1

Formation Kinematics

This chapter introduces the notation to be used in the book, as well as the subject of vectorial kinematics, which is frequently used to derive equations of motion.

1.1 Notation

$\rightarrow$	tends (or converges) to
$\Rightarrow$	implies
$\equiv$	identically equals (or equal)
$\triangleq$	defined as
$\ll$	much smaller than
$\gg$	much greater than
$\forall$	for all
$\exists$	(if) there exists
$\in$	belongs to
$\notin$	does not belong to
$\subset$	a strict subset of
$\subseteq$	a subset of
$\cap$	intersection
$\cup$	union
$\emptyset$	empty set
$\mapsto$	maps to
Σ	summation
Π	left product
$\otimes$	Kronecker product
∞	positive infinity

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$\mathbb{R}$	set of real numbers
$\mathbb{R}^k$	set of $k \times 1$ real vectors
$\mathbb{R}^{m\times n}$	set of $m \times n$ real matrices
$\mathbb{C}$	set of complex numbers
$\mathbb{C}^k$	set of $k \times 1$ complex vectors
$\mathbb{C}^+$	set of complex numbers with positive real parts
$\mathbb{C}^-$	set of complex numbers with negative real parts
$B(x, \epsilon)$	open ball centered at x with radius ϵ
$\|z\|$	amplitude (or absolute value) of number z
$Re\,(z)$	real part of number z
$Im\,(z)$	imaginary part of number z
x^T	transpose of a real vector x
$\\|x\\|$	2-norm of a real vector x
$\\|x\\|_p$	p-norm of a real vector x
A^T	transpose of a real matrix A
$\\|A\\|$	induced 2-norm of a real matrix A
$\\|A\\|_p$	induced p-norm of a real matrix A
$\|A\|$ or $\det(A)$	the determinant of a square matrix A
$A > 0$	a positive matrix A
$A \geq 0$	a nonnegative matrix A
e^z	exponential of a real or complex number z
e^A	exponential of a real matrix A
$\rho(A)$	spectral radius of matrix A
$\lambda_i(A)$	the i-th eigenvalue of matrix A
$\lambda_{\max}(A)$	the maximum eigenvalue of a real symmetric matrix A
$\lambda_{\min}(A)$	the minimum eigenvalue of a real symmetric matrix A
$rank(A)$	rank of matrix A
$\mathrm{diag}\,(a_1, \cdots, a_k)$	a diagonal matrix with diagonal entries a_1 to a_k
$\mathrm{diag}\,(A_1, \cdots, A_k)$	a block diagonal matrix with diagonal blocks A_1 to A_k
max	maximum
min	minimum
sup	supremum, the least upper bound
inf	infimum, the greatest lower bound
sin	sine function

cos	cosine function
sign	signum function
tanh	tangent hyperbolic function
sec	tangent hyperbolic function
$\mathbf{1}_k$	$k \times 1$ column vector of all ones
$\mathbf{0}_k$	$k \times 1$ column vector of all zeros
I_m	$m \times m$ identity matrix
i	imaginary unit
$\mathbf{0}_{m\times n}$	$m \times n$ zero matrix

1.2 Vectorial Kinematics

The motion of an individual vehicle, is a six degree-of-freedom movement in space with respect to time. If we borrow the concept of rigid-body dynamic behaviour, such movement is often captured by a translational movement of a mass point (e.g. the centre of mass) and a rotational movement about an instantaneous axis through that point. Therefore, a distinctive description of translational or rotational dynamic behaviour is often developed through vectorial kinematics and dynamics.

1.2.1 Frame Rotation

It is essential to know how to deal with several reference frames and the transformation of the matrix representations of a vector (since the representation depends on the specific reference frame) from one frame to another. Only relative *rotation* (orientation change) between reference frames is important when considering representation of vectors. The relative translation does not affect the components of a vector since neither direction nor magnitude depends on the placement of the frame's origin. Translational motion between frames can be treated in the same way as Galilean transformation.

The physical description of motion of a mass point requires an origin to construct a vector. It is different from the general statement of independence of a vector from a reference frame origin.

Rotation Matrix

A vector $\underset{\rightarrow}{\boldsymbol{v}}$ has different expressions under two different frames $\mathcal{F}_a$ and $\mathcal{F}_b$:

$$\underset{\rightarrow}{\boldsymbol{v}} = \underset{\rightarrow a}{\boldsymbol{F}}^T \underline{\boldsymbol{v}}_a = \underset{\rightarrow b}{\boldsymbol{F}}^T \underline{\boldsymbol{v}}_b \tag{1.1}$$

where $\underline{\boldsymbol{v}}_a$ and $\underline{\boldsymbol{v}}_b$ are numerical expressions of vector $\underset{\rightarrow}{\boldsymbol{v}}$ under frames F_a and F_b respectively, sometimes referred to as numerical vectors. The vector-like $\underset{\rightarrow a}{\boldsymbol{F}}$ and $\underset{\rightarrow b}{\boldsymbol{F}}$ are vectorized representations of frame axes, $\underset{\rightarrow a}{\boldsymbol{F}} = [\underset{\rightarrow 1}{\boldsymbol{a}}\ \underset{\rightarrow 2}{\boldsymbol{a}}\ \underset{\rightarrow 3}{\boldsymbol{a}}]^T$, $\underset{\rightarrow b}{\boldsymbol{F}} = [\underset{\rightarrow 1}{\boldsymbol{b}}\ \underset{\rightarrow 2}{\boldsymbol{b}}\ \underset{\rightarrow 3}{\boldsymbol{b}}]^T$. We simply call these vectrices, a made-up name for axis vectors presented in matrix format. It is obvious that the relationship between two expressions lies in the relationship between these two frames.

Consider two reference frames $\mathcal{F}_a : \{\underrightarrow{\boldsymbol{a}}_1, \underrightarrow{\boldsymbol{a}}_2, \underrightarrow{\boldsymbol{a}}_3\}$ and $\mathcal{F}_b : \{\underrightarrow{\boldsymbol{b}}_1, \underrightarrow{\boldsymbol{b}}_2, \underrightarrow{\boldsymbol{b}}_3\}$. Rotating from $\mathcal{F}_a$ to $\mathcal{F}_b$ means that $\underrightarrow{\boldsymbol{F}}_a$ to $\underrightarrow{\boldsymbol{F}}_b$: $\underrightarrow{\boldsymbol{F}}_a \Rightarrow \underrightarrow{\boldsymbol{F}}_b$.

From (1.1) we have

$$\underline{\boldsymbol{v}}_b = \underrightarrow{\boldsymbol{F}}_b \cdot \underrightarrow{\boldsymbol{v}} = \underrightarrow{\boldsymbol{F}}_b \cdot \underrightarrow{\boldsymbol{F}}_a^T \underline{\boldsymbol{v}}_a \stackrel{\Delta}{=} \underline{\boldsymbol{C}}_{ba} \underline{\boldsymbol{v}}_a \tag{1.2}$$

where

$$\underrightarrow{\boldsymbol{F}}_b \cdot \underrightarrow{\boldsymbol{F}}_a^T = \begin{bmatrix} \underrightarrow{\boldsymbol{b}}_1 \cdot \underrightarrow{\boldsymbol{a}}_1 & \underrightarrow{\boldsymbol{b}}_1 \cdot \underrightarrow{\boldsymbol{a}}_2 & \underrightarrow{\boldsymbol{b}}_1 \cdot \underrightarrow{\boldsymbol{a}}_3 \\ \underrightarrow{\boldsymbol{b}}_2 \cdot \underrightarrow{\boldsymbol{a}}_1 & \underrightarrow{\boldsymbol{b}}_2 \cdot \underrightarrow{\boldsymbol{a}}_2 & \underrightarrow{\boldsymbol{b}}_2 \cdot \underrightarrow{\boldsymbol{a}}_3 \\ \underrightarrow{\boldsymbol{b}}_3 \cdot \underrightarrow{\boldsymbol{a}}_1 & \underrightarrow{\boldsymbol{b}}_3 \cdot \underrightarrow{\boldsymbol{a}}_2 & \underrightarrow{\boldsymbol{b}}_3 \cdot \underrightarrow{\boldsymbol{a}}_3 \end{bmatrix} \stackrel{\Delta}{=} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{13} \end{bmatrix} = \underline{\boldsymbol{C}}_{ba}$$

The short expression then becomes

$$\underline{\boldsymbol{u}}_b = \underline{\boldsymbol{C}}_{ba} \underline{\boldsymbol{u}}_a \tag{1.3}$$

$$\underrightarrow{\boldsymbol{F}}_b = \underline{\boldsymbol{C}}_{ba} \underrightarrow{\boldsymbol{F}}_a \tag{1.4}$$

Switching the letters a and b,

$$\underline{\boldsymbol{C}}_{ab} = \underrightarrow{\boldsymbol{F}}_a \cdot \underrightarrow{\boldsymbol{F}}_b^T = \begin{bmatrix} \underrightarrow{\boldsymbol{a}}_1 \cdot \underrightarrow{\boldsymbol{b}}_1 & \underrightarrow{\boldsymbol{a}}_1 \cdot \underrightarrow{\boldsymbol{b}}_2 & \underrightarrow{\boldsymbol{a}}_1 \cdot \underrightarrow{\boldsymbol{b}}_3 \\ \underrightarrow{\boldsymbol{a}}_2 \cdot \underrightarrow{\boldsymbol{b}}_1 & \underrightarrow{\boldsymbol{a}}_2 \cdot \underrightarrow{\boldsymbol{b}}_2 & \underrightarrow{\boldsymbol{a}}_2 \cdot \underrightarrow{\boldsymbol{b}}_3 \\ \underrightarrow{\boldsymbol{a}}_3 \cdot \underrightarrow{\boldsymbol{b}}_1 & \underrightarrow{\boldsymbol{a}}_3 \cdot \underrightarrow{\boldsymbol{b}}_2 & \underrightarrow{\boldsymbol{a}}_3 \cdot \underrightarrow{\boldsymbol{b}}_3 \end{bmatrix} \tag{1.5}$$

Orthonormality

It can be shown that matrices $\underline{\boldsymbol{C}}_{ab}$ and $\underline{\boldsymbol{C}}_{ba}$ are orthonormal when both frames of reference $\mathcal{F}_a$ and $\mathcal{F}_b$ are orthonormal; in other words they have orthonormal basis vectors.

$$\underline{\boldsymbol{C}}_{ab}^T \underline{\boldsymbol{C}}_{ab} = \underline{\mathbf{1}} \tag{1.6}$$

$$\underline{\boldsymbol{C}}_{ba}^T \underline{\boldsymbol{C}}_{ba} = \underline{\mathbf{1}} \tag{1.7}$$

$$\underline{\boldsymbol{C}}_{ab}^T = \underline{\boldsymbol{C}}_{ba} = \underline{\boldsymbol{C}}_{ab}^{-1} \tag{1.8}$$

$$\det \underline{\boldsymbol{C}}_{ba} = 1 \tag{1.9}$$

Principal Rotations

There are three principal (basic) rotations of our interest:

- $\underrightarrow{\boldsymbol{F}}_a \Rightarrow \underrightarrow{\boldsymbol{F}}_b$ **about** $\underrightarrow{\boldsymbol{a}}_1$ **or** $\underrightarrow{\boldsymbol{x}}$

$$[\underrightarrow{\boldsymbol{b}}_1 \quad \underrightarrow{\boldsymbol{b}}_2 \quad \underrightarrow{\boldsymbol{b}}_3] = [\underrightarrow{\boldsymbol{a}}_1 \quad \underrightarrow{\boldsymbol{a}}_2 \quad \underrightarrow{\boldsymbol{a}}_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\sigma & -s_\sigma \\ 0 & s_\sigma & c_\sigma \end{bmatrix}$$

$$\underrightarrow{\boldsymbol{F}}_b^T \stackrel{\Delta}{=} \underrightarrow{\boldsymbol{F}}_a^T \underline{\boldsymbol{C}}_1^T(\sigma) \tag{1.10}$$

where $s_\sigma = \sin\sigma, c_\sigma = \cos\sigma$.

- $\vec{\boldsymbol{F}}_a \Rightarrow \vec{\boldsymbol{F}}_b$ **about** $\vec{\boldsymbol{a}}_2$ **or** $\vec{\boldsymbol{y}}$

$$[\vec{\boldsymbol{b}}_1 \quad \vec{\boldsymbol{b}}_2 \quad \vec{\boldsymbol{b}}_3] = [\vec{\boldsymbol{a}}_1 \quad \vec{\boldsymbol{a}}_2 \quad \vec{\boldsymbol{a}}_3]\begin{bmatrix} c_\sigma & 0 & s_\sigma \\ 0 & 1 & 0 \\ -s_\sigma & 0 & c_\sigma \end{bmatrix}$$

$$\vec{\boldsymbol{F}}_b^T \stackrel{\Delta}{=} \vec{\boldsymbol{F}}_a^T \underline{\boldsymbol{C}}_2^T(\sigma) \tag{1.11}$$

- $\vec{\boldsymbol{F}}_a \Rightarrow \vec{\boldsymbol{F}}_b$ **about** $\vec{\boldsymbol{a}}_3$ **or** $\vec{\boldsymbol{z}}$

$$[\vec{\boldsymbol{b}}_1 \quad \vec{\boldsymbol{b}}_2 \quad \vec{\boldsymbol{b}}_3] = [\vec{\boldsymbol{a}}_1 \quad \vec{\boldsymbol{a}}_2 \quad \vec{\boldsymbol{a}}_3]\begin{bmatrix} c_\sigma & -s_\sigma & 0 \\ s_\sigma & c_\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{\boldsymbol{F}}_b^T \stackrel{\Delta}{=} \vec{\boldsymbol{F}}_a^T \underline{\boldsymbol{C}}_3^T(\sigma) \tag{1.12}$$

1.2.2 The Motion of a Vector

The motion of a vector represents its rate of change with time, which is described as the *time derivative* of the vector. Consider a vector $\vec{\boldsymbol{u}}$ and its expressions $\underline{\boldsymbol{u}}_a$ and $\underline{\boldsymbol{u}}_b$ with respect to the reference frame $\mathcal{F}_a : \{\vec{\boldsymbol{a}}_1, \vec{\boldsymbol{a}}_2, \vec{\boldsymbol{a}}_3\}$ and $\mathcal{F}_b : \{\vec{\boldsymbol{b}}_1, \vec{\boldsymbol{b}}_2, \vec{\boldsymbol{b}}_3\}$ respectively; that is,

$$\vec{\boldsymbol{u}} = \vec{\boldsymbol{F}}_a^T \underline{\boldsymbol{u}}_a = \vec{\boldsymbol{a}}_1 u_{a,1} + \vec{\boldsymbol{a}}_2 u_{a,2} + \vec{\boldsymbol{a}}_3 u_{a,3} \tag{1.13}$$

$$= \vec{\boldsymbol{F}}_b^T \underline{\boldsymbol{u}}_b = \vec{\boldsymbol{b}}_1 u_{b,1} + \vec{\boldsymbol{a}}_2 u_{b,2} + \vec{\boldsymbol{b}}_3 u_{b,3}. \tag{1.14}$$

The time derivative is a vector itself, and also has different expressions under these two frames of reference:

$$\frac{d}{dt}\vec{\boldsymbol{u}} = \left(\frac{d}{dt}\vec{\boldsymbol{F}}_a\right)^T \underline{\boldsymbol{u}}_a + \vec{\boldsymbol{F}}_a^T \dot{\underline{\boldsymbol{u}}}_a \tag{1.15}$$

$$\frac{d}{dt}\vec{\boldsymbol{u}} = \left(\frac{d}{dt}\vec{\boldsymbol{F}}_b\right)^T \underline{u}_b + \vec{\boldsymbol{F}}_b^T \dot{\underline{\boldsymbol{u}}}_b \tag{1.16}$$

$$\frac{d}{dt}\vec{\boldsymbol{u}} \stackrel{\Delta}{=} \vec{\boldsymbol{v}} = \vec{\boldsymbol{F}}_a^T \underline{\boldsymbol{v}}_a = \vec{\boldsymbol{F}}_b^T \underline{\boldsymbol{v}}_b \tag{1.17}$$

Obviously, the expression of time derivative vector $\vec{\boldsymbol{v}}$ depends on the time derivative of the vectrix, or the change of rate of basis vectors of the associated frame of reference.

Absolute and Relative Time Derivatives

Assume $\mathcal{F}_a$ represents an inertial space (a Newtonian absolute space). To an observer in $\mathcal{F}_a$, the basis vectors of $\mathcal{F}_a$, or the vectrix $\vec{\boldsymbol{F}}_a$, will remain unchanged (no orientation change, and no magnitude change of course, since they are unit vectors). In other words, the first term in (1.15) is zero. Since $\mathcal{F}_a$ is an inertial frame, we define the time derivative of a vector $\vec{\boldsymbol{u}}$ in $\mathcal{F}_a$ as an *absolute time derivative*, denoted by a bullet • superscript:

$$\overset{\bullet}{\vec{\boldsymbol{u}}} \stackrel{\Delta}{=} \frac{d}{dt}(\vec{\boldsymbol{u}})|_{\mathcal{F}_a} = \vec{\boldsymbol{F}}_a^T \dot{\underline{u}}_a = \vec{\boldsymbol{a}}_1 \dot{u}_{a,1} + \vec{\boldsymbol{a}}_2 \dot{u}_{a,2} + \vec{\boldsymbol{a}}_3 \dot{u}_{a,3} \tag{1.18}$$

Assume $\mathcal{F}_b$ is a moving frame of reference relative to $\mathcal{F}_a$. Denote the moving (rotating) rate of change with time by $\underset{\rightarrow}{\boldsymbol{\omega}}_{ba}$. Similarly, to an observer in $\mathcal{F}_b$, the vectrix $\underset{\rightarrow}{\boldsymbol{F}}_b$ also remains unchanged. In other words, the first term in (1.16) is zero. Therefore, the time derivative of a vector $\underset{\rightarrow}{\boldsymbol{u}}$ in the moving frame $\mathcal{F}_b$ is defined as a *relative time derivative*, denoted by a circle ∘ superscript:

$$\overset{\circ}{\underset{\rightarrow}{\boldsymbol{u}}} \overset{\Delta}{=} \frac{d}{dt}(\underset{\rightarrow}{\boldsymbol{u}})|_{\mathcal{F}_b} = \underset{\rightarrow}{\boldsymbol{F}}_b^T \dot{\underline{u}}_b = \underset{\rightarrow}{\boldsymbol{b}}_1 \dot{u}_{b,1} + \underset{\rightarrow}{\boldsymbol{a}}_2 \dot{u}_{b,2} + \underset{\rightarrow}{\boldsymbol{b}}_3 \dot{u}_{b,3} \tag{1.19}$$

We here note that $\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{u}}}$ and $\overset{\circ}{\underset{\rightarrow}{\boldsymbol{u}}}$ are two different vectors,

$$\underset{\rightarrow}{\boldsymbol{v}} = \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{u}}} \qquad \underline{\boldsymbol{v}}_a = \dot{\underline{u}}_a \tag{1.20}$$

$$\underset{\rightarrow}{\boldsymbol{v}} \neq \overset{\circ}{\underset{\rightarrow}{\boldsymbol{u}}} \qquad \underline{\boldsymbol{v}}_b \neq \dot{\underline{u}}_b \tag{1.21}$$

However, they are closely related to each other.

The definitions of absolute and relative derivatives are special cases of the following general expressions for time derivatives in a frame of reference. To an observer in a frame of reference $\mathcal{F}_x$ ($x = a, b, c, \ldots$) the basis vectors of the frame, or the vectrix, remain unchanged. Hence,

$$\left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{F}}_x\right)_x \overset{\Delta}{=} \frac{d}{dt}(\underset{\rightarrow}{\boldsymbol{F}}_x)|_{\mathcal{F}_x} = \underset{\rightarrow}{\boldsymbol{O}} \tag{1.22}$$

and the special cases are:

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}_a = \underset{\rightarrow}{\boldsymbol{O}} \qquad \overset{\circ}{\underset{\rightarrow}{\boldsymbol{F}}}_b = \underset{\rightarrow}{\boldsymbol{O}}$$

The time derivative of a vector $\underset{\rightarrow}{\boldsymbol{u}}$ in the frame of reference $\mathcal{F}_x$ becomes:

$$\left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}}\right)_x = \left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{F}}_x\right)_x^T \underline{\boldsymbol{u}}_x + \underset{\rightarrow}{\boldsymbol{F}}_x^T \dot{\underline{\boldsymbol{u}}}_x = \underset{\rightarrow}{\boldsymbol{F}}_x^T \dot{\underline{\boldsymbol{u}}}_x \tag{1.23}$$

However, when dealing with multiple frames of reference in one frame $\mathcal{F}_x$, the basis vector of another frame $\mathcal{F}_y$ is no longer unchanged with time. Therefore, it leads to

$$\left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}}\right)_x = \left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{F}}_y\right)_x^T \underline{\boldsymbol{u}}_y + \underset{\rightarrow}{\boldsymbol{F}}_y^T \dot{\underline{\boldsymbol{u}}}_y = \left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{F}}_y\right)_x^T \underline{\boldsymbol{u}}_y + \left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}}\right)_y \tag{1.24}$$

In other words, the motion of a vector in frame $\mathcal{F}_x$ consists of the motion of this vector in frame $\mathcal{F}_y$ and the motion of frame $\mathcal{F}_y$ relative to frame $\mathcal{F}_x$.

Returning to our previous special cases (absolute and relative derivatives), (1.15) and (1.16):

$$\left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}}\right)_a = \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}_a^T \underline{\boldsymbol{u}}_a + \underset{\rightarrow}{\boldsymbol{F}}_a^T \dot{\underline{\boldsymbol{u}}}_a = \underset{\rightarrow}{\boldsymbol{F}}_a^T \dot{\underline{\boldsymbol{u}}}_a \tag{1.25}$$

$$\left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}}\right)_a = \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}_b^T \underline{\boldsymbol{u}}_b + \underset{\rightarrow}{\boldsymbol{F}}_b^T \dot{\underline{\boldsymbol{u}}}_b \tag{1.26}$$

$$\left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}}\right)_b = \overset{\circ}{\underset{\rightarrow}{\boldsymbol{F}}}_a^T \underline{\boldsymbol{u}}_a + \underset{\rightarrow}{\boldsymbol{F}}_a^T \dot{\underline{\boldsymbol{u}}}_a \tag{1.27}$$

$$\left(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}}\right)_b = \overset{\circ}{\underset{\rightarrow}{\boldsymbol{F}}}{}_b^T \underline{\boldsymbol{u}}_b + \underset{\rightarrow}{\boldsymbol{F}}{}_b^T \underline{\dot{\boldsymbol{u}}}_b = \underset{\rightarrow}{\boldsymbol{F}}{}_b^T \underline{\dot{\boldsymbol{u}}}_b \tag{1.28}$$

For the absolute derivative $(\frac{d}{dt}\underset{\rightarrow}{\boldsymbol{u}})_a$, we often drop the subscript a.

The focus now is placed on the rate of change of frame $\mathcal{F}_b$ relative to another frame $\mathcal{F}_a$.

Relative Rotation

To begin with, we look at the rotation of $\mathcal{F}_b$ relative to $\mathcal{F}_a$ about a *fixed* axis $\underset{\rightarrow}{\boldsymbol{a}}$,

$$\underset{\rightarrow}{\boldsymbol{\omega}}_{ba} = \underset{\rightarrow}{\boldsymbol{a}}\dot{\theta}(t) \tag{1.29}$$

One can conclude that, for a unit vector $\underset{\rightarrow}{\boldsymbol{b}}$ such as the axes of $\mathcal{F}_b$,

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{b}}} = \underset{\rightarrow}{\boldsymbol{\omega}}_{ba} \times \underset{\rightarrow}{\boldsymbol{b}} \tag{1.30}$$

where the symbol $\times$ between two vectors denotes the the cross product or vector product in three-dimensional space (see Section 4.5.3 in the book by Polyanin and Manzhirov [1] for a definition).

General Rotation

Generally speaking, the angular velocity not only changes with the magnitude $\dot{\theta}$, but also changes its rotational orientation (there is no fixed axis). In other words, we are looking for a general description for angular velocity (*general angular velocity*), one that it is not associated with $\underset{\rightarrow}{\boldsymbol{a}}$ like the one we had before: we are dealing with the general rotation case. From the previous case, we would like to see a general description of angular velocity, such that for an unit vector $\underset{\rightarrow}{\boldsymbol{d}}$, the following equation still holds:

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{d}}} = \underset{\rightarrow}{\boldsymbol{\omega}}_{ba} \times \underset{\rightarrow}{\boldsymbol{d}} \tag{1.31}$$

On the one hand,

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{d}}} = \frac{d}{dt}(\underset{\rightarrow}{\boldsymbol{F}}{}_b^T \underline{\boldsymbol{d}}_b) = \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}{}_b^T \underline{\boldsymbol{d}}_b + \underset{\rightarrow}{\boldsymbol{F}}{}_b^T \underline{\dot{\boldsymbol{d}}}_b = \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}{}_b^T \underline{\boldsymbol{d}}_b, \qquad \underline{\dot{\boldsymbol{d}}}_b = \underline{\boldsymbol{0}} \tag{1.32}$$

and on the other hand,

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{d}}} = \underset{\rightarrow}{\boldsymbol{\omega}}_{ba} \times \underset{\rightarrow}{\boldsymbol{d}} = \underset{\rightarrow}{\boldsymbol{F}}{}_b^T \underline{\boldsymbol{\omega}}_b^{ba\times} \underline{\boldsymbol{d}}_b \tag{1.33}$$

Therefore,

$$\underline{\boldsymbol{\omega}}_b^{ba\times} \underline{\boldsymbol{d}}_b = \underset{\rightarrow}{\boldsymbol{F}}_b \cdot \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{d}}} = \underset{\rightarrow}{\boldsymbol{F}}_b \cdot \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}{}_b^T \underline{\boldsymbol{d}}_b \tag{1.34}$$

Consider the arbitrary unit vector $\underset{\rightarrow}{\boldsymbol{d}}$. We must have

$$\underline{\boldsymbol{\omega}}_b^{ba\times} = \underset{\rightarrow}{\boldsymbol{F}}_b \cdot \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}{}_b^T \tag{1.35}$$

We define *general angular velocity* as:

$$\underset{\rightarrow ba}{\boldsymbol{\omega}} \overset{\Delta}{=} \underset{\rightarrow b}{\boldsymbol{F}}^T \underline{\boldsymbol{\omega}}_b^{ba} \tag{1.36}$$

where the expression matrix $\underline{\boldsymbol{\omega}}_b^{ba}$ is given by

$$\underline{\boldsymbol{\omega}}_b^{ba\times} \overset{\Delta}{=} \underset{\rightarrow b}{\boldsymbol{F}} \cdot \underset{\rightarrow b}{\dot{\boldsymbol{F}}}^T \tag{1.37}$$

By that definition, we have the conclusion of an expression for general rotation (1.31).

Proof: First, we show that $\underline{\boldsymbol{\omega}}_b^{ba\times}$ is *skew-symmetric*:

$$\underline{\boldsymbol{\omega}}_b^{ba\times} = \underset{\rightarrow b}{\boldsymbol{F}} \cdot \underset{\rightarrow b}{\dot{\boldsymbol{F}}}^T = \begin{bmatrix} \underset{\rightarrow 1}{\boldsymbol{b}} \\ \underset{\rightarrow 2}{\boldsymbol{b}} \\ \underset{\rightarrow 3}{\boldsymbol{b}} \end{bmatrix} \cdot \begin{bmatrix} \underset{\rightarrow 1}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 2}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 3}{\dot{\boldsymbol{b}}} \end{bmatrix}$$

$$= \begin{bmatrix} \underset{\rightarrow 1}{\boldsymbol{b}} \cdot \underset{\rightarrow 1}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 1}{\boldsymbol{b}} \cdot \underset{\rightarrow 2}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 1}{\boldsymbol{b}} \cdot \underset{\rightarrow 3}{\dot{\boldsymbol{b}}} \\ \underset{\rightarrow 2}{\boldsymbol{b}} \cdot \underset{\rightarrow 1}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 2}{\boldsymbol{b}} \cdot \underset{\rightarrow 2}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 2}{\boldsymbol{b}} \cdot \underset{\rightarrow 3}{\dot{\boldsymbol{b}}} \\ \underset{\rightarrow 3}{\boldsymbol{b}} \cdot \underset{\rightarrow 1}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 3}{\boldsymbol{b}} \cdot \underset{\rightarrow 2}{\dot{\boldsymbol{b}}} & \underset{\rightarrow 3}{\boldsymbol{b}} \cdot \underset{\rightarrow 3}{\dot{\boldsymbol{b}}} \end{bmatrix}$$

From the orthonormal basis vector characteristics: $\underset{\rightarrow i}{\boldsymbol{b}} \cdot \underset{\rightarrow i}{\boldsymbol{b}} = 1, \underset{\rightarrow i}{\boldsymbol{b}} \cdot \underset{\rightarrow j}{\boldsymbol{b}} = 0$, we can draw the conclusion that $\underset{\rightarrow i}{\boldsymbol{b}} \cdot \underset{\rightarrow i}{\dot{\boldsymbol{b}}} = 0, \underset{\rightarrow i}{\boldsymbol{b}} \cdot \underset{\rightarrow j}{\dot{\boldsymbol{b}}} = -\underset{\rightarrow i}{\dot{\boldsymbol{b}}} \cdot \underset{\rightarrow j}{\boldsymbol{b}}$ and find the skew-symmetric matrix.

Secondly, we find the derivative of basis vector $\underset{\rightarrow}{\boldsymbol{b}}$, (or any arbitrary vector $\underset{\rightarrow}{\boldsymbol{u}}$ fixed with $\mathcal{F}_b$):

$$\underset{\rightarrow ba}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\boldsymbol{b}} = \underset{\rightarrow b}{\boldsymbol{F}}^T \underline{\boldsymbol{\omega}}_b^{ba\times} \underline{\boldsymbol{b}}_b = \underbrace{\underset{\rightarrow b}{\boldsymbol{F}}^T \underset{\rightarrow b}{\boldsymbol{F}}}_{\text{identity dyadic}} \cdot \underset{\rightarrow b}{\dot{\boldsymbol{F}}}^T \underline{\boldsymbol{b}}_b = \underset{\rightarrow b}{\dot{\boldsymbol{F}}}^T \underline{\boldsymbol{b}}_b + \underset{\rightarrow b}{\boldsymbol{F}}^T \underbrace{\dot{\underline{\boldsymbol{b}}}_b}_{\text{basis unit length}} = \underset{\rightarrow}{\dot{\boldsymbol{b}}}$$

□

To conclude, for the general rotation of a basis vector, we have

$$\underset{\rightarrow}{\dot{\boldsymbol{b}}} = \underset{\rightarrow ba}{\boldsymbol{\omega}} \times \underset{\rightarrow}{\boldsymbol{b}} \quad \text{where} \quad \underline{\boldsymbol{\omega}}_b^{ba\times} \overset{\Delta}{=} \underset{\rightarrow b}{\boldsymbol{F}} \cdot \underset{\rightarrow b}{\dot{\boldsymbol{F}}}^T$$

$$\underset{\rightarrow b}{\dot{\boldsymbol{F}}} = \underset{\rightarrow ba}{\boldsymbol{\omega}} \times \underset{\rightarrow b}{\boldsymbol{F}} \quad \underset{\rightarrow b}{\dot{\boldsymbol{F}}}^T = \underset{\rightarrow ba}{\boldsymbol{\omega}} \times \underset{\rightarrow b}{\boldsymbol{F}}^T$$

Matrix Expression

From definition (1.37) and relationship equation (1.4), we have

$$\begin{aligned} \underline{\boldsymbol{\omega}}_b^{ba\times} &= \underset{\rightarrow b}{\boldsymbol{F}} \cdot \underset{\rightarrow b}{\dot{\boldsymbol{F}}}^T \\ &= \underline{\boldsymbol{C}}_{ba} \underset{\rightarrow a}{\boldsymbol{F}} \cdot (\underset{\rightarrow a}{\boldsymbol{F}}^T \dot{\underline{\boldsymbol{C}}}_{ab} + \underset{\rightarrow a}{\dot{\boldsymbol{F}}}^T \underline{\boldsymbol{C}}_{ab}) \\ &= \underline{\boldsymbol{C}}_{ba} \dot{\underline{\boldsymbol{C}}}_{ab} \\ &= -\dot{\underline{\boldsymbol{C}}}_{ba} \underline{\boldsymbol{C}}_{ba}^T \end{aligned}$$

From a different perspective, for a unit vector $\underset{\rightarrow}{\boldsymbol{d}}$

$$\begin{aligned}\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{d}}} &= \underset{\rightarrow}{\boldsymbol{F}}_a^T \underline{\dot{\boldsymbol{d}}}_a = \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\boldsymbol{C}}_{ba} \frac{d}{dt}(\underline{\boldsymbol{C}}_{ab}\underline{\boldsymbol{d}}_b) \\ &= \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\boldsymbol{C}}_{ba}\underline{\boldsymbol{C}}_{ab}\underline{\dot{\boldsymbol{d}}}_b + \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\boldsymbol{C}}_{ba}\underline{\dot{\boldsymbol{C}}}_{ab}\underline{\boldsymbol{d}}_b, \quad \underline{\dot{\boldsymbol{d}}}_b = \underline{\boldsymbol{0}} \\ &= \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\boldsymbol{C}}_{ba}\underline{\dot{\boldsymbol{C}}}_{ab}\underline{\boldsymbol{d}}_b\end{aligned}$$

On the other hand,

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{d}}} = \underset{\rightarrow}{\boldsymbol{\omega}}_{ba} \times \underset{\rightarrow}{\boldsymbol{d}} = \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\boldsymbol{\omega}}_b^{ba\times} \underline{\boldsymbol{d}}_b$$

Therefore, $\underline{\boldsymbol{\omega}}_b^{ba\times} = \underline{\boldsymbol{C}}_{ba}\underline{\dot{\boldsymbol{C}}}_{ab}$

In summary,

$$\underline{\boldsymbol{\omega}}_b^{ba\times} = \underline{\boldsymbol{C}}_{ba}\underline{\dot{\boldsymbol{C}}}_{ab} \tag{1.38}$$

$$\underline{\dot{\boldsymbol{C}}}_{ba} + \underline{\boldsymbol{\omega}}_b^{ba\times}\underline{\boldsymbol{C}}_{ba} = \underline{\boldsymbol{0}} \tag{1.39}$$

1.2.3 The First Time Derivative of a Vector

Consider the time derivative of an arbitrary vector,

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{u}}} = \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}{}_b^T \underline{\boldsymbol{u}}_b + \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\dot{\boldsymbol{u}}}_b = \underset{\rightarrow}{\boldsymbol{\omega}}_{ba} \times \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\boldsymbol{u}}_b + \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\dot{\boldsymbol{u}}}_b$$

Since

$$\overset{\circ}{\underset{\rightarrow}{\boldsymbol{u}}} = \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\dot{\boldsymbol{u}}}_b$$

we have:

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{u}}} = \overset{\circ}{\underset{\rightarrow}{\boldsymbol{u}}} + \underset{\rightarrow}{\boldsymbol{\omega}}_{ba} \times \underset{\rightarrow}{\boldsymbol{u}} \tag{1.40}$$

Furthermore,

$$\begin{aligned}\underset{\rightarrow}{\boldsymbol{F}}_a^T \underline{\dot{\boldsymbol{u}}}_a &= \overset{\bullet}{\underset{\rightarrow}{\boldsymbol{F}}}{}_b^T \underline{\boldsymbol{u}}_b + \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\dot{\boldsymbol{u}}}_b \\ &= \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\boldsymbol{\omega}}_b^{ba\times} \underline{\boldsymbol{u}}_b + \underset{\rightarrow}{\boldsymbol{F}}_b^T \underline{\dot{\boldsymbol{u}}}_b \\ &= \underset{\rightarrow}{\boldsymbol{F}}_a^T \underline{\boldsymbol{C}}_{ab}\left[\underline{\boldsymbol{\omega}}_b^{ba\times} \underline{\boldsymbol{u}}_b + \underline{\dot{\boldsymbol{u}}}_b\right]\end{aligned}$$

In summary

$$\overset{\bullet}{\underset{\rightarrow}{\boldsymbol{u}}} = \overset{\circ}{\underset{\rightarrow}{\boldsymbol{u}}} + \underset{\rightarrow}{\boldsymbol{\omega}}^{ba} \times \underset{\rightarrow}{\boldsymbol{u}} \tag{1.41}$$

$$\underline{\dot{u}}_a = \underline{\boldsymbol{C}}_{ab}\left[\underline{\boldsymbol{\omega}}_b^{ba\times} \underline{\boldsymbol{u}}_b + \underline{\dot{\boldsymbol{u}}}_b\right] \tag{1.42}$$

Pure Translation

We say $\mathcal{F}_b$ is in *pure translation* with respect to $\mathcal{F}_a$ if the rotation matrix $\underline{\boldsymbol{C}}_{ba}$ is constant in time; in other words, if the orientation of the basis vectors of one frame relative to the basis vectors of the other frame remains fixed. In other words, there might be rotation from $\mathcal{F}_a$ to $\mathcal{F}_b$, but there is *no change of that rotation in time*. Then,

$$\overset{\bullet}{\underrightarrow{\boldsymbol{F}}}_b = \frac{d}{dt}\left(\underline{\boldsymbol{C}}_{ba}\underrightarrow{\boldsymbol{F}}_a\right) = \dot{\underline{\boldsymbol{C}}}_{ba}\underrightarrow{\boldsymbol{F}}_a + \underline{\boldsymbol{C}}_{ba}\overset{\bullet}{\underrightarrow{\boldsymbol{F}}}_a = \underrightarrow{\boldsymbol{O}} \tag{1.43}$$

leading to

$$\overset{\bullet}{\underrightarrow{\boldsymbol{u}}} = \overset{\circ}{\underrightarrow{\boldsymbol{u}}} \tag{1.44}$$

1.2.4 The Second Time Derivative of a Vector

We can treat the second derivative as the first derivative of vector $\overset{\bullet}{\underrightarrow{\boldsymbol{u}}}$,

$$\begin{aligned}
\overset{\bullet\bullet}{\underrightarrow{\boldsymbol{u}}} &= \overset{\bullet}{\overset{\bullet}{\underrightarrow{\boldsymbol{u}}}} \\
&= \overset{\circ}{\overset{\bullet}{\underrightarrow{\boldsymbol{u}}}} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times \overset{\bullet}{\underrightarrow{\boldsymbol{u}}} \\
&= \frac{d}{dt}\Big|_{\mathcal{F}_b}(\overset{\circ}{\underrightarrow{\boldsymbol{u}}} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times \underrightarrow{\boldsymbol{u}}) + \underrightarrow{\boldsymbol{\omega}}_{ba} \times (\overset{\circ}{\underrightarrow{\boldsymbol{u}}} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times \underrightarrow{\boldsymbol{u}}) \\
&= \overset{\circ\circ}{\underrightarrow{\boldsymbol{u}}} + \overset{\circ}{\underrightarrow{\boldsymbol{\omega}}}_{ba} \times \underrightarrow{\boldsymbol{u}} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times \overset{\circ}{\underrightarrow{\boldsymbol{u}}} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times \overset{\circ}{\underrightarrow{\boldsymbol{u}}} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times (\underrightarrow{\boldsymbol{\omega}}_{ba} \times \underrightarrow{\boldsymbol{u}})
\end{aligned}$$

In summary

$$\begin{aligned}
\overset{\bullet\bullet}{\underrightarrow{\boldsymbol{u}}} &= \overset{\circ\circ}{\underrightarrow{\boldsymbol{u}}} + 2\underrightarrow{\boldsymbol{\omega}}_{ba} \times \overset{\circ}{\underrightarrow{\boldsymbol{u}}} + \overset{\circ}{\underrightarrow{\boldsymbol{\omega}}}_{ba} \times \underrightarrow{\boldsymbol{u}} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times (\underrightarrow{\boldsymbol{\omega}}_{ba} \times \underrightarrow{\boldsymbol{u}}) \\
\underline{\ddot{u}}_a &= \underline{\boldsymbol{C}}_{ab}(\underline{\ddot{u}}_b + 2\underline{\boldsymbol{\omega}}_b^{ba^\times}\underline{\dot{u}}_b + \underline{\dot{\omega}}_b^{ba^\times}\underline{\boldsymbol{u}}_b + \underline{\boldsymbol{\omega}}_b^{ba^\times}\underline{\boldsymbol{\omega}}_b^{ba^\times}\underline{\boldsymbol{u}}_b)
\end{aligned}$$

Another observation is:

$$\overset{\bullet}{\underrightarrow{\boldsymbol{\omega}}}_{ba} = \overset{\circ}{\underrightarrow{\boldsymbol{\omega}}}_{ba} + \underrightarrow{\boldsymbol{\omega}}_{ba} \times \underrightarrow{\boldsymbol{\omega}}_{ba} = \overset{\circ}{\underrightarrow{\boldsymbol{\omega}}}_{ba} \tag{1.45}$$

1.2.5 Motion with Respect to Multiple Frames

Here, we will formally prove that

$$\underrightarrow{\boldsymbol{\omega}}_{ca} = \underrightarrow{\boldsymbol{\omega}}_{cb} + \underrightarrow{\boldsymbol{\omega}}_{ba} \tag{1.46}$$

$$\underrightarrow{\boldsymbol{\omega}}_{ab} + \underrightarrow{\boldsymbol{\omega}}_{ba} = \underrightarrow{\boldsymbol{0}} \tag{1.47}$$

Proof:

$$\begin{aligned}
\underline{\boldsymbol{\omega}}_c^{ca^\times} &= -\dot{\underline{\boldsymbol{C}}}_{ca}\underline{\boldsymbol{C}}_{ac} \\
&= -\frac{d}{dt}\left(\underline{\boldsymbol{C}}_{cb}\underline{\boldsymbol{C}}_{ba}\right)\underline{\boldsymbol{C}}_{ab}\underline{\boldsymbol{C}}_{bc}
\end{aligned}$$

$$= -\dot{\underline{C}}_{cb}\underbrace{\underline{C}_{ba}\underline{C}_{ab}}_{\underline{1}}\underline{C}_{bc} - \underline{C}_{cb}\dot{\underline{C}}_{ba}\underline{C}_{ab}\underline{C}_{bc}$$
$$= -\dot{\underline{C}}_{cb}\underline{C}_{bc} + \underline{C}_{cb}(-\dot{\underline{C}}_{ba}\underline{C}_{ab})\underline{C}_{bc}$$
$$= \underline{\omega}_c^{cb\times} + \underline{C}_{cb}\underline{\omega}_b^{ba\times}\underline{C}_{bc}$$
$$= \underline{\omega}_c^{cb\times} + (\underline{C}_{cb}\underline{\omega}_b^{ba})^{\times}$$

Consider

$$\underline{C}_{cb}\underline{\omega}_b^{ba\times}\underline{C}_{bc} = (\underline{C}_{cb}\underline{\omega}_b^{ba})^{\times}$$

Then we can say

$$\underline{\omega}_c^{ca} = \underline{\omega}_c^{cb} + \underline{C}_{cb}\underline{\omega}_b^{ba}$$

which leads to the relationship equation

$$\underset{\rightarrow}{\omega}_{ca} = \underset{\rightarrow}{\omega}_{cb} + \underset{\rightarrow}{\omega}_{ba}.$$

With the fact that $\underset{\rightarrow}{\omega}_{aa} = \underset{\rightarrow}{\mathbf{0}}$, we can readily derive from the above equation that $\underset{\rightarrow}{\mathbf{0}} = \underset{\rightarrow}{\omega}_{ab} + \underset{\rightarrow}{\omega}_{ba}$. □

Time Derivatives

Assume a fixed (inertial, absolute) frame of reference $\mathcal{F}_a$ and two moving frames of reference $\mathcal{F}_b$ and $\mathcal{F}_c$, each rotating relative to $\mathcal{F}_a$ at rates of $\underset{\rightarrow}{\omega}_{ba}$ and $\underset{\rightarrow}{\omega}_{ca}$ respectively. Then we have

$$\left(\frac{d}{dt}\underset{\rightarrow}{u}\right)_a = \left(\frac{d}{dt}\underset{\rightarrow}{u}\right)_b + \underset{\rightarrow}{\omega}_{ba} \times \underset{\rightarrow}{u} \tag{1.48}$$

$$\left(\frac{d}{dt}\underset{\rightarrow}{u}\right)_a = \left(\frac{d}{dt}\underset{\rightarrow}{u}\right)_c + \underset{\rightarrow}{\omega}_{ca} \times \underset{\rightarrow}{u} \tag{1.49}$$

If we denote the absolute derivative by •, the relative derivative in $\mathcal{F}_b$ by ∘, and the relative derivative in $\mathcal{F}_c$ by ∗, then the above equations become

$$\overset{\bullet}{\underset{\rightarrow}{u}} = \overset{\circ}{\underset{\rightarrow}{u}} + \underset{\rightarrow}{\omega}_{ba} \times \underset{\rightarrow}{u}$$

$$\overset{\bullet}{\underset{\rightarrow}{u}} = \overset{*}{\underset{\rightarrow}{u}} + \underset{\rightarrow}{\omega}_{ca} \times \underset{\rightarrow}{u}$$

Note that the absolute time derivative formulae looks the same as each other, but the rotating rates are different.

1.3 Euler Parameters and Unit Quaternion

Euler's theorem states that any rotation of an object in 3-D space leaves some axis fixed: this is the rotation axis. As a result, any rotation can be described by a unit vector $\underline{a}$ (satisfying $\underline{a}^T\underline{a} = 1$) in the direction of the rotational axis, and the angle of rotation, ϕ, about $\underline{a}$. The rotation matrix is represented by

$$\underline{C}(\underline{a}, \phi) = \cos\phi\underline{1} + (1 - \cos\phi)\underline{a}\underline{a}^T - \sin\phi\underline{a}^{\times} \tag{1.50}$$

where the set $(\underline{\boldsymbol{a}}, \phi)$ is often called the Euler axis/angle variables.

To avoid a triangular calculation, these variables can be replaced by the so-called Euler parameters:

$$\eta = \cos\frac{\phi}{2} \tag{1.51}$$

$$\underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{a}}\sin\frac{\phi}{2} \tag{1.52}$$

Note that $\underline{\boldsymbol{\varepsilon}}^T\underline{\boldsymbol{\varepsilon}} + \eta^2 = 1$. Then the rotation matrix becomes

$$\underline{\boldsymbol{C}} = (2\eta^2 - 1)\underline{\boldsymbol{1}} + 2\underline{\boldsymbol{\varepsilon}}\underline{\boldsymbol{\varepsilon}}^T - 2\eta\underline{\boldsymbol{\varepsilon}}^\times \stackrel{\Delta}{=} \underline{\boldsymbol{C}}(\underline{\boldsymbol{\varepsilon}}, \eta) \tag{1.53}$$

Now, let us take a look at the rotation matrix corresponding to two consecutive rotations, represented by either their Euler axis/angle variables, or the Euler parameters,

$$\underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}_3, \phi_3) = \underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}_2, \phi_2)\underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}_1, \phi_1) \tag{1.54}$$

$$\underline{\boldsymbol{C}}(\underline{\boldsymbol{\varepsilon}}_3, \eta_3) = \underline{\boldsymbol{C}}(\underline{\boldsymbol{\varepsilon}}_2, \eta_2)\underline{\boldsymbol{C}}(\underline{\boldsymbol{\varepsilon}}_1, \eta_1) \tag{1.55}$$

After some tedious matrix algebraic manipulation, this leads to the following relationship:

$$\begin{cases} \cos\frac{\phi_3}{2} & = \cos\frac{\phi_2}{2}\cos\frac{\phi_1}{2} - \sin\frac{\phi_2}{2}\sin\frac{\phi_1}{2}\underline{\boldsymbol{a}}_1^T\underline{\boldsymbol{a}}_2 \\ \sin\frac{\phi_3}{2}\underline{\boldsymbol{a}}_3 & = \cos\frac{\phi_2}{2}\sin\frac{\phi_1}{2}\underline{\boldsymbol{a}}_1 + \sin\frac{\phi_2}{2}\cos\frac{\phi_1}{2}\underline{\boldsymbol{a}}_2 + \underline{\boldsymbol{a}}_1^\times\underline{\boldsymbol{a}}_2\sin\frac{\phi_2}{2}\sin\frac{\phi_1}{2} \end{cases} \tag{1.56}$$

or

$$\begin{cases} \eta_3 & = \eta_2\eta_1 - \underline{\boldsymbol{\varepsilon}}_1^T\underline{\boldsymbol{\varepsilon}}_2 \\ \underline{\boldsymbol{\varepsilon}}_3 & = \eta_2\underline{\boldsymbol{\varepsilon}}_1 + \eta_1\underline{\boldsymbol{\varepsilon}}_2 + \underline{\boldsymbol{\varepsilon}}_1^\times\underline{\boldsymbol{\varepsilon}}_2 \end{cases} \tag{1.57}$$

In matrix format, we obtain the following

$$\begin{bmatrix} \underline{\boldsymbol{\varepsilon}}_3 \\ \eta_3 \end{bmatrix} = \left[\begin{array}{c|c} \eta_2\underline{\boldsymbol{1}} - \underline{\boldsymbol{\varepsilon}}_2^\times & \underline{\boldsymbol{\varepsilon}}_2 \\ -\underline{\boldsymbol{\varepsilon}}_2^T & \eta_2 \end{array}\right] \begin{bmatrix} \underline{\boldsymbol{\varepsilon}}_1 \\ \eta_1 \end{bmatrix} \stackrel{\Delta}{=} \underline{\boldsymbol{L}}(\underline{\boldsymbol{\varepsilon}}_2, \eta_2)\begin{bmatrix} \underline{\boldsymbol{\varepsilon}}_1 \\ \eta_1 \end{bmatrix} \tag{1.58}$$

$$\begin{bmatrix} \underline{\boldsymbol{\varepsilon}}_3 \\ \eta_3 \end{bmatrix} = \left[\begin{array}{c|c} \eta_1\underline{\boldsymbol{1}} + \underline{\boldsymbol{\varepsilon}}_1^\times & \underline{\boldsymbol{\varepsilon}}_1 \\ -\underline{\boldsymbol{\varepsilon}}_1^T & \eta_1 \end{array}\right] \begin{bmatrix} \underline{\boldsymbol{\varepsilon}}_2 \\ \eta_2 \end{bmatrix} \stackrel{\Delta}{=} \underline{\boldsymbol{R}}(\underline{\boldsymbol{\varepsilon}}_1, \eta_1)\begin{bmatrix} \underline{\boldsymbol{\varepsilon}}_2 \\ \eta_2 \end{bmatrix} \tag{1.59}$$

The matrix representation of $\{\underline{\boldsymbol{\varepsilon}}, \eta\}$ is one of the expressions of the so-called *unit quaternion*, denoted by

$$\overline{\boldsymbol{q}} \stackrel{\Delta}{=} \begin{bmatrix} \underline{\boldsymbol{q}} \\ q \end{bmatrix} \tag{1.60}$$

In the current case,

$$\overline{\boldsymbol{q}} = \begin{bmatrix} \underline{\boldsymbol{\varepsilon}} \\ \eta \end{bmatrix}. \tag{1.61}$$

It is obvious that $|\overline{\boldsymbol{q}}|^2 = \underline{\boldsymbol{\varepsilon}}^T\underline{\boldsymbol{\varepsilon}} + \eta^2 = 1$. Therefore the Euler parameter is considered as a unit quaternion.

We define the quaternion multiplication as

$$\overline{\boldsymbol{q}} \otimes \overline{\boldsymbol{p}} \stackrel{\Delta}{=} \left[\begin{array}{c|c} q\underline{\boldsymbol{1}} - \underline{\boldsymbol{q}}^\times & \underline{\boldsymbol{q}} \\ \hline -\underline{\boldsymbol{q}}^T & q \end{array}\right] \overline{\boldsymbol{p}} = \underline{\boldsymbol{L}}(\overline{\boldsymbol{q}})\overline{\boldsymbol{p}} \tag{1.62}$$

$$\overline{\boldsymbol{q}} \otimes \overline{\boldsymbol{p}} \overset{\Delta}{=} \left[\begin{array}{c|c} p\underline{\mathbf{1}} + \underline{\boldsymbol{p}}^{\times} & \underline{\boldsymbol{p}} \\ \hline -\underline{\boldsymbol{p}}^{T} & p \end{array}\right] \overline{\boldsymbol{q}} = \underline{\boldsymbol{R}}(\overline{\boldsymbol{p}})\overline{\boldsymbol{q}} \tag{1.63}$$

Its inverse (representing a reverse rotation of angle $-\phi$ about unit axis $\underline{\boldsymbol{a}}$) is defined by

$$\overline{\boldsymbol{q}}^{-1} \overset{\Delta}{=} \begin{bmatrix} -\underline{\boldsymbol{q}} \\ q \end{bmatrix} \tag{1.64}$$

and one can prove that

$$\overline{\boldsymbol{q}} \otimes \overline{\boldsymbol{q}}^{-1} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} \overset{\Delta}{=} \overline{\mathbf{1}} \tag{1.65}$$

Using these definitions, we can obtain the formulation of attitude error. Assume $(\underline{\boldsymbol{a}}_d, \phi_d)$ and $(\underline{\boldsymbol{a}}, \phi)$ represent the desired attitude and actual attitude, respectively. The error between the actual and the desired attitudes can be treated as a consecutive rotation from the desired attitude,

$$\underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}, \phi) = \underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}_e, \phi_e)\underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}_d, \phi_d) \tag{1.66}$$

This leads to

$$\underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}_e, \phi_e) = \underline{\boldsymbol{C}}(\underline{\boldsymbol{a}}, \phi)\underline{\boldsymbol{C}}^{-1}(\underline{\boldsymbol{a}}_d, \phi_d)$$

In other words,

$$\overline{\boldsymbol{q}}_e = \overline{\boldsymbol{q}} \otimes \overline{\boldsymbol{q}}_d^{-1} = \left[\begin{array}{c|c} \eta_d \underline{\mathbf{1}} - \underline{\boldsymbol{\varepsilon}}_d^{\times} & -\underline{\boldsymbol{\varepsilon}}_d \\ \hline \underline{\boldsymbol{\varepsilon}}_d^{T} & \eta_d \end{array}\right] \begin{bmatrix} \underline{\boldsymbol{\varepsilon}} \\ \eta \end{bmatrix} = \underline{\boldsymbol{R}}(\overline{\boldsymbol{q}}_d^{-1})\overline{\boldsymbol{q}} \tag{1.67}$$

This expression shows that the error attitude (quaternion) is now represented by the actual attitude and the desired attitude. Further, as $\overline{\boldsymbol{q}} \to \overline{\boldsymbol{q}}_d$, the steady error attitude $\overline{\boldsymbol{q}}_e \to \overline{\mathbf{1}}$.

Under the Euler parameters or unit quaternion, the angular rate vector $\underset{\rightarrow}{\boldsymbol{\omega}}$ is described as

$$\underline{\boldsymbol{\omega}} = \dot{\phi}\underline{\boldsymbol{a}} - (1 - \cos\phi)\underline{\boldsymbol{a}}^{\times}\dot{\underline{\boldsymbol{a}}} + \sin\phi\dot{\underline{\boldsymbol{a}}} \tag{1.68}$$

and

$$\dot{\overline{\boldsymbol{q}}} = \begin{bmatrix} \dot{\underline{\boldsymbol{\varepsilon}}} \\ \dot{\eta} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -\underline{\boldsymbol{\omega}}^{\times} & \underline{\boldsymbol{\omega}} \\ -\underline{\boldsymbol{\omega}}^{T} & 0 \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{\varepsilon}} \\ \eta \end{bmatrix} \overset{\Delta}{=} \frac{1}{2}\underline{\boldsymbol{\Omega}}(\underline{\boldsymbol{\omega}})\overline{\boldsymbol{q}} \tag{1.69}$$