CHAPTER 1

INTRODUCTION

Matrix, vector, and tensor algebras are often used in the theory of continuum mechanics in order to have a simpler and more tractable presentation of the subject. In this chapter, the mathematical preliminaries required to understand the matrix, vector, and tensor operations used repeatedly in this book are presented. Principles of mechanics and approximation methods that represent the basis for the formulation of the kinematic and dynamic equations developed in this book are also reviewed in this chapter. In the first two sections of this chapter, matrix and vector notations are introduced and some of their important identities are presented. Some of the vector and matrix results are presented without proofs with the assumption that the reader has some familiarity with matrix and vector notations. In Section 3, the summation convention, which is widely used in continuum mechanics texts, is introduced. This introduction is made despite the fact that the summation convention is rarely used in this book. Tensor notations, on the other hand, are frequently used in this book and, for this reason, tensors are discussed in Section 4. In Section 5, the *polar decomposition theorem*, which is fundamental in continuum mechanics, is presented. This theorem states that any nonsingular square matrix can be decomposed as the product of an orthogonal matrix and a symmetric matrix. Other matrix decompositions that are used in computational mechanics are also discussed. In Section 6, D'Alembert's principle is introduced, while Section 7 discusses the virtual work principle. The finite element method is often used to obtain finite dimensional models of continuous systems that in reality have infinite number of degrees of freedom. To introduce the reader to some of the basic concepts used to obtain finite dimensional models, discussions of approximation methods are included in Section 8. The procedure for developing the discrete equations of motion is outlined in Section 9, while the principle of conservation of momentum and the principle of work and energy are discussed in Section 10. In continuum mechanics,

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the gradients of the position vectors can be determined by differentiation with respect to different parameters. The change of parameters can lead to the definitions of strain components in different directions. This change of parameters, however, does not change the coordinate system in which the gradient vectors are defined. The effect of the change of parameters on the definitions of the gradients is discussed in Section 11.

1.1 MATRICES

In this section, some identities, results, and properties from matrix algebra that are used repeatedly in this book are presented. Some proofs are omitted, with the assumption that the reader is familiar with the subject of linear algebra.

Definitions

An $m \times n$ matrix $\mathbf{A}$ is an ordered rectangular array, which can be written in the following form:

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \tag{1.1}$$

where a_{ij} is the ijth element that lies in the ith row and jth column of the matrix. Therefore, the first subscript i refers to the row number and the second subscript j refers to the column number. The arrangement of Equation 1 shows that the matrix $\mathbf{A}$ has m rows and n columns. If $m = n$, the matrix is said to be *square*; otherwise, the matrix is said to be *rectangular.* The *transpose* of an $m \times n$ matrix $\mathbf{A}$ is an $n \times m$ matrix, denoted as $\mathbf{A}^{\mathrm{T}}$, which is obtained from $\mathbf{A}$ by exchanging the rows and columns, that is, $\mathbf{A}^{\mathrm{T}} = (a_{ji})$.

A *diagonal matrix* is a square matrix whose only nonzero elements are the diagonal elements, that is, $a_{ij} = 0$ if $i \neq j$. An *identity* or *unit matrix,* denoted as $\mathbf{I}$, is a diagonal matrix that has all its diagonal elements equal to one. The *null* or *zero matrix* is a matrix that has all its elements equal to zero. The *trace* of a square matrix $\mathbf{A}$ is the sum of all its diagonal elements, that is,

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} \tag{1.2}$$

This equation shows that $\mathrm{tr}(\mathbf{I}) = n$, where $\mathbf{I}$ is the identity matrix and n is the dimension of the matrix.

A square matrix $\mathbf{A}$ is said to be *symmetric* if

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}, \quad a_{ij} = a_{ji} \tag{1.3}$$

A square matrix is said to be *skew symmetric* if

$$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}, \quad a_{ij} = -a_{ji} \tag{1.4}$$

This equation shows that all the diagonal elements of a skew-symmetric matrix must be equal to zero. That is, if $\mathbf{A}$ is a skew-symmetric matrix with dimension n, then $a_{ii} = 0$ for $i = 1, 2, \ldots, n$. Any square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix. For example, if $\mathbf{B}$ is a square matrix, $\mathbf{B}$ can be written as

$$\mathbf{B} = \bar{\mathbf{B}} + \widetilde{\mathbf{B}} \tag{1.5}$$

where $\bar{\mathbf{B}}$ and $\widetilde{\mathbf{B}}$ are, respectively, symmetric and skew-symmetric matrices defined as

$$\bar{\mathbf{B}} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^{\mathrm{T}}), \quad \widetilde{\mathbf{B}} = \frac{1}{2}(\mathbf{B} - \mathbf{B}^{\mathrm{T}}) \tag{1.6}$$

Skew-symmetric matrices are used in continuum mechanics to characterize the rotations of the material elements.

Determinant

The *determinant* of an $n \times n$ square matrix $\mathbf{A}$, denoted as $|\mathbf{A}|$ or $\det(\mathbf{A})$, is a scalar quantity. In order to be able to define the unique value of the determinant, some basic definitions have to be introduced. The *minor* M_{ij} corresponding to the element a_{ij} is the determinant of a matrix obtained by deleting the ith row and jth column from the original matrix $\mathbf{A}$. The *cofactor* C_{ij} of the element a_{ij} is defined as

$$C_{ij} = (-1)^{i+j} M_{ij} \tag{1.7}$$

Using this definition, the determinant of the matrix $\mathbf{A}$ can be obtained in terms of the cofactors of the elements of an arbitrary row j as follows:

$$|\mathbf{A}| = \sum_{k=1}^{n} a_{jk} C_{jk} \tag{1.8}$$

One can show that the determinant of a diagonal matrix is equal to the product of the diagonal elements, and the determinant of a matrix is equal to the determinant of its transpose; that is, if $\mathbf{A}$ is a square matrix, then $|\mathbf{A}| = |\mathbf{A}^{\mathrm{T}}|$. Furthermore, the interchange of any two columns or rows only changes the sign of the determinant. It can also be shown that if the matrix has linearly dependent rows or linearly dependent columns, the determinant is equal to zero. A matrix whose determinant is equal to zero is called a *singular matrix*. For an arbitrary square matrix, singular or nonsingular, it can be shown that the value of the determinant does not change if any row or column is added or subtracted from another. It can be also shown that the determinant of the product of two matrices is equal to the product of their determinants. That is, if $\mathbf{A}$ and $\mathbf{B}$ are two square matrices, then $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$.

As will be shown in this book, the determinants of some of the deformation measures used in continuum mechanics are used in the formulation of the energy expressions. Furthermore, the relationship between the volumes of a continuum in the undeformed state and the deformed state is expressed in terms of the determinant of the matrix of position vector gradients. Therefore, if the elements of a square matrix depend on a parameter, it is important to be able to

determine the derivatives of the determinant with respect to this parameter. Using Equation 8, one can show that if the elements of the matrix **A** depend on a parameter t, then

$$\frac{d}{dt}|\mathbf{A}| = \sum_{k=1}^{n} \dot{a}_{1k} C_{1k} + \sum_{k=1}^{n} \dot{a}_{2k} C_{2k} + \cdots + \sum_{k=1}^{n} \dot{a}_{nk} C_{nk} \tag{1.9}$$

where $\dot{a}_{ij} = da_{ij}/dt$. The use of this equation is demonstrated by the following example.

Example 1.1

Consider the matrix **J** defined as

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

where $J_{ij} = \partial r_i/\partial x_j$, and **r** and **x** are the vectors

$$\mathbf{r}(x_1, x_2, x_3, t) = [r_1 \quad r_2 \quad r_3]^{\mathbf{T}}, \quad \mathbf{x} = [x_1 \quad x_2 \quad x_3]^{\mathrm{T}}$$

That is, the elements of the vector **r** are functions of the coordinates x_1, x_2, and x_3 and the parameter t. If $J = |\mathbf{J}|$ is the determinant of **J**, prove that

$$\frac{dJ}{dt} = \left(\frac{\partial \dot{r}_1}{\partial r_1} + \frac{\partial \dot{r}_2}{\partial r_2} + \frac{\partial \dot{r}_3}{\partial r_3} \right) J$$

where $\partial \dot{r}_i/\partial r_j = (\partial/\partial r_j)(dr_i/dt)$, $i, j = 1, 2, 3$.

Solution: Using Equation 9, one can write

$$\frac{dJ}{dt} = \sum_{k=1}^{3} \dot{J}_{1k} C_{1k} + \sum_{k=1}^{3} \dot{J}_{2k} C_{2k} + \sum_{k=1}^{3} \dot{J}_{3k} C_{3k}$$

where C_{ij} is the cofactor associated with element J_{ij}. Note that the preceding equation can be written as

$$\frac{dJ}{dt} = \begin{vmatrix} \dot{J}_{11} & \dot{J}_{12} & \dot{J}_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ \dot{J}_{21} & \dot{J}_{22} & \dot{J}_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} + \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ \dot{J}_{31} & \dot{J}_{32} & \dot{J}_{33} \end{vmatrix}$$

In this equation,

$$\dot{J}_{ij} = \frac{\partial \dot{r}_i}{\partial x_j} = \frac{\partial \dot{r}_i}{\partial r_1}\frac{\partial r_1}{\partial x_j} + \frac{\partial \dot{r}_i}{\partial r_2}\frac{\partial r_2}{\partial x_j} + \frac{\partial \dot{r}_i}{\partial r_3}\frac{\partial r_3}{\partial x_j} = \sum_{k=1}^{3} \frac{\partial \dot{r}_i}{\partial r_k} J_{kj}$$

Using this expansion, one can show that

$$\begin{vmatrix} \dot{J}_{11} & \dot{J}_{12} & \dot{J}_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} = \left(\frac{\partial \dot{r}_1}{\partial r_1} \right) J$$

Similarly, one can show that

$$\begin{vmatrix} J_{11} & J_{12} & J_{13} \\ \dot{J}_{21} & \dot{J}_{22} & \dot{J}_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} = \left(\frac{\partial \dot{r}_2}{\partial r_2} \right) J, \quad \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ \dot{J}_{31} & \dot{J}_{32} & \dot{J}_{33} \end{vmatrix} = \left(\frac{\partial \dot{r}_3}{\partial r_3} \right) J$$

Using the preceding equations, it is clear that

$$\frac{dJ}{dt} = \left(\frac{\partial \dot{r}_1}{\partial r_1} + \frac{\partial \dot{r}_2}{\partial r_2} + \frac{\partial \dot{r}_3}{\partial r_3} \right) J$$

This matrix identity is important and is used in this book to evaluate the rate of change of the determinant of the matrix of position vector gradients in terms of important deformation measures.

Inverse and Orthogonality

A square matrix $\mathbf{A}^{-1}$ that satisfies the relationship

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \tag{1.10}$$

where $\mathbf{I}$ is the identity matrix, is called the *inverse* of the matrix $\mathbf{A}$. The inverse of the matrix $\mathbf{A}$ is defined as

$$\mathbf{A}^{-1} = \frac{\mathbf{C}_t}{|\mathbf{A}|} \tag{1.11}$$

where $\mathbf{C}_t$ is the *adjoint* of the matrix $\mathbf{A}$. The adjoint matrix $\mathbf{C}_t$ is the transpose of the matrix of the cofactors (C_{ij}) of the matrix $\mathbf{A}$. One can show that the determinant of the inverse $|\mathbf{A}^{-1}|$ is equal to $1/|\mathbf{A}|$.

A square matrix is said to be *orthogonal* if

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{I} \tag{1.12}$$

Note that in the case of an orthogonal matrix $\mathbf{A}$, one has

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{-1} \tag{1.13}$$

That is, the inverse of an orthogonal matrix is equal to its transpose. One can also show that if $\mathbf{A}$ is an orthogonal matrix, then $|\mathbf{A}| = \pm 1$; and if $\mathbf{A}_1$ and $\mathbf{A}_2$ are two orthogonal matrices that have the same dimensions, then their product $\mathbf{A}_1\mathbf{A}_2$ is also an orthogonal matrix.

Examples of orthogonal matrices are the 3×3 transformation matrices that define the orientation of coordinate systems. In the case of a right-handed coordinate system, one can show that the determinant of the transformation matrix is $+1$; this is *a proper orthogonal transformation.* If the right-hand rule is not followed, the determinant of the resulting orthogonal transformation is equal to -1, which is an *improper orthogonal transformation,* such as in the case of a reflection.

Matrix Operations

The sum of two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ is defined as

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij}) \tag{1.14}$$

In order to add two matrices, they must have the same dimensions. That is, the two matrices **A** and **B** must have the same number of rows and same number of columns in order to apply Equation 14.

The product of two matrices **A** and **B** is another matrix **C** defined as

$$\mathbf{C} = \mathbf{AB} \tag{1.15}$$

The element c_{ij} of the matrix **C** is defined by multiplying the elements of the ith row in **A** by the elements of the jth column in **B** according to the rule

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_k a_{ik}b_{kj} \tag{1.16}$$

Therefore, the number of columns in **A** must be equal to the number of rows in **B**. If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, then **C** is an $m \times p$ matrix. In general, $\mathbf{AB} \neq \mathbf{BA}$. That is, matrix multiplication is not commutative. The *associative law* for matrix multiplication, however, is valid; that is, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \mathbf{ABC}$, provided consistent dimensions of the matrices **A**, **B**, and **C** are used.

1.2 VECTORS

Vectors can be considered special cases of matrices. An n-dimensional vector **a** can be written as

$$\mathbf{a} = (a_i) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^{\mathrm{T}} \tag{1.17}$$

Therefore, it is assumed that the vector is a column, unless it is transposed to make it a row.

Because vectors can be treated as columns of matrices, the addition of vectors is the same as the addition of column matrices. That is, if $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$ are two n-dimensional vectors, then $\mathbf{a} + \mathbf{b} = (a_i + b_i)$. Three different types of products, however, can be used with vectors.

These are the *dot product*, the *cross product,* and the *outer* or *dyadic product.* The result of the dot product of two vectors is a scalar, the result of the cross product is a vector, and the result of the dyadic product is a matrix. These three different types of products are discussed in the following sections.

Dot Product

The *dot, inner,* or *scalar product* of two vectors **a** and **b** is defined as

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i \tag{1.18}$$

Note that the two vectors **a** and **b** must have the same dimension. The two vectors **a** and **b** are said to be *orthogonal* if $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = 0$. The *norm, magnitude,* or *length* of an *n*-dimensional vector is defined as

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}} = \sqrt{\mathbf{a}^{\mathrm{T}}\mathbf{a}} = \sqrt{\sum_{i=1}^{n} (a_i)^2} \tag{1.19}$$

It is clear from this definition that the norm is always a positive number, and it is equal to zero only when **a** is the zero vector, that is, all the components of **a** are equal to zero.

In the special case of three-dimensional vectors, the dot product of two arbitrary three-dimensional vectors **a** and **b** can be written in terms of their norms as $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos \alpha$, where α is the angle between the two vectors. A vector is said to be a *unit vector* if its norm is equal to one. It is clear from the definition of the norm given by Equation 19 that the absolute value of any element of a unit vector must not exceed one. A unit vector $\hat{\mathbf{a}}$ along the vector **a** can be simply obtained by dividing the vector by its norm. That is, $\hat{\mathbf{a}} = \mathbf{a}/|\mathbf{a}|$. The dot product $\mathbf{b} \cdot \hat{\mathbf{a}} = |\mathbf{b}| \cos \alpha$ defines the component of the vector **b** along the unit vector $\hat{\mathbf{a}}$, where α is the angle between the two vectors. The projection of the vector **b** on a plane perpendicular to the unit vector $\hat{\mathbf{a}}$ is defined by the equation $\mathbf{b} - (\mathbf{b} \cdot \hat{\mathbf{a}})\,\hat{\mathbf{a}}$, or equivalently by $\mathbf{b} - (|\mathbf{b}| \cos \alpha)\,\hat{\mathbf{a}}$.

Cross Product

The vector cross product is defined for three-dimensional vectors only. Let **a** and **b** be two three-dimensional vectors defined in the same coordinate system. Unit vectors along the axes of the coordinate system are denoted by the vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$. These base vectors are *orthonormal,* that is,

$$\mathbf{i}_i \cdot \mathbf{i}_j = \delta_{ij} \tag{1.20}$$

where δ_{ij} is the *Kronecker delta* defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.21}$$

The *cross product* of the two vectors **a** and **b** is defined as

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{i}_1 + (a_3 b_1 - a_1 b_3)\mathbf{i}_2 + (a_1 b_2 - a_2 b_1)\mathbf{i}_3 \tag{1.22}$$

which can be written as

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{1.23}$$

This equation can be written as

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = \tilde{\mathbf{a}}\mathbf{b} \tag{1.24}$$

where $\tilde{\mathbf{a}}$ is the skew-symmetric matrix associated with the vector **a** and is defined as

$$\tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{1.25}$$

One can show that the determinant of the skew-symmetric matrix $\tilde{\mathbf{a}}$ is equal to zero. That is, $|\tilde{\mathbf{a}}| = 0$. One can also show that

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = -\tilde{\mathbf{b}}\mathbf{a} \tag{1.26}$$

In this equation, $\tilde{\mathbf{b}}$ is the skew-symmetric matrix associated with the vector **b**. If **a** and **b** are two parallel vectors, it can be shown that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. That is, the cross product of two parallel vectors is equal to zero.

Dyadic Product

Another form of vector product used in this book is the *dyadic* or *outer product.* Whereas the dot product leads to a scalar and the cross product leads to a vector, the dyadic product leads to a matrix. The dyadic product of two vectors **a** and **b** is written as $\mathbf{a} \otimes \mathbf{b}$ and is defined as

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^{\mathrm{T}} \tag{1.27}$$

Note that, in general, $\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a}$. One can show that the dyadic product of two vectors satisfies the following identities:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}), \quad \mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c}^{\mathrm{T}} \tag{1.28}$$

In Equation 28, it is assumed that the vectors have the appropriate dimensions. The dyadic product satisfies the following additional properties for any arbitrary vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ and a square matrix $\mathbf{A}$:

$$\left.\begin{aligned}
&(\mathbf{u}\otimes\mathbf{v})^{\mathrm{T}} = \mathbf{v}\otimes\mathbf{u}\\
&\mathbf{A}(\mathbf{u}\otimes\mathbf{v}) = (\mathbf{A}\mathbf{u}\otimes\mathbf{v})\\
&(\mathbf{u}\otimes\mathbf{v})\mathbf{A} = (\mathbf{u}\otimes\mathbf{A}^{\mathrm{T}}\mathbf{v})\\
&\mathbf{u}\otimes(\mathbf{v}_1+\mathbf{v}_2) = \mathbf{u}\otimes\mathbf{v}_1+\mathbf{u}\otimes\mathbf{v}_2
\end{aligned}\right\} \tag{1.29}$$

The second and third identities of Equation 29 show that $(\mathbf{A}\mathbf{u}\otimes\mathbf{A}\mathbf{v}) = \mathbf{A}(\mathbf{u}\otimes\mathbf{v})\mathbf{A}^{\mathrm{T}}$. This result is important in understanding the rule of transformation of the second-order tensors that will be used repeatedly in this book. It is left to the reader as an exercise to verify the identities of Equation 29.

Example 1.2

Consider the two vectors $\mathbf{a} = [a_1 \quad a_2]^{\mathrm{T}}$ and $\mathbf{b} = [b_1 \quad b_2 \quad b_3]^{\mathrm{T}}$. The dyadic product of these two vectors is given by

$$\mathbf{a}\otimes\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \end{bmatrix}$$

For a given vector $\mathbf{c} = [c_1 \quad c_2 \quad c_3]^{\mathrm{T}}$, one has

$$\begin{aligned}
(\mathbf{a}\otimes\mathbf{b})\mathbf{c} &= \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_3b_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1b_1 \\ a_2b_1 \end{bmatrix} c_1 + \begin{bmatrix} a_1b_2 \\ a_2b_2 \end{bmatrix} c_2 + \begin{bmatrix} a_1b_3 \\ a_2b_3 \end{bmatrix} c_3 \\
&= \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} b_1c_1 + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} b_2c_2 + \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} b_3c_3 = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})
\end{aligned}$$

Also note that the dyadic product $\mathbf{a}\otimes\mathbf{b}$ can be written as

$$\mathbf{a}\otimes\mathbf{b} = \left[\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} b_1 \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} b_2 \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} b_3 \right] = [\mathbf{a}b_1 \quad \mathbf{a}b_2 \quad \mathbf{a}b_3]$$

It follows that if $\mathbf{R}$ is a 2×2 matrix, one has

$$\begin{aligned}
\mathbf{R}(\mathbf{a}\otimes\mathbf{b}) &= \mathbf{R}[\mathbf{a}b_1 \ \mathbf{a}b_2 \ \mathbf{a}b_3] = [(\mathbf{R}\mathbf{a})b_1 \ (\mathbf{R}\mathbf{a})b_2 \ (\mathbf{R}\mathbf{a})b_3] \\
&= (\mathbf{R}\mathbf{a}\otimes\mathbf{b})
\end{aligned}$$

Several important identities can be written in terms of the dyadic product. Some of these identities are valuable in the computer implementation of the dynamic formulations presented in this book because the use of these identities can lead to significant simplification of the computational algorithms. By using these identities, one can avoid rewriting codes that perform the same mathematical operations, thereby saving effort and time by producing a manageable computer code. One of these identities that can be written in terms of the dyadic product is obtained in the following example.

Example 1.3

In the computer implementation of the formulations presented in this book, one may require differentiating a unit vector $\hat{\mathbf{r}}$ along the vector $\mathbf{r}$ with respect to the components of the vector $\mathbf{r}$. Such a differentiation can be written in terms of the dyadic product. To demonstrate this, we write

$$\hat{\mathbf{r}} = \frac{1}{\sqrt{\mathbf{r}^{\mathrm{T}}\mathbf{r}}}\mathbf{r} = \frac{1}{|\mathbf{r}|}\mathbf{r}$$

where $|\mathbf{r}| = \sqrt{\mathbf{r}^{\mathrm{T}}\mathbf{r}}$. It follows that

$$\frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}} = \frac{1}{\sqrt{\mathbf{r}^{\mathrm{T}}\mathbf{r}}}\left(\mathbf{I} - \frac{1}{\mathbf{r}^{\mathrm{T}}\mathbf{r}}\mathbf{r}\mathbf{r}^{\mathrm{T}}\right)$$

This equation can be written in terms of the dyadic product as

$$\frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{r}} = \frac{1}{\sqrt{\mathbf{r}^{\mathrm{T}}\mathbf{r}}}\left(\mathbf{I} - \frac{1}{\mathbf{r}^{\mathrm{T}}\mathbf{r}}\mathbf{r} \otimes \mathbf{r}\right) = \frac{1}{|\mathbf{r}|}(\mathbf{I} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}})$$

Projection

If $\hat{\mathbf{a}}$ is a unit vector, the component of a vector $\mathbf{b}$ along the unit vector $\hat{\mathbf{a}}$ is defined by the dot product $\mathbf{b} \cdot \hat{\mathbf{a}}$. The projection of $\mathbf{b}$ along $\hat{\mathbf{a}}$ is then defined as $(\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}}$, which can be written using Equation 28 as $(\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} = (\hat{\mathbf{a}} \otimes \hat{\mathbf{a}})\mathbf{b}$. The matrix $\mathbf{P} = \hat{\mathbf{a}} \otimes \hat{\mathbf{a}}$ defines *a projection matrix*. For an arbitrary integer n, one can show that the projection matrix $\mathbf{P}$ satisfies the identity $\mathbf{P}^n = \mathbf{P}$. This is an expected result because the vector $(\hat{\mathbf{a}} \otimes \hat{\mathbf{a}})\mathbf{b} = \mathbf{P}\mathbf{b}$ is defined along $\hat{\mathbf{a}}$ and has no components in other directions. Other projections should not change this result.

The projection of the vector $\mathbf{b}$ on a plane perpendicular to the unit vector $\hat{\mathbf{a}}$ is defined as $\mathbf{b} - (\mathbf{b} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}}$, which can be written using the dyadic product as $(\mathbf{I} - \hat{\mathbf{a}} \otimes \hat{\mathbf{a}})\mathbf{b}$. This equation defines another projection matrix $\mathbf{P}_p = \mathbf{I} - \hat{\mathbf{a}} \otimes \hat{\mathbf{a}}$, or simply $\mathbf{P}_p = \mathbf{I} - \mathbf{P}$. For an arbitrary integer n, one can show that the projection matrix $\mathbf{P}_p$ satisfies the identity $\mathbf{P}_p^n = \mathbf{P}_p$. Furthermore, $\mathbf{P}\mathbf{P}_p = \mathbf{0}$ and $\mathbf{P} + \mathbf{P}_p = \mathbf{I}$.

Example 1.4

Consider the vector $\mathbf{a} = [1 \quad 2 \quad 0]^{\mathrm{T}}$. A unit vector along $\mathbf{a}$ is defined as

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & 2 & 0\end{bmatrix}^{\mathrm{T}}$$

The projection matrix $\mathbf{P}$ associated with this unit vector can be written as

$$\mathbf{P} = \hat{\mathbf{a}} \otimes \hat{\mathbf{a}} = \frac{1}{5}\begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 0\\ 0 & 0 & 0\end{bmatrix}$$

It follows that

$$\mathbf{P}^2 = \frac{1}{25}\begin{bmatrix}5 & 10 & 0\\ 10 & 20 & 0\\ 0 & 0 & 0\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 0\\ 0 & 0 & 0\end{bmatrix} = \mathbf{P}$$

The projection matrix $\mathbf{P}_p$ is defined in this example as

$$\mathbf{P}_p = \mathbf{I} - \hat{\mathbf{a}} \otimes \hat{\mathbf{a}} = \mathbf{I} - \mathbf{P} = \frac{1}{5}\begin{bmatrix}4 & -2 & 0\\ -2 & 1 & 0\\ 0 & 0 & 5\end{bmatrix}$$

Note that $\mathbf{P}_p^2 = (\mathbf{I} - \mathbf{P})^2 = \mathbf{I} - 2\mathbf{P} + \mathbf{P}^2 = \mathbf{I} - \mathbf{P} = \mathbf{P}_p$. Successive application of this equation shows that $\mathbf{P}_p^n = \mathbf{P}_p$. The reader can verify this fact by the data given in this example.

1.3 SUMMATION CONVENTION

In this section, another convenient notational method, the *summation convention*, is discussed. The summation convention is used in most books on the subject of continuum mechanics. According to this convention, summation over the values of the indices is automatically assumed if an index is repeated in an expression. For example, if an index j takes the values from 1 to n, then in the summation convention, one has

$$a_{jj} = a_{11} + a_{22} + \cdots + a_{nn} \tag{1.30}$$

and

$$a_{ijj} = a_{i11} + a_{i22} + \cdots + a_{inn} \tag{1.31}$$

The repeated index used in the summation is called the *dummy index*, an example of which is the index j used in the preceding equation. If the index is not a dummy index, it is called *a free index*, an example of which is the index i used in Equation 31. It follows that the trace of a matrix $\mathbf{A}$ can be written using the summation convention as $\mathrm{tr}(\mathbf{A}) = a_{ii}$. The dot product between two n-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ can be written using the summation convention as $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = a_i b_i$. The product of a matrix $\mathbf{A}$ and a vector $\mathbf{b}$ is another vector $\mathbf{c} = \mathbf{A}\mathbf{b}$ whose

components can be written using the summation convention as $c_i = a_{ij}b_j$. Here, i is the free index and j is the dummy index.

Unit Dyads

The dyadic product between two vectors can also be written using the summation convention. For example, in the case of three-dimensional vectors, one can define the base vectors $\mathbf{i}_k$, $k = 1$, 2, 3. Any three-dimensional vector can be written in terms of these base vectors using the summation convention as $\mathbf{a} = a_i\mathbf{i}_i = a_1\mathbf{i}_1 + a_2\mathbf{i}_2 + a_3\mathbf{i}_3$. The dyadic product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can then be written as

$$\mathbf{a} \otimes \mathbf{b} = (a_i\mathbf{i}_i) \otimes (b_j\mathbf{i}_j) = a_ib_j(\mathbf{i}_i \otimes \mathbf{i}_j) \tag{1.32}$$

For example, if $\mathbf{i}_i = \mathbf{i}_1 = [1\ 0\ 0]^{\mathrm{T}}$, $\mathbf{i}_j = \mathbf{i}_2 = [0\ 1\ 0]^{\mathrm{T}}$, and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary three-dimensional vectors, one can show that the dyadic product of the preceding equation can be written in the following matrix form:

$$\mathbf{a} \otimes \mathbf{b} = a_ib_j(\mathbf{i}_i \otimes \mathbf{i}_j) = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix} \tag{1.33}$$

The dyadic products of the base vectors $\mathbf{i}_i \otimes \mathbf{i}_j$ are called the *unit dyads*. Using this notation, the dyadic product can be generalized to the products of three or more vectors. For example, the *triadic product* of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ can be written as $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} = (a_i\mathbf{i}_i) \otimes (b_j\mathbf{i}_j) \otimes (c_k\mathbf{i}_k) = a_ib_jc_k(\mathbf{i}_i \otimes \mathbf{i}_j \otimes \mathbf{i}_k)$. In this book, the familiar summation sign $\sum$ will be used for the most part, instead of the summation convention.

1.4 CARTESIAN TENSORS

It is clear from the preceding section that a dyadic product is a linear combination of unit dyads. The *second-order Cartesian tensor* is defined as a linear combination of dyadic products. A second-order Cartesian tensor $\mathbf{A}$ takes the following form:

$$\mathbf{A} = \sum_{i,j=1}^{3} a_{ij}(\mathbf{i}_i \otimes \mathbf{i}_j) \tag{1.34}$$

where a_{ij} are called the components of $\mathbf{A}$. Using the analysis presented in the preceding section, one can show that the second-order tensor can be written in the matrix form of Equation 33. Nonetheless, for a given second-order tensor $\mathbf{A}$, one cannot in general find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{A} = \mathbf{a} \otimes \mathbf{b}$.

The *unit* or *identity tensor* can be written in terms of the base vectors as

$$\mathbf{I} = \sum_{i=1}^{3} \mathbf{i}_i \otimes \mathbf{i}_i \tag{1.35}$$

Using the definition of the second-order tensor as a linear combination of dyadic products, one can show, as previously mentioned, that the components of any second-order tensor can be arranged in the form of a 3×3 matrix. In continuum mechanics, the elements of tensors represent physical quantities such as moments of inertia, strains, and stresses. These elements can be defined in any coordinate system. The coordinate systems used depend on the formulation used to obtain the equilibrium equations. It is, therefore, important that the reader understands the rule of the coordinate transformation of tensors and recognizes that such a transformation leads to the definition of the same physical quantities in different frames of reference or different directions. One must also distinguish between the transformation of vectors and the change of parameters. The latter does not change the coordinate system in which the vectors are defined. This important difference will be discussed in more detail before concluding this chapter.

A tensor that has the same components in any coordinate system is called an *isotropic tensor.* An example of isotropic tensors is the unit tensor. It can be shown that second-order isotropic tensors take only one form and can be written as $\alpha\mathbf{I}$, where α is a scalar and $\mathbf{I}$ is the unit or the identity tensor. Second-order isotropic tensors are sometimes called *spherical tensors.*

Double Product or Double Contraction

If $\mathbf{A}$ is a second-order tensor, the *contraction* of this tensor to a scalar is defined as $\sum_{i=1}^{3} a_{ii} = a_{11} + a_{22} + a_{33} = \text{tr}(\mathbf{A})$, where tr denotes the trace of the matrix (sum of the diagonal elements) (Aris 1962). It can be shown that the trace of a second-order tensor is invariant under orthogonal coordinate transformations. In addition to the trace, the determinant of $\mathbf{A}$ is invariant under orthogonal coordinate transformation. This important result can also be obtained in the case of second-order tensors using the facts that the determinant of an orthogonal matrix is equal to ± 1 and the determinant of the product of matrices is equal to the product of the determinants of these matrices.

If $\mathbf{A}$ and $\mathbf{B}$ are second-order tensors, the *double product* or *double contraction* is defined as

$$\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^{\mathbf{T}}\mathbf{B}) \tag{1.36}$$

Using the properties of the trace, one can show that

$$\mathbf{A} : \mathbf{B} = \text{tr}(\mathbf{A}^{\mathbf{T}}\mathbf{B}) = \text{tr}(\mathbf{B}\mathbf{A}^{\mathbf{T}}) = \text{tr}(\mathbf{B}^{\mathbf{T}}\mathbf{A}) = \text{tr}(\mathbf{A}\mathbf{B}^{\mathbf{T}}) = \sum_{i,j=1}^{3} a_{ij}b_{ij} \tag{1.37}$$

where a_{ij} and b_{ij} are, respectively, the elements of the tensors $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$, and $\mathbf{v}$ are arbitrary vectors and $\mathbf{A}$ is a second-order tensor, one can show that the double contraction has the following properties:

$$\left.\begin{aligned} &\text{tr}(\mathbf{A}) = \mathbf{I} : \mathbf{A} \\ &\mathbf{A} : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{u} \cdot (\mathbf{A}\mathbf{v}) \\ &(\mathbf{a} \otimes \mathbf{b}) : (\mathbf{u} \otimes \mathbf{v}) = (\mathbf{a} \cdot \mathbf{u})(\mathbf{b} \cdot \mathbf{v}) \end{aligned}\right\} \tag{1.38}$$

It can also be shown that if **A** is a symmetric tensor and **B** is a skew-symmetric tensor, then $\mathbf{A}:\mathbf{B}=0$. It follows that if **A** is a symmetric tensor and **B** is an arbitrary tensor, the definition of the double product can be used to show that $\mathbf{A}:\mathbf{B}=\mathbf{A}:\mathbf{B}^{\mathrm{T}}=\mathbf{A}:(\mathbf{B}+\mathbf{B}^{\mathrm{T}})/2$.

If **A** and **B** are two symmetric tensors, one can show that

$$\mathbf{A}:\mathbf{B}=a_{11}b_{11}+a_{22}b_{22}+a_{33}b_{33}+2(a_{12}b_{12}+a_{13}b_{13}+a_{23}b_{23}) \tag{1.39}$$

The preceding equation will be used in this book in the formulation of the elastic forces of continuous bodies. These forces are expressed in terms of the strain and stress tensors. As will be shown in Chapters 2 and 3, the strain and stress tensors are symmetric and are given, respectively, in the following form:

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13}\\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23}\\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}\end{bmatrix},\quad \boldsymbol{\sigma}=\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{12} & \sigma_{22} & \sigma_{23}\\ \sigma_{13} & \sigma_{23} & \sigma_{33}\end{bmatrix} \tag{1.40}$$

Using Equation 39, one can write the double contraction of the strain and stress tensors as

$$\boldsymbol{\varepsilon}:\boldsymbol{\sigma}=\varepsilon_{11}\sigma_{11}+\varepsilon_{22}\sigma_{22}+\varepsilon_{33}\sigma_{33}+2(\varepsilon_{12}\sigma_{12}+\varepsilon_{13}\sigma_{13}+\varepsilon_{23}\sigma_{23}) \tag{1.41}$$

Because a second-order symmetric tensor has six independent elements, vector notations, instead of tensor notations, can also be used to define the strain and stress components of the preceding two equations. In this case, six-dimensional strain and stress vectors can be introduced as follows:

$$\left.\begin{aligned}\boldsymbol{\varepsilon}_v&=[\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ \varepsilon_{12}\ \varepsilon_{13}\ \varepsilon_{23}]^T\\ \boldsymbol{\sigma}_v&=[\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{12}\ \sigma_{13}\ \sigma_{23}]^T\end{aligned}\right\} \tag{1.42}$$

where subscript v is used to denote a vector. The dot product of the strain and stress vectors is given by

$$\boldsymbol{\varepsilon}\cdot\boldsymbol{\sigma}=\boldsymbol{\varepsilon}^{\mathbf{T}}\boldsymbol{\sigma}=\varepsilon_{11}\sigma_{11}+\varepsilon_{22}\sigma_{22}+\varepsilon_{33}\sigma_{33}+\varepsilon_{12}\sigma_{12}+\varepsilon_{13}\sigma_{13}+\varepsilon_{23}\sigma_{23} \tag{1.43}$$

Note the difference between the results of the double contraction and the dot product of Equations 41 and 43, respectively. There is a factor of 2 multiplied by the term that includes the off-diagonal elements in the double contraction of Equation 41. Equation 41 arises naturally when the elastic forces are formulated, as will be shown in Chapter 3. Therefore, it is important to distinguish between the double contraction and the dot product despite the fact that both products lead to scalar quantities.

Invariants of the Second-Order Tensor

Under an orthogonal transformation that represents rotation of the axes of the coordinate systems, the components of the vectors and second-order tensors change. Nonetheless, certain vector and tensor quantities do not change and remain invariant under such an orthogonal transformation. For example, the norm of a vector and the dot product of two three-dimensional vectors remain invariant under a rigid-body rotation.

For a second-order tensor **A**, one has the following three invariants that do not change under an orthogonal coordinate transformation:

$$\left.\begin{aligned} I_1 &= \text{tr}(\mathbf{A}) \\ I_2 &= \frac{1}{2}\{(\text{tr}(\mathbf{A}))^2 - \text{tr}(\mathbf{A}^2)\} \\ I_3 &= \det(\mathbf{A}) = |\mathbf{A}| \end{aligned}\right\} \tag{1.44}$$

These three invariants can also be written in terms of the *eigenvalues* of the tensor **A**. For a given tensor or a matrix **A**, the eigenvalue problem is defined as

$$\mathbf{A}\mathbf{y} = \lambda\mathbf{y} \tag{1.45}$$

where λ is called the eigenvalue and **y** is the *eigenvector* of **A**. Equation 45 shows that the direction of the vector **y** is not affected by multiplication with the tensor **A**. That is, **Ay** can change the length of **y**, but such a multiplication does not change the direction of **y**. For this reason, **y** is called a principal direction of the tensor **A**. The preceding eigenvalue equation can be written as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{y} = \mathbf{0} \tag{1.46}$$

For this equation to have a nontrivial solution, the determinant of the coefficient matrix must be equal to zero, that is,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \tag{1.47}$$

This equation is called the *characteristic equation,* and in the case of a second-order tensor it has three roots λ_1, λ_2, and λ_3. Associated with these three roots, there are three corresponding eigenvectors $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ that can be determined to within an arbitrary constant using Equation 46. That is, for a root λ_i, $i = 1, 2, 3$, one can solve the system of homogeneous equations $(\mathbf{A} - \lambda_i\mathbf{I})\mathbf{y}_i = \mathbf{0}$ for the eigenvector $\mathbf{y}_i$ to within an arbitrary constant, as demonstrated by the following example.

Example 1.5

Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic equation of this matrix can be obtained using Equation 47 as

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (1 - \lambda)(3 - \lambda)(2 - \lambda) = 0$$

The roots of this characteristic equation define the following three eigenvalues of the matrix **A**:

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 3$$

Associated with these three eigenvalues, there are three eigenvectors, which can be determined using Equation 46 as

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{y}_i = \mathbf{0}, \quad i = 1, 2, 3$$

or

$$\begin{bmatrix} 1-\lambda_i & -1 & 2 \\ 0 & 3-\lambda_i & 1 \\ 0 & 0 & 2-\lambda_i \end{bmatrix} \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{bmatrix} = \mathbf{0}, \quad i = 1, 2, 3$$

This equation can be used to solve for the eigenvectors associated with the three eigenvalues λ_1, λ_2, and λ_3. For $\lambda_1 = 1$, the preceding equation yields the following system of algebraic equations:

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \end{bmatrix} = \mathbf{0}$$

This system of algebraic equations defines the first eigenvector to within an arbitrary constant as

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 2$, one has

$$\mathbf{y}_2 = \begin{bmatrix} y_{21} \\ y_{22} \\ y_{23} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$$

The eigenvector associated with $\lambda_3 = 3$ can also be determined as

$$\mathbf{y}_3 = \begin{bmatrix} y_{31} \\ y_{32} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Symmetric Tensors

In the special case of a *symmetric tensor*, one can show that the eigenvalues are real and the eigenvectors are orthogonal. Because the eigenvectors can be determined to within an arbitrary constant, the eigenvectors can be normalized as unit vectors. For a symmetric tensor, one can then write

$$\left.\begin{aligned} \mathbf{A}\mathbf{y}_i &= \lambda_i \mathbf{y}_i, \quad i = 1, 2, 3 \\ \mathbf{y}_i^{\mathbf{T}} \mathbf{y}_j &= \delta_{ij}, \quad i, j = 1, 2, 3 \end{aligned}\right\} \tag{1.48}$$

If $\mathbf{y}_i$, $i = 1, 2, 3$, are selected as orthogonal unit vectors, one can form the orthogonal matrix $\boldsymbol{\Phi}$ whose columns are the orthonormal eigenvectors, that is,

$$\boldsymbol{\Phi} = [\mathbf{y}_1 \quad \mathbf{y}_2 \quad \mathbf{y}_3] \tag{1.49}$$

It follows that

$$\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\lambda} \tag{1.50}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.51}$$

Using the orthogonality property of $\boldsymbol{\Phi}$, one has

$$\mathbf{A} = \boldsymbol{\Phi}\boldsymbol{\lambda}\boldsymbol{\Phi}^{\mathrm{T}} = \sum_{i=1}^{3} \lambda_i (\mathbf{y}_i \otimes \mathbf{y}_i) \tag{1.52}$$

This equation, which defines the *spectral decomposition* of $\mathbf{A}$, shows that the orthogonal transformation $\boldsymbol{\Phi}$ can be used to transform the tensor $\mathbf{A}$ to a diagonal matrix as

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.53}$$

That is, the matrices $\mathbf{A}$ and $\boldsymbol{\lambda}$ have the same determinant and the same trace. This important result is often used in continuum mechanics to study the invariant properties of different tensors.

Let $\mathbf{R}$ be an orthogonal transformation matrix. Using the transformation $\mathbf{y} = \mathbf{R}\mathbf{z}$ in Equation 46 and premultiplying by $\mathbf{R}^{\mathrm{T}}$, one obtains

$$(\mathbf{R}^{\mathrm{T}}\mathbf{A}\mathbf{R} - \lambda\mathbf{I})\mathbf{z} = \mathbf{0} \tag{1.54}$$

This equation shows that the eigenvalues of a tensor or a matrix do not change under an orthogonal coordinate transformation. Furthermore, as previously discussed, the determinant and trace of the tensor or the matrix do not change under such a coordinate transformation. One then concludes that the invariants of a symmetric second-order tensor can be expressed in terms of its eigenvalues as follows:

$$\left.\begin{aligned} I_1 &= \mathrm{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2 &= \frac{1}{2}\{(\mathrm{tr}(\mathbf{A}))^2 - \mathrm{tr}(\mathbf{A}^2)\} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ I_3 &= \det(\mathbf{A}) = \lambda_1\lambda_2\lambda_3 \end{aligned}\right\} \tag{1.55}$$

Some of the material constitutive equations used in continuum mechanics are formulated in terms of the invariants of the strain tensor. Therefore, Equation 55 will be used in later chapters of this book.

For a general second-order tensor $\mathbf{A}$ (symmetric or nonsymmetric), the invariants are $I_1 = \text{tr}(\mathbf{A})$, $I_2 = \frac{1}{2}\{(\text{tr}(\mathbf{A}))^2 - \text{tr}(\mathbf{A}^2)\}$, and $I_3 = \det(\mathbf{A})$, as previously presented. One can show that the characteristic equation of a second-order tensor can be written in terms of these invariants as $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$. Furthermore, by repeatedly multiplying Equation 45 n times by $\mathbf{A}$, one obtains $\mathbf{A}^n\mathbf{y} = \lambda^n\mathbf{y}$. Using this identity after multiplying the characteristic equation $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$ by $\mathbf{y}$, one obtains $\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\mathbf{I} = \mathbf{0}$, which is the mathematical statement of the *Cayley–Hamilton theorem,* which states that a second-order tensor satisfies its characteristic equation. The simple proof provided here for the Cayley–Hamilton theorem is based on the assumption that the eigenvectors are linearly independent. A more general proof can be found in the literature.

For a second-order skew-symmetric tensor $\mathbf{W}$, one can show that the invariants are given by $I_1 = I_3 = 0$ and $I_2 = w_{12}^2 + w_{13}^2 + w_{23}^2$, where w_{ij} is the ijth element of the tensor $\mathbf{W}$. Using these results, the characteristic equation of a second-order tensor $\mathbf{W}$ can be written as $\lambda^3 + I_2\lambda = 0$. This equation shows that $\mathbf{W}$ has only one real eigenvalue, $\lambda = 0$, whereas the other two eigenvalues are imaginary.

Higher-Order Tensors

In continuum mechanics, the stress and strain tensors are related using the constitutive equations that define the material behavior. This relationship can be expressed in terms of a fourth-order tensor whose components are material coefficients. In general, a tensor $\mathbf{A}$ of order n is defined by 3^n elements, which can be written as $a_{ijk\ldots n}$. A lower-order tensor can be obtained as a special case by reducing the number of indices. A zero-order tensor is represented by a scalar, a first-order tensor is represented by a vector, and a second-order tensor is represented by a matrix. A tensor of order n is said to be *symmetric* with respect to two indices if the interchange of these two indices does not change the value of the elements of the tensor. The tensor is said to be *antisymmetric* or *skew symmetric* with respect to two indices if the interchange of these two indices changes only the sign of the elements of the tensor.

As in the case of the second-order tensors, higher-order tensors can be defined using outer products. For example, a third-order tensor $\mathbf{T}$ can be defined as the outer product of three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as follows:

$$\mathbf{T} = (\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}) = (t_{ijk}) \tag{1.56}$$

An element of the tensor $\mathbf{T}$ takes the form $u_i v_j w_k$. Roughly speaking, in the case of three-dimensional vectors, one may consider the third-order tensor a linear combination of a new set of unit dyads that consist of 27 elements (3 layers, each of which has 9 elements). Recall that the multiplication $\mathbf{Sb}$ of a second-order tensor $\mathbf{S} = (s_{ij})$ and a vector $\mathbf{b} = (b_i)$ defines a vector $\mathbf{c} = (c_i)$ according to $c_i = \sum_{j=1}^{3} s_{ij}b_j = s_{i1}b_1 + s_{i2}b_2 + s_{i3}b_3$, which represents a *single contraction*. Similarly, the multiplication $\mathbf{Tb}$ of a third-order tensor $\mathbf{T} = (t_{ijk})$ with a vector $\mathbf{b} = (b_i)$ is a single contraction that defines a second-order tensor $\mathbf{S} = (s_{ij})$ such

that

$$s_{ij} = \sum_{k=1}^{3} t_{ijk} b_k = t_{ij1} b_1 + t_{ij2} b_2 + t_{ij3} b_3 \tag{1.57}$$

Using this definition, it follows that the elements of layer or matrix l, $l = 1, 2, 3$ are given by $w_l(\mathbf{u} \otimes \mathbf{v}) = \mathbf{T}\mathbf{i}_l$. Using this definition of the product or following a procedure similar to the one used to define the elements of the second-order tensor, one can show that the elements of the third-order tensor are defined as

$$t_{ijk} = (\mathbf{i}_i \otimes \mathbf{i}_j) : \mathbf{T}\mathbf{i}_k \tag{1.58}$$

In this equation, the third-order tensor is defined such that it maps an arbitrary vector $\mathbf{b}$ according to $(\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w})\mathbf{b} = (\mathbf{w} \cdot \mathbf{b})(\mathbf{u} \otimes \mathbf{v})$. Note that whereas a third-order tensor can in general be written as $\mathbf{T} = (t_{ijk})$, one cannot always, as in the case of second-order tensors, find vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ such that $\mathbf{T} = (\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w})$.

Example 1.6

Let $\mathbf{T} = (t_{ijk}) = (\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w})$ be a third-order tensor and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three-dimensional vectors. The third-order tensor $\mathbf{T}$ has 27 elements defined by the products $w_1(\mathbf{u} \otimes \mathbf{v})$, $w_2(\mathbf{u} \otimes \mathbf{v})$, and $w_3(\mathbf{u} \otimes \mathbf{v})$. It follows that

$$\mathbf{T}\mathbf{i}_k = w_k(\mathbf{u} \otimes \mathbf{v}) = w_k \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix}$$

The element t_{13k} of the tensor $\mathbf{T}$ can be defined as

$$t_{13k} = (\mathbf{i}_1 \otimes \mathbf{i}_3) : \mathbf{T}\mathbf{i}_k = \mathrm{tr}\{(\mathbf{i}_1 \otimes \mathbf{i}_3)^{\mathrm{T}} \mathbf{T}\mathbf{i}_k\}$$

where

$$(\mathbf{i}_1 \otimes \mathbf{i}_3) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the preceding three equations, one obtains

$$t_{13k} = (\mathbf{i}_1 \otimes \mathbf{i}_3) : \mathbf{T}\mathbf{i}_k = \mathrm{tr}\{(\mathbf{i}_1 \otimes \mathbf{i}_3)^{\mathrm{T}} \mathbf{T}\mathbf{i}_k\} = u_1 v_3 w_k$$

Other elements of the tensor $\mathbf{T}$ can be determined in a similar manner.

The *double product or double contraction* can also be applied to third-order tensors. Let $\mathbf{T}$ and $\mathbf{S}$ be, respectively, third- and second-order tensors and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{u}$, and $\mathbf{v}$ be arbitrary vectors. For such tensors, one can verify the following properties based on the double contraction

(Bonet and Wood, 1997):

$$\left.\begin{aligned}
&\mathbf{T}:(\mathbf{a}\otimes\mathbf{b})=(\mathbf{T}\mathbf{b})\mathbf{a}\\
&(\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}):(\mathbf{u}\otimes\mathbf{v})=(\mathbf{u}\cdot\mathbf{b})(\mathbf{v}\cdot\mathbf{c})\mathbf{a}\\
&(\mathbf{a}\otimes\mathbf{S}):\mathbf{T}=(\mathbf{S}:\mathbf{T})\otimes\mathbf{a}\\
&(\mathbf{S}\otimes\mathbf{a}):\mathbf{T}=\mathbf{S}(\mathbf{T}\mathbf{a})
\end{aligned}\right\} \tag{1.59}$$

Using the first of these equations, one can show that the double contraction of a third-order tensor $\mathbf{T}$ by a second-order tensor $\mathbf{S}$ can be evaluated in terms of their components as

$$\mathbf{T}:\mathbf{S}=\sum_{i,j,k=1}^{3} t_{ijk}s_{jk}\mathbf{i}_i \tag{1.60}$$

An important example of a third-order tensor is the *alternating tensor* $\boldsymbol{\Gamma}$, which, when applied to a vector $\mathbf{v}=[v_1 \quad v_2 \quad v_3]^{\mathrm{T}}$, maps this vector to a skew-symmetric matrix associated with this vector, that is,

$$\boldsymbol{\Gamma}\mathbf{v}=-\tilde{\mathbf{v}} \tag{1.61}$$

where

$$\tilde{\mathbf{v}}=\begin{bmatrix} 0 & -v_3 & v_2\\ v_3 & 0 & -v_1\\ -v_2 & v_1 & 0\end{bmatrix} \tag{1.62}$$

The components Γ_{ijk} of $\boldsymbol{\Gamma}$ are defined as

$$\Gamma_{ijk}=\mathbf{i}_i\cdot(\mathbf{i}_j\times\mathbf{i}_k) \tag{1.63}$$

From this equation, it is clear that $\Gamma_{ijk}=0$ if any indices are repeated; $\Gamma_{ijk}=1$ for an even permutation of i, j, and k; $\{(i,j,k),(j,k,i),(k,i,j)\}$; and $\Gamma_{ijk}=-1$ for any other permutation. Using the first equation in Equation 59, one can show that, for any two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$, the alternating tensor $\boldsymbol{\Gamma}$ can be introduced using another definition as

$$\boldsymbol{\Gamma}:(\mathbf{u}\otimes\mathbf{v})=(\boldsymbol{\Gamma}\mathbf{v})\mathbf{u}=-\tilde{\mathbf{v}}\mathbf{u}=\mathbf{u}\times\mathbf{v} \tag{1.64}$$

Equation 61 can also be written in component form as $\tilde{v}_{ij}=\sum_{k=1}^{3}\Gamma_{ijk}v_k$ because for a fixed k, only terms that do not have repeated indices will appear. For example, if the summation convention is used, $\Gamma_{ij2}\tilde{v}_{ij}=\Gamma_{132}\tilde{v}_{13}+\Gamma_{312}\tilde{v}_{31}=-\tilde{v}_{13}+\tilde{v}_{31}=2v_2$. One can follow the same procedure for the other two components for the vector $\mathbf{v}$ and show that $v_k=-\frac{1}{2}\sum_{i,j=1}^{3}\Gamma_{ijk}\tilde{v}_{ij}=-\frac{1}{2}\sum_{i,j=1}^{3}\Gamma_{kij}\tilde{v}_{ij}=\frac{1}{2}\sum_{i,j=1}^{3}\Gamma_{ikj}\tilde{v}_{ij}$ (Aris, 1962). The alternating tensor $\boldsymbol{\Gamma}$ is another example of an isotropic tensor. Furthermore, one can show that the cross product between the two vectors in Equation 64 can be written as $\mathbf{u}\times\mathbf{v}=\sum_{i,j=1}^{3}u_iv_j(\mathbf{i}_i\times\mathbf{i}_j)=\sum_{i,j,k=1}^{3}\Gamma_{ijk}u_iv_j\mathbf{i}_k$.

In a similar manner to the third-order tensor, a *fourth-order tensor* $\mathbf{F}$ can be defined as

$$\mathbf{F}=(\mathbf{u}_1\otimes\mathbf{u}_2\otimes\mathbf{u}_3\otimes\mathbf{u}_4)=(f_{ijkl}) \tag{1.65}$$

where $\mathbf{u}_m$, $m = 1, 2, 3, 4$, is an arbitrary vector. As in the case of third-order tensors, one can write $u_{4m}\,(\mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \mathbf{u}_3) = \mathbf{F}\mathbf{i}_m$, where u_{4m} is the mth component of the vector $\mathbf{u}_4$**.** It can then be shown that the coefficients f_{ijkl} can be written as

$$f_{ijkl} = (\mathbf{i}_i \otimes \mathbf{i}_j) : \mathbf{F} : (\mathbf{i}_k \otimes \mathbf{i}_l) \tag{1.66}$$

More generally, if $\mathbf{F} = (f_{ijkl})$ is a fourth-order tensor and $\mathbf{v} = (v_i)$ is a vector, one has the single contraction, $\mathbf{F}\mathbf{v} = \mathbf{T}$, where $\mathbf{T} = (t_{ijk})$ is a third-order tensor whose elements are defined as

$$t_{ijk} = \sum_{l=1}^{3} f_{ijkl} v_l = f_{ijk1} v_1 + f_{ijk2} v_2 + f_{ijk3} v_3 \tag{1.67}$$

Similarly, the double contraction $\mathbf{F} : \mathbf{S} = \mathbf{B}$ of the fourth-order tensor $\mathbf{F} = (f_{ijkl})$ and a second-order tensor $\mathbf{S} = (s_{ij})$ defines a second-order tensor $\mathbf{B} = (b_{ij})$ whose elements are defined as $b_{ij} = \sum_{k,l=1}^{3} f_{ijkl} b_{kl}$. Using this definition, one can show that

$$\mathbf{F} : (\mathbf{u}_1 \otimes \mathbf{u}_2) = (\mathbf{F}\mathbf{u}_2)\mathbf{u}_1 \tag{1.68}$$

which results in a second-order tensor. The double product of fourth- and second-order tensors is important because it will be used to define the product that appears in the constitutive equations of the materials.

Alternatively, a fourth-order tensor $\mathbf{F}$ can be defined as the outer product of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ as $\mathbf{F} = \mathbf{A} \otimes \mathbf{B}$. In this case, the elements of the tensor $\mathbf{F}$ can be defined in terms of the elements of the tensors $\mathbf{A}$ and $\mathbf{B}$ as $f_{ijkl} = A_{ij} B_{kl}$. Furthermore, $\mathbf{F}^{\mathrm{T}} = (\mathbf{A} \otimes \mathbf{B})^{\mathrm{T}} = (\mathbf{B} \otimes \mathbf{A})$, and for arbitrary second-order tensors $\mathbf{S}_1$ and $\mathbf{S}_2$, one has the following identity: $\mathbf{S}_1 : \mathbf{F}^{\mathrm{T}} : \mathbf{S}_2 = \mathbf{S}_2 : \mathbf{F} : \mathbf{S}_1 = (\mathbf{F} : \mathbf{S}_1) : \mathbf{S}_2$.

1.5 POLAR DECOMPOSITION THEOREM

The *polar decomposition* theorem states that any square nonsingular matrix can be decomposed as the product of an orthogonal matrix and a symmetric matrix. According to this theorem, the square matrix $\mathbf{J}$ can have one of the following two decompositions:

$$\mathbf{J} = \mathbf{R}\mathbf{U}, \quad \mathbf{J} = \mathbf{V}\mathbf{R} \tag{1.69}$$

where $\mathbf{R}$ is an orthogonal matrix and $\mathbf{U}$ and $\mathbf{V}$ are nonsingular symmetric matrices. Note that if the decomposition in the first equation is proved, the proof of the decomposition of the second equation follows because $\mathbf{V} = \mathbf{J}\mathbf{R}^{\mathrm{T}} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}$. Therefore, it is sufficient to prove the first decomposition.

Although the proof of the polar decomposition theorem is outlined in this section for 3×3 matrices, the generalization to square matrices with higher dimensions is straightforward. In order to prove the polar decomposition theorem, we define the following symmetric matrix:

$$\mathbf{C} = \mathbf{J}^{\mathrm{T}}\mathbf{J} \tag{1.70}$$

Because $\mathbf{C}$ is symmetric, its eigenvalues are real and its eigenvectors are orthogonal. Furthermore, in addition to the symmetry property, the property of *positive definiteness* of $\mathbf{C}$ is required. The matrix $\mathbf{C}$ is said to be positive definite if for all nonzero vectors $\mathbf{a}$, $\mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a}$ is positive. It can be shown that the eigenvalues of a positive definite matrix are positive. Let λ_1, λ_2, and λ_3 be the eigenvalues of the symmetric positive definite matrix $\mathbf{C}$ and let $\mathbf{\Phi}$ be the matrix whose columns are the eigenvectors of $\mathbf{C}$ associated with the eigenvalues λ_1, λ_2, and λ_3. Using the orthogonality property of the eigenvectors, one has

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{1.71}$$

One can define the following matrix:

$$\mathbf{U} = \mathbf{\Phi} \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix} \mathbf{\Phi}^{\mathrm{T}} \tag{1.72}$$

Again using the orthogonality of the eigenvectors, one can show that

$$\mathbf{U}^2 = \mathbf{\Phi} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \mathbf{\Phi}^{\mathrm{T}} = \mathbf{C}, \quad \mathbf{U}^{-1} = \mathbf{\Phi} \begin{bmatrix} \dfrac{1}{\sqrt{\lambda_1}} & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{\lambda_2}} & 0 \\ 0 & 0 & \dfrac{1}{\sqrt{\lambda_3}} \end{bmatrix} \mathbf{\Phi}^{\mathrm{T}} \tag{1.73}$$

The matrix $\mathbf{R}$ that appears in the polar decomposition theorem can now be defined as

$$\mathbf{R} = \mathbf{J}\mathbf{U}^{-1} \tag{1.74}$$

One can show that this matrix is an orthogonal matrix. To this end, we write

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{U}^{-1}\mathbf{J}^{\mathrm{T}}\mathbf{J}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{C}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I} \tag{1.75}$$

which shows that $\mathbf{R}$ is indeed an orthogonal matrix. Using the positive definiteness property, one can show that the matrices $\mathbf{R}$, $\mathbf{U}$, and $\mathbf{V}$ that appear in the polar decomposition theorem are unique.

The result of the polar decomposition theorem, which states that a matrix can be written as the product of an orthogonal matrix and a symmetric matrix, can be used to explain some of the fundamental problems associated with some finite element formulations. In continuum mechanics, as will be discussed, the position field can be used to define the matrix of *position vector gradients*. This matrix can be written as the product of an orthogonal matrix and a symmetric matrix. The orthogonal matrix defines the rotation of the material elements. That is, the

rotation field can be defined using the matrix of position vector gradients, which is determined from the displacement or position field. In some finite element formulations, the displacement and rotation fields arc interpolated independently, and as a consequence, the geometry is not uniquely defined (Ding et al., 2014). The result of the polar decomposition theorem shows that the use of such independent interpolations for the displacements and rotations can lead to a redundancy problem. In computational mechanics, such a redundancy can lead to serious fundamental and numerical problems. In Chapters 5 and 6, two nonlinear finite element formulations are introduced. The two formulations define unique displacement and rotation fields, and therefore, the problem of coordinate redundancy is not an issue when these two nonlinear formulations are used.

Other Decompositions

There are several other techniques that can be used in the decomposition of matrices. One of these techniques is the **QR** decomposition, which is based on the *Householder transformation*. Using this technique, a square matrix $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is an orthogonal matrix and $\mathbf{R}$ is an upper triangular matrix. The Householder transformation operates on the columns of the matrix $\mathbf{A}$ to produce a set of orthonormal vectors. For example, if $\mathbf{A}$ is a 3×3 matrix, one can make the first column a unit vector and use this unit vector with the other two columns to produce an orthogonal triad that consists of three orthonormal vectors. This triad defines a coordinate system and the orthogonal matrix $\mathbf{Q}$. One can show that the use of this procedure leads to the upper triangular matrix $\mathbf{R}$ defined as $\mathbf{R} = \mathbf{Q}^{\mathbf{T}}\mathbf{A}$. The way the orthogonal matrix $\mathbf{Q}$ is defined here gives a physical interpretation for the **QR** decomposition of 3×3 matrices. In the next chapter, it will be shown that the matrix of *position vector gradients* plays an important role in the formulation of the kinematic and strain equations. One can show that the matrix $\mathbf{Q}$, which results from the **QR** decomposition of the matrix of position vector gradients, is associated with a coordinate system, called the *tangent frame*, frequently used in computational mechanics (Sugiyama et al., 2006). Furthermore, the triangular matrix $\mathbf{R}$, whose diagonal elements define the principal values, provides an alternate upper triangular form instead of the symmetric matrix $\mathbf{U}$ that results from the polar decomposition theorem. The matrix $\mathbf{U}$ in continuum mechanics, when obtained from the decomposition of the matrix of position vector gradients, describes the deformations of the continuum and can be used to determine the strain components because such strain components are not affected by the orthogonal matrix that results from the polar decomposition or the **QR** decomposition. The **QR** decomposition and the Householder transformation have been used in other areas of computational mechanics as reported in the literature (Kim and Vanderploeg, 1986; Shabana, 2001).

1.6 D'ALEMBERT'S PRINCIPLE

The virtual work method represents a powerful technique that can be used to formulate the equations of motion of the continuum. This method is based on *D'Alembert's principle*, which is the foundation for the skillful approaches developed by Lagrange. In this section, we review the particle and rigid-body mechanics to demonstrate the use of D'Alembert's principle in formulating the dynamic equations of motion.

Particle Mechanics

A continuum consists of an infinite number of particles or material points that can move relative to each other if the rigid-body assumptions cannot be applied. The dynamic equations of particles can be obtained using *Newton's second law of motion*, which states that the force acting on a particle is equal to the rate of change of momentum. Newton's second law can be written in vector form as $\mathbf{F} = \dot{\mathbf{p}}$. In this equation, $\mathbf{F}$ is the resultant of the forces acting on the particle and $\mathbf{p}$ is the particle momentum defined as $\mathbf{p} = m\mathbf{v}$, where m and $\mathbf{v}$ are, respectively, the mass and the absolute velocity vector of the particle. If the mass m is assumed to be constant, the preceding two equations lead to

$$\mathbf{F} = \dot{\mathbf{p}} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \tag{1.76}$$

In this equation, $\mathbf{a}$ is the absolute acceleration vector of the particle. In general, three scalar equations are required to describe the particle dynamics. This is mainly due to the fact that, in the case of unconstrained motion, the particle has three degrees of freedom in the spatial analysis because it is represented by a point that has no dimensions. In the case of planar motion, only two equations are required because, in this case, the particle has only two degrees of freedom.

Rigid-Body Kinematics

Rigid bodies are assumed to have dimensions, and therefore, they differ from particles. Nonetheless, a rigid body can be assumed to consist of an infinite number of particles. In this special case of continuum, the distances between the particles of the rigid body remain constant. As a consequence, the displacements of the points on the rigid body are constrained such that there is no relative motion between two points along the line joining them. Using this condition of rigidity, the number of degrees of freedom of a continuum can be significantly reduced. In the case of spatial analysis, a rigid body has six degrees of freedom that describe three independent translations and three independent rotations. In the case of planar motion, the rigid body has only three degrees of freedom: two describe the body translation and one describes the rotation of the body. For instance, as shown in Figure 1, the configuration of the rigid body in planar motion can be described using the vector $\mathbf{r}_O$ and the angle θ. The vector $\mathbf{r}_O$ defines the location of the reference point that represents the origin of a selected body coordinate system, whereas the angle θ defines the body rotation. Using these three coordinates, one can show that the global position vector of an arbitrary point on the body can be written as

$$\mathbf{r} = \mathbf{r}_O + \mathbf{u} \tag{1.77}$$

In this equation, $\mathbf{u}$ is the vector that defines the position of the point with respect to the reference point $\bar{O}$. Because the coordinates of the arbitrary point in the body coordinate system remain constant by virtue of the rigidity assumption, the vector $\mathbf{u}$ can be expressed in terms of these constant coordinates as

$$\mathbf{u} = \begin{bmatrix} \bar{x}_1 \cos\theta - \bar{x}_2 \sin\theta \\ \bar{x}_1 \sin\theta + \bar{x}_2 \cos\theta \end{bmatrix} \tag{1.78}$$

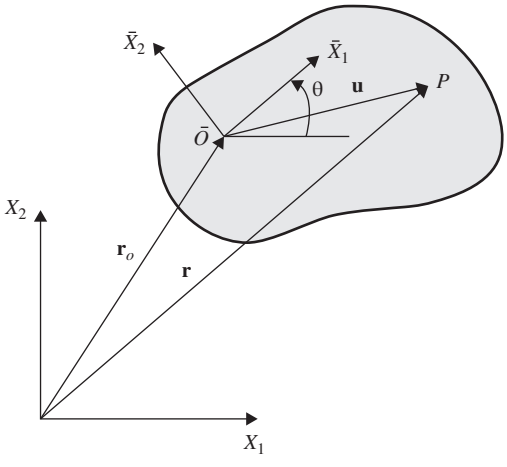

Figure 1.1 Rigid-body coordinates

where $\bar{x}_1$ and $\bar{x}_2$ are the constant coordinates of the arbitrary point defined in the body coordinate system. The preceding equation can be written in matrix form as

$$\mathbf{u} = \mathbf{A}\bar{\mathbf{u}} \tag{1.79}$$

where

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \bar{\mathbf{u}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{1.80}$$

Substituting Equation 79 into Equation 77 one obtains

$$\mathbf{r} = \mathbf{r}_O + \mathbf{A}\bar{\mathbf{u}} \tag{1.81}$$

In this equation, $\mathbf{A}$ represents the transformation matrix that defines the orientation of the selected body coordinate system. This transformation matrix is orthogonal, that is, $\mathbf{A}\mathbf{A}^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}$. Equation 81 shows that the position vector of an arbitrary point on the body is a function of the three coordinates $\mathbf{r}_O$ and θ that can change throughout the body motion. Therefore, these coordinates depend on time. If these coordinates are determined, the global position of any point on the body, or equivalently the body configuration, can be determined using the preceding equation. An equation in the same form as Equation 81 can be obtained in the case of spatial motion of rigid bodies, as will be demonstrated in Chapter 6. In the case of spatial motion, three-dimensional vectors instead of two-dimensional vectors are used, and the transformation matrix $\mathbf{A}$ is expressed in terms of three independent rotation parameters instead of one parameter.

The absolute velocity of an arbitrary point on the rigid body can be obtained by differentiating Equation 81 with respect to time. This leads to

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_O + \dot{\theta}\mathbf{A}_\theta\bar{\mathbf{u}} \tag{1.82}$$

where $\mathbf{A}_\theta = \partial \mathbf{A}/\partial\theta$ is the partial derivative of the transformation matrix $\mathbf{A}$ with respect to the angle θ. In deriving Equation 82, $\dot{\bar{\mathbf{u}}}$ is assumed to be zero because the case of a rigid body is considered. One can define the following *angular velocity vector:*

$$\bar{\boldsymbol{\omega}} = \begin{bmatrix} 0 & 0 & \dot{\theta} \end{bmatrix}^{\mathrm{T}} \tag{1.83}$$

Using this definition, the absolute velocity vector of Equation 82 can be written, after extending the vectors to three-dimensional form by adding zeros, as

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_O + \mathbf{A}(\bar{\boldsymbol{\omega}} \times \bar{\mathbf{u}}) \tag{1.84}$$

The planar transformation matrix is defined when the preceding equation is used as

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.85}$$

Equation 84 can also be rewritten using vectors defined in the global coordinate system as

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_O + \boldsymbol{\omega} \times \mathbf{u} \tag{1.86}$$

In this equation, $\boldsymbol{\omega} = \mathbf{A}\bar{\boldsymbol{\omega}}$, and $\mathbf{u} = \mathbf{A}\bar{\mathbf{u}}$. In the case of the simple planar motion, $\boldsymbol{\omega} = \bar{\boldsymbol{\omega}}$. In the more general case of spatial rigid-body motion, the absolute velocity vector takes the same form as Equations 84 and 86 except for the definition of the transformation matrix $\mathbf{A}$ and the angular velocity vectors $\boldsymbol{\omega}$ and $\bar{\boldsymbol{\omega}}$, which must be formulated using three rotation parameters instead of one, as described in Chapter 6.

The absolute acceleration of an arbitrary point on a rigid body in a planar motion can be obtained by differentiating Equation 82 with respect to time. This leads to

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_O + \ddot{\theta}\mathbf{A}_\theta\bar{\mathbf{u}} - \dot{\theta}^2\mathbf{A}\bar{\mathbf{u}} \tag{1.87}$$

In deriving this equation, the fact that $\mathbf{A}_{\theta\theta} = \partial^2\mathbf{A}/\partial\theta^2 = -\mathbf{A}$ is utilized. This identity applies only to planar transformation; it is a special case of a more general identity that applies to spatial transformation matrices (Roberson and Schwertassek, 1988; Shabana, 2013). Using three-dimensional vectors to represent this planar motion and introducing the following definition for the *angular acceleration:*

$$\bar{\boldsymbol{\alpha}} = \begin{bmatrix} 0 & 0 & \ddot{\theta} \end{bmatrix}^{\mathrm{T}}, \tag{1.88}$$

one can show that the absolute acceleration vector of Equation 87 can be written as

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_O + \mathbf{A}(\bar{\boldsymbol{\alpha}} \times \bar{\mathbf{u}}) + \mathbf{A}\{\bar{\boldsymbol{\omega}} \times (\bar{\boldsymbol{\omega}} \times \bar{\mathbf{u}})\} \tag{1.89}$$

Alternatively, this equation can be written using vectors defined in the global coordinate system as

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_O + \boldsymbol{\alpha} \times \mathbf{u} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{u}) \tag{1.90}$$

In this equation, $\boldsymbol{\alpha} = \mathbf{A}\bar{\boldsymbol{\alpha}}$, and other vectors are as defined previously in this section. Again, Equations 89 and 90 are also applicable to the spatial rigid-body motion. The only difference is in the definition of the transformation matrix and the angular velocity and angular acceleration vectors, which depend on three rotation parameters instead of one as will be discussed in Chapter 6.

Application of D'Alembert's Principle

D'Alembert's principle is the foundation for the skillful development of the principle of virtual work made by Lagrange. D'Alembert's principle states that the inertia forces can be treated as the applied external forces. This principle can be used to conveniently derive the equations of motion of rigid bodies by invoking Newton's second law and assuming that the rigid body consists of a large number of particles. To demonstrate the use of this principle, we consider the planar motion of a rigid body. Assuming that the body consists of a large number of particles, the equations of motion of an infinitesimal material volume on the rigid body can be written as

$$(\rho dV)\ddot{\mathbf{r}} = d\mathbf{F} \tag{1.91}$$

In this equation, $dm = \rho dV$ is the mass of the infinitesimal volume dV, ρ is the mass density of the body, $\ddot{\mathbf{r}}$ is the absolute acceleration vector defined by Equation 87, and $d\mathbf{F}$ is the body force per unit volume. In the *Newton–Euler formulation* of the equations of motion, the origin of the body coordinate system (reference point) is assumed to be attached to the body center of mass. In this case, the vector $\bar{\mathbf{u}}$ defines the position of the arbitrary point with respect to the body center of mass. It follows that

$$\int_V \rho\bar{\mathbf{u}}dV = \mathbf{0} \tag{1.92}$$

Using this identity and the fact that the angular velocity and angular acceleration do not depend on the spatial coordinates, substitution of Equation 87 into Equation 91, and integration leads to

$$m\ddot{\mathbf{r}}_O = \mathbf{F} \tag{1.93}$$

where m is the total mass of the body, and $\mathbf{F}$ is the vector of resultant forces acting on the body. Both are defined as

$$m = \int_V \rho dV, \quad \mathbf{F} = \int_V d\mathbf{F} \tag{1.94}$$

Equation 93 is the *Newton equation* for the rigid body. The vector of resultant forces $\mathbf{F}$ also includes the effect of other concentrated forces. Equation 93 for the planar motion includes two scalar equations. Because the unconstrained body in planar motion has three degrees of freedom, an additional moment equation is needed. Because D'Alembert's principle states that the inertia forces can be treated as the external forces, one can equate the moment of the inertia force $\rho dV\ddot{\mathbf{r}}$ with the moment of the applied forces $d\mathbf{F}$ about any point, we select to be the center of mass. Following this procedure and integrating, one obtains

$$\int_V \mathbf{u} \times (\rho dV\ddot{\mathbf{r}}) = \int_V \mathbf{u} \times d\mathbf{F} + \mathbf{M} \tag{1.95}$$

In this equation, $\mathbf{M}$ is the external moment applied to the body. Using Equations 87 and 92, one can show that the preceding equation reduces to one nontrivial equation associated with the rotation about the X_3-axis and is given by

$$I_O\ddot{\theta} = M_O \tag{1.96}$$

In this equation, $I_O = \int_V \rho(\bar{x}_1^2 + \bar{x}_2^2)dV$ defines the *mass moment of inertia* of the body about its center of mass, and M_O is the third component of the vector $\int_V \mathbf{u} \times d\mathbf{F} + \mathbf{M}$. Equation 96 is called *Euler equation.* Equations 93 and 96 are the two equations that govern the planar motion of the rigid body.

A similar procedure based on D'Alembert's principle can be used to obtain the equations that govern the spatial motion of rigid bodies. These equations are called the *Newton–Euler equations* and are given by

$$\begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{I}}_{\theta\theta} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_O \\ \bar{\boldsymbol{\alpha}} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \bar{\mathbf{M}}_O - \bar{\boldsymbol{\omega}} \times (\bar{\mathbf{I}}_{\theta\theta}\bar{\boldsymbol{\omega}}) \end{bmatrix} \tag{1.97}$$

In this equation, m is the total mass of the body, $\mathbf{I}$ is the identity matrix, $\mathbf{F}$ is the vector of the resultant forces defined in the global coordinate system, $\bar{\mathbf{M}}_O$ is the vector of the resultant moments defined in the body coordinate system, and $\bar{\mathbf{I}}_{\theta\theta}$ is the constant symmetric inertia tensor defined in the body coordinate system. The inertia tensor $\bar{\mathbf{I}}_{\theta\theta}$ is defined as

$$\bar{\mathbf{I}}_{\theta\theta} = \begin{bmatrix} \int_V \rho(\bar{x}_2^2 + \bar{x}_3^2)dV & \text{Symmetric} & \\ -\int_V \rho\bar{x}_1\bar{x}_2 dV & \int_V \rho(\bar{x}_1^2 + \bar{x}_3^2)dV & \\ -\int_V \rho\bar{x}_1\bar{x}_3 dV & -\int_V \rho\bar{x}_2\bar{x}_3 dV & \int_V \rho(\bar{x}_1^2 + \bar{x}_2^2)dV \end{bmatrix} \tag{1.98}$$

where $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ are the components of the vector $\bar{\mathbf{u}}$ that defines the position of the arbitrary point with respect to the origin of the body coordinate system. Equation 97 shows that Newton–Euler equations do not include inertia coupling between the body translation and rotation. This is mainly due to the use of the body center of mass as the reference point.

The analysis presented in this section shows that starting with Newton's second law, D'Alembert's principle can be used to obtain Euler equations of motion of the rigid body by equating the moments of the inertia forces to the moments of the applied forces. The application of D'Alembert's principle also allows for systematically eliminating the constraint forces, thereby obtaining a minimum number of motion equations equal to the number of degrees of freedom of the system. This subject is covered in more detail in books on computational and analytical dynamics.

The analysis presented in this section also shows that in the special case of rigid-body motion, precise description of the finite rotation of the body is important in the formulation of the dynamic equations. This description of the finite rotation becomes even more important when the body undergoes deformation coupled with a rigid-body motion. Therefore, in the case of deformable bodies, it is important to select a set of coordinates that correctly describe the rigid-body motion when computational methods are used to develop finite dimensional

models. The coordinates selected must define a unique displacement and rotation field and must lead to zero strain under an arbitrary rigid-body displacement. This subject will be discussed in more detail when large displacement finite element formulations are introduced in later chapters.

Continuum Forces

In the case of rigid bodies, the mass density and volume remain constant. However, in the case of a continuum subjected to arbitrary displacements, the mass density and volume change. For this reason, in continuum mechanics, it is important to distinguish between the mass density ρ_o and the volume V in the reference undeformed configuration and the mass density ρ and the volume v in the current deformed configuration. Nevertheless, the definitions and basic principles used in rigid-body dynamics can be generalized to the case of a general continuum by considering the continuum to consist of an infinite number of points that can move relative to each other. For example, the inertia force of a material point of an infinitesimal mass ρdv on the continuum in the current configuration can be written as $(\rho dv)\ddot{\mathbf{r}}$, where $\ddot{\mathbf{r}}$ is the absolute acceleration vector of the material point. Using this expression for the inertia force of the material point, the inertia force for the continuum can be defined as $\int_v \rho \ddot{\mathbf{r}} dv$. Using a similar procedure, the kinetic energy of the continuum can be written as $(1/2)\int_v \rho \dot{\mathbf{r}}^{\mathrm{T}} \dot{\mathbf{r}} dv$. In the next chapter, the relationship between the mass density and volume in the reference configuration and in the current configuration will be defined. This relationship will allow us to carry out the integration using the known properties and dimensions in the undeformed reference configuration. For instance, the use of the *principle of conservation of mass* obtained in the next chapter allows one to write the inertia force as $\int_V \rho_o \ddot{\mathbf{r}} dV$, where ρ_o and V are, respectively, the mass density and volume in the undeformed reference configuration.

In a similar manner, one can define the body forces acting on the continuum. Let $\mathbf{f}_b$ be the distributed body force per unit volume acting on the continuum at the material points. Examples of these forces are gravity and magnetic forces. The sum of the body forces acting on the continuum can be defined as $\int_v \mathbf{f}_b dv$. The effect of the traction forces that include tangential friction forces and normal reaction and pressure can be obtained by integration over the area. If $\mathbf{f}_s$ is the vector of distributed surface forces that act on a surface area s, the surface forces acting on the continuum can be defined using the integral $\int_s \mathbf{f}_s ds$.

Expressions for the internal elastic forces due to the continuum deformations will be developed in this book. These expressions will be written in terms of the strain and stress components, which are introduced in Chapters 2 and 3, respectively. As will be shown, D'Alembert's principle and Lagrange's techniques can still be used to obtain the equations of motion of the continuum. Nonetheless, the internal elastic forces can be expressed using different deformation and stress measures. It is important, however, that the resulting work of the elastic forces and the strain energy remain constant under an arbitrary rigid-body transformation. For this reason, the important concept of the *objectivity* will be a subject of discussion in this book.

1.7 VIRTUAL WORK PRINCIPLE

As previously mentioned, D'Alembert's principle represents the foundation for the skillful virtual work method developed by Lagrange. The virtual work principle can be used to

systematically derive the equations of motion of complex systems. In this principle, the concept of the virtual displacement, which represents an infinitesimal change in coordinates that is consistent with the constraints imposed on the motion of the system, is important. During this virtual change, time is assumed to be frozen. For a vector $\mathbf{r}$, the virtual change is denoted as $\delta\mathbf{r}$. Given a system that consists of n_p particles, the equations of motion of a particle i in the system can be written using Newton's second law as

$$m^i\ddot{\mathbf{r}}^i = \mathbf{F}_e^i + \mathbf{F}_c^i, \quad i = 1, 2, \ldots, n_p \tag{1.99}$$

In this equation, m^i is the mass of the particle, $\mathbf{r}^i$ is the global position vector, $\mathbf{F}_e^i$ is the vector of applied forces acting on the particle, and $\mathbf{F}_c^i$ is the vector of constraint forces. Multiplying the preceding equation by the virtual change in the position vector of the particle, one obtains

$$m^i\ddot{\mathbf{r}}^i \cdot \delta\mathbf{r}^i = \mathbf{F}_e^i \cdot \delta\mathbf{r}^i + \mathbf{F}_c^i \cdot \delta\mathbf{r}^i, \quad i = 1, 2, \ldots, n_p \tag{1.100}$$

This equation, which is called the *Lagrange–D'Alembert equation*, can be written as

$$\delta W_i^i = \delta W_e^i + \delta W_c^i \tag{1.101}$$

where δW_i^i is the virtual work of the inertia forces, δW_e^i is the virtual work of the applied forces, and δW_c^i is the virtual work of the constraint forces. These expressions for the virtual work are defined for particle i as

$$\delta W_i^i = m^i\ddot{\mathbf{r}}^i \cdot \delta\mathbf{r}^i, \quad \delta W_e^i = \mathbf{F}_e^i \cdot \delta\mathbf{r}^i, \quad \delta W_c^i = \mathbf{F}_c^i \cdot \delta\mathbf{r}^i \tag{1.102}$$

Using Equation 101, one can write

$$\sum_{i=1}^{n_p} \delta W_i^i = \sum_{i=1}^{n_p} \delta W_e^i + \sum_{i=1}^{n_p} \delta W_c^i \tag{1.103}$$

which can be written as

$$\delta W_i = \delta W_e + \delta W_c \tag{1.104}$$

where

$$\delta W_i = \sum_{i=1}^{n_p} \delta W_i^i, \quad \delta W_e = \sum_{i=1}^{n_p} \delta W_e^i, \quad \delta W_c = \sum_{i=1}^{n_p} \delta W_c^i \tag{1.105}$$

represent, respectively, the virtual work of the system inertia forces, the virtual work of the system applied forces, and the virtual work of the system constraint forces. Because the constraint forces acting on two particles are equal in magnitude and opposite in direction and because the virtual change in a specified (prescribed) coordinate is equal to zero, one must have $\delta W_c = 0$. This equation and Equation 104 lead to the principle of virtual work, which can be stated mathematically as

$$\delta W_i = \delta W_e \tag{1.106}$$

This principle states that the virtual work of the system inertia forces must be equal to the virtual work of the system applied forces. Note that in Equation 106, the constraint forces are systematically eliminated. Note also that although $\delta W_c = 0$, $\delta W_c^i \neq 0$ if the particle i is subjected to constraints.

Relationship with D'Alembert's Principle

A simple example can be used to demonstrate the relationship between the principle of virtual work and D'Alembert's principle. To this end, we consider the derivation of the equations of motion of a planar rigid body. The virtual work of the inertia forces of the rigid body can be written as

$$\delta W_i = \int_V \rho \ddot{\mathbf{r}} \cdot \delta \mathbf{r} dV \tag{1.107}$$

where $\mathbf{r}$ and $\ddot{\mathbf{r}}$ are given, respectively, by Equations 81 and 87. The virtual change $\delta\mathbf{r}$ can be written as

$$\delta \mathbf{r} = \delta \mathbf{r}_O + \mathbf{A}_\theta \bar{\mathbf{u}} \delta\theta \tag{1.108}$$

This equation can be written using matrix notation as

$$\delta \mathbf{r} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_\theta \bar{\mathbf{u}} \end{bmatrix} \begin{bmatrix} \delta \mathbf{r}_O \\ \delta\theta \end{bmatrix} \tag{1.109}$$

where $\mathbf{I}$ is the 2×2 identity matrix. The acceleration vector of Equation 87 can also be written as

$$\ddot{\mathbf{r}} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_\theta \bar{\mathbf{u}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_O \\ \ddot{\theta} \end{bmatrix} - \dot{\theta}^2 \mathbf{A}\bar{\mathbf{u}} \tag{1.110}$$

Substituting Equations 109 and 110 into Equation 107 and using Equation 92, which is the result of using the center of mass as the reference point, one obtains

$$\delta W_i = \begin{bmatrix} \delta \mathbf{r}_O^T & \delta\theta \end{bmatrix} \begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & I_O \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_O \\ \ddot{\theta} \end{bmatrix} \tag{1.111}$$

where m is the mass of the rigid body and I_O is the mass moment of inertia about the center of mass that was used in Equation 96.

The virtual work of all the applied forces and moments acting on the body can be written as

$$\delta W_e = \mathbf{F}^{\mathrm{T}} \delta \mathbf{r}_O + M_O \delta\theta = \begin{bmatrix} \delta \mathbf{r}_O^{\mathrm{T}} & \delta\theta \end{bmatrix} \begin{bmatrix} \mathbf{F} \\ M_O \end{bmatrix} \tag{1.112}$$

Substituting Equations 111 and 112 into the principle of virtual work of Equation 106, one obtains

$$\begin{bmatrix} \delta \mathbf{r}_O^{\mathrm{T}} & \delta\theta \end{bmatrix} \left\{ \begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & I_O \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_O \\ \ddot{\theta} \end{bmatrix} - \begin{bmatrix} \mathbf{F} \\ M_O \end{bmatrix} \right\} = 0 \tag{1.113}$$

In the case of unconstrained motion of the planar rigid body, $\mathbf{r}_O$ and θ represent three independent coordinates, and therefore, the coefficients of their virtual change in the preceding

equation must be identically equal to zero. This leads to the following system of equations of motion:

$$\begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & I_O \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}}_O \\ \ddot{\theta} \end{bmatrix} - \begin{bmatrix} \mathbf{F} \\ M_O \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix} \tag{1.114}$$

This matrix equation has three scalar equations, which are the planar Newton–Euler equations previously obtained in this chapter (see Equations 93 and 96) using D'Alembert's principle.

Although D'Alembert's principle and the virtual work principle lead to the same equations, it is important to note that in the virtual work principle, scalar quantities are used and there is no need to use cross products to define moments. The forces and moments are defined using the scalar virtual work expressions. In the case of constrained motion, D'Alembert's principle and the virtual work principle also lead to the same results. Both principles can be used to systematically eliminate the constraint forces and obtain a number of equations equal to the number of degrees of freedom of the system. Nonetheless, the principle of virtual work is much easier to use, particularly when complex systems are considered.

1.8 APPROXIMATION METHODS

Solids and fluids have an infinite number of degrees of freedom because their particles can have arbitrary displacements with respect to each other. The dynamics of such systems is described using partial differential equations that depend on time and the spatial coordinates. These general partial differential equations, which are applicable to any solid or fluid material, are derived in Chapter 3. Only for very simple problems, one can find a closed-form solution for these partial differential equations. For most problems, however, one resorts to numerical methods to obtain the solution of the partial differential equations. Approximation methods such as the *finite difference* and *finite element methods* are often used to solve the partial differential equations by transforming these equations into a finite set of ordinary differential or algebraic equations that can be solved using computer and numerical methods. In some of the numerical techniques based on the *Rayleigh–Ritz method,* physical variables such as position, displacement, velocity, and/or acceleration are approximated using interpolation functions that have finite order. The coefficients of the interpolating functions in the case of dynamics can be expressed in terms of coordinates that depend on time. In order to demonstrate the use of this procedure, the two-dimensional beam shown in Figure 2 is considered. It is assumed that the position of the material points on the beam can be described using the following polynomials:

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2 + a_4 (x_1)^2 + a_5 (x_1)^3 \\ b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1 x_2 + b_4 (x_1)^2 + b_5 (x_1)^3 \end{bmatrix} \tag{1.115}$$

where x_1 and x_2 are the beam local coordinates defined in the beam coordinate system shown in Figure 2 and a_i and b_i, $i = 0, 1, 2, \ldots, 5$, are the polynomial coefficients. The procedure used in computational methods such as the finite element method is to replace the polynomial coefficients with a set of coordinates that have physical meaning. This can be accomplished by developing a set of algebraic equations that relate the polynomial coefficients to the new set of coordinates. These algebraic equations can be solved to determine the polynomial coefficients in terms of the new coordinates. In the beam example considered in this section, there are

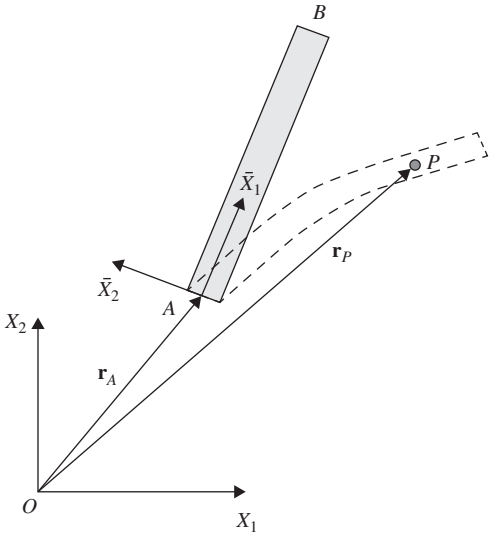

Figure 1.2 Two-dimensional beam

12 polynomial coefficients, and therefore, 12 coordinates can be used to replace the coefficients a_i and b_i in Equation 115. To this end, the preceding equation can be written as

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_2 & x_1x_2 & (x_1)^2 & (x_1)^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_1 & x_2 & x_1x_2 & (x_1)^2 & (x_1)^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \tag{1.116}$$

The coefficients a_i and b_i, $i = 0, 1, 2, \ldots, 5$, can be replaced by coefficients that represent position and gradient coordinates. To this end, we choose a set of coordinates associated with the position and gradient coordinates of the two endpoints of the beam. For the first endpoint at A, we use

$$\mathbf{r}^1 = \mathbf{r}(0,\ 0) = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \quad \mathbf{r}^1_{x_1} = \mathbf{r}_{x_1}(0,\ 0) = \begin{bmatrix} e_3 \\ e_4 \end{bmatrix}, \quad \mathbf{r}^1_{x_2} = \mathbf{r}_{x_2}(0,\ 0) = \begin{bmatrix} e_5 \\ e_6 \end{bmatrix} \tag{1.117}$$

and for the second endpoint at B, we use

$$\mathbf{r}^2 = \mathbf{r}(l,\ 0) = \begin{bmatrix} e_7 \\ e_8 \end{bmatrix}, \quad \mathbf{r}^2_{x_1} = \mathbf{r}_{x_1}(l,\ 0) = \begin{bmatrix} e_9 \\ e_{10} \end{bmatrix}, \quad \mathbf{r}^2_{x_2} = \mathbf{r}_{x_2}(l,\ 0) = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} \tag{1.118}$$

In this equation, l is the length of the beam and $\mathbf{r}_{x_k} = \partial \mathbf{r}/\partial x_k$, $k = 1, 2$. There are 12 coordinates in Equations 117 and 118, and therefore, these coordinates, which have physical meaning, can be used to replace the coefficients a_i and b_i, $i = 0, 1, 2, \ldots, 5$, in Equation 116. Using Equations 116–118, one can show that the position vector $\mathbf{r}$ can be written in terms of the coordinates of Equations 117 and 118 as (Omar and Shabana, 2001):

$$\mathbf{r} = \mathbf{S}(\mathbf{x})\mathbf{e}(t) \tag{1.119}$$

where $\mathbf{e} = \mathbf{e}(t) = [e_1 \quad e_2 \quad \ldots \quad e_{12}]^{\mathrm{T}}$ is a vector of time-dependent coefficients or coordinates, which consist of position and gradient coordinates, t is time, $\mathbf{x} = [x_1 \quad x_2]^{\mathrm{T}}$, and $\mathbf{S} = \mathbf{S}(\mathbf{x})$ is a matrix called the *shape function matrix* that depends on the local coordinates x_1 and x_2 and is given by

$$\mathbf{S} = \begin{bmatrix} s_1 & 0 & s_2 & 0 & s_3 & 0 & s_4 & 0 & s_5 & 0 & s_6 & 0 \\ 0 & s_1 & 0 & s_2 & 0 & s_3 & 0 & s_4 & 0 & s_5 & 0 & s_6 \end{bmatrix} \tag{1.120}$$

The elements s_i, $i = 1, 2, \ldots, 6$, which appear in this equation, are given by

$$\left.\begin{array}{lll} s_1 = 1 - 3\xi^2 + 2\xi^3, & s_2 = l(\xi - 2\xi^2 + \xi^3), & s_3 = l\eta(1 - \xi) \\ s_4 = 3\xi^2 - 2\xi^3, & s_5 = l(-\xi^2 + \xi^3), & s_6 = l\xi\,\eta \end{array}\right\} \tag{1.121}$$

where $\xi = x_1/l$ and $\eta = x_2/l$.

The procedure described in this section to approximate a field using polynomials and replace the polynomial coefficients using coordinates that can have physical meaning is a fundamental step in the finite element formulation. The choice of the coordinates to consist of absolute position and gradient coordinates is the basis of a finite element formulation called the *absolute nodal coordinate formulation* (ANCF). This formulation, which is discussed in Chapter 5, can be used to correctly describe an arbitrary rigid-body displacement including arbitrary rotations. Using the absolute position and gradient coordinates, ANCF does not impose any restriction on the amount of rotation or deformation within the element, and therefore, it is suited for the large-deformation analysis. Efficient solution of small-deformation problems, on the other hand, requires the use of a more elaborate procedure in order to eliminate displacement modes, which have a negligible effect on the solution. In the literature, small-deformation problems are often solved using the *floating frame of reference* (FFR) formulation, which is discussed in Chapter 6. In the FFR formulation, which can be considered a generalization of the Newton–Euler equations used in rigid-body dynamics, a different set of coordinates is used to define a local linear deformation problem that allows reducing systematically the number of degrees of freedom.

1.9 DISCRETE EQUATIONS

The principle of virtual work and approximation techniques can be used to determine a set of discrete ordinary differential equations that govern the dynamics of the continuum. As will be shown in Chapter 3, the motion of the continuum is governed by partial differential equations that depend on the spatial coordinates $\mathbf{x}$ and time t. The principle of virtual work

and the approximation techniques can be used to systematically convert the partial differential equations to a set of discrete ordinary differential equations. To demonstrate this standard procedure, consider a deformable body that may undergo arbitrary displacements. The global position vector of an arbitrary point on the body is defined by the vector $\mathbf{r}$. The mass of an infinitesimal volume dv of the body is ρdv, where ρ is the mass density in the current configuration. It follows that the inertia force of this infinitesimal volume is $(\rho dv)\ddot{\mathbf{r}}$, where $\ddot{\mathbf{r}}$ is the absolute acceleration vector. Therefore, the virtual work of the inertia forces of the body can be written as follows:

$$\delta W_i = \int_v \rho \ddot{\mathbf{r}}^{\mathrm{T}} \delta \mathbf{r} dv \tag{1.122}$$

Using the approximation techniques, the virtual displacement $\delta\mathbf{r}$ can be expressed in terms of the virtual changes of a finite set of coordinates $\mathbf{q}$, as previously demonstrated in the case of rigid bodies. That is,

$$\delta \mathbf{r} = \mathbf{S}\delta\mathbf{q} \tag{1.123}$$

In this equation, $\mathbf{S}$ is an appropriate matrix that relates the virtual change of $\mathbf{r}(\mathbf{x}, t)$ to the virtual change in the coordinates $\mathbf{q}(t)$. Note that, whereas $\mathbf{r}(\mathbf{x}, t)$ depends on both coordinates $\mathbf{x}$ and time t, the coordinates $\mathbf{q}(t)$ depend only on time. As will be shown in this book, the matrix $\mathbf{S}$ depends, in general, on both $\mathbf{x}$ and $\mathbf{q}$, as in the case of the small-deformation FFR formulation discussed in Chapter 6. That is, one can write $\mathbf{S} = \mathbf{S}(\mathbf{x}, \mathbf{q}(t))$. However, in other formulations, such as ANCF discussed in Chapter 5, one can write $\mathbf{S}$ as a function of $\mathbf{x}$ only, as demonstrated in the preceding section. Therefore, in large displacement formulations, one can in general write the absolute velocity and acceleration vectors as

$$\dot{\mathbf{r}} = \mathbf{S}\dot{\mathbf{q}}, \quad \ddot{\mathbf{r}} = \mathbf{S}\ddot{\mathbf{q}} + \boldsymbol{\gamma} \tag{1.124}$$

In this equation, $\boldsymbol{\gamma} = \dot{\mathbf{S}}\dot{\mathbf{q}}$ is a vector, which is quadratic in the first derivatives of the coordinates. If the matrix $\mathbf{S}$ is only a function of $\mathbf{x}$, the vector $\boldsymbol{\gamma}$ is identically zero. Substituting Equations and 123 and 124 into Equation 122, one obtains

$$\delta W_i = \int_v \rho\, (\mathbf{S}\ddot{\mathbf{q}} + \boldsymbol{\gamma})^{\mathrm{T}} (\mathbf{S}\delta\mathbf{q}) dv \tag{1.125}$$

The terms in this equation can be rearranged. This leads to

$$\delta W_i = \int_v \rho \{\ddot{\mathbf{q}}^{\mathrm{T}} (\mathbf{S}^{\mathrm{T}}\mathbf{S}) + \boldsymbol{\gamma}^{\mathrm{T}}\mathbf{S}\} dv \delta\mathbf{q} \tag{1.126}$$

This equation can be written as

$$\delta W_i = (\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}_v)^{\mathrm{T}} \delta\mathbf{q} \tag{1.127}$$

where

$$\mathbf{M} = \int_v \rho \mathbf{S}^{\mathrm{T}}\mathbf{S} dv, \quad \mathbf{Q}_v = -\int_v \rho \mathbf{S}^{\mathrm{T}}\boldsymbol{\gamma} dv \tag{1.128}$$

In this equation, $\mathbf{M}$ is the symmetric mass matrix, and $\mathbf{Q}_v$ is the vector of Coriolis and centrifugal forces. Depending on the set of coordinates selected, some nonlinear finite element formulations, as will be discussed in this book, lead to a constant mass matrix and zero centrifugal and Coriolis forces, while other formulations lead to a nonlinear mass matrix and nonzero centrifugal and Coriolis forces. The use of Equation 128 to evaluate the inertia forces leads to what is known in the literature as a *consistent mass formulation*. In some structural finite element formulations, *lumped mass techniques* are used to formulate the inertia forces by representing the inertia of the body using discrete bodies or masses instead of using the distributed inertia representation of Equation 128. In the finite element formulations discussed in this book, the mass matrix cannot, in general, be diagonal, even in the case in which lumped mass techniques are used. Furthermore, in the large-deformation finite element formulation presented in Chapter 5, one cannot use lumped masses, because the use of such a lumping scheme does not lead to correct modeling of the rigid-body dynamics.

Similarly, by using Equation 123, the virtual work of the applied forces can be written as

$$\delta W_e = \mathbf{Q}_e^{\mathrm{T}} \delta \mathbf{q} \tag{1.129}$$

Using Equations 127 and 129 and the principle of virtual work, which states that $\delta W_i = \delta W_e$, one obtains the following equation:

$$(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}_e - \mathbf{Q}_v)^{\mathrm{T}} \delta \mathbf{q} = 0 \tag{1.130}$$

If the elements of the vector $\mathbf{q}$ are independent, the preceding equation leads to the discrete ordinary differential equations of the system given as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}_e + \mathbf{Q}_v \tag{1.131}$$

However, if the elements of the vector $\mathbf{q}$ are not totally independent because of kinematic relationships between the coordinates, one can always write the coordinates $\mathbf{q}$ in terms of a reduced set of independent coordinates $\mathbf{q}_i$. In this case, one can write the following relationship between the virtual changes of the system coordinates and the virtual changes of the independent coordinates:

$$\delta \mathbf{q} = \mathbf{B} \delta \mathbf{q}_i \tag{1.132}$$

In this equation, $\mathbf{B}$ is a velocity transformation matrix that can be defined using the kinematic relationship between the coordinates. Substituting the preceding equation into Equation 130 and using the argument that the elements of the vector $\mathbf{q}_i$ are independent, one obtains the following reduced system of ordinary differential equations:

$$\mathbf{B}^{\mathrm{T}}(\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}_e - \mathbf{Q}_v) = \mathbf{0} \tag{1.133}$$

The vector of accelerations can also be expressed in terms of the independent accelerations, leading to a number of equations equal to the number of independent coordinates (degrees of freedom). To this end, one can use Equation 132 to write

$$\dot{\mathbf{q}} = \mathbf{B}\dot{\mathbf{q}}_i, \quad \ddot{\mathbf{q}} = \mathbf{B}\ddot{\mathbf{q}}_i + \dot{\mathbf{B}}\dot{\mathbf{q}}_i \tag{1.134}$$

Substituting these equations into Equation 133, one obtains a set of equations solely expressed in terms of the independent accelerations. These equations are given as

$$(\mathbf{B}^{\mathrm{T}}\mathbf{M}\mathbf{B})\ddot{\mathbf{q}}_i = \mathbf{B}^{\mathrm{T}}(\mathbf{Q}_e + \mathbf{Q}_v - \mathbf{M}\dot{\mathbf{B}}\dot{\mathbf{q}}_i) \tag{1.135}$$

In this equation, $(\mathbf{B}^{\mathrm{T}}\mathbf{M}\mathbf{B})$ is the generalized inertia matrix associated with the independent coordinates and $\mathbf{B}^{\mathrm{T}}(\mathbf{Q}_e + \mathbf{Q}_v - \mathbf{M}\dot{\mathbf{B}}\dot{\mathbf{q}}_i)$ is the vector of generalized forces that include applied, centrifugal, and Coriolis forces.

The procedure described in this section for writing the dynamic equations in terms of the independent coordinates or the degrees of freedom is called the *embedding technique*. The dynamic equations can also be formulated in terms of *redundant coordinates* using the technique of *Lagrange multipliers*. The subject of constrained dynamics is discussed in more detail in the multibody system dynamics literature (Roberson and Schwertassek, 1988; Shabana, 2013).

1.10 MOMENTUM, WORK, AND ENERGY

The study of the computational finite element method requires a sound understanding of the basic analytical mechanics principles. If correctly derived, the equations of motion must satisfy these principles. Violations of these mechanics principles are a clear indication of inconsistencies in formulating the dynamic equations of motion. Among the principles of mechanics that will be discussed in this section are the principle of linear and angular momentum and the principle of work and energy. Correct and consistent formulations of the finite element equations of motion must automatically satisfy these principles. When these equations of motion are numerically solved, the violations in these principles must be within the range of the error tolerance of the numerical integration method used. Nonetheless, some finite element formulations lead to energy drift as a result of being inconsistent. For example, the use of a nonunique rotation field can lead to violations of the principle of work and energy as evident by many of the results presented in the finite element literature.

Linear and Angular Momentum

The *linear momentum* of a body is defined as

$$\mathbf{M}_l = \int_v \rho \dot{\mathbf{r}} dv \tag{1.136}$$

where $\mathbf{M}_l$ is the vector of linear momentum, ρ and v are, respectively, the mass density and volume of the body in the current configuration, and $\mathbf{r}$ is the position vector of an arbitrary point on the body. Newton's second law states that the rate of change of the linear momentum is equal to the resultant of the forces acting on the body, that is,

$$\dot{\mathbf{M}}_l = \frac{d}{dt}\left(\int_v \rho \dot{\mathbf{r}} dv\right) = \mathbf{F} \tag{1.137}$$

where $\mathbf{F}$ is the resultant of the forces acting on the body. In the case of rigid bodies, the volume is assumed to remain constant; as a result, the preceding equation, when a centroidal body coordinate system is used, defines the Newton equation, $m\ddot{\mathbf{r}}_O = \mathbf{F}$, where m is the total mass of the body and $\ddot{\mathbf{r}}_O$ is the acceleration of the body center of mass. If the resultant of the forces acting on the body is equal to zero, one obtains the *principle of conservation of the linear momentum*, which is written as

$$\mathbf{M}_l = \mathbf{c}_l \tag{1.138}$$

In this equation, $\mathbf{c}_l$ is a constant vector.

The vector of the *angular momentum* of the body is defined as

$$\mathbf{M}_a = \int_v \rho(\mathbf{r} \times \dot{\mathbf{r}})\, dv \tag{1.139}$$

In the case of rigid bodies, the rate of change of the angular momentum is given by

$$\dot{\mathbf{M}}_a = \frac{d}{dt}\left(\int_v \rho(\mathbf{r} \times \dot{\mathbf{r}})\, dv\right) = \int_v \rho(\mathbf{r} \times \ddot{\mathbf{r}})\, dv \tag{1.140}$$

Using Newton's equation and D'Alembert's principle, one can show that, in the case of rigid bodies, the rate of change of angular momentum is equal to the resultant of the moment applied to the body. If the resultant of the moments is equal to zero, one obtains the *principle of conservation of angular momentum*, which is expressed mathematically as

$$\mathbf{M}_a = \mathbf{c}_a \tag{1.141}$$

where $\mathbf{c}_a$ is a constant vector.

Work and Energy

According to D'Alembert's principle, the work of the inertia forces is equal to the work of the applied forces. This statement can be written mathematically in the following form:

$$\int_v \rho(\ddot{\mathbf{r}} \cdot d\mathbf{r})\, dv = \mathbf{F} \cdot d\mathbf{r} \tag{1.142}$$

Using the identity $\ddot{r}_k = \dot{r}_k(d\dot{r}_k/dr_k)$, $k = 1, 2, 3$, one can show that

$$\ddot{\mathbf{r}} \cdot d\mathbf{r} = \dot{\mathbf{r}} \cdot d\dot{\mathbf{r}} \tag{1.143}$$

Substituting this equation into Equation 142 one obtains

$$\int_v \rho(\dot{\mathbf{r}} \cdot d\dot{\mathbf{r}})dv = \mathbf{F} \cdot d\mathbf{r} \tag{1.144}$$

Integrating this equation from t_0 to t and using the rigid-body assumption, one can show that

$$\frac{1}{2}\int_v \rho(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}})dv \; - \; T_0 = \int_{t_0}^{t} \mathbf{F}\cdot d\mathbf{r} \tag{1.145}$$

where T_0 is the kinetic energy of the body at the initial configuration. The preceding equation is a statement of the *principle of work and energy*. This equation shows that the change in the body kinetic energy is equal to the work done by the applied forces. This principle is derived using the equations of motion of the body. Therefore, any set of equations of motion, if correctly and consistently derived, must satisfy the principle of work and energy. As previously mentioned, some of the nonlinear finite element formulations proposed in the literature for the large displacement analysis fail to automatically satisfy this principle. The nonlinear finite element formulations presented in later chapters of this book automatically satisfy the principle of work and energy, and their use does not require taking special measures when the equations of motion are integrated numerically.

1.11 PARAMETER CHANGE AND COORDINATE TRANSFORMATION

In continuum mechanics, it is important to differentiate between two different transformations. The first is the *change of parameters* and the second is the *coordinate transformation of vectors*. Understanding the difference between these two transformations is crucial in understanding the definitions of the strain components that will be introduced in the next chapter. This subject is also important in the large-deformation finite element formulation discussed in this book.

Change of Parameters

In order to explain the difference between the change of parameters and the coordinate transformation of vectors, we consider the vector $\mathbf{r}$, which is expressed in terms of three coordinates x_1, x_2, and x_3. The vector $\mathbf{r}$ can then be written as

$$\mathbf{r} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^{\mathrm{T}} = \mathbf{r}(x_1,\, x_2,\, x_3) \tag{1.146}$$

Assume that the components of this vector are defined in the coordinate system $X_1X_2X_3$. The matrix of gradients of this vector obtained by differentiation with respect to the parameters x_1, x_2, and x_3 is given by

$$\mathbf{J} = \begin{bmatrix} \mathbf{r}_{x_1} & \mathbf{r}_{x_2} & \mathbf{r}_{x_3} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial r_1}{\partial x_1} & \dfrac{\partial r_1}{\partial x_2} & \dfrac{\partial r_1}{\partial x_3} \\ \dfrac{\partial r_2}{\partial x_1} & \dfrac{\partial r_2}{\partial x_2} & \dfrac{\partial r_2}{\partial x_3} \\ \dfrac{\partial r_3}{\partial x_1} & \dfrac{\partial r_3}{\partial x_2} & \dfrac{\partial r_3}{\partial x_3} \end{bmatrix} \tag{1.147}$$

It is important to realize that x_1, x_2, and x_3 represent coordinate lines, and the vector $\mathbf{r}_{x_i}$, $i = 1$, 2, 3, which is a gradient vector defined by differentiation with respect to the coordinate x_i,

represents the change in the vector $\mathbf{r}$ as a result of a small change in the coordinate x_i. The vector $\mathbf{r}_{x_i}$ is not necessarily a unit vector, and a measure of the deviation from a unit vector is defined as $(\mathbf{r}_{x_i}^{\mathrm{T}}\mathbf{r}_{x_i} - 1)$. A measure of the angle between two gradient vectors can be obtained using the dot product $\mathbf{r}_{x_i}^{\mathrm{T}}\mathbf{r}_{x_j}$, $i \neq j$. Therefore, equations such as $(\mathbf{r}_{x_i}^{\mathrm{T}}\mathbf{r}_{x_i} - 1)$ and $\mathbf{r}_{x_i}^{\mathrm{T}}\mathbf{r}_{x_j}$, $i \neq j$, can be used to measure the deformation and shear effects at material points of the continuum.

The vector $\mathbf{r}$ defined in the same coordinate system $X_1X_2X_3$ can be written in terms of another set of parameters $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$, which are related to the parameters x_1, x_2, and x_3 by the relation

$$\mathbf{x} = \bar{\mathbf{x}}(x_1,\ x_2,\ x_3) \tag{1.148}$$

where $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^{\mathrm{T}}$ and $\bar{\mathbf{x}} = \begin{bmatrix}\bar{x}_1 & \bar{x}_2 & \bar{x}_3\end{bmatrix}^{\mathrm{T}}$. Using differentiation with respect to the parameters $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$, the matrix of gradients can be written as

$$\bar{\mathbf{J}} = \begin{bmatrix}\mathbf{r}_{\bar{x}_1} & \mathbf{r}_{\bar{x}_2} & \mathbf{r}_{\bar{x}_3}\end{bmatrix} = \begin{bmatrix} \dfrac{\partial r_1}{\partial \bar{x}_1} & \dfrac{\partial r_1}{\partial \bar{x}_2} & \dfrac{\partial r_1}{\partial \bar{x}_3} \\ \dfrac{\partial r_2}{\partial \bar{x}_1} & \dfrac{\partial r_2}{\partial \bar{x}_2} & \dfrac{\partial r_2}{\partial \bar{x}_3} \\ \dfrac{\partial r_3}{\partial \bar{x}_1} & \dfrac{\partial r_3}{\partial \bar{x}_2} & \dfrac{\partial r_3}{\partial \bar{x}_3} \end{bmatrix} \tag{1.149}$$

In this case, the vector $\mathbf{r}_{\bar{x}_i}$, $i = 1, 2, 3$, represents a gradient vector obtained by differentiation with respect to the coordinate $\bar{x}_i$. Again, the vector $\mathbf{r}_{\bar{x}_i}$ is not necessarily a unit vector, and a measure of the deviation from a unit vector is given by $(\mathbf{r}_{\bar{x}_i}^{\mathrm{T}}\mathbf{r}_{\bar{x}_i} - 1)$. A measure of the angle between two gradient vectors can be obtained using the dot product $\mathbf{r}_{\bar{x}_i}^{\mathrm{T}}\mathbf{r}_{\bar{x}_j}$, $i \neq j$. Note that the relationship between $\mathbf{J}$ and $\bar{\mathbf{J}}$ is given by

$$\bar{\mathbf{J}} = \frac{\partial \mathbf{r}}{\partial \bar{\mathbf{x}}} = \frac{\partial \mathbf{r}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}} = \mathbf{J}\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}} \tag{1.150}$$

Although we assume in this section that parameters are defined along the orthogonal axes of coordinate systems, the relationship of Equation 150 is general and governs the definition of the gradients when defined using different parameters, including the case of curvilinear coordinates. It is important, however, to point out that the use of different sets of parameters leads to the definition of different gradient vectors. Nonetheless, these gradient vectors, regardless of what set of parameters is used, are defined in the same coordinate system in which the vector $\mathbf{r}$ is defined. That is, the differentiation of a vector does not change the coordinate system in which this vector is defined.

As a special case, the curve shown in Figure 3 is considered. The position of points on this curve can be defined in the coordinate system $X_1X_2X_3$ and can be written in terms of one parameter s as $\mathbf{r} = \mathbf{r}(s)$. Because there is only one parameter, there is only one gradient vector, defined as

$$\mathbf{r}_s = \frac{\partial \mathbf{r}}{\partial s} \tag{1.151}$$

This gradient vector defines the tangent vector, as shown in Figure 3, and if s is selected as the arc length, $\mathbf{r}_s$ is a unit vector. One may choose other parameters, such as α_1 and α_2, and define

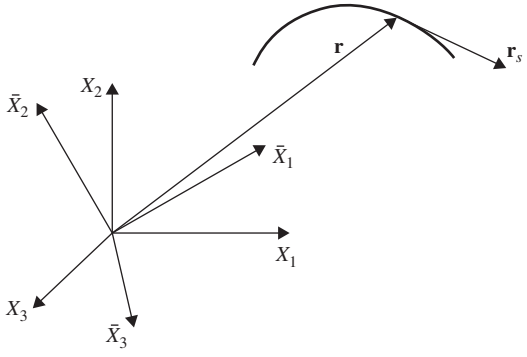

Figure 1.3 Space curve

the gradient vector by differentiation with respect to these new parameters. This leads to other definitions of the gradient vector as

$$\mathbf{r}_{\alpha_1} = \frac{\partial \mathbf{r}}{\partial \alpha_1}, \quad \mathbf{r}_{\alpha_2} = \frac{\partial \mathbf{r}}{\partial \alpha_2} \tag{1.152}$$

Clearly, the three gradient vectors defined in Equations 151 and 152 are different expressions for the tangent vector, for example,

$$\mathbf{r}_{\alpha_1} = \frac{\partial \mathbf{r}}{\partial \alpha_1} = \frac{\partial \mathbf{r}}{\partial s}\frac{\partial s}{\partial \alpha_1} \tag{1.153}$$

which shows that $\mathbf{r}_{\alpha_1}$ and $\mathbf{r}_s$ are two parallel vectors that differ by a scalar multiplier that depends on the relationship between the two parameters α_1 and s. This simple example shows that the change of parameters does not lead to a change in the coordinate system because the resulting gradient vectors are defined in the $X_1X_2X_3$ coordinate system in which the vector $\mathbf{r}$ is defined.

Although in the simple one-dimensional example the change of parameters does not change the orientation of the gradient vector, when two or more parameters are used, the change of parameters can lead to a change of the orientation of the gradient vectors but does not change the coordinate system in which these gradient vectors are defined. This is clear from Equation 150, which shows that the columns of the gradient matrix $\bar{\mathbf{J}}$ are linear combinations of the columns of the gradient matrix J.

Coordinate Transformation

The analysis presented thus far in this section shows that the change of parameters does not imply a change of the coordinate system in which the gradient vectors are defined. That is, if x_1, x_2, and x_3 are coordinates along the orthogonal axes of the coordinate system $X_1X_2X_3$ and $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ are the coordinates along the axes of another coordinate system $\bar{X}_1\bar{X}_2\bar{X}_3$, then, in general, $\mathbf{r}_{x_i} = \partial \mathbf{r}/\partial x_i \neq \mathbf{R}\mathbf{r}_{\bar{x}_i}$, $i = 1, 2, 3$, where $\mathbf{R}$ is the matrix that defines the orientation

of the coordinate system $\bar{X}_1\bar{X}_2\bar{X}_3$ with respect to the coordinate system $X_1X_2X_3$. The gradient vector $\mathbf{r}_{x_i}$ can be written in terms of components defined in the coordinate system $\bar{X}_1\bar{X}_2\bar{X}_3$ as

$$\bar{\mathbf{r}}_{x_i} = \mathbf{R}^{\mathrm{T}}\mathbf{r}_{x_i}, \quad i = 1,\ 2,\ 3 \tag{1.154}$$

In the analysis presented in this book, it is important to understand the difference between the change of parameters and the transformation between two coordinate systems, particularly when the large-deformation ANCF finite elements presented in Chapter 5 are discussed.

Deformation and Strains

In continuum mechanics, strains are used as measure of the deformation. The strains at a point on the continuum are defined using dot products of gradient vectors, and therefore, their values in a certain direction are independent of the coordinate system used. As will be shown in the following chapter, the strain components are used as measure of the stretch and shear. For example, a stretch strain component is defined as $(\mathbf{r}_{x_i} \cdot \mathbf{r}_{x_i} - 1)/2$, where $\mathbf{r}_{x_i} = \partial\mathbf{r}/\partial x_i$ is the gradient vector in the direction of the parameter x_i. Clearly, the strain measure $(\mathbf{r}_{x_i} \cdot \mathbf{r}_{x_i} - 1)/2$ is independent of the coordinate system as well as of a rigid-body coordinate transformation; this is clear from the dot product definition. Furthermore, this definition of the strain is unique since it measures the change at a point on the continuum in a specific direction defined by the coordinate line x_i, which can represent a straight or curved line.

The deformation on the other hand is not unique and depends on the choice of the coordinate system. For this reason, deformations are not often used in the general continuum mechanics developments. In order to provide an explanation, one can consider the slider crank mechanism shown in Figure 4. This mechanism is widely used in many applications including engines. As the crankshaft of the mechanism rotates and the piston moves, the connecting rod can be subjected to excessive forces that will produce deformation. A magnification of the deformation of the connecting is shown in Figure 4b. Regardless of the coordinate system used, the strains at an arbitrary point on the connecting rod have unique values since they are defined in terms of the dot product of gradient vectors. The deformation definition, however, is not unique. As shown in Figure 5, different coordinate systems can be used to measure the deformation. The figure shows the deformation δ of the mid-point defined in different coordinate systems. It is clear that these deformation measures have very different values. In fact, the deformation of some points, such as the end points of the connecting rod, can be zero in some coordinate

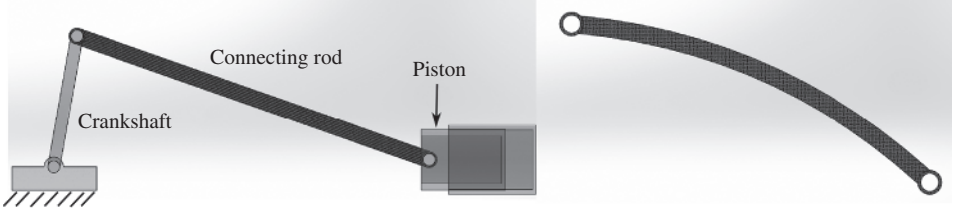

Figure 1.4 Slider crank mechanism

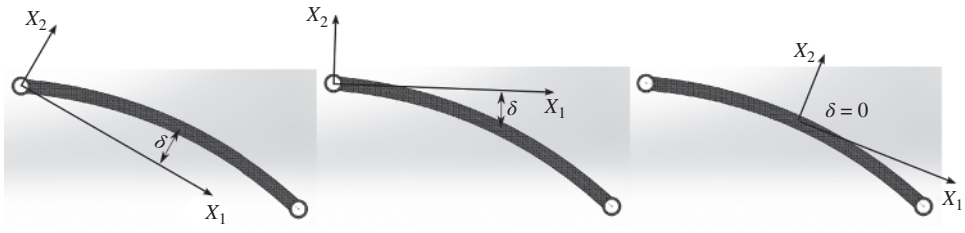

Figure 1.5 Deformation measurement.

systems and can assume a very large value in other coordinate systems. The strain measures, on the other hand, are absolute and do not depend on the choice of the coordinate systems.

Position Vector Gradients and Rigid Body Kinematics

It can be shown that the rigid body kinematic equations can be written as a linear polynomial in the spatial coordinates if the concept of the position vector gradients is used. That is, a polynomial similar to Equation 115 with a lower order can be used to describe the rigid body motion provided appropriate constraints are imposed on the position vector gradients. Using Equations 81 and 85, one can show that Equation 81 can be written as

$$\mathbf{r} = \mathbf{r}_O + \mathbf{A}\bar{\mathbf{u}} = \begin{bmatrix} a_0 + a_1 x_1 + a_2 x_2 \\ b_0 + b_1 x_1 + b_2 x_2 \end{bmatrix} \tag{1.155}$$

In this equation,

$$\mathbf{r}_O = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}, \quad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{r}_{x_1} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{r}_{x_2} = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \tag{1.156}$$

This equation shows that the rigid body kinematic equations can be written in the form of the linear polynomials of Equation 115. Since in the case of unconstrained planar motion the rigid body has three degrees of freedom, the six coefficients a_0, a_1, a_2, b_0, b_1, and b_2 must be related by three algebraic equations. It is clear from Equation 156 that these three algebraic constraint equations imposed on the gradient vectors are

$$\mathbf{r}_{x_1}^{\mathrm{T}} \mathbf{r}_{x_1} = 1, \quad \mathbf{r}_{x_2}^{\mathrm{T}} \mathbf{r}_{x_2} = 1, \quad \mathbf{r}_{x_1}^{\mathrm{T}} \mathbf{r}_{x_2} = 0 \tag{1.157}$$

These constraint equations imply that in the case of rigid body motion, the position vector gradients remain orthogonal unit vectors, and in this special case, the transformation matrix that defines the orientation of the body coordinate system can be written as $\mathbf{A} = \begin{bmatrix} \mathbf{r}_{x_1} & \mathbf{r}_{x_2} \end{bmatrix}$. A similar procedure can be used in the spatial analysis to show that the three-dimensional rigid body kinematic equations can be written in the form of linear polynomials in the spatial coordinates. It is important to note that the spatial coordinates x_1 and x_2 are defined with respect to the body coordinate system and assume their constant initial values regardless of the amount of displacement and rotation of the rigid body. That is, when the polynomial representation is used, the spatial coordinates always assume constant values and the motion of the body

or the finite element is defined by the time dependent coefficients of the polynomial used, or alternatively, by the element nodal coordinates used to replace the polynomial coefficients. It is also clear from Equations 155 and 156 that the rigid body kinematics can be expressed as linear functions of the angles. Furthermore, the position vector gradients in Equation 156 are written in terms of trigonometric functions which have infinite orders. Therefore, finite elements, in which the element displacement field is written as linear functions of angles, cannot correctly describe rigid body motion and such finite elements cannot be used with the nonincremental solution procedures often used in multibody system dynamics.

PROBLEMS

1. Find the sum of the following two matrices:

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & -1 \\ 2 & 0 & 5 \\ -4 & 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 4 \\ -4 & -2 & -3 \end{bmatrix}$$

Find also the trace of these two matrices as well as the trace of their product.

2. Find the product of the following three matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Find the determinant and the inverse of the following two matrices:

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

4. Show that the following two matrices are orthogonal:

$$\mathbf{A}_1 = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \mathbf{A}_2 = \mathbf{I} + \tilde{\mathbf{v}}\sin\theta + 2\tilde{\mathbf{v}}^2\sin^2\left(\frac{\theta}{2}\right)$$

where θ is an angle, $\mathbf{v}$ is a three-dimensional unit vector, $\tilde{\mathbf{v}}$ is the skew-symmetric matrix associated with the unit vector $\mathbf{v}$, and $\mathbf{I}$ is the identity matrix.

5. Show that the determinant of a 3×3 matrix does not change if a row or a column is subtracted or added to another.

6. Show that the matrices in Problem 2 are orthogonal and that their product is an orthogonal matrix.

7. Find the norm of the columns of the matrices of Problem 3.

8. Prove the identities given in Equations 28 and 29 for three-dimensional vectors.

9. Prove Equation 32 for three-dimensional vectors.

10. Show that if $\mathbf{A}$ is a symmetric tensor and $\mathbf{B}$ is a skew-symmetric tensor, then $\mathbf{A} : \mathbf{B} = 0$.

11. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are second-order tensors, show that the double product satisfies the identity $\mathbf{A} : (\mathbf{BC}) = (\mathbf{AC}^{\mathrm{T}}) : \mathbf{B} = (\mathbf{B}^{\mathrm{T}}\mathbf{A}) : \mathbf{C}$.

12. If $\mathbf{A}$ and $\mathbf{B}$ are two symmetric tensors, show that

$$\mathbf{A} : \mathbf{B} = A_{11}B_{11} + A_{22}B_{22} + A_{33}B_{33} + 2(A_{12}B_{12} + A_{13}B_{13} + A_{23}B_{23})$$

13. Find the invariants, eigenvalues, and eigenvectors of the following two matrices:

$$\mathbf{A} = \begin{bmatrix} -3 & 4 & -1 \\ 4 & 0 & 5 \\ -1 & 5 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -2 \\ 0 & -2 & -3 \end{bmatrix}$$

Verify that the three invariants of each of these matrices can be written in terms of the eigenvalues.

14. Find the projection matrices $\mathbf{P}$ and $\mathbf{P}_p$ associated with the unit vector $\hat{\mathbf{a}} = \left[1/\sqrt{3} \;\; 1/\sqrt{3} \;\; 1/\sqrt{3}\right]^{\mathrm{T}}$.

15. Show that the components of a third-order tensor $\mathbf{T}$ can be written as $t_{ijk} = (\mathbf{i}_i \otimes \mathbf{i}_j) : \mathbf{T}\mathbf{i}_k$, where $\mathbf{i}_i$, $\mathbf{i}_j$, and $\mathbf{i}_k$ are base vectors.

16. Prove the properties of Equation 59.

17. Show that the component v_k of any vector $\mathbf{v}$ can be written as $v_k = -(1/2)\Gamma_{ijk}\tilde{v}_{ij} = -(1/2)\Gamma_{kij}\tilde{v}_{ij} = (1/2)\Gamma_{ikj}\tilde{v}_{ij}$, where $\mathbf{\Gamma} = (\Gamma_{ijk})$ is the third-order alternating tensor and $\tilde{\mathbf{v}} = (\tilde{v}_{ij})$ is the skew-symmetric matrix associated with the vector $\mathbf{v}$.

18. Show that the cross product between the two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written as $\mathbf{u} \times \mathbf{v} = \sum_{i,\,j=1}^{3} u_i v_j (\mathbf{i}_i \times \mathbf{i}_j) = \sum_{i,\,j,\,k=1}^{3} \Gamma_{ijk} u_i v_j \mathbf{i}_k$, where $\mathbf{i}_i$, $\mathbf{i}_j$, and $\mathbf{i}_k$ are base vectors and $\mathbf{\Gamma} = (\Gamma_{ijk})$ is the third-order alternating tensor.

19. Show that the components of a fourth-order tensor $\mathbf{F}$ can be written as $f_{ijkl} = (\mathbf{i}_i \otimes \mathbf{i}_j) : \mathbf{F} : (\mathbf{i}_k \otimes \mathbf{i}_l)$, where $\mathbf{i}_i$, $\mathbf{i}_j$, $\mathbf{i}_k$, and $\mathbf{i}_l$ are base vectors.

20. Find the polar decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

21. Derive Euler equation of motion using Equation 95.

22. Using D'Alembert's principle, derive the equation of motion of a pendulum connected to the ground at one of its ends by a pin joint. Assume that the pendulum rod has length l, mass m, and mass moment of inertia about the center of mass I_O. The pendulum is subjected to an external moment M. Consider the effect of gravity. Explain how D'Alembert's principle can be used to systematically eliminate the reaction forces in this problem.

23. Solve Problem 22 using the principle of virtual work. Discuss the relationship between D'Alembert's principle and the principle of virtual work.

24. Verify the shape functions of Equation 121.