Part I

MEASURE-THEORETICAL FOUNDATIONS OF PROBABILITY THEORY

1

Measure

In this chapter, we introduce the concept of a measure and other closely related notions. We start with some examples and then introduce the concept of a σ-*algebra*, which is crucial in measure theory and probability theory. At first glance this concept seems to be a pure technical construction, which is usually not dealt with in textbooks on 'Probability and Statistics' for empirical sciences. However, a σ-algebra turned out to be the natural domain for a measure, including probability measures. Moreover, in probability theory, a σ-algebra is not only the domain of probability measures. The σ-algebra generated by a random variable can be interpreted as the set of events that is represented by this random variable. This is treated in more detail in chapter 2 on measurable mappings, which provides the general theory of random variables because random variables are measurable mappings. The virtues of σ-algebras will become fully apparent in chapter 10 on conditional expectations and its subsequent chapters. The pair $(\Omega, \mathscr{A})$ consisting of a nonempty set Ω and a σ-algebra $\mathscr{A}$ on Ω is called a *measurable space*. Such a measurable space is crucial for the definition of a *measure*. Next, we treat some important examples of measures, including the *counting measure*, the *Dirac measure*, and the *Lebesgue measure*. Finally, we turn to *continuity* and *uniqueness* properties of a measure.

1.1 Introductory examples

Consider Figure 1.1 showing the set Ω of all points (x, y) inside the rectangle and the sets A and B of all points (x, y) inside the two ellipses, respectively. These three sets are subsets of the plane $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ denotes the *set of all real numbers*, and $\mathbb{R} \times \mathbb{R} := \{(a, b)\colon a, b \in \mathbb{R}\}$ is the set of all ordered pairs (a, b) with $a, b \in \mathbb{R}$, called the *Cartesian product* or *product set* of $\mathbb{R}$ with itself. In Figure 1.1, the sets A and B have a nonempty intersection. Now let $area\,(A)$ and $area\,(B)$ denote their areas and $area\,(A \cap B)$ the area of their intersection. Inspecting this figure reveals:

$$area\,(A \cup B) = area\,(A) \; + \; area\,(B) \; - \; area\,(A \cap B).$$

Probability and Conditional Expectation: Fundamentals for the Empirical Sciences, First Edition. Rolf Steyer and Werner Nagel.

Companion website: http://www.probability-and-conditional-expectation.de

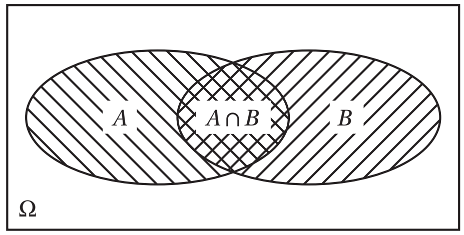

Figure 1.1 A Venn diagram of two sets and their intersection.

This example illustrates three important points:

(a) A measure such as *area* is a function on a *set system on* Ω, (i.e., on a *set of subsets* of a set Ω such as A, B, and $A \cap B$).

(b) If *area* is defined for the *subsets* $A, B \subset \Omega$, then it is also defined for their *intersection* $A \cap B$ and for their *union* $A \cup B$.

(c) Measures are *additive*. In other words, if A and B are *disjoint* subsets of Ω (i.e., if $A \cap B = \emptyset$), then *area* $(A \cup B)$ = *area* (A) + *area* (B).

Note that, in the example presented in Figure 1.1, the sets A and B are *not disjoint*, and this is why *area* $(A \cap B)$ has to be subtracted in the equation displayed above. Points (a) to (c) also apply to other measures such as *length* and *volume* as well as to *probability measures*. Therefore, we adopt a more general language and talk about subsets A, B of a set Ω (or *measurable sets* A, B) and their *measure* μ instead of lines and their lengths, rectangles and their areas, cubes and their volume, or events and their probabilities.

For example, if $\Omega = \{1, \dots, 6\}$ denotes the set of possible outcomes of tossing a fair dice, $A = \{1, 6\}$ and $B = \{2, 4, 6\}$ denote the events of tossing a 1 *or* a 6 and tossing an *even number*, respectively. Furthermore, $A \cap B = \{6\}$ and the probability of tossing a 1 or a 6 or an even number – the event $A \cup B$ – is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{2}{6} + \frac{3}{6} - \frac{1}{6} = \frac{4}{6}.$$

In the first example, the measure *area* assigns a real number to a subset of $\mathbb{R}^2$. In the second example, the measure P assigns a real number to a subset of $\Omega = \{1, \dots, 6\}$. This suggests that a measure should be defined such that it assigns a real number *to all subsets* of a set (i.e., to all elements of the power set). Unfortunately, this may lead to contradictions (see, e.g., Georgii, 2008). In contrast, when defining a measure on a σ-algebra, such contradictions can be avoided.

1.2 σ-Algebra and measurable space

In Definition 1.1, we consider a set system $\mathscr{A}$ on Ω, a sequence $A_1, A_2, \dots$ of subsets of Ω, and their countable union. Remember, a *set system on a set* Ω is a set of subsets of Ω presuming that Ω is not empty. A *sequence of subsets of a set* Ω is a function from the set $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

or $\mathbb{N} = \{1, 2, \dots\}$ or a subset of these sets to $\mathscr{P}(\Omega)$, the *power set* of Ω. Furthermore, the *finite union* of the sets $A_1, \dots, A_n$ and the *countable union* of the sets $A_1, A_2, \dots$ are defined by

$$\bigcup_{i=1}^{n} A_i := \{a \in \Omega\colon \exists\, i \in \{1, \dots, n\}\colon a \in A_i\} \tag{1.1}$$

and

$$\bigcup_{i=1}^{\infty} A_i := \{a \in \Omega\colon \exists\, i \in \mathbb{N}\colon a \in A_i\}, \tag{1.2}$$

respectively. Hence, by definition, $\bigcup_{i=1}^{n} A_i$ is the set of all elements that are an element of at least one of the sets A_i, $i = 1, \dots, n$, and $\bigcup_{i=1}^{\infty} A_i$ is the set of all elements that are an element of at least one of the sets A_i, $i \in \mathbb{N}$. Finally, $A^c := \Omega \setminus A$ denotes the complement of A (with respect to Ω).

Definition 1.1 [σ-Algebra]
A set $\mathscr{A}$ of subsets of a nonempty set Ω is called a σ-algebra (or σ-field) on Ω, if the following three conditions hold:

(a) $\Omega \in \mathscr{A}$.

(b) If $A \in \mathscr{A}$, *then* $A^c \in \mathscr{A}$.

(c) If $A_1, A_2, \dots \in \mathscr{A}$, *then* $\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$.

An element of a σ-algebra is called a measurable set.

Remark 1.2 [Closure with respect to set operations] Condition (c) postulates that σ-algebras are closed with respect to *countable* unions of sets $A_1, A_2, \dots \in \mathscr{A}$. However, in conjunction with (a) and (b), this implies that a σ-algebra is also closed with respect to *finite* unions of sets $A_1, \dots, A_n \in \mathscr{A}$, because every finite union of sets $A_1, \dots, A_n \in \mathscr{A}$ can be represented as a countable union of the sets that are elements of $\mathscr{A}$, for example:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \dots \cup A_n \cup \emptyset \cup \emptyset \cup \dots\ . \tag{1.3}$$

Note that (a) and (b) imply $\emptyset \in \mathscr{A}$, because $\Omega^c = \emptyset$.

Furthermore, although condition (c) only requires explicitly that σ-algebras are closed with respect to countable unions, Definition 1.1 implies that a σ-algebra is closed also with respect to intersections such as $A_1 \cap A_2$ and set differences $A_1 \setminus A_2$. In other words, if A_1 and A_2 are elements of $\mathscr{A}$, then $A_1 \cup A_2$, $A_1 \cap A_2$, and $A_1 \setminus A_2$ are elements of $\mathscr{A}$ as well, provided that $\mathscr{A}$ is a σ-algebra. The same is true for countable intersections $A_1 \cap A_2 \cap \dots$ of elements of $\mathscr{A}$. In more formal terms: If $\mathscr{A}$ is a σ-algebra, then,

$$A_1, A_2, \dots \in \mathscr{A} \ \Rightarrow\ \bigcap_{i=1}^{\infty} A_i \in \mathscr{A} \tag{1.4}$$

(see Exercise 1.1), where $\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \ldots$ is defined by

$$\bigcap_{i=1}^{\infty} A_i := \{a \in \Omega\colon \forall\, i \in \mathbb{N}\colon a \in A_i\}. \tag{1.5}$$

Because

$$\bigcap_{i=1}^{n} A_i = A_1 \cap \ldots \cap A_n \cap \Omega \cap \Omega \cap \ldots\ , \tag{1.6}$$

we can also conclude

$$A_1, \ldots, A_n \in \mathscr{A} \;\Rightarrow\; \bigcap_{i=1}^{n} A_i \in \mathscr{A}, \tag{1.7}$$

where $\bigcap_{i=1}^{n} A_i$, the *finite intersection* of the sets $A_1, \ldots, A_n$, is defined by

$$\bigcap_{i=1}^{n} A_i := \{a \in \Omega\colon \forall\, i \in \{1, \ldots, n\}\colon a \in A_i\}. \tag{1.8}$$

⊲

Remark 1.3 [Countable and uncountable unions] Defining a σ-*algebra,* we use the symbol σ in order to emphasize that unions of finitely or countably many sets are considered, *but not other unions of sets*. For example, the *closed interval* $[a, b] := \{x \in \mathbb{R}\colon a \le x \le b,\ a, b \in \mathbb{R}\}$ on the real axis is identical to the union of singletons $\{x\}$ that contain only one single element $x \in \mathbb{R}$, that is,

$$[a, b] = \bigcup_{a \le x \le b} \{x\}. \tag{1.9}$$

This union is neither finite nor countable. Hence, condition (c) of Definition 1.1 does *not* imply that this union is necessarily an element of a σ-algebra $\mathscr{A}$ on $\mathbb{R}$, even if all singletons $\{x\}$, $x \in \mathbb{R}$, are elements of $\mathscr{A}$. ⊲

The following notion of a *measurable space* proves to be convenient in measure theory.

Definition 1.4 [Measurable space]
If Ω *is a nonempty set and* $\mathscr{A}$ *a* σ*-algebra on* Ω*, then the pair* $(\Omega, \mathscr{A})$ *is called a* *measurable space.*

Example 1.5 [The smallest σ-algebra] The smallest σ-algebra on a nonempty set Ω is $\mathscr{A} = \{\Omega, \emptyset\}$. It contains only the elements Ω and the empty set $\emptyset$. As is easily seen, $\Omega \cup \emptyset = \Omega$, $\Omega^c = \emptyset$, and $\emptyset^c = \Omega$ are elements of $\mathscr{A}$. This shows that $\mathscr{A} = \{\Omega, \emptyset\}$ is closed with respect to union and complement. ⊲

Example 1.6 [Power set] The *power set* $\mathscr{P}(\Omega)$ of a nonempty set Ω (i.e., the *set of all subsets* of Ω) is a σ-algebra on Ω. It is the largest σ-algebra on a nonempty set Ω. All other σ-algebras on Ω are subsets of $\mathscr{P}(\Omega)$. ⊲

Example 1.7 [A small σ-algebra] If A is a subset of a nonempty set Ω, then $\mathscr{A} = \{\Omega, \emptyset, A, A^c\}$ is always a σ-algebra on Ω (see Exercise 1.2). Again, it is easily seen that this set system is closed with respect to union and complement. ⊲

Remark 1.8 [Motivation for σ-algebras] These examples show that there can be many different σ-algebras on a nonempty set Ω. Why not simply always use the largest one, the power set $\mathscr{P}(\Omega)$? In fact, this would be possible as long as Ω is finite or countable. There are at least three reasons for using σ-algebras. First, there are important sets Ω (e.g., $\Omega = \mathbb{R}$) such that measures of interest (e.g., *length* — which is the Lebesgue measure pertaining to $\Omega = \mathbb{R}$) cannot be defined on $\mathscr{P}(\Omega)$ (see e.g., Wise and Hall, 1993, counterexample 1.25). These measures can be defined, however, on other σ-algebras, such as the Borel-σ-algebra [see Eq. (1.18)]. (For an example in which the power set is 'too large', see Georgii, 2008.) Second, in some sense, σ-algebras contain those elements of a larger set system that are relevant for a particular question. In probability theory, together with Ω and a probability measure, each σ-algebra on Ω represents a random experiment that is in some sense contained in an (often larger) random experiment. For example, if we consider the random experiment of tossing a dice, then we may focus on whether or not the number of points is even. Together with Ω and the probability measure, the corresponding σ-algebra represents a 'new' random experiment contained in the random experiment of tossing a dice (see Exercise 1.3). Third, using different σ-algebras is indispensable for introducing conditional expectations, conditional independence, and conditional distributions (see chs. 9 to 17). ⊲

Example 1.9 [Joe and Ann] Consider the following random experiment: First, we sample a unit u from the set $\Omega_U := \{Joe, Ann\}$. Second, each unit receives (*yes*) or does not receive a treatment (*no*). Third, it is observed whether (+) or not (−) a success criterion is reached (see Fig. 1.2). Defining $\Omega_X := \{yes, no\}$ and $\Omega_Y := \{+, -\}$, the Cartesian product

$$\Omega := \Omega_U \times \Omega_X \times \Omega_Y = \{(Joe, no, -), (Joe, no, +), \ldots, (Ann, yes, +)\}$$

is the set of possible outcomes ω of this random experiment. It has eight elements, namely the triples $(Joe, no, -)$, $(Joe, no, +)$, …, $(Ann, yes, +)$ (see all eight leaves of Fig. 1.2 for a complete list of these elements).

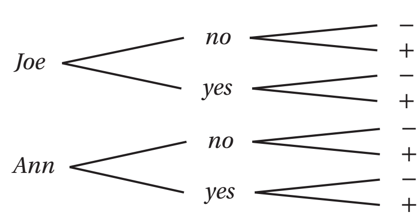

Figure 1.2 Example of a tree representation of a Cartesian product.

In this example, a first σ-algebra $\mathscr{A}$ we may consider is *the set of all subsets of* Ω, the power set $\mathscr{P}(\Omega)$. This set has $2^8 = 256$ elements, where 8 is the number of elements, that is, the cardinality of Ω (see Kheyfits, 2010, Th. 1.1.37). Among these elements is the set

$$A := \{(Joe, no, -), (Joe, no, +), (Joe, yes, -), (Joe, yes, +)\} = \{Joe\} \times \Omega_X \times \Omega_Y.$$

In the context of probability theory, it is also called the event that *Joe is drawn*. Other elements of $\mathscr{A}$ are the events

$$B := \{(Joe, yes, -), (Joe, yes, +), (Ann, yes, -), (Ann, yes, +)\} = \Omega_U \times \{yes\} \times \Omega_Y$$

that the *drawn person is treated*, and

$$C := \{(Joe, no, +), (Joe, yes, +), (Ann, no, +), (Ann, yes, +)\} = \Omega_U \times \Omega_X \times \{+\}$$

that *success* (+) occurs, irrespective of which person is drawn and whether or not the person is treated.

Aside from the power set of Ω, we could also consider the σ-algebras $\mathscr{A}_1 := \{\Omega, \emptyset, A, A^c\}$, $\mathscr{A}_2 := \{\Omega, \emptyset, B, B^c\}$, and $\mathscr{A}_3 := \{\Omega, \emptyset, C, C^c\}$, to name just three. (For another one, see Exercise 1.4.) In a sense, $\mathscr{A}_1$ represents the information regarding which person is drawn. In contrast, $\mathscr{A}_2$ contains the information regarding whether or not the drawn person is treated, and $\mathscr{A}_3$ whether or not the drawn person is successful. Of course, all these σ-algebras are subsets of $\mathscr{P}(\Omega)$, the power set of Ω. ◁

Example 1.10 [Trace of a set system and trace σ-algebra] Let Ω and Ω_0 be nonempty sets. If $\mathscr{E}$ is a set system on Ω and $\Omega_0 \subset \Omega$, then

$$\mathscr{E}|_{\Omega_0} := \{\Omega_0 \cap A : A \in \mathscr{E}\}$$

is a set system on Ω_0. It is called the *trace of* $\mathscr{E}$ *in* Ω_0. Furthermore, if $\mathscr{A}$ is a σ-algebra on Ω and $\Omega_0 \subset \Omega$, then the set system

$$\mathscr{A}|_{\Omega_0} := \{\Omega_0 \cap A : A \in \mathscr{A}\}$$

is a σ-algebra on Ω_0 (see Exercise 1.5). If $\Omega \neq \Omega_0$, then the trace $\mathscr{A}|_{\Omega_0}$ is a σ-algebra on Ω_0, but not on Ω, because $\Omega \notin \mathscr{A}|_{\Omega_0}$. ◁

Example 1.11 [Joe and Ann – continued] In Example 1.9, we defined the event A that Joe is drawn, the event B that the drawn person is treated, and the σ-algebra $\mathscr{A}_2 = \{\Omega, \emptyset, B, B^c\}$. The trace of $\mathscr{A}_2$ in A is

$$\mathscr{A}_2|_A = \{A, \emptyset, A \cap B, A \cap B^c\}.$$

Obviously, just like all elements of $\mathscr{A}_2$ are subsets of Ω, all elements of $\mathscr{A}_2|_A$ are subsets of A. From an application point of view, considering $\mathscr{A}_2|_A$ means to presume that Joe is drawn and consider the events that he is treated or not treated, respectively. ◁

1.2.1 σ-Algebra generated by a set system

The concept of a σ-*algebra generated by a set system* is useful in order to define important σ-algebras. It is also useful for specifying certain measures (see section 1.6). Theorem 1.12 prepares Definition 1.13. Reading this theorem, remember that a σ-algebra on a set Ω is itself a set (of subsets of Ω), so that we can consider the intersection of σ-algebras.

Theorem 1.12 [Intersection of σ-algebras is a σ-algebra]
Let I be a nonempty (finite, countable, or uncountable) index set, and let all $\mathscr{A}_i$, $i \in I$, be σ-algebras on Ω. Then, $\bigcap_{i \in I} \mathscr{A}_i$ is also a σ-algebra on Ω.

(Proof p. 28)

This theorem allows us to define the σ-*algebra generated by a set system* on Ω.

Definition 1.13 [σ-Algebra generated by a set system]
Let $\mathscr{E}$ be a set system on a nonempty set Ω, and let $(\mathscr{A}_i, i \in I)$ be the family of all σ-algebras on Ω that contain $\mathscr{E}$ as a subset. Then, we define

$$\sigma(\mathscr{E}) := \bigcap_{i \in I} \mathscr{A}_i \tag{1.10}$$

and call it the σ-algebra generated by $\mathscr{E}$. *The set $\mathscr{E}$ is also called a* generating system of $\sigma(\mathscr{E})$.

Remark 1.14 [Smallest σ-algebra containing $\mathscr{E}$ as a subset] According to Theorem 1.12, every set system $\mathscr{E}$ on Ω generates a uniquely defined σ-algebra $\sigma(\mathscr{E})$ on Ω. Note that the σ-algebra $\sigma(\mathscr{E})$ is the *smallest* σ-algebra on Ω containing $\mathscr{E}$ as a subset, that is,

$$\mathscr{C} \text{ is a } \sigma\text{-algebra on } \Omega \text{ and } \mathscr{E} \subset \mathscr{C} \;\Rightarrow\; \sigma(\mathscr{E}) \subset \mathscr{C}. \tag{1.11}$$

Furthermore,

$$\sigma[\sigma(\mathscr{E})] = \sigma(\mathscr{E}). \tag{1.12}$$

⊲

Lemma 1.15 immediately follows from (1.11). It can be used in proofs of the identity of two σ-algebras.

Lemma 1.15 [Smallest σ-algebra containing $\mathscr{E}$ as a subset]
Let $(\Omega, \mathscr{A})$ be a measurable space and $\mathscr{E}$ a set system on Ω with $\sigma(\mathscr{E}) = \mathscr{A}$. If $\mathscr{C}$ is a σ-algebra on Ω with $\mathscr{E} \subset \mathscr{C} \subset \mathscr{A}$, then $\mathscr{C} = \mathscr{A}$.

(Proof p. 29)

Remark 1.16 [σ-Algebra generated by unions of set systems] Let $\mathscr{D}$, $\mathscr{E}$, $\mathscr{F}$ be set systems on a nonempty set Ω. Then,

$$\sigma(\mathscr{D} \cup \mathscr{E} \cup \mathscr{F}) = \sigma[\mathscr{D} \cup \sigma(\mathscr{E} \cup \mathscr{F})] \tag{1.13}$$

(see Exercise 1.6). ◁

Example 1.17 [Several set systems may generate the same σ-algebra] If A is a subset of a nonempty set Ω, then the set system $\{A\}$ generates the σ-algebra $\{\Omega, \emptyset, A, A^c\}$. Note that $\{\Omega, \emptyset, A, A^c\}$ is also generated by the set systems $\{A^c\}$ and $\{A, A^c\}$, for instance. Hence,

$$\sigma(\{A\}) = \sigma(\{A^c\}) = \sigma(\{A, A^c\}) = \sigma(\{\Omega, \emptyset, A, A^c\}) = \{\Omega, \emptyset, A, A^c\}.$$

In contrast, if $\emptyset \neq A \neq \Omega$, then the σ-algebra $\{\Omega, \emptyset, A, A^c\}$ is neither generated by the set system $\{\Omega\}$ nor by $\{\Omega, \emptyset\}$. Instead,

$$\sigma(\{\emptyset\}) = \sigma(\{\Omega\}) = \sigma(\{\Omega, \emptyset\}) = \{\Omega, \emptyset\},$$

that is, $\{\Omega\}$, $\{\emptyset\}$, and $\{\Omega, \emptyset\}$ generate the σ-algebra $\{\Omega, \emptyset\}$. ◁

Example 1.18 [A generator of the power set] Let $\Omega \neq \emptyset$ be finite or countable, and let $\mathscr{E} := \{\{\omega\}: \omega \in \Omega\}$. Then, $\sigma(\mathscr{E}) = \mathscr{P}(\Omega)$ (see Exercise 1.7). ◁

This example is generalized in Lemma 1.20.

Remark 1.19 [Partition] Reading Lemma 1.20, remember that a set system $\mathscr{E}$ on a nonempty set Ω is called a *partition* of Ω if

(a) $\forall\, B \in \mathscr{E}$: $B \neq \emptyset$.

(b) $\forall\, B, C \in \mathscr{E}$: $B \neq C \;\Rightarrow\; B \cap C = \emptyset$.

(c) $\bigcup_{B \in \mathscr{E}} B = \Omega$.

◁

Lemma 1.20 [An element of a σ-algebra generated by a partition]
Let $\mathscr{E} := \{B_1, \dots, B_n\}$ or $\mathscr{E} := \{B_1, B_2, \dots\}$ be a finite or countable partition of Ω, respectively. Then, for all $C \in \sigma(\mathscr{E})$, there is an $I(C) \subset \mathbb{N}$ such that

$$C = \bigcup_{i \in I(C)} B_i = \bigcup_{B_i \subset C} B_i, \tag{1.14}$$

where, by convention, $\bigcup_{i \in \emptyset} B_i := \emptyset$.

(Proof p. 29)

Remark 1.21 [Constructing a σ-algebra] If $\mathscr{E} = \{A_1, \dots, A_m\}$ is a finite set of subsets of Ω, then there is a finite partition $\mathscr{F} = \{B_1, \dots, B_n\}$ of Ω with $\sigma(\mathscr{E}) = \sigma(\mathscr{F})$. Furthermore, if

$\mathscr{E}$ is a finite set of subsets of Ω, then each element of $\sigma(\mathscr{E})$ is obtained by finitely many unions, intersections, or complements of elements of $\mathscr{E}$ (see Exercise 1.8). ◁

Example 1.22 [Joe and Ann – continued] In Example 1.11, we already considered the event A that Joe is drawn and noted that the trace of the σ-algebra $\mathscr{A}_2 = \{\Omega, \emptyset, B, B^c\}$ in A is $\mathscr{A}_2|_A = \{A, \emptyset, A \cap B, A \cap B^c\}$. In contrast, the σ-algebra on Ω generated by the trace $\mathscr{A}_2|_A$ is

$$\sigma(\mathscr{A}_2|_A) = \{\Omega, \emptyset, A, A^c, A \cap B, A \cap B^c, (A \cap B) \cup A^c, (A \cap B^c) \cup A^c\},$$

where $(A \cap B) \cup A^c = A^c \cup B$ and $(A \cap B^c) \cup A^c = A^c \cup B^c$. ◁

Remark 1.23 [Monotonicity of generated σ-algebras] Let $\mathscr{E}_1, \mathscr{E}_2$ be set systems on a nonempty set Ω with $\mathscr{E}_1 \subset \mathscr{E}_2$. Then, $\sigma(\mathscr{E}_1) \subset \sigma(\mathscr{E}_2)$ (see Exercise 1.9). ◁

An important kind of σ-algebras are those for which there is a countable set system that generates them.

Definition 1.24 [Countably generated σ-algebra]
Let $(\Omega, \mathscr{A})$ be a measurable space. Then, $\mathscr{A}$ is called c o u n t a b l y g e n e r a t e d *if there is a finite or countable set $\mathscr{E} \subset \mathscr{A}$ such that $\sigma(\mathscr{E}) = \mathscr{A}$.*

Example 1.25 [Some countably generated σ-algebras] Examples of countably generated σ-algebras are:

(a) All σ-algebras on a finite nonempty set Ω.

(b) $\mathscr{P}(\mathbb{N}_0^n)$, $n \in \mathbb{N}$.

(For a proof, see Exercise 1.10. For another example, see Remark 1.28.) ◁

Remark 1.26 [A caveat] Note that there are countably generated σ-algebras for which not all of their elements can be constructed by countably many unions, intersections, or complements of elements of the generating system. An example in case are Borel σ-algebras on $\mathbb{R}$ or $\mathbb{R}^n$ (see Rem. 1.28 and Michel, 1978, sect. I.4). ◁

Lemma 1.27 [σ-Algebra generated by the trace of a set system]
Let $A \subset \Omega$ be nonempty, $\mathscr{E} \subset \mathscr{P}(\Omega)$, and $\mathscr{E}|_A := \{C \cap A\colon C \in \mathscr{E}\}$. Then,

$$\sigma(\mathscr{E}|_A) = \sigma(\mathscr{E})|_A, \tag{1.15}$$

where $\sigma(\mathscr{E}|_A)$ denotes the σ-algebra generated on A, whereas $\sigma(\mathscr{E})$ is a σ-algebra on Ω. Furthermore, if $\mathscr{C}$ is a σ-algebra on Ω and $A \in \mathscr{E}$ such that

$$\forall\, C \in \mathscr{E}\colon \quad C \neq A \;\Rightarrow\; A \cap C = \emptyset, \tag{1.16}$$

(i.e., A does not intersect with any other element of $\mathscr{E}$), then

$$\sigma(\mathscr{C} \cup \mathscr{E})|_A = \mathscr{C}|_A. \tag{1.17}$$

(Proof p. 30)

Hence, according to Equation (1.15), the σ-algebra generated by the trace of a set system $\mathscr{E}$ is the trace of the σ-algebra generated by $\mathscr{E}$; and, according to Equation (1.17), the trace of the σ-algebra $\sigma(\mathscr{C} \cup \mathscr{E})$ in the set A is identical to the trace of the σ-algebra $\mathscr{C}$ in A, if (1.16) holds.

1.2.2 σ-Algebra of Borel sets on $\mathbb{R}^n$

For $a, b \in \mathbb{R}$ with $a < b$, let us consider a *half-open interval* $]a, b]$ in $\mathbb{R}$, which is defined by

$$]a, b] := \{x \in \mathbb{R}: a < x \leq b\},$$

and the *set system*

$$\mathscr{I}_1 := \{]a, b]: a, b \in \mathbb{R} \text{ and } a < b\}$$

of all half-open intervals in $\mathbb{R}$. The σ-algebra generated by this set system is called the *Borel σ-algebra* on $\mathbb{R}$. It is denoted by $\mathscr{B}$ or $\mathscr{B}_1$. The elements of $\mathscr{B}$ are called the *Borel sets* of $\mathbb{R}$. In formal terms,

$$\mathscr{B} := \mathscr{B}_1 := \sigma(\mathscr{I}_1). \tag{1.18}$$

Note that there are several sets systems generating the Borel σ-algebra (see, e.g., Klenke, 2013, Th. 1.23). In particular,

$$\mathscr{B}_1 = \sigma(\{]-\infty, b]: b \in \mathbb{R}\}) \tag{1.19}$$

(see Georgii, 2008). Similarly, we define the *Borel σ-algebra on* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ to be the σ-algebra generated by the set system $\mathscr{I}_2$ of all half-open *rectangles* in $\mathbb{R}^2$, whose sides are parallel to the axes (see Fig. 1.3). These rectangles are defined by

$$]a_1, b_1] \times]a_2, b_2] = \{(x_1, x_2) \in \mathbb{R}^2: a_1 < x_1 \leq b_1, a_2 < x_2 \leq b_2\}.$$

The σ-algebra $\sigma(\mathscr{I}_2)$ is denoted by $\mathscr{B}_2$, that is, $\mathscr{B}_2 := \sigma(\mathscr{I}_2)$, and its elements are called the *Borel sets of* $\mathbb{R}^2$.

This definition is easily generalized: The *Borel σ-algebra on* $\mathbb{R}^n$ is defined by $\mathscr{B}_n := \sigma(\mathscr{I}_n)$, $n \in \mathbb{N}$, where $\mathscr{I}_n$ is the system of all half-open *cuboids* in $\mathbb{R}^n$, whose sides are parallel to the axes. Such a cuboid is a set

$$]a_1, b_1] \times \ldots \times]a_n, b_n] = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_1 < x_1 \leq b_1, \ldots, a_n < x_n \leq b_n\}, \tag{1.20}$$

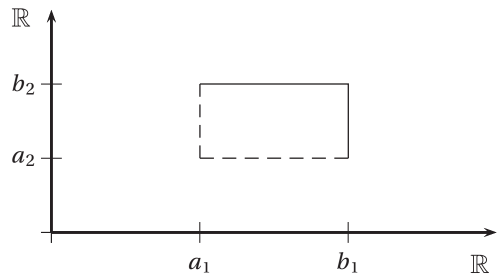

Figure 1.3 A half-open rectangle in the plane $\mathbb{R}^2$.

where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. Just like $\mathscr{B}_1$, the σ-algebra $\mathscr{B}_n$ has several generating systems, one of which is used in the equation

$$\mathscr{B}_n = \sigma(\{]-\infty, b_1] \times \ldots \times]-\infty, b_n]: b_1, \ldots, b_n \in \mathbb{R}\}) \tag{1.21}$$

(see Exercise 1.11).

Note that not every subset of $\mathbb{R}^n$ is a Borel set. In other words, $\mathscr{B}_n$ is not the power set of $\mathbb{R}^n$ (see Rem. 1.60). However, for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the singleton $\{x\}$ *is* a Borel set of $\mathbb{R}^n$, that is,

$$\{x\} \in \mathscr{B}_n, \quad \forall\, x \in \mathbb{R}^n \tag{1.22}$$

(see Exercise 1.12).

Furthermore, if $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the *extended set of real numbers*, then

$$\overline{\mathscr{B}} := \sigma(\mathscr{B} \cup \{\{-\infty\}, \{+\infty\}\})$$

is a σ-algebra on $\overline{\mathbb{R}}$, and it is called the *Borel σ-algebra on* $\overline{\mathbb{R}}$. Similarly, $\overline{\mathscr{B}}_n$ is called the *Borel σ-algebra on* $\overline{\mathbb{R}}^n$. It is defined as the product σ-algebra of $\overline{\mathscr{B}}$ with itself (n times) (see Def. 1.31). Finally, we may sometimes consider $\overline{\mathscr{B}}_n|_{\Omega_0}$, the *trace of the Borel σ-algebra on* $\overline{\mathbb{R}}^n$ *in* $\Omega_0 \subset \overline{\mathbb{R}}^n$.

Remark 1.28 [The Borel σ-algebra is countably generated] Note that

$$\mathscr{B} = \sigma(\{]a, b]: a, b \in \mathbb{Q}, a < b\}),$$

where $\mathbb{Q}$ denotes the set of rational numbers. Because $\mathbb{Q}$ is countable, the set of intervals $\{]a, b]: a, b \in \mathbb{Q}, a < b\}$ is countable as well. Therefore, the Borel σ-algebra $\mathscr{B}$ is countably generated. This also holds for $\mathscr{B}_n$, $n \in \mathbb{N}$ (see Klenke, 2013, Th. 1.23). ⊲

Remark 1.29 [Trace of the Borel σ-algebra in a countable subset of ℝ] Let $\mathscr{B}$ denote the Borel σ-algebra on $\mathbb{R}$. If $\Omega_0 \subset \mathbb{R}$ is finite or countable, then $\mathscr{B}|_{\Omega_0} = \mathscr{P}(\Omega_0)$, where $\mathscr{B}|_{\Omega_0}$ is the trace of the Borel σ-algebra on $\mathbb{R}$ in $\Omega_0 \subset \mathbb{R}$ (see Exercise 1.13). ⊲

1.2.3 σ-Algebra on a Cartesian product

In section 1.2.2, we defined a σ-algebra on $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ (n-times). Now we consider σ-algebras on general Cartesian products. We start with an example.

Example 1.30 [Joe and Ann – continued] In Example 1.9, we already considered the Cartesian product

$$\Omega := \Omega_U \times \Omega_X \times \Omega_Y,$$

which consists of the eight triples (*Joe*, *no*, −), (*Joe*, *no*, +), … , (*Ann*, *yes*, +) (see again Fig. 1.2). Now consider the σ-algebras $\mathscr{A}_1 := \mathscr{P}(\Omega_U)$, $\mathscr{A}_2 := \mathscr{P}(\Omega_X)$, and $\mathscr{A}_3 := \mathscr{P}(\Omega_Y)$, as well as the set

$$\mathscr{E} := \{A_1 \times A_2 \times A_3 \colon A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2, A_3 \in \mathscr{A}_3\},$$

which is a set system on Ω consisting of $4 \cdot 4 \cdot 4 = 64$ elements. For example, the set system $\mathscr{E}$ contains the elements

$$A := \{Joe\} \times \{no\} \times \{-\} = \{(Joe, no, -)\}$$

and

$$B := \{Ann\} \times \{yes\} \times \{+\} = \{(Ann, yes, +)\}.$$

However, $\mathscr{E}$ does not contain

$$A \cup B = \{(Joe, no, -), (Ann, yes, +)\}$$

as an element. The only product set $A_1 \times A_2 \times A_3$ with $A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2, A_3 \in \mathscr{A}_3$ that contains $A \cup B$ as a subset is $\Omega_U \times \Omega_X \times \Omega_Y = \Omega$. However, $A \cup B \neq \Omega$. Therefore, $\mathscr{E}$ is not a σ-algebra [cf. condition (c) of Rem. 1.2]. In this example, the σ-algebra generated by $\mathscr{E}$ is the power set of Ω, that is, $\sigma(\mathscr{E}) = \mathscr{P}(\Omega)$. It consists of $2^8 = 256$ elements. According to the following definition, $\sigma(\mathscr{E})$ is denoted by $\mathscr{A}_1 \otimes \mathscr{A}_2 \otimes \mathscr{A}_3$ and called the *product σ-algebra of* $\mathscr{A}_1$, $\mathscr{A}_2$, *and* $\mathscr{A}_3$. ⊲

Definition 1.31 [Product σ-algebra]
Let $(\Omega_1, \mathscr{A}_1), \ldots, (\Omega_n, \mathscr{A}_n)$ *be measurable spaces and* $\Omega := \Omega_1 \times \ldots \times \Omega_n$. *Then*

$$\mathscr{A}_1 \otimes \ldots \otimes \mathscr{A}_n := \bigotimes_{i=1}^{n} \mathscr{A}_i := \sigma\left(\left\{ \bigtimes_{i=1}^{n} A_i \colon A_i \in \mathscr{A}_i,\, i = 1, \ldots, n \right\}\right) \tag{1.23}$$

is called the p r o d u c t *σ-*a l g e b r a *of the σ-algebras* $\mathscr{A}_i$, $i = 1, \ldots, n$.

Note that the product σ-algebra $\mathscr{A}_1 \otimes \ldots \otimes \mathscr{A}_n$ *is not* the Cartesian product $\mathscr{A}_1 \times \ldots \times \mathscr{A}_n$. Instead, the product σ-algebra is generated by the set system of all Cartesian products of elements of the σ-algebras $\mathscr{A}_1, \ldots, \mathscr{A}_n$. In Lemma 2.42, we give an equivalent specification of a product σ-algebra, using projection mappings.

Lemma 1.32 provides a relationship between the generating systems of the σ-algebras $\mathscr{A}_i$, $i = 1, \ldots, n$, and the generating system of the product σ-algebra.

Lemma 1.32 [Generating system of a product σ-algebra]
For $i = 1, \ldots, n$, let $(\Omega_i, \mathscr{A}_i)$ be measurable spaces and $\mathscr{E}_i \subset \mathscr{A}_i$ with $\sigma(\mathscr{E}_i) = \mathscr{A}_i$. Then,

$$\bigotimes_{i=1}^{n} \mathscr{A}_i = \sigma\left(\left\{ \bigtimes_{i=1}^{n} A_i \colon A_i \in \mathscr{E}_i, i = 1, \ldots, n \right\}\right). \tag{1.24}$$

For a proof, see Klenke [2013, Th. 14.12 (i)].

This lemma implies

$$\mathscr{B}_n = \bigotimes_{i=1}^{n} \mathscr{B} = \mathscr{B} \otimes \ldots \otimes \mathscr{B} \ (n\text{-times})$$

for the Borel σ-algebra on $\mathbb{R}^n$. This lemma also implies the following corollary:

Corollary 1.33 [Countable generating system of a product σ-algebra]
Let $(\Omega_i, \mathscr{A}_i)$, $i = 1, \ldots, n$, be measurable spaces, where all $\mathscr{A}_i$ are countably generated. Then $\bigotimes_{i=1}^{n} \mathscr{A}_i$ is countably generated as well.

Example 1.34 [Countable sets and product σ-algebra] Let $\Omega_1, \ldots, \Omega_n$ be finite or countable nonempty sets and $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be their power sets. Then,

$$\bigotimes_{i=1}^{n} \mathscr{A}_i = \mathscr{P}\left(\bigtimes_{i=1}^{n} \Omega_i\right),$$

that is, $\bigotimes_{i=1}^{n} \mathscr{A}_i$ is the power set of $\Omega := \Omega_1 \times \ldots \times \Omega_n$ (see Exercise 1.14). ◁

Remark 1.35 [Complement of a Cartesian product] Let $(\Omega_1 \times \Omega_2, \mathscr{A}_1 \otimes \mathscr{A}_2)$ be a measurable space, $A \in \mathscr{A}_1$, and $B \in \mathscr{A}_2$. Then $(A \times B)^c \in \mathscr{A}_1 \otimes \mathscr{A}_2$, and this set can be written as:

$$(A \times B)^c = (A^c \times B) \cup (\Omega_1 \times B^c), \tag{1.25}$$

which is a union of disjoint sets (see Exercise 1.15). ◁

1.2.4 ∩-Stable set systems that generate a σ-algebra

For many proofs, generating set systems are useful, which are ∩-stable.

Definition 1.36 [∩-Stability]
Let Ω denote a nonempty set. A set $\mathscr{E}$ of subsets of Ω is called ∩-stable *(or* ∩-closed*) if $A \cap B \in \mathscr{E}$ for all $A, B \in \mathscr{E}$.*

Example 1.37 [Set system with one single element] A set system $\{A\}$ that has only a single element $A \subset \Omega \neq \emptyset$ is ∩-stable (cf. Example 1.17). ⊲

Example 1.38 [Partition and ∩-stability] If $\mathscr{E}$ is a partition of the set Ω, then $\mathscr{D} := \mathscr{E} \cup \{\emptyset\}$ is ∩-stable. ⊲

Example 1.39 [A ∩-stable generating system of a product σ-algebra] Consider the measurable spaces $(\Omega_i, \mathscr{A}_i)$, $i = 1, \dots, n$. The set

$$\{A_1 \times \ldots \times A_n : A_i \in \mathscr{A}_i,\ i = 1, \dots, n\},$$

is a ∩-stable generating system of $\bigotimes_{i=1}^n \mathscr{A}_i$ (see Exercise 1.16). ⊲

Another type of a set system is a Dynkin system. It can be used to show that a specific set system is a σ-algebra.

Definition 1.40 [Dynkin system]
A set $\mathscr{D}$ of subsets of a set Ω is called a Dynkin system *on Ω, if the following three conditions hold:*

(a) $\Omega \in \mathscr{D}$.

(b) If $A \in \mathscr{D}$, then $A^c \in \mathscr{D}$.

(c) If $A_1, A_2, \ldots \in \mathscr{D}$ and they are pairwise disjoint, then $\bigcup_{i=1}^\infty A_i \in \mathscr{D}$.

In the definition of a σ-algebra $\mathscr{A}$, we require $\bigcup_{i=1}^\infty A_i \in \mathscr{A}$ for all sequences $A_1, A_2, \ldots \in \mathscr{A}$, whereas for a Dynkin system the corresponding requirement is only made for all sequences $A_1, A_2, \ldots \in \mathscr{D}$ of pairwise disjoint sets. Analogously to Definition 1.13, for a set system $\mathscr{E}$ on Ω, $\delta(\mathscr{E})$ is defined as the Dynkin system generated by $\mathscr{E}$, that is, as the intersection of all Dynkin systems containing $\mathscr{E}$. According to Theorem 1.41, a Dynkin system is also a σ-algebra if and only if it is ∩-stable.

Theorem 1.41 [Dynkin system and σ-algebra]
Let Ω be a nonempty set.

(i) A Dynkin system $\mathscr{D}$ on Ω is a σ-algebra if and only if it is ∩-stable.

(ii) If $\mathscr{E}$ is a ∩-stable set of subsets of Ω, then $\delta(\mathscr{E}) = \sigma(\mathscr{E})$.

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For a proof, see Bauer (2001, Ths. 2.3 and 2.4). According to proposition (i) of this theorem, we can prove that a set system is a σ-algebra by showing that it is a ∩-stable Dynkin system, and proposition (ii) can be applied to show that the Dynkin system generated by a ∩-stable set system is a σ-algebra.

1.3 Measure and measure space

A measure assigns to all elements of a σ-algebra an element of the closed interval

$$[0, \infty] := \{x \in \mathbb{R}: 0 \leq x\} \cup \{\infty\},$$

that is, a nonnegative real number or the element ∞.

Example 1.42 [A first example] Let $\Omega = \mathbb{R}$, and assume that the closed interval $[3, 9] = \{x \in \mathbb{R}: 3 \leq x \leq 9\}$ as well as the union $[3, 9] \cup [10, 12]$ are elements of a σ-algebra on Ω. If the measure is *length*, then

$$\textit{length}\,([3, 9]) = 9 - 3 = 6$$

and

$$\begin{aligned} \textit{length}\,([3, 9] \cup [10, 12]) &= \textit{length}\,([3, 9]) + \textit{length}\,([10, 12]) \\ &= (9 - 3) + (12 - 10) = 6 + 2 = 8, \end{aligned}$$

because the two intervals are disjoint (i.e., their intersection is the empty set Ø). In this case, the lengths of the intervals [3, 9] and [10, 12] add up to the length of their union $[3, 9] \cup [10, 12]$. In Definition 1.43 (c), we require not only additivity but also σ-additivity. ◁

Reading Definition 1.43, remember that, for a sequence $a_1, a_2, \ldots$ of nonnegative real numbers, $\sum_{i=1}^{\infty} a_i$ is defined by

$$\sum_{i=1}^{\infty} a_i := \lim_{n \to \infty} \sum_{i=1}^{n} a_i.$$

Definition 1.43 [Measure and measure space]
Let $(\Omega, \mathscr{A})$ be a measurable space. A function $\mu: \mathscr{A} \to \overline{\mathbb{R}}$ is called a m e a s u r e *and the triple $(\Omega, \mathscr{A}, \mu)$ is called a* m e a s u r e s p a c e, *if*

(a) $\mu(\emptyset) = 0$.

(b) $\mu(A) \geq 0, \ \forall A \in \mathscr{A}$. *(nonnegativity)*

(c) If $A_1, A_2, \ldots \in \mathscr{A}$ are pairwise disjoint, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$. *(σ-additivity)*

1.3.1 σ-Additivity and related properties

Remark 1.44 [σ-Additivity implies finite additivity] Note that σ-additivity of a measure implies finite additivity, that is, it implies

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i), \quad \text{if } A_1, \dots, A_n \in \mathscr{A} \text{ are pairwise disjoint} \tag{1.26}$$

[see Rule (ii) of Box 1.1 and its proof in Exercise 1.18]. ◁

Remark 1.45 [σ-Additivity] Using the term σ-*additivity* signals that unions of finitely or countably many sets are considered, but not other unions of sets. If, instead of σ-additivity, we would require additivity for *any kind of unions*, including uncountable unions, then the Lebesgue measure λ on $(\mathbb{R}, \mathscr{B})$ – the measure representing *length* – could not be constructed anymore. This is explained in more detail in Remark 1.71. ◁

Remark 1.46 [Representation of a union as a union of pairwise disjoint sets] Let $(\Omega, \mathscr{A})$ be a measurable space. If $A_1, A_2, \dots \in \mathscr{A}$ is a sequence of subsets of Ω, then there is a sequence $B_1, B_2, \dots \in \mathscr{A}$ of pairwise disjoint sets with

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i. \tag{1.27}$$

One way to construct $B_1, B_2, \dots$ is to define $B_1 := A_1$ and

$$B_i := A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j\right), \quad \text{for } i > 1, \tag{1.28}$$

(see Exercise 1.17). ◁

Remark 1.47 [Additivity of measures for partitions] Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, $B \in \mathscr{A}$, and assume

(a) $A_1, \dots, A_n \in \mathscr{A}$ are pairwise disjoint,

(b) $B \subset \bigcup_{i=1}^{n} A_i$.

Then,

$$\mu(B) = \sum_{i=1}^{n} \mu(B \cap A_i). \tag{1.29}$$

Analogously, if

(c) $A_1, A_2, \dots \in \mathscr{A}$ are pairwise disjoint,

(d) $B \subset \bigcup_{i=1}^{\infty} A_i$, then

$$\mu(B) = \sum_{i=1}^{\infty} \mu(B \cap A_i) \tag{1.30}$$

(see Exercise 1.19). ◁

1.3.2 Other properties

Other important properties of a measure are displayed in Box 1.1. Some of these properties can intuitively be understood by inspecting the Venn diagram presented in Figure 1.1. These

Box 1.1 Rules of computation for measures.

Let $(\Omega, \mathscr{A}, \mu)$ be a measure space.

If $A_1, A_2, \ldots \in \mathscr{A}$ are pairwise disjoint, then,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \qquad (\sigma\text{-}additivity) \quad \text{(i)}$$

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i), \quad \forall\, n \in \mathbb{N}. \qquad (finite\ additivity) \quad \text{(ii)}$$

If $A, B \in \mathscr{A}$, then,

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B). \quad \text{(iii)}$$

$$\mu(\Omega) = \mu(B) + \mu(B^c). \quad \text{(iv)}$$

$$\mu(A) \leq \mu(B), \quad \text{if } A \subset B. \qquad (monotonicity) \quad \text{(v)}$$

$$\mu(A \setminus B) = \mu(A) - \mu(A \cap B), \quad \text{if } \mu(A \cap B) < \infty. \quad \text{(vi)}$$

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B), \quad \text{if } \mu(A \cap B) < \infty. \quad \text{(vii)}$$

$$\mu(A) = \mu(\Omega) < \infty \;\Rightarrow\; \mu(A \cap B) = \mu(B). \quad \text{(viii)}$$

$$\mu(A) = 0 \;\Rightarrow\; \mu(A \cup B) = \mu(B). \quad \text{(ix)}$$

Let $A \in \mathscr{A}$ and let $\Omega_0 \subset \Omega$ and be finite or countable with $\mu(\Omega \setminus \Omega_0) = 0$.

If, for all $\omega \in \Omega_0$, $\{\omega\} \in \mathscr{A}$, then

$$\mu(A) = \sum_{\omega \in A \cap \Omega_0} \mu(\{\omega\}). \quad \text{(x)}$$

If $A_1, A_2, \ldots \in \mathscr{A}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i). \qquad (\sigma\text{-}subadditivity) \quad \text{(xi)}$$

properties always hold with the conventions $+\infty + \infty = +\infty$ and $\alpha + \infty = +\infty$, for $\alpha \in \mathbb{R}$. However, note that the term $+\infty + (-\infty)$ or $+\infty - (+\infty)$ cannot meaningfully be defined. Therefore, properties (vi) and (vii) only hold if we assume $\mu(A \cap B) < \infty$. For proofs of all these properties, see Exercise 1.18.

Remark 1.48 [Finite additivity and σ-additivity applied to singletons] If Ω is finite or countable, then each $A \subset \Omega$ is finite or countable as well. Hence, for any measure μ on the measurable space $(\Omega, \mathscr{P}(\Omega))$,

$$\mu(A) = \mu\left(\bigcup_{\omega \in A} \{\omega\}\right) = \sum_{\omega \in A} \mu(\{\omega\}), \quad \forall A \subset \Omega. \tag{1.31}$$

This means that a measure on $(\Omega, \mathscr{P}(\Omega))$ is already uniquely defined if its values $\mu(\{\omega\})$ are uniquely defined for all $\omega \in \Omega$, provided that Ω is finite or countable. Rule (x) of Box 1.1 extends this result to a more general measure space $(\Omega, \mathscr{A}, \mu)$. This rule shows that a measure on $(\Omega, \mathscr{A})$ is already uniquely defined if its values $\mu(\{\omega\})$ are uniquely defined for all $\omega \in \Omega_0$, provided that Ω_0 is finite or countable with $\mu(\Omega \setminus \Omega_0) = 0$ and $\{\omega\} \in \mathscr{A}$ for all $\omega \in \Omega_0$. ◁

1.4 Specific measures

Now we consider some examples of measures, all of which are used later on in this volume in order to introduce still other measures. For some of these examples, we use the *indicator* of a set A.

Definition 1.49 [Indicator]
Let Ω be a set and $A \subset \Omega$. Then, the function $1_A \colon \Omega \to \mathbb{R}$ defined by

$$1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A, \end{cases} \tag{1.32}$$

is called the indicator *of A.*

Remark 1.50 [Sums and products of indicators] If $1_A, 1_B \colon \Omega \to \mathbb{R}$ are the indicators of two sets $A, B \subset \Omega$, then,

$$1_A \cdot 1_B = 1_{A \cap B} \tag{1.33}$$

and

$$1_A + 1_B - 1_{A \cap B} = 1_A + 1_B - 1_A \cdot 1_B = 1_{A \cup B}. \tag{1.34}$$

Equation (1.33) immediately implies

$$1_A + 1_B = 1_{A \cup B}, \quad \text{if } A \cap B = \emptyset. \tag{1.35}$$

More generally, if $A_1, \dots, A_n$ is a finite sequence of pairwise disjoint subsets of Ω, then,

$$\sum_{i=1}^{n} 1_{A_i} = 1_{\bigcup_{i=1}^{n} A_i}, \tag{1.36}$$

that is, then the sum of the indicators of the sets $A_1, \dots, A_n$ is the indicator of the union $\bigcup_{i=1}^{n} A_i$. Finally, if $A_1, A_2, \dots$ is a sequence of pairwise disjoint subsets of Ω, then,

$$\sum_{i=1}^{\infty} 1_{A_i} = 1_{\bigcup_{i=1}^{\infty} A_i}. \tag{1.37}$$

◁

Remark 1.51 [Indicators of products sets] Let Ω_1, Ω_2 be nonempty sets, $A \subset \Omega_1$ and $B \subset \Omega_2$. Then,

$$1_A(\omega_1) \cdot 1_B(\omega_2) = 1_{A \times B}(\omega_1, \omega_2), \quad \forall\, (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2. \tag{1.38}$$

This equation follows from the definitions of the product set and the indicator. ◁

1.4.1 Dirac measure and counting measure

Example 1.52 [Dirac measure] Let $(\Omega, \mathscr{A})$ be a measurable space, let $\omega \in \Omega$, and consider the function $\delta_\omega: \mathscr{A} \to \{0, 1\}$ defined by

$$\delta_\omega(A) := 1_A(\omega), \quad \forall\, A \in \mathscr{A}. \tag{1.39}$$

Then δ_ω is a measure on $(\Omega, \mathscr{A})$ (see Exercise 1.20). ◁

Definition 1.53 [Dirac measure]
The function δ_ω defined by Equation (1.39) is called the D i r a c m e a s u r e a t (p o i n t) ω.

Example 1.54 [Counting measure] Let $(\Omega, \mathscr{A})$ be a measurable space, and define the function $\mu_\#: \mathscr{A} \to \overline{\mathbb{R}}$ by

$$\mu_\#(A) := \begin{cases} \sum_{\omega \in \Omega} 1_A(\omega), & \text{if } A \text{ is finite} \\ \infty, & \text{if } A \text{ is infinite,} \end{cases} \quad \forall\, A \in \mathscr{A}. \tag{1.40}$$

Then $\mu_\#$ is a measure on $(\Omega, \mathscr{A})$ (see Exercise 1.21). ◁

Definition 1.55 [Counting measure]
The function $\mu_\#$ defined by Equation (1.40) is called the c o u n t i n g m e a s u r e *on* $(\Omega, \mathscr{A})$.

Remark 1.56 [Cardinality of a set] If A is finite, then $\mu_\#(A)$ is called the *cardinality* of A, that is, $\mu_\#(A)$ simply counts the number of elements ω of the set A. Furthermore, for finite or countable Ω and $A \subset \Omega$,

$$\mu_\#(A) = \sum_{\omega \in \Omega} 1_A(\omega) = \sum_{\omega \in \Omega} \delta_\omega(A). \tag{1.41}$$

◁

Example 1.57 [Sum of Dirac measures] Let $(\Omega, \mathscr{A})$ be a measurable space. If $B \subset \Omega$ is finite or countable and δ_ω is the Dirac measure on $(\Omega, \mathscr{A})$ at point ω, then $\sum_{\omega \in B} \delta_\omega$: $\mathscr{A} \to [0, \infty]$ defined by

$$\left(\sum_{\omega \in B} \delta_\omega\right)(A) := \sum_{\omega \in B} \delta_\omega(A), \quad \forall\, A \in \mathscr{A}, \tag{1.42}$$

is a measure on $(\Omega, \mathscr{A})$ (see Exercise 1.22). Hence, if Ω itself is finite or countable, then $\sum_{\omega \in \Omega} \delta_\omega$ is a measure on $(\Omega, \mathscr{A})$, and it is identical to the counting measure defined in Example 1.54, because, for $A \in \mathscr{A}$,

$$\begin{aligned} \left(\sum_{\omega \in \Omega} \delta_\omega\right)(A) &= \sum_{\omega \in \Omega} \delta_\omega(A) && [(1.42)] \\ &= \sum_{\omega \in \Omega} 1_A(\omega) && [(1.39)] \\ &= \mu_\#(A). && [(1.41)] \end{aligned} \tag{1.43}$$

◁

1.4.2 Lebesgue measure

Consider the *half-open interval* $]a, b]$. Then,

$$\lambda_1(]a, b]) = b - a \tag{1.44}$$

is the *length* of the interval $]a, b]$. Next consider a *rectangle* $]a_1, b_1] \times]a_2, b_2]$ in $\mathbb{R}^2$ with $a_1 < b_1$ and $a_2 < b_2$. This set can be visualized by the set of all points inside the rectangle presented in Figure 1.3 (excluding the lower and left boundary). Obviously,

$$\lambda_2(]a_1, b_1] \times]a_2, b_2]) = (b_1 - a_1) \cdot (b_2 - a_2) \tag{1.45}$$

is the *area* of this rectangle.

According to Theorem 1.58, there is one and only one measure on $(\mathbb{R}, \mathscr{B})$ satisfying (1.44) for all such intervals. This measure is called the *Lebesgue measure on* $(\mathbb{R}, \mathscr{B})$ and is denoted by λ or λ_1. Similarly, there is one and only one measure on $(\mathbb{R}^2, \mathscr{B}_2)$ satisfying (1.45) for all such rectangles. It is called the *Lebesgue measure on* $(\mathbb{R}^2, \mathscr{B}_2)$ and is denoted by λ_2. Theorem 1.58 deals with the general case.

Theorem 1.58 [Existence and uniqueness of the Lebesgue measure]
For all $n \in \mathbb{N}$, there is a uniquely defined measure λ_n on $(\mathbb{R}^n, \mathscr{B}_n)$ satisfying

$$\lambda_n(]a_1, b_1] \times \ldots \times]a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i), \quad \forall\, a_i, b_i \in \mathbb{R} \text{ with } a_i < b_i, i = 1, \ldots, n. \tag{1.46}$$

For a proof, see Klenke (2013, Th. 1.55).

Definition 1.59 [Lebesgue measure]
The measure λ_n satisfying Equation (1.46) *is called the* Lebesgue measure *on* $(\mathbb{R}^n, \mathscr{B}_n)$.

Remark 1.60 [Sets of real numbers that are not Lebesgue measurable] Hence, the Lebesgue measure λ_n is defined on $(\mathbb{R}^n, \mathscr{B}_n)$. Note, however, that this measure space $(\mathbb{R}^n, \mathscr{B}_n, \lambda_n)$ can be completed by additionally including all subsets of sets $A \in \mathscr{B}_n$ with $\lambda_n(A) = 0$. In Wise and Hall (1993, counterexample 1.25), it is shown for $n = 1$ that there are subsets $B \subset \mathbb{R}$ that are not elements of the completed σ-algebra. Therefore, $B \notin \mathscr{B}$, and this implies $\mathscr{B} \neq \mathscr{P}(\mathbb{R})$. ⊲

1.4.3 Other examples of a measure

Example 1.61 [Restriction of a measure to a sub-σ-algebra] Suppose $(\Omega, \mathscr{A}, \mu)$ is a measure space and $\mathscr{C} \subset \mathscr{A}$ a σ-algebra. Then the function $\nu\colon \mathscr{C} \to \overline{\mathbb{R}}$ defined by

$$\nu(A) := \mu(A), \quad \forall\, A \in \mathscr{C}, \tag{1.47}$$

is a measure on $(\Omega, \mathscr{C})$ (see Exercise 1.23). ⊲

Example 1.62 [Weighted sum of measures] If $\mu_1, \mu_2, \ldots$ are measures on $(\Omega, \mathscr{A})$ and $0 \leq \alpha_1, \alpha_2, \ldots \in \mathbb{R}$, then $\sum_{i=1}^{\infty} \alpha_i \mu_i\colon \mathscr{A} \to [0, \infty]$ defined by

$$\left(\sum_{i=1}^{\infty} \alpha_i \mu_i\right)(A) := \sum_{i=1}^{\infty} \alpha_i \mu_i(A), \quad \forall\, A \in \mathscr{A}, \tag{1.48}$$

is again a measure on $(\Omega, \mathscr{A})$ (see Exercise 1.24). For $0 = \alpha_{n+1} = \alpha_{n+2} = \ldots$ this implies: If $\mu_1, \ldots, \mu_n$ are measures on $(\Omega, \mathscr{A})$ and $\alpha_1, \ldots, \alpha_n$ are nonnegative, then the function $\sum_{i=1}^{n} \alpha_i \mu_i$ defined by

$$\left(\sum_{i=1}^{n} \alpha_i \mu_i\right)(A) := \sum_{i=1}^{n} \alpha_i \mu_i(A), \quad \forall\, A \in \mathscr{A}, \tag{1.49}$$

is also a measure on $(\Omega, \mathscr{A})$. ⊲

1.4.4 Finite and σ-finite measures

A measure μ on a measurable space $(\Omega, \mathscr{A})$ is called *finite* if $\mu(\Omega) < \infty$. Otherwise, it is called *infinite*. Within the class of infinite measures, there is a subclass with an important property, called σ-*finiteness*. Many fundamental propositions of measure and integration theory only hold for measures that are σ-finite.

Definition 1.63 [σ-Finite measure]
Let μ be a measure on a measurable space $(\Omega, \mathscr{A})$. Then μ is called σ-finite if there is a sequence $A_1, A_2, \ldots \in \mathscr{A}$ with $\bigcup_{i=1}^{\infty} A_i = \Omega$ and, for all $i = 1, 2, \ldots$, $\mu(A_i) < \infty$.

To emphasize, even if $\mu(\Omega) = \infty$, the measure μ can be σ-finite (see Examples 1.64 and 1.65). Note that any finite measure is also σ-finite.

Example 1.64 [σ-Finiteness of the Lebesgue measure] The Lebesgue measure λ on $(\mathbb{R}, \mathscr{B})$ is σ-finite, because $\mathbb{R} = \bigcup_{i=1}^{\infty}[-i, i]$ and $\lambda([-i, i]) = 2 \cdot i < \infty$, for all $i \in \mathbb{N}$. ⊲

Example 1.65 [A σ-finite counting measure] Consider the measurable space $(\mathbb{R}, \mathscr{B})$ and the measure $\mu\colon \mathscr{B} \to [0, \infty]$, where $\mu = \sum_{i=0}^{\infty} \delta_i$ and δ_i denotes the Dirac measure at i on $(\mathbb{R}, \mathscr{B})$ with $\delta_i(A) = 1_A(i)$, $A \in \mathscr{B}$, $i \in \mathbb{N}_0$ (see Example 1.57). Then μ is σ-finite because $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n, n]$ and $\mu([-n, n]) = n + 1$, for all $n \in \mathbb{N}_0$. This measure simply counts the number of elements $i \in \mathbb{N}_0$ in a Borel set A. In other words, for all finite $A \in \mathscr{B}$, $\mu(A)$ is the cardinality of the set $A \cap \mathbb{N}_0$. ⊲

1.4.5 Product measure

In section 1.4.2, we considered the Lebesgue measure on $(\mathbb{R}^n, \mathscr{B}_n)$ that is specified for n-dimensional cuboids by Equation (1.46) using the *product* of one-dimensional Lebesgue measures on $(\mathbb{R}, \mathscr{B})$. Now we introduce the general concept of a product measure. Lemma 1.66 shows that σ-finiteness of measures is sufficient for the existence and uniqueness of such a measure. Hence, this lemma shows that presuming finite measures is sufficient but not necessary for the definition of the product measure.

Lemma 1.66 [Existence and uniqueness]
Let $(\Omega_i, \mathscr{A}_i, \mu_i)$ be measure spaces with σ-finite measures μ_i, $i = 1, \ldots, n$. Then there is a uniquely defined measure, denoted $\mu_1 \otimes \ldots \otimes \mu_n$, on the product space

$$\left(\bigtimes_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathscr{A}_i \right),$$

satisfying

$$\begin{aligned} &\forall (A_1, \ldots, A_n) \in \mathscr{A}_1 \times \ldots \times \mathscr{A}_n: \\ &\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \cdot \ldots \cdot \mu_n(A_n). \end{aligned} \tag{1.50}$$

This measure is σ-finite as well.

For a proof, see Bauer (2001, Th. 23.9). Hence, $\mu := \mu_1 \otimes \ldots \otimes \mu_n$ is a measure on the product space $\left(\times_{i=1}^{n} \Omega_i, \bigotimes_{i=1}^{n} \mathscr{A}_i\right)$ with

$$\mu(A_1 \times \ldots \times A_n) := \mu_1(A_1) \cdot \ldots \cdot \mu_n(A_n), \quad \forall\, (A_1, \ldots, A_n) \in (\mathscr{A}_1 \times \ldots \times \mathscr{A}_n). \tag{1.51}$$

Definition 1.67 [Product measure]
The measure $\mu_1 \otimes \ldots \otimes \mu_n$ defined by Equation (1.50) is called the product measure *of $\mu_1, \ldots, \mu_n$.*

1.5 Continuity of a measure

The term σ-additivity refers to *countable* unions of pairwise disjoint sets and it implies finite additivity, which involves *finite* unions of pairwise disjoint sets. Furthermore, σ-additivity implies the following continuity properties of a measure, which are essential for the definition of the integral (see ch. 3).

Theorem 1.68 [Continuity of a measure]
Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, and let $A_1, A_2, \ldots \in \mathscr{A}$.

(i) If $A_1 \subset A_2 \subset \ldots$, then,

$$\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right). \qquad \textit{(continuity from below)}$$

(ii) If $A_1 \supset A_2 \supset \ldots$ and there is an $n \in \mathbb{N}$ with $\mu(A_n) < \infty$, then,

$$\lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right). \qquad \textit{(continuity from above)}$$

For a proof, see Klenke (2013, Theorem 1.36).

Remark 1.69 [Finite case] If $A_1, \ldots, A_n \in \mathscr{A}$ is a finite sequence with $A_1 \subset \ldots \subset A_n$, then $\bigcup_{i=1}^{n} A_i = A_n$ and

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \mu(A_n). \tag{1.52}$$

This is a trivial case of Theorem 1.68 (i) (with $A_n = A_{n+1} = A_{n+2} = \ldots$). ◁

Example 1.70 [Geometric examples] Figures 1.4 and 1.5 illustrate this theorem for the Lebesgue measure λ_2 on $(\mathbb{R}^2, \mathscr{B}_2)$, the *area* of a set O and the sets A_i, $i \in \mathbb{N}$. In this example, A_1 is the open rectangle in the open (i.e., the set without its boundary) egg-shaped set O displayed in Figure 1.4, A_2 the union of A_1 with two other rectangles in the middle

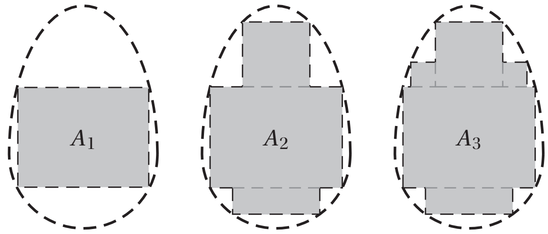

Figure 1.4 Approximation of an open egg-shaped set O from below.

figure, and A_3 the union of A_2 with two additional rectangles in the right figure. Adding more and more rectangles, it is plausible that $A_1 \subset A_2 \subset \dots \subset O$ and that their union approximates O (i.e., $\bigcup_{i=1}^{\infty} A_i = O$). Under these premises, Theorem 1.68 (i) yields the conclusion $\lim_{i\to\infty} \lambda_2(A_i) = \lambda_2\left(\bigcup_{i=1}^{\infty} A_i\right) = \lambda_2(O)$. Figure 1.5 illustrates the same principle. However, now the area of the egg-shaped set O is approximated from above by subtracting the areas of appropriate rectangles.

As a second example, consider the Lebesgue measure λ on $(\mathbb{R}, \mathscr{B})$ and the intervals $A_i =]x - \frac{1}{i}, x]$, $i \in \mathbb{N}$. Obviously, $A_1 \supset A_2 \supset \dots$ and $\lambda(A_i) = \frac{1}{i} < \infty$, for all $i \in \mathbb{N}$ (see also Exercise 1.12). Hence, for all $x \in \mathbb{R}$,

$$\lambda(\{x\}) = \lambda\left(\bigcap_{i=1}^{\infty}]x - \frac{1}{i}, x]\right) = \lim_{i\to\infty} \lambda\left(]x - \frac{1}{i}, x]\right) = \lim_{i\to\infty} \frac{1}{i} = 0. \tag{1.53}$$

This is an implication of continuity from above, and it implies

$$\begin{aligned} \forall\, a, b \in \mathbb{R}\colon a < b \;\Rightarrow\; \lambda(]a, b]) = \lambda([a, b]) &= \lambda([a, b[) = \lambda(]a, b[) \\ &= b - a. \end{aligned} \tag{1.54}$$

⊲

Remark 1.71 [A motivation for σ-additivity] As already mentioned in Remark 1.45, σ-additivity refers to unions of finitely or countably many sets. Now consider $\bigcup_{1\le x\le 2}\{x\} = [1, 2] \in \mathscr{B}$ [see Eq. (1.9)]. According to Equation (1.53), $\lambda(\{x\}) = 0$, for all $x \in [1, 2]$, and hence $\lambda(\{x \in [1, 2]\colon x \in \mathbb{Q}\}) = 0$, because the set of rational numbers is countable. In other words, the Lebesgue measure λ of the set of all rational numbers in the closed interval $[1, 2]$ is zero, and this is not a contradiction to

$$\lambda\left(\bigcup_{1\le x\le 2} \{x\}\right) = \lambda([1, 2]) = 2 - 1 = 1,$$

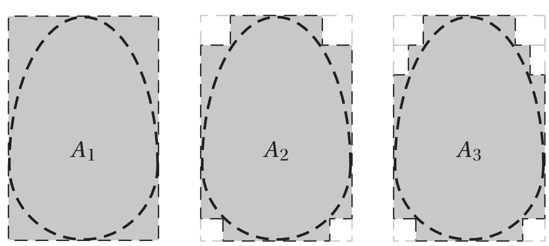

Figure 1.5 Approximation of an open egg-shaped set O from above.

because $\bigcup_{1 \le x \le 2} \{x\}$ is an uncountable union. This illustrates that additivity for uncountable unions can be meaningless. ⊲

1.6 Specifying a measure via a generating system

Given a measurable space $(\Omega, \mathscr{A})$, a measure is a function that is defined on $\mathscr{A}$. In many situations, such as when $\mathscr{A} = \sigma(\mathscr{E})$ can only be described by a generating set system $\mathscr{E}$ (e.g., the set system $\mathscr{I}_1$ generating the Borel σ-algebra on $\mathbb{R}$), it is important to answer the following questions:

(a) *Existence*: If there is a set function $\tilde{\mu}: \mathscr{E} \to \overline{\mathbb{R}}$, is there also a measure $\mu: \sigma(\mathscr{E}) \to \overline{\mathbb{R}}$ such that $\mu(A) = \tilde{\mu}(A), \forall A \in \mathscr{E}$?

(b) *Uniqueness*: Is a measure μ on $(\Omega, \sigma(\mathscr{E}))$ already uniquely defined by its values $\mu(A)$, $A \in \mathscr{E}$?

(Sufficient conditions for the existence of such a measure μ are formulated in Klenke, 2013, Theorem 1.53.)

The following uniqueness theorem for finite measures provides an answer to these questions, which suffices for our purposes. (A more general formulation for σ-finite measures with additional assumptions and a proof of Theorem 1.72 is found in Klenke, 2013, Lemma 1.42.)

Theorem 1.72 [Generating system and uniqueness of a measure]
Let $(\Omega, \mathscr{A})$ be a measurable space and let $\mathscr{E} \subset \mathscr{A}$, where $\mathscr{E}$ is ∩-stable and $\sigma(\mathscr{E}) = \mathscr{A}$. If μ_1 and μ_2 are finite measures on $(\Omega, \mathscr{A})$ (i.e., measures with $\mu_1(\Omega), \mu_2(\Omega) < \infty$), then,

$$\forall A \in \mathscr{E}: \mu_1(A) = \mu_2(A) \;\Rightarrow\; \forall A \in \mathscr{A}: \mu_1(A) = \mu_2(A).$$

Example 1.73 [Countable Ω] Let Ω be a nonempty finite or countable set, and let $\mathscr{A} = \mathscr{P}(\Omega)$. Then the set system

$$\mathscr{E}_1 = \{\emptyset\} \cup \{\{\omega\}: \omega \in \Omega\}$$

is ∩-stable and $\sigma(\mathscr{E}_1) = \mathscr{A}$. As already noted in Remark 1.48, a finite measure μ on $(\Omega, \mathscr{A})$ is uniquely defined by its values $\mu(\{\omega\}), \omega \in \Omega$. ⊲

Example 1.74 [Measures on $(\mathbb{R}, \mathscr{B})$] The set system

$$\mathscr{E}_2 = \{]a, b]: a < b, a, b \in \mathbb{R}\} \cup \{\emptyset\}$$

is ∩-stable and $\sigma(\mathscr{E}_2) = \mathscr{B}$ [see Eq. (1.18) and section 1.2.4]. Another ∩-stable set system $\mathscr{E}_3$ with $\sigma(\mathscr{E}_3) = \mathscr{B}$ is

$$\mathscr{E}_3 = \{]-\infty, b]: b \in \mathbb{R}\}$$

(cf. Klenke, 2013). This set system is crucial for the definition of a cumulative distribution function (see section 5.7.1). ⊲

1.7 σ-Algebra that is trivial with respect to a measure

All σ-algebras treated in section 1.2 have been defined without reference to a measure. Now we define the concept of a *trivial* σ-*algebra*, which is defined referring to a measure. We start with a lemma about the set of all subsets of a set Ω with $\mu(A) = 0$ or $\mu(A) = \mu(\Omega)$ (i.e., the set of all sets that are *trivial* with respect to the measure μ). Hence, the set of μ-trivial sets includes all *null sets* that is, all sets $A \in \mathscr{A}$ with $\mu(A) = 0$, and all sets $A \in \mathscr{A}$ with $\mu(A) = \mu(\Omega)$.

Lemma 1.75 [The set of all trivial sets is a σ-algebra]
Let $(\Omega, \mathscr{A}, \mu)$ *be a measure space, and assume that* μ *is finite. Then,*

$$\mathscr{T}_\mu := \{A \in \mathscr{A}\colon \mu(A) = 0 \text{ or } \mu(A) = \mu(\Omega)\} \tag{1.55}$$

is a σ-*algebra.*

(Proof p. 30)

This lemma allows for Definition 1.76:

Definition 1.76 [Trivial σ-algebra with respect to a measure]
Let $(\Omega, \mathscr{A}, \mu)$ *be a measure space, assume that* μ *is finite, and let* $\mathscr{T}_\mu$ *be defined by (1.55). Then each* σ-*algebra* $\mathscr{C} \subset \mathscr{T}_\mu$ *is called a* μ-*trivial* σ-*algebra and its elements* μ-*trivial sets.*

Obviously, $\{\Omega, \emptyset\}$ is a trivial σ-algebra with respect to all measures on $(\Omega, \mathscr{A})$. Hence, we can call it a *trivial* σ-*algebra* without reference to a specified measure.

1.8 Proofs

Proof of Theorem 1.12

(a)

$$\begin{aligned} \forall\, i \in I\colon \mathscr{A}_i \text{ is a σ-algebra on } \Omega \;&\Rightarrow\; \forall\, i \in I\colon \Omega \in \mathscr{A}_i \qquad &&[\text{Def. 1.1 (a)}] \\ &\Rightarrow\; \Omega \in \bigcap_{i \in I} \mathscr{A}_i. \end{aligned}$$

(b)

$$\begin{aligned} A \in \bigcap_{i \in I} \mathscr{A}_i \;&\Rightarrow\; \forall\, i \in I\colon A \in \mathscr{A}_i \\ &\Rightarrow\; \forall\, i \in I\colon A^c \in \mathscr{A}_i \qquad &&[\text{Def. 1.1 (b)}] \\ &\Rightarrow\; A^c \in \bigcap_{i \in I} \mathscr{A}_i. \end{aligned}$$

(c)

$$\begin{aligned} A_1, A_2, \ldots \in \bigcap_{i \in I} \mathscr{A}_i &\Rightarrow \ \forall\, i \in I\colon A_1, A_2, \ldots \in \mathscr{A}_i \\ &\Rightarrow \ \forall\, i \in I\colon \bigcup_{j=1}^{\infty} A_j \in \mathscr{A}_i \qquad \text{[Def. 1.1 (c)]} \\ &\Rightarrow \ \bigcup_{j=1}^{\infty} A_j \in \bigcap_{i \in I} \mathscr{A}_i. \end{aligned}$$

Proof of Lemma 1.15

If $\mathscr{C}$ is a σ-algebra with $\mathscr{E} \subset \mathscr{C}$ and $\mathscr{A} = \sigma(\mathscr{E})$, then (1.11) and the assumption $\mathscr{C} \subset \mathscr{A}$ imply $\mathscr{A} = \sigma(\mathscr{E}) \subset \mathscr{C} \subset \mathscr{A}$. Hence, $\mathscr{C} = \mathscr{A}$.

Proof of Lemma 1.20

Define $\mathscr{D} := \left\{C = \bigcup_{i \in I(C)} B_i\colon I(C) \subset \mathbb{N}\right\}$.

$\mathscr{E} \subset \mathscr{D}$: For $B_j \in \mathscr{E}$, choose $I(B_j) = \{j\}$. Then, $B_j = \bigcup_{i \in I(B_j)} B_i$.

$\mathscr{D} \subset \sigma(\mathscr{E})$: Because $\mathbb{N}$ is countable, any $I(C) \subset \mathbb{N}$ is finite or countable, and this implies that $C = \bigcup_{i \in C} B_i$ is an element of $\sigma(\mathscr{E})$ [see Def. 1.1 (c), (1.3)].

Checking the three conditions defining a σ-algebra (see Def. 1.1), we show that $\mathscr{D}$ is a σ-algebra.

(a)

$$\Omega = \begin{cases} \bigcup_{i=1}^{n} B_i, & \text{if } \mathscr{E} = \{B_1, \ldots, B_n\} \\ \bigcup_{i=1}^{\infty} B_i, & \text{if } \mathscr{E} = \{B_1, B_2, \ldots\}, \end{cases}$$

because $\mathscr{E}$ is assumed to be a partition. This shows that $\Omega \in \mathscr{D}$.

(b) The equation for Ω in (a) also implies $I(C^c) = I(C)^c$. Therefore, $C^c \in \mathscr{D}$ if $C \in \mathscr{D}$.

(c) If $C_1, C_2, \ldots \in \mathscr{D}$, then,

$$\bigcup_{j=1}^{\infty} C_j = \bigcup_{j=1}^{\infty} \ \bigcup_{i \in I(C_j)} B_i = \bigcup_{i \in \bigcup_{j=1}^{\infty} I(C_j)} B_i \in \mathscr{D},$$

because $\bigcup_{j=1}^{\infty} I(C_j) \subset \mathbb{N}$.

Finally, we prove the second equation in (1.14). If $j \in I(C)$ and $C = \bigcup_{i \in I(C)} B_i$, then $B_j \subset C$, which implies

$$\bigcup_{i \in I(C)} B_i \subset \bigcup_{B_i \subset C} B_i.$$

Vice versa, if $B_j \subset C$, then $j \in I(C)$, because for any $\omega \in B_j$, there is no $i \neq j$ such that $\omega \in B_i$ [see condition (b) of Rem. 1.19]. Hence,

$$\bigcup_{B_i \subset C} B_i \subset \bigcup_{i \in I(C)} B_i,$$

which proves the second equation in (1.14).

Proof of Lemma 1.27

In this proof, we use $\sigma_\Omega(\mathscr{E})$ to denote the σ-algebra on Ω generated by $\mathscr{E} \subset \mathscr{P}(\Omega)$. Similarly, $\sigma_A(\mathscr{D})$ denotes the σ-algebra on A generated by $\mathscr{D} \subset \mathscr{P}(A)$.

(1.15) $\sigma_\Omega(\mathscr{E})$ is a σ-algebra on Ω and $\mathscr{E} \subset \sigma_\Omega(\mathscr{E})$, by definition of $\sigma_\Omega(\mathscr{E})$. Hence, $\mathscr{E}|_A \subset \sigma_\Omega(\mathscr{E})|_A$, and $\sigma_\Omega(\mathscr{E})|_A$ is a σ-algebra on A (see Exercise 1.5). Therefore, the definition (1.10) yields

$$\sigma_A(\mathscr{E}|_A) \subset \sigma_\Omega(\mathscr{E})|_A.$$

Furthermore, $\mathscr{E} \subset \sigma_\Omega(\mathscr{E}|_A \cup \mathscr{E}|_{A^c})$, which implies

$$\begin{aligned}
\sigma_\Omega(\mathscr{E}) &\subset \sigma_\Omega(\mathscr{E}|_A \cup \mathscr{E}|_{A^c}) && \text{[Rem. 1.23]}\\
&\subset \sigma_\Omega(\sigma_A(\mathscr{E}|_A) \cup \sigma_{A^c}(\mathscr{E}|_{A^c})) && \text{[Rem. 1.23]}\\
&= \{C \cup D\colon C \in \sigma_A(\mathscr{E}|_A),\ D \in \sigma_{A^c}(\mathscr{E}|_{A^c})\}. && \text{[This set system is a σ-algebra]}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sigma_\Omega(\mathscr{E})|_A &\subset \{C \cup D\colon C \in \sigma_A(\mathscr{E}|_A),\ D \in \sigma_{A^c}(\mathscr{E}|_{A^c})\}|_A\\
&= \{(C \cup D) \cap A\colon C \in \sigma_A(\mathscr{E}|_A),\ D \in \sigma_{A^c}(\mathscr{E}|_{A^c})\}\\
&= \{C \cap A\colon C \in \sigma_A(\mathscr{E}|_A)\} && [D \subset A^c]\\
&= \sigma_A(\mathscr{E}|_A). && [C \subset A]
\end{aligned}$$

Hence, we have shown $\sigma_A(\mathscr{E}|_A) \subset \sigma_\Omega(\mathscr{E})|_A$ and $\sigma_\Omega(\mathscr{E})|_A \subset \sigma_A(\mathscr{E}|_A)$, which is equivalent to $\sigma_A(\mathscr{E}|_A) = \sigma_\Omega(\mathscr{E})|_A$.

(1.17)

$$\begin{aligned}
\sigma_\Omega(\mathscr{C} \cup \mathscr{E})|_A &= \sigma_A(\mathscr{C} \cup \mathscr{E}|_A) && \text{[(1.15)]}\\
&= \sigma_A(\mathscr{C}|_A \cup \mathscr{E}|_A) && \text{[See def. of the trace in Example 1.10]}\\
&= \sigma_A(\mathscr{C}|_A \cup \{\emptyset, A\}) && \text{[(1.16)]}\\
&= \sigma_A(\mathscr{C}|_A) && [\{\emptyset, A\} \subset \mathscr{C}|_A]\\
&= \mathscr{C}|_A. && \text{[Exercise 1.5, (1.12)]}
\end{aligned}$$

Proof of Lemma 1.75

(a) $\Omega \in \mathscr{T}_\mu$ by definition of $\mathscr{T}_\mu$.

(b) If $A \in \mathscr{T}_\mu$, then Rules (iv), (v) of Box 1.1 and finiteness of μ yield

$$\mu(A^c) = \mu(\Omega) - \mu(A) = \begin{cases} \mu(\Omega), & \text{if } \mu(A) = 0\\ 0, & \text{if } \mu(A) = \mu(\Omega), \end{cases}$$

which implies $A^c \in \mathscr{T}_\mu$.

(c) Let $A_1, A_2, \ldots \in \mathscr{A}$. We consider two cases. *First*, if $\mu(A_i) = 0$, for all A_i, $i \in \mathbb{N}$, then Rule (xi) of Box 1.1 yields $\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) = 0$ (i.e., $\bigcup_{i=1}^{\infty} A_i \in \mathscr{T}_\mu$). *Second*, if there is a $j \in \mathbb{N}$ such that $\mu(A_j) = \mu(\Omega)$, then Rule (v) of Box 1.1 yields

$$\mu(\Omega) = \mu(A_j) \leq \mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \mu(\Omega),$$

which implies $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu(\Omega)$. Therefore, $\bigcup_{i=1}^{\infty} A_i \in \mathscr{T}_\mu$.

Exercises

1.1 Let $\mathscr{A}$ be a σ-algebra of subsets of a nonempty set Ω, and let $A_1, A_2, \ldots \in \mathscr{A}$. Show: (a) $A_1 \cap A_2 \cap \ldots \in \mathscr{A}$, (b) $A_1 \cap A_2 \in \mathscr{A}$, and (c) $A_1 \setminus A_2 \in \mathscr{A}$.

1.2 Show that the set system $\mathscr{A} = \{\Omega, \emptyset, A, A^c\}$ is stable (closed) with respect to union of elements of $\mathscr{A}$.

1.3 Consider the set $\Omega = \{\omega_1, \ldots, \omega_6\}$ representing the set of all possible outcomes of tossing a dice and the power set $\mathscr{P}(\Omega)$, which, in probability theory, represents the set of all possible events (including the 'impossible' event Ø) in this random experiment. Specify the σ-algebra on Ω that represents all possible events if we only distinguish between even and uneven number of points.

1.4 Consider the random experiment that has been described in Example 1.9. Aside from the power set of Ω, we already considered the σ-algebras $\mathscr{A}_1 = \{\Omega, \emptyset, A, A^c\}$, $\mathscr{A}_2 = \{\Omega, \emptyset, B, B^c\}$, and $\mathscr{A}_3 = \{\Omega, \emptyset, C, C^c\}$. Define another σ-algebra not yet mentioned.

1.5 Prove: If $\mathscr{A}$ is a σ-algebra on Ω and $\Omega_0 \subset \Omega$, then $\mathscr{A}|_{\Omega_0} = \{\Omega_0 \cap A\colon\ A \in \mathscr{A}\}$ is a σ-algebra on Ω_0.

1.6 Prove the proposition of Remark 1.16.

1.7 Show that $\sigma(\mathscr{E}) = \mathscr{P}(\Omega)$ if Ω is finite or countable and $\mathscr{E} := \{\{\omega\}\colon\ \omega \in \Omega\}$.

1.8 Prove the proposition of Remark 1.21.

1.9 Let $\mathscr{E}_1, \mathscr{E}_2$ be set systems on Ω with $\mathscr{E}_1 \subset \mathscr{E}_2$. Show that $\sigma(\mathscr{E}_1) \subset \sigma(\mathscr{E}_2)$.

1.10 Prove propositions (a) and (b) of Example 1.25.

1.11 Prove Equation (1.21).

1.12 Show that $\{x\} \in \mathscr{B}_n$ for all $x \in \mathbb{R}^n$, where $\mathscr{B}_n$ is the Borel σ-algebra on $\mathbb{R}^n$.

1.13 Let $\mathscr{B}$ be the Borel σ-algebra on $\mathbb{R}$, and let $\Omega_0 \subset \mathbb{R}$ be finite or countable. Show that $\mathscr{B}|_{\Omega_0} = \mathscr{P}(\Omega_0)$.

1.14 Prove the proposition of Example 1.34.

1.15 Prove the proposition of Remark 1.35.

1.16 Let $(\Omega_i, \mathscr{A}_i)$, $i = 1, \ldots, n$, be measurable spaces. Show that the set system $\mathscr{E} := \{A_1 \times \ldots \times A_n\colon A_i \in \mathscr{A}_i,\ i = 1, \ldots, n\}$ is ∩-stable.

1.17 Prove the proposition of Remark 1.46.

1.18 Prove the rules of Box 1.1.

1.19 Prove the propositions of Remark 1.47.

1.20 Show that $\delta_\omega\colon \mathscr{A} \to \{0, 1\}$ in Example 1.52 is a measure.

1.21 Prove that the function defined by Equation (1.40) is a measure on $(\Omega, \mathscr{A})$.

1.22 Show that $\sum_{\omega\in B} \delta_\omega$ in Example 1.57 is a measure.

1.23 Show that $\nu\colon \mathscr{C} \to \overline{\mathbb{R}}$ defined in Example 1.61 is a measure on $(\Omega, \mathscr{C})$.

1.24 Prove that the function $\sum_{i=1}^{\infty} \alpha_i\mu_i$ defined in Example 1.62 is a measure on $(\Omega, \mathscr{A})$.

Solutions

1.1 (a) If $A_1, A_2, \ldots \in \mathscr{A}$, then $A_1^c, A_2^c, \ldots \in \mathscr{A}$ [see Def. 1.1 (b)]. Hence,

$$\bigcap_{i=1}^{\infty} A_i = \left[\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right]^c = \left[\bigcup_{i=1}^{\infty} A_i^c\right]^c \qquad \text{[de Morgan]}$$
$$\in \mathscr{A}. \qquad \text{[Def. 1.1 (c), (b)]}$$

(b) Let $A_1, A_2 \in \mathscr{A}$ and choose $A_3, A_4, \ldots$ such that $\Omega = A_i$, for all $i \geq 3$, $i \in \mathbb{N}$. Then, according to Definition 1.1 (a),

$$A_1 \cap A_2 = A_1 \cap A_2 \cap \Omega = \bigcap_{i=1}^{\infty} A_i \in \mathscr{A}.$$

(c) $A_1 \setminus A_2 = A_1 \cap A_2^c \in \mathscr{A}$ [see (b) and Def. 1.1 (b)].

1.2 The unions $\Omega \cup A = \Omega$, $\Omega \cup A^c = \Omega$, and $\Omega \cup \emptyset = \Omega$ are all elements of $\mathscr{A}$, and the same is true for $\emptyset \cup A = A$, $\emptyset \cup A^c = A^c$, and $A \cup A^c = \Omega$. Furthermore, $B \cup B = B$ for all $B \in \mathscr{A}$.

1.3 The σ-algebra on Ω that only distinguishes between an even and uneven number of points is $\mathscr{A}_1 := \{\{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4, \omega_6\}, \Omega, \emptyset\}$. This is a sub-σ-algebra of $\mathscr{P}(\Omega)$. Therefore, $\mathscr{A}_1$ represents the set of all possible events of a random experiment that is, in a sense, contained in the original random experiment.

1.4 Consider the set system that contains as elements $A, A^c, B, B^c, \Omega, \emptyset$, all unions and all intersections of these sets as well as the unions and intersections of the resulting sets such as $(A^c \cup B^c) \cap (A \cup B)$ and $(A^c \cup B^c) \cup (A \cup B)$. Altogether, these are 16 sets. This is $\sigma(\mathscr{A}_1 \cup \mathscr{A}_2)$, the σ-algebra generated by $\mathscr{A}_1 \cup \mathscr{A}_2 = \{A, A^c, B, B^c, \Omega, \emptyset\}$ (see Def. 1.13 and Rem. 1.21).

1.5 (a) $\Omega_0 \cap \Omega = \Omega_0$. This implies $\Omega_0 \in \mathscr{A}|_{\Omega_0}$.

(b)

$$A^* \in \mathscr{A}|_{\Omega_0} \;\Rightarrow\; \exists\, A \in \mathscr{A}\colon\; A^* = \Omega_0 \cap A.$$

With this set A and using B^c for the complement of a set B with respect to Ω,

$$\begin{aligned} \Omega_0 \setminus A^* &= \Omega_0 \setminus (\Omega_0 \cap A) \\ &= \Omega_0 \cap (\Omega_0 \cap A)^c \\ &= \Omega_0 \cap (\Omega_0^c \cup A^c) \\ &= (\Omega_0 \cap \Omega_0^c) \cup (\Omega_0 \cap A^c) \\ &= \Omega_0 \cap A^c \in \mathscr{A}|_{\Omega_0}. \end{aligned}$$

(c)

$$A_1^*, A_2^*, \ldots \in \mathscr{A}|_{\Omega_0} \Rightarrow \exists A_1, A_2, \ldots \in \mathscr{A}\colon A_i^* = \Omega_0 \cap A_i, i \in \mathbb{N}.$$

Hence,

$$A_1^* \cup A_2^* \cup \ldots = (\Omega_0 \cap A_1) \cup (\Omega_0 \cap A_2) \cup \ldots = \Omega_0 \cap (A_1 \cup A_2 \cup \ldots) \in \mathscr{A}|_{\Omega_0}.$$

1.6 If $\mathscr{G}$ is a σ-algebra on Ω, then

$$\mathscr{E} \cup \mathscr{F} \subset \mathscr{G} \Leftrightarrow \sigma(\mathscr{E} \cup \mathscr{F}) \subset \mathscr{G}. \qquad [(1.11)] \tag{1.56}$$

Furthermore, for three sets A, B, C,

$$A \cup B \subset C \Leftrightarrow A \subset C \wedge B \subset C. \tag{1.57}$$

Hence,

$$\begin{aligned} \mathscr{D} \cup \mathscr{E} \cup \mathscr{F} \subset \mathscr{G} &\Leftrightarrow (\mathscr{D} \subset \mathscr{G}) \wedge (\mathscr{E} \cup \mathscr{F} \subset \mathscr{G}) && [(1.57)] \\ &\Leftrightarrow (\mathscr{D} \subset \mathscr{G}) \wedge (\sigma(\mathscr{E} \cup \mathscr{F}) \subset \mathscr{G}) && [(1.56)] \\ &\Leftrightarrow \mathscr{D} \cup \sigma(\mathscr{E} \cup \mathscr{F}) \subset \mathscr{G}. && [(1.57)] \end{aligned}$$

Now Definition 1.13 yields the proposition.

1.7 If Ω is finite or countable, then each of its subsets A is finite or countable as well. Therefore,

$$\forall A \subset \Omega\colon A = \bigcup_{\omega \in A} \{\omega\} \in \sigma(\mathscr{E}). \qquad [\text{Def. 1.1 (c), Rem. 1.2}]$$

Because each element A of $\mathscr{P}(\Omega)$ is a union $\bigcup_{\omega \in A}\{\omega\}$ of singletons $\{\omega\}$, $\omega \in A$, this implies $\mathscr{P}(\Omega) \subset \sigma(\mathscr{E})$. Hence, $\mathscr{E} \subset \mathscr{P}(\Omega) \subset \sigma(\mathscr{E})$. Therefore, Lemma 1.15 implies $\sigma(\mathscr{E}) = \mathscr{P}(\Omega)$.

1.8 Suppose that $\mathscr{E} = \{A_1, \ldots, A_m\}$ and $A_j^1 := A_j$ and let A_j^c denote the complement of A_j. Then, for all $(k_1, \ldots, k_m) \in \{1, c\}^m$ define

$$B_{(k_1,\ldots,k_m)} := \bigcap_{j=1}^{m} A_j^{k_j}.$$

Then

$$\mathscr{F} := \{B_{(k_1,\dots,k_m)}: (k_1, \dots, k_m) \in \{1, c\}^m, B_{(k_1,\dots,k_m)} \neq \emptyset\}$$

is a finite partition of Ω. Note that $\mathscr{F}$ contains all nonempty intersections of sets A_j or their complements, respectively, where $j = 1, \dots, m$. Now Lemma 1.20 implies the proposition.

1.9 If $\mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{P}(\Omega)$, then for any σ-algebra $\mathscr{A}$ on Ω with $\mathscr{E}_2 \subset \mathscr{A}$ also $\mathscr{E}_1 \subset \mathscr{A}$. Remember, if $J \subset I$, then $\bigcap_{i \in I} B_i \subset \bigcap_{i \in J} B_i$, for any sets B_i, $i \in I$. Therefore, $\sigma(\mathscr{E}_1)$, which is the intersection of all σ-algebras containing $\mathscr{E}_1$, is a subset of the intersection of all σ-algebras containing $\mathscr{E}_2$, which is $\sigma(\mathscr{E}_2)$.

1.10 (a) If Ω is finite, then $\mathscr{P}(\Omega)$ is a finite set system. Therefore, each σ-algebra $\mathscr{A}$ on Ω is a finite set system. Because $\mathscr{A} = \sigma(\mathscr{A})$, this σ-algebra is countably generated.

(b) The set $\mathbb{N}_0$ is countable and therefore also $\mathbb{N}_0^n$ for $n \in \mathbb{N}$. Example 1.18 then implies that $\mathscr{P}(\mathbb{N}_0^n)$ is countably generated.

1.11 Let $\mathscr{H}_n = \{]-\infty, b_1] \times \dots \times]-\infty, b_n]: b_1, \dots, b_n \in \mathbb{R}\}$.

(i) For all $(b_1, \dots, b_n) \in \mathbb{R}^n$ and all $m \in \mathbb{N}$ with $m < b_i$, $i = 1, \dots, n$,

$$B_m :=]-m, b_1] \times \dots \times]-m, b_n] \in \mathscr{I}_n.$$

According to Definition 1.1 (c) this implies

$$\bigcup_{\substack{m \in \mathbb{N} \\ m < b_i,\, i = 1, \dots, n}} B_m =]-\infty, b_1] \times \dots \times]-\infty, b_n] \in \sigma(\mathscr{I}_n).$$

Hence, $\mathscr{H}_n \subset \sigma(\mathscr{I}_n)$, which, according to (1.11) and (1.12), implies

$$\sigma(\mathscr{H}_n) \subset \sigma(\mathscr{I}_n) = \mathscr{B}_n.$$

(ii) For all $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$, with $a_i < b_i$, $i = 1, \dots, n$,

$$]a_1, b_1] \times \dots \times]a_n, b_n] =]-\infty, b_1] \times \dots \times]-\infty, b_n] \setminus \left(\bigcup_{j=1}^{n} H_j \right),$$

where $H_j :=]-\infty, b_1] \times \dots \times]-\infty, b_{j-1}] \times]-\infty, a_j] \times]-\infty, b_{j+1}] \times \dots \times]-\infty, b_n]$. Hence, according to Remark 1.2, $]a_1, b_1] \times \dots \times]a_n, b_n] \in \sigma(\mathscr{H}_n)$ and $\mathscr{I}_n \subset \sigma(\mathscr{H}_n)$, which, according to (1.11) and (1.12), implies

$$\mathscr{B}_n = \sigma(\mathscr{I}_n) \subset \sigma(\mathscr{H}_n).$$

1.12 If $x \in \mathbb{R}$, then $\{x\} = \bigcap_{i=1}^{\infty}]x - 1/i, x]$. According to Equation (1.18), the intervals $]x - 1/i, x]$ are elements of the generating set system of $\mathscr{B}$, the Borel σ-algebra on

$\mathbb{R}$. Therefore, their countable intersection is an element of $\mathscr{B}$. If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then

$$\{x\} = \bigcap_{i=1}^{\infty} \left(\bigtimes_{j=1}^{n}]x_j - \frac{1}{i}, x_j] \right).$$

According to Equation (1.20), the cuboids $\bigtimes_{j=1}^{n}]x_j - \frac{1}{i}, x_j]$ are elements of the set system $\mathscr{I}_n$ and $\sigma(\mathscr{I}_n) = \mathscr{B}_n$.

1.13 Because $\{x\} \in \mathscr{B}$ for all $x \in \mathbb{R}$ (see Exercise 1.12), we can conclude: $\{x\} \in \mathscr{B}|_{\Omega_0}$ for all $x \in \Omega_0$. Hence, if Ω_0 is finite or countable, Example 1.18 implies $\mathscr{B}|_{\Omega_0} = \mathscr{P}(\Omega_0)$.

1.14 Let $\Omega_1, \dots, \Omega_n$ be finite or countable sets, and let $\mathscr{A}_1, \dots, \mathscr{A}_n$ be their power sets. Then $\omega_1 \in \Omega_1, \dots, \omega_n \in \Omega_n$ implies $\{\omega_1\} \in \mathscr{A}_1, \dots, \{\omega_n\} \in \mathscr{A}_n$. Therefore,

$$\{(\omega_1, \dots, \omega_n)\} = \{\omega_1\} \times \dots \times \{\omega_n\} \in \left\{ \bigtimes_{i=1}^{n} A_i \colon A_i \in \mathscr{A}_i, i \in \{1, \dots, n\} \right\}.$$

Hence,

$$\sigma(\{(\omega_1, \dots, \omega_n)\} \colon \omega_1 \in \Omega_1, \dots, \omega_n \in \Omega_n) \subset \bigotimes_{i=1}^{n} \mathscr{A}_i.$$

With Ω_i being finite or countable, $\Omega = \Omega_1 \times \dots \times \Omega_n$ is finite or countable. Therefore,

$$\sigma(\{(\omega_1, \dots, \omega_n)\} \colon \omega_1 \in \Omega_1, \dots, \omega_n \in \Omega_n) = \mathscr{P}(\Omega)$$

(see Example 1.18). Because $\bigotimes_{i=1}^{n} \mathscr{A}_i \subset \mathscr{P}(\Omega)$, we can conclude

$$\bigotimes_{i=1}^{n} \mathscr{A}_i = \mathscr{P}(\Omega_1 \times \dots \times \Omega_n) = \mathscr{P}\left(\bigtimes_{i=1}^{n} \Omega_i \right).$$

1.15

$$\begin{aligned}(A \times B)^c &= \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \colon \omega_1 \notin A \text{ or } \omega_2 \notin B\} \\ &= \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \colon (\omega_1 \notin A, \omega_2 \in B) \text{ or } \omega_2 \notin B\} \\ &= (A^c \times B) \cup (\Omega_1 \times B^c)\end{aligned}$$

and

$$\begin{aligned}&(A^c \times B) \cap (\Omega_1 \times B^c) \\ &= \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \colon \omega_1 \notin A, \omega_2 \in B, \omega_2 \notin B\} \\ &= \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \colon \omega_1 \notin A, \omega_2 \in B \cap B^c = \emptyset\} \\ &= \emptyset.\end{aligned}$$

1.16 Remember that $(a \in A,\ b \in B)$ means $(a \in A$ and $b \in B)$ and that $(a \in A$ and $b \in B)$ and $(b \in B$ and $a \in A)$ are equivalent. Let $A_1, B_1 \in \mathscr{A}_1, \dots, A_n, B_n \in \mathscr{A}_n$. Then $A_1 \cap B_1 \in$

$\mathscr{A}_1, \dots, A_n \cap B_n \in \mathscr{A}_n$. Hence, $A_1 \times \dots \times A_n \in \mathscr{E}, B_1 \times \dots \times B_n \in \mathscr{E}$ and $(A_1 \cap B_1) \times \dots \times (A_n \cap B_n) \in \mathscr{E}$. Furthermore,

$$\begin{aligned} &(A_1 \times \dots \times A_n) \cap (B_1 \times \dots \times B_n) \\ &= \{(\omega_1, \dots, \omega_n) : \omega_1 \in A_1, \dots, \omega_n \in A_n, \omega_1 \in B_1, \dots, \omega_n \in B_n\} \\ &= \{(\omega_1, \dots, \omega_n) : \omega_1 \in (A_1 \cap B_1), \dots, \omega_n \in (A_n \cap B_n)\} \\ &= (A_1 \cap B_1) \times \dots \times (A_n \cap B_n) \in \mathscr{E}. \end{aligned}$$

1.17 Let B_i denote the sets defined in Remark 1.46.

(i) $B_1 = A_1 \in \mathscr{A}$. For all $i \in \mathbb{N}, i > 1, B_i \in \mathscr{A}$:

$$B_i = A_i \setminus \left(\bigcup_{j=1}^{i-1} A_j \right) = A_i \cap \left(\bigcup_{j=1}^{i-1} A_j \right)^c \in \mathscr{A}. \qquad \text{[Def. 1.1 (b), Rem. 1.2]}$$

(ii) For any sequence $C_1, C_2, \dots \subset \Omega$, define

$$\bigcup_{j=m}^{n} C_j := \emptyset, \quad \text{if } m > n, \qquad \text{and} \qquad \bigcap_{j=m}^{n} C_j := \Omega, \quad \text{if } m > n.$$

Then, using associativity and commutativity of the intersection, for $1 \le k < l$,

$$\begin{aligned} B_k \cap B_l &= \left[A_k \setminus \left(\bigcup_{j=1}^{k-1} A_j \right) \right] \cap \left[A_l \setminus \left(\bigcup_{j=1}^{l-1} A_j \right) \right] \\ &= A_k \cap \left(\bigcup_{j=1}^{k-1} A_j \right)^c \cap A_l \cap \left(\bigcup_{j=1}^{l-1} A_j \right)^c && [A \setminus B = A \cap B^c] \\ &= A_k \cap \left(\bigcap_{j=1}^{k-1} A_j^c \right) \cap A_l \cap \left(\bigcap_{j=1}^{l-1} A_j^c \right) && \text{[de Morgan]} \\ &= A_k \cap A_l \cap \left(\bigcap_{j=1}^{k-1} A_j^c \right) \cap \left(\bigcap_{j=1}^{k-1} A_j^c \right) \cap A_k^c \cap \left(\bigcap_{j=k+1}^{l-1} A_j^c \right) \\ &= \emptyset. && [A_k \cap A_k^c = \emptyset] \end{aligned}$$

(iii) The sets B_i are defined such that $B_i \subset A_i$, for all $i \in I$. Therefore, $\bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} A_i$. Furthermore, for all $\omega \in \Omega$,

$$\begin{aligned} \omega \in \bigcup_{i=1}^{\infty} A_i &\Rightarrow \exists\, i \in \mathbb{N}: \omega \in A_i \wedge (\forall j < i: \omega \notin A_j) \\ &\Rightarrow \exists\, i \in \mathbb{N}: \omega \in A_1^c \cap \dots \cap A_{i-1}^c \cap A_i = B_i \\ &\Rightarrow \omega \in \bigcup_{i=1}^{\infty} B_i. \end{aligned}$$

Hence, $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} B_i$, and this implies $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$.

1.18 (i) This is condition (c) of Definition 1.43.

(ii) If $A_1, \ldots, A_n \in \mathscr{A}$ are pairwise disjoint, then $A_1, A_2, \ldots$ with $\emptyset = A_{n+1} = A_{n+2} = \ldots$ is a sequence of pairwise disjoint measurable sets. Therefore, conditions (a) and (c) of Def. 1.43 imply

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{n} \mu(A_i) + \sum_{i=n+1}^{\infty} \mu(\emptyset) = \sum_{i=1}^{n} \mu(A_i).$$

(iii) For $A, B \subset \Omega$,

$$A = (A \cap B) \cup (A \cap B^c) = (A \cap B) \cup (A \setminus B)$$

and

$$(A \cap B) \cap (A \cap B^c) = A \cap B \cap B^c = \emptyset.$$

Hence, for sets $A, B \in \mathscr{A}$, Rule (ii) (finite additivity of μ) implies proposition (iii).

(iv) This proposition is a special case of (iii) with $A = \Omega$.

(v) Exchanging the roles of A and B in (iii), we obtain

$$\mu(B) = \mu(A \cap B) + \mu(B \setminus A).$$

If $A \subset B$, then $A \cap B = A$; and, because $\mu(B \setminus A) \geq 0$,

$$\mu(A) = \mu(A \cap B) \leq \mu(A \cap B) + \mu(B \setminus A) = \mu(B).$$

(vi) This rule immediately follows from proposition (iv) for $\mu(A \cap B) < \infty$. [Note that $\mu(A) - \mu(A \cap B)$ is not defined if $\mu(A) = \mu(A \cap B) = \infty$.]

(vii) For $A, B \subset \Omega$,

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A).$$

Because the right-hand side is a union of pairwise disjoint sets, finite additivity of μ yields

$$\begin{aligned} \mu(A \cup B) + \mu(A \cap B) &= \mu(A \setminus B) + \mu(A \cap B) + \mu(B \setminus A) + \mu(A \cap B) \\ &= \mu(A) + \mu(B). \qquad \text{[Box 1.1 (iii)]} \end{aligned}$$

(viii) $\mu(\Omega) = \mu(A \cup A^c) = \mu(A) + \mu(A^c)$. Hence, if $\mu(\Omega) = \mu(A) < \infty$, then $\mu(A^c) = 0$. Therefore, for all $B \in \mathscr{A}$, (v) implies $\mu(A^c \cap B) = 0$. Furthermore, $B = (A \cap B) \cup (A^c \cap B)$ and $(A \cap B) \cap (A^c \cap B) = \emptyset$. Hence, $\mu(B) = \mu(A \cap B) + \mu(A^c \cap B) = \mu(A \cap B)$. Note that, in general, $\mu(A) = \mu(\Omega)$ does not imply $A = \Omega$.

(ix) $\mu(A) = 0$ implies

$$\begin{aligned} \mu(B) &= \mu(A) + \mu(B) \\ &\geq \mu(A \cup B) && [(\text{xi})] \\ &\geq \mu(B). && [(\text{v})] \end{aligned}$$

Note that, in general, $\mu(A) = 0$ does not imply $A = \emptyset$.

(x) Let $B := \Omega \setminus \Omega_0$. Then $\mu(B) = 0$ as well as $\mu(A \cap B) = 0$ for all $A \in \mathscr{A}$ [see Box 1.1 (v)]. Furthermore, for $A \in \mathscr{A}$: $A = (A \cap \Omega_0) \cup (A \cap B)$, where $A \cap \Omega_0$ and $A \cap B$ are disjoint. Now, the sets $A \cap \Omega_0$, $A \in \mathscr{A}$, are the elements of the trace σ-algebra and $(\Omega_0, \mathscr{A}|_{\Omega_0}) = [\Omega_0, \mathscr{P}(\Omega_0)]$. Therefore, we can apply Equation (1.31). Hence, for all $A \in \mathscr{A}$,

$$\begin{aligned} \mu(A) &= \mu(A \cap \Omega_0) + \mu(A \cap B) && [\text{Box 1.1 (ii)}] \\ &= \sum_{\omega \in A \cap \Omega_0} \mu(\{\omega\}) + \mu(A \cap B) && [(1.31)] \\ &= \sum_{\omega \in A \cap \Omega_0} \mu(\{\omega\}). && [\mu(A \cap B) = 0] \end{aligned}$$

(xi) Let $A_1, A_2, \ldots \in \mathscr{A}$ and define $B_1, B_2, \ldots \in \mathscr{A}$ by $B_1 = A_1$, and $B_i = A_i \setminus \bigcup_{j=1}^{i-1} B_j$ for $i > 1$ (see Rem. 1.46). Then $B_1, B_2, \ldots$ is a sequence of pairwise disjoint sets with $B_i \subset A_i$ for all $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$. Hence,

$$\begin{aligned} \mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} B_i\right) \\ &= \sum_{i=1}^{\infty} \mu(B_i) && [\text{Def. 1.43 (c)}] \\ &\leq \sum_{i=1}^{\infty} \mu(A_i). && [\text{Box 1.1 (v)}] \end{aligned}$$

1.19 If the $A_1, \ldots, A_n \in \mathscr{A}$ are pairwise disjoint and $B \in \mathscr{A}$, then, for $i \neq j$, $i, j = 1, \ldots, n$,

$$(B \cap A_i) \cap (B \cap A_j) = B \cap (A_i \cap A_j) = B \cap \emptyset = \emptyset.$$

Hence, the sets $B \cap A_1, \ldots, B \cap A_n$ are pairwise disjoint. Furthermore, condition (b) of Remark 1.47 implies

$$\bigcup_{i=1}^{n} (B \cap A_i) = B \cap \bigcup_{i=1}^{n} A_i = B.$$

Therefore, additivity of μ yields

$$\mu(B) = \mu\left(\bigcup_{i=1}^{n} (B \cap A_i)\right) = \sum_{i=1}^{n} \mu(B \cap A_i),$$

which is Equation (1.29). The proof of Equation (1.30) is literally the same except for replacing $\bigcup_{i=1}^{n}$ by $\bigcup_{i=1}^{\infty}$, $\sum_{i=1}^{n}$ by $\sum_{i=1}^{\infty}$, and additivity of μ by σ-additivity.

1.20 Let $\omega \in \Omega$.

(a) According to Equation (1.32), $\delta_\omega(\emptyset) = 1_\emptyset(\omega) = 0$.

(b) According to Equation (1.32), $\delta_\omega(A) = 1_A(\omega) \in \{0, 1\}$, for all $A \in \mathscr{A}$, and this implies $\delta_\omega(A) \geq 0$, for all $A \in \mathscr{A}$.

(c) If $A_1, A_2, \ldots \in \mathscr{A}$ are pairwise disjoint, then

$$\begin{aligned} \delta_\omega\left(\bigcup_{i=1}^{\infty} A_i\right) &= 1_{\bigcup_{i=1}^{\infty} A_i}(\omega) && [(1.32)] \\ &= \sum_{i=1}^{\infty} 1_{A_i}(\omega) && [(1.37)] \\ &= \sum_{i=1}^{\infty} \delta_\omega(A_i). && [(1.32)] \end{aligned}$$

1.21 (a) According to Equation (1.40), $\mu_\#(\emptyset) = \sum_{\omega \in \Omega} 1_\emptyset(\omega) = 0$.

(b) According to Equation (1.40), $\mu_\#(A) = \sum_{\omega \in \Omega} 1_A(\omega)$, for all finite $A \in \mathscr{A}$, and $\mu_\#(A) = \infty$, if A is infinite. This implies $\mu_\#(A) \geq 0$, for all $A \in \mathscr{A}$.

(c) If $A_1, A_2, \ldots \in \mathscr{A}$ are pairwise disjoint and all A_i are finite, then

$$\begin{aligned} \mu_\#\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{\omega \in \Omega} 1_{\bigcup_{i=1}^{\infty} A_i}(\omega) && [(1.40)] \\ &= \sum_{\omega \in \Omega} \sum_{i=1}^{\infty} 1_{A_i}(\omega) && [(1.37)] \\ &= \sum_{i=1}^{\infty} \sum_{\omega \in \Omega} 1_{A_i}(\omega) \\ &= \sum_{i=1}^{\infty} \mu_\#(A_i). && [(1.40)] \end{aligned}$$

Note that the set $\bigcup_{i=1}^{\infty} A_i$ can be countably infinite, even if all A_i are finite. In this case, $\mu_\#\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty = \sum_{i=1}^{\infty} \mu_\#(A_i)$. If at least one of the A_i is infinite, then $\bigcup_{j=1}^{\infty} A_j \supset A_i$ is an infinite set and $\mu_\#\left(\bigcup_{j=1}^{\infty} A_j\right) \geq \mu_\#(A_i)$ is infinite as well.

1.22 (a) Using Equations (1.42) and (1.39),

$$\left(\sum_{\omega \in B} \delta_\omega\right)(\emptyset) = \sum_{\omega \in B} \delta_\omega(\emptyset) = \sum_{\omega \in B} 1_\emptyset(\omega) = \sum_{\omega \in B} 0 = 0.$$

(b) Using Equations (1.42) and (1.39),

$$\forall A \in \mathscr{A} : \left(\sum_{\omega \in B} \delta_\omega\right)(A) = \sum_{\omega \in B} \delta_\omega(A) = \sum_{\omega \in B} 1_A(\omega) \geq 0.$$

(c) If $A_1, A_2, \ldots \in \mathscr{A}$ are pairwise disjoint, then

$$\begin{aligned}
\left(\sum_{\omega \in B} \delta_\omega\right)\left(\bigcup_{i=1}^{\infty} A_i\right) &= \sum_{\omega \in B} \delta_\omega\left(\bigcup_{i=1}^{\infty} A_i\right) && [(1.42)] \\
&= \sum_{\omega \in B} 1_{\bigcup_{i=1}^{\infty} A_i}(\omega) && [(1.39)] \\
&= \sum_{\omega \in B} \sum_{i=1}^{\infty} 1_{A_i}(\omega) && [(1.37)] \\
&= \sum_{\omega \in B} \sum_{i=1}^{\infty} \delta_\omega(A_i) && [(1.39)] \\
&= \sum_{i=1}^{\infty} \left(\left(\sum_{\omega \in B} \delta_\omega\right)(A_i)\right). && [(1.42)]
\end{aligned}$$

1.23 (a) Equation (1.47) yields: $\nu(\emptyset) = \mu(\emptyset) = 0$.

(b) Equation (1.47) also yields: $\nu(A) = \mu(A) \geq 0$, for all $A \in \mathscr{C}$.

(c) If $A_1, A_2, \ldots \in \mathscr{C}$ are pairwise disjoint, then

$$\begin{aligned}
\nu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mu\left(\bigcup_{i=1}^{\infty} A_i\right) && [\text{Def. 1.1 (c), (1.47)}] \\
&= \sum_{i=1}^{\infty} \mu(A_i) && [\text{Def. 1.43 (c)}] \\
&= \sum_{i=1}^{\infty} \nu(A_i). && [(1.47)]
\end{aligned}$$

1.24 (a) Using Equation (1.48) and Definition 1.43 (a) yields

$$\left(\sum_{i=1}^{\infty} \alpha_i \mu_i\right)(\emptyset) = \sum_{i=1}^{\infty} \alpha_i \mu_i(\emptyset) = \sum_{i=1}^{\infty} 0 = 0.$$

(b) Similarly, using Equation (1.48) yields, for all $A \in \mathscr{A}$,

$$\left(\sum_{i=1}^{\infty} \alpha_i \mu_i\right)(A) = \sum_{i=1}^{\infty} \alpha_i \mu_i(A) = \lim_{n \to \infty} \sum_{i=1}^{n} \alpha_i \mu_i(A) \geq 0,$$

because $\mu_i(A) \geq 0$, and we assume $\alpha_i \geq 0$.

(c) If $A_1, A_2, \ldots \in \mathscr{A}$ are pairwise disjoint, then

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} \alpha_i \mu_i\right)\left(\bigcup_{j=1}^{\infty} A_j\right) &= \sum_{i=1}^{\infty} \alpha_i \mu_i \left(\bigcup_{j=1}^{\infty} A_j\right) && [(1.48)] \\
&= \sum_{i=1}^{\infty} \alpha_i \sum_{j=1}^{\infty} \mu_i(A_j) && [\text{Def. } 1.43\ (c)] \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \alpha_i \mu_i(A_j) \\
&= \sum_{j=1}^{\infty} \left(\left(\sum_{i=1}^{\infty} \alpha_i \mu_i\right)(A_j)\right). && [(1.48)]
\end{aligned}$$

Note that the last but one equation holds, because rearranging summands does not change the sum if the terms α_i and $\mu_i(A_j)$ are nonnegative.