Solve the following equation:

$$\frac{X}{2} + \frac{3}{4} = 5.$$

SOLUTION: Rewrite the given equation as

$$\frac{X}{2} = 5 - \frac{3}{4} = \frac{20-3}{4} = \frac{17}{4},$$
$$X = \frac{17}{4} \times 2 = \frac{17}{2} = 8.5.$$

e. Solve the following equation:

$$0.2X + 10.4 = 6.2.$$

SOLUTION: Rewrite the given equation as

$$0.2X = -10.4 + 6.2 = -4.2,$$
$$X = \frac{-4.2}{0.2} = \frac{-42}{2} = -21.$$

■

1.3.1 Solving Two Simultaneous Linear Equations for Two Unknown Variables

Solving simultaneous linear equation sets involves finding the values of two unknown variables that satisfy each equation at the same time. Graphically, the solution values of both the unknown variables indicate the point of intersection of two lines. This is useful in break-even analysis, market equilibrium analysis, economic order quantity (EOQ) calculations, and finding corner points in linear programming applications. For example, let us solve the following set of equations for X and Y:

$$\begin{aligned} 2X + 3Y &= 13, \\ X - 2Y &= 17. \end{aligned}$$

Although there is more than one method of solving for two unknowns X and Y, we adopt the *method of elimination* as follows:

Denote the equation $2X + 3Y = 13$ as (1) and $X - 2Y = 17$ as (2).

The first step in this procedure is to make the coefficients on one of the variables, say, X, in both equations the same by appropriate arithmetic operations. Thus, multiplying equation (2) by 2 we get $2X - 4Y = 34$. Call this equation (2′). Subtract equation (2′) from equation (1), that is,

$$2X + 3Y = 13 \quad (1)$$
$$\underline{2X - 4Y = 34} \quad (2')$$
$$0 + 7Y = -21 \quad (3)$$

Then, solving equation (3) renders $Y = (-21)/7 = -3$. We have "eliminated" X.

The second step is to substitute $Y = -3$ into equation (1).
Thus, $2X + 3(-3) = 13$, and so

$$2X - 9 = 13,$$
$$2X = 13 + 9 = 22,$$

and thus $X = 22/2 = 11$. Thus, the simultaneous solution is $X = 11$ and $Y = -3$. Equation systems that have at least one solution are termed *consistent*.

A simple *test for consistency* is the following: For the system

$$aX + bY = e,$$
$$cX + dY = f,$$

if $ad - cb \neq 0$, then this equation system is consistent.

EXAMPLE 1.2

a. Solve the following equation set:

$$X_1 + X_2 = 0,$$
$$2X_1 + (1/3)X_2 = 5.$$

SOLUTION: Designate the first equation as (1) and the second equation as (2). Multiply equation (2) by 3 to make the coefficient on X_2 the same in both equations. Then subtract equation (1) from this new equation. Thus,

3 × equation (2)	$6X_1 + X_2 = 15$
subtract equation (1)	$\underline{X_1 + X_2 = 0}$
	$5X_1 + 0 = 15$

(Here variable X_2 has been eliminated.) Solving this resulting equation for X_1 we get

$$X_1 = 15/5 = 3.$$

Then substituting $X_1 = 3$ into equation (1) yields $3 + X_2 = 0$ or $X_2 = -3$.

b. Solve the following equation set:

$$0.4P + 0.2Q = 7,$$
$$7P - 0.2Q = -2.$$

SOLUTION: Designating the first equation as (1) and the second equation as (2) and adding the two equations, we get

$7.4P = 5$. Solving this single equation in one unknown gives

$$P = \frac{5}{7.4} = 0.6757.$$

Substituting P = 0.6757 into equation (1) we get

$$0.4(0.6757) + 0.2Q = 7,$$
$$0.2703 + 0.2Q = 7,$$
$$0.2Q = 7 - 0.2703 = 6.7297,$$

and thus $Q = \frac{6.7297}{0.2} = 33.6485$.

Thus, the simultaneous solution is P = 0.6757 and Q = 33.6485.

c. Solve the following equation set:

$$3X_1 - 6X_2 = 1. \quad (1)$$
$$6X_1 - 12X_2 = 2. \quad (2)$$

Multiplying equation (1) by 2 and subtracting equation (2) from 2 × equation (1) yields "0 = 0." What happened? This example reveals that an equation system may not possess a simultaneous solution? Here, the two equations are parallel lines—they do not intersect. Such equations are known as *dependent* equations. (Note that via the consistency test, $3(-12) - 6(-6) = 0$.)

In sum, to solve two simultaneous linear equations in two unknowns:

Step 1: Perform the above test for *consistency*. If the system is *consistent* or *independent*, go to step 2.

Step 2: Use the method of elimination to solve for the unknowns. ■

1.4 SUMMATION NOTATION

Since the operation of addition occurs frequently in statistics, the special Greek symbol, $\sum$(pronounced "sigma"), is used to denote a sum. For example, if we have a set of n values for a variable X, the expression, $\sum_{i=1}^{n} X_i$ means that these n values, running from i = 1, 2, 3, ...,n, are to be added together. Thus,

$$\sum_{i=1}^{n} X_i = X_1 + X_2 + X_3 + \cdots + X_n.$$

Mathematically, "Σ" is an *operator*—It operates only on those terms with an i index. And the operation itself is addition.

The use of summation notation is illustrated in the following examples.

EXAMPLE 1.3

Assume that we have five values of a variable X, as given below:

$$X_1 = 12,$$
$$X_2 = 0,$$
$$X_3 = -1,$$
$$X_4 = -5,$$
$$X_5 = 3.$$

Then, $\sum_{i=1}^{5} X_i = X_1 + X_2 + X_3 + X_4 + X_5 = 12 + 0 + (-1) + (-5) + 3 = 9.$

In statistics, we frequently deal with summing the squared values of a variable. Thus, in our example,

$$\sum_{i=1}^{5} X_i^2 = X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = 12^2 + 0^2 + (-1)^2 + (-5)^2 + 3^2$$
$$= 144 + 0 + 1 + 25 + 9 = 179.$$

We should realize that $\sum_{i=1}^{n} X_i^2$, the *summation of the squares*, is not the same as $\left(\sum_{i=1}^{n} X_i\right)^2$, the *square of the sum*, that is,

$$\sum_{i=1}^{n} X_i^2 \neq \left(\sum_{i=1}^{n} X_i\right)^2.$$

In our example, the summation of squares is equal to 179. This is not equal to the square of the sum, which is $9^2 = 81$.

Another frequently used operation involves the summation of the product of sets of values. That is, suppose that we have two variables, X and Y, each having n observations. Then,

$$\sum_{i=1}^{n} X_i Y_i = X_1 Y_1 + X_2 Y_2 + X_3 Y_3 + \cdots + X_n Y_n.$$

■

EXAMPLE 1.4

Continuing with our previous example involving five values, suppose that there is also a second variable Y whose five values are

$$Y_1 = 1,$$
$$Y_2 = 3,$$
$$Y_3 = -2,$$
$$Y_4 = 4,$$
$$Y_5 = 6.$$

Then, $\sum_{i=1}^{5} X_i Y_i = X_1 Y_1 + X_2 Y_2 + X_3 Y_3 + X_4 Y_4 + X_5 Y_5 = 12(1) + (0)(3) + (-1)(-2)$

$$+(-5)(4) + 3(6) = 12 + 0 + 2 - 20 + 18 = 12.$$

In computing $\sum_{i=1}^{n} X_i Y_i$, we must realize that the first value of X is multiplied by the first value of Y, the second value of X is multiplied by second value of Y, and so on. These cross-products are then summed in order to obtain the desired result. However, we should note here that the summation of cross-products is not equal to the product of the individual sums, that is,

$$\sum_{i=1}^{n} X_i Y_i \neq \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right).$$

Table 1.1a Summation Values

Observation	X_i	X_i^2	Y_i	Y_i^2	X_iY_i
1	12	144	1	1	12
2	0	0	3	9	0
3	−1	1	−2	4	2
4	−5	25	4	16	−20
5	3	9	6	36	18
Total	$\sum_{i=1}^{5} X_i = 9$	$\sum_{i=1}^{5} X_i^2 = 179$	$\sum_{i=1}^{5} Y_i = 12$	$\sum_{i=1}^{5} Y_i^2 = 66$	$\sum_{i=1}^{5} X_iY_i = 12$

In our example, $\sum_{i=1}^{5} X_i = 9$ and $\sum_{i=1}^{5} Y_i = 1 + 3 + (-2) + 4 + 6 = 12$, so that

$$\left(\sum_{i=1}^{5} X_i\right)\left(\sum_{i=1}^{5} Y_i\right) = 9 \times 12 = 108.$$

This is not the same as $\sum_{i=1}^{5} X_iY_i$, which equals 12.

Before studying the four basic rules of performing operations with summation notation, it would be helpful to present the values for each of the five observations of X and Y in a tabular format (see Table 1.1a).

Rule 1: The sum of the set of sums of two variables is equal to the sum of their individual sums. That is,

$$\sum_{i=1}^{n} (X_i + Y_i) = \left(\sum_{i=1}^{n} X_i\right) + \left(\sum_{i=1}^{n} Y_i\right).$$

Thus, in our example,

$$\sum_{i=1}^{5} (X_i + Y_i) = (12 + 1) + (0 + 3) + (-1 + (-2)) + (-5 + 4) + (3 + 6) = 21,$$

$$\text{which is equal to} \left(\sum_{i=1}^{n} X_i\right) + \left(\sum_{i=1}^{n} Y_i\right) = 9 + 12 = 21.$$

Rule 2: The sum of the set of differences between two variables is equal to the difference between their individual sums, or

$$\sum_{i=1}^{n} (X_i - Y_i) = \left(\sum_{i=1}^{n} X_i\right) - \left(\sum_{i=1}^{n} Y_i\right).$$

Thus, in our example,

$$\sum_{i=1}^{5} (X_i - Y_i) = (12 - 1) + (0 - 3) + (-1 - (-2)) + (-5 - 4) + (3 - 6) = -3,$$

$$\text{which is equal to} \left(\sum_{i=1}^{n} X_i\right) - \left(\sum_{i=1}^{n} Y_i\right) = 9 - 12 = -3.$$

Rule 3: The sum of a constant times a variable equals that constant times the sum.

$$\sum_{i=1}^{n} cX_i = c\sum_{i=1}^{n} X_i, \quad \text{where c is the constant.}$$

Thus, in our example, if c = 3,

$$\sum_{i=1}^{5} cX_i = 3(12) + 3(0) + 3(-1) + 3(-5) + 3(3) = 36 + 0 - 3 - 15 + 9 = 27,$$

$$\text{which is equal to } 3\left(\sum_{i=1}^{5} X_i\right) = 3(9) = 27.$$

Rule 4: The sum of a constant taken n times equals n times the constant, that is,

$$\sum_{i=1}^{n} c = nc, \quad \text{where c is the constant.}$$

Thus, if the constant c = 3 is summed five times, we would have

$$\sum_{i=1}^{5} c = 3 + 3 + 3 + 3 + 3 = 5(3) = 15.$$

To illustrate how these summation rules are used, we may demonstrate one of the mathematical properties pertaining to the average or arithmetic mean of a sample of X values. To this end, if the sample mean is $\overline{X} = \sum_{i=1}^{n} X_i/n$, then

$$\sum_{i=1}^{n}(X_i - \overline{X}) = 0.$$

This property states that the summation of the differences between each observation and the arithmetic mean is zero.

This can be proven mathematically by the following steps:

Step 1: As just indicated, the arithmetic mean of a sample of X values can be defined as

$$\overline{X} = \left(\sum_{i=1}^{n} X_i\right)/n.$$

Using summation rule 2, we have

$$\sum_{i=1}^{n}(X_i - \overline{X}) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \overline{X}.$$

Step 2: Since, for any fixed set of data, $\overline{X}$ can be considered a constant, then, and from summation rule 4, we have

$$\sum_{i=1}^{n} \overline{X} = n\overline{X}.$$

$$\text{Therefore,} \quad \sum_{i=1}^{n}(X_i - \overline{X}) = \sum_{i=1}^{n} X_i - n\overline{X}.$$

Step 3: However, from step 1, since

$$\overline{X} = \left(\sum_{i=1}^{n} X_i\right)/n \quad \text{and} \quad n\overline{X} = \sum_{i=1}^{n} X_i,$$

we consequently obtain

$$\sum_{i=1}^{n}(X_i - \overline{X}) = \sum_{i=1}^{n} X_i - \sum_{i=1}^{n} X_i = 0.$$

We have thus shown that $\sum_{i=1}^{n}(X_i - \overline{X}) = 0$, and this is true for any sample data set. Why? Because $\overline{X}$ is the "center of gravity" of the X values—It is where the X values would "balance."

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EXAMPLE 1.5

Using the data in Table 1.1a, namely, $X_1 = 12$, $X_2 = 0$, $X_3 = -1$, $X_4 = -5$, $X_5 = 3$ and $Y_1 = 1$, $Y_2 = 3$, $Y_3 = -2$, $Y_4 = 4$, $Y_5 = 6$,

(a) Evaluate the Pearson correlation coefficient

$$r = \sum_{i=1}^{5} \frac{\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sqrt{\sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 \sum_{i=1}^{5}\left(Y_i - \overline{Y}\right)^2}},$$

where $\overline{X} = \frac{\sum X_i}{5}$ and $\overline{Y} = \frac{\sum Y_i}{5}$.

(b) Also, reevaluate the correlation coefficient using the following "short formula":

$$r = \frac{\sum X_i Y_i - \frac{\sum X_i \sum Y_i}{n}}{\left(\left(\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}\right)\left(\sum Y_i^2 - \frac{\left(\sum Y_i\right)^2}{n}\right)\right)^{1/2}}.$$

SOLUTION:

(a)

$$r = \sum_{i=1}^{5} \frac{\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sqrt{\sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 \sum_{i=1}^{5}\left(Y_i - \overline{Y}\right)^2}}$$

$$= \frac{-9.6}{[(162.8)(37.2)]^{1/2}}$$

$$= -0.1104 = -0.1104.$$

From the short formula,

(b)

$$r = \frac{12 - \frac{(9)(12)}{5}}{\left[\left(179 - \frac{9^2}{5}\right)\left(66 - \frac{12^2}{5}\right)\right]^{1/2}}$$

$$= \frac{-9.6}{[(162.8)(37.2)]^{1/2}}$$

$$= -0.1104.$$

■

EXAMPLE 1.6

Using the data and the results in Example 1.5, compute $\frac{r \times \sqrt{3}}{\sqrt{1-r^2}}$.

SOLUTION:

$$\frac{r \times \sqrt{3}}{\sqrt{1 - r^2}} = \frac{-0.1104 \times 1.4422}{\sqrt{1 - (-0.1104)^2}} = \frac{-0.1592}{\sqrt{1 - 0.0122}} = \frac{-0.1592}{\sqrt{0.9878}} = -0.1602.$$

■

EXAMPLE 1.7

Using the data and results in Example 1.5:

(a) Compute the slope value, $b = \frac{\sum_{i=1}^{5}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{5}(X_i - \overline{X})^2}$, and the intercept value, $a = \overline{Y} - b\overline{X}$, of the linear equation $Y = a + bX$. Using the resulting equation, predict the value of Y when $X = 14$.

(b) Also, reevaluate the slope value b using the following "short formula":

$$b = \frac{\sum X_i Y_i - \frac{\sum X_i Y_i}{n}}{\left(\sum X_i^2 - \frac{(\sum X_i)^2}{n}\right)}.$$

SOLUTION:

(a) Referring to Table 1.1b,

$$b = \frac{\sum_{i=1}^{5}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{5}(X_i - \overline{X})^2} = \frac{-9.6}{162.8} = -0.0589.$$

$$a = \overline{Y} - b\overline{X} = 2.4 - (-0.0589)(1.8) = 2.5060.$$

$$Y = a + bX = 2.5060 + (-0.0589)(14) = 1.6814.$$

Table 1.1b Summation Values

X_i	$X_i - \overline{X}$	$(X_i - \overline{X})^2$	Y_i	$Y_i - \overline{Y}$	$(Y_i - \overline{Y})^2$	$(X_i - \overline{X})(Y_i - \overline{Y})$
12	10.2	104.04	1	−1.4	1.96	−14.28
0	−1.8	3.24	3	0.6	0.36	−1.08
−1	−2.8	7.84	−2	−4.4	19.36	12.32
−5	−6.8	46.24	4	1.6	2.56	−10.88
3	1.2	1.44	6	3.6	12.96	4.32
$\sum_{i=1}^{5} X_i = 9$	0	$\sum (X_i - \overline{X})^2 = 162.8$	$\sum_{i=1}^{5} Y_i = 12$		$\sum (Y_i - \overline{Y})^2 = 37.2$	$\sum (X_i - \overline{X})(Y_i - \overline{Y}) = -9.6$

Table 1.2 Summation Values

X_i	X_i^2	Y_i	Y_i^2	$X_i Y_i$
12	144	1	1	12
0	0	3	9	0
−1	1	−2	4	2
−5	25	4	16	−20
3	9	6	36	18
$\sum_{i=1}^{5} X_i = 9$	$\sum_{i=1}^{5} X_i^2 = 179$	$\sum_{i=1}^{5} Y_i = 12$	$\sum_{i=1}^{5} Y_i^2 = 66$	$\sum_{i=1}^{5} X_i Y_i = 12$

(b) Referring to Table 1.2,

$$\begin{aligned} b &= \frac{\sum X_i Y_i - \frac{\sum X_i \sum Y_i}{n}}{\left[\sum X_i^2 - \frac{(\sum X_i)^2}{n}\right]} \\ &= \frac{12 - \frac{(9)(12)}{5}}{179 - \frac{9^2}{5}} \\ &= \frac{-9.6}{162.8} \\ &= -0.0589. \end{aligned}$$

■

1.5 SETS

A *set* is a collection or grouping of items (called *elements*) without regard to structure or order. The elements may be people, sheep, desks, cars, files in a cabinet, or even numbers. We may define our sets as the collection of all sheep in a given pasture, all people in a room, all cars in a given parking lot at a given time, all values between 0 and 1, or all integers. The number of elements in a set may be infinite. For example, the numbers in the set of integers are called *countably infinite*, while the numbers between any two points on the real line, such as 0 and 1, are *uncountably infinite*.

A type of notation used to define a set is the *roster method*. In the roster method, we simply list the names of the members separated by commas. Then we enclose the names with braces. The names can be written in any order. For example, if we want to construct the set A consisting of the first five nonnegative integers using the roster method, we simply write

$$A = \{0,1,2,3,4\}.$$

For some infinite (unending) sets, we can write names (or numbers) for some of the members and then write dots to indicate the rest of the numbers. For example, if we want to write a set B of nonnegative even integers using the roster method, we form

$$B = \{0,2,4,6,8,\ldots\}.$$

When we solve an equation, we look for the set of all solutions. We call such a set the *solution set*. For example, the solution set of the equation $2x + 7 = 15$ is $\{4\}$. In some contexts, one might just list the solution as 4. Similarly, the solutions of $x(x - 2) = 0$ are 0 and 2. One might use the roster method and say that the solution set is $\{0, 2\}$. The symbol "$\in$" is used to denote membership in a set or *element inclusion*. For example, the expression $3 \in$ "the set of natural numbers" is true because 3 is a natural number. But the expression, Hence $2/3 \notin \{0, 3, 7, 9\}$ is false, because $2/3$ is not a member of the given set.

1.5.1 Subset, Empty Set, Universal Set, and Complement of A Set

When all the members of one set are also members of another, we say that the first set is a *subset* of the second. For example, $A \subset B$ means that "A is a subset of B." The symbol $\subset$ is used to indicate *set inclusion*. It can also be written as $B \supset A$, which again means that "A is a subset of B."

The set without any members is called the *empty set*, and is denoted by "$\varnothing$" (also called the *null set*). The empty set is a subset of every set.

A *universal set* is the set containing everything in a given context. We denote the universal set by U. For example, all integers comprise a universal set. All registered voters in a given election form a universal set. As another example, the set of all whole numbers, both positive and negative, form a universal set.

Given a set A, we may define its *complement* as the set containing all the elements in the universal set U that are not members of set A (Figure 1.1a). We denote the complement of A by $\overline{A}$ or $\tilde{A}$, often called "not A." With regard to die tossing, all odd integers form the complement of the set of all even integers. In a deck of playing cards, the suit "spades" forms the complement of the three remaining suits (hearts, diamonds, and clubs).

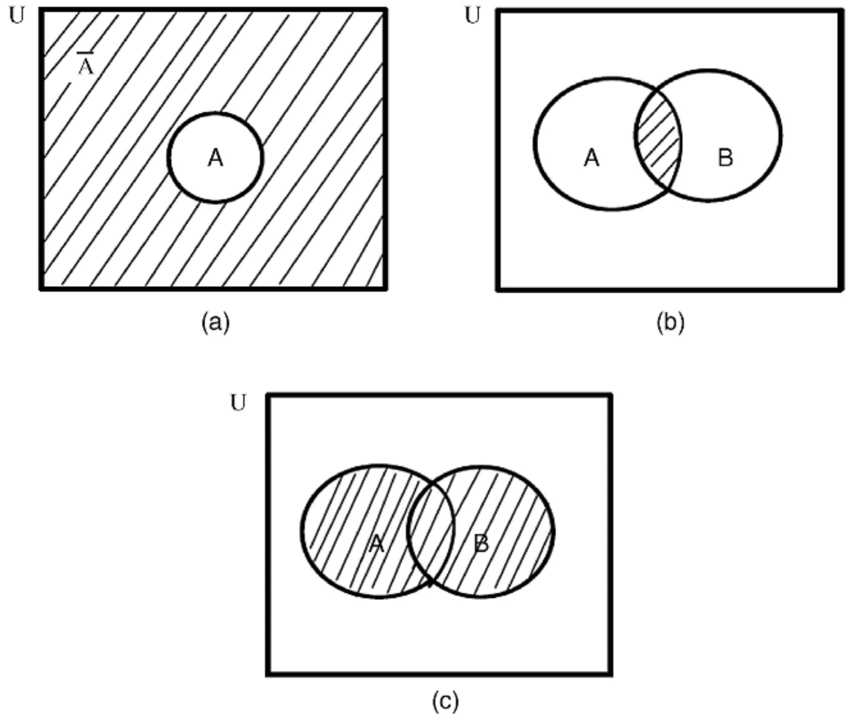


Figure 1.1 Set operations. (a) $\overline{A}$ is the complement of A. (b) Intersection of A and B. (c) Union of A and B.

EXAMPLE 1.8

Determine whether the following is true or false:

$$\{1,2,3,4\} \subset \{0,1,2,3,4,5\}.$$

This is true because every member of the first set is also a member of the second. ■

EXAMPLE 1.9

List all the subsets of $\{1,2,3\}$.

$\{1,2,3\}$	Every set is a subset of itself.
$\{1,2\},\{1,3\},\{2,3\}$	There are three subsets with two members each.
$\{1\},\{2\},\{3\}$	There are three subsets with one member each.
$\varnothing$	The empty set is a subset of any set.

■

1.5.2 Intersection and Union

The *intersection* of two sets A and B is the set of all members that are common to A and B or reside in both A and B (Figure 1.1b). We denote the intersection of sets A and B as $A \cap B$, where "$\cap$" stands for "and." For example, the intersection of the two sets $A = \{1,2,3,4,5\}$ and $B = \{-2,-1,0,1,2,3\}$ is

$$A \cap B = \{1,2,3,4,5\} \cap \{-2,-1,0,1,2,3\} = \{1,2,3\}.$$

The numbers 1, 2, and 3 are common to both sets, so the intersection is $\{1,2,3\}$.

The *union* of two sets A and B is formed by pooling their elements and is defined as those that are in A, in B, or in both A and B (Figure 1.1c). We denote the union of sets A and B as $A \cup B$, where the symbol "$\cup$" stands for "or"/"both." For example, the union of the two sets $A = \{1,2,3\}$ and $B = \{3,4,5\}$ is

$$A \cup B = \{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}.$$

The numbers in either or both sets are 1,2,3,4, and 5, so the union is $\{1,2,3,4,5\}$.

Finally, four specialized sets that will be useful in subsequent sections are as follows:

Open interval from a to b

$$(a,b) = \{a < X < b\}.$$

Closed interval from a to b

$$[a,b] = \{a \le X \le b\}.$$

Left open, right closed from a to b

$$(a,b] = \{a < X \le b\}.$$

Left closed, right open from a to b

$$[a,b) = \{a \le X < b\}.$$

1.6 FUNCTIONS AND GRAPHS

Given variables x and y, let f be a "rule" or "law of correspondence" that associates with each value of x a unique value of y. Then, y is said to be a *function* of x and written $y = f(x)$. Here, y is termed the *image* of x under rule f. The expression $f(x)$ does not mean "f times x." Instead, $f(x)$ is the symbol for the indicated correspondence.

For the function $y = f(x)$, the set of x values is called the *domain of the function*, and the set corresponding to the y values, which are the image of at least one x, is known as the *range of the function*. Here x and y are, respectively, known as the *independent* and *dependent variables* in the context of the equation $y = f(x)$.

For the function $y = 3 + 2x$, what is the "rule" that tells us how to get a y from x? For each x value, there is one and only one y value. However, a y value may be the image of more than one x. For example, let $y = x^2$ and let $x = \pm 2$ (Figure 1.2). But if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$, then f is said to be *one-to-one*.

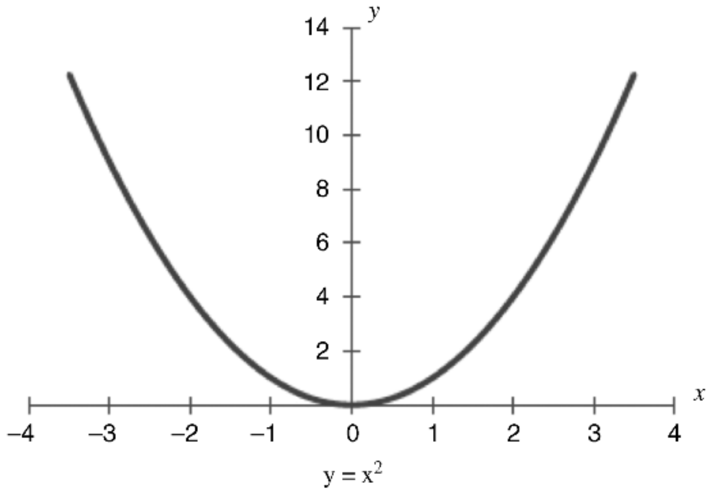


Figure 1.2 Graph of $y = x^2$.

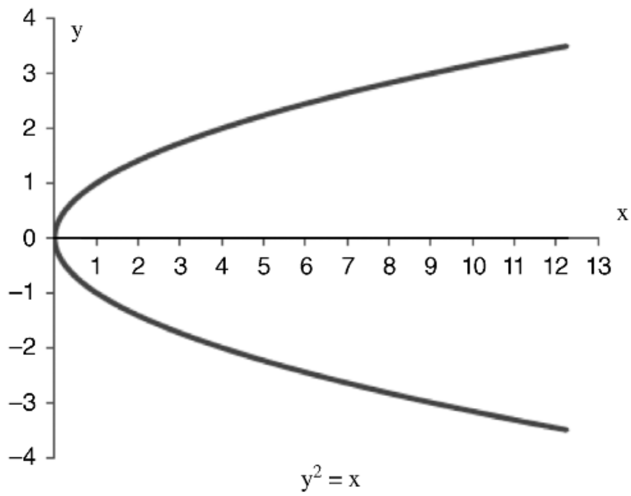


Figure 1.3 Graph of $y = \pm\sqrt{x}$.

It should be evident that $y^2 = x$ is not a function. This is because, if we solve the equation for y, we obtain $y = \pm\sqrt{x}$. For a given value of x, y can be both positive and negative. So, there is not one and only one y value for a given x. The graph is shown in Figure 1.3.

1.6.1 Vertical Line Test

To determine whether or not a particular graph represents a function, we can use the *vertical line test*. For a given value of x, find y by drawing a vertical line at x. If the vertical line cuts the graph at more than one value of y, then the graph does not represent a function.

For instance, if in Figure 1.2 we draw a vertical line at $x = 2$, the said line cuts the graph at $y = 4$ only. Thus, $y = x^2$ is a function. In a similar way, we can easily find that in Figure 1.3, $y^2 = x$ is not a function.

Why are functions important? What is their place in decision making? As we shall see later on, when we undertake the process of model building and we deal with a concept such as "cost," this notion will be represented by a "cost function"; and if we desire to maximize "profit," then to do so we shall employ a "profit function," and so on. Hence, a function is an analytical device useful for representing an abstract concept.

EXAMPLE 1.10

Show that

$x^2 + y^2 = k^2$, where k, a constant, is not a function (it actually represents a circle). From $y^2 = k^2 - x^2$, we obtain

$y = \pm\sqrt{(k^2 - x^2)}$. This shows that for a given value of x, there is more than one value of y. ■

1.7 WORKING WITH FUNCTIONS

Evaluating functions given by a formula can involve algebraic simplification, as the following examples show. Similarly, solving for the input or independent variable involves solving an equation algebraically.

1.7.1 Evaluating Functions

A formula like $f(x) = \left(x^2 + 2/6 + x\right)$ is a rule that tells us what the function f does with its input value x. In the formula, the letter x is a "placeholder" for the input value. Thus, to evaluate f(x), we replace each occurrence of x in the formula with the value of the input.

EXAMPLE 1.11

Let $g(x) = \frac{2x^2+3}{4+x}$. Evaluate the following expressions. Some of your answers will contain c, a constant.

a. $g(3)$

b. $g(-3)$

c. $g(c)$

d. $g(c - 2)$

e. $g(c) - 4$

f. $g(c) - g(3)$

SOLUTIONS:

a. To evaluate g(3), replace every x in the formula with 3:

$$g(3) = \frac{2\left(3^2\right) + 3}{4 + 3} = \frac{21}{7} = 3.$$

b. To evaluate g(−3), replace every x in the formula with (−3):

$$g(-3) = \frac{2(-3)^2 + 3}{4 - 3} = 21.$$

c. To evaluate g(c), replace every x in the formula with c:

$$g(c) = \frac{2c^2 + 3}{4 + c}.$$

d. To evaluate g(c − 2), replace every x in the formula with c − 2:

$$g(c-2) = \frac{2(c-2)^2 + 3}{4 + (c-2)} = \frac{2\left(c^2 - 4c + 4\right) + 3}{2 + c} = \frac{2c^2 - 8c + 11}{2 + c}.$$

e. To evaluate g(c) −4, first evaluate g(c) and then subtract 4:

$$g(c) - 4 = \frac{2c^2 + 3}{4 + c} - 4 = \frac{2c^2 + 3}{4 + c} - 4\left(\frac{4 + c}{4 + c}\right) = \frac{2c^2 + 3}{4 + c} - \left(\frac{16 + 4c}{4 + c}\right)$$
$$= \frac{2c^2 + 3 - 16 - 4c}{4 + c} = \frac{2c^2 - 4c - 13}{4 + c}.$$

f. To evaluate g(c) −g(3), subtract g(3) from g(c):

$$g(3) = 3 \quad \text{from part (a).}$$

$$\text{From part c,} \quad g(c) = \frac{2c^2 + 3}{4 + c}.$$

Thus,

$$g(c) - g(3) = \frac{2c^2 + 3}{4 + c} - 3 = \frac{\left(2c^2 + 3\right) - 3(4 + c)}{(4 + c)} = \frac{2c^2 - 3c - 9}{(4 + c)}.$$

■

1.7.2 Graphing Functions

For the sake of clarity, a function can also be described in graphical form. Generally, the independent variable is shown on the x-axis (horizontal axis), the dependent variable is shown on they-axis (vertical axis), and the points are plotted and connected. Such a diagram is called a *graph*. Graphs may include lines and curves that consist of sets of points. For each x value, there is a value for f(x), and each ordered pair (x,f(x)) constitutes a point on the function's graph. To draw a linear function, the easiest way to proceed is to find the points where the line cuts the x- and y-axes. To find those values, we have to solve the function for x when y = 0 and for y when

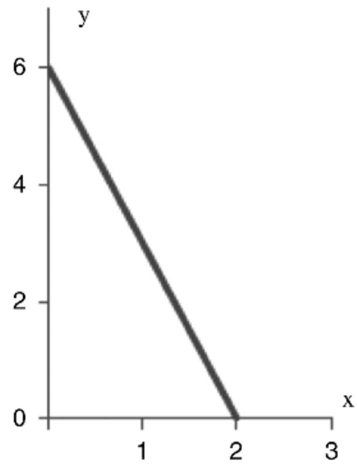


Figure 1.4 Graph of the straight line $3x + y = 6$ using *basic points*.

$x = 0$. As those values are, respectively, *horizontal* and *vertical intercepts*, this method is called the *intercept method*. For example, to draw the function $3x + y = 6$, we proceed as follows:

$$\text{If } x = 0, \text{ then } y = 6 \text{ and so 6 is the y intercept.}$$
$$\text{If } y = 0, \text{ then } 3x = 6 \text{ and so 2 is the x intercept.}$$

The straight-line graph is obtained by joining the two points $(0, 6)$ and $(2, 0)$ (called *basic points*). See Figure 1.4.

***Note*:** Two points are enough to draw a unique straight line.

Another way to graph a function is to connect more than two points. This method is called the *point method* and can be used for both linear and nonlinear functions. The resulting graph is smoother and more accurate when dealing with nonlinear functions. To illustrate the point method, let us draw the same function, $3x + y = 6$.

First, find y values for different x values (Table 1.3). Here the x values are selected arbitrarily around the origin, and selecting such x values depends on the nature of the given function.

Next, plot the points and connect them as shown in Figure 1.5.

Table 1.3 Values of y for Selected x Values

x	y
2	0
1	3
0	6
−1	9
−2	12

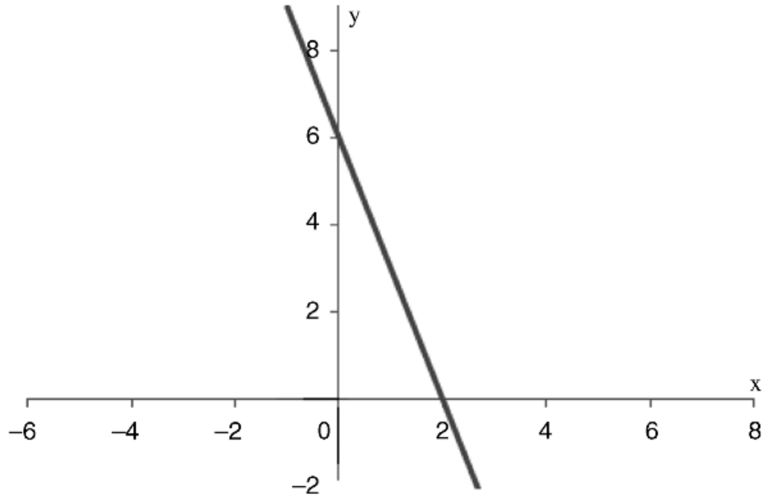


Figure 1.5 Graphing 3x + y = 6 using the *point method*.

EXAMPLE 1.12

Graph y = |x| (the absolute value of x). Here, |x| = x when x > 0; |x| = −x when x < 0. *Note*: |7| = 7; |−7| = 7.

First, select a set of x values and then construct Table 1.4.

Then, draw the graph corresponding to the points in this table (Figure 1.6).

Table 1.4 x and y (= |x|) Values

x	y
3	3
2	2
1	1
0	0
−1	1
−2	2
−3	3

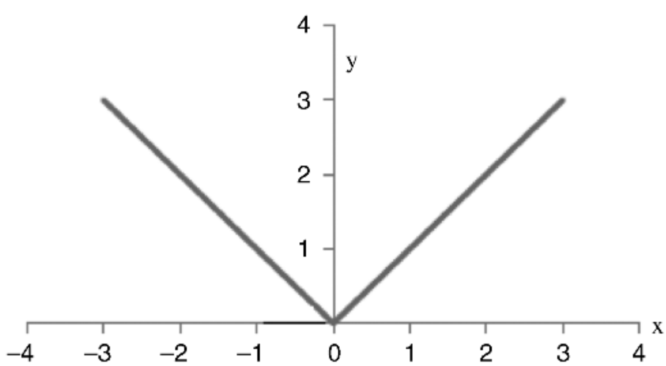


Figure 1.6 Graph of y = |x|. ■

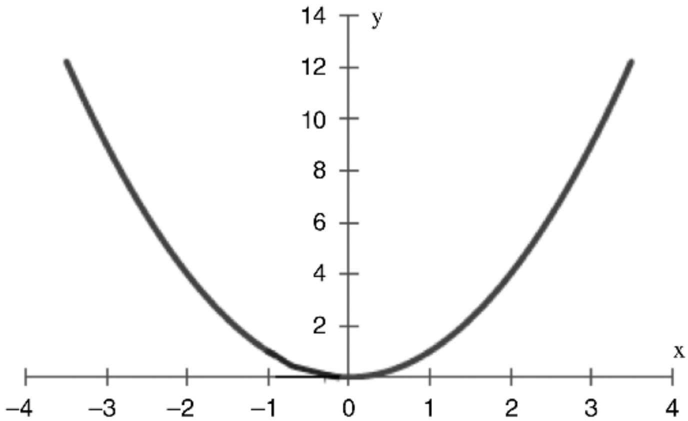


Figure 1.7 Graph of $y = x^2$.

Table 1.5 x and y ($=x^2$) Values

x	y
3	9
2	4
1	1
0	0
−1	1
−2	4
−3	9

EXAMPLE 1.13

Graph $y = x^2$ (Figure 1.7, Table 1.5). ■

EXAMPLE 1.14

Graph the exponential function $y = 2^x$ (Figure 1.8, Table 1.6). ■

Table 1.6 x and y ($= 2^x$) Values

x	y
3	8
2	4
1	2
0	1
−1	1/2
−2	1/4
−3	1/8

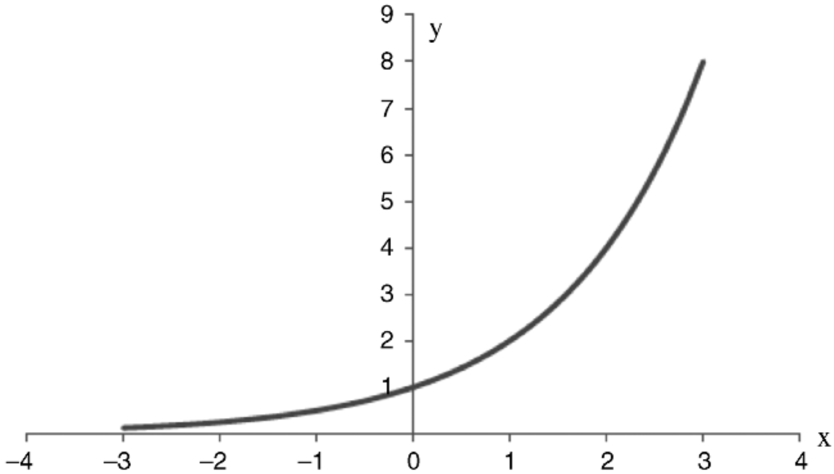


Figure 1.8 Graph of $y = 2^x$.

1.8 DIFFERENTIATION AND INTEGRATION

In many business applications, we encounter a variety of instances where we have to deal with *rates of change*. For example, we may want to know the effect of a change in interest rates on business investments or we may want to determine the effect of change in the disposable income of families on their consumption. Rates of change such as these can be described by the concept of a “derivative.” Likewise, “integration” is used to compute a sum. For example, integration can be used to find the sum of accumulated net returns of an investment over a given period of time.

1.8.1 Derivative

The function $y = f(x)$ is said to approach the *limit* L as x tends to a if “f(x) is near L” whenever “x is near a.” This statement is symbolized as

$$\lim_{x \to a} f(x) = L.$$

Now, if it is also the case that $L = f(a)$, then f is said to be *continuous* at $x = a$ and is written as

$$\lim_{x \to a} f(x) = f(a).$$

And if f is continuous at each point of its domain, then f is called a *continuous function*. Intuitively, think of a function as being continuous if its graph is without breaks.

Next, if $x = a$ is any point of the interval (x_1, x_2), and if a function $y = f(x)$ is defined over (x_1, x_2), then

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

is called the *derivative of f(x) at x = a* and denoted f′(a). In general, *the derivative of f with respect to x* is defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x) = dy/dx.$$

Here dy/dx is the instantaneous rate of change in y per unit change in x as Δx gets smaller and smaller.

For instance, let us find the derivative of $y = f(x) = x^2$ at x = 2. We first find

$$\frac{\Delta y}{\Delta x} = \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \frac{(2 + \Delta x)^2 - f(2)}{\Delta x}$$
$$= \frac{4 + 4\Delta x + (\Delta x)^2 - 4}{\Delta x} = \frac{4\Delta x + (\Delta x)^2}{\Delta x} = 4 + \Delta x.$$

Then,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4 + \Delta x) = 4 = dy/dx.$$

So, when Δx is near zero, 4 + Δx is near 4—which is the derivative of $y = f(x) = x^2$ at x = 2.

Note that if f is differentiable (it has a derivative) at x = a, then it is also continuous at x = a. However, the converse does not hold.

Geometrically, the derivative f′(a) represents the *slope* of the function y = f(x) at x = a (it is the slope of the line tangent to f(x) at x = a). In various (business) applications, dy/dx depicts the *marginal change in y with respect to x.*

Since the direct calculation of the derivative of a function via the limit process can be quite tedious, various rules of differentiation will now be offered as a convenience. These rules will expedite the calculation of derivatives.

Some Rules of Differentiation

1. *Power rule:* If $y = f(x) = x^n$, then

$$\frac{dy}{dx} = nx^{n-1}.$$

EXAMPLE 1.15

Find $\frac{dy}{dx}$ for the following:

a. $y = x^3$.

b. y = x.

c. $y = \frac{1}{x} = x^{-1}, \quad x \neq 0.$

d. $y = \frac{1}{\sqrt{x}} = x^{-(1/2)}, \quad x \neq 0.$

SOLUTIONS:

a. $\frac{dy}{dx} = 3x^{3-1} = 3x^2$.

b. $\frac{dy}{dx} = 1$.

c. $\frac{dy}{dx} = -1x^{-1-1} = -1x^{-2} = \frac{-1}{x^2}$.

d. $\frac{dy}{dx} = -\frac{1}{2}x^{\{-(1/2)-1\}} = -\frac{1}{2}x^{\{-(3/2)\}} = -\frac{1}{2}\left(\frac{1}{\sqrt{x^3}}\right)$. ■

2. The derivative of a constant is zero. If $y = f(x) = k$, where k is a constant, then $dy/dx = 0$.

EXAMPLE 1.16

If $y = 10$, then $dy/dx = 0$. It is obvious that the graph of $y = 10$ is a straight line parallel to x-axis, and its slope dy/dx is 0. This is due to the fact that even when x changes, y does not change ■

3. *Coefficient rule:* If $y = f(x) = kg(x)$, where k is a constant, then

$$\frac{dy}{dx} = kg'(x) = k\frac{dg}{dx}.$$

EXAMPLE 1.17

Find $\frac{dy}{dx}$ for the following:

a. $y = 5x^4$.

b. $y = \left(\frac{1}{2}\right)x^2$.

c. $y = \left(\frac{1}{2x^2}\right), \quad x \neq 0$.

SOLUTION:

a. $\frac{dy}{dx} = 5\left(\frac{d(x^4)}{dx}\right) = 5 \times 4x^3 = 20x^3$.

b. $\frac{dy}{dx} = \left(\frac{1}{2}\right)\left(\frac{d(x^2)}{dx}\right) = \left(\frac{1}{2}\right) \times 2x = x$.

c. $\frac{dy}{dx} = \left(\frac{1}{2}\right)\left(\frac{d(x^{-2})}{dx}\right) = \left(\frac{1}{2}\right) \times (-2)x^{-2-1} = -x^{-3} = -\left(\frac{1}{x^3}\right)$. ■

4. *Sum (difference) rule:* If $y = f(x) = g(x) \pm h(x)$, then

$$\frac{dy}{dx} = g'(x) \pm h'(x) = \frac{dg}{dx} \pm \frac{dh}{dx}.$$

EXAMPLE 1.18

Find the derivatives of the following:

a. $y = 5x^3 + 7x^4$.

b. $y = 2x^2 - \left(\frac{3}{x^2}\right), \quad x \neq 0$.

c. $y = 3x + 10$.

SOLUTION:

a. $\frac{dy}{dx} = 5 \times 3x^{3-1} + 7 \times 4x^{4-1} = 15x^2 + 28x^3$.

b. $\frac{dy}{dx} = 2 \times 2x^{2-1} - 3(-2)x^{-2-1} = 4x + 6x^{-3} = 4x + \left(\frac{6}{x^3}\right)$.

c. $\frac{dy}{dx} = 3 + 0 = 3$. ■

5. *Product rule:* If $y = f(x) \times g(x)$, then

$$\frac{dy}{dx} = f'(x) \times g(x) + f(x) \times g'(x) = \frac{df}{dx} \times g(x) + f(x) \times \frac{dg}{dx}.$$

EXAMPLE 1.19

Find the derivative of $y = f(x) \times g(x)$, where (a) $f(x) = 7x^2$ and $g(x) = 3x^3$, and (b) $f(x) = 9x^3$ and $g(x) = 2x^5$.

SOLUTION:

$\frac{dy}{dx} = f'(x) \times g(x) + f(x) \times g'(x)$.

(a) $\frac{dy}{dx} = (14x)\left(3x^3\right) + \left(7x^2\right)\left(9x^2\right) = 42x^4 + 63x^4 = 105x^4$.

(b) $\frac{dy}{dx} = 27x^2(2x^5) + 9x^3(10x^4) = 54x^7 + 90x^7 = 144x^7$. ■

6. *Quotient rule:* If $y = f(x)/g(x)$, $g(x) \neq 0$, then

$$\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

EXAMPLE 1.20

Find the derivative of $y = \frac{f(x)}{g(x)}$, where

(a) $f(x) = 3x^2 + 2$ and $g(x) = 5x^3 + 7$.

(b) $f(x) = 7x^3 + 3$ and $g(x) = 6x^2 - 5x$.

SOLUTION:

(a)

$$\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2},$$

$$\frac{dy}{dx} = \frac{\left(5x^3 + 7\right)(6x) - \left(3x^2 + 2\right)\left(15x^2\right)}{\left(5x^3 + 7\right)^2},$$

$$\frac{dy}{dx} = \frac{\left(30x^4 + 42x\right) - \left(45x^4 + 30x^2\right)}{25x^6 + 70x^3 + 49},$$

$$\frac{dy}{dx} = \frac{-15x^4 - 30x^2 + 42x}{25x^6 + 70x^3 + 49}.$$

(b) $$\frac{dy}{dx} = \frac{(6x^2 - 5x)(21x^2) - (7x^3 + 3)(12x - 5)}{(6x^2 - 5x)^2}$$

$$= \frac{126x^4 - 105x^3 - 84x^4 - 36x + 35x^3 + 15}{36x^4 - 60x^3 + 25x^2} = \frac{42x^4 - 70x^3 - 36x + 15}{36x^4 - 60x^3 + 25x^2}.$$

■

1.8.2 Derivatives of Logarithmic and Exponential Functions

If $y = \log_a x$, then

$$\frac{dy}{dx} = \frac{1}{x} \times \log_a e = \frac{1}{x} \times \frac{\ln e}{\ln a} = \frac{1}{x \ln a}.$$

This result follows from the rule

$$\log_a b = \frac{\ln b}{\ln a} \quad \text{and} \quad \ln e = 1.$$

$$\text{If } y = \ln x, \quad \text{then} \quad \frac{dy}{dx} = \frac{1}{x}.$$

$$\text{If } y = e^x, \quad \text{then} \quad \frac{dy}{dx} = e^x.$$

Note**:** For proofs of the above rules of differentiation (including other rules of differentiation), see any standard calculus textbook.

1.8.3 Higher Order Derivatives

Given $y = f(x)$, we previously denoted the first derivative of f with respect to x as dy/dx or $f'(x)$. For instance, if $y = f(x) = 2x^3 + 4x^2 - 2x + 10$, then

$$\frac{dy}{dx} = f'(x) = 6x^2 + 8x - 2.$$

Since dy/dx is also a function of x, we can find its derivative with respect to x. That is, we desire to find the derivative of the first derivative, which we shall term the *second derivative* of f and denote as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} \quad \text{or} \quad f''(x).$$

So given dy/dx above,

$$\frac{d^2y}{dx^2} = f''(x) = 12x + 8.$$

Note that the same basic rules of differentiation still apply—only the notation has changed.

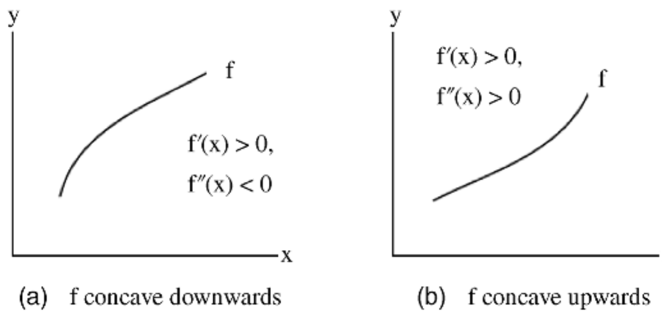


Figure 1.9 Behavior of the second derivative of f. (a) f concave downward. (b) f concave upward.

Geometrically, the second derivative of f is the rate of change of the slope of f at a particular point on the graph of f. For example, in Figure 1.9a, as we move along the curve from left to right, $f' > 0$ with $f'' < 0$, that is, the slope of f is decreasing as x increases. In Figure 1.9b, again $f' > 0$, but now $f'' > 0$, that is, the slope of f is increasing as x increases. In the former case we say that f is *concave downward*; in the latter case f is termed *concave upward*. (As an exercise, the reader should draw comparable graphs for $f' < 0$. Also, describe the behavior of the slope of f in terms of f'' and the concavity of f for each case.)

Furthermore, since d^2y/dx^2 also depends on x, we can find its derivative with respect to x, which we shall call the *third derivative* of f, by determining the derivative of the second derivative. Our notation for the third derivative of f is

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} \quad \text{or} \quad f'''(x).$$

From d^2y/dx^2 above,

$$\frac{d^3y}{dx^3} = f'''(x) = 12.$$

In general, the *nth order derivative* of f with respect to x is denoted as

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \quad \text{or} \quad f^n(x).$$

For our purposes, we shall mostly rely on the first and second derivatives of a function in our applications to profit (and to revenue) maximization or to cost minimization.

To gain some practice with the calculation of higher order derivatives, let us find f', f'', and f''' when $y = f(x) = x^6 - x^{-1} + 2x - 6$, $x \neq 0$. It is straightforward to show that

$$f' = 6x^5 + x^{-2} + 2,$$

$$f'' = 30x^4 - 2x^{-3},$$

$$f''' = 120x^3 + 6x^{-4}.$$

EXAMPLE 1.21

Find f′, f″ when

(a) $f(x) = 3x^4 + 9x^{-1}, \quad x \neq 0.$
(b) $f(x) = x^3 + 5x^{-2} + 3x^2.$

SOLUTION:

(a)
$$\begin{aligned} f(x) &= 3x^4 + 9x^{-1}, \\ f'(x) &= 12x^3 - \frac{9}{x^2}, \\ f''(x) &= 36x^2 + 18x^{-3} \\ &= 36x^2 + \frac{18}{x^3}. \end{aligned}$$

(b)
$$\begin{aligned} f(x) &= x^3 + 5x^{-2} + 3x^2, \\ f'(x) &= 3x^2 - 10x^{-3} + 6x, \\ f''(x) &= 6x + 30x^{-4} + 6, \\ &= 6x + \frac{30}{x^4} + 6. \end{aligned}$$

■

1.8.4 Integration

There are two distinct ways to define the concept of integration. First, an *integral* is the limit of a specific summation or addition process. Geometrically, it corresponds to the area under a given curve. This type of integral is termed a *definite integral*.

Second, an integral can be viewed as the result of reversing the process of differentiation. That is, if the derivative of $y = F(x)$ exists, then this derivative is also a function of x and can be denoted as f(x). The reverse problem then amounts to finding, from the given function f(x), a new function F(x) (or functions that have f(x) as their derivative). If F(x) can be found, then F(x) is termed the *indefinite integral* of f(x).

Specifically, suppose we are given the derivative $dy/dx = f(x)$, $a < x < b$, and asked to find $y = F(x)$. The function $y = F(x)$ is a solution to $dy/dx = f(x)$ if, over the domain $a < x < b$, F(x) is differentiable and

$$\frac{dF(x)}{dx} = f(x).$$

Hence, F(x) is termed the *integral of f(x)* with respect to x. In fact, if $y = F(x)$ is a particular solution of $dy/dx = f(x)$, then "all" solutions are subsumed within the expression

$$y = F(x) + c, \quad \text{where c is an arbitrary constant.}$$

This is indicated by writing

$$\int f(x)dx = F(x) + c,$$

where the symbol "$\int$" is termed the "integral sign."

Let us take a closer look at these two types of integrals.

1.8.5 The Definite Integral

Suppose we wish to find the area under the curve $y = f(x)$ and above the x-axis between $x = a$ and $x = b$. For example, we may be interested in determining the area under $y = f(x) = 10 - x$ and above the x-axis between $x = 2$ and $x = 7$ (Figure 1.10). A first approximation to this total area can be obtained by calculating the areas of a finite set of rectangles and then summing these separate areas. That is, we first consider the rectangle whose base (Δx) is from $x = 2$ to $x = 3$. The exact area of this rectangle is height x base $= f(2) \times \Delta x = 8 \times 1 = 8$. Clearly, the area of this rectangle is only an approximation to the actual area under f(x) between $x = 2$ and $x = 3$ since an error component (an overestimate) corresponding to the shaded triangular portion of the rectangle occurs.

Next, consider the area of the approximating rectangle between $x = 3$ and $x = 4$. It amounts to $f(3) \times \Delta x = 7 \times 1 = 7$ and also admits an error (it is shaded) or overestimate of the true area under f(x) between $x = 3$ and $x = 4$. Continuing this process three more times (we determine rectangular areas from $x = 4$ to $x = 5$, from $x = 5$ to $x = 6$, and from $x = 6$ to $x = 7$) enables us to approximate the entire desired area as the sum of the areas of the five approximating rectangles as

$$\sum_{x=2}^{7} f(x)(\Delta x) = f(2) \times 1 + f(3) \times 1 + f(4) \times 1 + f(5) \times 1 + f(6) \times 1$$
$$= 8 + 7 + 6 + 5 + 4 = 30.$$

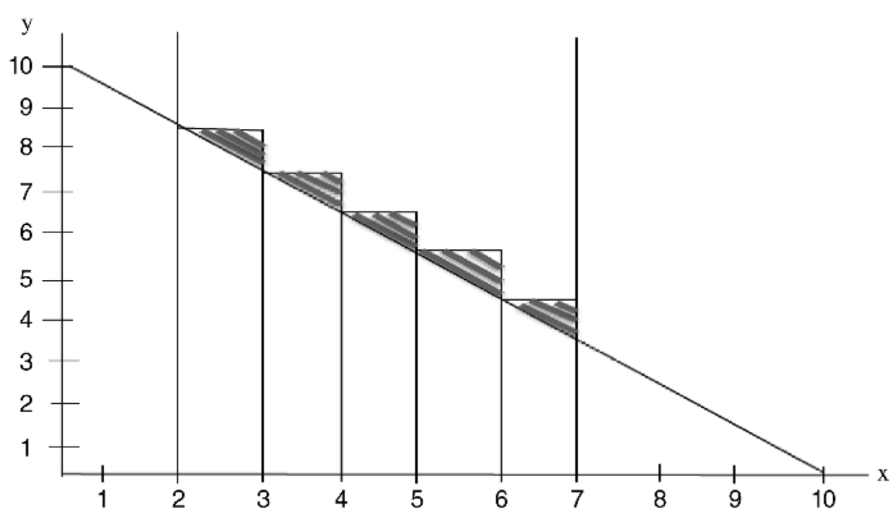


Figure 1.10 Area between $y = 10 - x$ and the x-axis from $x = 2$ to $x = 7$.

Thus, 30 is our approximation to the total area under $y = f(x) = 10 - x$ and above the x-axis between $x = 2$ and $x = 7$.

Let us now generalize this procedure. Given a function $y = f(x)$, we seek to find the approximate area under $f(x)$ and above the x-axis between $x = a$ and $x = b$ or over $[a,b]$. First, determine the base Δx of each approximating rectangle by dividing $[a,b]$ into n equal parts or

$$\Delta x = \frac{b - a}{n}.$$

Thus, the interval is subdivided according to

$$\begin{aligned} x_0 &= a, \\ x_1 &= a + \Delta x, \\ x_2 &= a + 2(\Delta x), \\ \vdots &\ \ \vdots \ \ \vdots \\ x_n &= a + n(\Delta x) = b. \end{aligned}$$

Next, the sum of the areas of the n approximating rectangles is given by

$$\sum_{i=1}^{n} f(x_i)(\Delta x) = f(x_1)(\Delta x) + f(x_2)(\Delta x) + \cdots + f(x_n)(\Delta x).$$

A glance back at Figure 1.10 reveals that if n increases or Δx decreases in size, the error incurred in using the sum of the rectangular areas to estimate the true area also decreases. And if, "in the limit," we let $\Delta x \to 0$, then the sum of the errors also tends to zero. Hence, an exact measure of the area under $y = f(x)$ and above the x-axis over $[a,b]$ is given by

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(\Delta x) &= \lim_{\Delta x\to 0} \sum_{i=1}^{n} f(x_i)(\Delta x) \\ &= \int_a^b f(x)dx, \end{aligned}$$

the *definite integral* of $f(x)$. Note that "$\int$" is an "operator" and the operation is summation. In sum, the definite integral is the limiting sum of a set of rectangles as the number of rectangles increases without bound; it amounts to the area under a curve over a closed interval. Note also that the above limit exists provided f is continuous.

How is a definite integral computed? To answer this question, we need to only look to the *Fundamental Theorem of Integral Calculus*: If $f(x)$ is a given continuous function on $[a, b]$ and F is any differentiable function such that $F'(x) = f(x), x \in [a,b]$, then

$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a).$$

To facilitate the implementation of this theorem, we need to first look to how indefinite integrals are evaluated. Why? Because the fundamental theorem utilizes

F(x) and F(x) is obtained via the indefinite integral. As was the case with differentiation, we shall rely upon a set of rules for integration.

1.8.6 Some Rules of Integration

1. Power rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c.$$

Here, c is the constant and $n \neq -1$. If $n = -1$, then

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln x + c.$$

EXAMPLE 1.22

$$\int x^5 dx = \frac{x^6}{6} + c.$$

■

2. The integral of zero is a constant, including zero:

$$\int 0\, dx = c.$$

3. The integral of a constant times a function is the constant times the integral of the function:

$$\int kg(x)dx = k\int g(x)dx, \quad \text{k is a constant.}$$

EXAMPLE 1.23

a. $\int 6x^3 dx = 6\frac{x^4}{4} + c = \frac{3x^4}{2} + c.$
b. $\int \frac{\sqrt{3}}{x} dx = \sqrt{3}\int \frac{1}{x} dx = \sqrt{3}\ln x + c.$
c. $\int 3x^2(y-1)^2 dx = 3(y-1)^2 \int x^2 dx = 3(y-1)^2\frac{x^3}{3} + c = x^3(y-1)^2 + c.$ ■

4. Integral of a sum or difference rule:

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx.$$

EXAMPLE 1.24

a. Integrate $\int (f(x) + g(x))dx$, where $f(x) = 10ax$ and $g(x) = 4bx^3$.

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx = \int 10axdx + \int 4bx^3 dx$$
$$= 10a\frac{x^2}{2} + 4b\frac{x^4}{4} + c = 5ax^2 + bx^4 + c.$$

b. $$\int (y-1)^2 dy = \int \left(y^2 - 2y + 1\right) dy = \int y^2 dy - \int 2y dy + \int 1 dy$$
$$= \frac{y^3}{3} - \frac{2y^2}{2} + y + c = \frac{1}{3}y^3 - y^2 + y + c.$$ ■

5. Integration by parts:

Recall from the product rule for differentiation that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides, we obtain

$$f(x)g(x) = \int f'(x)g(x) + \int f(x)g'(x),$$

or

$$\int f(x)g'(x) = f(x)g(x) - \int f'(x)g(x).$$

Generally, most textbooks present this rule in an abbreviated form. That is, let $u = f(x)$ and $v = g(x)$. Then,

$$\int u\, dv = uv - \int v\, du.$$

EXAMPLE 1.25

Solve the following using integration by parts.

$$\int x(x-2)^2 dx.$$

SOLUTION: Let $u = x$ and $dv = (x-2)^2 dx$,

$$du = 1$$

$$v = \int (x-2)^2 dx = \frac{(x-2)^3}{3}.$$

Using the formula

$$\int udv = uv - \int vdu, \text{ we obtain}$$

$$\int x(x-2)^2 dx = x\left[\frac{(x-2)^3}{3}\right] - \int \frac{(x-2)^3}{3} dx$$

$$= \frac{x(x-2)^3}{3} - \frac{(x-2)^4}{3(4)} + c$$

$$= \left[\frac{(x-2)^3}{3}\right]\left(x - \frac{x-2}{4}\right) + c$$

$$= \left[\frac{(x-2)^3}{3}\right]\left(\frac{3x+2}{4}\right) + c.$$

■

6. Integrals of logarithmic and exponential functions:

$$\int \ln x\, dx = x(\ln x - 1) + c,$$
$$\int e^x dx = e^x + c,$$
$$\int e^{kx} dx = \frac{1}{k} e^{kx} + c.$$

EXAMPLE 1.26a

$$\int (3e^x - 2x + e)dx = 3\int e^x dx - 2\int x dx + e\int dx = 3e^x - 2\frac{x^2}{2} + ex + c$$
$$= 3e^x - x^2 + ex + c.$$

■

EXAMPLE 1.26b

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + c.$$

■

Armed with the essentials of indefinite integration, we are now ready to explore the computation of definite integrals.

EXAMPLE 1.27

Let us evaluate the area under $y = f(x) = 10 - x$ and above the x-axis between $x = 2$ and $x = 7$ (see the preceding introduction to definite integrals). To this end, we have

$$\int_2^7 (10 - x)dx = \left(10x - \frac{1}{2}x^2\right)\Big]_2^7 = F(7) - F(2)$$
$$= \left[70 - \frac{1}{2}(49)\right] - \left[20 - \frac{1}{2}(4)\right]$$
$$= \frac{91}{2} - 18 = 55/2.$$

■

EXAMPLE 1.28

Find

$$\int_1^2 (2x + 5)dx = \int_1^2 2xdx + \int_1^2 5dx$$

$$= x^2\Big]_1^2 + 5x\Big]_1^2 = (4 - 1) + (10 - 5) = 8.$$

■

EXAMPLE 1.29

Evaluate

$$\int_0^1 \left(x^2 - 2x + 3\right)dx \int_0^1 x^2dx - \int_0^1 2xdx + \int_0^1 3dx$$

$$= \frac{1}{3}x^3\Big]_0^1 - x^2\Big]_0^1 + 3x\Big]_0^1 = \left(\frac{1}{3} - 0\right) - (1 - 0) + (3 - 0)$$

$$= \frac{7}{3}.$$

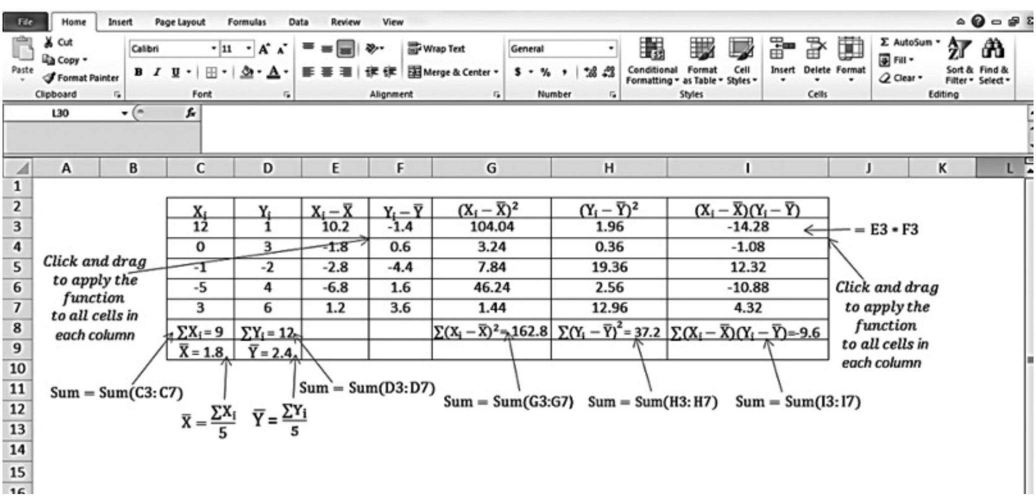
■

1.9 EXCEL APPLICATIONS

EXAMPLE 1.30

For the values of X and Y given in Example 1.5, evaluate r using the long and short formulas with EXCEL.

SOLUTION: Using Excel and the long formula, r is evaluated as follows:

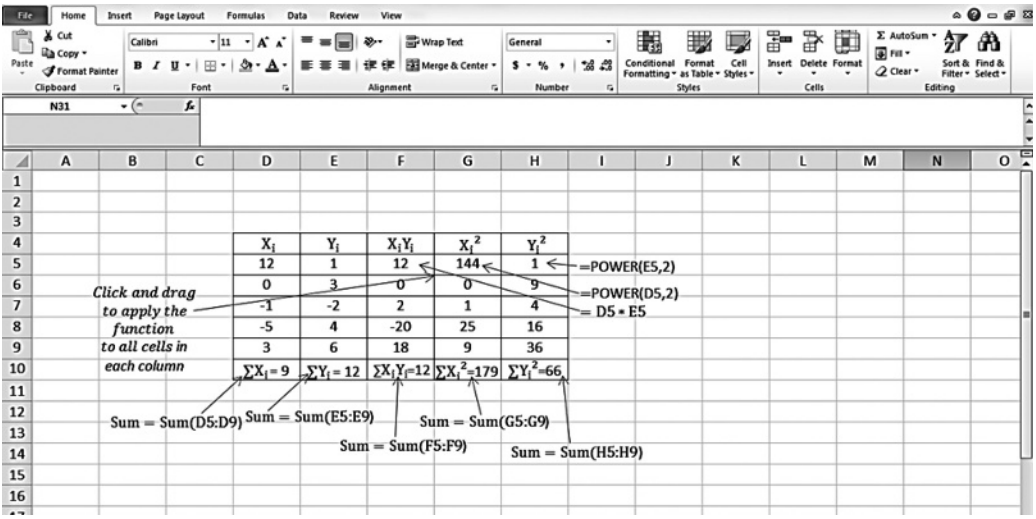


X_i	Y_i	$X_i - \bar{X}$	$Y_i - \bar{Y}$	$(X_i - \bar{X})^2$	$(Y_i - \bar{Y})^2$	$(X_i - \bar{X})(Y_i - \bar{Y})$
12	1	10.2	-1.4	104.04	1.96	-14.28
0	3	-1.8	0.6	3.24	0.36	-1.08
-1	-2	-2.8	-4.4	7.84	19.36	12.32
-5	4	-6.8	1.6	46.24	2.56	-10.88
3	6	1.2	3.6	1.44	12.96	4.32
$\Sigma X_i = 9$	$\Sigma Y_i = 12$			$\Sigma(X_i - \bar{X})^2 = 162.8$	$\Sigma(Y_i - \bar{Y})^2 = 37.2$	$\Sigma(X_i - \bar{X})(Y_i - \bar{Y}) = -9.6$
$\bar{X} = 1.8$	$\bar{Y} = 2.4$					

$$r = \sum_{i=1}^{5} \frac{\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sqrt{\sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 \sum_{i=1}^{5}\left(Y_i - \overline{Y}\right)^2}}$$

$$= \frac{-9.6}{\sqrt{(162.8)(37.2)}} = -0.1104.$$

Using Excel and the short formula, r is evaluated as follows:



$$r = \frac{\sum X_i Y_i - \frac{\sum X_i \sum Y_i}{n}}{\left[\left(\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}\right)\left(\sum Y_i^2 - \frac{\left(\sum Y_i\right)^2}{n}\right)\right]^{1/2}}$$

$$= \frac{12 - \frac{(9)(12)}{5}}{\left[\left(179 - \frac{9^2}{5}\right)\left(66 - \frac{12^2}{5}\right)\right]^{1/2}}$$

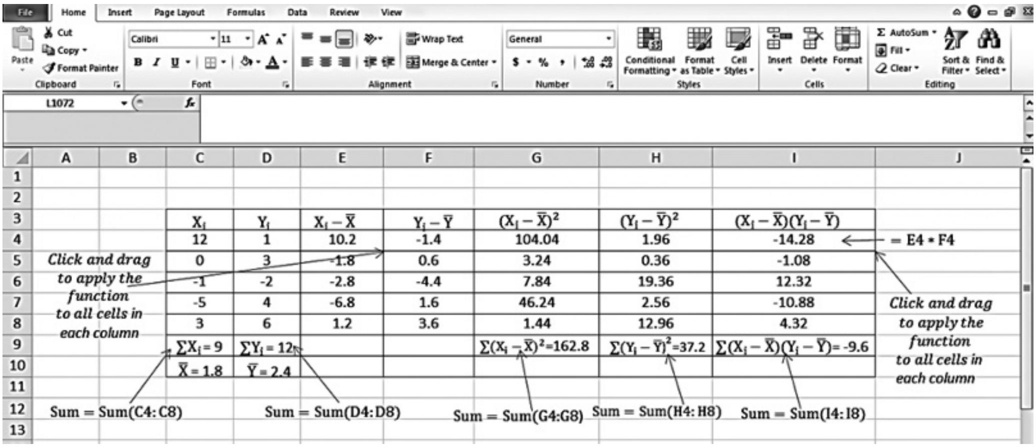
$$= \frac{-9.6}{[(162.8)(37.2)]^{1/2}} = -0.1104.$$

■

EXAMPLE 1.31

Using the X and Y values given in Example 1.5, evaluate the slope coefficient b by the long formula with Excel. Also, predict the value of Y when X = 11, assuming that the linear relationship between X and Y is Y = a + bX.

SOLUTION: Using Excel, the slope coefficient b is evaluated by the long formula as follows:



X_i	Y_i	$X_i - \overline{X}$	$Y_i - \overline{Y}$	$(X_i - \overline{X})^2$	$(Y_i - \overline{Y})^2$	$(X_i - \overline{X})(Y_i - \overline{Y})$
12	1	10.2	-1.4	104.04	1.96	-14.28
0	3	-1.8	0.6	3.24	0.36	-1.08
-1	-2	-2.8	-4.4	7.84	19.36	12.32
-5	4	-6.8	1.6	46.24	2.56	-10.88
3	6	1.2	3.6	1.44	12.96	4.32
$\Sigma X_i = 9$	$\Sigma Y_i = 12$			$\Sigma(X_i - \overline{X})^2 = 162.8$	$\Sigma(Y_i - \overline{Y})^2 = 37.2$	$\Sigma(X_i - \overline{X})(Y_i - \overline{Y}) = -9.6$
$\overline{X} = 1.8$	$\overline{Y} = 2.4$					

$$b = \frac{\sum_{i=1}^{5} (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{5} (X_i - \overline{X})^2}$$

$$= \frac{-9.6}{162.8} = -0.058968.$$

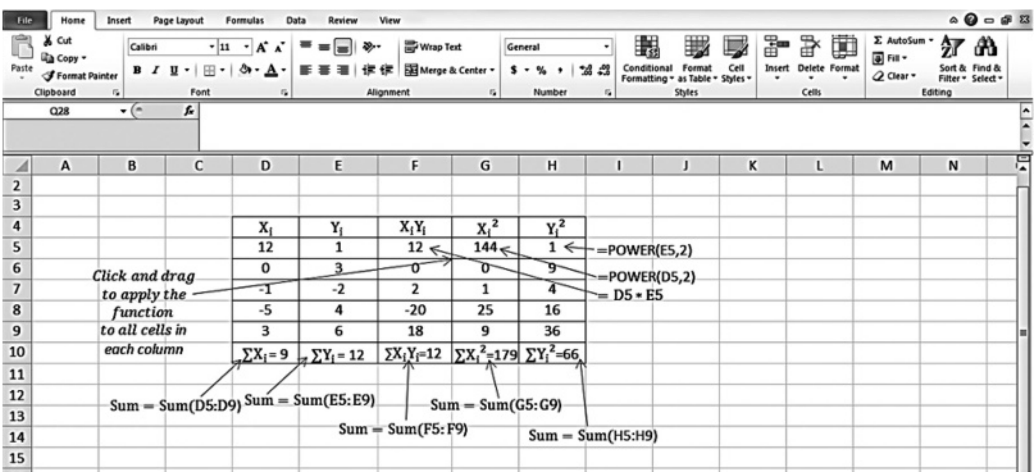
$$a = \overline{Y} - b\overline{X}$$

$$= 2.4 - (-0.058968)(1.8)^{\circ} = 2.5061424.$$

$$Y = a + bX$$

$$= 2.5061424 + (-0.058968)(11) = 1.8574944.$$

Using Excel, the slope coefficient b is evaluated by the short formula below:



X_i	Y_i	X_iY_i	X_i^2	Y_i^2
12	1	12	144	1
0	3	0	0	9
-1	-2	2	1	4
-5	4	-20	25	16
3	6	18	9	36
$\Sigma X_i = 9$	$\Sigma Y_i = 12$	$\Sigma X_iY_i = 12$	$\Sigma X_i^2 = 179$	$\Sigma Y_i^2 = 66$

$$b = \frac{\sum X_i Y_i - \dfrac{\sum X_i \sum Y_i}{n}}{\left[\sum X_i^2 - \dfrac{\left(\sum X_i\right)^2}{n}\right]}$$

$$= \frac{12 - \dfrac{(9)(12)}{5}}{179 - \dfrac{9^2}{5}}$$

$$= \frac{-9.6}{162.8} = -0.058968.$$

■

EXAMPLE 1.32

Solve the following simultaneous linear equations using the Solver Add-in in Excel.

$$3P + Q = 26 \text{ and } P - Q = 2.$$

SOLUTION: INSTRUCTIONS:

A. Excel spread sheet setup of the problem:
Enter data in Excel spread sheet.

a. Show variables P in B1 and Q in C1. Trial value 1 in B2 and 1 in C2. Type "equation 1" in A3 and enter the coefficient 3 of the variable P in B3 and 1 of the variable Q in C3. In the cell D3, enter the Array formula "=Sumproduct (highlight the trial values 1 and 1, highlight 3 and 1) and hit Enter. You will see 4 in D3, which is the left-hand side value of the equation 1 with the given trial values. In the cell E3, type "equal" (not the symbol = since Excel expects a formula to follow) and enter 26 in cell F3, which is the right-hand side value of equation 1.

b. Repeat the above step for equation 2 in cell A4. This will result in 0 in D4, "equal" in E4 and 2 in F4.

c. Click any empty cell below the setup table as follows:

	A	B	C	D	E	F
1	Variables	P	Q			
2	Trial values	1	1			
3	Equation 1	3	1	4	equal	26
4	Equation 2	1	-1	0	equal	2

B. Using Solver Add-In*

Click on DATA in the menu bar and then Solver in the extreme right side of the ribbon at the top of your screen.

a. In the Solver parameter dialog box, at set objective, select Target Cell D3, where 4 is obtained by the Array formula.
b. At "to" select the button "value of" and enter 26 in white bar.
c. Under "By Changing Variable Cells," click on the white bar and highlight both the trial value cells at the same time.
d. Click inside the lower bigger rectangular box and click on "Add" on the right-hand side to get "Add Constraint" dialog box.
e. Under Cell Reference: Highlight Cell, D4 where "0" is displayed by the formula.
f. Change the middle symbol to "=".
g. Constraint: Highlight Cell where "2" is typed.
h. Click "OK" since you don't have a third constraint equation.
i. Uncheck "Make Unconstrained Variables Non-Negative" box and select solving method as "Simplex LP."
j. Click on "Solve"; you see a circle marked "Keep Solver Solution" in the Solver Results dialog box.
k. Click OK to see the solution as in the table below:

	A	B	C	D	E	F
1	Variables	P	Q			
2	Trial values	7	5			
3	Equation 1	3	1	26	equal	26
4	Equation 2	1	-1	2	equal	2

Note: The trial values of variables P and Q are changed by solver program to P = 7 and Q = 5, which satisfy both the equations. Therefore, the final solution of the two equations is P = 7 and Q = 5.

* If you do not find Solver when you click on "Data" in the menu bar in Microsoft Excel (Office 2007 and onward),

1. Click *the MICROSOFT ICON or FILE* (at the top left of your screen).

2. Click *Excel Options* (at the bottom of the menu).

3. Click *ADD-INS.*

4. Click *GO* (at the bottom of the window).

5. Check the box with "Solver Add-in."

6. Click on *OK* and see *"Solver"* at the top right of your screen. ■

EXAMPLE 1.33

Solve the following simultaneous linear equations using the Solver Add-in in Excel.

$$2X + 3Y = 18 \quad \text{and} \quad X + 3Y = 15.$$

SOLUTION: Let $2X + 3Y = 18$ be equation 1 and $X + 3Y = 15$ be equation 2.
INSTRUCTIONS:

A. Excel spread sheet setup of the problem:
Enter data in Excel spread sheet.

a. Show variables X in B1 and Y in C1. Trial value 1 in B2 and 1 in C2.
Type "equation 1" in A3 and enter the coefficient, 2, of the variable X in B3 and 3 of the variable Y in C3. In the cell D3, enter the Array formula "=Sumproduct(highlight the trial values 1 and 1, highlight 2 and 3)" and hit Enter. You will see 5 in D3, which is the left-hand side value of equation 1 with the given trial values. In the cell E3, type "equal" (not the symbol = since Excel expects a formula to follow) and enter 18 in cell F3, which is the right-hand side value of equation 1.
b. Repeat the above step for equation 2 in cell A4. This will result in 4 in D4, "equal" in E4 and 15 in F4.
c. Click any empty cell below the setup table as follows:

	A	B	C	D	E	F
1	Variables	X	Y			
2	Trial values	1	1			
3	Equation 1	2	3	5	equal	18
4	Equation 2	1	3	4	equal	15

B. Using Solver Add-In*
Click on DATA in the menu bar and then Solver in the extreme right side of the ribbon at the top of your screen.

a. In the Solver parameter dialog box, at "set objective," select Target Cell D3, where 5 is obtained by the Array formula.
b. At "to" select the button "value of" and enter 18 in white box.
c. Under "By Changing Variable Cells," click on the white box and highlight both the trial value cells at the same time.
d. Click inside the lower bigger rectangular box and click on "Add" on the right-hand side to get "Add Constraint" dialog box.
e. Under Cell Reference: Highlight Cell, D4 where "4" is displayed by the formula.
f. Change the middle symbol to "=".
g. Constraint: Highlight Cell, where "15" is typed.
h. Click "OK" since you do not have a third constraint equation.
i. Uncheck "Make Unconstrained Variables Non-Negative" box and select solving method as "Simplex LP."
j. Click on "Solve"; you see a circle marked "Keep Solver Solution" in the Solver Results dialog box.
k. Click OK to see the solution as in the table below:

	A	B	C	D	E	F
1	Variables	X	Y			
2	Trial values	3	4			
3	Equation 1	2	3	18	equal	18
4	Equation 2	1	3	15	equal	15
5						

Note: The trial values of variables X and Y are changed by solver program to X = 3 and Y = 4, which satisfy both the equations. Therefore, the final solution of the two equations is X = 3 and Y = 4.

* If you do not find "Solver" when you click on "Data" in the menu bar in Microsoft Excel (Office 2007 and onward),

1. Click *the MICROSOFT ICON or FILE* (at the top left of your screen).
2. Click *Excel Options* (at the bottom of the menu).
3. Click *ADD-INS.*
4. Click *GO* (at the bottom of the window).
5. Check the box with "Solver Add-in."
6. Click on *OK* and see *"Solver"* at the top right of your screen. ■

CHAPTER 1 REVIEW

You should be able to:

1. Distinguish between a linear and nonlinear function.
2. Distinguish between dependent and independent variables.
3. Distinguish between the slope and Y intercept of a linear function.
4. Solve two simultaneous linear equations in two unknowns using the method of elimination.
5. Distinguish between consistent and dependent linear equation systems.
6. Explain the notation $\sum_{i=1}^{n} X_1$.
7. Calculate a Pearson correlation coefficient r.
8. Distinguish between the empty set and the universal set.
9. Identify the complement of a set.
10. Distinguish between the set operations of union and intersection.
11. Distinguish between open and closed intervals.
12. Distinguish between the domain and range of a function.
13. Determine if a particular graph represents a function using the vertical line test.

14. Graph a linear function using the intercept method.
15. Give an intuitive definition of a continuous function.
16. Apply the power, product, and quotient rules of differentiation.
17. Characterize the concavity of a function in terms of the sign of its second derivative.
18. Distinguish between a definite and indefinite integral.
19. Explain why marginal values are slopes of functions.

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EXERCISES

Solve the following.

1. $\frac{1}{3}X + 7 = 51.$
2. $\frac{2}{3X} + 9 = 12.$
3. $2.3X - 8.9 = -4.3.$

Solve the following sets of simultaneous equations:

4. $$\begin{aligned} 7X + 3Y &= 14, \\ X - Y &= 12. \end{aligned}$$

5. $2X + 3Y = 18,$
$X + 3Y = 15.$

6. $5X - 7Y = 28,$
$2X + 5Y = 19.$

7. $2X_1 - X_2 = 3,$
$7X_1 + 2X_2 = 27.$

8. **a.** $3P + Q = 26,$
$P - 5Q = 2.$

b. $-6X + 3Y = 0,$
$3X + Y = 0.$

c. $P + 2Q = 10,$
$4P + 8Q = 20.$

Suppose that there are six observations for the variables X and Y as given below:

$X_1 = 2,\quad X_2 = 1,\quad X_3 = 3,\quad X_4 = -5,\quad X_5 = 1,\quad X_6 = -2$
$Y_1 = 4,\quad Y_2 = 0,\quad Y_3 = -1,\quad Y_4 = 2,\quad Y_5 = 7,\quad Y_6 = -3$

For Exercises 9–19, compute the following:

9. $\sum_{i=1}^{6} X_i.$

10. $\sum_{i=1}^{6} Y_i.$

11. $\sum_{i=1}^{6} X_i^2.$

12. $\sum_{i=1}^{6} Y_i^2.$

13. $\sum_{i=1}^{6} X_i Y_i.$

14. $\sum_{i=1}^{6}(X_i + Y_i).$

15. $\sum_{i=1}^{6}(X_i - Y_i).$

16. $\sum_{i=1}^{6}(X_i - 3Y_i + 2X_i^2).$

17. $\sum_{i=1}^{6} cX_i,$ where $c = -1.$

18. $\sum_{i=1}^{6}(X_i - 3Y_i + c),$ where $c = 3.$

19. $\sum_{i=1}^{6} (X_i - Y_i)^2$
$= \sum_{i=1}^{6} X_i^2 - \sum_{i=1}^{6} 2X_i Y_i + \sum_{i=1}^{6} Y_i^2.$

20. Write set notation for the set of all whole numbers less than 8.

21. Write set notation for the set of odd whole numbers greater than 0.

22. Write set notation for the solution set of equation $(x - 2)(x + 3) = 0.$

Determine whether true or false in Exercises 23–25.

23. $2 \in \{-3,2,4,5,7\}.$

24. $3 \in \{0,2,4,6,8,\ldots\}.$

25. $\{0,1,7\} \subset \{0,1,8,13\}.$

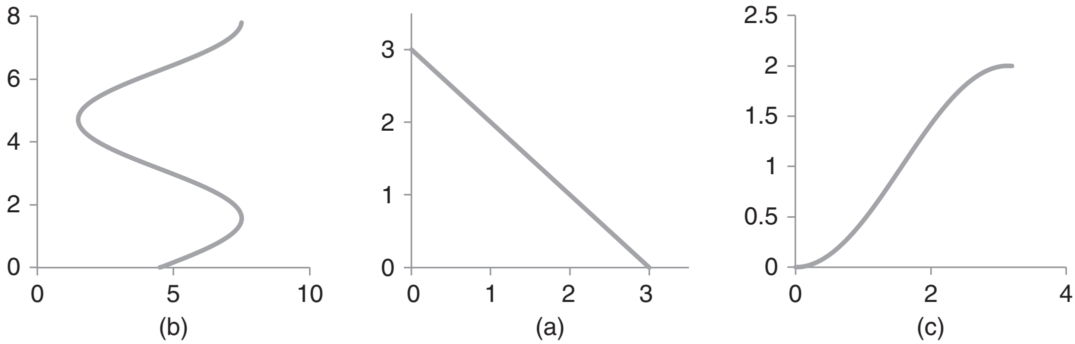
26. Find $\{0,3,5,7\} \cap \{0,1,3,11\}.$

27. Find $\{3,5,9\} \cup \{1,6,7\}.$

28. If U = {3 red marbles, 7 blue marbles, 8 white marbles} and A = {8 white marbles}, find $\overline{A}$. What is $\overline{\overline{A}}$ (the complement of $\overline{A}$)?

29. Using the information in Exercise 28, find $A \cup \overline{A}$, $A \cap \overline{A}$, $\overline{U}$, and $\overline{\phi}$.

30. Determine which of the following graphs are also functions:



Graph the following functions:

31. $2x + y - 10 = 0.$

32. $2y = \frac{1}{x}, \quad x \neq 0.$

33. $y - 3\sqrt{x} = 5.$

Find dy/dx for the functions given in Exercises 34–43.

34. $y = x^{3/4}.$

35. $y = \frac{1}{x^3}, \quad x \neq 0.$ Evaluate $\frac{dy}{dx}$ at $x = 1.$

36. $y = 15.$

37. $y = \left(\frac{1}{2}\right).$

38. $y = 5x^3.$ Evaluate $\frac{dy}{dx}$ at $x = 2.$

39. $y = \left(\frac{1}{5}\right)x^5.$

40. $y = 6x + 7.$

41. $y = -0.5x + 3.$

42. $y = 3x^2 + 4\left(\frac{1}{x^2}\right), x \neq 0.$

43. $y = 5x^3 + 4x^2 + 3x + 2.$

44. **a.** If $f(x) = 2x^2+1$, $g(x) = 7x^3$, and $y = f(x)g(x)$, find $\frac{dy}{dx}$.
b. If $f(x) = 5x^3$, $g(x) = 6+4x^5$, and $y=f(x)g(x)$, find $\frac{dy}{dx}$.

45. **a.** If $f(x) = 3x - 6$, $g(x) = 4x^2 + x$, and $y = f(x)/g(x)$, find $\frac{dy}{dx}$.
b. If $f(x) = 5x^3 - 10$, $g(x) = -x^2 + x$, and $y = \frac{f(x)}{g(x)}$, find $\frac{dy}{dx}$.

46. **a.** If $f(x) = 5x^7 + 2x^2$, find $f'''(x)$.
b. If $f(x) = 9x^2 - 11x^5 + 7x$, find $f'''(x)$.

47. $\int \frac{1}{x^2} dx.$

48. $\int x^3 dx.$

49. $\int \left(3x^2 - 4x + 8\right) dx.$

50. $\int xe^{2x}dx$.

(*Hint:* Use integration by parts.)

51. $\int_a^b \frac{1}{(b-a)}dx$.

52. $\int_0^t 100e^{-0.1x}dx$.

53. Given

$$\begin{array}{lllll} X_1 = -2, & X_2 = -3, & X_3 = 0, & X_4 = 5, & X_5 = 9 \\ Y_1 = 5, & Y_2 = 9, & Y_3 = 6, & Y_4 = 8, & Y_5 = 7 \end{array}$$

a. Evaluate the Pearson correlation coefficient

$$r = \sum_{i=1}^{5} \frac{\left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sqrt{\sum_{i=1}^{5}\left(X_i - \overline{X}\right)^2 \sum_{i=1}^{5}\left(Y_i - \overline{Y}\right)^2}}, \quad (1 \le r \le 1),$$

where $\overline{X} = \frac{\sum X_i}{5}$ and $\overline{Y} = \frac{\sum Y_i}{5}$. How is r interpreted?

b. Also, reevaluate r using the following "short formula":

$$r = \frac{\sum X_i Y_i - \frac{\sum X_i \sum Y_i}{n}}{\left(\left(\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}\right)\left(\sum Y_i^2 - \frac{\left(\sum Y_i\right)^2}{n}\right)\right)^{\frac{1}{2}}}.$$

(*Hint:* To facilitate your calculations, construct a table.)

54. Using the data and the results in Exercise 53, compute $\frac{r \times \sqrt{3}}{\sqrt{1-r^2}}$.

55. Using the data and results in Exercise 53,

a. Compute the slope value, $b = \frac{\sum_{i=1}^{5}(X_i-\overline{X})(Y_i-\overline{Y})}{\sum_{i=1}^{5}(X_i-\overline{X})^2}$, and the intercept value, $a = \overline{Y} - b\overline{X}$, of the linear equation $Y = a + bX$. Use this estimated equation to predict the value of Y when $X = 11$.

b. Also, reevaluate the slope value b using the following "short formula":

$$b = \frac{\sum X_i Y_i - \frac{\sum X_i \sum Y_i}{n}}{\left[\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}\right]}.$$

56. Repeat Exercise 53 with the following values of the variables:

$$\begin{array}{llll} X_1 = 7, & X_2 = 1, & X_3 = 4, & X_4 = 8 \\ Y_1 = 5, & Y_2 = 0, & Y_3 = 10, & Y_4 = 1 \end{array}$$

57. Find the derivative of $y = f(x)g(x)$, where

a. $f(x) = 9x+7$, $g(x) = 2x^3 + x$.

b. $f(x) = 4x^2-3x$, $g(x) = x^7 - 7$.

58. If $f(x) = 3x^6 - 2x^2$, $g(x) = 4x^3 + x^2$, and $y = f(x)\,g(x)$, find $\frac{dy}{dx}$.

59. Find the derivative of $y = \frac{f(x)}{g(x)}$, where

a. $f(x) = x^2 - 9$, $g(x) = x + 3$.

b. $f(x) = \frac{1}{X} + x$, $x \neq 0$, $g(x) = 2x^3 + 3$.

60. **a.** If $f(x) = 6x^9 - 5x^{-2}$, find $f'(x)$, $f''(x)$, and $f'''(x)$.
b. If $f(x) = 8x^3 - 7x^2$, find $f'(x)$, $f''(x)$, and $f'''(x)$.

EXCEL APPLICATIONS

61. Answer Exercise 53, both parts (a) and (b), by formulating appropriate Excel spread sheets.
62. Also answer Exercise 55, both parts (a) and (b), by formulating appropriate Excel spread sheets.
63. Solve the simultaneous equations given in Exercises 4, 6, and 8 (a) earlier using the Solver Add-in in Excel.

APPENDIX 1.A: A REVIEW OF BASIC MATHEMATICS

1.A.1 Order of Arithmetic Operations

In simplifying arithmetic expressions, it is conventional to follow the order given below:

1. Evaluate items in parentheses (•) first, items in braces {•} second, and items in large brackets [•] next.
2. Exponents.
3. Multiplication or division, whichever comes first, left to right.
4. Addition or subtraction, whichever comes first, left to right.

EXAMPLE A.1

$$\text{Solve}: \left[\{(20-12)\div(7-5)\}+2\right]\times 7+\sqrt{36}+9^2+11-4\times\left(\frac{6}{\sqrt{144}}\right)$$

$$= \left[\{8\div 2\}+2\right]\times 7+\sqrt{36}+9^2+11-4\times\left(\frac{6}{\sqrt{144}}\right)\text{(parentheses first)}$$

$$= [4+2]\times 7+6+81+11-4\times\frac{6}{12}\text{(powers and/or square roots next)}$$

$$= 6\times 7+6+81+11-4\times\frac{1}{2}\text{(large brackets next)}$$

$$= 42+6+81+11-2\text{(multiplication)}$$

$$= 138\text{(addition and subtraction, left to right).}$$

Note : Define the positive square root (denoted $\sqrt{\cdot}$) of a positive number as positive.

For example, $\sqrt{25} = 5$.

■

EXAMPLE A.2

$$\text{Solve}: \left[\left(8^2\right) \times 4\right] \times 7 + 16$$
$$= [64 \times 4] \times 7 + 16$$
$$= 256 \times 7 + 16$$
$$= 1792 + 16$$
$$= 1808.$$

■

1.A.2 Exponents

An *exponent* or *power* is a symbol placed at the upper right of the *base* (a number or variable), which indicates the number of times the latter is to be multiplied by itself. For example, $3^4 = 3 \times 3 \times 3 \times 3 = 81$. Here, 3 is called the base and 4 is called the power or exponent. In general, $a^p = a \times a \times a \times a \cdots$ (p times). With base e, the exponential function $y = e^x$ has several applications. Some of these are shown later in Chapter 2. Here, e is an irrational number (it can be expressed as a nonrepeating and unending decimal) introduced by Leonhard Euler (1707–1783). It is expressed as

$$e = 1 + \left(\frac{1}{1}\right) + \left(\frac{1}{1 \times 2}\right) + \left(\frac{1}{1 \times 2 \times 3}\right) + \left(\frac{1}{1 \times 2 \times 3 \times 4}\right) + \cdots$$

and approximately equals to 2.7182818. Figure A.1 shows $y = e^x$.

Exponents are extremely useful in simplifying expressions in multiplication and division involving a combination of algebraic and numerical values. There are several applications of exponents in economics and business dealing with growth rates, especially continuous compounding. Exponents are useful in simplifying both linear and nonlinear functions and comparing their graphs. They are also used in the smoothing of functions in forecasting.

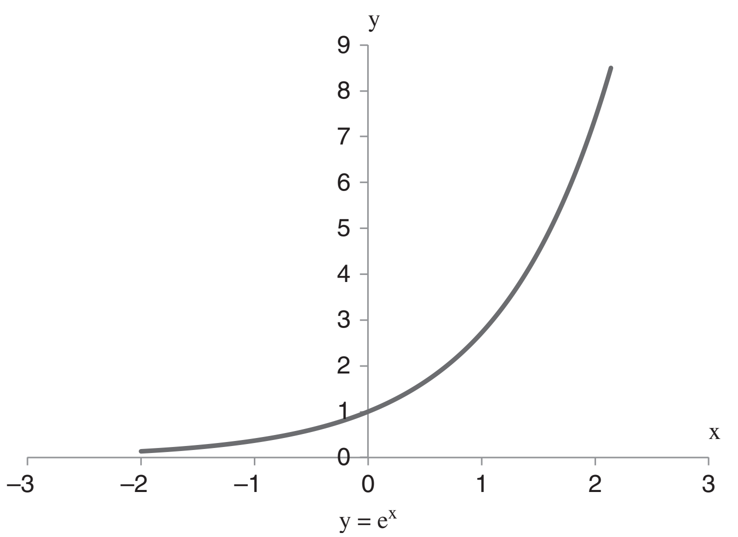


Figure A.1 Graph of the exponential function e^x.

Rules of Exponents

1. $a^p \times a^q = a^{(p+q)}$.
Example: $3^3 \times 3^2 = 3^{3+2} = 3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243$.

2. $(a^p)^q = a^{p \times q}$.
Example: $\left(3^3\right)^2 = 3^{(3 \times 2)} = 3^6 = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$.

3. $(a \times b)^p = a^p \times b^p$.
Example: $(2 \times 4)^3 = 2^3 \times 4^3 = 8 \times 64 = 512$.

4. $\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$.
Example: $\left(\frac{2}{3}\right)^3 = \left(\frac{2^3}{3^3}\right) = \left(\frac{8}{27}\right) = 0.2963$.

5. $\frac{a^p}{a^q} = a^{p-q}$.
Example: $\left(\frac{6^4}{6^2}\right) = 6^{4-2} = 6^2 = 36$.

6. $a^{1/p} = \sqrt[p]{a}$
Example: $64^{1/6} = \sqrt[6]{64} = 2$.
Note: If p is even, a has to be positive to get a real number. If a is negative, $\sqrt{a}$ is not a real number. If p is odd, then $\sqrt[p]{a}$ is defined for both positive and negative values of a. For example, $\sqrt[6]{-64}$ is not a real number, while $\sqrt[3]{-64} = -4$.

7. $a^{p/q} = \sqrt[q]{a^p}$.
Example: $(64)^{5/6} = \sqrt[6]{64^5} = \left(\sqrt[6]{64}\right)^5 = 2^5 = 32$.

8. $a^{-p} = \frac{1}{a^p}$.
Example: $3^{-3} = \left(\frac{1}{3^3}\right) = \left(\frac{1}{27}\right) = 0.0370$.

9. $a^1 = a$.
Example: $6^1 = 6$.

10. $a^0 = 1 (a \neq 0)$.
0° is not defined.
Example: $10^\circ = 1$.

11. $1^p = 1$.
Example: $1^4 = 1$.

EXAMPLE A.3

$$\text{Simplify}: \frac{\left(a^3\right)^2 \times \left(b^{(4/5)}\right)^3}{a^4 \times b^4}$$
$$= \left(\frac{a^6 \times b^{(12/5)}}{a^4 \times b^4}\right)$$
$$= a^{6-4} \times b^{(12/5)-4}$$
$$= a^2 \times b^{-((8/5))}$$
$$= \frac{a^2}{\sqrt[5]{b^8}} \quad \text{or} \quad \frac{a^2}{b\sqrt[5]{b^3}}.$$

■

EXAMPLE A.4

$$\text{Simplify}: \frac{\left(x^3\right)^{1/9} \times y^{3/4}}{\left(\frac{x}{y}\right)^{3/4}}$$

$$= \frac{x^{3/9} \times y^{3/4}}{x^{3/4} \times y^{-(3/4)}}$$

$$= x^{(3/9)-(3/4)} \times y^{(3/4)+(3/4)}$$

$$= x^{-(15/36)} \times y^{6/4}$$

$$= \frac{y^{3/2}}{x^{5/12}}.$$

■

1.A.3 Logarithms

The *logarithm* of a positive number y is the power to which the base must be raised to equal the given number. In general, if $y = a^x$, then$x = \log_a y$, where a is the base. For example, the logarithm of 8 to the base 2 is 3, because 3 is the power to which 2 must be raised to produce 8 ($2^3 = 8, \log_2 8 = 3$). If $y = 1$, $\log y = 0\left(1 = a^0 \text{ or } x = 0\right)$. Clearly, a logarithm is a fancy way to write an exponent.

The base a cannot be 1. If $a = 1$, the logarithm is still defined but is trivial. If we want to solve $x = \log_1 y$, we can convert the log to exponential form, $y = 1^x$. In this case, y must be 1 because any real power of 1 is 1, and x can be any real number. So, there is no certain answer when $a = 1$.

There is a one-to-one relationship between the logarithmic functions and exponential functions. Table A.1 shows some examples of this relationship.

Special Cases in Logarithms

Common logs:	If the base a is 10, the logarithms are called *common logs*.
Natural (Napierian) logs:	If the base a is the mathematical constant e, the logarithms are called *natural logs* and denoted "ln" for short. These logs are used in the natural and social sciences to represent growth or decay.

Table A.1 Equivalence of Exponential and Logarithmic Functions: Some Examples

Exponential	Logarithmic
$y = a^x$	$x = \log_a y$
$8 = 2^3$	$3 = \log_2 8$
$16 = \left(\frac{1}{2}\right)^{-4}$	$-4 = \log_{1/2} 16$
$\frac{1}{25} = 5^{-2}$	$-2 = \log_5 \left(\frac{1}{25}\right)$
$1000 = 10^3$	$3 = \log_{10} 1000$
$81 = 3^4$	$4 = \log_3 81$

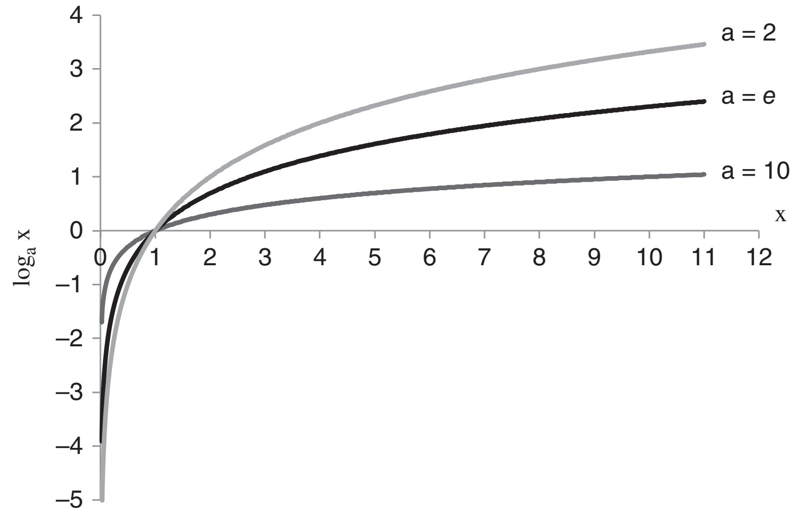


Figure A.2 Graphs of logarithmic functions with three different bases.

Binary (Base 2) logs: If the base a is 2, the logarithms are called *binary logs* and denoted "lb." Binary logs are commonly used in computer science and information theory.

Figure A.2 depicts common, natural, and binary logs.

Most calculators can be used to evaluate common logs as well as natural logs. For example, we can find the common log of 0.00127 using a calculator's "log function" as $\log_{10} 0.00127 = -2.896$. Scientific calculators typically have both "log" and "ln" buttons. For example, if we want to find the natural logs of 157.2 and 0.2759, we can use "ln function." So, $\log_e 157.2 = \ln 157.2 = 5.0575$. Similarly, $\ln 0.2759 = -1.2877$.

Tables provided in some textbooks can also be used to find the logs of numbers or numbers from logs (also called antilogs).

Rules of Logarithms

1. *Product Rule:* If x and y are positive numbers, then $\log_a xy = \log_a x + \log_a y$

The logarithm of a product of two numbers equals the sum of logarithms of the individual numbers. This is similar to the rule that the exponent of the product of two numbers to the same base equals the sum of the exponents of the same base. (See rule 1, Section 1.2.)

EXAMPLE A.5

a. $\log_{10}(3 \times 7) = \log_{10}3 + \log_{10}7$.

b. $\log_4 7 + \log_4 5 = \log_4(7 \times 5) = \log 35$.

c. $\log_7 7x = \log_7 7 + \log_7 x = 1 + \log_7 x$, since $\log_7 7 = 1$.

d. $\log_{10}(357 \times 269) = \log_{10}357 + \log_{10}269$.

■

2. *Quotient Rule:* If *x* and *y* are positive numbers, then $\log_a(x/y) = \log_a x - \log_a y$.

EXAMPLE A.6

a. $\log_4(7/3) = \log_4 7 - \log_4 3$.

b. $\log_3(y/8) = \log_3 y - \log_3 8$. ■

3. *Power Rule:* $\log_a x^k = k \log_a x$.

This rule follows from the product rule above. For example, if $y = x$ in the product rule, then

$$\log_a xy = \log_a x^2 = \log_a x + \log_a x = 2 \log_a x.$$

EXAMPLE A.7

a. $\log_3 x^7 = 7 \log_3 x$.

b. $2\log_a x^n = 2n\log_a x$. ■

4. *Rule of Addition or Subtraction:* $\log_a(x \pm y) \neq \log_a x \pm \log_a y$.

This rule follows the one-to-one relationship between logs and exponents: $a^{x+y} \neq a^x + a^y$.

EXAMPLE A.8

Find the common log of 100.

Any positive number can be represented as a power of 10. Thus, $100 = 10^2$. Therefore, $\log_{10} 100 = 2$. ■

1.A.4 Prime Factorization

Factoring to prime numbers is a process by which a number is written as a product of its prime factors. A *prime number* is a natural number that can only be divided by 1 and itself without a remainder. Prime factorization is a widely used process in dealing with fractions. The following are the smallest prime numbers less than 50:

2,3,5,7,11,13,17,19,23,29,31,37,41,43,47.

The number 2 is the only even prime number because all other nonzero even numbers can be divided by 2.

EXAMPLE A.9

Find the prime factors of 18, 44, 54, and 120.

$$18 = 2 \times 3^2$$
$$44 = 2^2 \times 11$$
$$54 = 2 \times 3^3$$
$$120 = 2^3 \times 3 \times 5$$ ■

1.A.5 Factoring

Monomial Factoring

Monomial factoring is a process by which an expression such as (ab + ac) is written as a product of its factors a and (b + c). For example, to factorize the expression 8xy – 4x, first we need to find the greatest common factor of 8xy and 4x, which is 4x. Then rewrite the expression as (4x)(2y – 1). Similarly, the factors of the expression 5ax – 20bx are 5x and (a – 4b).

EXAMPLE A.10

Factorize the following expressions:

a. $3x^2 + 5xy$.

b. $6x^3y^2 - 24x^2y + 18x^4y^3$.

SOLUTIONS:

a. $3x^2 + 5xy = x(3x + 5y)$.

b. $6x^3y^2 - 24x^2y + 18x^4y^3 = 6x^2y(xy) - 6x^2y(4) + 6x^2y\left(3x^2y^2\right) = 6x^2y\left(xy - 4 + 3x^2y^2\right)$.

Here, $6x^2y$ is the greatest common factor. ■

Binomial Factoring

Binomial factoring is used when the expression is a quadratic. A *quadratic expression* is a second-degree polynomial equation that has the form $y = ax^2 + bx + c$. When a is 1, the expression has the form of $y = x^2 + bx + c$, and factoring is simpler. In this case, find two integers, K and L, such that $KL = c$ and $K + L = b$. In case a is not equal to 1, find two integers K and L such that $KL = ac$ and $K + L = b$.

For example, if we want to factorize $y = x^2 - 5x + 6$, we need find two integers, K and L, such that $KL = 6$, and $K + L = -5$. In this case, factors are $(x + K)$ and $(x + L)$, such that $(x + K)(x + L) = x^2 + (K + L)x + KL$. By trial and error, we find $K = -2$ and $L = -3$. Thus, $x^2 - 5x + 6 = (x - 2)(x - 3)$.

EXAMPLE A.11

a. Factorize the expression $y = 2x^2 + 6x - 8$.

The expression can be written as $y = 2\left(x^2 + 3x - 4\right)$. As $x^2 + 3x - 4$ has the form of $y = x^2 + bx + c$, we need to find two integers, K and L, such that $KL = -4$ and $K + L = 3$. In this case, K and L are –1 and 4, respectively. So we can write the expression as $y = 2(x - 1)(x + 4)$. Note that $y = 2x^2 + 6x - 8$ has three factors.

b. Find the factors of $2x^2 + x - 6$.

We need to find two integers, K and L, such that $KL = 2(-6) = -12$, the product of the coefficient of x^2, a, and the constant c, and $K + L = 1$, the coefficient of x, b. By trial and error, we see that $K = 4$ and $L = -3$. Then, we rewrite the given expression as $2x^2 + 4x - 3x - 6$ and factor by grouping $2x(x + 2) - 3(x + 2)$ or $(x + 2)(2x - 3)$. ■

EXAMPLE A.12

Find the factors of $y = 3x^2 - 11x - 4$.

First, we need to find two integers K and L, such that $KL = 3 \times -4 = -12$ and $K + L = -11$. In this case, K and L are −12 and +1 ($(-12) \times 1 = -12$ and $(-12) + 1 = -11$). So, we can rewrite the expression as

$$\begin{aligned} y &= 3x^2 + x - 12x - 4, \\ y &= x(3x + 1) - 4(3x + 1), \\ y &= (3x + 1)(x - 4). \end{aligned}$$

■

1.A.6 Fractions: Ratios

A *fraction* is a ratio of the form x/y, where y is different from zero and x and y are both whole numbers. x is known as the *numerator* and y the *denominator*. Ratios $\frac{13}{4}$, $\frac{4}{9}$, and $\frac{-6}{7}$ are examples of such fractions. If the numerator of a fraction is less than the denominator, the fraction is called a *proper fraction*. When the numerator is greater than or equal to the denominator, the fraction is called an *improper fraction*. Thus, $\frac{4}{9}$ and $\frac{-6}{7}$ are proper fractions and $\frac{13}{4}$ is an improper fraction. Note that the ratio becomes a percent when the denominator becomes 100. For example, 67% means a ratio of $\frac{67}{100}$.

1.A.6.1 Greatest Common Factor The *greatest common factor* (GCF), also known as the greatest common denominator or greatest common divisor, is the largest number that is a common divisor of two or more non-zero numbers with zero remainder. GCF is used to reduce fractions to their lowest terms, which are called *irreducible fractions*. When the fractions are irreducible, the greatest common factor is 1.

To find the greatest common factor, prime factors of each number should be listed. Then the numbers that are common should be multiplied to find the GCF.

EXAMPLE A.13

Reduce the fraction 36/48.

To reduce, we need to find the greatest common factor of 36 and 48.

$$\begin{aligned} &\text{Prime factors of } 36: \quad 2 \times 2 \times 3 \times 3. \\ &\text{Prime factors of } 48: \quad 2 \times 2 \times 2 \times 2 \times 3. \end{aligned}$$

Here, $2 \times 2 \times 3$ is common in both numbers. So, $2 \times 2 \times 3 = 12$ is the greatest common factor.

$$\text{GCF}(36, 48) = 12.$$

So, the fraction can be written as

$$\frac{36}{48} = \frac{(3 \times 12)}{(4 \times 12)} = \frac{3}{4}.$$

■

1.A.6.2 Least Common Multiple The *least common multiple* (LCM) or least common denominator is the smallest positive number that is the common multiple of two or more nonzero numbers. LCM of two or more numbers can be divided by these numbers with no remainder. It is required to find the lowest common denominator when adding or subtracting fractions.

There are several ways to find the least common multiple. One way is to list the multiples of the numbers and then find the smallest common one. The other way is to prime factorize the numbers, and then multiply all the primes at their highest powers.

EXAMPLE A.14

Find the least common multiple of 6 and 8.

Method 1
Multiples of 6: 0,6,12,18,24,30,36, . . .
Multiples of 8: 0,8,16,24,32,40,48, . . .

Here, 24 is the lowest nonzero multiple for both numbers; 24 can be divided by both 6 and 8. So,

$$\text{LCM}(6, 8) = 24.$$

Method 2
Prime factors of 6: 2×3
Prime factors of 8: $2 \times 2 \times 2 = 2^3$.

Here, there are two primes, 2 and 3. To find the LCM, we should take the primes at their highest powers and multiply them. So, we have 2^3 and 3^1. If we multiply these, we will find

$$\text{LCM}(6, 8) = 2^3 \times 3 = 8 \times 3 = 24.$$

■

1.A.6.3 Addition and Subtraction of Fractions

If the denominator is common to the fractions, which are called *like fractions*, add or subtract the numerators and divide by the common denominator to obtain the result.

EXAMPLE A.15

a. $\frac{15}{19} + \frac{6}{19} = \frac{15 + 6}{19} = \frac{21}{19}$.

b. $\frac{15}{19} - \frac{6}{19} = \frac{15 - 6}{19} = \frac{9}{19}$.

However, if the denominator is not common to the fractions (we have *unlike fractions*), then the following procedure is used to obtain the least common denominator (LCD). First, note that multiplying the numerator and denominator by the same nonzero number does not change the value of the fraction. Second, the usual procedure for obtaining the common denominator is to multiply the numerator and the denominator of each fraction by the ratio of the LCM and the denominator of each fraction. ■

EXAMPLE A.16a

Evaluate $\frac{7}{6} + \frac{11}{4}$.

We first find LCD.

Step 1: Factor each denominator as a product of primes.

Step 2: Figure out the greatest number of times that each prime factor occurs in any single factorization.

Step 3: LCM is the product of the factors found in step 2.

Thus, $6 = 3 \times 2$ and $4 = 2 \times 2$.

$$\text{LCM} = 3 \times 2 \times 2 = 12.$$

Thus, 12 is the LCM. In relation to fractions, note that LCD is the same as LCM.

With $12 \div 6 = 2$, the fraction $\frac{7}{6} = \frac{7 \times 2}{6 \times 2} = \frac{14}{12}$.

And with $12 \div 4 = 3$, the fraction $\frac{11}{4} = \frac{11 \times 3}{4 \times 3} = \frac{33}{12}$.

$$\frac{7}{6} + \frac{11}{4} = \frac{14}{12} + \frac{33}{12} = \frac{47}{12}.$$ ■

EXAMPLE A.16b

Find $\frac{8}{6} + \frac{21}{18} - \frac{11}{12}$.

To solve this expression, we need to find the least common multiple of 6, 18, and 12.

$$\begin{aligned} 6 &= 3 \times 2 \\ 18 &= 3 \times 3 \times 2 \\ 12 &= 3 \times 2 \times 2 \end{aligned}$$

In this case, $\text{LCD} = \text{LCM} = 36 = 3 \times 3 \times 2 \times 2$, so that

$$\begin{aligned} \frac{8}{6} &= \frac{8 \times 6}{6 \times 6} = \frac{48}{36}, \\ \frac{21}{18} &= \frac{21 \times 2}{18 \times 2} = \frac{42}{36}, \\ \frac{11}{12} &= \frac{11 \times 3}{12 \times 3} = \frac{33}{36}. \end{aligned}$$

$$\text{Thus, } \frac{8}{6} + \frac{21}{18} - \frac{11}{12} = \frac{48}{36} + \frac{42}{36} - \frac{33}{36} = \frac{48 + 42 - 33}{36} = \frac{57}{36}.$$ ■

1.A.6.4 Multiplication and Division of Fractions

The product of any two fractions is obtained by the ratio of the product of the two numerators and the product of the two denominators.

When dividing a fraction by another fraction, take the reciprocal of the second action and multiply it by the first fraction, following the previous procedure.

EXAMPLE A.17

a. $\frac{8}{5} \times \frac{10}{3} = \frac{8 \times 10}{5 \times 3} = \frac{80}{15} = \frac{16}{3}$.

b. $\frac{-9}{12} \times \frac{2}{3} = \frac{(-9) \times 2}{12 \times 3} = \frac{-18}{36} = \frac{-1}{2}$. ■

EXAMPLE A.18

a. $\frac{2}{7} \div \frac{9}{5} = \frac{2}{7} \times \frac{5}{9} = \frac{2 \times 5}{7 \times 9} = \frac{10}{63}$.

b. $\left(\frac{4}{7}\right) \div \left(\frac{3}{(-5)}\right) = \left(\frac{4}{7}\right) \times \left(\frac{(-5)}{3}\right) = \left(\frac{4 \times (-5)}{7 \times 3}\right) = \left(\frac{-20}{21}\right)$.

c. $10 \div \left(\frac{3}{2}\right) = \left(\frac{10}{1}\right) \times \left(\frac{2}{3}\right) = \left(\frac{10 \times 2}{1 \times 3}\right) = \left(\frac{20}{3}\right)$. ■

1.A.6.5 Mixed Fractions A *mixed fraction* is the sum of a whole number and a fraction less than 1.

EXAMPLE A.19

Express in terms of thirds.

$$4\frac{2}{3} = 4 + \frac{2}{3} = \frac{4}{1} + \frac{2}{3} = \frac{4 \times 3}{1 \times 3} + \frac{2 \times 1}{3 \times 1} = \frac{12}{3} + \frac{2}{3} = \frac{12+2}{3} = \frac{14}{3}.$$ ■

EXAMPLE A.20

Express $2\frac{3}{7}$ as ratio.

An easy way to do this is to multiply 2 by 7 and add 3 to get the numerator of the ratio and the denominator remains the same. Thus,

$$2\frac{3}{7} = \frac{(2 \times 7) + 3}{7} = \frac{17}{7}.$$

When adding or subtracting mixed fractions, whole parts and fractional parts can be added or subtracted separately, as shown in the following example. ■

EXAMPLE A.21

Simplify, $$7\frac{1}{3} + 2\frac{3}{4} + \frac{1}{2} - 1\frac{7}{12}.$$

The expression can be written as $\left(7 + \frac{1}{3}\right) + \left(2 + \frac{3}{4}\right) + \left(\frac{1}{2}\right) - \left(1 + \frac{7}{12}\right)$.

If we move the whole numbers, $(7 + 2 - 1) + \left(\frac{1}{3} + \frac{3}{4} + \frac{1}{2} - \frac{7}{12}\right)$,

LCD for 3,4,2, and 12 is 12. So, $8 + \left(\frac{4}{12} + \frac{9}{12} + \frac{6}{12} - \frac{7}{12}\right)$

$$= 8 + \frac{12}{12} = 8 + 1 = 9.$$ ■

1.A.7 Decimals

A *decimal* is a number that involves the following three parts:

i. Whole number

ii. Decimal point

iii. Numbers whose denominators are powers of 10

EXAMPLE A.22

Show the position of each digit in 15.231.

Here, 15 is the whole number and "." is the decimal point. The numbers after decimal are, respectively, 2 in "tenths",$\left(\frac{1}{10}\right)$, 3 in "hundredths", $\left(\frac{1}{100}\right)$, and 1 in "thousandths,"$\left(\frac{1}{1000}\right)$. Thus,

$$\begin{aligned}15.231 &= 15 + \left(\frac{2}{10}\right) + \left(\frac{3}{100}\right) + \left(\frac{1}{1000}\right)\\ &= (1 \times 10) + (5 \times 1) + 2 \times \left(\frac{1}{10}\right) + 3 \times \left(\frac{1}{100}\right) + 1 \times \left(\frac{1}{1000}\right)\\ &= 10 + 5 + 0.2 + 0.03 + 0.001.\end{aligned}$$

■

1.A.7.1 Operations with Decimals

Addition or Subtraction These operations are carried out after aligning the numbers in their appropriate positions.

EXAMPLE A.23

a. $175.31 + 21.5692 = 175.31 + 21.5692 = 196.8792.$

b. $80.05 - 108.5381 = -108.5381 + 80.05 = -28.4881.$ ■

Multiplication

To multiply two numbers with decimals, first obtain the product of the two numbers ignoring the decimal points. Next, count the "total" number of positions after the decimal points in both the numbers and place the decimal point in the product number after the "total" number of positions counting from right to left.

EXAMPLE A.24

a. Simplify 3.13×0.002.

Step 1: $313 \times 2 = 626$.

Step 2: Total number of positions after the decimal in both the numbers, $2 + 3 = 5$.

Step 3: 5 positions, 0.00626.

b. Simplify 7.47×5.0239.

Step 1: $747 \times 50239 = 37528533$.
Step 2: Total number of positions after the decimal in both the numbers, $2 + 4 = 6$.
Step 3: 6 positions, 37.528533. ■

Division To divide a decimal by another, first multiply both the numerator and the denominator by a power of 10 until they become whole integers. The power of 10 is decided based on the maximum number of digits after the decimal. The following example demonstrates this particular procedure.

EXAMPLE A.25

Simplify $125,488 \div 2.48$.

To find $\frac{125,488}{2.48}$, solve $\frac{12,548,800}{248} = 50,600$. Then,

$125,488 \div 2.48 = 50,600$. ■

Rounding Off Decimals Sometimes, when there are a large number of decimal places in a number and an exact answer is not needed, you can round off decimals to a specified place, depending upon the desired accuracy needed.

Rule: If the digit to the right of the digit to be rounded off is 5 or more, then you round off the digit by 1 and drop the digits to the right of the rounded off digit. Thus, 3.142857 rounded off to the third decimal place is 3.143, since the fourth number after the decimal, 8, is greater than 5, and the other digits beyond 8 are dropped. Similarly, if the digit to the right of the digit to be rounded off is less than 5, do not change the digit to be rounded off and drop all the digits to the right of the digit to be rounded off. Thus, 3.142857 rounded off to two decimal places is 3.14, since the decimal number after the second decimal, 2, is less than 5, and all digits to the right of second decimal place are dropped.

EXAMPLE A.26

Round off the following to four decimal places:

a. 1.4142136, Ans : 1.4142.
b. 2.3178623, Ans : 2.3179. ■

1.A.8 Percent

Percent is a ratio of two numbers, where the denominator is 100. A percent is shown with the symbol "%", which means $1/100$. For example, 20% means $20/100 = 1/5$. So 1 out of 5 units is 20%. Similarly, -1.5% is $-1.5/100 = -15/1000 = -3/200$. This concept is useful in the comparison of several situations involving data. It is to be noted, however, that the denominator 100 as a base is arbitrarily used for convenience, and the actual percentage figures are based on the actual data values.

EXAMPLE A.27

In a class of 45, there are 27 male and 18 female students. What are the percentages of the female and male students in the class?

The percentage of female students is

$$\frac{18}{45} = 0.4 = \frac{40}{100} = 40\%.$$

Similarly, the percentage of male students is

$$\frac{27}{45} = 0.6 = \frac{60}{100} = 60\%.$$

Note that sum of the percentages of male and female students is $40\% + 60\% = 100\%$. ■

1.A.8.1 Finding Percent (%) of a Number To determine percent of a number, change the percent to a fraction or decimal (whichever is easier for you) and multiply this fraction (or the decimal) with the number. Remember, the word "of" means multiply.

Note: Remember that the symbol % means $\frac{1}{100}$.

You may want to adapt the following rules to find answers. Turn the question word-for-word into an equation. For "what" substitute the letter x; for "is", substitute an equal sign; and for "of", substitute a multiplication sign. Change percent to decimals or fractions, whichever you find easier. Then solve the equation.

EXAMPLE A.28

a. What is 20% of 80?

$$\frac{20}{100} \times 80 = \frac{1600}{100} \text{ or } 0.20 \times 80 = 16.00 = 16.$$

b. What is 12% of 50?

$$\frac{12}{100} \times 50 = \frac{600}{100} = 6 \quad \text{or} \quad 0.12 \times 50 = 6.00 = 6.$$

c. What is $\frac{1}{2}$% of 18?

$$\frac{1/2}{100} \times 18 = \frac{1}{200} \times 18 = \frac{18}{200} = \frac{9}{100} \quad \text{or} \quad 0.005 \times 18 = 0.09.$$

d. What is 0.4% of 80?

$$\frac{0.4}{100} \times 80 = 0.004 \times 80 = 0.32.$$

e. 18 is what percent of 90?

Suppose x is the percent of 90. Then

$$18 = \frac{x}{100}(90), \quad x = \left(\frac{18}{90}\right) \times 100 = \frac{100}{5} = 20.$$

f. 10 is 0.5% of what number?

$$10 = \frac{0.50}{100}(x), \quad \left(\frac{1000}{0.50}\right) = x, \quad 2000 = x.$$

g. What is 15% of 60?

$$x = \left(\frac{15}{100}\right) \times 60 = \left(\frac{90}{10}\right) = 9 \quad \text{or} \quad 0.15(60) = 9.$$

■

1.A.8.2 Finding Percentage Increase (+) or Percentage Decrease (−) To find the *percentage change* (increase or decrease), use the following formula:

$$\frac{\text{Actual change}}{\text{Starting value}} \times 100\% = \text{percentage change} = \left(\frac{\Delta X}{X_1}\right) \times 1,$$

where $\Delta X = X_2 - X_1$ denotes the actual change in X, X_1 = starting value, and X_2 = end value.

EXAMPLE A.29

f. What is the percentage change of a $500 item that is sold for $400?

Here $X_1 = \$500$,

$X_2 = \$400$,

$\Delta X = \$400 - \$500 = -\$100.$

$$\text{Percentage change} = \left(\frac{-100}{500}\right) \times 100\% = -\left(\frac{1}{5}\right) \times 100\% = -20\%.$$

Therefore, X exhibits a decrease of 20%.

g. What is the percentage change of Jon's salary if it went from $150 a week to $200 a week?

Here $X_1 = \$150$,

$X_2 = \$200$,

$\Delta X = \$200 - \$150 = \$50.$

$$\text{Percentage change} = \left(\frac{50}{150}\right) \times 100\% = \left(\frac{1}{3}\right) \times 100\% = 33.33\%.$$

■

1.A.9 Conversions between Decimals (D), Fractions (F), and Percents (P)

1.A.9.1 Changing Decimals to Fractions (D → F) To change a decimal with two decimal places to a fraction:

Step 1: Move the decimal point two places to the right.

Step 2: Put that number over 100 as the numerator.

Step 3: Reduce if necessary.

EXAMPLE A.30

a. $0.65 = \left(\frac{65}{100}\right) = \left(\frac{13}{20}\right).$

b. $0.05 = \left(\frac{5}{100}\right) = \left(\frac{1}{20}\right).$

c. $0.75 = \left(\frac{75}{100}\right) = \left(\frac{3}{4}\right).$

This rule can be generalized to a number with any number of decimal places. For example, with three decimals, 0.075 becomes $0.075 \times \left(10^3/10^3\right) = (75/1000) = (3/40)$. The procedure can be summarized as shown in the following example:

Read it:	0.8
Write it:	8/10
Reduce it:	4/5

■

1.A.9.2 Changing Fraction to Decimals: (F → D) Any fraction can be converted to an equivalent decimal. To change a fraction to a decimal, simply do what the operation says. Since the fraction (a/b) means a ÷ b, we can divide the numerator of a fraction by its denominator to convert the fraction to a decimal. For example, to convert 3/8 to a decimal, divide 3 by 8 by long division as follows: $3 \div 8 = 0.375$.

EXAMPLE A.31

a. Thefraction $\frac{13}{20}$ means 13 dividedby 20(insertdecimalpointsandzerosaccordingly).

$$\text{So,}\quad 13.00 \div 20 = 0.65.$$

b. $5.000 \div 8 = 0.625.$ ■

1.A.9.3 Changing Decimals to Percents (D → P) To change a decimal to a percent:

Step 1: Move the decimal point two places to the right.

Step 2: Insert a percent sign.

EXAMPLE A.32

a. 0.75 = 75%.

b. 0.05 = 5%. ■

1.A.9.4 Changing Percents to Decimals (P → D) To change percent to decimals:

Step 1: Divide by 100.

Step 2: Move the decimals to the left by two places.

EXAMPLE A.33

a. $64\% = \frac{64}{100} = 0.64.$

b. $2.5\% = \frac{2.5}{100} = 0.025.$

c. $0.1\% = \frac{0.1}{100} = 0.001$(one tenth of 1%). ■

1.A.9.5 Changing Fractions to Percents (F → P) To change a fraction to a percent:

Step 1: Multiply by 100.

Step 2: Insert a percent sign.

EXAMPLE A.34

Change the following fractions to percents:

a. $\frac{1}{2}$

b. $\frac{2}{5}$

c. $\frac{3}{4}$

d. $\frac{7}{5}$

SOLUTIONS:

a. $\frac{1}{2} \times 100 = \frac{100}{2} = 50\%.$

b. $\frac{2}{5} \times 100 = \frac{200}{5} = 40\%.$

c. $\frac{3}{4} \times 100 = \frac{300}{4} = 75\%$.

d. $\frac{7}{5} \times 100 = \frac{700}{5} = 140\%$. ■

1.A.9.6 Changing Percents to Fractions (P → F) To change percents to fractions:

Step 1: Divide the percent by 100.

Step 2: Eliminate the percent sign.

Step 3: Reduce if necessary.

EXAMPLE A.35

a. $60\% = \left(\frac{60}{100}\right) = \left(\frac{3}{5}\right)$.

b. $13\% = \left(\frac{13}{100}\right)$.

c. One tenth of 1% : $0.1\% = \left(\frac{0.1}{100}\right) = 0.001$.

d. Two tenths of 1% : $0.2\% = \left(\frac{0.2}{100}\right) = 0.002$. ■

1.A.10 Ratio and Proportion

A *proportion* is, by definition, the equality of two ratios. For example, $\frac{3}{5} = \frac{6}{10}$.

In general, $\frac{x}{y} = \frac{z}{w}$, where y and w are each different from zero.

Multiplying both sides by the common denominator, yw, we obtain

$$yw \cdot \left(\frac{x}{y}\right) = yw \cdot \left(\frac{z}{w}\right), \quad \text{which in turn gives } wx = yz.$$

This result can also be obtained by multiplying diagonally, known as *cross-multiplication*. The terms wx and yz are known as *cross-products*. Using any three values in the proportion, we can always find the fourth value.

For example, if $\frac{3}{4} = \frac{x}{12}$, then $4x = 3 \times 12$ and $x = \frac{3 \times 12}{4} = \frac{36}{4} = 9$.

EXAMPLE A.36

A day trip to a Stock Exchange for a class of 25 students costs $1250. How much does it cost for the entire class of 150 freshmen to visit the Stock Exchange?

SOLUTION: Let us assume it costs $x, then using the concept of proportion, we obtain

$$\frac{25}{1250} = \frac{150}{x}.$$

Cross-multiplication produces, $25x = 150 \times 1250$.

Dividing both sides by 25,

$$x = \frac{150 \times 1250}{25} = 7500 \text{ dollars.}$$ ■

EXAMPLE A.37

In a business school, the student to faculty ratio is found to be 1100 to 77. If the university on the whole maintains a student body of 2900, find the actual number of faculty needed in order to keep the same ratio as in the business school.

SOLUTION: Let x stand for the number of faculty needed at the university level to maintain the same student faculty ratio as in the business school. Then

$77/1100 = x/2900$. Cross-multiplying, we obtain $1100x = 2900 \times 77$

Dividing both sides by 1100, $x = 2900 \times 77/1100 = 203$. ■

Exercises

Simplify the following:

A.1 $(18 \div 12) \times 8^2 - 88$.

A.2 $20 - 6 \times 12 + 20^2 + (12 + 2) \times 6$.

A.3 $[(4)^2 \times 4] \times 16 + 10$.

A.4 $\frac{18}{12} \times 3^2 - 10$.

A.5 $X^2YXY^3X^2Y^2$.

A.6 $\frac{X^2Y^3}{\sqrt{X}}$.

A.7 $\left(\frac{X^3}{-64Y^6}\right)^{1/3}$.

A.8 $\left[4\sqrt{\frac{625}{81}}\right]3$.

A.9 $\left[3\sqrt{\frac{1}{27}}\right]^2$.

A.10 $40 - 1.96\frac{20}{\sqrt{16}}$.

A.11 $0.30 + 1.645\sqrt{\frac{0.3(1-0.3)}{100}}$.

A.12 $(0.1)^3$.

A.13 $\{(0.3)xyz\}^0$.

A.14 $\left(\frac{0.03}{0.004}\right)^4$.

A.15 $\frac{(0.3)^4}{(0.2)^3} \times 10^3$.

A.16 $0.027^{2/3}$.

A.17 $\left(\frac{1.24(0.25)}{0.62}\right)^{1/2}$.

A.18 $x \cdot x \cdot x^4$.

A.19 $x \cdot x \cdot y^2$.

A.20 $\frac{x^2 \cdot x^4}{y^4 \cdot y^2}$.

A.21 $(x^4)^4$.

A.22 $\left((x^3)^4\right)^5$.

A.23 $(4 - 2^2)^0$.

A.24 $\frac{x^4}{x^3}$.

A.25 $\left(\frac{x^3}{y^4}\right)^5$.

A.26 $\left(\frac{x^{-3}}{y^{-4}}\right)^{-6}$.

A.27 $\left(\frac{x^4}{x^3}\right)^5$.

A.28 $\left(\frac{15}{3q}\right)^4$.

A.29 $\frac{12^3}{12^0}$.

A.30 $\frac{x^{-5}}{x^3}$.

A.31 $\left(\frac{9}{16}\right)^{1/2}$.

A.32 $\frac{1}{3}y^3\left(2x^{-3}\right)^4$.

A.33 $27^{4/3}$.

A.34 $\frac{5}{3}\left(2\frac{5}{3}\right)^{-1}$.

A.35 $\frac{x^8/y^4}{8/y^2}$.

A.36 $\frac{(625)^{1/4}}{25^2}$.

A.37 $\left(\frac{1}{3}\right)^3$.

A.38 $\frac{-(2.5)(1.2)-(3.4)(0.2)}{30}$.

A.39 Evaluate $y = \left(\frac{1}{x}\right)^{-(4/6)}$, at $x = 27$.

A.40 Find the prime factors of 36, 48, 112, and 192.

Factor the following:

A.41 $x^2 - 7x + 12$.

A.42 $x^2 + 5x - 24$.

A.43 $x^2 + x - 12$.

A.44 $3x^2 - 4x + 1$.

A.45 $5x^2 + 7x + 2$.

A.46 $2x^2 + x - 6$.

A.47 $x^2 + 5x + 6$.

Evaluate the following expressions:

A.48 $\frac{1}{3}+\frac{1}{5}+\frac{1}{2}$.

A.49 $\frac{7}{8}-\frac{3}{24}$.

A.50 $-\frac{5}{8}+\frac{1}{6}-\frac{3}{2}$.

A.51 $\frac{1}{2}-\frac{1}{4}$.

Solve the following:

A.52 $\left(\frac{7}{3}-\frac{2}{3}+\frac{1}{5}\right)/20$.

A.53 $\frac{(1/3)-(1/5)}{(2/3)-(1/6)}$.

A.54 $\frac{2}{7}\times\frac{14}{4}$.

A.55 $\frac{3}{5}\times\frac{-2}{7}$.

A.56 $\frac{1}{4}\div\frac{2}{3}$.

A.57 $\frac{-2}{3}\div\frac{3}{5}$.

A.58 $0\div\frac{1}{3}$.

A.59 $\frac{0}{5}\div\frac{5}{6}$.

Simplify the following:

A.60 $3\frac{5}{8}+2\frac{1}{4}-7\frac{2}{3}$.

A.61 $2\frac{3}{5}+\frac{1}{3}-\frac{2}{7}$.

A62 $\left(1\frac{1}{4}\right)\left(\frac{3}{4}\right)$.

A.63 $\left(2\frac{3}{4}\right)\left(4\frac{1}{2}\right)$.

A.64 $\frac{1}{3}\div\frac{2}{3}$.

A.65 $2\frac{3}{4}\div 4\frac{1}{3}$.

Show the position of each digit in the following:

A.66 4.76.

A.67 0.873.

A.68 125.007.

Find:

A.69 $1.273 + 3.2$.

A.70 $0.25 + 0.5$.

A.71 $0.34 - 0.07 + 1.7$.

Evaluate:

A.72 0.02×0.7.

A.73 $0.1 \times 0.01 \times 0.005$.

A.74 $0.09 \div 0.1$.

A.75 $\frac{(1.5)(-0.6)-(2.5)(1.7)}{20}$.

A.76 $\frac{-(2.5)(1.2)-(3.4)(0.2)}{30}$.

A.77 $\frac{0.4}{0.04}-\frac{6(3.6)}{0.072}$.

Round off the resulting decimal to the given place:

A.78 $\frac{1}{3}$, to the nearest tenth.

A.79 $\frac{5}{7}$, to the nearest hundredths

A.80 $\frac{2}{3}$, to the nearest thousandth

A.81 $\frac{22}{7}$, to the nearest ten thousandth.

A.82 An investor bought 125 shares of a company stock at a price of $\$79\frac{5}{16}$ a share. What is the cost of the stock? If this represents (3/5) of the total amount of investment, find the total amount of investment.

A.83 In a business school, the ratio of female to male undergraduate students is 1 to 4. After 200 female students are admitted, the ratio of female to male students became $\frac{2}{3}$. Find the total number of students after admitting 200 female students in the school.

A.84 A U.K. store sells an item of jewelry for $1725.00, which includes all taxes at the rate of 17.5%. For export purposes, to earn foreign exchange, if the government exempts taxes on all exports to other countries, what price should be paid for the jewelry piece by a foreign visitor if taxes are excluded?

A.85 Mr. John Smith receives $1400.00 as salary every week after taxes. If he is in 30% tax bracket, what is his salary per week before taxes?

A.86 Movie tickets are sold for $7.50 per adult and $6.50 per child. A party of 10 paid a total of $71.00. How many adults and children are there in the party?

A.87 Fill in the blanks:

Row	Fraction	Decimal	%, Percentage
1	. . .	0.02	. . .
2	. . .	0.125	. . .
3	. . .	3.750	. . .
4	$\frac{1}{16}$	. . .	. . .
5	$\frac{1}{8}$	. . .	. . .
6	$\frac{3}{2}$	. . .	. . .
7	. . .	. . .	30
8	. . .	. . .	50
9	. . .	. . .	10
10	. . .	. . .	200
11	$\left(\frac{1}{4}\right)^2$	. . .	. . .
12	$\left(\frac{1}{10}\right)^5$	. . .	. . .

A.88 The *Wall Street Journal* reported on Friday, May 25, 2012 a high of 12,539.59 for the Dow Jones Industrial Average Index and a low of 12,419.63. Find the percentage increase from low to high.

A.89 The *Wall Street Journal*, Friday, May 25, 2012, reported that the 52 week crude oil high and low prices per barrel were, respectively, $109.77 and $75.67. Find the percentage decrease from high price to low price.