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Basic Concepts of Reliability Engineering

This chapter reviews basic concepts and common reliability engineering practices in the manufacturing industry. In addition, we briefly introduce the history of Bayesian statistics and how it relates to advances in the field of reliability engineering.

Experienced reliability engineers who are very familiar with reliability basics and would like to start learning Bayesian statistics right away, may skip this chapter and start with Chapter 2. Bayesian statistics has unique advantages for reliability estimations and predictive analytics in complex systems. In other cases, Bayesian methods may provide flexible solutions to aggregate various sources of information to potentially reduce necessary sample sizes and therefore achieve cost effectiveness. The following chapters provide more specific discussions and case study examples to expand on these topics.

1.1 Introduction

High product quality and reliability are critical to any industry in today's competitive business environment. In addition, predictable development time, efficient manufacturing with high yields, and exemplary field reliability are all hallmarks of a successful product development process.

Some of the popular best practices in industry include Design for Reliability and Design for Six Sigma programs to improve product robustness during the design phase. One core competency in these programs is to adopt advanced predictive analytics early in the product development to ensure first-pass success, instead of over-reliance on physical testing at the end of the development phase or on field performance data after product release.

The International Organization for Standardization (ISO) defines reliability as the "ability of a structure or structural member to fulfil the specified requirements, during the working life, for which it has been designed" (ISO 2394:2015 General principles on reliability for structures, Section 2.1.8). Typically, reliability is stated in terms of probability and associated confidence level. As an example, the reliability of a light bulb can be stated as the probability that the light bulb will last 5000 hours under normal operating conditions is 0.95 with 95% confidence.

Accurate and timely reliability prediction during the product development phase provides inputs for the design strategy and boosts understanding and confidence in product reliability before products are released to the market. It is also desirable to utilize and aggregate information from different sources in an effective way for reliability predictions.

Practical Applications of Bayesian Reliability, First Edition. Yan Liu and Athula I. Abeyratne.

Companion website: www.wiley.com/go/bayesian20

Textbooks on reliability engineering nowadays are dominated by frequentist statistics approaches for reliability modeling and predictions. In a frequentist/classical framework, it is often difficult or impossible to propagate individual component level classical confidence intervals to a complex system comprising many components or subsystems. In a Bayesian framework, on the other hand, posterior distributions are true probability statements about unknown parameters, so they may be easily propagated through these system reliability models. Besides, it is often more flexible to use Bayesian models to integrate different sources of information, and update inferences when new data becomes available.

Given the benefits mentioned above, potential applications of Bayesian methods on reliability prediction are quite extensive. Historically, Bayesian methods for reliability engineering were applied on component reliability assessment where conjugate prior (will be discussed in Chapter 2) distributions were widely used due to mathematical tractability. Recent breakthroughs in computational algorithms have made it feasible to solve more complex Bayesian models, which have greatly boosted advancement and applications of Bayesian modeling. One popular algorithm is Markov chain Monte Carlo (MCMC) sampling, a method of simulating from a probability distribution based on constructing a Markov chain. MCMC methods along with rapid advancement in high-speed computing have made it possible for building and solving complex Bayesian models for system reliability.

Over the past one or two decades, Bayesian statistics books have appeared in different scientific fields. However, most existing Bayesian statistics books do not focus on reliability analysis/predictions, thus real-life practical examples on reliability modeling are often absent. This challenge prevents reliability engineers from adopting the Bayesian approach to solve real-life problems. The goal of our book is to address this gap.

A few general topics covered in this book are:

- Design for reliability
- Basic concepts of Bayesian statistics and models
- Bayesian models for component reliability estimation
- Bayesian models for system reliability estimation
- Bayesian networks
- Advanced Bayesian reliability models.

Specifically, the topics covered are:

- Design for reliability
  This topic includes reliability definition, basic probability theory and computations, statistical models, basics of component reliability prediction, basics of system reliability prediction, critical feature capability prediction, Monte Carlo simulations, and accelerated life testing (ALT), etc.
- Basic concepts of Bayesian statistics and models
  This topic includes Bayes' theorem and history, Bayesian inference vs. frequentist inference, basic statistical concepts: point estimate, confidence interval, discrete and continuous probability distributions, censored data, and selection of prior distributions (conjugate priors, non-informative priors, and informative priors), likelihood function, model selection criteria, introduction of MCMC algorithms and sampling methods, and Bayesian computation software (WinBUGS, OpenBUGS, Just Another Gibbs Sampler (JAGS), R, etc.).

- Bayesian models for component reliability estimation
  This topic includes component level reliability prediction from reliability life testing, binomial distribution, Poisson distribution, exponential distribution, Weibull distribution, normal distribution, log-normal distribution, and reliability prediction from ALT (Arrhenius model, inverse power law model, etc.).
- Bayesian models for system reliability estimation
  This topic includes reliability block diagram, series system, parallel system, mixed series and parallel system, fault tree analysis with uncertainty, process capability or design capability analysis with uncertainty, Monte Carlo simulation, and two-level nested Monte Carlo simulation and examples (strength-stress interference, tolerance stack up, etc.).
- Bayesian networks
  This topic includes basics of conditional probability, joint probability distributions, marginal probability distributions, structures of a Bayesian network, examples, and basic steps to construct a Bayesian network model.
- Advanced Bayesian reliability models
  This topic includes using hierarchical Bayesian models to predict reliability during iterative product development, to predict reliability of specific failure mechanisms, to aggregate different sources of imperfect data, to aggregate component level and system level data for system reliability prediction, and to borrow partial strength from historical product reliability information.

The first three chapters introduce commonly used reliability engineering methods and basics of Bayesian concepts and computations. The following chapters focus more on applications related to the individual topics introduced above. Readers are free to tailor their reading to specific chapters according to their interests and objectives.

1.1.1 Reliability Definition

In reliability engineering, product reliability is defined as the probability that a component or a system performs a required function under specified use conditions for a stated period of time. Note that the three key elements in the reliability definition are probability, use condition, and duration. Probability measures the likelihood of something happening. For example, when tossing a fair coin there is a 50% probability of the coin landing heads. When throwing a six-faced fair dice, the probability of observing each of the six outcomes (1, 2, 3, 4, 5, 6) is 1/6. Use conditions describe the conditions a product is operated under, e.g. temperature, humidity, pressure, voltage. Duration is usually related to the lifetime of a product. Reliability is usually estimated based on time to failure data from bench tests, accelerated life tests, or field service.

In engineering practices, it is common to define design requirements and use different types of tests, such as design verification tests or qualification tests, to ensure the product or the incoming parts meet these requirements. Here quality is measured by the probability of meeting a certain requirement, which can be thought of as reliability at time zero. Though these are quality assurance practices, the term "reliability" is sometimes used to refer to the probability of meeting a certain requirement.

Often in design verification tests, the samples are preconditioned through an equivalent lifecycle under specified stress conditions (to ensure reliable products, the stress conditions applied in the tests are usually as aggressive as or more aggressive than the actual use conditions in the field) before being tested against a requirement. In such cases, the probability of meeting the requirement can be thought of as reliability at one lifecycle. However, this may not be the case for every requirement. To ensure quality and reliability, requirements may be classified at different levels based on importance and risk. Specified confidence/reliability requirements (e.g. 95%/99%) are assigned to key product characteristics according to their risk level. A specified 95%/99% confidence/reliability requirement means that the probability of meeting a requirement shall be at least 99% at 95% confidence level. The concept of confidence interval in these reliability requirements will be elaborated on in Chapter 2. The required sample sizes needed to meet these requirements are typically stated in terms of a frequentist statistical approach. We will explore the Bayesian solution to these problems in this book.

Given these practices in the industry, in this book we use the term reliability to generally refer to both cases described above (i.e. the traditional reliability definition and the broader applications in quality assurance). To avoid confusion, in each chapter when we go through a specific topic or example, we reiterate the definition of reliability in the context of that topic or example.

1.1.2 Design for Reliability and Design for Six Sigma

One way to assess reliability is the analysis of field data to estimate the product life expectancy and the probability of failure. This approach is appropriate for estimating reliability and monitoring performance trending of released products. However, this approach is not applicable for decision making related to new design/parts, especially when there are significant changes in the design. Quality and reliability issues caught after product release can be extremely costly to both customers and manufacturers. Product recalls due to reliability issues often result in customer dissatisfaction, huge financial losses due to repair and "firefighting" expenses, and brand degradation. Some product failures can even result in safety issues depending on the risk level of the failure mode.

Another approach in traditional industry processes is that the reliability analysis generally occurs at the end of product development, after the design is complete. One challenge in this product development process is that at the early phase of design, normally there's no adequate reliability analysis to drive decisions. This leads to potential risks of over or under design. Subsequent design changes may vary component use conditions, resulting in different reliability and design margins. Failures caught later in the development process may require many iterations of design change, which are costly and time consuming. For example, design-related issues such as power density and variability over time become a concern with advanced technology in the electronics industry (Turner 2006). As a result, there is a large demand to shift responsibility for reliability assurance to designers (Turner 2006) or collaboration between reliability engineers and designers.

In industry, the Design for Reliability philosophy is often combined with a broader quality improvement initiative called Design for Six Sigma, which is a program to ensure high-quality design and manufacturing, and to minimize design iterations. Design for Six Sigma adopted many statistical techniques (including design of experiments, control charts, reliability testing, etc.) in product development processes to promote first-pass design success, to reduce manufacturing defects, to increase design robustness to environmental factors, to reduce waste, and to increase product lifetime.

Though initially the Design for Six Sigma program was invented as an initiative to improve quality, industry practices in various corporations have demonstrated that its main value is beyond quality improvement and is more on time delivery and cost savings (Hindo 2007). The cost of poor quality was estimated to be as high as 15–30% of the entire cost (Defeo 2001). Using the cost of poor quality as the driver of the project selection in the Six Sigma program, various corporations including Honeywell, General Electric, Black & Decker, and 3M reported cost savings as high as hundreds of millions of dollars or even a few billion dollars (Hindo 2007, Defeo 2001) after implementing the programs. It was estimated that corporations that have implemented Six Sigma programs spent less than 5% of revenue fixing problems, much less than the cost in other corporations who spent 25–40% of revenue fixing problems (Pyzdek 2003).

1.2 Basic Theory and Concepts of Reliability Statistics

In this chapter, some commonly used concepts and practices in reliability engineering are briefly introduced/reviewed. We will use R scripts for basic reliability analysis. R is a language and environment for statistical computing and graphics. It is a free software and can be downloaded from the website https://www.r-project.org. R is now widely used in academia and industry. R Studio is an open source software that provides a friendly user interface for R. The instructions for installing R and R Studio are provided in Appendix A. Commonly used R commands are provided in Appendix B.

1.2.1 Random Variables

A random variable (r.v.) maps an outcome of an experiment to real number. For example, if a coin is tossed the outcome is either a head or a tail. We can define a random variable, X in this experiment such that $X = 1$ if a head turns up and $X = 0$ if a tail turns up. The sample space, S of an experiment is defined as the set of all possible outcomes of the experiment. In this case the sample space is

$$S = \{H, T\}.$$

The sample space for the r.v. X is $\{0, 1\}$.

In another example, suppose we are interested in time to failure of an electronic circuit. In this case the random variable, T, is the lifetime of the electronic circuit. This is a continuous random variable and all possible outcomes consist of all non-negative real numbers. Probability distributions can be defined on the random variables to account for the uncertainty associated with the outcome of the experiment that generated the random variable. As an example, if we toss a fair coin then the probability of observing a head is 0.5. This can be stated as $P(X = 1) = 0.5$ and $P(X = 0) = 0.5$. Since X is a discrete random variable, these probabilities describe its probability mass function (PMF).

Properties of a random variable can be described by its probability density (mass for discrete r.v.) function, cumulative distribution function (CDF), reliability function, and the hazard function. The use of these functions depends on the question that we are trying to answer.

1.2.2 Discrete Probability Distributions

Discrete probability distributions are used for attribute data (binary data, e.g. good/bad, yes/no, pass/fail, etc.) or count data (the observations are non-negative integer values). Though generally continuous variable data are preferred in engineering practices, they may not always be available. In other cases, continuous data from product testing results are sometimes converted to the pass/fail type of attribute data. This type of converting is highly inefficient and results in loss of useful information, but it can still be seen in the industry in quality assurance as part of a tradition.

Commonly used discrete probability distributions include the binomial distribution and the Poisson distribution. In quality assurance, binomial distributions are often used for pass/fail data. Poisson distribution can be used for count data, e.g. to measure the distribution of the number of defects per unit area. We will revisit these distributions with examples in Chapters 2 and 4.

For a discrete random variable X with a sample space S, a PMF, $m(x)$ can be defined as

$$m(x) \geq 0, \quad x \in S,$$

and

$$\sum_{x \in S} m(x) = 1. \tag{1.1}$$

1.2.3 Continuous Probability Distributions

In reliability engineering, continuous probability distributions are often used to describe continuous data, such as time to failure, cycles to failure, etc. A Weibull distribution is often used to model time to failures. In other engineering practices, continuous probability distributions are used for dimensions, voltages, and any other continuous variables. Normal distribution is often used to model dimensions. Appendix C introduces commonly used discrete and continuous probability distributions. More details are discussed in Chapter 4.

For a continuous random variable, X, with possible values on the real line, a probability density function (PDF), $f(x)$, can be defined as

$$f(x) \geq 0, \quad -\infty < x < \infty,$$

and

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1. \tag{1.2}$$

With this definition any non-negative function that integrates to 1 over the real line can be considered to be a PDF. Which PDF to use is dependent on the type of data being analyzed.

1.2.4 Properties of Discrete and Continuous Random Variables

1.2.4.1 Probability Mass Function

Probability mass function (PMF), $m(x)$, is the probability that a discrete random variable X takes the value x. As an example, the following R code and Figure 1.1 show the

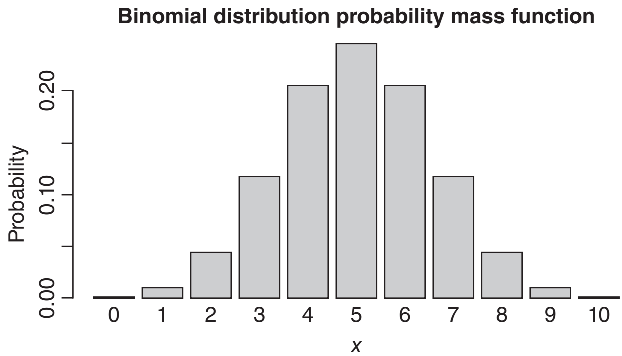

Figure 1.1 Probability mass function of a binomial distribution (size = 10, probability = 0.5).

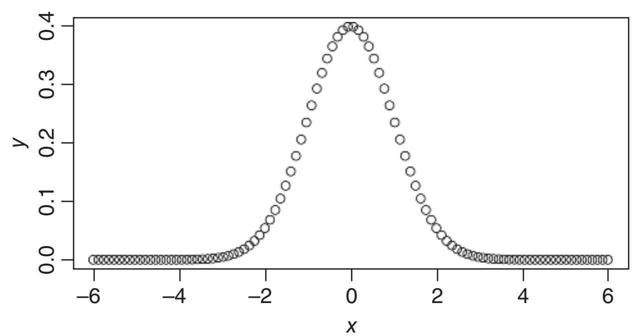

Figure 1.2 Probability density function of a normal distribution (mean = 1, sd = 1).

bar plot of a binomial distribution PMF with number of trials = 10 and probability of success = 0.5.

```
(1.2.4_Probability_Computations.R)
        ## probability mass function of a Binomial distribution

        x <- seq(0,10,by=1)
        y <- dbinom(x, size=10, prob=0.5)
        barplot(y, names=x, xlab = "x", ylab="Probability", main =
"Binomial distribution probability mass function")
```

1.2.4.2 Probability Density Function

The probability density function (PDF), $f(t)$, of a random variable T is non-negative real valued function that integrates to 1 over the range it is defined. As an example, normal distribution with mean 0 and standard deviation 1 is a PDF. In Chapter 4 we will discuss various probability distribution functions in detail.

The following R script shows how to plot the PDF of a normal distribution with mean = 0 and standard deviation = 1. Figure 1.2 shows the plotted PDF.

```
        x <- seq(-6,6,length=100)
        y <- dnorm(x,mean=0,sd=1)  # calculate the PDF of a Normal
distribution
        plot(x,y)  # generate PDF plot
```

1.2.4.3 Cumulative Distribution Function

The cumulative distribution function (CDF), $F(x)$, of a discrete random variable X at value k is the sum of all probabilities up to and including the value k, which is given by

$$F(k) = P(X \leq k) = \sum_{x \leq k} m(x), \tag{1.3}$$

where $m(x)$ is the PMF.

The CDF $F(t)$ of a continuous random variable X at value t is the cumulative probability of X having values less than or equal to t, i.e.

$$F(t) = P(x <= t) = \int_0^t f(x)\ \mathrm{d}x, \tag{1.4}$$

where $f(x)$ is the PDF. If the random variable T is the time to failure of a particular component then $F(t)$ provides the cumulative probability that the component fails on or before time t.

The R script to calculate the CDF of a normal distribution with mean = 0 and standard deviation = 1 is shown below. The CDF plot is shown in Figure 1.3.

```
        y1 <- pnorm(x,mean=0,sd=1) # calculate the CDF of a Normal
distribution
        plot(x,y1)  # generate CDF plot
```

1.2.4.4 Reliability or Survival Function

The reliability or survival function $R(t)$ is the probability of survival beyond time t. A reliability or survival function measures the percentage of products that survive a certain period of time without failures, i.e.

$$R(t) = 1 - F(t). \tag{1.5}$$

The R script to calculate the CDF of a normal distribution with mean = 0 and standard deviation = 1 is shown below. The reliability/survival curve is shown in Figure 1.4.

```
y2 <- 1-y1  # calculate 1-CDF (reliability)
plot(x,y2)  # generate reliability plot
```

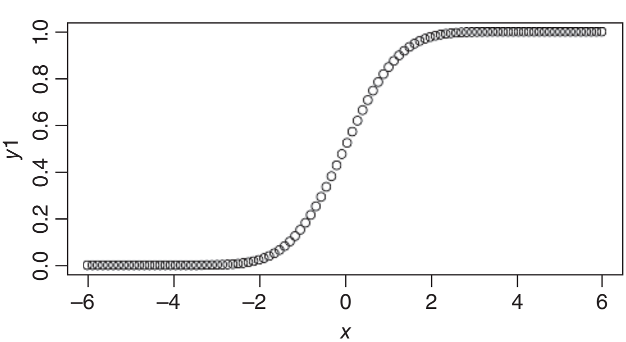

Figure 1.3 Cumulative distribution function of a normal distribution (mean = 1, sd = 1).

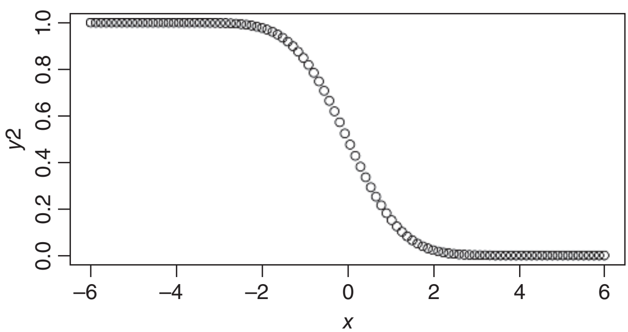

Figure 1.4 Reliability function of a normal distribution (mean = 1, sd = 1).

1.2.4.5 Hazard Rate or Instantaneous Failure Rate

The hazard rate or instantaneous failure rate, $h(t)$, for a continuous random variable, T, is the failure rate of the survivors at time t in the next moment following time t. Let $f(t)$ and $R(t)$ be the PDF and the reliability functions, respectively, then the hazard rate function is given by

$$h(t) = f(t)/R(t). \tag{1.6}$$

In this book we will use the terms "hazard rate" and "failure rate" to mean the instantaneous failure rate. Let's investigate the hazard rate through an exponential distribution. The lifetime distribution of certain electrical components such as light bulbs tends to follow exponential distributions. The PDF of an exponential distribution is given by

$$f(t) = \lambda e^{-\lambda t}, \tag{1.7}$$

where λ is the number of failures per unit time. If time is measured in hours, then it is the number of failures per hour. The reliability function of the above exponential distribution is given by

$$R(t) = P(T > t) = \int_t^\infty \lambda e^{-\lambda x} \mathrm{d}x = e^{-\lambda t}. \tag{1.8}$$

Combining Eqs. (1.6)–(1.8) we get $h(t) = \lambda$. Therefore, the hazard rate for an exponential distribution is the same as the number of failures per unit time. Theoretically λ could be any positive real number. Therefore, hazard rate is not a probability. One of the defining characteristics of the exponential distribution is that it has a constant hazard rate which is the same as its parameter, λ.

There are other lifetime distributions, such as a Weibull distribution, that are more flexible than an exponential distribution to model failure rates that are varying over time. We will discuss these in Chapter 4.

A product could have different stages of failure rates. If $h(t)$ is decreasing, then the failure mechanism is called "infant mortality." A product subjected to decreasing failure rate, typically experiences more failures early on, so it is in the best interest of the manufacturer not to release such a product to the market. To get rid of the products that fail early on, they can be subjected to accelerated use conditions for a short period in the final testing. A good example of this is battery burn-in. When $h(t)$ is increasing it is an

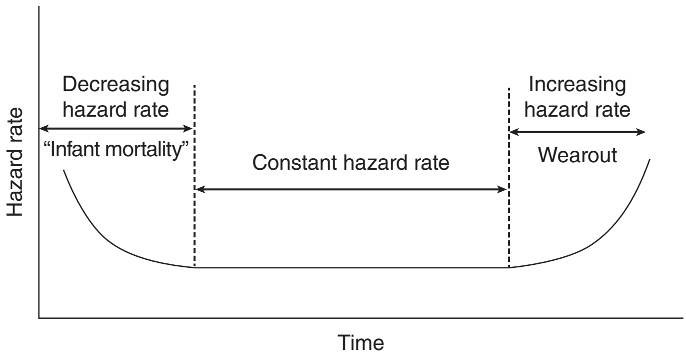

Figure 1.5 A bathtub curve showing the three regions of failure mechanism.

indication that the product has reached the wear-out phase. When $h(t)$ is a constant, the product is in useful life and the failures are considered to be random. A bathtub curve is commonly used to describe the three regions of hazard rate (Figure 1.5).

1.2.4.6 Cumulative Hazard Function

The cumulative hazard function (CHF), $H(t)$, is obtained by integrating the hazard function and is given by

$$\begin{aligned} H(t) &= \int_0^t h(x)\mathrm{d}x \\ &= \int_0^t \frac{f(x)}{R(x)}\mathrm{d}x \\ &= -\log(R(t)). \end{aligned} \tag{1.9}$$

Equation (1.9) yields an important relationship between the CHF and the reliability function, which is $R(t) = e^{-H(t)}$. For an exponential distribution, $H(t) = \int_0^t h(x)\mathrm{d}x = \int_0^t \lambda \mathrm{d}x = \lambda t$.

Therefore, the reliability function, $R(t) = e^{-\lambda t}$, which is what we have from (1.8).

1.2.4.7 The Average Failure Rate Over Time

The failure rate of a product changes over time. Therefore, the average failure rate is quoted as the typical failure rate between two time points. The average failure rate between times t_1 and t_2 is given by

$$AFR(t_1, t_2) = \frac{\int_{t_1}^{t_2} h(t)\mathrm{d}t}{(t_2 - t_1)} = \frac{H(t_2) - H(t_1)}{(t_2 - t_1)} = \frac{\log(R(t_1)) - \log(R(t_2))}{(t_2 - t_1)}. \tag{1.10}$$

1.2.4.8 Mean Time to Failure

For a continuous random variable T with a PDF $f(x)$, the mean time to failure (MTTF) is given by

$$MTTF = \int_{-\infty}^{\infty} t f(t)\mathrm{d}t. \tag{1.11}$$

1.2.4.9 Mean Number of Failures

For a discrete random variable X with a PMF $m(x)$ and sample space S, the mean number of failures (MNFs) is given by

$$\text{MNF} = \sum_{x \in S} x\, m(x). \tag{1.12}$$

As an example, for an exponential failure time distribution with a failure rate of λ per unit time, the MTTF is given by

$$\text{MTTF} = \int_0^{\infty} t\, \lambda e^{-\lambda t} \mathrm{d}t = \frac{1}{\lambda}. \tag{1.13}$$

Suppose a product has an exponential failure time distribution with a failure rate of 0.025 failures a month. What is the MTTF for this product? From Eq. (1.123) we obtain MTTF = 1/0.025 = 40 hours.3

1.2.5 Censored Data

The collect time to failure data in a bench test is a common way to estimate MTTF and reliability at the specified lifetime. However, for highly reliable products, it may not be feasible to run everything to failure, due to test time constraints. For example, it might take years to run all the parts in a test to failure. A reliability engineer may have to terminate the test after six months in order to have a feasible estimation in time for a project. The engineer may observe that some of the parts have failed but some have not. In this case, the data collected is censored data. There are several different types of censored data. The concepts of different types of censored data are shown in Figure 1.6.

Right censoring means that the failure has not occurred by a certain time point (censoring time). In this case the failure would occur after the censoring time but it is not known when, for example a certain part in a reliability test survives 1000 hours and the test terminates at this point. Then the time to failure data of this part is right censored, with the censoring time being 1000 hours.

Left censoring means that the failure has occurred prior to a certain time point but it is not known exactly when. For example, a test is terminated after testing a part for

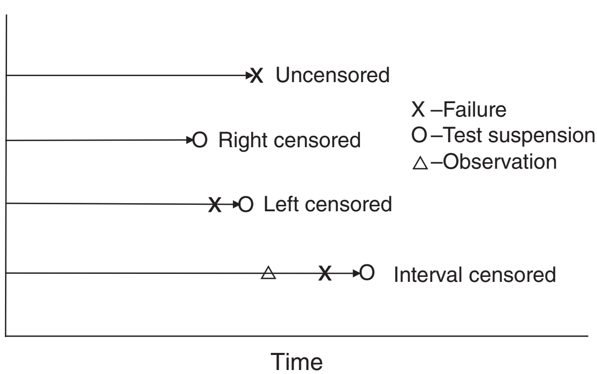

Figure 1.6 Various types of censored data.

1000 hours. An engineer only examines the part after the termination of the test and finds that the part has failed. Then the time to failure data of this part is left censored data, with the censoring time being 1000 hours.

Interval censoring means that the failure has occurred between a certain time interval but the exact time to failure is unknown. For example, a part is tested under stress conditions continuously. An engineer examines the part at 500 hours and the part has not failed. The part continues to be tested. The engineer examines the part again at 1000 hours and the part is found to have failed. The engineer thus know that the lifetime of this part is between 500 and 1000 hours, though the exact time to failure is unknown. Then the lifetime of this part is interval censored.

There are other types of censoring which are based on the cause of censoring. Type I or time censoring occurs when each unit in a fixed sample is tested for a prespecified length of time. All units not failed at the end of the test are time censored. For example, if 100 light bulbs are tested for 1000 hours each and only 60 of them have failed at the end of the test, then the 40 bulbs that have not failed are time censored.

Failure censoring or Type II censoring occurs when a test terminates after a prespecified number of failures has occurred, for example, we are test 100 light bulbs and the test terminates at the failure of the 35th light bulb. The data for the 65 light bulbs that have not failed at the test termination are called failure censored.

Type III censoring is a combination of Type I and Type II censoring. Type III censoring occurs when a test terminates if either the time criterion or failure criterion is met, for example, we test 100 light bulbs and the test terminates when 35 bulbs have failed, or each bulb is tested for 1000 hours, whichever comes first. The bulbs that have not failed at the termination of the test are called Type III censored.

Regardless of which censoring mechanism was used in generating lifetime data, how the data is handle in the analysis falls into one of the three main censoring categories mentioned: right, left, or interval censoring. The main classical approach for analyzing reliability data is called the maximum likelihood method. The likelihood function plays a very important role in estimating the unknown parameters of lifetime probability models in both classical (frequentist) and Bayesian statistics.

The contribution of each data point to the likelihood function varies by whether the data point is uncensored, right censored, left censored, or interval censored. In the following we will define the likelihood function in different censoring scenarios. Let $t_1, t_2 \ldots, t_n$ be a random sample of lifetime data obtained in a reliability test.

For uncensored data, the likelihood function is the product of the PDF for each failure time, i.e.

$$L = \prod_{i=1}^{n} f(t_i),$$

where

n is the number of data points
t_i is the observed failure time of the ith unit
$f(t_i)$ is the PDF evaluated at the ith failure time.

For a combination of uncensored and right censored data, the likelihood function is the product of the PDF for each failure time and the product of the reliability function at each right censored time, i.e.

$$L = \prod_{i=1}^{k} f(x_i) \prod_{j=k+1}^{n} R(t_j),$$

where

n is the number of data points, where the failure and censoring times are organized such that the first k data points are failures and the last $n-k$ data points are right censored data
x_i is the observed time to failure of the ith data point
$f(x_i)$ is the PDF evaluated at the ith failure time;
t_j is the recorded operating time (censoring limit) of the jth time point;
$R(t_j)$ is the reliability/survival function evaluated at the jth censoring time.

For a combination of uncensored and left censored data, the likelihood function is the product of the PDF for each failure data and the product of CDF for each left censored data, i.e.

$$L = \prod_{i=1}^{k} f(x_i) \prod_{j=k+1}^{n} F(t_j),$$

where

n is the number of data points, where the failure and censoring times are organized such that the first k data points are failures and the last $n-k$ data points are left censored data
x_i is the observed time to failure of the ith failure time
$f(x_i)$ is the PDF evaluated at the ith failure time
t_j is the censoring limit of the jth censoring time
$F(t_j)$ is the CDF evaluated at the jth censoring limit.

For a combination of uncensored data and interval censored data, the likelihood function is

$$L = \prod_{i=1}^{k} f(x_i) \prod_{j=k+1}^{n} \left[F(t_{upper\_j}) - F(t_{lower\_j})\right],$$

where

n is the number of data points, where the failure and censoring times are organized such that the first k data points are failures and the last $n-k$ data points are interval censored data
x_i is the observed time to failure of the ith failure time
$f(x_i)$ is the PDF evaluated at the ith failure time
t_{upper_j} and t_{lower_j} are the upper and lower censoring limits of the jth data point, respectively
$F(t_{upper\_j}) - F(t_{lower\_j})$ is the probability for values of time between t_{upper_j} and t_{lower_j}.

1.2.6 Parametric Models of Time to Failure Data

Parametric models use probability distributions of time to failure data to estimate reliability, CDF, and other metrics of interest. Examples are shown in Section 1.2.3. When using a distribution to model time to failure data, methods such as the maximum likelihood method is often used to estimate the unknown parameters of the distribution. Besides point estimates of the model parameters, their confidence intervals are also computed.

In addition, reliability practitioners are usually interested in point estimates and confidence intervals associated with reliability at a given time and various percentiles of the distribution. Some of the parametric reliability models we consider in this book include binomial, Poisson, exponential, gamma, beta, Weibull, normal, and log-normal. Details of these topics are further discussed in Chapters 2 and 4.

The maximum likelihood method estimates unknown parameters of a distribution by maximizing the likelihood function. An example of finding the maximum likelihood estimate of the success probability in a binomial distribution will be introduced in Section 2.5.

1.2.7 Nonparametric Estimation of Survival

Unlike parametric models, nonparametric methods don't assume any parametric probability distribution of time to failure data. These models utilize the observed number of failures (or deaths) and non-failures to determine the survivor function.

Frequently used nonparametric methods include the Kaplan–Meier (Kaplan and Meier 1958), Nelson–Aalen, and Cutler–Ederer (CE) life-table (also known as actuarial) methods. Another nonparametric method for estimating survival function is Bernard's median rank method, which is based on the rank of the ordered observations rather than the observation itself. These methods will be discussed briefly in this section using simple examples.

The main advantage of using nonparametric methods is to eliminate the need for having the data fit a specified probability distribution. The main drawback of using nonparametric methods is the inability to predict reliability beyond the time range of the data collected.

The Kaplan–Meier estimator is the most commonly used nonparametric method to estimate reliability without assuming distributions. When the data is a combination of uncensored and censored data, the Kaplan–Meier reliability estimation $\widehat{R}(t_i)$ at the failure time t_i is given by

$$\widehat{R}(t_i) = \prod_{j=1}^{i} \left(\frac{n_i - d_i}{n_i} \right), \tag{1.14}$$

where

t_i is the event time of the ith ordered failure time
n_i is the number of survivors just before the failure time t_i
d_i is the number of failures occurring at time t_i
$\widehat{R}(t_i) = \widehat{R}(t_{i-1})$, if no failures are recorded at time t_i (i.e. the event corresponds to censored data).

Example 1.1 The failure and censored times (hours) of 10 key components of a machine are 385, 450+, 475, 500, 575, 600+, 750, 750, 875, and 900+. For simplicity, the times have been ordered from low to high. A + sign next to the number indicates a right censored time. We will use the Kaplan–Meier method to estimate the survivor function of this component.

Table 1.1 Kaplan–Meier estimate of machine component survival.

Time (hours)	Number failed	Number at risk	K–M estimate
385	1	10	(10−1)/10 = 0.9
450+	0		0.9
475	1	8	0.9*(8−1)/8 = 0.787 5
500	1	7	0.787 5*(7−1)/7 = 0.675
575	1	6	0.675*(6−1)/6 = 0.562 5
600+	0		0.562 5
750	2	4	0.562 5*(4−2)/4 = 0.281 25
875	1	2	0.281 25*(2−1)/2 = 0.140 625
900+	0		0.140 625

Table 1.2 Kaplan–Meier estimates and 95% pointwise confidence intervals (generated from R-code).

Time	N.risk	N.event	Survival	Std err	Lower 95% CI	Upper 95% CI
385	10	1	0.900	0.0949	0.7320	1.000
475	8	1	0.787	0.1340	0.5641	1.000
500	7	1	0.675	0.1551	0.4303	1.000
575	6	1	0.562	0.1651	0.3165	1.000
750	4	2	0.281	0.1631	0.0903	0.876
875	2	1	0.141	0.1286	0.0234	0.844

Table 1.1 shows the Kaplan–Meier survival function estimation steps. The first column provides ordered failure and censored times. The second column from the left shows the number failed (1) or number censored (0) at each time point. The third column shows the number of units at risk just before each failure time. The fourth column shows the Kaplan–Meier estimate of the survival function using formula (1.13).

The R script below will generate the Kaplan–Meier estimates and 95% pointwise confidence intervals given in Table 1.2, and the survival plot shown in Figure 1.7 for the machine component failure time data.

(1.2.7_Kaplan_Meier_Surv_Anal_Table_1_2.R)

```
## Kaplan-Meier Survival Analysis of the data in Table 1.1

# load the package "survival"
library(survival)
# Create a time to failure data vector
TimeToEvent <- c(385,450,475,500,575,600,750,750,875,900)
# Note that censoring times are 450, 600, and 900
# Create a vector of 0s and 1s where 0 indicates censoring event
# and 1 indicates a failure event
```

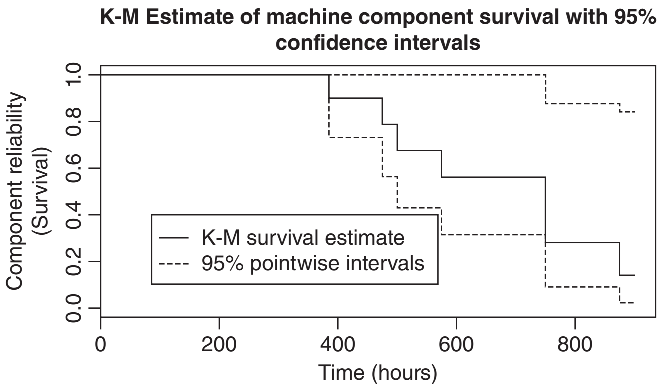

Figure 1.7 Kaplan–Meier estimate of machine component reliability (survival).

```
        Censor <- c(1,0,1,1,1,0,1,1,1,0)
        #Create a survival object
        SurvObj <- Surv(TimeToEvent, Censor)
        # Compute Kaplan-Meier survival estimates
        KMEst <- survfit(SurvObj ~ 1)
        summary(KMEst)
        cb <- confBands(y_bmt, type = "hall")
        # Generate K-M Survival plot with 95% pointwise confidence bounds
        # PDF(file="Figure1_7.pdf",width=7,height=5)
        # jpeg("Fig 1.7_Kaplan_Meier_Survival.jpeg", width = 6, height = 4,
units = 'in', res = 600)  # save the plot as jpeg format
        plot(KMEst,main = 'K-M Estimte of Machine Component Survival
with 95% Confidence Intervals',xlab="Time (Hours)",
             ylab="Component Reliability (Survival)",cex.main=0.9,
cex.lab=0.9)
        legend(100, 0.40, legend = c('K-M survival estimate','95%
pointwise intervals'), lty = 1:2)
        # dev.off()
```

Figure 1.7 shows the Kaplan–Meier survival estimates and 95% pointwise confidence interval. Notice that the survival plot is a step function and it only changes at failure times and remain constant through censored times.

1.2.8 Accelerated Life Testing

Using time to failure data to estimate product reliability from bench tests under regular use conditions can be very time consuming and costly. An alternative method, ALT, can be used instead to test products under increased level of stress conditions and thus failures can be generated in a shorter period of time (Figure 1.8).

ALT is used to address wear-out failure mechanisms. The assumption is that a product will exhibit the same failure mode and mechanism under increased stress levels in ALT as it would exhibit under actual use conditions in the field. Data generated in ALT can be used to estimate reliability or life under field use conditions. In a ALT, time to failure data at a specific stress level are fit to a life distribution. Time to failure data collected under different stress levels can be fit to a stress-life model (Figure 1.9), which is used to estimate the life distribution under the field use conditions.

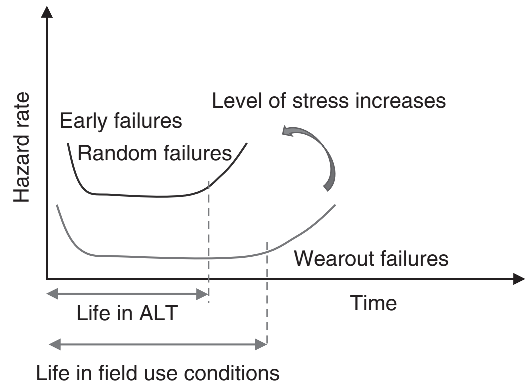

Figure 1.8 Accelerated life tests.

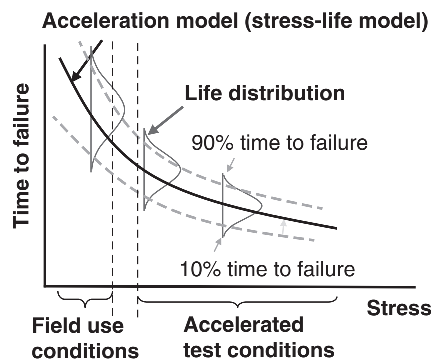

Figure 1.9 Acceleration model and life distribution.

Methods of acceleration include increasing the usage frequency, the stress levels, or both. Typical stresses in ALT include voltage, current, temperature, humidity, vibration, or a combination of various types of stresses. Here we review two commonly used acceleration models: (i) the temperature acceleration model (law of Arrhenius) and (ii) the inverse power law model.

In the temperature acceleration model, time to failure can be expressed as Eq. (1.15), which models the lifetime under a particular temperature T as

$$t = A_0 \exp(E_a/kT), \tag{1.15}$$

where

t is time to failure
A_0 is a constant
E_a is the activation energy for the failure mechanism
k is Boltzmann's constant (8.617×10^{-5} eV K^{-1})
T is the temperature measured in degrees of Kelvin (K).

In this model, the acceleration factor, AF, between two temperatures T_1 and T_2 can be obtained from Eq. (1.15) as follows

$$AF = \frac{t_1}{t_2} = \frac{A_0 e^{(E_a/kT_1)}}{A_0 e^{(E_a/kT_2)}} = e^{(E_a/k)[1/T_1 - 1/T_2]}. \tag{1.16}$$

In the inverse power law model, time to failure can be expressed as

$$t = \frac{a}{S^c}, \tag{1.17}$$

where

S is stress or load (e.g. voltage)
a and c are positive constants.

There are other acceleration models, such as the Eyring model, the degradation model, the step-stress model, etc. A good discussion of various ALT models can be found in Nelson (2004) and Tobias and Trindade (1995). We will discuss some of these models in detail in Chapter 10.

The ALT models and the corresponding acceleration stresses are specific to the underlying failure mechanism. Examples of some commonly encountered failure mechanisms in the semiconductor industry include electro-migration (Black 1974), hot carrier injection, and time-dependent dielectric breakdown. Although there is some agreement, there could be a number of different models used for some ALT failure models. Even for well-accepted models, the parameters can sometimes have a fairly wide range, depending on materials and manufacturing process, etc.

Another form of ALT is highly accelerated life testing (HALT). The main objective of HALT is to quickly identify the weakest links by testing the products to failure under intense stress conditions, thus the product quality can be improved by eliminating the problems. Stresses applied in HALT are usually well beyond normal use conditions. When failures are precipitated, the design problem is identified and fixed, and the products will continue to be tested to identify the next weakest link. With several rounds of reinforcing designs, the product will become more and more robust until it is too costly to continue fixing problems or when there are sufficient design margins. The limitation of HALT is that the test results cannot be used to predict the product reliability. In other words, HALT can be used to ensure that the product quality and reliability are better, but there is little information about how much better.

Burn-in is another type of accelerated testing intended to screen out/precipitate early failures or infant mortality type of failures usually caused by manufacturing defects. For components like integrated circuit boards, lithium/iodine batteries, and so on, it is common practice to apply burn-in tests under elevated stress conditions (voltages, current, humidity, and temperature, etc.). One debate of using burn-in is that those accelerated stress conditions should not be so high that they result in product damage.

1.3 Bayesian Approach to Reliability Inferences

1.3.1 Brief History of Bayes' Theorem and Bayesian Statistics

In the 1740s, English Reverend and mathematician Thomas Bayes developed Bayes' theorem to answer the question about the probability of a cause given an effect (called "inverse probability"). In Bayes' solution, a thought experiment is used which is described as follows: an assistant randomly throws balls on a flat square table and tells whether every new ball stops to the left or right of the first ball. With this information, how can he tell the position of the first ball? Bayes realized that this is a learning process: the more information we have on the new balls, the more we know about the first ball's position (McGrayne 2011). In brief, Bayes' theorem can be described as

$$\text{initial belief(called prior)} + \text{new data/evidence} = \text{improved belief(called posterior)}$$

Though a significant discovery, Bayes didn't publish his idea. Instead, two years after his death, in 1763, Bayes' friend Richard Price found Bayes' paper, realized the significance of this solution for "one of the most difficult problems in the doctrine of chances," and helped publish Bayes' paper entitled "An Essay towards solving a Problem in the Doctrine of Chances" in the Royal Society's *Philosophical Transactions* (McGrayne 2011).

In 1774, French mathematician Pierre-Simon Laplace independently developed and published Bayes' theorem. In the early and mid-20th century, due to the doubt about subjective probability to quantify our initial belief and the idea that science cannot allow subjectivity, Bayesian inference was attacked by statisticians including R.A. Fisher and J. Neyman, and was replaced with frequentist inference. Later, in Section 2.5, we will discuss the difference between Bayesian inference and frequentist inference. One major difference is that uncertainties of the model parameters are quantified by probability distributions in Bayesian inference and by confidence intervals in frequentist inference.

Meanwhile during early and mid-20th century, a few statisticians continued to make progress on Bayesian statistics. The subjective interpretation of probability (probability was based on personal beliefs which can be quantified) was developed by Ramsey (1927) and Finetti (1937). The objective form of Bayesian inference was further developed by Jeffreys (1946) by devising rules for selecting priors.

The revival of Bayesian statistics is related to the early development of modern computer science. During World War II, German U-boats cut off British sources of food. The German codes produced by Enigma machines were able to change rapidly and were considered unbreakable. Alan Turing, father of computer science and artificial intelligence, built a machine and used Bayesian methods to crack the Enigma code (McGrayne 2011).

However, Bayesian statistics remain dismissed in academia until in the 1990s when personal computers became popular and the MCMC was developed. MCMC is a method of simulating from a probability distribution based on constructing a Markov chain. Details of this sampling algorithm will be discussed in Chapter 3.

Historically, applications of Bayesian reliability in industry were limited to special cases where conjugate prior distributions are used, due to mathematical tractability. MCMC algorithms and the increased computational power of personal computers have made it easier to develop and solve more advanced statistical models for complex problems, which is not be feasible using the traditional statistical methods of the past.

These breakthroughs in computational algorithms have greatly boosted advancement and applications of Bayesian modeling. In the last two decades, extensive research for new Bayesian methodologies generated the practical application of complicated models in a wide range of science.

1.3.2 How Does Bayesian Statistics Relate to Other Advances in the Industry?

Two advances in the industry are observed or anticipated.

1.3.2.1 Advancement of Predictive Analytics

Predictive reliability analysis in advance provides inputs for the design strategy and boosts understanding and confidence in product reliability before products are released to the market. A Bayesian framework provides a straightforward solution for product reliability prediction with uncertainty quantified, even in complex systems, which could have been challenging using traditional reliability analysis methods (Martz and Waller 1990).

1.3.2.2 Cost Reduction

One challenge that many industries face is economic pressure to develop highly reliable products while simultaneously lowering costs. With technology maturing, the reliability of many products is increasing, and so is customer expectation. To demonstrate high reliability based on traditional methods might be an extremely expensive task with an incredibly large sample size.

In addition, Bayesian modeling enables aggregating information from different sources (e.g. historical data and bench test data) to predict reliability, which may provide unique benefits for sample size reduction. Some examples to demonstrate this benefit are included in Chapters 7 and 8. The cost associated with testing could be reduced potentially, and time to market could be improved without compromising performance and reliability.

In the following chapters, various case examples are provided to help engineers gain insights into these new capabilities, to learn how to adopt Bayesian models to solve engineering prediction problems, and to be creative in developing advanced Bayesian models to solve complex problems.

1.4 Component Reliability Estimation

Component reliability estimation often involves estimating reliability from lifetime data using the parametric or nonparametric methods introduced above. Lifetime data could be collected from the characterization test or from the field. When lifetime data are absent, component reliability can also be estimated from the data gathered in accelerated life tests. The results of these analysis are the point estimate and the specified (e.g. 95%) confidence interval (called credible intervals in Bayesian world) of the component reliability. In Chapter 4, we will go through the commonly used probability distributions one by one to discuss in detail how to estimate component reliability based on lifetime data using the Bayesian approach.

1.5 System Reliability Estimation

System reliability is the probability of the system to operate without failures for a specified period of time under specific use conditions. Since the components or subsystems in a system may experience different stress conditions, it may not be feasible to obtain lifetime data at the system level from one type of bench test during product development. Instead, reliability at the component or subsystem level is often assessed individually from different reliability tests or different sources of data.

With component/subsystem level reliability estimated individually, system level reliability is then estimated by aggregating reliability from the component/subsystem level, based on certain system reliability models. Commonly used system reliability prediction methods include the reliability block diagram, fault tree analysis, and Bayesian network, etc. which will be discussed in Chapter 7.

The results of the system reliability estimation are the point estimate and the specified (e.g. 95%) confidence interval of the system reliability. In a frequentist/classical framework, except in the simplest cases, it is often difficult or impossible to propagate classical confidence intervals though complex system models. In a Bayesian framework, on the other hand, posterior distributions are true probability statements about unknown parameters, so they may be easily propagated through these system reliability models (Hamada et al. 2008). In Chapter 7, we will discuss this topic in more detail.

1.6 Design Capability Prediction (Monte Carlo Simulations)

Monte Carlo simulation is a method that samples random values repeatedly to solve problems. Nowadays Monte Carlo simulations are widely used among engineers for reliability analysis and design capability analysis. In a design capability analysis, a design characteristic/performance is compared against the requirement to estimate the probability of the design meeting the requirement. In a Six Sigma design, a typical requirement is to demonstrate that the process capability index, $C_{\mathrm{pk}} > 1.5$. This requirement indicates that both the lower specification limit (LSL) and the upper specification limit (USL) are at least 4.5 standard deviations from the mean. If the data is normally distributed, this implies about 7 ppm is not meeting the specification requirements. Monte Carlo simulations can also be applied to many other engineering cases to quantify the probability of an outcome/output variable given the known uncertainties of input variables and a model describing the relationship between the output variable and input variables.

Figure 1.10 shows the general flow of how Monte Carlo simulation works to estimate the probability of meeting a design requirement. Assume

- y is the output variable to be determined and to be compared against a requirement
- $x_1, x_2, \cdots, x_n$ are n known input variables
- uncertainty of each of the input variable can be described by a probability distribution
- $y = f(x_1, x_2, \cdots, x_n)$ is the transfer function that describes the relationship between the output and inputs. Note that the transfer function is a deterministic model that can be based on first principles (e.g. Newton's laws, Ohm's law, etc.) or empirical equations (e.g. linear regression, nonlinear regression, etc.).

The following steps are then performed in Monte Carlo simulations:

1) A random value is generated for each input variable x_i $(i = 1, \ldots, n)$ based on the predefined probability distribution of x_i.
2) Based on the set of values for $x_1, x_2, \cdots, x_n$ and the transfer function $y = f(x_1, x_2, \cdots, x_n)$, the y value is calculated.
3) Steps 1 and 2 are repeated k times (k is usually a large number, e.g. 100 000) to collect a set of y values.
4) The summary statistics for the set of y values (mean, standard deviation, etc.) is estimated and each of the y values compared to the requirement.

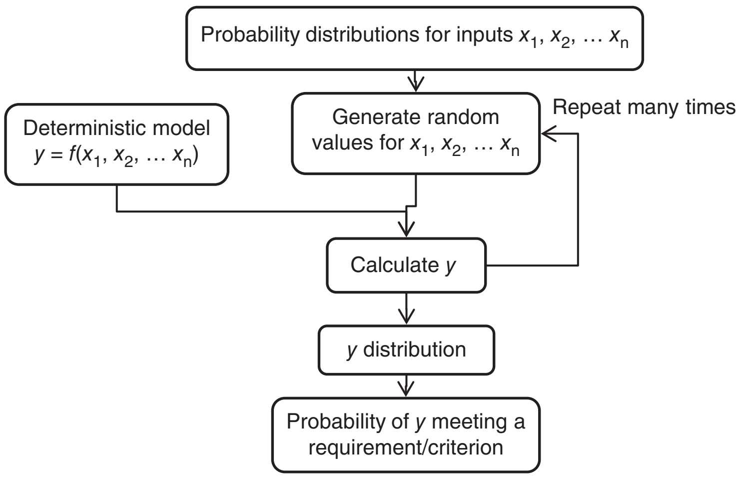

Figure 1.10 Flow chart of Monte Carlo simulations.

5) If out of all the y values collected in step 3 m out of k times the y value meet the requirement, then we say the probability of meeting the requirement based on the Monte Carlo simulation is $\frac{m}{k} \times 100\%$.

Examples and more discussions on Monte Carlo simulations to estimate design capability are included in Chapter 6.

1.7 Summary

This book introduces advanced Bayesian models to engineers and scientists with many real-life reliability case studies. In this chapter, we review basic concepts and existing commonly applied practices in reliability engineering. Good discussion of classical approach to reliability analysis can be found in Meeker and Escobar (1998) and Tobias and Trindade (1995).

We also briefly introduce the history of Bayesian inference and how Bayesian analysis relates to other advances in the industry. Classical reliability techniques were limited to reliability models where closed form solution or normal approximation was available for the parameters or statistics of interest. This limitation is no longer applied to Bayesian methods since the distributions of quantities of interest are obtained from the simulated posterior samples.

Bayesian statistics has unique advantages for reliability estimations and predictive analytics in complex systems and can allow flexible modeling solutions to reduce sample sizes and testing cost. However, historically, Bayesian modeling and analysis are limited in application due to mathematical tractability. Recent advancements in high-speed computing and breakthroughs in computational algorithms have made it feasible to solve more complex Bayesian models, which has greatly increased applications of Bayesian modeling. One popular algorithm is MCMC sampling, which will be introduced in Chapter 3. In later chapters, we will discuss these topics in detail and provide case study examples to demonstrate these benefits.

References

Black, J.R. (1974) *Physics of Electromigration*. IEEE Proceedings of the International Reliability Physics Symposium, April 2–4, 1974, Las Vegas, NV, USA.

Defeo, J.A. (2001). The tip of the iceberg – when accounting for quality, don't forget the often hidden costs of poor quality. *Quality Progress* 34 (5): 29–37.

Finetti, B. (1937, 1964). *La Prévision: ses lois logiques, ses sources subjectives*. Annales de l'Institut Henri Poincaré [*Foresight: its Logical Laws, Its Subjective Sources* (translation of the 1937 article in French)] (ed. H.E. Kyburg and H.E. Smokler) Studies in Subjective Probability. New York: Wiley.

Hamada, M.S., Wilson, A.G., Shane Reese, C., and Martz, H.F. (2008). *Bayesian Reliability*. New York: Springer.

Hindo, B. (2007) At 3M, A struggle between efficiency and creativity. *Inside Innovation – In Depth*, June 11, 2007.

ISO 2394:2015(en) (2015) General principles on reliability for structures, section 2.1.8, https://www.iso.org/obp/ui/#iso:std:iso:2394:ed-4:v1:en, accessed April 16, 2018.

Jeffreys, H. (1946). *An Invariant Form for the Prior Probability in Estimation Problems. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences.* 186 (1007): 453–461.

Kaplan, E.L. and Meier, P. (1958). Nonparametric estimation from incomplete observations. *Journal of America Statistics Association* 53 (282): 457–481.

Martz, H.F. and Waller, R.A. (1990). Bayesian reliability analysis of complex series/parallel systems of binomial subsystems and components. *Technometrics* 32 (4): 407–416.

McGrayne, S.B. (2011). *The Theory that Would not Die: How Bayes' Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two Centuries of Controversy*. New Haven, CT: Yale University Press.

Meeker, W.Q. and Escobar, L.A. (1998). *Statistical Methods for Reliability Data*. Hoboken, NJ: Wiley-Interscience.

Nelson, W.B. (2004). *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*. Hoboken, NJ: Wiley.

Pyzdek, T. (2003). *The Six Sigma Handbook*. New York: The McGraw-Hill Companies, Inc.

Ramsey, F.P. (1927). Facts and propositions. *Aristotelian Society Supplementary* 7: 153–170.

Tobias, P.A. and Trindade, D.C. (1995). *Applied Relibility*, 2e. New York: Van Nostrand Reinhold.

Turner, T.E. (2006) *Design for reliability*. Proceedings of 13th IPFA 2006, Singapore, 257–263.