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Introduction

In this chapter, the fundamental concepts of mathematical optimization and multi-parametric programming will be presented. Such concepts will be the foundation towards the development of state-of-the-art multi-parametric programming strategies and applications, which will appear in this book in the next chapters.

1.1 Concepts of Optimization

1.1.1 Convex Analysis

This section presents the idea of convex sets and introduces function convexity. Convexity plays a vital role to establish the required properties which will enable a multi-parametric solution to hold. In this setting, the following definitions are established.

Definition 1.1 (Line). Consider the points x_1 and $x_2 \in \mathbb{R}^n$. Then the line that passes through these points is defined as

$$\{x | x = (1 - \gamma)x_1 + \gamma x_2, \forall \gamma \in \mathbb{R}\}. \tag{1.1}$$

Definition 1.2 (Line Segment). The closed line segment joining the points x_1,$x_2 \in \mathbb{R}^n$ is defined as:

$$\{x | x = (1 - \gamma)x_1 + \gamma x_2, 0 \leq \gamma \leq 1\}. \tag{1.2}$$

Multi-parametric Optimization and Control, First Edition.
Efstratios N. Pistikopoulos, Nikolaos A. Diangelakis, and Richard Oberdieck.

Definition 1.3 (Convex Set). A set $C \in \mathbb{R}^n$ is a convex set, if the closed line segment joining any two points in the set C belongs to the set C for each γ such that $0 \leq \gamma \leq 1$.

1.1.1.1 Properties of Convex Sets

Let C_1 and C_2 be convex sets defined in $\mathbb{R}^n$. Then

(1) The intersection of $C_1 \cap C_2$ is a convex set.
(2) The summation $C_1 + C_2$ of two convex sets is a convex set.
(3) Let α be a real number. The product αC_1 is a convex set.

Examples of convex sets include lines, polytopes and polyhedra, and open and closed halfspaces.

Definition 1.4 (Convex Function). Let $C \in \mathbb{R}^n$ be a convex subset, and the real function $f(x)$ defined in C. The function $f(x)$ is a convex function if for any $x_1,x_2 \in C$,

$$f\left[(1-\gamma)x_1 + \gamma x_2\right] \leq (1-\gamma)f(x_1) + \gamma f(x_2), 0 \leq \gamma \leq 1. \quad (1.3)$$

If strict inequality holds in expression (1.3) for $0 < \gamma < 1$, then $f(x)$ is a strictly convex function.

Definition 1.5 (Concave Function). Let $C \in \mathbb{R}^n$ be a convex subset, and the real function $f(x)$ defined in C. The function $f(x)$ is a concave function if for any $x_1,x_2 \in C$,

$$f\left[(1-\gamma)x_1 + \gamma x_2\right] \geq (1-\gamma)f(x_1) + \gamma f(x_2), 0 \leq \gamma \leq 1. \quad (1.4)$$

If strict inequality holds in expression (1.4) for $0 < \gamma < 1$, then $f(x)$ is a strictly concave function.

1.1.1.2 Properties of Convex Functions

(1) Let $f_1(x), \dots, f_n(x)$ be convex functions defined on a convex subset C. Their summation

$$f_1(x) + \cdots + f_n(x) \quad (1.5)$$

is convex, and if at least of one $f_i(x)$ is a strictly convex function, then their summation is strictly convex.

(2) Let a γ be a positive number and $f(x)$ be a (strictly) convex function defined in a convex subset $C \in \mathbb{R}^n$. Then the product $\gamma f(x)$ is (strictly) convex.

(3) Let $f(x)$ be a (strictly) convex function defined in $C \in \mathbb{R}^n$, and $g(y)$ be an increasing convex function defined on the range of $f(x)$ in $\mathbb{R}$. Then, the composite function $g[f(x)]$ defined in C is a (strictly) convex function.

(4) Let $f_1(x), \ldots, f_n(x)$ be convex functions defined on a convex subset C. If these functions are bounded from above, their pointwise supremum

$$f(x) = \max\{f_1(x), \ldots, f_n(x)\} \tag{1.6}$$

is a convex function on C.

(5) Let $f_1(x), \ldots, f_n(x)$ be concave functions defined on a convex subset C. If these functions are bounded from below, their pointwise infimum

$$f(x) = \min\{f_1(x), \ldots, f_n(x)\} \tag{1.7}$$

is a concave function on C.

1.1.2 Optimality Conditions

We introduce the following definitions for the solution of general nonlinear optimization problems:

Definition 1.6 (Local Minimum). $x^* \in \mathbb{R}^n$ is called a local minimum if there exists ball of radius ϵ around x^*, $B(x^*)$, such that

$$f(x^*) \leq f(x), \ \ \forall x \in B(x^*). \tag{1.8}$$

Definition 1.7 (Global Minimum). $x^* \in \mathbb{R}^n$ is called a global minimum if

$$f(x^*) \leq f(x), \ \ \forall x \in \mathbb{R}^n. \tag{1.9}$$

A constrained nonlinear optimization problem, which aims to minimize a real valued function $f(x)$ subject to the inequality constraints $g(x) = \{g_i(x) \leq 0, i \in \mathbb{I}\}$ and equality constraints

$h(x) = \{h_j(x) = 0, j \in \mathbb{J}\}$ is denoted as

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) \leq 0 \\ & h(x) = 0 \cdot \\ & x \in \mathbb{R}^n \end{aligned} \tag{1.10}$$

Problem (1.10) is a nonlinear optimization problem, if and only if, at least one of $f(x), g_i(x), h_j(x)$ is a nonlinear function. We assume that the aforementioned functions are continuous and differentiable.

Definition 1.8 (Active Constraints). An inequality constraint $g_i(x)$ is called active at a point $\bar{x} \in X$ if $g_i(\bar{x}) = 0$. Conversely, $g_i(x)$ is called inactive if $g_i(\bar{x}) < 0$.

Remark 1.1 If one step of the dual simplex algorithm consists of changing one element of the active set, i.e. let $k_1 = \{i_1, \dots, i_{n-1}, i_n\}$, then the dual pivot involving the constraint i_n yields $k_2 = \{i_1, \dots, i_{n-1}, i_{n+1}\}$.

The first-order constraint qualifications that will be presented in the following text are necessary prerequisites to identify whether a feasible point $\bar{x}$ is a local optimum of the function $f(x)$.

- **Linear independence constraint qualification**: The gradients $\nabla g_j(\bar{x})$ for all $i \in I$ and $\nabla h_i(\bar{x})$ for all $j \in J$ are linearly independent.
- **Slater constraint qualification**: The constraints $g_i(\bar{x})$ for all $i \in I$ are pseudo-convex[1] at $\bar{x}$, while the constraints $h_j(\bar{x})$ for all $j \in J$ are quasi-convex or quasi-concave.[2] In addition, the gradients $\nabla h_j(\bar{x})$ are linearly independent and there exists $\tilde{x}$ such that $g_i(\tilde{x}) < 0$ and $h_j(\tilde{x}) = 0$.

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[1] A function $f(x)$ is called pseudo-convex if for all feasible x, y where $\nabla f(x)(x - y) \geq 0$ we have $f(y) \geq f(x)$.

[2] A function $f(x)$ is called quasi-convex if for all feasible x, y and $\gamma \in [0, 1]$ we have $f(\gamma x + (1 - \gamma)y) \leq \max\{f(x), f(y)\}$. Note that a quasi-concave function is a function whose negative is quasi-convex.

1.1.2.1 Karush–Kuhn–Tucker Necessary Optimality Conditions

Let $f(x)$ and $g(x)$ be differentiable at a feasible solution $x^* \in X$, and let $h(x)$ have continuous partial derivatives at x^*. In addition, let p be the number of active inequality constraints at x^*. Then if one of the aforementioned constraint qualifications hold, there exist Lagrange multipliers λ, μ such that

$$\begin{aligned} \nabla_x f(x^*) + \mu^T \nabla_x h(x^*) + \lambda^T \nabla_x g(x^*) &= 0 \\ h(x^*) &= 0 \\ g(x^*) &\leq 0 \\ \lambda_i g_i(x^*) &= 0 \;\; i = 1, 2, \ldots, p \\ \lambda_i &\geq 0 \;\; i = i, 2, \ldots, p \end{aligned} \quad . \tag{1.11}$$

These conditions are the Karush–Kuhn–Tucker (KKT) Necessary Conditions and they are the basis for the solution of nonlinear optimization problems.

1.1.2.2 Karun–Kush–Tucker First-Order Sufficient Optimality Conditions

Consider the sets $J^+ = j : \mu_j > 0$ and $J^- = j : \mu_j < 0$. Then, if the following conditions hold:

- $f(x)$ is pseudo-convex at $\overline{x}$ with respect to all other feasible points x.
- $g_i(x)$ for all $j \in J$ are quasi-convex at $\overline{x}$ with respect to all other feasible points x.
- $h_j(x)$ for all $i \in J^+$ are quasi-convex at $\overline{x}$ with respect to all other feasible points x.
- $h_j(x)$ for all $i \in J^-$ are quasi-concave at $\overline{x}$ with respect to all other feasible points x.

then $\overline{x}$ is a global optimum of problem (1.10). If the aforementioned conditions hold only within a ball of radius ϵ around $\overline{x}$, then $\overline{x}$ is a local optimum of problem (1.10).

1.1.3 Interpretation of Lagrange Multipliers

Consider the following problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & h(x) = \beta \\ & x \in \mathbb{R}^n \end{aligned} \; . \tag{1.12}$$

Let x^* be the global minimum of problem (1.12), and that the gradient of the equality constraints are linearly independent. In addition, assume that the corresponding Lagrange multiplier is λ^*. The vector $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ is a perturbation vector. The solution of problem (1.12) is a function of the perturbation vector along with the multiplier. Hence, the Lagrange function can be written as

$$L(x, \lambda) = f(x(\beta)) + \lambda(\beta)^T [h(x(\beta)) - \beta]. \tag{1.13}$$

Calculating the partial derivative of the Lagrange function with respect to the perturbation vector, we have

$$\nabla_\beta L = \left[\frac{\partial x}{\partial \beta}\right]^T \left(\nabla_x f + \left[\frac{\partial x}{\partial b}\right]^T \lambda\right) + \left[\frac{\partial x}{\partial \beta}\right]^T [h(x)) - \beta] - \lambda \tag{1.14}$$

which yields

$$\nabla_\beta L(x^*, \lambda^*) = -\lambda^* \tag{1.15}$$

Hence, the Lagrange multipliers can be interpreted as a measure of sensitivity of the objective function with respect to the perturbation vector of the constraints at the optimum point x^*.

1.2 Concepts of Multi-parametric Programming

1.2.1 Basic Sensitivity Theorem

Having the essentials of optimization for the purposes of this book covered, the objective of this subchapter is to introduce the role of parameters in an optimization formulation. In this context, the following multi-parametric programming problem is considered:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x,\theta) \\ \text{subject to} \quad & g(x,\theta) \leq 0 \\ & h(x,\theta) = 0\,, \\ & x \in \mathbb{R}^n \\ & \theta \in \mathbb{R}^m \end{aligned} \tag{1.16}$$

where x is the vector of the continuous optimization variables, θ is the vector of the uncertain parameters, and the sets $i \in \mathbb{I}, j \in \mathbb{J}$ correspond to the inequality and equality constraint sets, respectively.

Theorem 1.1 ***(Basic Sensitivity Theorem, [1])*** *Let a general multi-parametric programming problem be described by (1.16). Assume that the functions defining problem (1.16) are twice differentiable in x and their gradients with respect to x and the constraints are once continuously differentiable in θ in a neighborhood of (x^*, θ^*). In addition, assume that the second-order sufficient conditions for a local minimum of the problem hold at x^* with associated Lagrange multipliers λ^* and μ^*. Lastly, let the gradients $\nabla g_i(x^*, \theta^*)$ (for $i \in \mathbb{I}$ such that $g_i(x^*, \theta^*) = 0$) and $\nabla h_j(x^*, \theta^*)$ be linearly independent (i.e. LICQ holds), and $\lambda_i \geq 0$ for $i \in \mathbb{I}$ such that $g_i(x^*, \theta^*) = 0$, i.e. strict complementary slackness (SCS) holds.*

Then, the first-order sensitivity results for a second-order local minimizing point x^* are known as the basic sensitivity theorem (BST), and the following properties hold:

- *x^* is a local isolated minimizing point of the problem and the associated Lagrange multipliers λ_i^* and μ_j^* are unique.*
- *For θ in the neighborhood or θ^*, there exists a unique, but continuously differentiable vector function $\eta(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]^T$ satisfying the second-order sufficient conditions for a local minimum of the problem with associated unique Lagrange multipliers $\lambda(\theta)$ and $\mu(\theta)$.*
- *For θ near θ^* the set of binding inequalities is unchanged, SCS holds and the binding constraint gradients are linearly independent at $x(\theta)$.*

Proof: See [1]. □

If there exist Lagrange multipliers, λ_i^* and μ_j^*, such that the first-order KKT conditions hold, then we have:

$$\begin{aligned}
L(x^*, \mu^*, \lambda^*, \theta^*) &= \nabla_x f(x^*, \theta^*) + \mu^T \nabla_x h(x^*, \theta^*) + \lambda^T \nabla_x g(x^*, \theta^*) = 0 \\
h(x^*, \theta^*) &= 0 \\
g(x^*, \theta^*) &\le 0 \\
\lambda_i g_i(x^*, \theta^*) &= 0, \quad i = 1, 2, \ldots, p \\
\lambda_i &\ge 0, \quad i = i, 2, \ldots, p
\end{aligned} \tag{1.17}$$

and the vector $F(x, \lambda, \mu, \theta)$ is defined as follows:

$$F(x, \lambda, \mu, \theta) = \begin{bmatrix} \nabla_x L(x, \lambda, \mu, \theta) \\ \lambda_i g_i(x, \theta) \\ h(x, \theta) \end{bmatrix} = 0 \tag{1.18}$$

Furthermore, if there exists $z(x)$ for which

$$z(x) \nabla_{xx} L(\eta, \theta) z(x) \ge 0, \quad \forall z \neq 0 \tag{1.19}$$

the Basic Sensitivity Theorem holds, and it is identically satisfied for a neighborhood θ around θ^* and can be differentiated with respect to θ to yield explicit expressions for the partial derivatives of the vector function $\eta(\theta) = [x(\theta), \lambda(\theta), \mu(\theta)]^T$.

The first-order estimate of the variation of an isolated local solution x(θ) of (1.16) and the associated unique Lagrange multipliers $\lambda(\theta)$ and $\mu(\theta)$ can be approximated, given that $\eta(\theta^*) = [x(\theta^*), \lambda(\theta^*), \mu(\theta^*)]^T$ is known and that $\nabla_\theta \eta(\theta^*)$ is available.

In particular, let a be the concatenation of the vectors η and θ $a = [\eta^T | \theta^T]^T$. The first-order Taylor expansion of the vector F around a^* can be expressed as follows:

$$F(a) = \nabla_a F(a^*)(a - a^*) + F(a^*) \tag{1.20}$$

Under the assumptions and the principles of the Basic Sensitivity Theorem, in a neighborhood of a^* the first-order KKT conditions hold and the value of F(a) around a^* remains zero. For systems that consist of polynomial objective functions of up to second degree and linear constraints, with respect to the optimization variables and the uncertain parameters, the first-order Taylor expansion is exact. Hence, the exact multi-parametric solution can be obtained for the following multi-parametric quadratic programming problem

$$\begin{array}{ll}\min\limits_{x} & \frac{1}{2}x^T Q x + x^T H^T \theta + c_x^T x \\ \text{s.t.} & A_i x \leq b_i + F_i \theta \\ & A_j x = b_j + F_j \theta \\ & \theta \in \Theta := \{\theta \in \mathbb{R}^m \mid CR_A \theta \leq CR_b\} \\ & x \in \mathbb{R}^n \end{array}, \tag{1.21}$$

where matrices $A_{i(1\times n)}$, and $F_{i(1\times m)}$, $A_{j(1\times n)}$, $F_{j(1\times m)}$ and the scalars b_i, b_j correspond to the ith and jth inequality and equality constraints of the sets $\mathbb{I}$ and $\mathbb{J}$, respectively. This problem serves as the basis that will be discussed in Part I, where its solution properties and solution strategies among other things are in focus. Part II then focusses on the application of such problems to optimal control, as the use of parameters enables the formulation of explicit model predictive control problems.

1.3 Polytopes

Multi-parametric programming is intimately related to the properties and operations applicable to polytopes. In the following, some basic definitions on polytopes are stated, which are used throughout the book.

Definition 1.9 A function $x(\theta) : \Theta \mapsto \mathbb{R}^n$, where $\Theta \in \mathbb{R}^q$ is a polytope, is called piecewise affine if it is possible to partition Θ into disjoint polytopes, called critical regions, CR_i and

$$x(\theta) = K_i \theta + r_i, \ \forall \theta \in CR_i. \tag{1.22}$$

Remark 1.2 The definition of piecewise quadratic is analogous.

Definition 1.10 The set $\mathcal{P}$ is called a n-dimensional polytope if it satisfies

$$\mathcal{P} := \left\{x \in \mathbb{R}^n | a_i^T x \leq b_i, i = 1, \dots, m\right\}, \tag{1.23}$$

where m is finite.

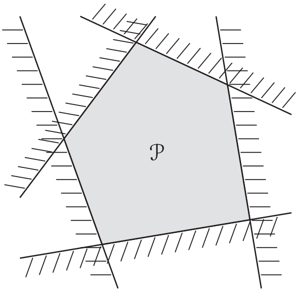

Figure 1.1 A schematic representation of a two-dimensional polytope $\mathcal{P}$.

A schematic representation of a polytope is given in Figure 1.1.

In addition to Definition (1.10), the following well-known characteristics of polytopes are considered:

- A polytope is called bounded if and only if there exists a finite $x_{\min} \in \mathbb{R}^n$ and $x_{\max} \in \mathbb{R}^n$ such $x_{\min} \leq x \leq x_{\max}$ for all $x \in \mathcal{P}$.
- A polytope, which is closed and bounded, is called compact.
- Let $\mathcal{P}$ be an n-dimensional polytope. Then, a subset of a polytope is called a face of $\mathcal{P}$ if it can be represented as

  $$\mathcal{F} = \mathcal{P} \cap \left\{x \in \mathbb{R}^n | a^T x = b\right\} \qquad (1.24)$$

  for some inequality $a^T x \leq b$, which holds for all $x \in \mathcal{P}$. The faces of polytopes of dimension $n-1, 1$, and 0 are referred to as facets, edges, and vertices, respectively.
- Two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ are called disjoint if $\mathrm{int}(\mathcal{P}_1) \cap \mathrm{int}(\mathcal{P}_2) = \emptyset$. Similarly, two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ are called overlapping if $\mathrm{int}(\mathcal{P}_1) \cap \mathrm{int}(\mathcal{P}_2) \neq \emptyset$. Lastly, two polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ are called adjacent or neighboring if $\mathcal{P}_1 \cap \mathcal{P}_2$ is a $n-1$-dimensional polytope.
- Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two adjacent polytopes. Then the facet-to-facet property is said to hold if $F = \mathcal{P}_1 \cap \mathcal{P}_2$ is a facet of both $\mathcal{P}_1$ and $\mathcal{P}_2$ (see Figure 1.2 for an illustration).
- Let $\mathcal{P}$ be an n-dimensional polytope. Then, there exists a series of k vertices $x_i \in \mathbb{R}^n$ such that

  $$\mathcal{P} := \left\{ x \in \mathbb{R}^n | x = \sum_{i=1}^{k} \lambda_i x_i, \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0 \right\}. \qquad (1.25)$$
- Eq. (1.23) is referred to the halfspace (or H) representation, while Eq. (1.25) denotes the vertex (or V) representation. The process

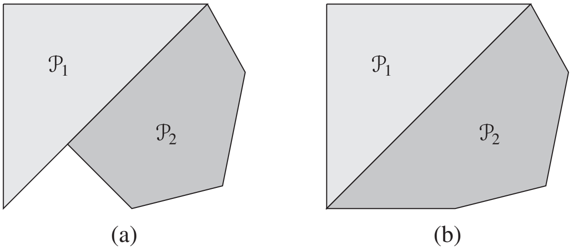

Figure 1.2 A schematic representation of the differences between two polytopes $\mathcal{P}$ and $\mathcal{P}_2$ (a), which are adjacent and (b) where the facet-to-facet property holds. Clearly, while all polytopes that satisfy the facet-to-facet property are adjacent, the opposite may not be true.

of moving from the halfspace to the vertex representation is referred to as vertex enumeration.

- The Chebyshev center of a polytope is given as the largest Euclidean ball that lies in a polytope [2]. It can be determined by solving the following linear programming (LP) problem:

$$\begin{aligned} R = \underset{x,r}{\text{minimize}} \ & -r \\ \text{subject to } \ & A_i x + r||A_i||_2 \leq b_i, \ \ \forall i = 1, \dots, m, \end{aligned} \tag{1.26}$$

  where the solution R denotes the radius of the largest Euclidean ball. Based on the solution of problem (1.26), the following conclusions can be drawn:
  - Problem (1.26) is infeasible: The polytope is empty.
  - $R = 0$: The polytope is lower-dimensional.
  - $R > 0$: The polytope is full-dimensional.

1.3.1 Approaches for the Removal of Redundant Constraints

A concept that is very important in multi-parametric programming is the aspect of redundancy:

Theorem 1.2 *(**[3]**) Consider an n-dimensional compact polytope $\mathcal{P}$ in halfspace representation. A constraint $a_i^T x \leq b_i$ is redundant if and only if*

$$\mathcal{P}_i = \{x \in \mathbb{R}^n | A_i x > b_i, A_k x \leq b_k, \forall k \neq i\} = \emptyset. \tag{1.27}$$

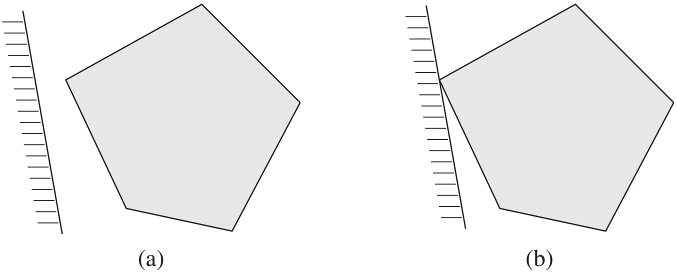

Figure 1.3 A schematic representation of (a) strongly and (b) weakly redundant constraints.

Additionally, a constraint $A_i x \leq b_i$ *is strongly redundant if and only if*

$$\mathcal{P}_i' = \{x \in \mathbb{R}^n | A_i x \geq b_i, A_k x \leq b_k, \forall k \neq i\} = \emptyset. \tag{1.28}$$

Remark 1.3 A constraint is called weakly redundant if it is redundant but not strongly redundant, i.e. Eq. (1.27) but not Eq. (1.28) holds. A schematic representation of weakly and strongly redundant constraints is shown in Figure 1.3.

If a polytope $\mathcal{P}$ does not feature any redundant constraints, it is said to be in minimal representation.

Consider an n-dimensional compact polytope $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following strategies aim at identifying the minimal representation of $\mathcal{P}$:

Remark 1.4 Here, two of the most common approaches used are reported. The field of the removal of redundant constraints has been widely studied, and its review is beyond the scope of this book. The reader is referred to [3, 4] for an interesting treatment of the matter.

1.3.1.1 Lower-Upper Bound Classification

Given the bounds $l_j \leq x_j \leq u_j, \forall j = 1, \dots, m$, a constraint $A_i x \leq b_i$ is redundant if

$$U_i \leq b_i, \tag{1.29}$$

where

$$U_i = \sum_{j \in P_j} A_{ij} u_j + \sum_{j \in N_j} A_{ij} l_j, \tag{1.30}$$

where $P_j = \{j | A_{ij} > 0\}$ and $N_j = \{j | A_{ij} < 0\}$. This approach relies on the identification of the worst-case scenario given the lower and upper bounds [5]. If these bounds are not available, they can be calculated by solving the following $2n$ LP problems [6]:

$$\begin{aligned} &\underset{x}{\text{Minimize}} \;\; \pm x_i \\ &\text{Subject to} \;\; x \in \mathcal{P}. \end{aligned} \tag{1.31}$$

1.3.1.2 Solution of Linear Programming Problem

Consider the following constraint-specific version of problem (1.26):

$$\begin{aligned} R_i = \; &\underset{x,r}{\text{minimize}} && -r \\ &\text{subject to} && Ax \leq (b - ||A^i||_2 r) \\ &&& A_i x = b_i \\ &&& ||A^i||_2 = ||1 - (AA_i^T)^2||_2 \\ &&& x \in \mathcal{P}, \;\; r \in \mathbb{R}, \end{aligned} \tag{1.32}$$

where $(\cdot)^2$ denotes the element-wise square of $(\cdot)$. Note that $Ax \leq b$ is assumed to be normalized such that $||a_i||_2 = 1$ for all $i = 1, \ldots, m$. Then the ith constraint is redundant if and only if $R_i \leq 0$. Note that this identifies weakly and strongly redundant constraints.

Remark 1.5 The solution of problem (1.32) identifies the largest Euclidean ball which on the set $\mathcal{K} = \{x | x \in \mathcal{P} \cup A_i x = b_i\}$, i.e. which lies on the ith constraint. Thus, the solution can be understood as the center of the ith constraint with respect to $\mathcal{P}$.

1.3.2 Projections

One of the operations used in this book is the (orthogonal) projection:

Definition 1.11 (Projection [7]).Let $P \subset \mathbb{R}^d \times \mathbb{R}^k$ be a polytope. Then the projection $\pi_d(P)$ of P onto $\mathbb{R}^d$ is defined as:

$$\pi_d(P) = \left\{ x \in \mathbb{R}^d | \exists y \in \mathbb{R}^k, (x, y) \in P \right\}. \tag{1.33}$$

Projecting polytopes is one of the fundamental operations in computational geometry and has many applications in control theory. Two commonly encountered strategies for the calculation of the projection are the following:

- Solving a multi-parametric linear programming (mp-LP) problem (see e.g. [8])
- Performing a Fourier–Motzkin (FM) elimination (see, e.g. [9])

In addition, the concept of a hybrid projection is introduced:

Definition 1.12 (Hybrid Projection). Consider the set $P \subset \mathbb{R}^d \times \mathbb{R}^k \times \{0,1\}^r$. Then, the hybrid projection $\tilde{\pi}_d(P)$ of P onto $\mathbb{R}^d$ is defined as

$$\tilde{\pi}_d(P) = \left\{x \in \mathbb{R}^d | \exists y \in \mathbb{R}^k \times \{0,1\}^r, (x,y) \in P\right\}. \tag{1.34}$$

By inspection it is clear that (i) $\tilde{\pi}_d(P)$ is obtained by performing at most 2^r projections, one for each combination of the binary variables and consequently (ii) $\tilde{\pi}_d(P)$ is generally a union of at most 2^r possibly overlapping polytopes.

A hybrid projection can thereby be performed by solving a multi-parametric mixed-integer programming problem purely based on feasibility requirements.

1.3.3 Modeling of the Union of Polytopes

The aim is to represent a union of polytopes $P = \bigcup_{i=1}^{p} \{x | G_i x \leq g_i\}$ as a single set of linear inequality constraints in order to seamlessly include them within multi-parametric programming problems. However, in order to address the possible non-convexity within unions of polytopes, the introduction of suitable binary variables is required. First, consider that a point $x \in P$ if and only if there exists at least one i such that $G_i x \leq g_i$. Thus, one binary variable y_i is defined such that

$$[G_i x \leq g_i] \rightarrow [y_i = 1] \tag{1.35a}$$

$$\sum_{i=1}^{p} y_i \geq 1. \tag{1.35b}$$

Let $G_{i,j}$ and $g_{i,j}$ denote the jth row and element of $G_i \in \mathbb{R}^{t_i \times n}$ and $g_i \in \mathbb{R}^{t_i}$, respectively. Then, the statement $G_i x \leq g_i$ holds if and only if $G_{i,j} x \leq g_{i,j}$, $\forall j$. Thus, one binary variable per row of G_i, y_i^j is defined such that

$$[G_{i,j} x \leq g_{i,j}] \leftrightarrow [y_i^j = 1] \tag{1.36a}$$

$$\left[\sum_{j=1}^{t_i} y_i^j = t_i\right] \rightarrow [y_i = 1] \tag{1.36b}$$

$$\sum_{i=1}^{p} y_i \geq 1. \tag{1.36c}$$

Based on [10, 11], Eqs. (1.1.36a)–(1.1.36c) are reformulated as

$$G_{i,j}^T x + M y_i^j \leq M + g_{i,j} \tag{1.37a}$$

$$G_{i,j}^T x - m y_i \geq g_{i,j} \tag{1.37b}$$

$$t_i y_i \leq \sum_{j=1}^{t_i} y_i^j \tag{1.37c}$$

$$y_i \geq \sum_{j=1}^{t_i} y_i^j + 1 - t_i, \tag{1.37d}$$

where $m \leq x \leq M$, $\forall x \in P$. Thus, the final formulation of the union as a set of linear inequality constraints featuring binary variables is given as

$$P = \bigcup_{i=1}^{p} \{x | G_i x \leq g_i\} \rightarrow \left\{ \begin{array}{rl} G_{i,j}^T x + M y_i^j & \leq M + g_{i,j} \\ -G_{i,j}^T x + m y_i^j & \leq -g_{i,j} \\ t_i y_i - \sum_{j=1}^{t_i} y_i^j & \leq 0 \\ -y_i + \sum_{j=1}^{t_i} y_i^j & \leq t_i - 1 \\ -\sum_{i=1}^{p} y_i & \leq -1 \end{array} \right. . \tag{1.38}$$

1.4 Organization of the Book

The remainder of this book is organized in two parts. In the first part, the theoretical and algorithmic essentials of multi-parametric programming problems will be established. These include algorithms for the solution of multi-parametric linear programming (mp-LP), multi-parametric quadratic programming (mp-QP), multi-parametric mixed-integer linear programming (mp-MILP), and multi-parametric mixed-integer quadratic programming (mp-MIQP) problems. On the other hand, the latter of these parts is focused on the applications of multi-parametric programming and specifically on its utilization to provide solutions to receding horizon optimization problems such as model predictive control.

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