1

Introduction

In this chapter, we introduce some necessary background about the active disturbance rejection control (ADRC). Some notation and preliminary results are also presented.

1.1 Problem Statement

In most control industries, it is hard to establish accurate mathematical models to describe the systems precisely. In addition, there are some terms that are not explicitly known in mathematical equations and, on the other hand, some unknown external disturbances exist around the system environment. The uncertainty, which includes internal uncertainty and external disturbance, is ubiquitous in practical control systems. This is perhaps the main reason why the proportional–integral–derivative (PID) control approach has dominated the control industry for almost a century because PID control does not utilize any mathematical model for system control. The birth and large-scale deployment of the PID control technology can be traced back to the period of the 1920s–1940s in response to the demands of industrial automation before World War II. Its dominance is evident even today across various sectors of the entire industry. It has been reported that 98% of the control loops in the pulp and paper industries are controlled by single-input single-output PI controllers [18]. In process control applications, more than 95% of the controllers are of the PID type [9].

Let us look at the structure of PID control first. For a control system, let the control input be $u(t)$ and let the output be $y(t)$. The control objective is to make the output $y(t)$ track a reference signal $v(t)$. Let $e(t) = y(t) - v(t)$ be the tracking error. Then PID control law is represented as follows:

$$u(t) = k_0 e(t) + k_1 \int_0^t e(\tau) d\tau + k_2 \dot{e}(t), \tag{1.1.1}$$

where k_0, k_1, and k_2 are tuning parameters. The PID control is a typical error-based control method, rather than a model-based method, which is seen from Figure 1.1.1 for its advantage

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Bao-Zhu Guo and Zhi-Liang Zhao.

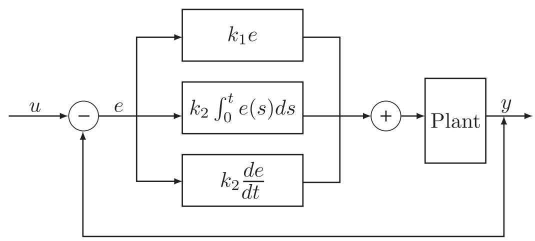

Figure 1.1.1 PID control topology.

of easy design. The nature of independent mathematical model and easy design perhaps have explained the partiality of control engineers to PID.

However, it is undeniable that PID is increasingly overwhelmed by the new demands in this era of modern industries where an unending efficiency is pursued for systems working in more complicated environments. In these circumstances, a new control technology named active disturbance rejection control (ADRC) was proposed by Jingqing Han in the 1980s and 1990s to deal with the control systems with vast uncertainty [58, 59, 60, 62, 63]. As indicated in Han's seminal work [58], the initial motivation for the ADRC is to improve the control capability and performance limited by PID control in two ways. One is by changing the linear PID (1.1.1) to nonlinear PID and the other is to make use of "derivative" in PID more efficiently because it is commonly recognized that, in PID, the "D" part can significantly improve the capability and transient performance of the control systems. However, the derivative of error is not easily measured and the classical differentiation most often magnifies the noise, which makes the PID control actually PI control in applications, that is, in (1.1.1), $k_2 = 0$.

In automatic principle of compensation, the differential signal for a given reference signal $v(t)$ is approximated by $y(t)$ in the following process:

$$\hat{y}(s) = \frac{s}{Ts+1}\hat{v}(s) = \frac{1}{T}\left(\hat{v}(s) - \frac{1}{Ts+1}\hat{v}(s)\right), \tag{1.1.2}$$

where $\hat{L}(s)$ represents the Laplace transform of $L(t)$, T is a constant, and $\frac{1}{Ts+1}\hat{v}(s)$ represents the inertial element with respect to T (see Figure 1.1.2).

The time domain realization of (1.1.2) is

$$y(t) = \frac{1}{T}(v(t) - v(t-T)). \tag{1.1.3}$$

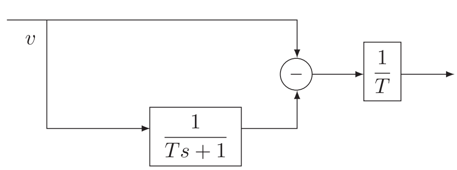

Figure 1.1.2 Classical differentiation topology.

If $v(t)$ is contaminated by a high-frequency noise $n(t)$ with zero expectation, the inertial element can filter the noise ([62], pp. 50–51):

$$y(t) = \frac{1}{T}(v(t) + n(t) - v(t-T)) \approx \dot{v}(t) + \frac{1}{T}n(t). \tag{1.1.4}$$

That is, the output signal contains the magnified noise $\frac{1}{T}n(t)$. If T is small, the differential signal may be overwhelmed by the magnified noise.

To overcome this difficulty, Han proposed a noise tolerant tracking differentiator:

$$\hat{y}(s) = \frac{1}{T_2 - T_1}\left(\frac{1}{T_1 s + 1} - \frac{1}{T_2 s + 1}\right)\hat{v}(s), \tag{1.1.5}$$

whose state-space realization is

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\dfrac{1}{T_1 T_2}(x_1(t) - v(t)) - \dfrac{T_2 - T_1}{T_1 T_2}x_2(t), \\ y(t) = x_2(t). \end{cases} \tag{1.1.6}$$

The smaller T_1/T_2 is, the quicker $x_1(t)$ tracks $v(t)$. The abstract form of (1.1.6) is formulated by Han as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = r^2 f\left(x_1(t) - v(t), \dfrac{x_2(t)}{r}\right), \end{cases} \tag{1.1.7}$$

where r is the tuning parameter and $f(\cdot)$ is an appropriate nonlinear function. Although a convergence of (1.1.7) is first reported in [59], it is lately shown to be true only for the constant signal $v(t)$. Nevertheless, the effectiveness of a tracking differentiator (1.1.7) has been witnessed by many numerical experiments and control practices [64, 147, 152, 153]. The convergence proof for (1.1.7) is finally established in [55 and 52]. In Chapter 2, we analyze this differentiator, and some illustrative numerical simulations and applications are also presented.

The second key part of the ADRC is the extended state observer (ESO). The ESO is an extension of the state observer in control theory. In control theory, a state observer is a system that provides an estimate of the internal state of a given real system from its input and output. For the linear system of the following:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{1.1.8}$$

where $x(t) \in \mathbb{R}^n (n \geq 1)$ is the state, $u(t) \in \mathbb{R}^m$ is the control (input), and $y(t) \in \mathbb{R}^l$ is the output (measurement). When $n = 1$, the whole state is measured and the state observer is unwanted. If $n > 1$, the Luenberger observer can be designed in the following way to recover the whole state by input and output:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)), \tag{1.1.9}$$

where the matrix L is chosen so that $A - LC$ is Hurwitz. It is readily shown that the observer error $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. The existence of the gain matrix L is guaranteed by the detectability of system (1.1.8). If it is further assumed that system (1.1.8) is stabilizable, then there exists a matrix K such that the closed-loop system under the state feedback $u(t) = Kx(t)$ is asymptotically stable: $x(t) \to 0$ as $t \to \infty$. In other words, $A + BK$ is Hurwitz. When the observer (1.1.9) exists, then under the observer-based feedback control $u(t) = K\hat{x}(t)$, the closed-loop system becomes

$$\begin{cases} \dot{x}(t) = (A + BK)x(t), \\ \dot{\hat{x}}(t) = (A - LC + BK)\hat{x}(t) + LCx(t). \end{cases} \tag{1.1.10}$$

It can be shown that $(x(t), \hat{x}(t)) \to 0$ as $t \to \infty$ and, moreover, the eigenvalues of (1.1.10) are composed of $\sigma(A + BK) \cup \sigma(A - LC)$, which is called the separation principle for the linear system (1.1.8). In other words, the matrices K and L can be chosen separately.

The observer design is a relatively independent topic in control theory. There are huge works attributed to observer design for nonlinear systems; see, for instance, the nonlinear observer with linearizable error dynamics in [87 and 88], the high-gain observer in [84], the sliding mode observer in [24, 26, and 130], the state observer for a system with uncertainty [22], and the high-gain finite-time observer in [103, 109, and 116]. For more details of the state observer we refer to recent monograph [14].

A breakthrough in observer design is the extended state observer, which was proposed by Han in the 1990s to be used not only to estimate the state but also the "total disturbance" that comes from unmodeled system dynamics, unknown coefficient of control and external disturbance. Actually, uncertainty is ubiquitous in a control system itself and the external environment, such as unmodeled system dynamics, external disturbance, and inaccuracy in control coefficient. The ubiquitous uncertainty in systems explains why the PID control technology is so popular in industry control because PID control is based mainly on the output error not on the systems' mathematical models. Since the ESO, the "total disturbance" and the state of the system are estimated simultaneously, we can design an output feedback control that is not critically reliant on the mathematical models. Let us start from an nth order SISO nonlinear control systems given by

$$\begin{cases} x^{(n)}(t) = f(t, x(t), \dot{x}(t), \ldots, x^{(n-1)}(t)) + w(t) + u(t), \\ y(t) = x(t), \end{cases}$$

which can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \quad \vdots \\ \dot{x}_n(t) = f(t, x_1(t), x_2(t), \ldots, x_n(t)) + w(t) + u(t), \\ y(t) = x_1(t), \end{cases} \tag{1.1.11}$$

where $u(t) \in C(\mathbb{R}, \mathbb{R})$ is the control (input), $y(t)$ is the output (measurement), $f \in C(\mathbb{R}^n, \mathbb{R})$ is the system function, which is possibly unknown, and $w \in C(\mathbb{R}, \mathbb{R})$ is unknown external disturbance; $f(\cdot, t) + w(t)$ is called the "total disturbance" or "extended state" and

$\alpha_i \in \mathbb{R}, i = 1, 2, \ldots, n+1$ are the tuning parameters. The ESO designed in [60] is as follows:

$$\begin{cases} \dot{\hat{x}}_1(t) = \hat{x}_2(t) - \alpha_1 g_1(\hat{x}_1(t) - y(t)), \\ \dot{\hat{x}}_2(t) = \hat{x}_3(t) - \alpha_2 g_2(\hat{x}_1(t) - y(t)), \\ \quad \vdots \\ \dot{\hat{x}}_n(t) = \hat{x}_{n+1}(t) - \alpha_n g_n(\hat{x}_1(t) - y(t)) + u(t), \\ \dot{\hat{x}}_{n+1}(t) = -\alpha_{n+1} g_{n+1}(\hat{x}_1(t) - y(t)). \end{cases} \tag{1.1.12}$$

By appropriately choosing the nonlinear functions $g_i \in C(\mathbb{R}, \mathbb{R})$ and tuning the parameters α_i, we expect that the states $\hat{x}_i(t), i = 1, 2, \ldots, n+1$ of the ESO (1.1.12) can approximately recover the states $x_i(t), i = 1, 2, \ldots, n$ and the extended state $f(\cdot, t) + w(t)$, that is,

$$\hat{x}_i(t) \approx x_i(t), i = 1, 2, \ldots, n, \hat{x}_{n+1}(t) \approx f(\cdot, t) + w(t).$$

In Chapter 3, we have a principle of choosing the nonlinear functions $g_i(\cdot)$ and tuning the gain parameters α_i. The convergence of the ESO is established. We also present some numerical results to show visually the estimations of state and extended state. In particular, if the functions $g_i(\cdot)$ in (1.1.12) are linear, the ESO is referred to as the linear extended state observer (LESO). The LESO is also called the extended high-gain observer in [35].

The final key part of the ADRC is the TD and the ESO-based feedback control. In the feedback loop, a key component is to compensate (cancel) the "total disturbance" by making use of its estimate obtained from the ESO. The topology of the active disturbance rejection control is blocked in Figure 1.1.3.

Now we can describe the whole picture of the ADRC for a control system with vast uncertainty that includes the external disturbance and unmodeled dynamics. The control purpose is to design an output feedback control law that drives the output of the system to track a given reference signal $v(t)$. Generally speaking, the derivatives of the reference $v(t)$ cannot be measured accurately due to noise. The first step of the ADRC is to design a tracking differentiator

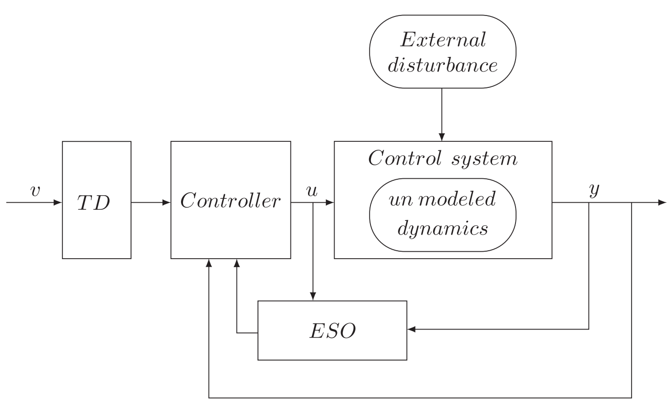

Figure 1.1.3 Topology of active disturbance rejection control.

(TD) to recover the derivatives of $v(t)$ without magnifying measured noise. Tracking differentiator also serves as transient profile for output tracking. The second step is to estimate, through the ESO, the system state and the "total disturbance" in real time by the input and output of the original system. The last step is to design an ESO-based feedback control that is used to compensate the "total disturbance" and track the estimated derivatives of $v(t)$. The whole ADRC design process and convergence are analyzed in Chapter 4.

The distinctive feature of the ADRC lies in its estimation/cancelation nature. In control theory, most approaches like high-gain control (HGC) and sliding mode control (SMC) are based on the worst case scenario, but there are some approaches that use the same idea of the ADRC to deal with the uncertainty. One popular approach is the internal model principle (IMP) [33, 34, 77, 99] and a less popular approach is the external model principle (EMP) [66, 104, 129, 149]. In the internal model principle and external model principle, the dynamic of the system is exactly known and the "external disturbance" is considered as a signal generated by the exogenous system, which follows exactly known dynamics. The unknown parts are initial states. However, in some complicated environments, it is very difficult to obtain the exact mathematical model of the exogenous system, which generates "external disturbance". In the ADRC configuration, we do not need a mathematical model of external disturbance and even most parts of the mathematical model of the control system itself can be unknown. This is discussed in Section 4.5.

The systems dealt with by the ADRC can also be coped with by high-gain control [128] and sometimes by sliding mode control [94, 130, 131]. However, control law by these approaches is designed in the worst case of uncertainty, which may cause unnecessary energy waste and may even be unrealizable in many engineering practices. In Section 4.6, three control methods are compared numerically by a simple example.

1.2 Overview of Engineering Applications

Nowadays, the ADRC is widely used in many engineering practices. It is reported in [166] that the ADRC control has been tested in the Parker Hannifin Parflex hose extrusion plant and across multiple production lines for over eight months. The product performance capability index (Cpk) is improved by 30% and the energy consumption is reduced by over 50%.

The Cleveland state university in the USA established a center for advanced control technologies (CACT) for further investigation of the ADRC technology. Under the cooperation of CACT and an American risk investment, the industrial giant Texas Instruments (TI) has adopted this method. In April 2013, TI issued its new motor control chips based on the ADRC. The control chips can be used in almost every motor such as washing machines, medical devices, electric cars and so on.

There is a lot of literature on the application of the ADRC. In what follows, we briefly overview some typical examples. In the flight and integrated control fields, an ESO and non-smooth feedback law is employed to achieve high performance of flight control [72]. In [126], the ADRC is adopted to tackle some problems encountered in pitch and roll altitude control. The ADRC is used for integrated flight-propulsion control in [135], and the coupling effects between altitude and velocity and attenuates measurement noise are eliminated by this method. In [169], the ADRC is applied to altitude control of a spacecraft model that is nonlinear in dynamics with inertia uncertainty and external disturbance. The ESO is applied to estimate the disturbance and the sliding mode control is designed based on the ESO to achieve the control purpose. The safe landing of unmanned aerial vehicles (UAVs) under various wind conditions has been a challenging task for decades. In [143], by using the ADRC method, an auto-landing control system consisting of a throttle control subsystem and an altitude control subsystem has been designed. It is indicated that this method can estimate directly in real time the UAV's internal and external disturbances and then compensate in the feedback. The simulation results show that this auto-landing control system can land the UAV safely under wide range wind disturbances (e.g., wind turbulence, wind shear). The application of the ADRC on this aspect can be found in monograph [139].

In the energy conversion and power plant control fields, [28] presents a controller for maximum wind energy capture of a wind power system by employing the ADRC method. The uncertainties in the torque of turbine and friction are both considered as an unknown disturbance to the system. The ESO is used to estimate the unknown disturbance. The maximum energy capture is achieved through the design of a tracking-differentiator. It is pointed out that this method has the merits of feasibility, adaptability, and robustness compared to the other methods. The paper [102] summarizes some methods for capturing the largest wind energy. It is indicated that the ADRC method captures the largest wind energy. The ADRC is used for a thermal power plant, which is characterized by nonlinearity, changing parameters, unknown disturbances, large time-delays, large inertia, and highly coupled dynamics among various control loops in [167]. In [121], the ADRC method is developed to cope with the highly nonlinear dynamics of the converter and the disturbances. The ADRC method is used for a thermal power generation unit in [69]. It is reported that the real-time dynamic linearization is implemented by disturbance estimation via the ESO and disturbance compensation via the control law, instead of differential geometry-based feedback linearization and direct feedback linearization theory, which need an accurate mathematical model of the plant. The decoupling for an MIMO coordinated system of boiler–turbine unit is also easily implemented by employing the ADRC. The simulation results on STAR-90 show that the ADRC coordinated control scheme can effectively solve problems of strong nonlinearity, uncertainty, coupling, and large time delays. It can also significantly improve the control performance of a coordinated control system. To eliminate the total disturbance effect on the active power filter (APF) performance, the ADRC is adopted in [95]. It is reported that the ADRC control has the merits of strong robustness, stability, and adaptability in dealing with the internal perturbation and external disturbance. In [151], the ADRC is used to regulate the frequency error for a three-area interconnected power system. As the interconnected power system transmits the power from one area to another, the system frequency will inevitably deviate from a scheduled frequency, resulting in a frequency error. A control system is essential to correct the deviation in the presence of external disturbances and structural uncertainties to ensure the safe and smooth operation of the power system. It is reported in [151] that the ADRC can extract the information of the disturbance from input and output data of the system and actively compensate for the disturbance in real time. Considering the difficulty of developing an accurate mathematical model for active power filters (APF), [168] uses the ADRC to parallel APF systems. It is reported that the analog signal detected in the ADRC controller is less than other control strategies. In [27], the ADRC is applied to an electrical power-assist steering system (EPAS) in automobiles to reduce the steering torque exerted by a driver so as to achieve good steering feel in the presence of external disturbances and system uncertainties. With the proposed ADRC, the driver can turn the steering wheel with the desired steering torque, which is independent of load torques, and tends to vary, depending on driving conditions.

As to motor and vehicle control, in [127], the ADRC is used to ensure high dynamic performance of a magnet synchronous motor (PMSM) servo system. It is concluded that the proposed topology produces better dynamic performance, such as smaller overshoot and faster transient time, than the conventional PID controller in its overall operating conditions. A matrix converter (MC) is superior to a drive induction motor since it has more attractive advantages than a conventional pulse width modulation (PWM) inverter such as the absence of a large dc-link capacitor, unity input power factor, and bidirectional power flow. However, due to the direct conversion characteristic of an MC, the drive performance of an induction motor is easily influenced by input voltage disturbances of the MC, and the stability of an induction motor drive system fed by an MC would be affected by a sudden change of load as well. In [105], the ADRC is applied to the MC fed induction motor drive system to solve the problems successfully. In [31], the ADRC is developed to ensure high dynamic performance of induction motors. In [123], the ADRC is developed to implement high-precision motion control of permanent-magnet synchronous motors. Simulations and experimental results show that the ADRC achieves a better position response and is robust to parameter variation and load disturbance. Furthermore, the ADRC is designed directly in discrete time with a simple structure and fast computation, which makes it widely applicable to all other types of drives. In [96], an ESO-based controller is designed for the permanent-magnet synchronous motor speed-regulator, where the ESO is employed to estimate both the states and the disturbances simultaneously, so that the composite speed controller can have a corresponding part to compensate the disturbances. Lateral locomotion control is a key technology for intelligent vehicles and is significant to vehicle safety itself. In [115], the ADRC is used for the lateral locomotion control. Simulation results show that, within the large velocity scale, the ADRC controllers can assist the intelligent vehicle to accomplish smooth and high precision on lateral locomotion, as well as remaining robust to system parameter perturbations and disturbances. In [146], the ADRC is applied to the anti-lock braking system (ABS) with regenerative braking of electric vehicles. Simulation results indicate that this method can regulate the slip rate at expired value in all conditions and, at the same time, it can restore the kinetic energy of a vehicle to an electrical source. In [142], the ADRC is applied to the regenerative retarding of a vehicle equipped with a new energy recovery retarder. Considering the railway restriction and comfort requirement, the ADRC is applied to the operation curve tracking of the maglev train in [100].

There is also a lot of literature on the ADRC's application in ship control. In [113], the ADRC is applied to the ship tracking control by considering the strong nonlinearity, uncertainty, and typical underactuated properties, as well as the restraints of the rudder. The simulation results show that the designed controller can achieve high precision on ship tracking control and has strong robustness to ship parameter perturbations and environment disturbances. In [108], the ADRC is used on the ship's main engine for optimal control under unmatched uncertainty. The simulation results show that the controller has strong robustness to parameter perturbations of the ship and environmental disturbances.

In robot control [73], the ESO is used to estimate and compensate the nonlinear dynamics of the manipulator and the external disturbances for a complex robot systems motion control. [120] applies the ADRC to the lateral control of tracked robots on stairs. The simulation results show that this algorithm can keep the robot smooth and precise in lateral control and effectively overcome the disturbance. In [114], the ADRC is applied to the rock drill robot joint hydraulic drive system. The simulation results show that the ADRC controller has ideal robustness to the system parameters' disturbances and the large load disturbance and a rapid and smooth control process and high steady precise performances can be implemented.

As to gyroscopes, [162] applies the ADRC to control two vibrating axes (or modes) of vibrational MEMS gyroscopes in the presence of the mismatch of natural frequencies between two axes, mechanical-thermal noises, quadrature errors, and parameter variations. The simulation results on a *Z*-axis MEMS gyroscope show that the controller is very effective by driving the output of the drive axis to a desired trajectory, forcing the vibration of the sense axis to zero for a force-to-rebalance operation, and precisely estimating the rotation rate. In [29], the ADRC is used for both vibrating axes (drive and sense) of vibrational gyroscopes, in both simulation and hardware tests on a vibrational piezoelectric beam gyroscope. The proposed controller proves to be robust against structural uncertainties and it also facilitates accurate sensing of time-varying rotation rates. [154] uses the ADRC and fuzzy control method for stabilizing circuits in platform inertial navigation systems (INS) based on fiber optic gyroscopes (FOGs).

1.3 Preliminaries

In this section, we first present a canonical form of active disturbance rejection control (ADRC). To make the book self-contained, we also present some notation and results about Lyapunov stability, asymptotical stability, finite-time stability, and weighted homogeneity.

1.3.1 Canonical Form of ADRC

As pointed out in the previous section, the ADRC can deal with nonlinear systems with vast uncertainty. However, for the sake of clarity, we first limit ourselves to a class of nonlinear systems that are canonical forms of the ADRC. Let us start with some engineering control systems.

Firstly, we consider micro-electro-mechanical systems (MEMS). The mechanical structure of the MEMS gyroscope can be understood as a proof mass attached to a rigid frame by springs and dampers, as shown in Figure 1.3.1. As the mass is driven to resonance along the drive (*X*) axis and the rigid frame is rotating along the rotation axis, a Coriolis acceleration will be produced along the sense (*Y*) axis, which is perpendicular to both drive and rotation axes. The Coriolis acceleration is proportional to the amplitude of the output of the drive axis and the unknown rotation rate. Therefore, we can estimate the rotation rate through measuring the vibration of the sense axis. To measure accurately the rotation rate, the vibration magnitude of the drive axis has to be regulated to a fixed level. Therefore, the controller of the drive axis is mainly used to drive the drive axis to resonance and to regulate the output amplitude.

The vibrational MEMS gyroscope can be modeled as follows:

$$\begin{cases} \ddot{x}(t) + 2\zeta\omega_n^2 x(t) + \omega_{xy} y(t) - 2\Omega\dot{y}(t) = \dfrac{k}{m}u(t) + N_x(t), \\ \ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_{xy} x(t) + 2\Omega\dot{x}(t) = N_y(t), \end{cases} \tag{1.3.1}$$

where $x(t)$ and $y(t)$ are the outputs of the drive and sense axes, $2\Omega\dot{x}(t)$ and $2\Omega\dot{y}(t)$ are the Coriolis accelerations, Ω is the rotation rate, ω_n is the natural frequency of the drive and sense axes, $\omega_{xy}y(t)$ and $\omega_{xy}x(t)$ are quadrature errors caused by spring couplings between two axes,

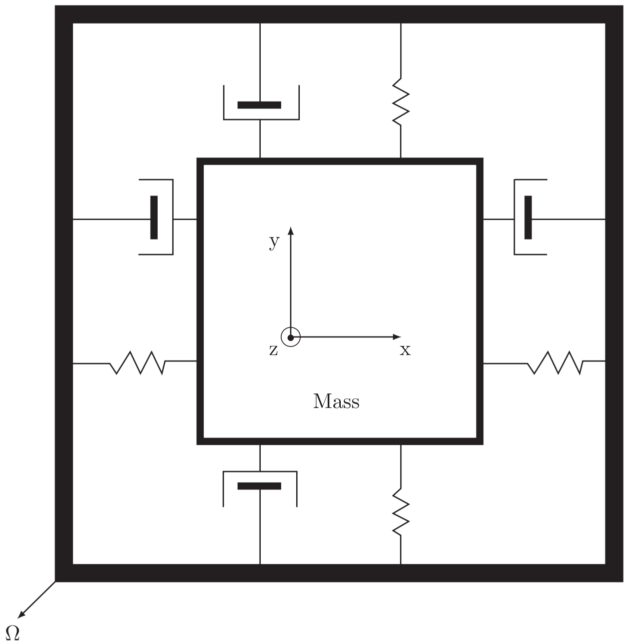

Figure 1.3.1 Mass–spring–damper structure of an MEMS gyroscope system.

ζ is the damping coefficient, m is the mass of the MEMS gyroscope, k is the control gain, and $u(t)$ is the control input for the drive axis. The $N_x(t)$ and $N_y(t)$ are external disturbances. We can rewrite system (1.3.1) as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = f(x_1(t), x_2(t), Y(t), N_x(t)) + bu(t), \\ \dot{Y}(t) = F_0(x_1(t), x_2(t), Y(t), N_y(t)), \end{cases} \tag{1.3.2}$$

where $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$, $Y(t) = (y(t), \dot{y}(t))^\top$,

$$f(x_1(t), x_2(t), Y(t), N_x(t)) = -2\zeta\omega_n^2 x_1(t) - \omega_{xy} y(t) + 2\Omega\dot{y}(t) + N_x(t),$$

and

$$F_0(x_1(t), x_2(t), Y(t), N_y(t)) = \begin{pmatrix} 0 \\ -2\zeta\omega_n\dot{y}(t) - \omega_{xy}x(t) - 2\Omega\dot{x}(t) + N_y(t) \end{pmatrix}.$$

Obviously, both the nonlinear functions $f(\cdot)$ and $F_0(\cdot)$ contain external disturbances. However, the external disturbances, $f(\cdot)$ and $F(\cdot)$, cannot always be accurately measured due to the possible deviation of parameters ζ, Ω, ω_n, and ω_{xy} away from their real values.

Next, consider an hydraulic system where an inertia load is driven by a servo-valve-controlled hydraulic rotary actuator. A schematic structure is presented on the right of Figure 1.3.2. The objective is to drive the inertia load to track a given smooth motion trajectory by the position

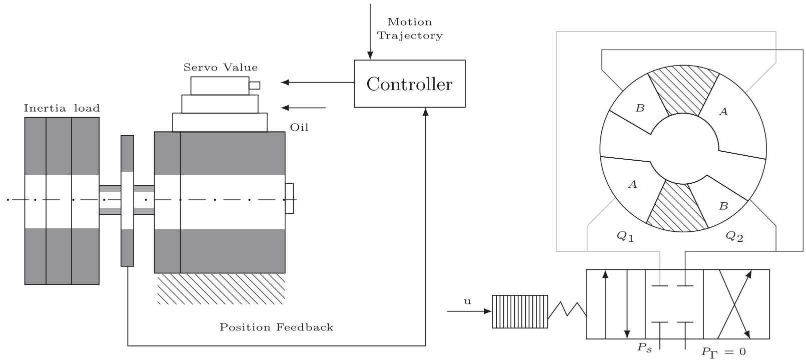

Figure 1.3.2 Architecture of the hydraulic system.

measurement. The motion dynamics of the inertia load can be described by the following equation:

$$J\ddot{x}(t) = P_L(t)D_m - F\dot{x}(t), \tag{1.3.3}$$

where J and $x(t)$ represent the moment of inertia and the angular displacement of the load, respectively, D_m is the radian displacement of the actuator, F represents the friction coefficient, $P_L(t) = P_1(t) - P_2(t)$ is the load pressure of the hydraulic actuator, and $P_1(t)$ and $P_2(t)$ are the pressures inside the two chambers of the actuator. The dynamics of load pressure can be written as

$$\frac{V_t}{4\beta_e}\dot{P}_L(t) = -D_m\dot{x}(t) - C_tP_L(t) + Q_0 + Q(t) + Q_L(t), \tag{1.3.4}$$

where V_t is the total control volume of the actuator, β_e is the effective oil bulk modulus, C_t is the coefficient of the total internal leakage of the actuator due to pressure, Q_0 is a constant modeling error and $Q(t)$ is the time-varying modeling error caused by complicated internal leakage, parameter deviations, unmodeled pressure dynamics, modeling error caused by the following flow equation, and so on, $Q_L(t) = (Q_1(t) + Q_2(t))/2$ is the load flow, Q_1 is the supplied flow rate to the forward chamber, and Q_2 is the return flow rate of the return chamber. $Q_L(t)$ is related to the spool valve displacement of the servovalve x_v by

$$Q_L(t) = k_qx_v(t)\sqrt{P_s - \mathrm{sign}(x_v(t))P_L(t)}, \quad k_q = C_d\omega\sqrt{1/\rho}, \tag{1.3.5}$$

where C_d is the discharge coefficient, ω is the spool valve area gradient, ρ is the density of oil, and P_s is the supply pressure of the fluid with respect to the return pressure P_r. The control applied to the servovalve is directly proportional to the spool position, that is, $x_v(t) = k_iu(t)$, where k_i is a positive constant; $u(t)$ is the control input voltage.

Let $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$, and $x_3(t) = \frac{D_m}{J}P_L(t) - F_2(x) + f(t, x_1(t), x_2(t))$. Then the control system can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = \varphi(t, x_1(t), x_2(t), x_3(t)) + b(x_2(t), x_3(t), u(t))u(t), \end{cases} \tag{1.3.6}$$

where

$$\varphi(t,x_1(t),x_2(t),x_3(t)) = -\left(2F^2+\frac{4\beta_e(D_m^2+CFJ)}{JV}\right)x_2(t)-\left(F+\tfrac{4\beta_e C}{V}\right)x_3(t) + \frac{4\beta_e D_m(Q_0+Q(t))}{JV} \tag{1.3.7}$$

and

$$b(x_2,x_3,u)=\frac{4\beta_e K_q D_m}{JV}\sqrt{P_S-\frac{J(Fx_2+x_3)}{D_m}\text{sign}(u)}. \tag{1.3.8}$$

In practice, there also exist internal and external disturbances in the system function $\varphi(\cdot)$.

Finally, we consider the dynamic of autonomous underwater vehicles (AUVs). The AUV can be modeled as follows:

$$\begin{cases}\dot{\mathbf{x}}(t)=J(\mathbf{x}(t))\mathbf{v}(t),\\ M\dot{\mathbf{v}}(t)+C(\mathbf{v}(t))\mathbf{v}(t)+D(\mathbf{v}(t))\mathbf{v}(t)+\mathbf{d}(t)=\mathbf{u}(t),\\ \mathbf{y}(t)=\mathbf{x}(t),\end{cases} \tag{1.3.9}$$

where

$$\mathbf{x}(t)=(x(t),y(t),z(t),\phi(t),\theta(t),\psi(t))^\top \tag{1.3.10}$$

is the vehicle location and orientation in the earth-fixed frame, $\mathbf{v}(t)$ is the vector of the vehicle's velocities expressed in the body-fixed frame, and $\mathbf{y}(t)$ is the output. The positive definite inertia matrix $M=M_{RB}+M_A$ includes the inertia M_{RB} of the vehicle as a rigid body and the added inertia M_A due to the acceleration of the wave. The matrix $C(\mathbf{v})\in\mathbb{R}^{6\times 6}$, which is skew-symmetrical groups of the coriolis and centripetal force. The hydrodynamic damping term $D(\mathbf{v})\in\mathbb{R}^{6\times 6}$ takes into account the dissipation due to the friction exerted by the fluid surrounding the AUV. The vector $g(\mathbf{x})\in\mathbb{R}^6$ is the combined gravitation and buoyancy forces in the body-fixed frame, $\mathbf{d}(t)\in\mathbb{R}^6$ is the external disturbance, and $J(\mathbf{x})$ is the kinematic transformation matrix expressing the transformation from the body-fixed frame to the earth-fixed frame:

$$J(\mathbf{x})=\begin{pmatrix}J_1(\mathbf{x}) & 0\\ 0 & J_2(\mathbf{x})\end{pmatrix} \tag{1.3.11}$$

with

$$J_1(\mathbf{x})=\begin{pmatrix}\cos\psi\cos\theta & -\sin\psi\cos\theta & \sin\psi\sin\phi+\cos\psi\cos\phi\sin\theta\\ \sin\psi\cos\theta & \cos\psi+\sin\phi\sin\theta\sin\psi & -\cos\psi\sin\phi+\sin\theta\sin\psi\cos\phi\\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta\end{pmatrix},$$

$$J_2(\mathbf{x})=\begin{pmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta}\end{pmatrix}. \tag{1.3.12}$$

The control purpose is to make the output $\mathbf{y}(t)$ track the desired trajectory $\mathbf{x}_d(t)$. Let $\mathbf{x}_1(t) = \mathbf{x}(t)$ and $\mathbf{x}_2(t) = J(\mathbf{x}(t))\dot{\mathbf{v}}(t)$. Then the dynamics (1.3.9) can be written as

$$\begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{x}_2(t), \\ \dot{\mathbf{x}}_2(t) = F(\mathbf{x}_1(t), \mathbf{x}_2(t), d(t)) + J(\mathbf{x}_1(t))M^{-1}\mathbf{u}(t), \end{cases} \tag{1.3.13}$$

where

$$\begin{aligned} F(\mathbf{x}_1, \mathbf{x}_2, \mathbf{d}) = &-J(\mathbf{x}_1)M^{-1}C(J^{-1}(\mathbf{x}_1)\mathbf{x}_2)J^{-1}(\mathbf{x}_1)\mathbf{x}_2 \\ &- J(\mathbf{x}_1)M^{-1}D(J^{-1}(\mathbf{x}_1)\mathbf{x}_2)J^{-1}(\mathbf{x}_1)\mathbf{x}_2 - J(\mathbf{x}_1)M^{-1}\mathbf{d}, \end{aligned} \tag{1.3.14}$$

and there are external disturbance and parameter uncertainty in $F(\cdot)$.

It is seen that all these systems, MEMS gyroscope (1.3.2), hydraulic system (1.3.6), and (1.3.13), are the special cases of the following nonlinear systems with vast uncertainty:

$$\begin{cases} \dot{x}_{i1}(t) = x_{i2}(t), \\ \dot{x}_{i2}(t) = x_{i3}(t), \\ \qquad \vdots \\ \dot{x}_{in_i}(t) = f_i(x(t), \zeta(t), w(t)) + b(x(t), \zeta(t), w(t))u_i(t), \\ \dot{\zeta}(t) = f_{i0}(x(t), \zeta(t), w(t)), \quad i = 1, 2, \ldots, r, \end{cases} \tag{1.3.15}$$

where $(x^\top(t), \zeta^\top(t))^\top = ((x_{11}(t), \ldots, x_{1n_1}(t), x_{21}(t), \ldots, x_{rn_r}(t))^\top, \zeta^\top(t)) \in \mathbb{R}^{nm+l}$ is the system state, $y(t) = (x_{11}(t), \ldots, x_{r1}(t))^\top \in \mathbb{R}^r$ is the output (measurement), $u(t) = (u_1(t), \ldots, u_r(t))^\top \in \mathbb{R}^r$ is the input (control), and $w(t) \in \mathbb{R}^k$ is the external disturbance. The system functions $f_i \in C(\mathbb{R}^{n_1+\cdots+n_r+l+k}, \mathbb{R})$ and $f_{i0} \in C(\mathbb{R}^{n_1+\cdots+n_r+l+k}, \mathbb{R}^k)$ are completely unknown or partially unknown. In addition, some uncertainties are allowed in functions $b_i \in C(\mathbb{R}^{n_1+\cdots+n_r+l+k}, \mathbb{R})$. In fact, except for the above examples, there are many other control systems that can be modeled as (1.3.15). In this book, we consider system (1.3.15) as the control canonical form of ADRC.

To discuss further the canonical form of ADRC, we introduce some background about linear MIMO systems as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ \ x(0) = x_0, \\ y(t) = Cx(t), \end{cases} \tag{1.3.16}$$

where $x(t) \in \mathbb{R}^n$ is the state, x_0 is the initial state, $u(t) \in \mathbb{R}^m$ is the input (control), $y(t) \in \mathbb{R}^p$ is the output (measurement), $A \in \mathbb{R}^{n\times n}$ is the system matrix, $B \in \mathbb{R}^{n\times m}$ is the control matrix, and $C \in \mathbb{R}^{p\times n}$ is the output matrix.

The concept of relative degree is useful for understanding the control structure of system (1.3.16).

Definition 1.3.1 *For system (1.3.16), let*

$$d_i = \begin{cases} \mu_i, \quad C_iA^kB = 0, \quad k = 0, 1, \ldots, \mu_i - 2(\mu_i \le n), \quad C_iA^{\mu_i - 1}B \neq 0, \\ n-1, \quad C_iA^kB = 0, \quad k = 0, 1, \ldots, n-1, \end{cases}$$

where C_i is the ith row of C, $i = 1, 2, \dots, p$. Then $\{d_1, d_2, \dots, d_p\}$ is called the relative degree of system (1.3.16) (or triple (A, B, C)).

Let

$$z_i(t) = \begin{pmatrix} z_{i1}(t) \\ z_{i2}(t) \\ \vdots \\ z_{id_i}(t) \end{pmatrix} = E_i x(t) = \begin{pmatrix} C_i x(t) \\ C_i A x(t) \\ \vdots \\ C_i A^{d_i-1} x(t) \end{pmatrix}, \quad i = 1, 2, \dots, m, \tag{1.3.17}$$

and assume that the following matrix E is full rank matrix, that is, $\text{rank}(E) = d_1 + d_2 + \cdots + d_m$,

$$E = (E_1, E_2, \dots, E_m)^\top. \tag{1.3.18}$$

Then there exists matrix $F \in \mathbb{R}^{s\times n}$ with rank $s = n - d_1 - \cdots - d_m$ such that $(E, F)^\top$ is invertible. Let $z(t) = (z_1(t), \dots, z_m(t))^\top$ with

$$\begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_m(t) \\ \zeta(t) \end{pmatrix} = Tx(t) = \begin{pmatrix} E \\ F \end{pmatrix} x(t). \tag{1.3.19}$$

It is obvious that the above transformation is invertible and under this transformation,

$$\begin{cases} \dot{z}_{i1}(t) = z_{i2}(t), \\ \dot{z}_{i2}(t) = z_{i3}(t), \\ \quad \vdots \\ \dot{z}_{id_i}(t) = c_i A^{d_i} T^{-1}(z(t), \zeta(t))^\top + C_i A^{d_i} B u(t), i = 1, 2, \dots, m, \\ \dot{\zeta}(t) = AFT^{-1}(z(t), \zeta(t))^\top + FBu(t). \end{cases} \tag{1.3.20}$$

Furthermore, if $FB = 0$, then system (1.3.20) is a special case of (1.3.15).

The following nonlinear system can also be transformed into a special case of (1.3.15) by a geometric method under some conditions:

$$\begin{cases} \dot{x}(t) = f(x(t)) + \displaystyle\sum_{i=1}^{m} g_i(x(t)) u_i(t), \ \ x(0) = x_0 \in \mathbb{R}^n, \\ y(t) = (y_1(t), y_2(t), \dots, y_m(t))^\top = h(x(t)), \end{cases} \tag{1.3.21}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) = (u_1(t), u_2(t), \dots, u_m(t))^\top \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^m$ is the output, $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is the system function, $g_i \in C(\mathbb{R}^n, \mathbb{R}^n)$ $(i = 1, 2, \dots, m)$ are control functions.

Now, we introduce the Lie derivative and Lie bracket in geometry.

Definition 1.3.2 *Suppose that* $h(x) = (h_1(x), h_2(x), \ldots, h_m(x))^\top \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^\top \in C(\mathbb{R}^n, \mathbb{R}^n)$. *The Lie derivative* $L_f h(x) : \mathbb{R}^n \to \mathbb{R}$ *of function* $h(x)$ *along vector field* $f(x)$ *is defined as*

$$L_f h(x) = \left(\frac{\partial h(x)}{\partial x_1}, \frac{\partial h(x)}{\partial x_2}, \ldots, \frac{\partial h(x)}{\partial x_n}\right)\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial h(x)}{\partial x_i} f_i(x). \tag{1.3.22}$$

If $L_f h \in C^1(\mathbb{R}^n, \mathbb{R})$, *then the Lie derivative of* $L_f h(x)$ *along the vector field* $f(x)$ *is denoted by* $L_f^2 h(x)$ *that is,* $L_f^2 h(x) = (L_f(L_f h))(x)$. *Generally, we denote* $L_f^0 h(x) = h(x)$ *and* $L_f^i h(x) = L_f(L_f^{(i-1)} h(x))$, $i = 1, 2, \ldots, n$. *Similarly,* $L_g L_f h(x)$ *is the symbol of* $L_g(L_f h(x))$, *where* $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))^\top \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *is another vector field.*

The Lie bracket of vector fields $f(x)$ *and* $g(x)$ *is a vector field denoted by* $[f, g](x)$ *given as*

$$[f, g](x) \triangleq \begin{pmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} & \cdots & \frac{\partial g_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n(x)}{\partial x_1} & \frac{\partial g_n(x)}{\partial x_2} & \cdots & \frac{\partial g_n(x)}{\partial x_n} \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} & \cdots & \frac{\partial f_2(x)}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \frac{\partial f_n(x)}{\partial x_2} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix}\begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_n(x) \end{pmatrix}. \tag{1.3.23}$$

Generally, we denote $ad_f^k g(x) \triangleq [f, ad_f^{k-1} g](x)$ *and* $ad_f^0 g(x) = g(x)$.

For the Lie derivatives and Lie brackets, we have the following basic properties.

Lemma 1.3.1 *For the vector fields* $f, g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ *and functions* $\alpha, \beta \in C(\mathbb{R}^n, \mathbb{R})$, $\lambda \in C^1(\mathbb{R}^n, \mathbb{R})$, *the following conclusions hold true.*

(i) $L_{\alpha f}\lambda(x) = \alpha(x) L_f \lambda(x)$.

(ii) If $\alpha, \beta \in C^1(\mathbb{R}^n, \mathbb{R})$, *then*

$$[\alpha f, \beta g](x) = \alpha(x)\beta(x)[f, g](x) + \alpha(x)(L_f \beta(x))g(x) - \beta(x)(L_g \alpha(x))f(x).$$

(iii) $L_{[f,g]}\lambda(x) = L_f L_g \lambda(x) - L_g L_f \lambda(x)$.

For the given smooth vector fields $f_i \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $i = 1, 2, \ldots, d$, the vector space (depending on x) spanned by $f_1(x), f_2(x), \ldots, f_d(x)$ is called the distribution of vector fields $f_i(x)$, $i = 1, 2, \ldots, d$. We use the symbol $\Delta(x)$ to denote the distribution, that is,

$$\Delta(x) = \text{span}\{f_1(x), f_2(x), \ldots, f_d(x)\}. \tag{1.3.24}$$

The distribution $\Delta(x)$ is called involutive if the Lie bracket $[f_i, f_j](x)$ of any pair of vector fields $f_i(\cdot)$ and $f_j(\cdot)$ is a vector field that belongs to $\Delta(x)$, that is, there exist functions $a_k \in C(\mathbb{R}^n, \mathbb{R})$, $k = 1, 2, \ldots, d$, such that

$$[f_i, g_j](x) = \sum_{i=1}^{n} a_k(x) f_k(x). \tag{1.3.25}$$

In order to transform system (1.3.21) into the canonical form, we introduce the Frobenius theorem.

Suppose that one is interested in solving the following differential equation:

$$\begin{cases} L_{f_1}\varphi(x) = f_{11}(x)\dfrac{\partial\varphi(x)}{\partial x_1} + f_{12}(x)\dfrac{\partial\varphi(x)}{\partial x_2} + \cdots + f_{1n}(x)\dfrac{\partial\varphi(x)}{\partial x_n} = 0, \\ L_{f_2}\varphi(x) = f_{21}(x)\dfrac{\partial\varphi(x)}{\partial x_1} + f_{22}(x)\dfrac{\partial\varphi(x)}{\partial x_2} + \cdots + f_{2n}(x)\dfrac{\partial\varphi(x)}{\partial x_n} = 0, \\ \qquad\vdots \\ L_{f_d}\varphi(x) = f_{d1}(x)\dfrac{\partial\varphi(x)}{\partial x_1} + f_{d2}(x)\dfrac{\partial\varphi(x)}{\partial x_2} + \cdots + f_{dn}(x)\dfrac{\partial\varphi(x)}{\partial x_n} = 0, \end{cases} \tag{1.3.26}$$

where $f_1, f_2, \ldots, f_d \in C^1(U \subset \mathbb{R}^n, \mathbb{R}^n)$ are vector fields that span a distribution $\Delta(x)$ for integer $d < n$, and $f_{ij}(\cdot)$ is the jth component of vector field $f_i(\cdot)$. The system of partial differential equations (1.3.26) or the d-dimensional distribution $\Delta(\cdot)$ is said to be *completely integrable* if there exist $n - d$ independent smooth functions $\varphi_i \in C^1(\mathbb{R}^n, \mathbb{R})$, $i = 1, 2, \ldots, n - d$, satisfying differential equations (1.3.26) on U. By "independent", we mean that the row vector group composed by gradients $\nabla\varphi_1(x), \nabla\varphi_2(x), \ldots, \nabla\varphi_{n-d}(x)$ are independent at every $x \in U$.

Lemma 1.3.2 *A distribution is completely integrable if and only if it is involutive.*

Now we give the definition of relative degree for nonlinear systems (1.3.21).

Definition 1.3.3 *Let $U \subset \mathbb{R}^n$ be a neighborhood near the initial state of system (1.3.21). If there exist positive integers r_i, $i = 1, 2, \ldots, m$ such that*

$$L_{g_j} L_f^k h_i(x) = 0 \quad \forall\ x \in U, \quad 0 \le k \le r_i - 2,\ \ i = 1, 2, \ldots,\ m, j = 1, 2, \ldots, m, \tag{1.3.27}$$

and the following matrix function $A(x)$ is invertible at x_0, then we say that system (1.3.21) has the relative degree $\{r_1, r_2, \dots, r_m\}$ at initial state x_0:

$$A(x) = \begin{pmatrix} L_{g_1}L_f^{r_1-1}h_1(x) & L_{g_2}L_f^{r_1-1}h_1(x) & \cdots & L_{g_m}L_f^{r_1-1}h_1(x) \\ L_{g_1}L_f^{r_2-1}h_2(x) & L_{g_2}L_f^{r_2-1}h_2(x) & \cdots & L_{g_m}L_f^{r_2-1}h_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1}L_f^{r_m-1}h_m(x) & L_{g_2}L_f^{r_m-1}h_m(x) & \cdots & L_{g_m}L_f^{r_m-1}h_m(x) \end{pmatrix}. \tag{1.3.28}$$

Lemma 1.3.3 *Suppose that system (1.3.21) has a (vector) relative degree $\{r_1, \dots, r_m\}$ at the initial state x_0. Then*

$$r_1 + r_2 + \cdots + r_m \le n.$$

Set, for $i = 1, 2, \dots, m$,

$$\begin{cases} \xi_{i1}(x) = \phi_{i1}(x) = h_i(x), \\ \xi_{i2}(x) = \phi_{i2}(x) = L_f h_i(x), \\ \qquad \vdots \\ \xi_{ir_i}(x) = L_f^{r_i-1}h_i(x). \end{cases} \tag{1.3.29}$$

We assume without loss of generality that $r = r_1 + r_2 + \cdots + r_i < n$ and there exist $n - r$ functions $\phi_{r+1}, \phi_{r+2}, \dots, \phi_n \in C(\mathbb{R}^n, \mathbb{R})$ such that the mapping

$$\Phi(x) = (\phi_{11}(x), \phi_{12}(x), \dots, \phi_{1r_i}(x), \dots, \phi_{mr_m}(x), \phi_{r+1}(x), \dots, \phi_n(x))^\top$$

has a Jacobian matrix that is nonsingular at x_0. Moreover, if the distribution

$$G(x) = \operatorname{span}\{g_1(x), g_2(x), \dots, g_m(x)\}$$

is involutive near x_0, then $\phi_{r+1}(x), \dots, \phi_n(x)$ satisfy

$$L_{g_j}\phi_i(x) = 0, \ \ j = 1, 2, \dots, m, \ \ j = r+1, \dots, n, \ \ x \in U,$$

where $U \subset \mathbb{R}^n$ is a neighborhood of initial state x_0.

Set

$$\xi_i(x) = \begin{pmatrix} \xi_{i1}(x) \\ \xi_{i2}(x) \\ \vdots \\ \xi_{ir_i}(x) \end{pmatrix} = \begin{pmatrix} \phi_{i1}(x) \\ \phi_{i2}(x) \\ \vdots \\ \phi_{ir_i}(x) \end{pmatrix}, \quad i = 1, 2 \dots, m, \tag{1.3.30}$$

$$\xi(x) = (\xi_1(x), \xi_2(x), \dots, \xi_m(x))^\top, \tag{1.3.31}$$

$$\eta(x) = \begin{pmatrix} \eta_1(x) \\ \eta_2(x) \\ \vdots \\ \eta_{n-r}(x) \end{pmatrix} = \begin{pmatrix} \phi_{r+1}(x) \\ \phi_{r+2}(x) \\ \vdots \\ \phi_n(x) \end{pmatrix}, \tag{1.3.32}$$

and

$$\begin{aligned} & b_{ij}(\xi,\eta) = L_{g_i} L_f^{r_i-1} h_i(\Phi^{-1}(\xi,\eta)), \ \ \forall\ 1 \le i,j \le m, \\ & \psi_i(\xi,\eta) = L_f^{r_i} h_i(\Phi^{-1}(\xi,\eta)), \ \ \forall\ 1 \le i \le m, \\ & F_0(\xi,\eta) = (L_f \phi_{r+1}(\Phi^{-1}(\xi,\eta)), L_f \phi_{r+2}(\Phi^{-1}(\xi,\eta)), \ldots, L_f \phi_n(\Phi^{-1}(\xi,\eta))^\top. \end{aligned} \tag{1.3.33}$$

Then the general nonlinear affine system (1.3.21) has the form

$$\begin{cases} \dot{\xi}_{i1}(t) = \xi_{i2}(t), \\ \dot{\xi}_{i2}(t) = \xi_{i3}(t), \\ \qquad \vdots \\ \dot{\xi}_{ir_i}(t) = \psi(\xi(t),\eta(t)) + \displaystyle\sum_{j=1}^{m} b_{ij}(\xi(t),\eta(t))u_j(t), \\ \dot{\eta}(t) = F_0(\xi(t),\eta(t)), \\ y_i(t) = \xi_{i1}(t), \end{cases} \tag{1.3.34}$$

which is also a special case of the canonical form of ADRC (1.3.15).

1.3.2 Stability for Nonlinear Systems

In this section, we give some basic notation and results about stability for nonlinear systems. In this book, the stability means Lyapunov stability, which is named after Aleksandr Lyapunov, a Russian mathematician who published his doctoral thesis, *The General Problem of Stability of Motion*, in 1892. It was becoming more interests during the Cold War period when it was found to be applicable to the stability of aerospace guidance systems, which typically contains strong nonlinearities that are not treatable by other methods.

In this book, we use $\|\cdot\|$ to denote the Euclidian norm of $\mathbb{R}^n$: $\| (\nu_1, \nu_2, \ldots, \nu_n) \| = \left(\sum_{i=1}^n |\nu_i|^2\right)^{1/2}$ and $\|\cdot\|_\infty$ the infinite norm of $\mathbb{R}^n$: $\| (\nu_1, \ldots, \nu_n) \|_\infty = \max_{i=1,\ldots,n} |\nu_i|$. It is well known that the two norms are equivalent; however, for the simplicity, we use different norms in different circumstances.

Consider the nonlinear system of the following:

$$\dot{x}(t) = f(t, x(t)), \tag{1.3.35}$$

where $f(t,x) = (f_1(t,x), f_2(t,x), \ldots, f_n(t,x))^\top \in C([0;\infty) \times \mathbb{R}^n, \mathbb{R}^n)$ with $f_i(t,x)$ being locally Lipschitz continuous with respect to x (i.e., $|f_i(t,x_1) - f_i(t,x_2)| \le L_t \|$

$x_1 - x_2 \parallel$, for some $L_t > 0$ and all $x_1, x_2 \in \mathbb{R}^n$) and $f_i(0) = 0, i = 1, 2, \ldots, n$. It is obvious that $x(t) \equiv 0$ is a trivial solution of system (1.3.35). The trivial solution is also said to be an equilibrium state of the system. To represent the dependence of a system solution with initial state, we denote, in this section, the solution of system (1.3.35) with the initial state $x(0) = x_0 \in \mathbb{R}^n$ as $x(t; x_0)$.

Definition 1.3.4 *If for any positive constant $\epsilon > 0$ there exists $\sigma > 0$ such that for any $x_0 \in \mathbb{R}^n$ satisfying $\| x_0 \| < \sigma$, the solution $x(t; x_0)$ of (1.3.35) satisfies $\| x(t; x_0) \| < \epsilon, \forall \ t \geq 0$, then the zero equilibrium of system (1.3.35) is said to be stable in the sense of Lyapunov.*

Definition 1.3.5 *A domain $\Omega \subset \mathbb{R}^n (0 \in \Omega^\circ)$ is said to be the attracting basin of the zero equilibrium state of (1.3.35), if for any $x_0 \in \Omega$, the solution with initial value x_0 tends to zero as time goes to ∞, that is, $\lim_{t\to\infty} \| x(t; x_0) \| = 0$, and for any $x \in \mathbb{R}^n \setminus \Omega$, $\lim_{t\to\infty} \| x(t; x_0) \| = 0$ is not valid any more. We say that the zero equilibrium of system (1.3.35) is attractive on Ω. Furthermore, if $\Omega = \mathbb{R}^n$, we say that the zero equilibrium of system (1.3.35) is globally attractive.*

Definition 1.3.6 *The zero equilibrium of system (1.3.35) is said to be asymptotically stable on attracting basin Ω if it is stable and attracting on Ω. If $\Omega = \mathbb{R}^n$, we say that the equilibrium is globally asymptotically stable.*

We point out that there is no implication relation between stability and attractiveness. For example, consider the following system:

$$\begin{cases} \dot{x}_1(t) = x_2(t), & x_1(0) = x_{10}, \\ \dot{x}_2(t) = x_1(t), & x_2(0) = x_{20}. \end{cases} \tag{1.3.36}$$

The solution of system (1.3.36) is

$$\begin{cases} x_1(t; x_{10}, x_{20}) = x_{10} \cos t - x_{20} \sin t, \\ x_2(t; x_{10}, x_{20}) = x_{10} \sin t + x_{20} \cos t. \end{cases} \tag{1.3.37}$$

A straightforward computation shows that

$$x_1^2(t; x_{10}, x_{20}) + x_2^2(t; x_{10}, x_{20}) = x_{10}^2 + x_{20}^2.$$

It is obvious that the zero equilibrium state of system (1.3.36) is stable, but not attractive. Also there exists an example where the zero equilibrium is attractive but not stable. Consider the following system:

$$\begin{cases} \dot{x}(t) = f(x(t)) + y(t), \\ \dot{y}(t) = -x(t), \end{cases} \qquad f(x) = \begin{cases} -4x, & x > 0; \\ 2x, & -1 \leq x \leq 0; \\ -x - 3, & x \leq -1. \end{cases} \tag{1.3.38}$$

If $x > 0$, the general solution of the system (1.3.38) is

$$\begin{cases} x(t) = c_1(2 - \sqrt{3})e^{(-2+\sqrt{3})t} + c_2(2 + \sqrt{3})e^{(-2-\sqrt{3})t}, \\ y(t) = c_1 e^{(-2+\sqrt{3})t} + c_2 e^{(-2-\sqrt{3})t}. \end{cases} \tag{1.3.39}$$

For $x \in [-1, 0]$, the general solution is

$$\begin{cases} x(t) = c_1 e^t + c_2 t e^t, \\ y(t) = (-c_1 + c_2) e^t - c_2 t e^t. \end{cases} \tag{1.3.40}$$

For $x < -1$, the general solution is

$$\begin{cases} x(t) = 1/2\, c_1 e^{-\frac{t}{2}} \left(\cos \frac{\sqrt{3}}{2} t + \sqrt{3} \sin \frac{\sqrt{3}}{2} t \right) + 1/2\, c_2 e^{-\frac{t}{2}} \left(\cos \frac{\sqrt{3}}{2} t - \sqrt{3} \sin \frac{\sqrt{3}}{2} t \right), \\ y(t) = c_1 e^{-\frac{t}{2}} \cos \dfrac{\sqrt{3}}{2} t + c_2 e^{-\frac{t}{2}} \sin \dfrac{\sqrt{3}}{2} t + 3. \end{cases} \tag{1.3.41}$$

The trajectories of system (1.3.38) are plotted in Figure 1.3.3.

By the general solution and Figure 1.3.3, we can obtain $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$ for each solution $(x(t), y(t))$ of system (1.3.38). Consider the solution of system (1.3.38) with the initial value

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -e^{-t_0} \\ e^{-t_0} \end{pmatrix}, \tag{1.3.42}$$

where $t_0 > 0$ is a positive constant. The solution of system (1.3.38) with initial value $(x_0, y_0)^\top$ in interval $[0, t_0]$ is

$$x(t; x_0) = -e^{t-t_0}, \;\; y(t) = e^{t-t_0}, \quad t \le t_0, \tag{1.3.43}$$

which satisfies $x(t_0) = -1$ and $y(t_0) = 1$. A simple computation shows that

$$\lim_{t_0\to\infty} x_0 = \lim_{t_0\to\infty} -e^{-t_0} = 0, \quad \lim_{t_0\to\infty} y_0 = \lim_{t_0\to\infty} e^{-t_0} = 0. \tag{1.3.44}$$

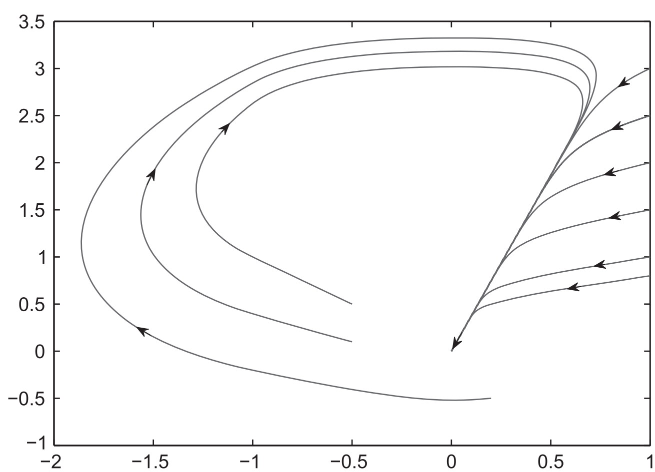

Figure 1.3.3 Orbit distribution of system (1.3.38).

This implies that when t_0 is large enough, $\| (x_0, y_0)^\top \|$ can be as small as expected. However, system (1.3.38) is not stable because no matter how small the norm of the initial state is, $\| (x(t_0; x_0), y(t_0; y_0))^\top \| = \sqrt{2}$ can not be always arbitrarily small.

The class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions and Lyapunov functions are important in stability analysis.

Definition 1.3.7 *The function $\varphi \in C([0, a), [0, \infty))$ is said to be a class $\mathcal{K}$ function if $\varphi(r)$ is strictly increasing on $[0, a)$ and $\varphi(0) = 0$. Furthermore, if $a = +\infty$ and $\lim_{r\to+\infty} \varphi(r) = \infty$, then $\varphi(r)$ is a class $\mathcal{K}_\infty$ function.*

Definition 1.3.8 *Let $\Omega \subset \mathbb{R}^n$ and $0 \in \Omega^\circ$. Function $V \in C(\Omega, [0, \infty))$ is said to be positive definite ($-V(x)$ is said to be negative definite) if, for any $x \in \Omega, V(x) \geq 0$ and $V(x) = 0$ if and only if $x = 0$. Furthermore, if $\Omega = \mathbb{R}^n$ and $\lim_{\|x\|\to+\infty} V(x) = +\infty$, then $V(x)$ is said to be a radially unbounded positive definite function. In stability analysis, the positive definite function is also said to be a Lyapunov function.*

Theorem 1.3.1 *Suppose that $V \in C(\Omega, [0, \infty))$ is a positive definite function on Ω, where $\Omega \subset \mathbb{R}^n (0 \in \Omega^\circ)$ is a connected domain and $B_r = \{x \in \mathbb{R}^n : \| x \| \leq r\} \subset \Omega$ for $r > 0$. Then there exist class $\mathcal{K}$ functions $\kappa_1, \kappa_2 \in C([0, r), \bar{\mathbb{R}}^+)$ such that*

$$\kappa_1(\| x \|) \leq V(x) \leq \kappa_2(\| x \|), \quad \forall\, x \in B_r.$$

Furthermore, if $V(x)$ is radially unbounded, then $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are class $\mathcal{K}_\infty$ functions.

Proof. Let $\kappa(\tau) = \inf_{\tau\leq\|x\|\leq r} W(x),\ \ \tau \in [0, r)$. It is easy to verify that for any $\tau \in (0, r)$, $\kappa(\tau) > 0$, $\kappa(0) = 0$, and $\kappa(\tau)$ is continuous on $(0, r)$.

Let $\kappa_1(\tau) = \frac{\tau\kappa(\tau)}{r}$. A direct computation shows that $\kappa_1(0) = 0$. For any $\tau_1, \tau_2 \in [0, r)$, if $\tau_1 < \tau_2$ then

$$\kappa_1(\tau_1) = \frac{\tau_1\kappa(\tau_1)}{r} \leq \frac{\tau_1\kappa(\tau_2)}{r} < \frac{\tau_2\kappa(\tau_2)}{r} = \kappa_1(\tau_2).$$

Let $\tilde{\kappa}(\tau) = \max\limits_{\|x\|\leq\tau} |V(\tau)|$, $\tau \in [0, r)$. Also, we can verify that $\tilde{\kappa}(\tau)$ is continuous on $[0, r)$, for any $\tau \in (0, r)$, $\tilde{\kappa}(\tau) > 0$. Therefore $\kappa_1(\tau)$ is a class $\mathcal{K}$ function.

Let $\kappa_2(\tau) = \tilde{\kappa}(\tau) + \tau, \tau \in [0, r)$. A simple computation shows that $\kappa_1(0) = 0$, for any $\tau \in (0, \tau)$, $\kappa_2(\tau) > 0$, and for any $\tau_1, \tau_2 \in [0, r)$, if $\tau_1 < \tau_2$; then

$$\kappa_2(\tau_1) = \tilde{\kappa}(\tau_1) + \tau_1 \leq \tilde{\kappa}(\tau_2) + \tau_1 < \tilde{\kappa}(\tau_2) + \tau_2 = \kappa_2(\tau_2).$$

Therefore κ_2 is a $\mathcal{K}$ function.

Finally, for any $x \in B_r$,

$$\kappa_1(\| x \|) \leq \kappa(\| x \|) = \inf_{\xi\in B_r,\, \|x\|\leq\|\xi\|\leq r} V(\xi) \leq V(x) \leq \max_{\xi\in B_r,\, 0\leq\|\xi\|\leq\|x\|} V(\xi)$$

$$= \tilde{\kappa}(\| x \|) \leq \kappa_2(\| x \|).$$

This completes the proof of the theorem. □

The next result is the Lyapunov theorem on stability for an autonomous system:

$$\dot{x}(t) = f(x(t)), \;\; f(x) = (f_1(x), f_2(x), \dots, f_n(x))^\top, \;\; f_i \in C(\mathbb{R}^n, \mathbb{R}), i = 1, 2, \dots, n. \tag{1.3.45}$$

Theorem 1.3.2 *Let $f(0) \equiv 0$ in (1.3.45) and hence the zero state is an equilibrium state of system (1.3.45). Let $\Omega = B_r(0) \subset \mathbb{R}^n$ and $V \in C^1(\Omega, \mathbb{R})$ is a positive definite Lyapunov function:*

1. *If for every $x \in \Omega$, the Lie derivative of $V(x)$:*
$$L_f V(x) = \left.\frac{dV(x)}{dt}\right|_{along\ (1.3.45)} = \sum_{i=1}^{n} \frac{\partial V(x)}{\partial x_i} f_i(x) \leq 0, \tag{1.3.46}$$
then the zero equilibrium state of system (1.3.45) is Lyapunov stable.
2. *If $-\frac{dV(x(t))}{dt}\Big|_{(1.3.45)}$ is positive definite on Ω, then the zero equilibrium of system (1.3.45) is asymptotically stable.*

Proof.

1. For the positive definite Lyapunov function $V(x)$, it follows from Theorem 1.3.1 that there exist class $\mathcal{K}$ functions $\kappa_1, \kappa_2 \in \mathbb{C}([0, r), [0, \infty))$ such that

$$\kappa_1(\| x \|) \leq V(x) \leq \kappa_2(\| x \|).$$

Let $x(t; x_0)$ be the solution of system (1.3.45) with initial condition $x(0) = x_0$. For any $\epsilon > 0$, let $\delta = \kappa_1(\kappa_2^{-1}(\epsilon))$. It follows from (1.3.46) that $V(x(t; x_0)) \leq V(x(t; x_0)) = V(x_0)$ for any $t > 0$. For any $x_0 \in \mathbb{R}^n$, if $\| x_0 \| < 0$, then

$$\| x(t; x_0) \| \leq \kappa_1^{-1}(V(x(t; x_0))) \leq \kappa_1^{-1}(V(x_0)) \leq \kappa_1^{-1}(\kappa_2(\| x_0 \|)) < \epsilon. \tag{1.3.47}$$

The Lyapunov stability of zero equilibrium of system (1.3.45) is proved.

2. Let

$$W(x(t; x_0)) = \left.\frac{dV(x(t; x_0))}{dt}\right|_{\text{along } (1.3.45)} = \sum_{i=1}^{n} \frac{\partial V(x(t; x_0))}{\partial x_i} f_i(x(t; x_0)). \tag{1.3.48}$$

From the positive definiteness of $W(x)$, there exist class $\mathcal{K}$ functions $\tilde{\kappa}_1(\cdot)$ and $\tilde{\kappa}_2(\cdot)$ such that

$$\tilde{\kappa}_1(\| x \|) \leq W(x) \leq \tilde{\kappa}_2(\| x \|), \;\; x \in \mathbb{R}^n.$$

Therefore if $V(x(t; x_0)) > \sigma$ for some $\sigma > 0$, then

$$\begin{aligned} \left.\frac{dV(x(t; x_0))}{dt}\right|_{\text{along } (1.3.45)} &\leq -W(x(t; x_0)) \leq -\tilde{\kappa}_1(\| x(t; x_0) \|) \leq -\tilde{\kappa}_1(\kappa_1^{-1}(V(x(t; x_0)))) \\ &\leq -\tilde{\kappa}_1(\kappa_1^{-1}(\sigma)) < 0. \end{aligned}$$

This implies that

$$\lim_{t \to \infty} V(x(t; x_0)) = 0,$$

which together with the Lyapunov stability proved in (1) yields the asymptotical stability. This completes the proof of the theorem. □

The following Theorem 1.3.3 is the inverse Lyapunov theorem 1.3.2.

Theorem 1.3.3 *Suppose that the zero equilibrium of system (1.3.35) is asymptotically stable and the attracting basin is $\Omega \subset \mathbb{R}^n$, where Ω is a connected domain and $0 \in \Omega^\circ$. If $f \in C(\Omega, \mathbb{R}^n)$ is locally Lipschitz continuous, then there exist Lyapunov functions $V \in C^1(\Omega, [0, \infty))$ and $W \in C(\Omega, [0, \infty))$ such that*

$$\left.\frac{dV(x)}{dt}\right|_{\text{along}(1.3.35)} \le -W(x), \quad \forall\ x \in \Omega,\ \lim_{x\to\partial\Omega} V(x) = +\infty.$$

The Theorem 1.3.3 is a special case of Theorem 1.3.11, which is proved in Section 1.3.6.

The well-known Lasalle invariance principle is a powerful tool for verifying stability for autonomous systems.

Theorem 1.3.4 *Suppose that in system (1.3.45), $f(0) = 0$ and $\Omega = B_r(0) \subset \mathbb{R}^n$ is a connected domain. The function $V \in C^1(\Omega, [0, \infty))$ is positive definite and satisfies*

$$L_f V(x) \le 0, \quad \forall x \in \Omega. \tag{1.3.49}$$

In addition, there is no nonzero solution of system (1.3.35) staying in the following set $L_f V^{-1}(0)$:

$$L_f V^{-1}(0) = \{x \in \Omega : L_f V(x) = 0\}.$$

Then the zero equilibrium of (1.3.35) is asymptotically stable.

As a preliminary of proving Theorem 1.3.4, we give Lemma 1.3.4.

Lemma 1.3.4 *Let $x(t; x_0)$ be the solution of system (1.3.45) and let x^* be the limit point of $x(t; x_0)$, that is, there exists series t_k: $t_k \to \infty$ as k goes to ∞ such that $\lim_{k\to\infty} x(t_k; x_0) = x^*$. Then any point in $E = \{x(t; x^*); t \ge 0\}$ is the limit point of $x(t; x_0)$.*

Proof of Theorem 1.3.4. Let $\Omega(x_0) = \{x^* \in \mathbb{R}^n |\ x^* \text{ is the limit point of } x(t; x_0)\}$. It follows from (1.3.49) that the equilibrium of system (1.3.45) is Lyapunov stable and $\Omega(x_0)$ is a bounded nonempty set. We show that $\Omega(x_0) = \{0\}$. If this is not true, then there exists a sequence t_n: $t_n \to \infty$ such that $\lim_{n\to\infty} x(t_n; x_0) = x^* \ne 0$. Again, by using (1.3.49), we can obtain that $V(x(t; x_0))$ is nonincreasing as t increases. This together with the positive definiteness and continuity of $V(x)$ gives

$$\lim_{t\to\infty} V(x(t; x_0)) = V(x^*) > 0. \tag{1.3.50}$$

Now consider the solution of (1.3.45) starting from x^*. From (1.3.49), we have $V(x(t; x_*)) < V(x^*)$ for all $t > 0$. If for any $t \ge 0$

$$V(x(t; x^*)) \equiv V(x^*), \tag{1.3.51}$$

then

$$\frac{dV(x(t;x^*))}{dt} = 0, \tag{1.3.52}$$

which implies that $\{x(t;x^*)|\ t \geq 0\} \subset L_f V^{-1}(0)$. This is a contradiction. Hence there exists $t_1 > 0$ such that $V(x(t_1;x^*)) < V(x^*)$. It follows from Lemma 1.3.4 that there exists a sequence $\{t_n^*\}$ such that

$$\lim_{n\to\infty} x(t_n^*;x_0) = x(t_1;x^*),$$

which yields

$$\lim_{n\to\infty} V(x(t_n^*;x_0)) = V(x(t_1;x^*)) < V(x^*).$$

This contradicts (1.3.50). The result is thus concluded. □

1.3.3 Stability of Linear Systems

Let $A \in \mathbb{R}^{n\times n}$. Consider the linear system of the following:

$$\dot{x}(t) = Ax(t),\ \ x(0) = x_0. \tag{1.3.53}$$

First of all, we introduce the Kronecker product and straightening operator of the matrices.

Definition 1.3.9 *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n},\quad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ b_{21} & b_{22} & \cdots & a_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{s1} & b_{s2} & \cdots & b_{sl} \end{pmatrix}_{s\times l}. \tag{1.3.54}$$

The Kronecker product of A and B is an $(ml)\times(ns)$ matrix, which is defined as follows:

$$A\otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}_{(ml)\times(ns)}. \tag{1.3.55}$$

The straightening operator is a $1\times(nm)$ matrix given by

$$\overrightarrow{A} = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots a_{2n}, \ldots, a_{n1}, \ldots, a_{nn})^\top. \tag{1.3.56}$$

We can verify that the Kronecker product and straightening operator have the following properties.

Property 1.3.1

(i) *If $m = n$ and $s = l$, then*

$$\det(A \otimes B) = (\det(A))^m (\det(B))^s.$$

(ii) *The Kronecker product $A \otimes B$ is invertible if and only if both A and B are invertible, and*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

(iii) *Let E_{ij} be an $m \times n$ matrix with the ith row and jth column component being one and other entries being identical to zero. Let $e_i \in \mathbb{R}^{1\times m}$ (or $e_i \in \mathbb{R}^{1\times n}$) with the ith component being one and other entries being zero. Then*

$$\begin{aligned} A = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}, \quad Ae_i = (a_{1i}, a_{2i}, \ldots, a_{mi})^\top, \\ e_i^\top A = (a_{i1}, a_{i2}, \ldots, a_{in}), E_{ij} = e_i e_j^\top, \quad \overrightarrow{E_{ij}} = e_i \otimes e_j. \end{aligned} \tag{1.3.57}$$

(iv) *Let $A \in \mathbb{R}^{n\times m}$, $B \in \mathbb{R}^{m\times s}$, and $C \in \mathbb{R}^{s\times l}$. Then $\overrightarrow{ABC} = (A \otimes C^\top)\overrightarrow{B}$.*

Let $A, C \in \mathbb{R}^{n\times n}$. The Lyapunov equation determined by unknown matrix X with respect to A and C is defined by

$$A^\top X + XA = C. \tag{1.3.58}$$

The following Property 1.3.2 is about solvability of the Lyapunov equation (1.3.58).

Property 1.3.2 *Let $A, C \in \mathbb{R}^{n\times n}$. The following conclusions are equivalent:*

(i) *There exists a unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying (1.3.58).*
(ii) *There exists a unique vector $x \in \mathbb{R}^{n^2}$ satisfying the linear equation*

$$(A^\top \otimes I_{n\times n} + I_{n\times n} \otimes A^\top)x = \overrightarrow{C}. \tag{1.3.59}$$

(iii) *The matrix $A^\top \otimes I_{n\times n} + I_{n\times n} \otimes A^\top$ is invertible, that is, rank$(A^\top \otimes I_{n\times n} + I_{n\times n} \otimes A^\top) = n^2$.*
(iv) $\prod_{i,j=1}^{n}(\lambda_i + \lambda_j) \neq 0$.

Based on Property 1.3.2, we have immediately Theorem 1.3.5.

Theorem 1.3.5 *If A is a Hurwitz matrix, that is, all the eigenvalues of A have the negative real part, then for any positive definite symmetrical matrix $C \in \mathbb{R}^{n\times n}$ there is a unique positive definite symmetrical matrix solution $X \in \mathbb{R}^{n\times n}$ to the Lyapunov equation:*

$$A^\top X + XA = -C.$$

Let $V(x) = x^\top Bx$. Then $V(x)$ can be written as

$$V(x) = \frac{1}{\det(\Delta)} \begin{vmatrix} 0 & X \\ \overrightarrow{C} & \Delta \end{vmatrix}, \tag{1.3.60}$$

where $\Delta = A^\top \otimes I_{n\times n} + I \otimes A^\top$, $X = (X_1, X_2, \ldots, X_n)$, and

$$X_1 = (x_1^2, 2x_1x_2, \ldots, 2x_1x_n), \quad X_2 = (0, x_2^2, \ldots, 2x_2x_n), \quad X_n = (0, 0, \ldots, x_n^2).$$

Proof. Let $B = (b_{ij})$ satisfy

$$(A^\top \otimes I + I \otimes A^\top)\overrightarrow{B} = \overrightarrow{C}. \tag{1.3.61}$$

It follows from the Gramer law that

$$b_{ij} = \frac{\det(\Delta_{ij})}{\det(\Delta)}, \quad i, j = 1, 2, \ldots, n,$$

where Δ_{ij} is the matrix where the $(i-1)n_j$th column (the number of b_{ij}'s coefficient column) in Δ is replaced by $\overrightarrow{C}$ and other columns are the same as in Δ. Then

$$V(x) = x^\top Bx = \sum_{i,j=1}^{n} b_{ij}x_ix_j. \tag{1.3.62}$$

On the other hand, a direct computation shows that

$$\frac{1}{\det(\Delta)} \begin{vmatrix} 0 & X \\ \overrightarrow{C} & \Delta \end{vmatrix} = \sum_{i,j=1}^{n} \frac{\det(\Delta_{ij})}{\det(\Delta)} x_ix_j = \sum_{i,j=1}^{n} b_{ij}x_ix_j. \tag{1.3.63}$$

□

The following stability theorem can be directly obtained as an application of Theorem 1.3.5.

Theorem 1.3.6 *If A is a Hurwitz matrix, then the zero equilibrium of system (1.3.53) is globally asymptotical stable.*

Proof. Since A is Hurwitz, it follows from Theorem 1.3.5 that there exists a positive definite symmetrical matrix P_A such that

$$A^\top P_A + P_A A = -I_{n\times n},$$

where $I_{n\times n}$ is the $n \times n$ identity matrix. Let $V(\nu) = \nu^\top P_A \nu$ for all $\nu \in \mathbb{R}^n$. A direct computation shows that

$$\left.\frac{dV(x(t;x_0))}{dt}\right|_{\text{along } (1.3.53)} = (x(t;x_0))^\top (A^\top P_A + P_A A)x(t;x_0) = -\parallel x(t;x_0)\parallel^2. \tag{1.3.64}$$

Theorem 1.3.6 then follows by setting $W(\nu) = \parallel \nu \parallel^2$ for all $\nu \in \mathbb{R}^n$. □

1.3.4 Finite-Time Stability of Continuous System

The finite-time stability for continuous systems has many investigations. Here we list some preliminary results.

Definition 1.3.10 *Let $\Omega \in \mathbb{R}^n$ be a connected domain, $0 \in \Omega^\circ$, and $f(t,\cdot) \in C(\Omega, \mathbb{R}^n)$, $f(t,0) \equiv 0$, that is, the zero state is the equilibrium state of the system (1.3.35). The zero state is finite stable on the attracting basin Ω if it is Lyapunov stable and, for every $x_0 \in \Omega$, there exits a positive constant $T(x_0) > 0$ such that the solution of (1.3.35) starting from x_0 satisfies*

$$\lim_{t \uparrow T(x_0)} x(t; x_0) = 0,$$

$$x(t) = 0, \ \forall\ t \in [T(x_0), \infty).$$

Furthermore, if $\Omega = \mathbb{R}^n$, then the zero equilibrium of system (1.3.35) is globally finite-time stable, where $T(\cdot) : \Omega \to \mathbb{R}$ is a positive-valued function defined on Ω, which is said to be a setting-time function.

Now we look at an example. Consider the differential equation

$$\dot{x}(t) = -|x(t)|^\alpha \text{sign}(x(t)), \ \ x(0) = x_0, \alpha \in (0,1). \tag{1.3.65}$$

If $x_0 > 0$, then the solution of (1.3.65) is

$$x(t; x_0) = \begin{cases} \left(x_0^{1-\alpha} - t\right)^{\frac{1}{1-\alpha}}, & t < x_0^{1-\alpha}, \\ 0 & t \geq x_0^{1-\alpha}, \end{cases} \tag{1.3.66}$$

while $x_0 < 0$, the solution is

$$x(t; x_0) = \begin{cases} \left(t - |x_0|^{1-\alpha}\right)^{\frac{1}{1-\alpha}}, & t < |x_0|^{1-\alpha}, \\ 0 & t \geq |x_0|^{1-\alpha}. \end{cases} \tag{1.3.67}$$

We can clearly see that the system (1.3.65) is finite-time stable.

For the zero equilibrium state of a nonlinear system, we can verify the finite-time stability by Theorem 1.3.7.

Theorem 1.3.7 *Suppose that there exists a positive definite function $V \in C^1(\Omega, [0,\infty))$ and positive constants $\alpha \in (0,1)$ and $C > 0$ such that*

$$L_f V(x) \leq -CV^\alpha(x).$$

Then the zero equilibrium of system (1.3.35) is finite-time stable on $\Omega \in \mathbb{R}^n$ and the setting time $T(x_0)$ satisfies

$$T(x_0) \leq \frac{1}{C(1-\alpha)} V^{1-\alpha}(x_0), \tag{1.3.68}$$

where x_0 is the initial state of the system.

Proof. Let $x(t; x_0)$ be the solution of the initial value problem following

$$\dot{x}(t) = f(x(t)), \ \ x(0) = x_0. \tag{1.3.69}$$

Then

$$\frac{dV(x(t;x_0))}{dt} = L_f V(x(t;x_0)) \leq -CV^{\alpha}(x(t;x_0)). \tag{1.3.70}$$

Solve the following initial value problem:

$$\dot{z}(t) = -C|z(t)|^{\alpha}\text{sign}(z(t)), \;\; z(0) = V(x_0), \tag{1.3.71}$$

to obtain

$$z(t) = \begin{cases} \left(t + \dfrac{1}{C(1-\alpha)}(V(x_0))^{1-\alpha}\right)^{\frac{1}{1-\alpha}}, & t < \dfrac{1}{C(1-\alpha)}(V(x_0))^{1-\alpha}, \\ 0, & t \geq \dfrac{1}{C(1-\alpha)}(V(x_0))^{1-\alpha}. \end{cases} \tag{1.3.72}$$

This together with the comparison principle of the ordinary differential equations gives

$$V(x(t;x_0)) = 0, \;\; \forall \; t \geq \frac{1}{C(1-\alpha)}(V(x_0))^{1-\alpha}. \tag{1.3.73}$$

This completes the proof of the theorem.

The most popular continuous finite-time stable systems are those weighted homogeneous systems.

Definition 1.3.11 *The function* $V : \mathbb{R}^n \to \mathbb{R}$ *is said to be* d*-degree weighted homogeneous with the weights* $\{r_i > 0\}_{i=1}^n$*, if there exist positive constant* $\lambda > 0$ *and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *such that*

$$V(\lambda^{r_1}x_1, \lambda^{r_2}x_2, \ldots, \lambda^{r_n}x_n) = \lambda^d V(x_1, x_2, \ldots, x_n). \tag{1.3.74}$$

A vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ *is said to be* d*-degree weighted homogeneous with weights* $\{r_i > 0\}_{i=1}^n$ *if, for every* $i = 1, 2, \ldots, n$*,* $\lambda > 0$*, and* $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$*,*

$$g_i(\lambda^{r_1}x_1, \lambda^{r_2}x_2, \ldots, \lambda^{r_n}x_n) = \lambda^{d+r_i} g_i(x_1, x_2, \ldots, x_n), \tag{1.3.75}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ *is the* i*th component of* $g(\cdot)$*.*

If the vector field $g : \mathbb{R}^n \to \mathbb{R}^n$ *is* d*-degree weighted homogeneous with weights* $\{r_i > 0\}_{i=1}^n$*, then we say that the system*

$$\dot{x}(t) = g(x(t))$$

is d*-degree weighted homogeneous with weights* $\{r_i > 0\}_{i=1}^n$*.*

Example 1.3.1 *The following nonlinear system*

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -|x_1(t)|^{\alpha}\text{sign}(x_1(t)) - |x_2(t)|^{\beta}\text{sign}(x_2(t)), \end{cases} \tag{1.3.76}$$

is weighted homogeneous if $\beta = \frac{2\alpha}{1+\alpha}, \;\; \alpha > 0$*. Actually, let* $r_1 = 1, \;\; r_2 = (\alpha+1)/2$*, and let*

$$f_1(x_1, x_2) = x_2, \;\; f_2(x_1, x_2) = -|x_1|^{\alpha}\ \text{sign}(x_1) - |x_2|^{\beta}\ \text{sign}(x_2). \tag{1.3.77}$$

For any vector $(x_1, x_2) \in \mathbb{R}^2$ *and positive constant* $\lambda > 0$,

$$\begin{cases} f_1(\lambda^{r_1}x_1, \lambda^{r_2}x_2) = \lambda^{r_2}x_2 = \lambda^{\frac{\alpha-1}{2}+r_1}f_1(x_1, x_2), \\ f_2(\lambda^{r_1}x_1, \lambda^{r_2}x_2) = -\lambda^{\alpha r_1}|x_1|^{\alpha}\text{sign}(x_1) - \lambda^{\beta r_2}|x_2|^{\beta}\text{sign}(x_2) = \lambda^{\frac{\alpha-1}{2}+r_2}f_2(x_1, x_2). \end{cases} \tag{1.3.78}$$

Therefore system (1.3.76) is $\frac{\alpha-1}{2}$ *degree homogeneous with weights* $\{r_1, r_2\}$.

The following system (1.3.79) is also weighted homogeneous.

Example 1.3.2

$$\begin{cases} \dot{x}_1(t) = x_2(t) - |x_1(t)|^{\theta}\text{sign}(x_1(t)), \\ \dot{x}_2(t) = -|x_1(t)|^{2\theta-1}\text{sign}(x_1(t)), \end{cases} \tag{1.3.79}$$

where $\theta > 0$. *Actually, let*

$$f_1(x_1, x_2) = x_2 + |x_1|^{\theta}\text{sign}(x_1), \ \ f_2(x_1, x_2) = |x_1|^{2\theta-1}\text{sign}(x_1).$$

Then for any vector $(x_1, x_2) \in \mathbb{R}^2$ *and positive constant* $\lambda > 0$,

$$\begin{cases} f_1(\lambda x_1, \lambda^{\theta}x_2) = \lambda^{\theta}x_2 + |x_1|^{\theta}\text{sign}(x_1) = \lambda^{\theta-1+1}f_1(x_1, x_2), \\ f_2(\lambda x_1, \lambda^{\theta}x_2) = \lambda^{2\theta-1}|x_1|^{2\theta-1}\text{sign}(x_1) = \lambda^{\theta-1+\theta}f_2(x_1, x_2). \end{cases} \tag{1.3.80}$$

This means that system (1.3.79) is $\theta - 1$ *degree homogeneous with weights* $\{1, \theta\}$.

Property 1.3.3 *Suppose that* $V_1, V_2 : \mathbb{R}^n \to \mathbb{R}$ *are continuous weighted homogeneous functions with the same weights* $\{r_i > 0\}_{i=1}^n$, *with degree* $l_1 > 0$ *and* $l_2 > 0$ *respectively. Assume that* $V_1(x)$ *is positive definite. Then for any* $x \in \mathbb{R}^n$,

$$\left(\min_{y \in V_1^{-1}(1)} V_2(y)\right)(V_1(x))^{l_2/l_1} \le V_2(x) \le \left(\max_{y \in V_1^{-1}(1)} V_2(y)\right)(V_1(x))^{l_2/l_1}, \tag{1.3.81}$$

where $V_1^{-1}(1) \triangleq \{x \in \mathbb{R}^n |\ V_1(x) = 1\}$.

Theorem 1.3.8 *If the matrix*

$$K = \begin{pmatrix} -k_1 & 1 & 0 & \cdots & 0 \\ -k_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_n & 0 & 0 & \cdots & 1 \\ -k_{n+1} & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{1.3.82}$$

is Hurwitz, then there exists $\theta^* \in \left(\frac{n}{n+1}, 1\right)$ *such that for any* $\theta \in (\theta^*, 1)$, *the system following is finite-time stable:*

$$\begin{cases} \dot{x}_1(t) = x_2(t) - k_1[x_1(t)]^{\theta}, \\ \dot{x}_2(t) = x_3(t) - k_2[x_1(t)]^{2\theta-1}, \\ \quad \vdots \\ \dot{x}_n(t) = x_{n+1}(t) - k_n[x_1(t)]^{n\theta-(n-1)}, \\ \dot{x}_{n+1}(t) = -k_{n+1}[x_1(t)]^{(n+1)\theta-n}. \end{cases} \tag{1.3.83}$$

Proof. We can verify that system (1.3.83) is $(\theta-1)$ degree homogeneous with weights $\{(i-1)\theta-(i-2)\}_{i=1}^{n+1}$. Let $q = \Pi_{i=1}^{n}((i-1)\theta-(i-2))$. Then

$$y(x) = \begin{pmatrix} [x_1]^{\frac{1}{q}} \\ [x_2]^{\frac{1}{\theta q}} \\ \vdots \\ [x_n]^{\frac{1}{((n-1)\theta-(n-2))q}} \\ [x_{n+1}]^{\frac{1}{(n\theta-(n-1))q}} \end{pmatrix} \tag{1.3.84}$$

and

$$V(\theta, x) = y^{\top} P y, \tag{1.3.85}$$

where P is the positive definite matrix solution to the Lyapunov equation $K^{\top}P + PK = -I_{(n+1)\times(n+1)}$. Let

$$\mathcal{S} = \{x \in \mathbb{R}^{n+1} : \ V(1, x) = 1\}. \tag{1.3.86}$$

It is easy to verify that $\mathcal{S}$ is a compact set. Let $\theta = 1$. Then system (1.3.83) becomes $\dot{x}(t) = Kx(t)$, which is asymptotically stable and

$$\frac{dV(1, x(t))}{dt} = -\parallel x(t)\|^2 < -a < 0, \ \ a > 0, \ \ \forall \ x \in \mathcal{S}.$$

By the continuity of $V(\theta, x)$ on θ, there exists $\theta^* \in \left(\frac{n}{n+1}, 1\right)$ such that, for any $\theta \in (\theta^*, 1)$,

$$\frac{dV(\theta, x(t))}{dt} < -\frac{a}{2} < 0, \ \ \forall \ x \in \mathcal{S}.$$

In addition, we can verify that for any $\theta \in (\theta^*, 1)$, $V(\theta, x)$ is $1/q^2$ degree homogeneous with weights $\{(i-1)\theta-(i-2)\}_{i=1}^{n+1}$. This implies that $\frac{dV(\theta,x(t))}{dt}$ is negative definite. By Theorem 1.3.7, system (1.3.83) is finite-time stable. □

The following Theorem 1.3.9 is about finite stability for the weighted homogeneous systems.

Theorem 1.3.9 *Suppose that the vector field* $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ *is d-degree homogeneous with weights* $\{r_i\}_{i=1}^{n}$, $f(0) = 0$.

(i) If the zero equilibrium of the system

$$\dot{x}(t) = f(x(t)) \tag{1.3.87}$$

is finite-time stable on attracting basin $\Omega \subset \mathbb{R}^n$, then it is asymptotically stable on Ω.

(ii) If the zero equilibrium of system (1.3.87) is asymptotically stable on attracting basin Ω, and the degree $d < 0$, then the zero equilibrium of system (1.3.87) is finite-stable on Ω. Furthermore, let $U \subset \Omega$ be an open neighborhood of zero state. Then for any integer $k > \max\{d, r_1, r_2, \ldots, r_n\}$ there exists a positive definite function $V \in C^1(U, [0, \infty))$, which is k-degree homogeneous with weights $\{r_i > 0\}_{i=1}^n$. In addition, if $\Omega = \mathbb{R}^n$ then the Lyapunov function $V(x)$ is radially unbounded.

Proof. We only need to prove (ii). For the sake of simplicity and without loss of generality, we may assume that $\Omega = \mathbb{R}^n$. Since the zero equilibrium state of system (1.3.87) is finite-time stable, it is asymptotically stable. By Theorem 1.3.3, there exists a positive definite Lyapunov function $\tilde{V} : \mathbb{R}^n \to \mathbb{R}$ such that $L_F \tilde{V}$ is negative definite on $\mathbb{R}^n$. Let $\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that

$$\alpha(s) = \begin{cases} 0, & s \in (-\infty, 1], \\ 1, & s \in [2, +\infty), \end{cases} \quad \text{and } \forall\ s \in \mathbb{R},\ \alpha'(s) > 0, \tag{1.3.88}$$

and

$$V(x) = \begin{cases} \displaystyle\int_0^{+\infty} \frac{1}{\mu^{k+1}} (\alpha \circ \tilde{V})(\mu^{r_1} x_1, \ldots, \mu^{r_n} x_n) d\mu, & x \in \mathbb{R}^n \setminus \{0\}, \\ 0, & x = 0. \end{cases} \tag{1.3.89}$$

Apparently, $V(x)$ is positive definite. For any $\lambda > 0$, $x \neq 0$,

$$\begin{aligned} V(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n) &= \int_0^{+\infty} \frac{1}{\mu^{k+1}} (\alpha \circ \tilde{V})((\lambda\mu)^{r_1} x_1, \ldots, (\lambda\mu)^{r_n} x_n) d\mu \\ &= \lambda^k \int_0^{+\infty} \frac{1}{(\lambda\mu)^{k+1}} (\alpha \circ \tilde{V})((\lambda\mu)^{r_1} x_1, \ldots, (\lambda\mu)^{r_n} x_n) d(\lambda\mu) \\ &= \lambda^k V(x). \end{aligned} \tag{1.3.90}$$

This shows that $V(x)$ is k-degree homogenous with weights $\{r_1, \ldots, r_n\}$. Furthermore, we can find that there exist $l, L > 0$ such that

$$\begin{aligned} &\tilde{V}(\mu^{r_1} x_1, \ldots, \mu^{r_n} x_n) \leq 1 \ \forall\ x \in \mathbb{R}^n, \tfrac{1}{2} \leq \| x \| \leq 2,\ \ \mu \leq l, \\ &\tilde{V}(\mu^{r_1} x_1, \ldots, \mu^{r_n} x_n) \geq 2 \ \forall\ x \in \mathbb{R}^n, \tfrac{1}{2} \leq \| x \| \leq 2,\ \ \mu \geq L. \end{aligned} \tag{1.3.91}$$

Therefore, for any $x \in \mathbb{R}^n$, $1/2 \leq \| x \| \leq 2$,

$$V(x) = \int_l^L \frac{1}{\mu^{k+1}} (\alpha \circ \tilde{V})(\mu^{r_1} x_1, \ldots, \mu^{r_n} x_n) d\mu + \frac{1}{kL^k}. \tag{1.3.92}$$

It is easy to see that $V(x)$ is of C^∞ on $\{x \in \mathbb{R} : 1/2 <\| x \|< 2\}$ and

$$\frac{\partial V(x)}{\partial x_i} = \int_l^L \frac{\mu^{r_i}}{\mu^{k+1}} \alpha'(\tilde{V}(y_1, \ldots, y_n)) \frac{\partial \tilde{V}(y_1, \ldots, y_n)}{\partial y_i} d\mu, \;\; y_i = \mu^{r_i} x_i. \tag{1.3.93}$$

It then follows that

$$\sum_{i=1}^n f_i(x) \frac{\partial V(x)}{\partial x_i} = \int_l^L \frac{1}{\mu^{d+k+1}} \alpha'(\tilde{V}(y_1, \ldots, y_n)) \times \left(\sum_{i=1}^n \left(f_i(y_1, \ldots, y_n) \frac{\partial V(y_1, \ldots, y_n)}{\partial y_i} \right) \right) d\mu, \;\; y_i = \mu^{r_i} x_i. \tag{1.3.94}$$

Since $\alpha'(s) > 0$, $L_f \tilde{V}(x) < 0$ for $x \in \mathbb{R}^n$ and $\frac{1}{2} \leq \| x \| \leq 2$. A straightforward computation shows that $L_F V(x)$ is homogeneous of degree $k + d$ with weights $\{r_i\}_{i=1}^n$. This together with (1.3.94) yields that $L_f V(x)$ is negative definite. By Property 1.3.3,

$$L_F V(x) \leq \left(\min_{y \in V^{-1}(1)} L_F V(y) \right) (V(x))^{\frac{k+d}{k}}.$$

Since $d < 0$, this together with Theorem 1.3.7 completes the proof of the theorem. □

1.3.5 Stability of Discontinuous Systems

In this section, we investigate stability for system (1.3.35), where $f(t, x)$ is not continuous with respect to x. In this case, we consider system (1.3.35) as the following differential inclusion:

$$\dot{x}(t) \in F(t, x), \tag{1.3.95}$$

where

$$F(t, x) = \mathbf{K}_x f(t, x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \overline{\text{co}}\{f(t, B_\delta(x) \setminus N)\}, \tag{1.3.96}$$

where $\overline{\text{co}}(\cdot)$ denotes the convex closure of a set, $B_\delta(x) = \{\nu \in \mathbb{R}^n | \;\; \| \nu - x \|_\infty < r\}$, and $\mu(\cdot)$ is the Lebesgue measure of $\mathbb{R}^n$. If $f(t, x)$ is Lebesgue measurable and locally bounded, then there exist t and $f(t, x)$-dependent zero measure subset N_0^t of $\mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$ and $N \subset \mathbb{R}^n : \;\; \mu(N) = 0$,

$$\mathbf{K}_x f(t, x) = \overline{co}\{v = \lim_{n \to \infty} f(t, x_n) : x_i \notin N_0^t \cup N, \lim_{i \to \infty} x_i = x\}. \tag{1.3.97}$$

We say that $x(t)$ is a generalized solution (or a Filippov solution) of (1.3.35) if $x(t)$ is absolutely continuous on each compact subinterval $I \subset [0, \infty)$ and

$$\dot{x}(t) \in F(t, x(t)) \text{ almost everywhere on } I. \tag{1.3.98}$$

The following Definition 1.3.12 defines stability for systems with discontinuous right-hand sides.

Definition 1.3.12 *Let $f(t,\cdot)$ be Lebesgue measurable and locally bounded, let $F(t,\cdot)$ be defined in (1.3.96), and $0 \in F(t,0)$ for almost all $t \geq 0$. For any $x_0 \in \mathbb{R}^n$, the set of solution of (1.3.35) (or (1.3.95)) with initial condition $x(0) = x_0$ is denoted by $\mathcal{S}_{t;x_0}$. The zero equilibrium of system (1.3.35) (or differential inclusion (1.3.95)) is uniformly globally asymptotically stable if*

(i) For any $\delta > 0$, $x_0 \in \mathbb{R}$, and $x(t;x_0) \in \mathcal{S}_{t;x_0}$, if $\| x_0 \|_\infty < \delta$, then for any $t > 0$, $\| x(t;x_0) \|_\infty < m(\delta)$, where $m \in C((0,+\infty),(0,+\infty))$ satisfies $\lim_{\delta \to 0^+} m(\delta) = 0$;
(ii) For any $R > 0$, $\epsilon > 0$, $x_0 \in \mathbb{R}^n$, and $x(t;x_0) \in \mathcal{S}_{t;x_0}$ if $\| x_0 \|_\infty \leq R$, then $\| x(t;x_0) \|_\infty < \epsilon$ for any $t > T(R,\epsilon)$, where $T(R,\epsilon)$ is an R and ϵ-dependent constant.

The following Theorem 1.3.10 is an extension of Theorem 1.3.2.

Theorem 1.3.10 *Let $f(\cdot,x)$ be Lebesgue measurable and locally bounded, and let $F(t,x)$ be defined in (1.3.96), and $0 \in F(t,0)$ for almost all $t \geq 0$. Assume that there exists a Lyapunov function $V(t,x)$ and the class $\mathcal{K}_\infty$ functions $\kappa_1(\cdot), \kappa_2(\cdot)$, and $\kappa_3(\cdot)$ such that*

$$\kappa_1(\| \nu \|_\infty) \leq V(t,\nu) \leq \kappa_2(\| \nu \|_\infty) \quad \forall\ t \in [0,\infty), \quad \nu \in \mathbb{R}^n, \tag{1.3.99}$$

and for any $0 < t_1 \leq t_2$

$$V(t_2, x(t_2;x_0)) - V(t_1, x(t_1;x_0)) \leq \int_{t_1}^{t_2} \kappa_3(\| x(\tau;x_0) \|_\infty) d\tau, \tag{1.3.100}$$

then the zero equilibrium state of system (1.3.35) (or differential inclusion (1.3.95)) is uniformly globally asymptotically stable.

When $V(t,\cdot)$ is of C^1 class, the inequality (1.3.100) can be obtained by the following infinitesimal decreasing condition: there exists a class $\mathcal{K}_\infty$ function $\kappa(\cdot)$ such that for almost all $t \geq 0$, all $x \in \mathbb{R}^n$, and $\nu \in F(t,x)$,

$$\frac{\partial V(t,x)}{\partial t} + \langle \nabla_x V(t,x), \nu \rangle \leq -\kappa(\| x \|_\infty). \tag{1.3.101}$$

The proof of Theorem 1.3.10 is similar to Theorem 1.3.2, and the details are omitted.
The following Theorem 1.3.11 is the converse of the Lyapunov theorem.

Theorem 1.3.11 ***(Converse of second Lyapunov theorem)*** *Let $F(t,x)$ be defined in (1.3.96) and $0 \in F(t,0)$ for almost all $t \geq 0$. Assume that the zero equilibrium state of system (1.3.35) (or differential inclusion (1.3.95)) is uniformly globally asymptotically stable and there exists a zero measure set $N_0 \subset [0,\infty)$ such that*

- *$F(t,x)$ is a nonempty convex compact set for any $(t,x) \in ([0,\infty) \setminus N_0) \times \mathbb{R}^n$.*
- *For any $R > 0$, if $\| x \|_\infty \leq R$ and $t \in [0,R] \setminus N_0$ then $F(t,x) \subset \overline{B}_M(0)$ for some $M > 0$.*
- *For any $(t_0,x_0) \in ([0,\infty) \setminus N_0) \times \mathbb{R}^n$ and $\epsilon > 0$, there exists $\delta > 0$ such that, for any $(t,x) \in ([0,\infty) \setminus N_0)$, if $\| (t-t_0, x-x_0) \|_\infty < \delta$ then $F(t,x) \subset F(t_0,x_0) + B_\epsilon(0)$.*

Then for any $\lambda > 0$, there exist $V \in C^\infty([0,\infty)\times\mathbb{R}^n,[0,\infty))$ and the class $\mathcal{K}_\infty$ functions $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ such that

$$\kappa_1(\| \nu \|_\infty) \le V(t,\nu) \le \kappa_2(\| \nu \|_\infty) \quad \forall\, t \ge 0,\ \nu \in \mathbb{R}^n \tag{1.3.102}$$

and

$$\frac{\partial V(t,x)}{\partial t} + \langle \nabla_x V(t,x), \nu\rangle \le -\lambda V(t,x),\ \ \forall\ t\in[0,\infty)\setminus N_0, x\in\mathbb{R}^n, \nu \in F(t,x). \tag{1.3.103}$$

The proof of Theorem 1.3.11 is presented in the next subsection.

1.3.6 Proof of Theorem 1.3.11

The proof of Theorem 1.3.11 is lengthy and we split the proof into three steps. In the first step, we show that the uniform global asymptotical stability also holds true for some perturbed system

$$\dot{x}(t) \in F_2(t,x(t)), \tag{1.3.104}$$

where $F(t,x)\subset F_2(t,x)$ for every x and almost all t, $F_2(t,x)$ is locally Lipschitz continuous in $[0,\infty)\times(\mathbb{R}^n\setminus\{0\})$, that is, for each $(t_0,x_0)\in[0,\infty)\times(\mathbb{R}^n\setminus\{0\})$, there exist $L>0$ and $\delta>0$ such that for any $(t_1,x_1),(t_2,x_2)\in B_\delta((t_0,x_0))$

$$\hbar(F_2(t_1,x_1),F_2(t_2,x_2)) \le \|(t_1,x_1)-(t_2,x_2)\|_\infty, \tag{1.3.105}$$

where $\hbar(\cdot,\cdot)$ is the Hausdorff distance between nonempty compact subsets of $\mathbb{R}^n$:

$$\hbar(A,B) = \max\left\{\sup_{a\in A}\mathrm{dis}(a,B),\ \ \sup_{b\in A}\mathrm{dis}(b,A)\right\} \tag{1.3.106}$$

with $\mathrm{dis}(a,B)=\inf_{b\in B}\|a-b\|_\infty$. In the second step, we construct a Lipschitz continuous function $V_L(t,x)$. In the final step, we smooth $V_L(t,x)$ to be a C^∞ function. All these steps are accomplished by a series of lemmas.

The following Lemma 1.3.5 gives an initial value continuous dependence on differential inclusion (1.3.95) for which the proof is omitted.

Lemma 1.3.5 *Suppose that $F(t,x)$ satisfies three conditions in Theorem 1.3.11. Let $y:[T_1,T_2]\to\mathbb{R}^n$ be the solution of (1.3.95) and $b>0$. Assume that $F(t,x)$ is Lipschitz continuous in $[T_1,T_2]$, $\|x-y(t)\|_\infty<b$, that is, there exists some constant $K>0$ such that, for any $t,\bar{t}\in[T_1,T_2]$ and any $x,\bar{x}\in\mathbb{R}^n$ with $\|x-y(t)\|_\infty\le b$, $\|\bar{x}-y(\bar{t})\|_\infty\le b$,*

$$\hbar(F(t,x),F(\bar{t},\bar{x})) \le K\|(t-\bar{t},x-\bar{x})\|_\infty. \tag{1.3.107}$$

Let $(t_0,x_0)\in[T_1,T_2]\times\mathbb{R}^n$ satisfy $\|x_0-y(t_0)\|_\infty\le b$. Then there exists a solution $x(t)$ of (1.3.95) with $x(t_0)=x_0$ satisfying

$$\|x(t)-y(t)|_\infty \le \|x_0-y(t_0)\|e^{K|t-t_0|} \tag{1.3.108}$$

as long as $\|x_0-y(t_0)\|e^{K|t-t|}\le b$.

To prove Theorem 1.3.11, we need firstly to regularize $F(t,x)$. To facilitate the construction of a Lyapunov function that is smooth up to $t=0$, we extend $F(t,x)$ on $[-1,0)\times\mathbb{R}^n$ by setting $F(t,x)=\{-x\}$. It is easy to verify that if $F(t,x)$ is a nonempty convex compact set for any $(t,x)\in([0,\infty)\setminus N_0)\times\mathbb{R}^n$, then, after the extension, it is also a nonempty convex compact set for any $(t,x)\in([-1,\infty)\setminus N_0)\times\mathbb{R}^n$. We need the following lemma to smooth $F(t,x)$.

Lemma 1.3.6 *Let $F(t,x)$ be a nonempty, compact, and convex subset on $[-1,\infty)\times\mathbb{R}^n$, R be a positive constant, and let $\{t_j^1\}_{j=1}^\infty$, $\{t_j^2\}_{j=1}^\infty$, and $\{\delta_j\}_{j=1}^\infty$ be sequences of numbers satisfying*

$$-1\le t_j^1\le t_j^2\le R,\ \ \forall\ j\in\mathbb{N}^+=\mathbb{N}\setminus\{0\},\quad \lim_{j\to\infty}\delta_j=0. \tag{1.3.109}$$

Let $\{x_j(t)\}_{j=1}^\infty$ be a sequence of absolutely continuous functions $x_j:[t_j^1,t_j^2]\to\overline{B}_R(0)$ such that for almost all $t\in[t_j^1,t_j^2]$

$$\dot{x}_j(t)\in\overline{\text{co}}\{F(\overline{B_{\delta^j}(t,x_j(t))}\cap(((-1,\infty)\setminus N_0)\times\mathbb{R}^n))\}. \tag{1.3.110}$$

Then there exist numbers $t_1,t_2\in[-1,R]$, function $x:[t_1,t_2]\to\overline{B_R(0)}$, and sequence $j_k\to\infty$ such that $x(t)$ is a solution of (1.3.95), and $t_{j_k}^1\to t_1$, $t_{j_k}^2\to t_2$ as $k\to\infty$ such that

$$\lim_{k\to\infty}x_{j_k}(t_{j_k}^1)=x(t_1),\ \ \lim_{k\to\infty}x_{j_k}(t_{j_k}^2)=x(t_2). \tag{1.3.111}$$

Proof. In what follows, we need to take frequently a convergent subsequence from a sequence of numbers or functions. For the sake of simplicity, we avoid using multiple indices and just assume that the given sequence itself is convergent. According to the second item of the conditions on $F(t,x)$, there exists $M>0$ such that for any $t\in[-1,R+1]\setminus N_0$ and $\|x\|_\infty\le R+1$, $F(t,x)\subset\overline{B_M(0)}$.

For any $j\ge 0$ and $t\in[-1,R]$, set

$$\tilde{x}_j(t)=\begin{cases}x_j(t_j^1), & \text{if}\quad -1\le t\le t_j^1,\\ x_j(t), & \text{if}\quad t_j^1\le t\le t_j^2,\\ x_j(t_j^2), & \text{if}\quad t_j^2\le t\le R.\end{cases} \tag{1.3.112}$$

Since for any $j\ge 0$ and $t\in[-1,R]$, $\|\tilde{x}_j(t)\|_\infty\le R$, and for almost all $t\in[-1,R]$ $\|\dot{\tilde{x}}_i(t)\|_\infty\le M$, we can obtain a sequence $\{\tilde{x}_j(t)\}_{j=1}^\infty$ that is bounded in the Sobolev space $H^1((-1,R),\mathbb{R}^n)$ and hence a subsequence (still denoted by itself) that is weakly convergent to a function $x(t)$. It follows that $x_j\to x$ in $C^0([-1,R],\mathbb{R}^n)$ and $\tilde{x}_j\rightharpoonup\dot{x}$ in $L^2((-1,R),\mathbb{R}^n)$. As a consequence, $\tilde{x}_j(t_j^1)\to x(t_1)$ and $\tilde{x}_j(t_j^2)\to x(t_2)$. To prove that $x(t)$ is a solution to (1.3.95), that is, $x(t)\in F(t,x(t))$ for almost all $t\in[t_1,t_2]$, we consider the functional $J(w)$ defined on $L^2((-1,\mathbb{R}),\mathbb{R}^n)$ by

$$J(w)=\int_{t_1}^{t_2}\text{dis}(w(t),F(t,x(t)))dt. \tag{1.3.113}$$

Since the non-negative map from t to $\text{dis}(w(t), F(t, x(t))$ is measurable, it is easy to verify that the functional $J(w)$ is well defined, convex, and continuous in the strong topology of $L^2((-1, \mathbb{R}), \mathbb{R}^n)$. Since $\dot{\tilde{x}}_j \rightharpoonup \dot{x}$ in $L^2((-1, \mathbb{R}), \mathbb{R}^n)$, we can obtain

$$0 \le J(\dot{x}(t)) \le \liminf_{j\to\infty} J(\dot{\tilde{x}}_j(t)). \tag{1.3.114}$$

To prove $J(\dot{x}) = 0$, we only need to show that $\lim_{j\to\infty} J(\dot{\tilde{x}}_j) = 0$. Noting $\text{dis}(\dot{\tilde{x}}_j(t), F(t, x(t))) \le M$ for every j and almost all $t \in (t_1, t_2)$, by the Lebesgue theorem, we only need to prove that

$$\lim_{j\to\infty} \text{dis}(\dot{\tilde{x}}_j(t), F(t, x(t))) = 0. \tag{1.3.115}$$

For almost all $t_0 \in (t_1, t_2) \setminus N_0$, there exists an integer $j_0 \ge 0$ such that, for any $j \ge j_0$ and $t_0 \in (t_1^j, t_2^j)$, $\dot{x}_j(t_0)$ exists and belongs to $\overline{co}\{F(\overline{B_{\delta_j}(t_0, x_j(0))} \cap (([-1, +\infty) \setminus N_0) \times \mathbb{R}^n))\}$. Let $\epsilon > 0$. By the third condition in Theorem 1.3.11, there exists $\delta > 0$ such that for any $(t, x) \in ([-1, +\infty) \setminus N_0) \times \mathbb{R}^n$, if $\|(t - t_0, x - x(t_0))\|_\infty \le \delta$, then $F(t, x) \subset F(t_0, x(t_0)) + \overline{B_\epsilon(0)}$. We may assume without loss of generality that for all $j \ge j_0$,

$$\delta_j < \frac{\delta}{2}, \ \ \|x_j(t_0) - x(t_0)\| < \frac{\delta}{2}. \tag{1.3.116}$$

This implies that for all $j > j_0$, $\overline{B_{\delta_j}(t_0, x_j(t_0))} \subset \overline{B_\delta(t_0, x(t_0))}$ and

$$\overline{co}\{F(\overline{B_{\delta_j}(t_0, x_j(t_0))} \cap (([-1, \infty) \setminus N_0) \times \mathbb{R}^n))\} \subset F(t_0, x(t_0)) + \overline{B_\epsilon(0)}. \tag{1.3.117}$$

It follows that for any $j \ge j_0$, $\text{dis}(\dot{\tilde{x}}_j(t_0), F(t_0, x(t_0))) \le \epsilon$. This completes the proof of the lemma. □

Now we use $\delta(t, x)$ to denote any continuous function defined on $[-1, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ such that, for any $(t, x) \in [-1, \infty) \times \mathbb{R}^n$, $\delta(t, x) \ge 0$ and $\delta(t, x) = 0$ if and only if $x = 0$. For such a given function $\delta(t, x)$, set

$$F_1(t, x) = \overline{co}\{F(\overline{B_{\delta(t,x)}} \cap E); \quad \forall \ (t, x) \in [-1, \infty) \times \mathbb{R}^n, \tag{1.3.118}$$

where $E = ([-1, \infty) \setminus N_0) \times \mathbb{R}^n$. We can verify that $F_1(t, x)$ also satisfies the condition of Theorem 1.3.11. Now we show that the following differential inclusion is globally asymptotically stable:

$$\dot{x}(t) \in F_1(t, x(t)), \quad t \ge -1. \tag{1.3.119}$$

From the uniform global asymptotical stability of (1.3.95), for any solution $x(t_0, x_0)$ of (1.3.95), there exists a class $\mathcal{KL}$ function $\beta : \ [0, \infty) \times [0, \infty) \to [0, \infty)$ such that $\|x(t_0 + h)\| \le \beta(h, \|x_0\|)$. We say that $\beta(t, s)$ is the class $\mathcal{KL}$ function if, for any given t, $\beta(t, s)$ is the class $\mathcal{K}_\infty$ function with respect to s and, for any given s, $\beta(t, s)$ is decreasing with respect to t and $\lim_{t\to\infty} \beta(t, s) = 0$. Let $\varphi_i(h) = \beta(h, 2^i)$. We can prove that the sequence $\{\varphi_i(h)\}_{i=-\infty}^{\infty}$ of positive continuous decreasing functions on $[0, \infty)$ satisfies:

(i) For any $(t^0, x_0) \in [-1, +\infty) \times \mathbb{R}^n$ and any solution $x(t)$ of (1.3.95) with $x(t_0) = x_0$, if $\|x^0\|_\infty \le 2^i$ then $\|x(t^0 + h)\|_\infty < \varphi_i(h)$for any $h \ge 0$.

(ii) $\lim_{h\to\infty} \varphi_i(h) = 0$ for any i.
(iii) $\{\varphi_i(0)\}_{i=-\infty}^{\infty}$ is a nondecreasing sequence such that $\lim_{i\to-\infty} \varphi_i(0) = 0$ and $\lim_{i\to\infty} \varphi_i(0) = \infty$.

For integer $i \in \mathbb{Z}$, let $p_i \in \mathbb{Z}$ be the greatest natural number such that $\varphi_{p_i}(0) \leq 2^{i-1}$. Choose $T_i \geq 1$ so that $\varphi_i(T_i) \leq 2^{i-1}$ and set $\hat{T}_i = \max\{T_i, \max\{T_j : \ p_j = i\}\}$.

Lemma 1.3.7 *Let $i \in \mathbb{Z}$ and $k \in \mathbb{N}^+ \cup \{-1\}$. Then there exists constant $\delta > 0$ such that for any solution $x(t)$ of the following differential inclusion:*

$$\dot{x}(t) \in \overline{\mathrm{co}}\{F(\overline{B_\delta(t,x)} \cap E)\} \tag{1.3.120}$$

with $\|x(t_0)\| \leq 2^i$, $t_0 \in [k, k+1]$, it has

$$\|x(t_0+h)\|_\infty \leq \varphi_i(h), \ \ \forall \ h \in [0, \hat{T}_i]. \tag{1.3.121}$$

Proof. Suppose that the conclusion is false. Then there exists a decreasing sequence of positive numbers $\{\delta_j\}_{j=1}^{\infty} : \lim_{j\to\infty} \delta_j = 0$ and a sequence of absolutely continuous functions $\{x_j(t)\}_{j=1}^{\infty}$ with $x_j : [t_j^0, t_j^1] \to \mathbb{R}^n$, $t_j^0 \leq t_j^1 \leq t_j^0 + \hat{T}_i$ such that

$$\begin{cases} \dot{x}_j(t) \in \overline{co}\{F(\overline{B_{\delta_j}(t, x_j(t))} \cap E)\} \text{ for almost all } \ t \in [t_j^0, t_j^1], \\ \|x_j(t_j^0)\|_\infty \leq 2^i, \ \ \|x_j(t)\|_\infty \leq \varphi_i(t - t_j^0) \text{ for all } \ t \in [t_j^0, t_j^1), \|x_j(t_j^1)\|_\infty = \varphi_i(t_j^1 - t_j^0). \end{cases} \tag{1.3.122}$$

By Lemma 1.3.6 and extracting a subsequence if necessary, we may also assume that for some $t^0, t^1 \in [k, k+1+\hat{T}_i]$ and some solution x: $[t^0, t^1] \to \mathbb{R}^n$ of (4.1):

$$\lim_{j\to\infty} (t_j^0, x_j(t_j^0)) = (t^0, x(t^0)), \ \ \lim_{j\to\infty} (t_j^1, x_j(t_j^1)) = (t^1, x(t^1)). \tag{1.3.123}$$

This yields $\|x(t^0)\|_\infty \leq 2^i$ and $\|x(t^1)\|_\infty = \varphi_i(t^1 - t^0)$, which contradicts the definition of $\varphi_i(t)$. This completes the proof of the lemma. □

For any $(i,k) \in \mathbb{Z} \times (-1 \cup \mathbb{N}^+)$, the number $\delta > 0$ in Lemma 1.3.7 related to i and k is denoted by δ_i^k. Let $\delta : [-1, \infty) \times \mathbb{R}^n \to [0, \infty)$ be a Lipschitz continuous function with a Lipschitz constant one, and satisfy

$$\delta(t,x) = 0 \ \text{ if and only if } \ x = 0 \tag{1.3.124}$$

and

$$\delta(t,x) < \min(\delta_i^k, \delta_{p_i}^k), \ \ \forall \ k \leq t, \ \ 2^{p_i} \leq \|x\| \leq \varphi_i(0). \tag{1.3.125}$$

Lemma 1.3.8 *Let $\delta(t)$ satisfy (1.3.124) and (1.3.125), and let $x(t)$ be any solution of (1.3.119). Then for $t_0 \geq -1$, $i \in \mathbb{Z}$ and $\|x(t_0)\| \leq 2^i$,*

(a) $\|x(t_0+h)\|_\infty < \varphi_i(0)$ *for any* $h \in [0, T_i]$.
(b) $\|x(t_0+T_i)\|_\infty \leq 2^{i-1}$.

Proof. If (a) is false, then there exist t_1, t_2: $t_0 < t_1 < t_2 \leq t_0 + T_i$ such that

$$2^i = \|x(t_1)\|_\infty < \|x(t)\|_\infty < \|x(t_2)\|_\infty = \varphi_i(0), \quad t \in (t_1, t_2). \tag{1.3.126}$$

Set $k = [t_1]$. Since $2^{p_i} \leq \|x(t)\| \leq \varphi_i(0)$ for $t \in [t_1, t_2]$, $x(t)$ is also a solution of the following differential inclusion:

$$\dot{x}(t) \in \overline{\text{co}}\{F(\overline{B_{\delta_i^k}(t,x)} \cap E)\}. \tag{1.3.127}$$

By Lemma 1.3.7, $\|x(t_2)\|_\infty \leq \varphi_i(t_2 - t_1) < \varphi_i(0)$, which contradicts $\|x(t_2)\|_\infty = \varphi_i(0)$. Therefore, (a) is valid.

Now we prove (b). Assume that $\|x(t_0 + T_i)\|_\infty > 2^{i-1}$. If $2^{p_i} \leq \|x(t_0 + h)\|_\infty \leq \varphi_i(0)$ for every $h \in [0, h_i]$, then $x(t)$ also satisfies (1.3.127) on $[t_0, t_0 + T_i]$ with $k = [t_0]$. By Lemma 1.3.7, $\|x(t_0 + T_i)\|_\infty < \varphi_i(T_i) \leq 2^{i-1}$, which is a contradiction. Therefore, there exist t_1 and t_2 such that $t_0 \leq t_1 < t_2 \leq t_0 + T_i$ and $2^{p_i} = \|x(t_1)\|_\infty < \|x(t)\|_\infty < \|x(t_2)\|_\infty = 2^{i-1}$ for $t \in (t_1, t_2)$. For $k = [t_1]$, consider $\delta(t, x(t)) < \delta_{p_i}^k$ for $t_1 < t < t_2$ and $t_2 - t_1 \leq T_i \leq \hat{T}_{p_i}$. Once again, by Lemma 1.3.7, $\|x(t_2)\|_\infty < \varphi_{p_i}(0) \leq 2^{i-1}$. The conclusion is obtained by the contradiction of the property of T_2. □

By Lemma 1.3.8, we can obtain that $\|x(t_0 + h)\|_\infty < \varphi_{i-l}(0)$ for any $l \in \mathbb{N}^+$ and $h \geq \sum_{j=i-l+1}^{i} T_j$. This means that (1.3.119) is uniformly globally asymptotically stable.

We are now in a position to enlarge and regularize the differential inclusion $\dot{x}(t) \in F_2(t, x(t))$. To this purpose, we need some suitable partition of unity. Set

$$U = (-1, +\infty) \times (\mathbb{R}^n \setminus \{0\}), \tag{1.3.128}$$

and for any $(t, x) \in U$,

$$W(t,x) = \left\{(s,y) \in U | \ \ \|(s-t, y-x)\| < \frac{1}{3}\delta(t,x)\right\}. \tag{1.3.129}$$

It is easy to see that the family $\{W(t,x)\}_{(t,x)\in U\cap E}$ is an open covering of U.

Let $\{\psi_i(t,x)\}_{i\in\mathbb{N}^+}$ be a C^∞-partition of unity on U subordinate to the open covering $\{W(t,x)\}_{(t,x)\in U\cap E}$ of U. It means that, firstly, each $\psi_i(t,x)$ is a nonnegative function of class C^∞ on $\mathbb{R}^{n+1}$, with support contained in $W(t_i, x_i)$ for $(t_i, x_i) \in U \cap E$; secondly, for any $(t,x) \in U$, $\sum_{i=1}^\infty \psi_i(t,x) = 1$; and lastly, for any $(t,x) \in U$, there exists a number $\rho > 0$ such that $\psi_i(t,x) \equiv 0$ on $\overline{B_\rho(t,x)}$ for all $i \in \mathbb{N}^+$ except finitely many $i's$.

For any $(t,x) \in (-1, +\infty) \times \mathbb{R}^n$, set

$$F_2(t,x) = \begin{cases} \displaystyle\sum_{i=1}^\infty \psi_i(t,x)\overline{co}\{F(\overline{B_{\frac{1}{3}\delta(t_i,x_i)}(t_i,x_i)} \cap E)\}, & x \neq 0, \\ F(t,0), & x = 0. \end{cases} \tag{1.3.130}$$

Since the summation in (1.3.130) is finite on the compact subset of U, we see that $F_2(t,x)$ is locally Lipschitz continuous in the Hausdorff distance on U. It is clear that for $x = 0$, $F(t,x) \subset F_2(t,x)$. Let $x \neq 0$ and $t \in (-1, +\infty) \setminus N_0$, $i \in \mathbb{N}^+$ such that $\psi_i(t,x) > 0$. By the definition of $\psi(t,x)$, $(t,x) \in W(t_i, x_i)$. This together with (1.3.129) yields $\|(t - t_i, x - x_i)\|_\infty < \frac{1}{3}\delta(t_i, x_i)$. Hence

$$F(t,x) \subset F(\overline{B_{\frac{1}{3}\delta(t_i,x_i)}(t_i,x_i)} \cap E), \tag{1.3.131}$$

which implies that $F(t,x) \subset F_2(t,x)$. Therefore, for every $(t,x) \in ((-1,+\infty) \setminus N_0) \times \mathbb{R}^n$, $F(t,x) \subset F_2(t,x)$.

Furthermore, for any $(t,x) \in U$ and $i \in \mathbb{N}^+$ satisfying $\psi_i(t,x) > 0$, since $\delta(t,x)$ is Lipschitz continuous with the Lipschitz constant one, we can obtain

$$\delta(t_i,x_i) - \delta(t,x) \le \|(t-t_i, x-x_i)\|_\infty \le \frac{1}{3}\delta(t_i,x_i). \tag{1.3.132}$$

This yields

$$\overline{B_{\frac{1}{3}\delta(t_i,x_i)}(t_i,x_i)} \subset \overline{B_{\frac{2}{3}\delta(t_i,x_i)}(t,x)} \subset \overline{B_{\delta(t,x)}(t,x)}, \tag{1.3.133}$$

and hence $F_2(t,x) \subset \overline{co}\{F(\overline{B_{\delta(t,x)}(t,x)} \cap E)\} \subset F_1(t,x)$. This together with the uniform global asymptotical stability of $\dot{x}(t) \in F_1(t,x(t))$ deduces that $\dot{x}(t) \in F_2(t,x(t))$ is also uniformly globally asymptotically stable.

Secondly, we construct a local Lipschitz continuous Lyapunov function.

For any $(t_0,x_0) \in (-1,+\infty) \times \mathbb{R}^n$, let $\mathcal{S}_{t_0;x_0}$ be the set of solutions $x(t)$ of the differential inclusion $\dot{x}(t) \in F_2(t,x(t))$ with initial condition $x(t_0) = x_0$. For any $q \in \mathbb{N}^+$, $r \in [0,\infty)$, and $(t,x) \in (-1,+\infty) \times \mathbb{R}^n$, set

$$G_q(r) = \max\left\{0,\ \ r - \frac{1}{q}\right\} \tag{1.3.134}$$

and

$$V_q(t,x) = \sup_{\varphi \in \mathcal{S}_{t;x}} \sup_{\tau \ge 0} e^{2\lambda\tau} G_q(\|\varphi(t+\tau)\|_\infty), \tag{1.3.135}$$

where λ is the positive number appearing in Theorem 1.3.11.

From the uniform global asymptotical stability of $\dot{x}(t) \in F_2(t,x(t))$, we can infer that, for any $R > 0$ and $q \in \mathbb{N}^+$, there exist the class $\mathcal{K}_\infty$ function $m(R)$ and nondecreasing function $T(R,q)$ such that, as long as $\|x_0\|_\infty \le R$, for each $(t_0,x_0) \in (-1,+\infty) \times \mathbb{R}^n$, $\varphi \in \mathcal{S}_{t_0;x_0}$, $\|\varphi(t_0+\tau)\|_\infty < m(R)$ for all $\tau \ge 0$, and $\|\varphi(t_0+\tau)\|_\infty < \frac{1}{q}$ for any $\tau > T(R,q)$.

The following Lemma 1.3.9 is a direct consequence of (1.3.134) and (1.3.135).

Lemma 1.3.9 *Let $R > 0$ and $(t,x) \in (-1,+\infty) \times \overline{B_R(0)}$. Then for any $q \in \mathbb{N}^+$,*

$$G_q(\|x\|_\infty) \le V_q(t,x) \le e^{2\lambda T(R,q)} m(R) < \infty. \tag{1.3.136}$$

Another important property of $V_q(t,x)$ is the local Lipschitz continuity.

Proposition 1.3.1 *Let $q \in \mathbb{N}^+$ and $R > 0$. Then there exists a positive constant $C_q(R)$ such that for any $t_1,t_2 \in [-R/(R+1), R]$ and $x_1,x_2 \in \overline{B_R(0)}$,*

$$|V_q(t_1,x_1) - V_q(t_2,x_2)| \le C_q(R)\|(t_1-t_2, x_1-x_2)\|_\infty. \tag{1.3.137}$$

We assume without loss of generality that, for every $q \in \mathbb{N}^+$, the function $C_q(R)$ is nondecreasing.

To prove Proposition 1.3.1, we need the following elementary lemma.

Lemma 1.3.10 *Let $V(x)$ be a function defined on a set $\mathbb{K} = \prod_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$. Assume that there exists a constant $L > 0$ such that for any $x_0 \in \mathbb{K}$ there exists $\eta_0 > 0$ satisfying*

$$|V(x) - V(x_0)| \le \|x - x_0\|_\infty, \quad \forall\ x \in \overline{B_{\eta_0}(x_0)} \cap \mathbb{K}. \tag{1.3.138}$$

Then $V(x)$ is Lipschitz continuous on $\mathbb{K}$ with the Lipschitz constant nL.

Proof. Let

$$x^1 = (x_1^1, x_2^1, \ldots, x_n^1) \in \mathbb{K},\ \ x^2 = (x_1^2, x_2^2, \ldots, x_n^2) \in \mathbb{K}. \tag{1.3.139}$$

Then

$$|V(x^1) - V(x^2)| \le \sum_{j=1}^{n} |V(x_1^1, \ldots, x_j^1, x_{j+1}^2, \ldots, x_n^2) - V(x_1^1, \ldots, x_{j-1}^1, x_j^2, \ldots, x_n^2)|. \tag{1.3.140}$$

By (1.3.138), it follows that, for any $j \in [1, n]$,

$$|V(x_1^1, \ldots, x_j^1, x_{j+1}^2, \ldots, x_n^2) - V(x_1^1, \ldots, x_{j-1}^1, x_j^2, \ldots, x_n^2)| \le L|x_j^1 - x_j^2|. \tag{1.3.141}$$

This completes the proof of the lemma. □

Proposition 1.3.1 can be obtained directly by Lemma 1.3.10 and the following Proposition 1.3.2.

Proposition 1.3.2 *Let $q \in \mathbb{N}^+$ and $R > 0$. Then there exists a positive constant $L > 0$ such that, for any $t_0 \in [-R/(R+1), R]$ and any $x_0 \in \overline{B_R(0)}$, there exists $\eta_0 > 0$ satisfying*

$$|V_q(t, x) - V_q(t_0, x_0)| \le L\|(t - t_0, x - x_0)\|_\infty,\ \ \forall\ (t, x) \in \overline{B_{\eta_0}(t_0, x_0)}. \tag{1.3.142}$$

Let $T = T(R + 1, q)$. By local Lipschitz continuity of $F_2(t, x)$ on U, there exists $K > 0$ such that for any (t_1, x_1) and (t_2, x_2) satisfying

$$-\frac{R+1}{R+2} \le t_i \le R + T + 1,\ \ \frac{1}{2}m^{-1}\left(\frac{1}{q}\right) \le \|x_1\|_\infty \le m(R + 2),\ \ i = 1, 2, \tag{1.3.143}$$

we have

$$\hbar(F_2(t_1, x_1), F_2(t_2, x_2)) \le \|(t_1 - t_2, x_1 - x_2)\|_\infty. \tag{1.3.144}$$

Let $M \ge 1$ be a constant such that for every $(t, x) \in (-(R+1)/(R+2), R+T+1) \setminus N_0 \times \overline{B_{m(R+2)}(0)}$,

$$F_2(t, x) \subset \overline{B_M(0)}. \tag{1.3.145}$$

Let

$$L = e^{2\lambda T}((M + 1)e^{K(T+1)} + 2\lambda m(R + 1)) \tag{1.3.146}$$

and let $\bar{\eta}_0$ be a constant satisfying

$$\bar{\eta}_0 \in \left(0,\ \min\left\{\frac{R+1}{R+2} - \frac{R}{R+1}, e^{-K(T+1)}\min\left\{m(R+2) - m(R+1), \frac{1}{2}m^{-1}\left(\frac{1}{q}\right)\right\}\right\}\right) \tag{1.3.147}$$

and

$$b = \bar{\eta}_0 e^{K(T+1)}. \tag{1.3.148}$$

The proof of Proposition 1.3.2 is lengthy. Before giving the proof, we present the following Lemma 1.3.11, which is useful in the proof of Proposition 1.3.2.

Lemma 1.3.11 *Let* $(t_0, x_0) \in [-R/(R+1), R] \times \overline{B_R(0)}$ *and* $(t_1, x_1) \times \overline{B_{\bar{\eta}_0}(t_0, x_0)}$. *If* $V_q(t_1, x_1) > 0$, *then for any* $\varphi_1 \in \mathcal{S}_{t_1;x_1}$ *satisfying* $\|\varphi_1(t_1+\tau)\|_\infty > 1/q$ *for some* $\tau \in [0, T]$,

$$m^{-1}\left(\frac{1}{q}\right) < \|\varphi_1(t_1+h)\|_\infty < m(R+1) \quad \forall\ h \in [0, \tau]. \tag{1.3.149}$$

Proof. By

$$\bar{\eta}_0 < \frac{R+1}{R+2} - \frac{R}{R+1} < 1, \tag{1.3.150}$$

it has

$$-\frac{R+1}{R+2} < t_1 \le t_1 + \tau < R+T+1. \tag{1.3.151}$$

Let $\varphi_1(t)$ satisfy the conditions of the lemma. Since $\|x_1\|_\infty \le \|x_0\|_\infty + \eta_0 < R+1$,

$$\|\varphi_1(t_1+h)\| < m(R+1), \ \ \forall\ h \ge 0. \tag{1.3.152}$$

Since $\|\varphi_1(t_1+\tau)\|_\infty > 1/q$, we have

$$\|\varphi_1(t_1+h)\|_\infty > m^{-1}\left(\frac{1}{q}\right), \ \ \forall\ h \in [0, \tau]. \tag{1.3.153}$$

The remaining proof of the lemma can be obtained from (1.3.147) to (1.3.149). □

Proof of Proposition 1.3.2. Let $(t_0, x_0) \in [-R/(R+1), R] \times \overline{B_R(0)}$ be fixed. In what follows, we always assume that

$$\eta_0 \in \left(0, \frac{\bar{\eta}_0}{2M+1}\right). \tag{1.3.154}$$

The proof is divided into two cases: $V_q(t_0, x_0) \ne 0$ and $V_q(t_0, x_0) = 0$.

Case 1: $V_q(t_0, x_0) \ne 0$. In this case, the proof is accomplished by the following two claims.

Claim 1: If η_0 *is small enough, then* $V_q(t, x) \ne 0$ *for any* $(t, x) \in \overline{B_{\eta_0}(t_0, x_0)}$.

Let $\varphi_0 \in \mathcal{S}_{t_0,x_0}$ satisfy

$$V_q(t_0, x_0) - e^{2\lambda\tau} G_q(\|\varphi_0(t_0+\tau)\|_\infty) < \frac{V_q(t_0, x_0)}{2}, \ \ \tau \in (0, T]. \tag{1.3.155}$$

Then $\|\varphi_0(t_0+\tau)\| > 1/q$. This together with Lemma 1.3.11 shows that

$$m^{-1}\left(\frac{1}{q}\right) < \|\varphi_0(t_0+h)\|_\infty < m(R+1), \ \ \forall\ h \in [0, \tau]. \tag{1.3.156}$$

We assume without loss of generality that $\eta_0 < \tau$. Then $\varphi_0(t)$ is defined on $[t_0 - \eta_0, t_0]$ and (1.3.156) holds true for $h \in [-\eta_0, \tau$ and $[t_0 - \eta_0, t_0 + \tau] \subset$

$[-(R+1)/(R+2), R+T+1]$. Let $(t,x) \in \overline{B_{\eta_0}}(t_0,x_0)$. Then $|t-t_0| \le \eta_0$. By (1.3.145),

$$\begin{aligned}\|\varphi_0(t)-x\|_\infty &\le \|\varphi_0(t)-\varphi_0(t_0)\|_\infty \le M|t-t_0|+\eta_0\\ &\le (M+1)\eta_0 < \overline{\eta}_0 < b.\end{aligned} \tag{1.3.157}$$

By Lemmas 1.3.5 and 1.3.12, there exists $\psi \in \mathcal{S}_{t;x}$ such that

$$|\varphi_0(t+s)-\psi(t+s)\|_\infty \le \|\varphi_0(t)-x\|_\infty e^{Ks} \tag{1.3.158}$$

as long as $m^{-1}(1/q) \le \|\varphi_0(t+s)\|_\infty \le m(R+1)$ and s is small enough so that $\|\varphi_0(t)-x\|_\infty e^{Ks} < b$.

It follows from (1.3.158) that if η_0 is sufficiently small, then

$$\begin{aligned}\|\psi(t_0+\tau)\|_\infty &\ge \|\varphi_0(t_0+\tau)\|_\infty-(M+1)e^{K(\eta_0+\tau)}\\ &\ge \|\varphi_0(t_0+\tau)\|_\infty-(M+1)e^{K(T+1)}\eta_0 > \frac{1}{q}.\end{aligned}$$

This yields $V_q(t,x) > 0$.

Let η_0 be the same as in Claim 1 and let $(t_1,x_1),(t_2,x_2) \in \overline{B_{\eta_0}(t_0,x_0)}$. The inequality (1.3.142) is a consequence of the following Claim 2.

Claim 2: $|V_q(t_1,x_1)-V_q(t_2,x_2)| \le L\|(t_1-t_2,x_1-x_2)\|_\infty$.

We may assume without loss of the generality that $t_1 \le t_2$. Firstly, we prove that

$$|V_q(t_1,x_1)-V_q(t_1,x_2)| \le e^{(2\lambda+K)T}\|x_1-x_2\|_\infty. \tag{1.3.159}$$

By the definition of $V_q(t,x)$, for every $\sigma \in (0, V_q(t_1,x_1))$, there exist $\varphi_1 \in \mathcal{S}_{t_1;x_1}$ and $\tau \in [0,T]$ such that

$$V_q(t_1,x_1)-\sigma < e^{2\lambda\tau}G_q(\|\varphi_1(t_1+\tau)\|_\infty) \le V_q(t_1,x_1). \tag{1.3.160}$$

Hence

$$V_q(t_1,x_1)-V_q(t_1,x_2) < e^{2\lambda\tau}G_q(\|\varphi_1(t_1+\tau)\|_\infty)-V_q(t_1,x_2)+\sigma. \tag{1.3.161}$$

Since $\|x_1-x_2\| \le 2\eta_0 < \bar{\eta}_0 < b$, we infer from Lemmas 1.3.5, 1.3.11, and Claim 1 that there exists a solution $\psi_2 \in \mathcal{S}_{t_1,x_2}$ such that $\|\varphi_1(t)-\varphi_2(t)\|_\infty \le \|x_1-x_2\|_\infty e^{K|t-t_1|}$ as long as $m^{-1}(1/q) \le \|\varphi_1(t)\|_\infty \le m(R+1)$ and $\|x_1-x_2\|_\infty e^{K|t_1-t|} \le b$. Since $V_q(t_1,x_2) \ge e^{2\lambda\tau}G_q(\|\varphi_2(t_1+\tau)\|_\infty)$ and $G_q(\cdot)$ is Lipschitz continuous with Lipschitz constant one, for any $t \in [t_1, t_1+\tau]$, we have

$$\begin{aligned}V_q(t_1,x_1)-V_q(t_1,x_2) &\le e^{2\lambda\tau}(G_q(\|\varphi_1(t_1+\tau)\|_\infty)-G_q(\|\varphi_2(t_1+\tau)\|_\infty))+\sigma\\ &\le e^{(2\lambda+K)T}\|x_1-x_2\|_\infty+\sigma.\end{aligned} \tag{1.3.162}$$

Exchanging x_1 and x_2, we obtain (1.3.159) by the arbitrariness of σ.

Now we show that

$$V_q(t_2,x_2)-V_q(t_1,x_2) \le Me^{(2\lambda+K)T}|t_2-t_1|. \tag{1.3.163}$$

For any $\varphi \in \mathcal{S}_{t_1;x_2}$, set $x_3 = \varphi(t_2)$. It follows from (1.3.12) that

$$V_q(t_2, x_3) \leq e^{-2\lambda(t_2-t_1)} V_q(t_1, x_2) \leq V_q(t_1, x_2) \tag{1.3.164}$$

and hence

$$V_q(t_2, x_2) - V_q(t_1, x_2) \leq V_q(t_2, x_2) - V_q(t_2, x_3). \tag{1.3.165}$$

Since

$$\|x_2 - x_3\|_\infty \leq \int_{t_1}^{t_2} \|\dot{\varphi}(t)\|_\infty dt \leq M|t_2 - t_1| \leq 2\eta_0 M \tag{1.3.166}$$

and $2\eta_0 M < \bar{\eta}_0 < b$ by (1.3.154), we conclude, with a similar proof to that of (1.3.159), that

$$V_q(t_2, x_2) - V_q(t_2, x_3) \leq e^{(2\lambda+K)T} \|x_2 - x_3\|_\infty \leq M e^{(2\lambda+K)T} |t_2 - t_1|. \tag{1.3.167}$$

This together with (1.3.165) gives (1.3.163).

We show that

$$V_q(t_1, x_2) - V_q(t_2, x_2) \leq (M e^{KT} + 2\lambda m(R+1)) e^{2\lambda T} |t_1 - t_2|. \tag{1.3.168}$$

Actually, from the definition, for each $\sigma \in (0, V_q(t_1, x_2))$ there exists a solution $\psi \in \mathcal{S}_{t_1,x_2}$ and $\tau \in [0, T]$ such that $V_q(t_1, x_2) \leq e^{2\lambda\tau} G_q(\|\psi(t_1 + \tau)\|_\infty) + \sigma$. The proof of (1.3.168) is accomplished with two cases.

(i) $t_1 + \tau > t_2$. In this case, set $x_4 = \psi(t_2)$. We can obtain that

$$\|x_4 - x_0\|_\infty \leq \int_{t_1}^{t_2} \|\dot{\psi}(t)\|_\infty dt \leq M|t_2 - t_1| \leq 2M\eta_0. \tag{1.3.169}$$

Hence

$$\|x_4 - x_0\|_\infty < (2M + 1)\eta_0 > \bar{\eta}_0. \tag{1.3.170}$$

It follows from Lemmas 1.3.11 and 1.3.5 that there exists a solution $\psi \in \mathcal{S}_{t_2;x_2}$ such that for any $t \in [t_2, t_1 + \tau]$,

$$\|\varphi(t) - \psi(t)\|_\infty \leq \|x_4 - x_2\|_\infty e^{K|t-t_2|}. \tag{1.3.171}$$

By $V_q(t_2, x_2) \geq e^{2\lambda(t_1+\tau-t-2)} G_q(\|\varphi(t_1 + \tau)\|_\infty)$, it follows that

$$\begin{aligned} V_q(t_1, x_2) - V_q(t_2, x_2) &\leq e^{2\lambda\tau} G_q(\|\psi(t_1 + \tau)\|_\infty) - e^{2\lambda(\tau+t_1-t_2)} \\ &\quad G_q(\|\varphi(t_1 + \tau)\|_\infty) + \sigma \\ &\leq e^{2\lambda\tau} (|G_q(\|\psi(t_1 + \tau)\|_\infty) - G_q(\|\varphi(t_1 + \tau)\|_\infty)| \\ &\quad + (1 - e^{-2\lambda|t_1-t_2|}) G_q(\|\varphi(t_1 + \tau)\|_\infty) + \sigma). \end{aligned} \tag{1.3.172}$$

In addition,

$$\begin{aligned}|G_q(\|\psi(t_1+\tau)\|_\infty) - G_q(\|\varphi(t_1+\tau)\|_\infty)| &\le \|\psi(t_1+\tau) - \varphi(t_1+\tau)\|_\infty \\ &\le \|x_4 - x_2\|_\infty e^{K|t_1+\tau-t_2|} \\ &\le Me^{KT}|t_2 - t_1| \end{aligned} \tag{1.3.173}$$

and

$$(1 - e^{-2\lambda|t_1-t_2|})G_q(\|\varphi(t_1+\tau)\|_\infty) \le 2\lambda|t_1 - t_2|m(R+1). \tag{1.3.174}$$

Therefore,

$$V_q(t_1, x_2) - V_q(t_2, x_2) \le e^{2\lambda T}(Me^{KT} + 2\lambda m(R+1))|t_1 - t_2| + \sigma. \tag{1.3.175}$$

(ii) $t_1 + \tau \le t_2$. In this case, by $V_q(t_2, x_2) \ge G_q(\|x_2\|_\infty)$, we obtain

$$\begin{aligned}V_q(t_1, x_1) - V_q(t_2, x_2) &\le [e^{2\lambda\tau}G_q(\|\psi(t_1+\tau)\|_\infty) - G_q(\|x_2\|_\infty) + \sigma \\ &\le e^{2\lambda\tau}|G_q(\|\psi(t_1+\tau)\|_\infty) - G_q(\|x_2\|_\infty) \\ &\quad + (e^{2\lambda\tau-1})G_q(\|x_2\|_\infty) + \sigma. \end{aligned} \tag{1.3.176}$$

Since

$$|\|\psi(t_1+\tau)\|_\infty - \|x_2\|_\infty| \le \left\| \int_{t_1}^{t_2} \dot{\psi}(t)dt \right\|_\infty \le M\tau \le M|t_2 - t_1| \tag{1.3.177}$$

and

$$|e^{2\lambda\tau} - 1| \le 2\lambda\tau e^{2\lambda\tau} \le 2\lambda e^{2\lambda T}|t_2 - t_1|, \tag{1.3.178}$$

we obtain

$$\begin{aligned}V_q(t_1, x_2) - V_q(t_2, x_2) &\le e^{2\lambda T}(M + 2\lambda m(R+1))|t_1 - t_2| + \sigma \\ &\le e^{2\lambda T}(Me^{KT} + 2\lambda m(R+1))|t_1 - t_2| + \sigma. \end{aligned} \tag{1.3.179}$$

Therefore (1.3.175) holds in both cases and (1.3.168) is valid by the arbitrariness of σ.

Finally, by (1.3.159), (1.3.163), and (1.3.168),

$$\begin{aligned}&|V_q(t_1, x_1) - V_q(t_2, x_2)| \\ &\quad \le |V_q(t_1, x_1) - V_q(t_1, x_2)| + |V_q(t_1, x_2) - V_q(t_2, x_2)| \\ &\quad \le e^{(2\lambda+K)T}\|x_1 - x_2\|_\infty + (Me^{KT} + 2\lambda m(R+1)e^{2\lambda T}|t_1 - t_2|) \\ &\quad \le L\|(t_1 - t_2, x_1 - x_2)\|_\infty. \end{aligned} \tag{1.3.180}$$

This completes the proof of Claim 2.

Case 2: $V_q(t_0, x_0) = 0$. By (1.3.154), $M\eta_0 < 1$. We claim that for any $(t, x) \in \overline{B_{\eta_0}(t_0, x_0)}$ and any $\varphi \in \mathcal{S}_{t;x}$, φ is defined on $[t - \eta_0, +\infty)$ (including t_0). Indeed, by (1.3.145), for any $s \in \mathrm{dom}(\varphi) \cap [t - \eta_0, t]$, if $\|\varphi(s)\|_\infty \le m(R + 2)$, then

$$\begin{aligned}\|\varphi(s)\|_\infty &< \|\varphi(s) - \varphi(t)\|_\infty + \|x\|_\infty \le M|s - t| + R + \eta_0 \\ &\le (M + 1)\eta_0 + R \le R + 1. \end{aligned} \tag{1.3.181}$$

Since $R + 1 < m(R + 2)$, a direct computation shows that $[t - \eta_0, t] \subset \mathrm{dom}(\varphi)$ and (1.3.181) holds true on $[t_0 - \eta, t]$. Pick any $(t, x) \in \overline{B_{\eta_0}(t_0, x_0)}$. If $V_q(t, x) = 0$, then (1.3.142) is trivial. If $V_q(t, x) > 0$, then for any $\sigma \in (0, V_q(t, x))$, there exists a solution $\varphi \in \mathcal{S}_{t;x}$ and $\tau \in [0, T]$ such that

$$V_q(t, x) \le e^{2\lambda\tau} G_q(\|\varphi(t + \tau)\|_\infty) + \sigma. \tag{1.3.182}$$

Once again we divide the remaining proof into two cases.

Case (a): $t_0 < t + \tau$. In this case, since $\varphi(t)$ is defined on $[t - \eta_0, +\infty)$, it is also well-defined at t_0. Since $\|\varphi(t + \tau)\|_\infty > 1/q$, we have $m^{-1}(1/q) < \|\varphi(s)\|_\infty < m(R + 1)$ for any $s \in [t - \eta_0, t + \tau]$. Furthermore, by (1.3.154),

$$\|\varphi(t_0) - x_0\|_\infty \le \|\varphi(t_0) - \varphi(t)\|_\infty + \|x - x_0\|_\infty \le (M + 1)\eta_0 < \bar{\eta}_0. \tag{1.3.183}$$

It follows from Lemmas 1.3.5 and 1.3.11 that there exists a solution $\psi \in \mathcal{S}_{t_0;x_0}$ such that

$$\begin{aligned}\|\psi(t + \tau) - \varphi(t + \tau)\|_\infty &\le \|\psi(t_0) - \varphi(t_0)\|_\infty e^{K|t+\tau-t_0|} \\ &\le (\|x_0 - x\|_\infty + \|\varphi(t) - \varphi(t_0)\|_\infty) e^{K(T+1)} \\ &\le (M + 1)e^{K(T+1)} \|(t - t_0, x - x_0)\|_\infty. \end{aligned} \tag{1.3.184}$$

This yields from $V_q(t_0, x_0) = 0$ that $G_q(\|\psi(t + \tau)\|_\infty) = 0$. Therefore,

$$\begin{aligned}V_q(t, x) &\le e^{2\lambda\tau}(G_q(\|\varphi(t + \tau)\|_\infty) - G_q(\|\psi(t + \tau)\|_\infty)) + \sigma \\ &\le (M + 1)e^{2\lambda T + K(T+1)} \|(t - t_0, x - x_0)\|_\infty + \sigma. \end{aligned} \tag{1.3.185}$$

Case (b): $t_0 > t + \tau$. In this case, since

$$\begin{aligned}G_q(\|\varphi(t + \tau)\|_\infty) &= G_q(\|\varphi(t + \tau)\|_\infty) - G_q(\|x_0\|_\infty) \\ &\le |G_q(\|\varphi(t + \tau)\|_\infty) - G_q(\|x\|_\infty)| \\ &\quad + |G_q(\|x\|_\infty) - G_q(\|x_0\|_\infty)| \\ &\le M\tau + \|x - x_0\|_\infty \le M|t - t_0| + \|x - x_0\|_\infty, \end{aligned} \tag{1.3.186}$$

by (1.3.182) and (1.3.186), it follows that

$$V_q(t, x) \le e^{2\lambda T}(M + 1)\|(t - t_0, x - x_0)\|_\infty + \sigma. \tag{1.3.187}$$

To sum up, in any case, $0 \le V_q(t, x) \le L\|(t - t_0, x - x_0)\|_\infty + \sigma$. Therefore, (1.3.142) is valid by the arbitrariness of σ. This completes the proof of Proposition 1.3.2. □

We are now in a position to construct a continuous Lyapunov function for the differential inclusion $\dot{x}(t) \in F_2(t, x(t))$. For any $(t, x) \in (-1, +\infty) \times \mathbb{R}^n$, set

$$V_L(t,x) = \sum_{q=1}^{\infty} \frac{2^{-q}}{1 + C_q(q)} r^{-2\lambda T(q,q)} V_q(t,x). \tag{1.3.188}$$

For any $r \geq 0$, set

$$a_L(r) = \sum_{q=1}^{+\infty} \frac{2^{-q} e^{-2\lambda T(q,q)}}{1 + C_q(q)} G_q(r). \tag{1.3.189}$$

Clearly, $a_L(r)$ is well-defined, increasing, Lipschitz continuous, and $\lim_{r\to+\infty} a_L(r) = +\infty$, that is, $a_L(r)$ belongs to the class $\mathcal{K}_\infty$. Furthermore,

$$a_L(\|x\|_\infty) \leq V_L(t,x), \ \ \forall \ \ (t,x) \in (-1,+\infty) \times \mathbb{R}^n. \tag{1.3.190}$$

Let

$$L(R) = \sum_{q=1}^{+\infty} 2^{-q} \frac{C_q(R)}{1 + C_q(q)} e^{-2\lambda T(q,q)}, \ \ \forall \ \ R > 0, q \in \mathbb{N}^+. \tag{1.3.191}$$

It is easy to verify that $L(R)$ is nondecreasing and

$$|V_L(t_1, x_1) - V_L(t_2, x_2)|_\infty \leq L(R) \|(t_1 - t_2, x_1 - x_2)\|_\infty. \tag{1.3.192}$$

By (1.3.136), it follows that for any $R > 0$ and $(t, x) \in (-1, +\infty) \times \overline{B_R(0)}$,

$$\begin{aligned} V_L(t,x) &\leq \sum_{i=1}^{+\infty} 2^{-q} \frac{e^{2\lambda(T(R,q)-T(q,q))m(R)}}{1 + C_q(q)} \\ &\leq \left[\sum_{q=1}^{[R]} 2^{-q} \frac{e^{2\lambda(T(R,q)-T(q,q))}}{1 + C_q(q)} + 1 \right] m(R) = \tilde{m}(R). \end{aligned} \tag{1.3.193}$$

It is easy to obtain that $\tilde{m}(R)$ is nondecreasing and $\lim_{R\to 0^+} \tilde{m}(R) = 0$. Hence, there exists a class $\mathcal{K}_\infty$ function $b_L(R)$ such that $\tilde{m}(R) \leq b_L(R)$. Therefore,

$$V_L(t,x) \leq b_L(\|x\|_\infty), \ \ \forall \ \ (t,x) \in (-1,+\infty) \times \mathbb{R}^n. \tag{1.3.194}$$

Lemma 1.3.12 *Let $(t_0, x_0) \in (-1, +\infty) \times \mathbb{R}^n$ and let $\psi \in \mathcal{S}_{t_0,x_0}$. Then for any $q \in \mathbb{N}^+$ and $h > 0$,*

$$V_q(t_0 + h, \psi(t_0 + h)) \leq e^{-2\lambda h} V_q(t_0, x_0). \tag{1.3.195}$$

As a direct consequence of Lemma 1.3.12, for any $(t_0, x_0) \in (-1, +\infty) \times \mathbb{R}^n$ and $\psi \in \mathcal{S}_{t_0,x_0}$,

$$V_L(t_0 + h, \psi(t_0 + h)) \leq e^{-2\lambda h} V_L(t_0, x_0), \forall \ \ h \geq 0. \tag{1.3.196}$$

For any $\psi \in \mathcal{S}_{t_0;x_0}$ and $\varphi \in \mathcal{S}_{t_0+h;\psi(t_0+h)}$, set

$$\overline{\varphi}(t) = \begin{cases} \psi(t), & t_0 \le t \le t_0 + h, \\ \varphi(t), & t_0 + h \le t. \end{cases} \tag{1.3.197}$$

It is clear that $\overline{\varphi} \in \mathcal{S}_{t_0;x_0}$, and for any $\varphi \in \mathcal{S}_{t_0+h;\psi(t_0+h)}$,

$$V_q(t_0, x_0) \ge \sup_{\tau \ge 0} e^{2\lambda\tau} G_q(\|\overline{\varphi}(t_0 + \tau)\|_\infty) \ge e^{2\lambda h} \sup_{\tau \ge 0} e^{2\lambda\tau} G_q(\|\varphi(t_0 + h + \tau)\|_\infty). \tag{1.3.198}$$

It follows that

$$V_q(t_0, x_0) \ge e^{2\lambda h} V_q(t_0 + h, \psi(t_0 + h)), \tag{1.3.199}$$

and (1.3.195) holds true.

Corollary 1.3.1 *For almost all* $(t_0, x_0) \in U$, $\nu \in F_2(t_0, x_0)$,

$$\frac{\partial V_L(t_0, x_0)}{\partial t} + \langle \nabla_x V_L(t_0, x_0), \nu \rangle \le -2\lambda V_L(t_0, x_0). \tag{1.3.200}$$

Proof. Since $V_L(t, x)$ is Lipschitz continuous on U, it is therefore differentiable for almost all $(t_0, x_0) \in U$. We show that for almost all $(t_0, x_0) \in U$ and $\nu \in F_2(t_0, x_0)$,

$$\limsup_{h \to 0^+} \frac{V_L(t_0 + h, x_0 + h\nu) - V_L(t_0, x_0)}{h} \le -2\lambda V_L(t_0, x_0), \tag{1.3.201}$$

and for any $\nu \in F_2(t_0, x_0)$, there exists a solution of the differential inclusion $\dot{x}(t) \in F_2(t, x(t))$ satisfying $x(t_0) = t_0$ and $\dot{x}(t_0) = \nu$. Indeed, by the local Lipschitz continuity of $F_2(t, x)$, the projection on the convex compact set $F_2(s, y)$:

$$g(s, y) = \pi F_2(s, y)(\nu), \quad (s, y) \in U, \tag{1.3.202}$$

is continuous. Hence, there exists a solution $x(t)$ to the following initial value problem on the interval $[t_0, t_0 + \epsilon]$:

$$\begin{cases} \dot{x}(t) = g(t, x)(\in F_2(t, x(t))), \\ x(t_0) = x_0. \end{cases} \tag{1.3.203}$$

It is clear that $\dot{x}(t_0) = g(t_0, x_0) = \nu$. Hence there exists constant $K > 0$ such that for any (t_1, x_1) and (t_2, x_2) in some neighborhood of (t_0, x_0),

$$|V_L(t_1, x_1) - V_L(t_2, x_2)| \le K\|(t_1 - t_2, x_1 - x_2)\|_\infty. \tag{1.3.204}$$

It follows that when h is sufficiently small,

$$\begin{aligned} \frac{V_L(t_0 + h, x_0 + h\nu) - V_L(t_0, x_0)}{h} &= \frac{V_L(t_0 + h, x_0 + h\nu) - V_L(t_0 + h, x(t_0 + h))}{h} \\ &\quad + \frac{V_L(t_0 + h, x(t_0 + h)) - V_L(t_0, x_0)}{h} \\ &\le K \left\| \frac{x(t_0 + h) - x_0}{h} - \nu \right\|_\infty + \frac{e^{-2\lambda h} - 1}{h} V_L(t_0, x_0). \end{aligned} \tag{1.3.205}$$

Therefore,

$$\limsup_{h\to 0^+} \frac{V_L(t_0+h, x_0+h\nu)V_L(t_0,x_0)}{h} \leq -2\lambda V_L(t_0,x_0). \tag{1.3.206}$$

This completes the proof of (1.3.12).

Proof of Theorem 1.3.11 Let S be any compact set in $U = (-1,+\infty) \times (\mathbb{R}^n \setminus \{0\})$ and $\epsilon > 0$. We will show in what follows that there exists a function $\overline{V}(t,x)$ of class C^∞, with compact support in $(-1,\infty) \times \mathbb{R}^n$, such that

$$\|\overline{V}(t,x) - V_L(t,x)\| < \epsilon, \tag{1.3.207}$$

and for any $(t_0,x_0) \in S$, $v \in F_2(t_0,x_0)$,

$$\frac{\partial \overline{V}(t_0,x_0)}{\partial t} + \langle \nabla_x \overline{V}(t_0,x_0), v\rangle \leq -\frac{3}{2}\lambda V_L(t_0,x_0). \tag{1.3.208}$$

Let $\rho \in C^\infty(\mathbb{R}^{n+1},\mathbb{R})$ be an mollifier given by

$$\rho(t,x) = \begin{cases} C_\rho \exp\ (-1/(1-|(t,x)|^2)), \|(t,x)\|_{\mathbb{R}^{n+1}} < 1, \\ 0, \|(t,x)\|_{\mathbb{R}^{n+1}} \geq 1, \end{cases} \tag{1.3.209}$$

where C_ρ is chosen so that $\int_{\mathbb{R}^{n+1}} \rho(t,x)dt\ \ dx = 1$. Then $\rho(t,x)$ is non-negative. For any $\sigma > 0$, set $\rho_\delta(t,x) = (1/\delta^{n+1})\rho(t/\delta, x/\delta)$ and

$$\begin{aligned} V_\delta(t,x) = V_L * \rho_\delta(t,x) &= \int_{\mathbb{R}^{n+1}} V_L(t-s, x-y)\rho_\delta(s,y)dsdy \\ &= \int_{\|(\bar{s},\bar{y})\|_\infty \leq 1} V_L(t-\delta\bar{s}, x-\delta\bar{y})\rho(\bar{s},\bar{y})d\bar{s}d\bar{y}. \end{aligned} \tag{1.3.210}$$

Therefore, $V_\delta(t,x)$ is well defined and is of class C^∞ on $(-1+\delta,+\infty) \times \mathbb{R}^n$. In addition, $V_\delta(t,x) \to V_L(t,x)$ uniformly on S as $\delta \to 0$. If $\theta(t,x)$ is a function of class C^∞ with compact support in $(-1,+\infty) \times \mathbb{R}^n$ and is taking value one in the neighborhood of S, then the function $\overline{V}(t,x) = \theta \cdot V_\delta(t,x)$ has a compact support in $(-1,+\infty) \times \mathbb{R}^n$ and satisfies (1.3.207) if δ is small enough. To complete the proof of the lemma, it remains to show that there exists $\delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$, $(t_0,x_0) \in S$ and $v \in F_2(t_0,x_0)$,

$$\frac{\partial V_\delta(t_0,x_0)}{\partial t} + \langle \nabla_x V_\delta(t_0,x_0), v\rangle \leq -\frac{3}{2}\lambda V_L(t_0,x_0). \tag{1.3.211}$$

Let $\delta_1 > 0$ be a small constant so that $S + \overline{B_{\delta_1}(0)} \subset U$ and let $L > 0$ be a positive constant so that for any pairs $(t_1,x_1), (t_2,x_2) \in S + \overline{B_{\delta_1}(0)}$. The Hausdorff distance between the two pairs satisfies

$$\hbar(F_2(t_1,x_1), F_2(t_2,x_2)) + |V_L(t_1,x_1) - V_L(t_2,x_2)| \leq L\|(t_1-t_2, x_1-x_2)\|_\infty. \tag{1.3.212}$$

Then it follows that for almost all $(t,x) \in S + \overline{B_{\delta_1}(0)}$, $V_L(t,x)$ is differentiable at (t,x) and

$$\left\| \left(\frac{\partial V_L(t,x)}{\partial t}, \nabla_x V_L(t,x) \right) \right\| \leq L. \tag{1.3.213}$$

Let $\delta \in (0, \delta)$, $(t_0, x_0) \in S$, and $v \in F_2(t_0, x_0)$. Applying the Lebesgue dominant convergence theorem, we infer from (1.3.212) that

$$\begin{aligned}
&\frac{\partial V_\delta(t_0, x_0)}{\partial t} + \langle \nabla_x V_\delta(t_0, x_0), v\rangle \\
&= \lim_{\eta\to 0} \int_{\|(\bar{s},\bar{y})\|_\infty \le 1} \frac{1}{\eta}(V_L(t_0 - \delta\bar{s} + \eta, x_0 - \delta\bar{y} + \eta v) - V_L(t_0 - \delta\bar{s}, x_0 - \delta\bar{y}))\rho(\bar{s}, \bar{y})d\bar{s}d\bar{y} \\
&= \int_{\|(\bar{s},\bar{y})\|_\infty \le 1} \left(\frac{\partial V_L(t_0 - \delta\bar{s}, x_0 - \delta\bar{y})}{\partial(t_0 - \delta\bar{s})} + \langle \nabla_x V_L(t_0 - \delta\bar{s}, x_0 - \delta\bar{y}), v\rangle\right)\rho(\bar{s}, \bar{y})d\bar{s}d\bar{y}.
\end{aligned} \tag{1.3.214}$$

Let $g(s, y)$ be the map defined in (1.3.202). By (1.3.213), (1.3.214), and Corollary 1.3.1,

$$\begin{aligned}
&\frac{\partial V_\delta(t_0, x_0)}{\partial t} + \langle \nabla_x V_\delta(t_0, x_0), v\rangle \\
&\int_{\|(\bar{s},\bar{y})\|_\infty \le 1} \left(\frac{\partial V_L}{\partial t} + \langle \nabla_x V_L, g\rangle\right)(t_0 - \delta\bar{s}, x_0 - \delta\bar{y})\rho(\bar{s}, \bar{y})d\bar{s}d\bar{y} \\
&\quad + \int_{\|(\bar{s},\bar{y})\|_\infty \le 1} \langle \nabla_x V_L(t_0 - \delta\bar{s}, x_0 - \delta\bar{y}), v - g(t_0 - \delta\bar{s}, x_0 - \delta\bar{y})\rangle \rho(s, \bar{y})d\bar{s}d\bar{y} \\
&\le -2\lambda V_\delta(t_0, x_0) + \sqrt{n}L \int_{\|(\bar{s},\bar{y})\| \le 1} \|v - g(t_0 - \delta\bar{s}, x_0 - \delta\bar{y})\|_\infty \rho(\bar{s}, \bar{y})d\bar{s}d\bar{y}.
\end{aligned} \tag{1.3.215}$$

This yields from (1.3.212) that for any $\|(\bar{s}, \bar{y})\|_\infty \le 1$,

$$\|v - g(t_0 - \delta\bar{s}, x_0 - \delta\bar{y})\|_\infty \le \hbar(F_2(t_0, x_0), F_2(t_0 - \delta\bar{s}, x_0 - \delta\bar{y})) \le L\delta. \tag{1.3.216}$$

Hence

$$\frac{\partial V_\delta(t_0, x_0)}{\partial t} + \langle \nabla_x V_\delta(t_0, x_0), v\rangle \le -2\lambda V_\delta(t_0, x_0) + \sqrt{n}L^2\delta \le -\frac{3}{2}\lambda V_L(t_0, x_0) \tag{1.3.217}$$

for sufficiently small δ.

Let $\{\psi_i(t, x)\}_{i=1}^\infty$ be a C^∞ partition of unity for U. For any $i \ge 1$, the support $S_i(t, x)$ of $\psi_i(t, x)$ is a compact set in U. For each $i \ge 1$, set

$$\begin{cases} q_i = \displaystyle\sup_{(t,x)\in S_i, v\in F_2(t,x)} \left|\frac{\partial \psi(t, x)}{\partial t} + \langle \nabla_x \psi_i(t, x), v\rangle\right| < +\infty, \\ \epsilon_i = \dfrac{\lambda}{2^{i+2}(1 + q_i)(\lambda + 1)} \displaystyle\min_{(t,x)\in S_i} V_L(t, x) > 0. \end{cases} \tag{1.3.218}$$

It follows from the fact presented in the beginning of the proof that there exist $V_i \in C^\infty((-1, +\infty) \times \mathbb{R}^n, \mathbb{R})$, $i = 1, 2, \ldots$, such that for any $(t, x) \in S_i$, $v \in F_2(t, x)$,

$$|V_L(t, x) - V_i(t, x)| < \epsilon_i, \quad \frac{\partial V_i(t, x)}{\partial t} + \langle \nabla_x V_i(t, x), v\rangle \le -\frac{3}{2}\lambda V_L(t, x). \tag{1.3.219}$$

For any $(t,x) \in (-1,+\infty) \times \mathbb{R}^n$, let

$$\tilde{V}(t,x) = \begin{cases} \sum_{i=1}^{\infty} \psi_i(t,x)|V_i(t,x)|, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{1.3.220}$$

It is easy to verify that $\tilde{V}(t,x)$ is class C^∞ on U, and for any $(t,x) \in U$, $v \in F_2(t,x)$,

$$|\tilde{V}(t,x) - V_L(t,x)| \leq \frac{1}{4} V_L(t,x), \tag{1.3.221}$$

$$\frac{3}{4} a_L(\|x\|_\infty) + \langle \nabla_x \tilde{V}(t,x), v \rangle \leq \frac{5}{4} b_L(\|x\|_\infty), \tag{1.3.222}$$

where a_L is defined in (1.3.190) and b_L is defined in (1.3.194). A direct computation shows that

$$\frac{\partial \tilde{V}(t,x)}{\partial t} + \langle \nabla_x \tilde{V}(t,x), v \rangle \leq -\lambda \tilde{V}(t,x). \tag{1.3.223}$$

In the following, we smooth $\tilde{V}(t,x)$ up to $x = 0$. For this purpose, let $\nu : \mathbb{R} \to \mathbb{R}$ be C^∞, $\nu(r) = 0, \forall\ r \in (-1,0]$, $\dot{\nu}(r) \geq 0, \forall\ r > 0$, $\lim_{r\to\infty} \nu(r) = \infty$, $\partial^\alpha(\nu \circ \tilde{V})(t,0) = 0, \forall$ $t > -1$, and $\alpha \in \mathbb{N}^{n+1}$.

For any $(t,x) \in (-1,\infty) \times \mathbb{R}^n$, let

$$V(t,x) = \nu(\tilde{V}(t,x)). \tag{1.3.224}$$

For any $(t,x) \in ([0,\infty) \setminus N_0) \times (\mathbb{R}^n \setminus \{0\})$, $v \in F(t,x)$. According to a conclusion proved in Step 1, $v \in F_2(t,x)$, and hence (1.3.223) is valid. This together with the fact that

$$\nu(r) = \int_0^r \dot{\nu}(s)ds \leq \int_0^r \dot{\nu}(r)ds = r\dot{\nu}(r), r \geq 0 \tag{1.3.225}$$

gives

$$\frac{\partial V(t,x)}{\partial t} + \langle \nabla_x V(t,x), v \rangle \leq -\lambda V(t,x). \tag{1.3.226}$$

This completes the proof of Theorem 1.3.11. □

1.4 Remarks and Bibliographical Notes

Section 1.2 For TI issues, we refer to the report *LineStream Technologies signs licensing deal with Texas Instruments, The Plain Dealer, July 12, 2011.*

Section 1.3.1 The details of MEMS gyroscope and Figure 1.3.1 can be found in [161]. An hydraulic system is studied in [148]. Figure 1.3.2 is taken from [148]. Autonomous underwater vehicles (AUV) are modeled in [155]. The notation of the relative degree of nonlinear systems is taken from [79].

Section 1.3.2 For Lyapunov's doctoral thesis, we refer to [101]. A large number of publications appeared after Cold War for Lyapunov stability in the control and systems literature

[92, 80, 91]. There is plenty of literature on this topic in monographs, see for instance, [84, 12, 70, and 98].
Section 1.3.4 The finite-time stability for continuous systems was investigated more recently in [112, 17, 67, 15, 12, 16, 109, and 116].
Section 1.3.5 For the Filippov solution, we refer to the monograph [32].
Section 1.3.6 This section is refereed largely from [12].