

## 1

## Manipulations on Matrixes

Even if matrix algebra requires some basic analytic tools in order to handle it with confidence, the benefits are huge compared to the price paid. This chapter, and those that follow, aim to make a review of these basic operations.

### 1.1 Matrix Properties

$\mathbb{R}^n$  and  $\mathbb{C}^n$  represent the ensemble of all the ordered tuples of  $n$  real/complex numbers  $(x_1, x_2, \dots, x_n)$ . Each of these tuples can be seen as the coordinate of a point in  $n$ -dimensional (or  $2n$ -dimensional) space. Normally these tuples of  $n$  numbers are ordered in column vectors  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ . The notation  $\mathbf{x} \in \mathbb{R}^n$  indicates that  $\mathbf{x}$  is an ordered tuple (column vector) of  $n$  real numbers that define a vector in  $\mathbb{R}^n$ ; similarly for  $\mathbf{x} \in \mathbb{C}^n$ . A matrix is obtained by juxtaposing a set of column vectors (tuples):  $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m]$  obtaining  $\mathbf{X} \in \mathbb{R}^{n \times m}$  (or  $\mathbf{X} \in \mathbb{C}^{n \times m}$  if complex-valued). The  $(i, j)$  entry of  $\mathbf{X}$  ( $i$  and  $j$  indicate the row and column index, respectively) is indicated by  $[\mathbf{X}]_{ij} = \mathbf{X}(i, j) = x_{ij}$ , depending on the context.

Typical matrixes or vectors:

$$\mathbf{X} = \text{diag}(x_1, x_2, \dots, x_n) = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} \quad (\text{diagonal matrix})$$

$$\mathbf{I}_n = \text{diag}(1, 1, \dots, 1) = \mathbf{I} \quad (n \times n \text{ identity matrix})$$

$$\mathbf{e}_k \in \mathbb{R}^n \text{ such that } [\mathbf{e}_k]_{il} = \delta_{i-k} \quad (k\text{th element selection vector})$$

$$\mathbf{1}_n = \mathbf{1} \in \mathbb{R}^n \text{ such that } [\mathbf{1}]_{il} = 1 \text{ for } \forall i \quad (\text{one vector})$$

$$\mathbf{X}(\mathbf{1} : n, k) = \mathbf{X}(:, k) = \begin{bmatrix} x_{1k} \\ x_{2k} \\ \vdots \\ x_{nk} \end{bmatrix} \quad (\text{colon notation for } k\text{th column extraction})$$

Function  $\delta_i$  is the Dirac delta ( $\delta_0 = 1$  and  $\delta_i = 0$  for  $\forall i \neq 0$ ); for signals the equivalent notation is used:  $\delta[i]$ .

### 1.1.1 Elementary Operations

There are a set of elementary operations for vectors and matrixes. These properties are collected for both real and complex valued matrixes, and differences are highlighted whenever necessary to distinguish operation on complex matrixes as different from real-valued ones. The reader is referred to [1, 2] or any textbook on linear algebra for extensive discussions or proofs.

- **Sum:** for  $\mathbf{A}$  and  $\mathbf{B} \in \mathbb{C}^{n \times m}$ , sum is  $\mathbf{C} = \mathbf{A} + \mathbf{B} \in \mathbb{C}^{n \times m}$  with entries:  $c_{ij} = a_{ij} + b_{ij}$ . Associative and commutative properties hold for the sum:  $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ , and  $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ .
- **Product:** for  $\mathbf{A} \in \mathbb{C}^{n \times l}$  and  $\mathbf{B} \in \mathbb{C}^{l \times m}$ , the matrix  $\mathbf{C} = \mathbf{AB} \in \mathbb{C}^{n \times m}$  has elements  $c_{ij} = \sum_{k=1}^l a_{ik}b_{kj}$ . Distributive and associative properties hold for matrix product:  $(\mathbf{A}(\mathbf{B} + \mathbf{C})) = \mathbf{AB} + \mathbf{AC}$ , and  $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ , but not commutative (assuming that matrixes dimensions are appropriate):  $\mathbf{AB} \neq \mathbf{BA}$ , in general. Elementwise (or *Hadamard*) multiplication for  $\mathbf{A}$  and  $\mathbf{B} \in \mathbb{C}^{n \times m}$  is  $\mathbf{C} = \mathbf{A} \odot \mathbf{B} \in \mathbb{C}^{n \times m}$  with  $c_{ij} = a_{ij}b_{ij}$ ; other equivalent notation is pointwise multiplication  $\mathbf{C} = \mathbf{A} * \mathbf{B}$  that resembles the Matlab coding.
- **Block-partitioned matrix:** when problems are large but still with some regularity, the matrix accounting for the specific problem can be arranged into sub-matrixes. Each of these sub-matrixes (or blocks) has a predefined regularity or homogeneity such as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \cdots & \mathbf{X}_{1n} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \cdots & \mathbf{X}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_{m1} & \mathbf{X}_{m2} & \cdots & \mathbf{X}_{mn} \end{bmatrix}$$

where the dimensions of blocks are appropriate for the problem at hand. Relevant is that block structure is preserved and all the matrix operations are equally defined on blocks. As an example, matrix-vector product for block-partitioned matrixes is

$$\mathbf{X}\mathbf{y} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \cdots & \mathbf{X}_{1n} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \cdots & \mathbf{X}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{X}_{m1} & \mathbf{X}_{m2} & \cdots & \mathbf{X}_{mn} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_n \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n \mathbf{X}_{1k}\mathbf{y}_k \\ \sum_{k=1}^n \mathbf{X}_{2k}\mathbf{y}_k \\ \vdots \\ \sum_{k=1}^n \mathbf{X}_{mk}\mathbf{y}_k \end{bmatrix}.$$

Usually, it is convenient (and *advisable*) to preserve the block-partitioned structure across all the matrix manipulation steps.

- **Ordering properties** for any  $\mathbf{A} \in \mathbb{R}^{n \times m}$ , or  $\mathbf{A} \in \mathbb{C}^{n \times m}$  if complex-valued:

**Transpose matrix:**  $\mathbf{A}^T$  is  $\mathbf{A}^T \in \mathbb{C}^{m \times n}$  such that  $[\mathbf{A}^T]_{ij} = [\mathbf{A}]_{ji} = a_{ji}$  (row/columns are exchanged). Notice that for the product,  $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ . For a *symmetric matrix* we have  $\mathbf{A}^T = \mathbf{A}$ .

**Conjugate matrix:**  $\mathbf{A}^*$  is the matrix in  $\mathbb{C}^{n \times m}$  such that  $[\mathbf{A}^*]_{ij} = a_{ij}^*$  (complex conjugate elementwise).

**Hermitian matrix:** for a square matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  for which the following property holds:  $a_{ij} = a_{ji}^*$ .

**Conjugate transpose (Hermitian transpose) matrix:**  $\mathbf{A}^H$  is  $\mathbf{A}^H \in \mathbb{C}^{m \times n}$  such that  $\mathbf{A}^H = (\mathbf{A}^*)^T$ ; therefore  $[\mathbf{A}^H]_{ij} = a_{ji}^*$ . As for transposition it holds that:  $(\mathbf{A}\mathbf{B})^H = \mathbf{B}^H\mathbf{A}^H$ .

A matrix for which  $\mathbf{A}^H = \mathbf{A}$  is a *Hermitian matrix* (it has Hermitian symmetry).

**Gram matrix:** for any arbitrary matrix  $\mathbf{A}$  the product  $\mathbf{A}^T\mathbf{A}$  (or  $\mathbf{A}^H\mathbf{A}$  for complex-valued matrix) is a Gram matrix, which is square, positive semidefinite (see below) and symmetric  $(\mathbf{A}^T\mathbf{A})^T = \mathbf{A}^T\mathbf{A}$  (or Hermitian symmetric  $(\mathbf{A}^H\mathbf{A})^H = \mathbf{A}^H\mathbf{A}$ ). Similarly  $\mathbf{A}\mathbf{A}^T$  (or  $\mathbf{A}\mathbf{A}^H$ ) is a Gram matrix.

- **Trace:** for a square matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  this is  $\text{tr}[\mathbf{A}] = \sum_{i=1}^n a_{ii}$  (sum along main diagonal). Assuming all square matrixes, the following properties hold true:  $\text{tr}[\mathbf{A}\mathbf{B}\mathbf{C}] = \text{tr}[\mathbf{C}\mathbf{A}\mathbf{B}] = \text{tr}[\mathbf{B}\mathbf{C}\mathbf{A}]$  (invariance for cyclic shift) that for the special case is  $\text{tr}[\mathbf{A}\mathbf{B}] = \text{tr}[\mathbf{B}\mathbf{A}]$ ; for the sum  $\text{tr}[\mathbf{A} + \mathbf{B}] = \text{tr}[\mathbf{A}] + \text{tr}[\mathbf{B}]$ .
- **Lp norm:** for a vector  $\mathbf{x} \in \mathbb{C}^n$ , the  $L_p$  norm is a scalar  $\geq 0$  defined as:

$$\|\mathbf{x}\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

there are the following special cases (or notations):

$$\|\mathbf{x}\|_2 = \|\mathbf{x}\|_F \text{ (Euclidean or Frobenius norm, see below)}$$

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$$

$$\|\mathbf{x}\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$$

as limit  $\|\mathbf{x}\|_0$  simply counts the number of non-zero elements of  $\mathbf{x}$  and is used to evaluate the sparseness of a vector.

- **Euclidean (or Frobenius) norm** (special case of  $L_p$  norm):

$$\mathbf{x} \in \mathbb{C}^n \rightarrow \|\mathbf{x}\|_F = \left( \sum_{i=1}^n |x_i|^2 \right)^{1/2} = (\mathbf{x}^H\mathbf{x})^{1/2} = \text{tr}[\mathbf{x}\mathbf{x}^H]^{1/2}$$

$$\mathbf{X} \in \mathbb{C}^{n \times n} \rightarrow \|\mathbf{X}\|_F = \left( \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2}$$

$$\|\mathbf{x}\|_{\mathbf{Q}}^2 = \mathbf{x}^T\mathbf{Q}\mathbf{x} : \text{weighted norm of } \mathbf{x} \text{ for a positive definite } \mathbf{Q}.$$

The square of the Euclidean norm for vectors is typically indicated in any of the equivalent forms:

$$\|\mathbf{x}\|^2 = \|\mathbf{x}\|_F^2 = \mathbf{x}^H\mathbf{x}.$$

- **Inner product:** for any two vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$  their inner product is a *scalar* defined as  $\mathbf{x}^H\mathbf{y} = \sum_{i=1}^n x_i^*y_i$ . The two vectors are orthogonal with respect to each other when  $\mathbf{x}^H\mathbf{y} = 0$ . Due to the fact that the inner product is a scalar, the following equalities hold true:  $\mathbf{x}^H\mathbf{y} = (\mathbf{x}^H\mathbf{y})^T = \mathbf{y}^T\mathbf{x}^*$  and  $\mathbf{x}^H\mathbf{y} = \text{tr}[\mathbf{x}^H\mathbf{y}] = \text{tr}[\mathbf{y}\mathbf{x}^H]$ . Recall that  $\|\mathbf{x}\|_F^2 = \mathbf{x}^H\mathbf{x}$  reduces to the inner product.

- **Outer Product:** for any two vectors  $\mathbf{x} \in \mathbb{C}^n$  and  $\mathbf{y} \in \mathbb{C}^m$  their outer product is an  $n \times m$  matrix defined as  $\mathbf{A} = \mathbf{xy}^H$ , therefore,  $a_{ij} = x_i y_j^*$ . When  $\mathbf{x} = \mathbf{y}$  the outer product is  $\mathbf{A} = \mathbf{xx}^H$  and it is a rank-1 Hermitian matrix. In the case where  $\mathbf{x}$  is a vector of random variables (Section 3.7), the correlation matrix follows from the outer product:  $\mathbf{R}_x = \mathbb{E}[\mathbf{xx}^H]$ .
- **Determinant** is a metric associated to any square matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$ . The geometric meaning of a determinant [3] denoted as  $\det[\mathbf{A}]$ , or  $|\mathbf{A}|$ , is the volume of the hypercube enclosed by the columns (or equivalently rows) of  $\mathbf{A}$ , and  $\det[\mathbf{A}] > 0$  only when columns (or rows) enclose a non-null volume, so columns (or rows) are linearly independent. For a general context  $\mathbf{A} \in \mathbb{C}^{n \times n}$  the following properties hold:  $\det[\mathbf{AB}] = \det[\mathbf{A}]\det[\mathbf{B}]$ ;  $\det[\mathbf{A}^T] = \det[\mathbf{A}]$ ;  $\det[\mathbf{A}^H] = \det[\mathbf{A}]^*$ ;  $\det[\alpha\mathbf{A}] = \alpha^n \det[\mathbf{A}]$ ;  $\det[\mathbf{A} + \mathbf{xy}^T] = (1 + \mathbf{y}^T \mathbf{A}^{-1} \mathbf{x}) \det[\mathbf{A}]$  (matrix determinant lemma).
- **Positive definite (Positive semidefinite) symmetric square matrix:** a symmetric (or Hermitian if complex-valued) matrix  $\mathbf{A}$  is positive definite if  $\mathbf{z}^T \mathbf{A} \mathbf{z} > 0$  (or  $\mathbf{z}^H \mathbf{A} \mathbf{z} > 0$ ) and positive semidefinite if  $\mathbf{z}^T \mathbf{A} \mathbf{z} \geq 0$  (or  $\mathbf{z}^H \mathbf{A} \mathbf{z} \geq 0$ ) for all non-zero vectors  $\mathbf{z}$ . Notice that for complex-valued matrix, the quadratic form (see Section 1.5)  $\mathbf{z}^H \mathbf{A} \mathbf{z}$  is always real as  $\mathbf{A}$  is Hermitian. A positive definite (or positive semidefinite) matrix is often indicated by the following notation:  $\mathbf{A} > 0$  (or  $\mathbf{A} \geq 0$ ), and if  $\mathbf{A} > 0 \Rightarrow \det[\mathbf{A}] > 0$  (or  $\mathbf{A} \geq 0 \Rightarrow \det[\mathbf{A}] \geq 0$ ).
- **Inverse matrix** of a square matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  is the matrix  $\mathbf{A}^{-1} \in \mathbb{C}^{n \times n}$  that satisfies  $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ . The inverse matrix exists iff  $\det[\mathbf{A}] \neq 0$  (that is equivalent to  $\text{rank}[\mathbf{A}] = n$ ). When  $\det[\mathbf{A}] = 0$  the matrix is singular and the inverse does not exist. The following properties hold (under the hypothesis that all the involved inverse matrixes exist):
  - $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$  and  $(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H$  (or  $\mathbf{A}^{-T}$  and  $\mathbf{A}^{-H}$  to ease the notation)
  - $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
  - For a positive definite matrix it holds that:

$$[\mathbf{A}^{-1}]_{ii} \geq \frac{1}{[\mathbf{A}]_{ii}}$$

- $\det[\mathbf{A}^{-1}] = 1/\det[\mathbf{A}]$
- Inversion lemma:

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{DA}^{-1}\mathbf{B} + \mathbf{C}^{-1})^{-1}\mathbf{DA}^{-1}$$

There are some special cases of practical interest:

$$\begin{aligned} (\mathbf{A} + \mathbf{B})^{-1} &= \mathbf{A}^{-1} - \mathbf{A}^{-1}(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1}\mathbf{A}^{-1} \\ (\sigma^2\mathbf{I} + \mathbf{V}\mathbf{S}\mathbf{V}^H)^{-1} &= \frac{1}{\sigma^2} (\mathbf{I} - \mathbf{V}(\mathbf{V}^H\mathbf{V} + \sigma^2\mathbf{S}^{-1})^{-1}\mathbf{V}^H) \\ (\mathbf{A} + \mu\mathbf{xx}^H)^{-1} &= \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{xx}^H\mathbf{A}^{-1}}{\frac{1}{\mu} + \mathbf{x}^H\mathbf{A}^{-1}\mathbf{x}} \text{ (Woodbury identity)} \\ (\mathbf{A}^{-1} + \mathbf{B}^H\mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{B}^H\mathbf{C}^{-1} &= \mathbf{AB}^H(\mathbf{BAB}^H + \mathbf{C})^{-1} \end{aligned}$$

- Inversion of a block-partitioned matrix (the dimension of the blocks are arbitrary but appropriate):

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}$$

where (provided that all the inverse matrixes exist)

$$\mathbf{E} = \mathbf{A}^{-1} + \mathbf{A}^{-1} \mathbf{B} \mathbf{H} \mathbf{C} \mathbf{A}^{-1} = (\mathbf{A} - \mathbf{B} \mathbf{D}^{-1} \mathbf{C})^{-1}$$

$$\mathbf{F} = -\mathbf{A}^{-1} \mathbf{B} \mathbf{H} = -\mathbf{E} \mathbf{B} \mathbf{D}^{-1}$$

$$\mathbf{G} = -\mathbf{H} \mathbf{C} \mathbf{A}^{-1} = -\mathbf{D}^{-1} \mathbf{C} \mathbf{E}$$

$$\mathbf{H} = \mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{C} \mathbf{E} \mathbf{B} \mathbf{D}^{-1} = (\mathbf{D} - \mathbf{C} \mathbf{A}^{-1} \mathbf{B})^{-1}$$

- Inequality of the sum:

$$\mathbf{A}^{-1} \geq (\mathbf{A} + \mathbf{B})^{-1} \geq \mathbf{0}$$

for symmetric positive definite  $\mathbf{A}$  and symmetric positive semidefinite  $\mathbf{B}$ .

- **Rank:** for a matrix  $\mathbf{A} \in \mathbb{C}^{n \times m}$  the rank  $[\mathbf{A}]$  denotes the maximum number of linearly independent columns or rows of the matrix: 1.  $\text{rank}[\mathbf{A}] \leq \min(n, m)$ ; 2.  $\text{rank}[\mathbf{A} + \mathbf{B}] \leq \text{rank}[\mathbf{A}] + \text{rank}[\mathbf{B}] \leq \min(n, m)$ ; 3.  $\text{rank}[\mathbf{A}\mathbf{B}] \leq \min(\text{rank}[\mathbf{A}], \text{rank}[\mathbf{B}])$ .
- **Orthogonal matrix.** A matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is orthogonal if  $\mathbf{A}^T \mathbf{A} = \mathbf{A} \mathbf{A}^T = \mathbf{I}$ , or equivalently the inner product of every pair of non-identical columns (or rows) is zero. Therefore, for an orthogonal matrix,  $\mathbf{A}^{-1} = \mathbf{A}^T$ . Since the columns (or rows) of orthogonal matrixes have unitary norm, these are called orthonormal matrixes. Similar definitions hold for orthogonal  $\mathbf{A} \in \mathbb{C}^{n \times n}$ , that is,  $\mathbf{A}^H \mathbf{A} = \mathbf{A} \mathbf{A}^H = \mathbf{I}$ .
- **Unitary matrix.** A matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  is called unitary if  $\mathbf{A}^H \mathbf{A} = \mathbf{A} \mathbf{A}^H = \mathbf{I}$ . Therefore for a unitary matrix it holds that  $\mathbf{A}^{-1} = \mathbf{A}^H$ . Sometimes unitary matrixes are called orthogonal/orthonormal matrixes with an extension of the terms used for real matrixes.
- **Square-root** of a matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  is  $\mathbf{B}$  so that  $\mathbf{B}^H \mathbf{B} = \mathbf{A}$ . In general, square-root is not unique as there could be any unitary matrix  $\mathbf{Q}$  such that  $\mathbf{B}^H \mathbf{I} \mathbf{B} = \mathbf{B}^H (\mathbf{Q}^H \mathbf{Q}) \mathbf{B} = (\mathbf{Q} \mathbf{B})^H (\mathbf{Q} \mathbf{B}) = \mathbf{A}$ . Common notation, with some abuse, is  $\mathbf{A}^{1/2}$  and  $\mathbf{A} = \mathbf{A}^{H/2} \mathbf{A}^{1/2}$ .
- **Triangular matrix.** A square matrix  $\mathbf{A} \in \mathbb{C}^{n \times n}$  with entry  $[\mathbf{A}]_{ij}$  is upper triangular if  $[\mathbf{A}]_{ij} = 0$  for  $j > i$  (lower triangular if  $[\mathbf{A}]_{ij} = 0$  for  $i > j$ ), strictly upper triangular if  $[\mathbf{A}]_{ij} = 0$  for  $j \geq i$  (lower triangular for  $i \geq j$ ). The inverse of an upper (lower) triangular matrix is upper (lower) triangular.
- **Kronecker (tensor) product** between  $\mathbf{A} \in \mathbb{C}^{m \times n}$  and  $\mathbf{B} \in \mathbb{C}^{k \times \ell}$  is the block-partitioned matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11} \mathbf{B} & a_{12} \mathbf{B} & \cdots & a_{1n} \mathbf{B} \\ a_{21} \mathbf{B} & a_{22} \mathbf{B} & \cdots & a_{2n} \mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \mathbf{B} & a_{m2} \mathbf{B} & \cdots & a_{mn} \mathbf{B} \end{bmatrix} \in \mathbb{C}^{mk \times n\ell}$$

that generalizes the matrix multiplication when handling multidimensional signals.

– *Properties:*

$$\begin{aligned}
 \mathbf{I}_m \otimes \mathbf{I}_n &= \mathbf{I}_{mn} \\
 \mathbf{A} \otimes (\alpha \mathbf{B}) &= (\alpha \mathbf{A}) \otimes \mathbf{B} = \alpha (\mathbf{A} \otimes \mathbf{B}) \\
 (\mathbf{A} \otimes \mathbf{B})^H &= \mathbf{A}^H \otimes \mathbf{B}^H \\
 (\mathbf{A} + \mathbf{B}) \otimes \mathbf{C} &= (\mathbf{A} \otimes \mathbf{C}) + (\mathbf{B} \otimes \mathbf{C}) \\
 \mathbf{A} \otimes (\mathbf{B} + \mathbf{C}) &= (\mathbf{A} \otimes \mathbf{B}) + (\mathbf{A} \otimes \mathbf{C}) \\
 \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) &= (\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} \\
 (\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) &= \mathbf{AC} \otimes \mathbf{BD} \\
 (\mathbf{A} \otimes \mathbf{B})^{-1} &= \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \\
 \det[\mathbf{A} \otimes \mathbf{B}] &= (\det[\mathbf{A}])^m (\det[\mathbf{B}])^n \text{ per } \mathbf{A} \in \mathbb{C}^{m \times m}, \mathbf{B} \in \mathbb{C}^{n \times n} \\
 \text{tr}[\mathbf{A} \otimes \mathbf{B}] &= \text{tr}[\mathbf{A}] \text{tr}[\mathbf{B}] \\
 \text{rank}[\mathbf{A} \otimes \mathbf{B}] &= \text{rank}[\mathbf{A}] \text{rank}[\mathbf{B}] \\
 &\text{if } \mathbf{A} > 0 \text{ and } \mathbf{B} > 0 \implies (\mathbf{A} \otimes \mathbf{B}) > 0
 \end{aligned}$$

- **Vectorization** is related to a matrix, it is based on  $\text{vec}[\cdot]$  operator that for a matrix  $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \dots, \ \mathbf{a}_n] \in \mathbb{C}^{m \times n}$  gives a vector with all columns ordered one after the other:  $\text{vec}[\mathbf{A}] = [\mathbf{a}_1^T \ \mathbf{a}_2^T \ \dots \ \mathbf{a}_n^T]^T \in \mathbb{C}^{mn}$ . There are some properties that are useful to maintain the block-partitioned structure of matrixes:

$$\begin{aligned}
 \text{vec}[\alpha \mathbf{A} + \beta \mathbf{B}] &= \alpha \text{vec}[\mathbf{A}] + \beta \text{vec}[\mathbf{B}] \\
 \text{vec}[\mathbf{ABC}] &= (\mathbf{C}^T \otimes \mathbf{A}) \text{vec}[\mathbf{B}] \\
 \text{vec}[\mathbf{AB}] &= (\mathbf{I} \otimes \mathbf{A}) \text{vec}[\mathbf{B}] = (\mathbf{B}^T \otimes \mathbf{I}) \text{vec}[\mathbf{A}]
 \end{aligned}$$

## 1.2 Eigen-Decompositions

Any matrix-vector multiplication  $\mathbf{Ax}$  represents a linear transformation for the vector  $\mathbf{x}$ . However, there are some “special” vectors  $\mathbf{x}$  such that their transformation  $\mathbf{Ax}$  preserves the direction of  $\mathbf{x}$  up to a scaling factor; these vectors are the eigenvectors. In detail, the eigenvectors of a symmetric matrix  $\mathbf{A} \in \mathbb{R}^{n \times n}$  (or a Hermitian symmetric  $\mathbf{A} \in \mathbb{C}^{n \times n}$ , such that  $\mathbf{A}^H = \mathbf{A}$ ) is a set of  $n$  vectors  $\mathbf{q}_1, \dots, \mathbf{q}_n$  such that:

$$\mathbf{Aq}_i = \lambda_i \mathbf{q}_i$$

where the  $\mathbf{q}_i$  and  $\lambda_i$  are the (right) eigenvectors and the eigenvalues of matrix  $\mathbf{A}$ . Collecting all these vectors into a block-matrix

$$\mathbf{A}[\mathbf{q}_1, \dots, \mathbf{q}_n] = [\mathbf{q}_1, \dots, \mathbf{q}_n] \text{diag}(\lambda_1, \dots, \lambda_n) \implies \mathbf{AQ} = \mathbf{QA}$$

with  $\mathbf{Q} \in \mathbb{C}^{n \times n}$  such that  $\mathbf{Q}^{-1} \mathbf{AQ} = \mathbf{\Lambda}$ . The eigenvalues are the solutions of the polynomial-type equation  $\det(\lambda \mathbf{I} - \mathbf{A}) = 0$  and, for symmetry, are real-valued ( $\lambda_i \in \mathbb{R}$ ). The decomposition

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T = \sum_{k=1}^n \lambda_k \mathbf{q}_k \mathbf{q}_k^T$$

shows that  $\mathbf{A}$  can be decomposed into the sum of rank-1 matrixes  $\mathbf{q}_k \mathbf{q}_k^T$  weighted by  $\lambda_k$  (similarly for  $\mathbf{A} \in \mathbb{C}^{n \times n}$ :  $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H = \sum_{k=1}^n \lambda_k \mathbf{q}_k \mathbf{q}_k^H$ ). The eigenvectors are orthonormal and this is stated as:

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{I} \Leftrightarrow \mathbf{q}_i^T \mathbf{q}_j = \delta_{i-j} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } \forall i \neq j \end{cases}$$

(and similarly  $\mathbf{q}_i^H \mathbf{q}_j = \delta_{i-j}$ ). The inverse of a matrix has the same eigenvectors, but inverse eigenvalues:

$$\mathbf{A}^{-1} = \mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^T$$

The  $k$ th power of a matrix is

$$\mathbf{A}^k = (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T)(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T) \dots (\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T) = \mathbf{Q}\mathbf{\Lambda}^k \mathbf{Q}^T$$

and the square-root is

$$\mathbf{A} = \mathbf{A}^{T/2} \mathbf{A}^{1/2} \rightarrow \mathbf{A}^{1/2} = \mathbf{\Lambda}^{1/2} \mathbf{Q}^T$$

but notice that the square-root factorization is not unique as any arbitrary  $n \times n$  orthonormal matrix  $\mathbf{B}$  (i.e.,  $\mathbf{B}^T \mathbf{B} = \mathbf{I}$ ) makes the square-root factorization  $\mathbf{A} = \mathbf{A}^{T/2} \mathbf{B}^T \mathbf{B} \mathbf{A}^{1/2}$  and thus  $\mathbf{A}^{1/2} = \mathbf{B} \mathbf{\Lambda}^{1/2} \mathbf{Q}^T$ . The geometric mean of the eigenvalues are related to the volume of the hypercube from the columns of  $\mathbf{A}$

$$\det[\mathbf{A}] = \prod_{i=1}^n \lambda_i ;$$

the volume  $\det[\mathbf{A}] = 0$  when at least one eigenvalue is zero. Another very useful property is the trace:

$$\text{tr}[\mathbf{A}] = \sum_{i=1}^n [\mathbf{A}]_{ii} = \sum_{i=1}^n \lambda_i$$

When a matrix is not symmetric, one can evaluate the right and left eigenvectors. For each eigenvalue  $\lambda_i$ , a vector  $\mathbf{w}_i \in \mathbb{R}^n$  can be associated such that  $\mathbf{w}_i^T \mathbf{A} = \lambda_i \mathbf{w}_i^T$ ; this is the left eigenvector of  $\mathbf{A}$  relative to the eigenvalue  $\lambda_i$ . Collecting all these eigenvalues/(left) eigenvectors of  $\mathbf{A}$  it follows that:  $\mathbf{W}^T \mathbf{A} = \mathbf{\Lambda} \mathbf{W}^T$ , where the columns of  $\mathbf{W}$  are the left eigenvectors of  $\mathbf{A}$ . Notice that  $(\mathbf{w}_i^T \mathbf{A})^T = \mathbf{A}^T \mathbf{w}_i = \lambda_i \mathbf{w}_i$ , thus the left eigenvector is the (right) eigenvector for the matrix  $\mathbf{A}^T$ . For symmetric matrixes,  $\mathbf{Q} = \mathbf{W}$ . The same properties hold for  $\mathbf{A} \in \mathbb{C}^{n \times n}$  with Hermitian transposition in place of transposition.

The most general decomposition of an arbitrary  $n \times m$  matrix into eigenvectors is Singular Value Decomposition (SVD). The decomposition can be stated as follows: any  $\mathbf{A} \in \mathbb{C}^{n \times m}$  can be decomposed into orthogonal matrixes

$$\mathbf{A} = \mathbf{U} \cdot \mathbf{\Sigma} \cdot \mathbf{V}^H$$

$(m,n)$     $(m,m)$     $(m,n)(n,n)$

where  $\mathbf{U}$  and  $\mathbf{V}$  are orthonormal and (real-valued) *singular values*  $\{\sigma_k\}$  usually ordered for decreasing values  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \sigma_{r+1} = \dots \sigma_p = 0$  with  $r \leq p = \min(m, n)$ :

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & \sigma_p \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

rectangular (for  $m \neq n$ ) and, in general, non-diagonal. Decomposition highlights that a matrix can be decomposed into the sum of (up to)  $r$  rank-1 matrixes

$$\mathbf{A} = \sum_{k=1}^r \sigma_k \mathbf{u}_k \mathbf{v}_k^H$$

so that  $\text{rank}(\mathbf{A}) = r$  (from the first  $r$  non-zero singular values). This is a relevant property exploited in many signal processing methods. The following main properties can be highlighted:

$$\begin{aligned} \mathbf{U}^H \mathbf{U} &= \mathbf{U} \mathbf{U}^H = \mathbf{I}_m \\ \mathbf{V}^H \mathbf{V} &= \mathbf{V} \mathbf{V}^H = \mathbf{I}_n \\ \mathbf{A}^H \mathbf{A} \mathbf{v}_i &= \sigma_i^2 \mathbf{v}_i \\ \mathbf{A} \mathbf{A}^H \mathbf{u}_i &= \sigma_i^2 \mathbf{u}_i \\ \lambda_i(\mathbf{A} \mathbf{A}^H) &= \lambda_i(\mathbf{A}^H \mathbf{A}) = \sigma_i^2 \end{aligned}$$

To summarize, columns of  $\mathbf{U}$  are the eigenvectors of the Gram matrix  $\mathbf{A} \mathbf{A}^H$  while columns of  $\mathbf{V}$  are the eigenvectors of the Gram matrix  $\mathbf{A}^H \mathbf{A}$ . Based on the orthogonality property:  $\mathbf{U}^H \cdot \mathbf{A} \cdot \mathbf{V} = \mathbf{\Sigma}$ . Since  $\text{rank}(\mathbf{A}) = r \leq \min(m, n)$ , the following equivalent representation is often useful to partition the columns of  $\mathbf{U}$  and  $\mathbf{V}$  related to the first non-zero singular values:

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^H = \left[ \underbrace{\mathbf{U}_1}_{(m,r)}, \underbrace{\mathbf{U}_2}_{(m,m-r)} \right] \begin{bmatrix} \mathbf{\Sigma}_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix} \begin{matrix} \} (r, n) \\ \} (n-r, n) \end{matrix} = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^H$$

Columns are orthogonal, so  $\mathbf{U}_1^H \mathbf{U}_1 = \mathbf{I}_r$  and  $\mathbf{V}_1^H \mathbf{V}_1 = \mathbf{I}_r$ , but  $\mathbf{U}_1 \mathbf{U}_1^H \neq \mathbf{U}_1^H \mathbf{U}_1$  and  $\mathbf{V}_1 \mathbf{V}_1^H \neq \mathbf{V}_1^H \mathbf{V}_1$ . SVD highlights the properties of the linear transformation  $\mathbf{A}$  as detailed in Section 2.4.



## 1.3 Eigenvectors in Everyday Life

The eigenvectors of a linear transformation highlight the structure of the transformation itself. However, there are engineering problems where analysis using eigenvectors computation is far more common than one might imagine; there follow some classical introductory examples.

### 1.3.1 Conversations in a Noisy Restaurant

In a restaurant there are  $N$  tables occupied by guests that are having conversations within each table; the mutual interference among tables induces each table to adapt the intensity of the voice to guarantee the intelligibility of the conversation and cope with the interference from the other tables. Let  $P_i$  be the power of the voice at each table with  $P_i \geq 0$ ; the aim here is to evaluate the optimum power values  $P_1, P_2, \dots, P_N$  at every table to guarantee intelligibility without saturating the powers (i.e., *avoid the guests having to unnecessarily talk too loudly*). Tables are referred as nodes in a *weighted graph* illustrated in Figure 1.1 (see Chapter 22) where weights account for their mutual interaction.

Assuming that the gain from  $j$ th node toward the  $i$ -th node is  $g_{ij} \leq 1$ , the overall interference experienced by the users of the  $i$ -th group is the sum of the signal generated by all the tables scaled by the corresponding attenuation  $\sum_{j \neq i} g_{ij} P_j$ . What matters for intelligibility is that the voices within the  $i$ th table should be large enough compared to the overall interference. Namely, the ratio between the power of the signal and the overall interference should be

$$\frac{P_i}{\sum_{j \neq i} g_{ij} P_j} \geq \gamma_i$$

where  $\gamma_i$  is the minimum value for the  $i$ -th node to have an acceptable quality—usually this depends on the context and is known. Rearranging the inequality for the  $N$  nodes,

$$\gamma_i \sum_{j \neq i} g_{ij} P_j \leq P_i$$

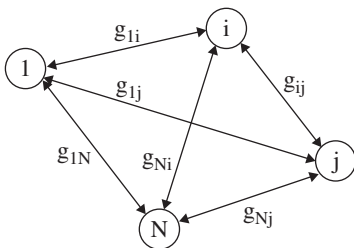


Figure 1.1 Graph representing tables mutually interfering with inter-table gain  $g_{ij} \leq 1$ .

that in matrix form becomes

$$\underbrace{\begin{pmatrix} \gamma_1 & 0 & \dots & 0 \\ 0 & \gamma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_N \end{pmatrix}}_{\mathbf{D}} \cdot \underbrace{\begin{pmatrix} 0 & g_{12} & \dots & g_{1N} \\ g_{21} & 0 & \dots & g_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N1} & g_{N2} & \dots & 0 \end{pmatrix}}_{\mathbf{G}} \cdot \underbrace{\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix}}_{\mathbf{p}} \leq \underbrace{\begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{pmatrix}}_{\mathbf{p}}$$

or equivalently

$$\mathbf{A}\mathbf{p} \leq \mathbf{p}$$

where

$$\mathbf{A} = \mathbf{D}\mathbf{G}$$

is a square matrix with known and positive entries (matrix  $\mathbf{A}$  is positive) accounting for link-gains  $\mathbf{G}$  and threshold signal to interference levels  $\mathbf{D}$ . To solve, the inequality can be rewritten as an equality with  $0 < \rho \leq 1$  such that

$$\mathbf{A}\mathbf{p} = \rho\mathbf{p}.$$

A trivial solution with all groups inactive ( $P_i = 0$  for  $\forall i$ ) should be avoided assuming that  $\rho \neq 0$ . The solution for the vector of powers  $\mathbf{p}$  is any eigenvector of the matrix  $\mathbf{A}$  associated to an eigenvalue smaller than 1. Notice that the eigenvectors are normalized to have unit-norm so that optimal power configuration across nodes is independent of any scaling factor of all nodes (called reference power). To avoid waste of power, it is advisable to choose a small reference power as this guarantees the minimum signal to interference level anyway. In practice, customers in a restaurant do not know  $\mathbf{A}$  (possibly a subset or blocks of  $\mathbf{G}$ ) and the power configuration including the scale factor is autonomously chosen by each group to reach the global equilibrium. The computation of the eigenvector is an iterative process (see Section 2.6) *implicitly carried out by mutual respect among guests (if any)* while fulfilling the minimal intra-group constraints on quality  $\mathbf{D}$  that is eased when customers are homogeneous, and all the groups have the same quality  $\gamma$  ( $\mathbf{D} = \gamma\mathbf{I}$ ).

Inspection of the problem shows some minimal conditions for the solution. The **Gershgorin theorem** helps to evaluate the conditions for the eigenvalues to be bounded by unitary value. Let

$$R_i = \sum_{j=1, j \neq i}^N a_{ij}$$

be the sum of the  $i$ th row except the diagonal; every eigenvalue of  $\mathbf{A}$  indicated as  $\lambda_k$  is fully contained within every Gershgorin disk

$$|\lambda_k - a_{ii}| \leq R_i = \sum_{j=1, j \neq i}^N a_{ij}$$

centered in  $a_{ii}$  with radius  $R_i$ . In other words, all the eigenvalues of  $\mathbf{A}$  lie in the union of the Gersgorin disks. Returning to the specific problem,  $a_{ii} = 0$  and thus

$$|\lambda_k| \leq \sum_{j=1, j \neq i}^N a_{ij} = \gamma_i \sum_{j \neq i} g_{ij}.$$

Assuming that the solution is for  $\lambda_{\max} = \rho \leq 1$  as this corresponds to the strict inequality,  $|\lambda_k| \leq \lambda_{\max} \leq 1$  and thus the condition

$$\gamma_i \sum_{j \neq i} g_{ij} \leq 1 \implies \gamma_i \leq \frac{1}{\sum_{j \neq i} g_{ij}}$$

guarantees that there exists one solution. The existence of at least one eigenvector  $\mathbf{p}$  with positive entries is based on the Perron–Frobenius theorem [21], fulfilled in this case.

Notice that when there are two sets of disjointed groups as shown in Figure 1.2, the gain matrix becomes

$$\mathbf{G} = \begin{pmatrix} 0 & g_{12} & 0 & 0 & g_{15} & 0 \\ g_{12} & 0 & 0 & 0 & g_{25} & 0 \\ 0 & 0 & 0 & g_{34} & 0 & g_{36} \\ 0 & 0 & g_{34} & 0 & 0 & g_{46} \\ g_{15} & g_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{36} & g_{46} & 0 & 0 \end{pmatrix}$$

but since the numbering of the nodes are just conventional, these gains can be reordered by grouping connected nodes:

$$\tilde{\mathbf{G}} = \begin{pmatrix} 0 & g_{12} & g_{15} & 0 & 0 & 0 \\ g_{12} & 0 & g_{25} & 0 & 0 & 0 \\ g_{15} & g_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{34} & g_{46} \\ 0 & 0 & 0 & g_{34} & 0 & g_{36} \\ 0 & 0 & 0 & g_{46} & g_{36} & 0 \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} 0 & g_{12} & 0 & 0 & g_{15} & 0 \\ g_{12} & 0 & 0 & 0 & g_{25} & 0 \\ 0 & 0 & 0 & g_{34} & 0 & g_{36} \\ 0 & 0 & g_{34} & 0 & 0 & g_{46} \\ g_{15} & g_{25} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{36} & g_{46} & 0 & 0 \end{pmatrix} \cdot \mathbf{L}^T$$

where the ordering matrix is symmetric

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and it swap the rows/columns  $3 \leftrightarrow 5$ . Gain matrix  $\tilde{\mathbf{G}}$  has a block-partitioned structure

$$\tilde{\mathbf{G}} = \begin{pmatrix} \tilde{\mathbf{G}}_{11} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}}_{22} \end{pmatrix}$$

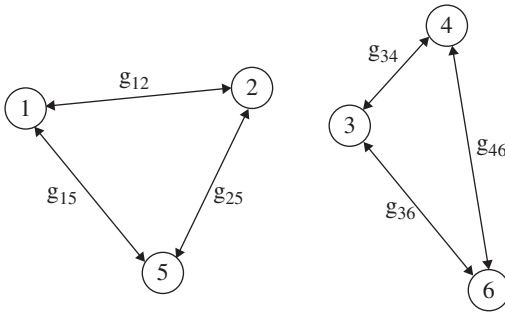


Figure 1.2 Graph for two disjoint and non-interfering sets.

this highlights that the computation of the optimal solution can be decoupled into two separated solutions for the two sets of nodes.

### 1.3.2 Power Control in a Cellular Communication System

In cellular systems, every mobile device (e.g., a smart phone) is connected to a base-station deployed at the center of every cell (see Figure 1.3) that in turn adjusts its transmission power to control both the received power and the interference toward the other cells (*inter-cell interference*). The interference superimposes on the background noise with power  $\sigma^2$ . The problem definition is the same as above except that every user with power  $P_i$  is connected to the serving base-station with intra-cell gain  $g_{ii}$  and is interfering with all the other cells. Once again, what matters is the ratio between the power of the signal reaching the serving base-station and the interference arising from all the other mobile devices (gray lines in figure), each camped in its own base-station, augmented by some background noise; the signal to interference and noise ratio (SINR) is

$$SINR_i = \frac{g_{ii}P_i}{\sum_{j \neq i} g_{ij}P_j + \sigma^2}.$$

To guarantee a reliable communication, the SINR should be above a predefined reliability value

$$SINR_i \geq \gamma_i.$$

The overall problem can be now defined as

$$\gamma_i \left( \sum_{j \neq i} G_{ij}P_j + q_i \right) \leq P_i$$

where  $G_{ij} = g_{ij}/g_{ii}$  and  $q_i = \sigma^2/g_{ii}$ . In compact notation, this is

$$\mathbf{D}(\mathbf{G}\mathbf{P} + \mathbf{q}) \leq \mathbf{p} \longrightarrow (\mathbf{I} - \mathbf{D}\mathbf{G})\mathbf{p} \geq \mathbf{D}\mathbf{q}$$

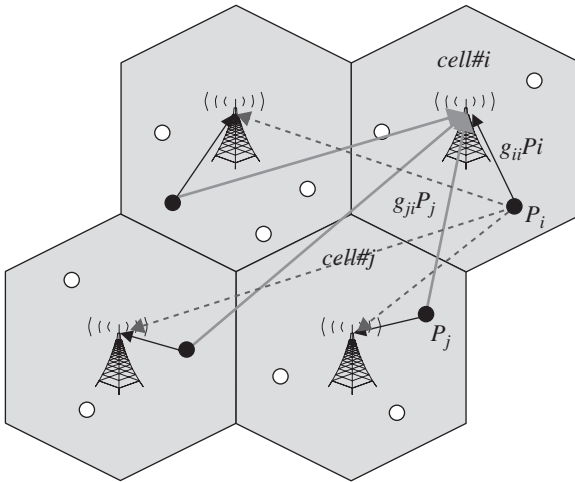


Figure 1.3 Mutual interference in a cellular communication system.

and the solution is

$$\mathbf{p} \geq (\mathbf{I} - \mathbf{DG})^{-1} \mathbf{Dq}$$

with positive values  $\mathbf{p}$  provided that the inverse  $(\mathbf{I} - \mathbf{DG})^{-1}$  is a positive matrix (proof now is not straightforward as Perron–Frobenius properties do not hold).

In practical systems, the solution is not obtained by directly solving the power distribution among the users within all the cells, but rather by iterative methods that assign the power to users in every cell and, based on this choice, the base-stations exchange the information on the experienced power and interference with others to change and refine the power assignments accordingly. All this must be carried out dynamically to compensate for the power losses/gains of the mobile devices  $g_{ij}$  that are changing their propagation settings. In other words, every cell measures the  $SINR_i$ , and the base-station, from the measurements of the ensemble of other  $SINR_j$  with  $j \neq i$  (typically the surrounding cells), orders a local correction of  $P_i$  for the devices within the serving cell (inner power control loop) while an exchange with the other base-stations defines the macro-corrections (outer power control loop). The overall system is quite complex, but the final scope is to solve the system  $\mathbf{p} \geq (\mathbf{I} - \mathbf{DG})^{-1} \mathbf{Dq}$  iteratively by message exchange. Sometimes the solution cannot be attained as there are conditions in the system of equations  $\mathbf{p} \geq (\mathbf{I} - \mathbf{DG})^{-1} \mathbf{Dq}$  that push some terminals with a very low SINR (being placed in a very bad propagation situation with  $g_{ii} \simeq 0$ —e.g., a mobile device in an elevator) to raise intolerably the power of the network to let these *bad-users* have positive values. Even if this is a solution of the system, it increases the interference for all the others and the average transmission power (and battery usage) due to a few *bad-users*. In these situations, it is convenient to drop the calls of these few users with a large benefit to all the others.

### 1.3.3 Price Equilibrium in the Economy

In a system with  $N$  manufacturing units that exchange their products, the equilibrium prices guarantee that every manufacturer has a profit. Let the  $j$ -th manufacturer sell a fraction  $a_{ij}$  to the  $i$ -th manufacturer; in closed system (without any exchange outside of the set of  $N$  entities), we expect that all production will be sold (including the internal-usage  $a_{ji}$ ) so that

$$\sum_{i=1}^N a_{ij} = 1,$$

and goods are preferably delivered to others ( $a_{ij} \geq 0$ ). If the selling-price is  $P_j$  then the  $i$ -th manufacturer spends  $a_{ij}P_j$  toward  $j$ -th; the total expenses by the  $i$ -th is the sum over all the acquired goods:  $\sum_{j=1}^N a_{ij}P_j$ . In the case of zero-profit, the expenses should equal the incomes so that

$$\sum_{j=1}^N a_{ij}P_j = P_i$$

In compact notation, this is

$$\mathbf{A}\mathbf{p} = \mathbf{p}$$

and the price of equilibrium is the eigenvector with positive entries associated to the unit eigenvalue of matrix  $\mathbf{A}$ . Any scaling factor is irrelevant as the model is zero-profit, provided that all the manufacturing units adapt the pricing accordingly, possibly converging to values that correspond to the same eigenvector.

To be more realistic, the model should account for some profit

$$P_i - \sum_{j=1}^N a_{ij}P_j \geq \gamma_i$$

so that

$$(\mathbf{I} - \mathbf{A})\mathbf{p} \geq \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_N \end{pmatrix} = \boldsymbol{\gamma}$$

with an analytical relationship similar to power control as  $\mathbf{p} \geq (\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\gamma}$ . Interestingly, when perturbing the solution by increasing (or decreasing) the profit in one of the manufacturers, all the others will change and the (new) equilibrium is attained iteratively by adapting prices and profits. Daily experience shows that a fast adaptation is mandatory when the market is changing, and some of the players are quick reacting at the expense of the others, pushing the profit of some to be  $\gamma_i < 0$ .

## 1.4 Derivative Rules

A set of variables  $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$  ordered in a vector can be used to represent the  $n \mapsto 1$  mapping that is indicated with the compact notation

$$\phi(\mathbf{x}) = \phi(x_1, x_2, \dots, x_n),$$

and similarly for  $\phi(\mathbf{X})$  if depending on the set of  $mn$  variables ordered in a matrix  $\mathbf{X}$  of size  $m \times n$ . It is convenient to preserve the compact matrix notation in optimization problems; below are the rules for evaluating derivatives. A few definitions on notation are mandatory:

**Derivative wrt a vector  $\mathbf{x}$**  (gradient wrt all the  $n$  variables)

$$\frac{\partial}{\partial \mathbf{x}} \phi(\mathbf{x}) = \begin{bmatrix} \frac{\partial \phi(\mathbf{x})}{\partial x_1} \\ \frac{\partial \phi(\mathbf{x})}{\partial x_2} \\ \vdots \\ \frac{\partial \phi(\mathbf{x})}{\partial x_n} \end{bmatrix},$$

is an  $n \times 1$  vector with all the partial derivatives with respect to the  $n$  variables. Assuming that there are  $m$  multivariate functions ordered into a vector  $\boldsymbol{\phi}(\mathbf{x}) = [\phi_1(\mathbf{x}) \ \phi_2(\mathbf{x}) \ \dots \ \phi_m(\mathbf{x})]^T$ , it is convenient to arrange the multivariate functions into a row and have the derivative wrt  $\mathbf{x}$  ordered column-wise into the  $n \times m$  matrix of partial derivative wrt  $\mathbf{x}$ :

$$\frac{\partial}{\partial \mathbf{x}} \boldsymbol{\phi}^T(\mathbf{x}) = \begin{bmatrix} \frac{\partial \phi_1(\mathbf{x})}{\partial \mathbf{x}} & \dots & \frac{\partial \phi_m(\mathbf{x})}{\partial \mathbf{x}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi_1(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial \phi_m(\mathbf{x})}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_1(\mathbf{x})}{\partial x_n} & \dots & \frac{\partial \phi_m(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

**Derivative wrt matrix  $\mathbf{X}$**  of size  $m \times n$ , in compact notation it is the matrix of derivatives

$$\frac{\partial}{\partial \mathbf{X}} \boldsymbol{\phi}(\mathbf{X}) = \begin{bmatrix} \frac{\partial \boldsymbol{\phi}(\mathbf{X})}{\partial x_{11}} & \dots & \frac{\partial \boldsymbol{\phi}(\mathbf{X})}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \boldsymbol{\phi}(\mathbf{X})}{\partial x_{m1}} & \dots & \frac{\partial \boldsymbol{\phi}(\mathbf{X})}{\partial x_{mn}} \end{bmatrix}$$

partial derivatives are ordered in the same way as the columns of  $\mathbf{X}$ .

### 1.4.1 Derivative with respect to $\mathbf{x} \in \mathbb{R}^n$

Two relevant cases of derivative are considered herein:

- $\phi(\mathbf{x}) = \mathbf{x}^T \mathbf{a}$  and  $\phi(\mathbf{x}) = \mathbf{a}^T \mathbf{x}$

$$\frac{\partial \mathbf{x}^T \mathbf{a}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{x}^T \mathbf{a}}{\partial x_1} \\ \frac{\partial \mathbf{x}^T \mathbf{a}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{x}^T \mathbf{a}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{a}$$

$$\frac{\partial \mathbf{a}^T \mathbf{x}}{\partial \mathbf{x}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{a}$$

*Special cases:* 1)  $\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^T = \frac{\partial}{\partial \mathbf{x}} \mathbf{x} = \mathbf{I}_n$ ; 2)  $\frac{\partial}{\partial \mathbf{x}} \text{tr}[\mathbf{x}\mathbf{a}^T] = \frac{\partial}{\partial \mathbf{x}} \text{tr}[\mathbf{a}\mathbf{x}^T] = \mathbf{a}$ .

- $\phi(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$  (quadratic function) with  $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n]$

$$\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_1} + \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_1} \mathbf{x} \\ \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_2} + \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_2} \mathbf{x} \\ \vdots \\ \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_n} + \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_n} \mathbf{x} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_1} \\ \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_2} \\ \vdots \\ \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_n} \end{bmatrix}}_{\mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial \mathbf{x}}} + \underbrace{\begin{bmatrix} \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_1} \mathbf{x} \\ \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_2} \mathbf{x} \\ \vdots \\ \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_n} \mathbf{x} \end{bmatrix}}_{\frac{\partial \mathbf{x}^T \mathbf{A}}{\partial \mathbf{x}} \mathbf{x}}$$

Considering the first term, it is a gradient of multiple functions wrt the vector so that

$$\mathbf{A} \mathbf{x} = \begin{bmatrix} \mathbf{A}(1, :)\mathbf{x} \\ \mathbf{A}(2, :)\mathbf{x} \\ \vdots \\ \mathbf{A}(n, :)\mathbf{x} \end{bmatrix}$$

$$\frac{\partial \mathbf{A} \mathbf{x}}{\partial x_k} = \mathbf{a}_k = \mathbf{A}(:, k)$$

Therefore

$$\begin{bmatrix} \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_1} \\ \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_2} \\ \vdots \\ \mathbf{x}^T \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^T \mathbf{a}_1 \\ \mathbf{x}^T \mathbf{a}_2 \\ \vdots \\ \mathbf{x}^T \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{x} \\ \mathbf{a}_2^T \mathbf{x} \\ \vdots \\ \mathbf{a}_n^T \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \mathbf{x} = \mathbf{A}^T \mathbf{x}$$



similarly for the second term

$$\frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_k} = \frac{\partial (\mathbf{A}^T \mathbf{x})^T}{\partial x_k} = \left( \frac{\partial \mathbf{A}^T \mathbf{x}}{\partial x_k} \right)^T \Rightarrow \begin{bmatrix} \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_1} \\ \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_2} \\ \vdots \\ \frac{\partial \mathbf{x}^T \mathbf{A}}{\partial x_n} \end{bmatrix} = \dots = \mathbf{A} \mathbf{x}.$$

To summarize,

$$\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^T) \mathbf{x}$$

Furthermore, if  $\mathbf{A}$  is symmetric ( $\mathbf{A} = \mathbf{A}^T$ ) it reduces to

$$\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = 2\mathbf{A} \mathbf{x}.$$

#### 1.4.2 Derivative with respect to $\mathbf{x} \in \mathbb{C}^n$

Considering the scalar complex-valued variable  $x = x_R + jx_I$ , any function  $\phi(x) : \mathbb{C} \mapsto \mathbb{C}$  could be managed by considering the variable as represented by a vector with two components (real and imaginary part of  $x$ )  $h(\cdot) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$  such that  $h(x_R, x_I) = \phi(x)$  and therefore the formalism defined for the derivative with respect to a  $2 \times 1$  vector in  $\mathbb{R}^2$  could be used (i.e., any complex-valued variable is represented as two real-valued variables). However, one could define a mapping  $g : \mathbb{C} \times \mathbb{C} \mapsto \mathbb{C}$  such that  $g(x, x^*) = \phi(x)$ , where the variables  $x$  and  $x^*$  are treated as two different variables and the function is analytic (see [4]). The derivative wrt a complex variable is <sup>1</sup>

$$\frac{\partial \phi(x)}{\partial x} = \frac{1}{2} \left( \frac{\partial \phi(x)}{\partial x_R} - j \frac{\partial \phi(x)}{\partial x_I} \right)$$

<sup>1</sup> Since

$$h(x_R, x_I) = g(x, x^*)$$

taking the partial derivative of both

$$\begin{cases} \frac{\partial h}{\partial x_R} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial x_R} + \frac{\partial g}{\partial x^*} \frac{\partial x^*}{\partial x_R} \\ \frac{\partial h}{\partial x_I} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial x_I} + \frac{\partial g}{\partial x^*} \frac{\partial x^*}{\partial x_I} \end{cases}$$

and the following equalities  $\frac{\partial x}{\partial x_R} = \frac{\partial x^*}{\partial x_R} = 1$ ,  $\frac{\partial x}{\partial x_I} = j$  and  $\frac{\partial x^*}{\partial x_I} = -j$ , solving for  $\frac{\partial g}{\partial x}$  and  $\frac{\partial g}{\partial x^*}$  gives:

$$\begin{cases} \frac{\partial g}{\partial x} = \frac{1}{2} \left( \frac{\partial h}{\partial x_R} - j \frac{\partial h}{\partial x_I} \right) \\ \frac{\partial g}{\partial x^*} = \frac{1}{2} \left( \frac{\partial h}{\partial x_R} + j \frac{\partial h}{\partial x_I} \right) \end{cases}$$

Derivative wrt  $x$  is equivalent to consider  $x^*$  as constant, similarly for  $x^*$ .

Below are a few simple cases of derivatives with respect a complex variable:

$$\begin{aligned}\frac{\partial x}{\partial x} &= \frac{1}{2} \left( \frac{\partial x}{\partial x_R} - j \frac{\partial x}{\partial x_I} \right) = \frac{1}{2}(1+1) = 1 \\ \frac{\partial x^*}{\partial x} &= \frac{1}{2} \left( \frac{\partial x^*}{\partial x_R} - j \frac{\partial x^*}{\partial x_I} \right) = \frac{1}{2}(1-1) = 0 \\ \frac{\partial |x|^2}{\partial x} &= \frac{\partial x x^*}{\partial x} = \frac{\partial x}{\partial x} x^* + x \frac{\partial x^*}{\partial x} = x^*\end{aligned}$$

Let us consider the case of  $n$  complex variables ( $\mathbf{x} \in \mathbb{C}^n$ ), and following the same ordering as for real-valued multivariate variables we have

- $\phi(\mathbf{x}) = \mathbf{a}^H \mathbf{x} = \sum_{i=1}^n a_i^* x_i$

$$\frac{\partial \mathbf{a}^H \mathbf{x}}{\partial x_k} = \frac{1}{2} \left( \frac{\partial}{\partial x_{R,k}} - j \frac{\partial}{\partial x_{I,k}} \right) \sum_{i=1}^n a_i^* x_i = a_k^* \rightarrow \frac{\partial \mathbf{a}^H \mathbf{x}}{\partial \mathbf{x}} = \begin{bmatrix} a_1^* \\ a_2^* \\ \vdots \\ a_n^* \end{bmatrix} = \mathbf{a}^*$$

- $\phi(\mathbf{x}) = \mathbf{x}^H \mathbf{a} = \sum_{i=1}^n a_i x_i^*$

$$\frac{\partial \mathbf{x}^H \mathbf{a}}{\partial x_k} = \frac{1}{2} \left( \frac{\partial}{\partial x_{R,k}} - j \frac{\partial}{\partial x_{I,k}} \right) \sum_{i=1}^n a_i x_i^* = 0 \rightarrow \frac{\partial \mathbf{x}^H \mathbf{a}}{\partial \mathbf{x}} = \mathbf{0}$$

- $\phi(\mathbf{x}) = \mathbf{x}^H \mathbf{A} \mathbf{x}$  where  $\mathbf{A}$  has Hermitian symmetry ( $\mathbf{A}^H = \mathbf{A}$ )

Due to the fact that the  $k$ th entry is:

$$\frac{\partial \phi(\mathbf{x})}{\partial x_k} = \mathbf{x}^H \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_k} + \frac{\partial \mathbf{x}^H}{\partial x_k} \mathbf{A} \mathbf{x} = \mathbf{x}^H \mathbf{a}_k + \mathbf{0}$$

we have:

$$\frac{\partial \mathbf{x}^H \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = \begin{bmatrix} \mathbf{x}^H \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_1} + \frac{\partial \mathbf{x}^H}{\partial x_1} \mathbf{A} \mathbf{x} \\ \mathbf{x}^H \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_2} + \frac{\partial \mathbf{x}^H}{\partial x_2} \mathbf{A} \mathbf{x} \\ \vdots \\ \mathbf{x}^H \frac{\partial \mathbf{A} \mathbf{x}}{\partial x_n} + \frac{\partial \mathbf{x}^H}{\partial x_n} \mathbf{A} \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^H \mathbf{a}_1 \\ \mathbf{x}^H \mathbf{a}_2 \\ \vdots \\ \mathbf{x}^H \mathbf{a}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{x}^* \\ \mathbf{a}_2^T \mathbf{x}^* \\ \vdots \\ \mathbf{a}_n^T \mathbf{x}^* \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^T \\ \mathbf{a}_2^T \\ \vdots \\ \mathbf{a}_n^T \end{bmatrix} \mathbf{x}^* = \mathbf{A}^T \mathbf{x}^* = (\mathbf{A} \mathbf{x})^*$$

the last equality holds for Hermitian symmetry.

### 1.4.3 Derivative with respect to the Matrix $\mathbf{X} \in \mathbb{R}^{m \times n}$

Derivation is essentially the same as for a vector as it is just a matter of ordering. However, there are some essential relationships that are often useful in optimization problems [9]:

$$\frac{\partial}{\partial \mathbf{X}} \text{tr}[\mathbf{X}] = \mathbf{I}$$

$$\begin{aligned}
 \frac{\partial}{\partial \mathbf{X}} \text{tr}[\mathbf{A}\mathbf{X}] &= \mathbf{A}^T \\
 \frac{\partial}{\partial \mathbf{X}} \text{tr}[\mathbf{A}\mathbf{X}^{-1}] &= -(\mathbf{X}^{-1}\mathbf{A}\mathbf{X}^{-1})^T \\
 \frac{\partial}{\partial \mathbf{X}} \text{tr}[\mathbf{X}^n] &= n(\mathbf{X}^{n-1})^T \\
 \frac{\partial}{\partial \mathbf{X}} \text{tr}[\exp[\mathbf{X}]] &= \exp[\mathbf{X}] \\
 \frac{\partial}{\partial \mathbf{X}} \det[\mathbf{X}] &= \det[\mathbf{X}](\mathbf{X}^{-1})^T \\
 \frac{\partial}{\partial \mathbf{X}} \ln \det[\mathbf{X}] &= (\mathbf{X}^{-1})^T \\
 \frac{\partial}{\partial \mathbf{X}} \det[\mathbf{X}^n] &= n \det[\mathbf{X}]^n (\mathbf{X}^{-1})^T \\
 \frac{\partial}{\partial \mathbf{X}} \text{tr}[\mathbf{X}^T \mathbf{A} \mathbf{X} \mathbf{B}] &= \mathbf{A} \mathbf{X} \mathbf{B} + \mathbf{A}^T \mathbf{X} \mathbf{B}^T \\
 \frac{\partial}{\partial \mathbf{X}} \text{tr}[\mathbf{X} \mathbf{X}^T] &= 2\mathbf{X}
 \end{aligned}$$

## 1.5 Quadratic Forms

A quadratic form is represented by the following expression:

$$\psi(\mathbf{x}) = \mathbf{x}^T \mathbf{R} \mathbf{x} = \text{tr}[\mathbf{R} \mathbf{x} \mathbf{x}^T] = \|\mathbf{x}\|_{\mathbf{R}}^2$$

with  $\mathbf{x} \in \mathbb{R}^N$  and  $\mathbf{R} \in \mathbb{R}^{N \times N}$ . Function  $\psi(\mathbf{x})$  is a scalar and therefore  $\psi(\mathbf{x})^T = (\mathbf{x}^T \mathbf{R} \mathbf{x})^T = \mathbf{x}^T \mathbf{R}^T \mathbf{x} = \psi(\mathbf{x}) = \mathbf{x}^T \mathbf{R} \mathbf{x}$ : in a quadratic form  $\mathbf{R}$  must be symmetric. Moreover  $\mathbf{R}$  is positive definite  $\mathbf{R} > \mathbf{0}$  if  $\psi(\mathbf{x}) > 0$  (or positive semidefinite if  $\mathbf{R} \geq \mathbf{0}$ ) for each  $\mathbf{x} \neq \mathbf{0}$ .

For any arbitrary block-partitioning of  $\mathbf{R}$ , it is possible to write:

$$\begin{aligned}
 \psi(\mathbf{x}) = \mathbf{x}^T \mathbf{R} \mathbf{x} &= [\mathbf{x}_1^T, \mathbf{x}_2^T] \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{12}^T & \mathbf{R}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\
 &= \mathbf{x}_1^T \mathbf{R}_{11} \mathbf{x}_1 + \mathbf{x}_1^T \mathbf{R}_{12} \mathbf{x}_2 + \mathbf{x}_2^T \mathbf{R}_{12}^T \mathbf{x}_1 + \mathbf{x}_2^T \mathbf{R}_{22} \mathbf{x}_2
 \end{aligned}$$

If  $\mathbf{R} \geq \mathbf{0}$ , this implies  $\mathbf{R}_{11} \geq \mathbf{0}$  and  $\mathbf{R}_{22} \geq \mathbf{0}$ ; it is easy to verify this assertion considering  $\mathbf{x}_1 = \mathbf{0}$  or  $\mathbf{x}_2 = \mathbf{0}$ .

When  $\mathbf{x}_2$  is fixed, the constrained quadratic form is minimized (with respect to  $\mathbf{x}_1$ ) for

$$\hat{\mathbf{x}}_1 = -\mathbf{R}_{11}^{-1} \mathbf{R}_{12} \mathbf{x}_2$$

while the value of the form for  $\hat{\mathbf{x}} = [\hat{\mathbf{x}}_1^T, \mathbf{x}_2^T]^T$  is

$$\psi(\hat{\mathbf{x}}) = \mathbf{x}_2^T (\mathbf{R}_{22} - \mathbf{R}_{12}^T \mathbf{R}_{11}^{-1} \mathbf{R}_{12}) \mathbf{x}_2 \geq 0$$

the quantity  $\mathbf{R}_{22} - \mathbf{R}_{12}^T \mathbf{R}_{11}^{-1} \mathbf{R}_{12}$  is indicated as the Shur complement of the block-partitioned matrix  $\mathbf{R}$ .

## 1.6 Diagonalization of a Quadratic Form

Consider the generic quadratic form (see Figure 1.4 for  $N = 2$ )

$$\psi(\mathbf{x}) = \mathbf{x}^T \mathbf{R} \mathbf{x} - 2\mathbf{p}^T \mathbf{x} + \psi_0$$

with  $\mathbf{R}^T = \mathbf{R}$  full-rank symmetric and positive definite, and  $\psi_0 \geq 0$  is a constant. By nullifying the gradient of the quadratic form it is possible to identify the coordinates of its minimum:

$$\frac{\partial \psi(\mathbf{x})}{\partial \mathbf{x}} = 2\mathbf{R}\mathbf{x} - 2\mathbf{p} = 0 \Rightarrow \mathbf{x}_{\min} = \mathbf{R}^{-1}\mathbf{p}$$

there corresponds a value of  $\psi$  as

$$\psi_{\min} = \psi(\mathbf{x} = \mathbf{R}^{-1}\mathbf{p}) = \mathbf{p}^T \mathbf{R}^{-1} \mathbf{R} \mathbf{R}^{-1} \mathbf{p} - 2\mathbf{p}^T \mathbf{R}^{-1} \mathbf{p} + \psi_0 = \psi_0 - \mathbf{p}^T \mathbf{R}^{-1} \mathbf{p}.$$

Often it is useful to refer the quadratic form to its principal axis with the origin placed at the coordinates of the minimum and an appropriate rotation to have  $\mathbf{A}$  replaced by a diagonal matrix.

Diagonalization is obtained as a two step procedure. First the new variable is the one wrt the translated reference

$$\mathbf{y} = \mathbf{x} - \mathbf{x}_{\min}$$

so that the quadratic form becomes

$$\begin{aligned} \psi(\mathbf{x}) &= (\mathbf{x} - \mathbf{x}_{\min})^T \mathbf{R} (\mathbf{x} - \mathbf{x}_{\min}) + \psi_0 - \mathbf{p}^T \mathbf{R}^{-1} \mathbf{p} \\ \psi(\mathbf{y}) &= \mathbf{y}^T \mathbf{R} \mathbf{y} + \psi_{\min} \end{aligned}$$

wrt the new variable  $\mathbf{y}$ . As second step, the eigenvector decomposition of  $\mathbf{R} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^T$  yields

$$\psi(\mathbf{y}) = \mathbf{y}^T (\mathbf{U} \mathbf{\Lambda} \mathbf{U}^T) \mathbf{y} + \psi_{\min} = (\mathbf{U}^T \mathbf{y})^T \mathbf{\Lambda} (\mathbf{U}^T \mathbf{y}) + \psi_{\min}$$

The transformation:

$$\mathbf{z} = \mathbf{U}^T \mathbf{y}$$

corresponds to a rotation of the reference system  $\mathbf{y}$  in order to align wrt the principal axes of the quadratic form as in Figure 1.4. In the new coordinate system, the quadratic form is diagonal:

$$\psi(\mathbf{z}) = \mathbf{z}^T \mathbf{\Lambda} \mathbf{z} + \psi_{\min} = \sum_{i=1}^m \lambda_i z_i^2 + \psi_{\min}.$$

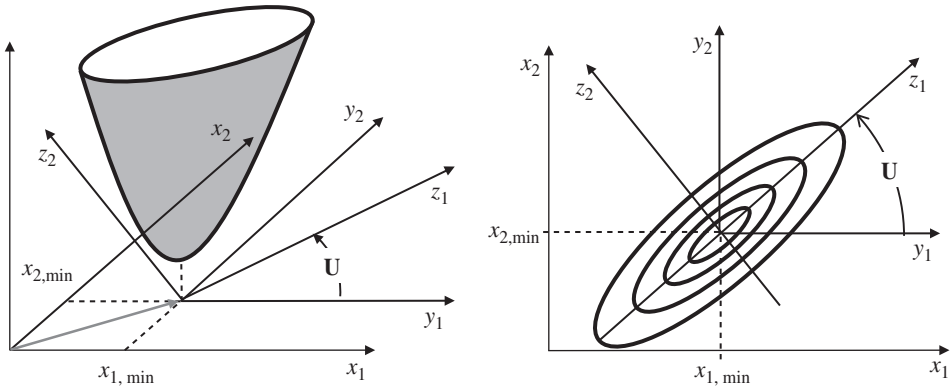


Figure 1.4 Quadratic form and its diagonalization.

This diagonalized form allows an interpretation of the eigenvalues of the quadratic form as being proportional to the curvature of the quadratic form when evaluated along the principal axes as

$$\frac{\partial^2 \psi(\mathbf{z})}{\partial z_k^2} = 2\lambda_k$$

Further deductions are possible based on this simple geometrical representation. Namely, for a positive definite matrix, all the eigenvalues are positive and the quadratic form is a paraboloid (or hyper-paraboloid for  $N > 2$ ); the contour lines (the convex hull for  $\psi(\mathbf{x}) \leq \tilde{\psi} = \text{cost}$ ) are ellipses (as in the Figure 1.4) that are stretched according to the ratio between the corresponding eigenvalues. When all eigenvalues are equal, these contour lines are circles.

## 1.7 Rayleigh Quotient

Given two quadratic forms  $\psi_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^H \mathbf{A} \mathbf{x}$  and  $\psi_{\mathbf{B}}(\mathbf{x}) = \mathbf{x}^H \mathbf{B} \mathbf{x}$  (for generality,  $\mathbf{x} \in \mathbb{C}^N$ ), their ratio

$$V(\mathbf{x}) = \frac{\mathbf{x}^H \mathbf{A} \mathbf{x}}{\mathbf{x}^H \mathbf{B} \mathbf{x}}$$

is indicated as a Rayleigh quotient. The optimization (max/min) of functions like  $V(\mathbf{x})$  is sometimes necessary and requires some specific considerations.

Let  $\mathbf{B} > \mathbf{0}$  be positive definite in order to avoid meaningless conditions (i.e.,  $\psi_{\mathbf{B}}(\mathbf{x}) = 0$  for some value of  $\mathbf{x}$ ); it is possible to have the square-root decomposition as  $\mathbf{B} = \mathbf{B}^{H/2} \mathbf{B}^{1/2}$ , and by using the transformation

$$\mathbf{y} = \mathbf{B}^{1/2} \mathbf{x}$$

(that is invertible  $\mathbf{x} = \mathbf{B}^{-1/2}\mathbf{y}$ ) we obtain:

$$V(\mathbf{x}) = \frac{\mathbf{x}^H \mathbf{A} \mathbf{x}}{(\mathbf{B}^{1/2} \mathbf{x})^H \mathbf{B}^{1/2} \mathbf{x}} = \frac{\mathbf{y}^H (\mathbf{B}^{-H/2} \mathbf{A} \mathbf{B}^{-1/2}) \mathbf{y}}{\mathbf{y}^H \mathbf{y}} = \frac{\mathbf{y}^H \mathbf{Q} \mathbf{y}}{\mathbf{y}^H \mathbf{y}}$$

The solution that maximizes/minimizes  $V(\mathbf{x})$  can be found by the optimization of  $\psi_{\mathbf{Q}}(\mathbf{y}) = \mathbf{y}^H \mathbf{Q} \mathbf{y}$  for  $\mathbf{Q} = \mathbf{B}^{-H/2} \mathbf{A} \mathbf{B}^{-1/2}$  subject to the constraints of  $\mathbf{y}$  with unitary norm (see also Section 1.8). Alternatively, by using the eigenvector decomposition of  $\mathbf{Q} = \mathbf{V} \mathbf{\Lambda} \mathbf{V}^H$  it is possible to write:

$$V(\mathbf{x}) = \frac{\mathbf{y}^H \mathbf{Q} \mathbf{y}}{\mathbf{y}^H \mathbf{y}} = \frac{\sum_k \lambda_k |\mathbf{y}^H \mathbf{v}_k|^2}{\mathbf{y}^H \mathbf{y}}$$

moreover, the following property holds:

$$\lambda_{\min}(\mathbf{Q}) \leq V(\mathbf{x}) \leq \lambda_{\max}(\mathbf{Q})$$

for the following choices

$$\begin{aligned} V(\mathbf{x}) = \lambda_{\min}(\mathbf{Q}) &\Leftrightarrow \mathbf{y} = \mathbf{v}_{\min} \rightarrow \mathbf{x}_{\min} = \mathbf{B}^{-1/2} \mathbf{v}_{\min} \\ V(\mathbf{x}) = \lambda_{\max}(\mathbf{Q}) &\Leftrightarrow \mathbf{y} = \mathbf{v}_{\max} \rightarrow \mathbf{x}_{\max} = \mathbf{B}^{-1/2} \mathbf{v}_{\max} \end{aligned}$$

To summarize, the max/min of  $V(\mathbf{x})$  is obtained from the eigenvectors corresponding to the max/min of  $\mathbf{B}^{-H/2} \mathbf{A} \mathbf{B}^{-1/2}$ :

$$\mathbf{x} = \mathbf{B}^{-1/2} \text{eig}_{\max/\min} \{ \mathbf{B}^{-H/2} \mathbf{A} \mathbf{B}^{-1/2} \}.$$

## 1.8 Basics of Optimization

Optimization problems are quite common in engineering and having an essential knowledge of the main tools is vital. To exemplify, several times it is necessary to identify the coordinates of minimum (or maximum) of a function  $\phi(\mathbf{x})$  wrt  $\mathbf{x} \in \mathbb{R}^N$  subject to (s.t.) some conditions like  $g_1(\mathbf{x}) = 0, \dots, g_M(\mathbf{x}) = 0$ . The overall minimization can be cast as follows:

$$\min_{\mathbf{x}} \{ \phi(\mathbf{x}) \} \text{ s.t. } \mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ \vdots \\ g_M(\mathbf{x}) \end{bmatrix} = \mathbf{0}$$

This is a constrained optimization that is solved by using the *Lagrange multipliers* method. The objective function is redefined including the constraints

$$L(\mathbf{x}, \boldsymbol{\lambda}) = \phi(\mathbf{x}) + \sum_{k=1}^M \lambda_k g_k(\mathbf{x}) = \phi(\mathbf{x}) + \boldsymbol{\lambda}^T \mathbf{g}(\mathbf{x})$$

where the set of (unknown) terms  $\lambda_k$  are called multipliers. The optimization of  $L(\mathbf{x}, \lambda)$  is now wrt a total of  $N + M$  unknowns and it is carried out by evaluating the gradient with respect to  $\mathbf{x}$  and  $\lambda$ :

$$\begin{cases} \frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = \mathbf{0} \\ \frac{\partial L(\mathbf{x}, \lambda)}{\partial \lambda} = \mathbf{0} \end{cases} \Rightarrow \begin{cases} \frac{\partial \phi(\mathbf{x})}{\partial \mathbf{x}} + \lambda^T \frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{0} \\ \mathbf{g}(\mathbf{x}) = \mathbf{0} \end{cases}$$

Notice that the second set of equations is the constraint condition. The solution of the system of equations should eliminate the multipliers and it requires some ability in manipulating equations. A comprehensive discussion on optimization methods can be found in the reference [16] with proof of a unique solution in the case of convex optimization. However, some examples widely used within the book can help to clarify the Lagrange multiplier method.

### 1.8.1 Quadratic Function with Simple Linear Constraint (M=1)

If  $\phi(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$  is quadratic function, the optimization with a linear constraint is formulated as:

$$\min_{\mathbf{x}} \{ \mathbf{x}^T \mathbf{A} \mathbf{x} \} \text{ s.t. } \mathbf{c}^T \mathbf{x} = 1$$

It can be shown that this problem is convex. The formulation with the Lagrangian multiplier is:

$$L(\mathbf{x}, \lambda) = \mathbf{x}^T \mathbf{A} \mathbf{x} + \lambda(\mathbf{c}^T \mathbf{x} - 1)$$

that optimized yields

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = 0 \Rightarrow 2\mathbf{A}\mathbf{x} + \lambda\mathbf{c} = 0 \Rightarrow \mathbf{x} = -\frac{\lambda}{2}\mathbf{A}^{-1}\mathbf{c}$$

Using the constraint  $\mathbf{c}^T \mathbf{x} = 1$ , there is the solution for the multipliers

$$\mathbf{c}^T \mathbf{x} = 1 \Rightarrow -\frac{\lambda}{2}\mathbf{c}^T \mathbf{A}^{-1}\mathbf{c} = 1 \Rightarrow \lambda = -\frac{2}{\mathbf{c}^T \mathbf{A}^{-1}\mathbf{c}}$$

(notice that  $\mathbf{c}^T \mathbf{A}^{-1}\mathbf{c}$  is a scalar) that substituted gives the solution:

$$\mathbf{x}_{\text{opt}} = \frac{\mathbf{A}^{-1}\mathbf{c}}{\mathbf{c}^T \mathbf{A}^{-1}\mathbf{c}}.$$

### 1.8.2 Quadratic Function with Multiple Linear Constraints

Let the quadratic form  $\phi(\mathbf{x}) = \mathbf{x}^H \mathbf{A} \mathbf{x}$  with complex-valued entries and Hermitian symmetric  $\mathbf{A}^H = \mathbf{A}$ ; the minimization is subject to  $M$  constraints  $\mathbf{c}_m^H \mathbf{x} = d_m$  with  $m = 1, \dots, M$ , and it is a quadratic function minimization with multiple linear constraints:

$$\min_{\mathbf{x}} \{ \mathbf{x}^H \mathbf{A} \mathbf{x} \} \text{ s.t. } \mathbf{C} \mathbf{x} = \mathbf{d}.$$

Due to the fact that the optimization is generalized here to the case of complex variables, the total number of constraints is  $2M$ . In fact the single constraint expression can be written as:

$$\delta_m = \delta_{R,m} + j\delta_{I,m} = \mathbf{c}_m^H \mathbf{x} - d_m = 0$$

The Lagrangian function with  $2M$  constraint conditions ( $\lambda \in \mathbb{C}^{M \times 1}$ ) is:

$$L(\mathbf{x}, \lambda) = \mathbf{x}^T \mathbf{A} \mathbf{x} + \sum_{m=1}^M (\lambda_{R,m} \delta_{R,m} + \lambda_{I,m} \delta_{I,m})$$

It is useful to recall that the constraints can be written as

$$(\lambda_{R,m} \delta_{R,m} + \lambda_{I,m} \delta_{I,m}) = \frac{1}{2} (\lambda_m^* \delta_m + \lambda_m \delta_m^*)$$

to highlight the formulas of the derivatives for complex-valued variables (see Section 1.4.2). Therefore

$$L(\mathbf{x}, \lambda) = \mathbf{x}^H \mathbf{A} \mathbf{x} + \frac{1}{2} (\lambda^H \boldsymbol{\delta} + \boldsymbol{\delta}^H \lambda) = \mathbf{x}^H \mathbf{A} \mathbf{x} + \frac{1}{2} \lambda^H (\mathbf{C} \mathbf{x} - \mathbf{d}) + \frac{1}{2} (\mathbf{C} \mathbf{x} - \mathbf{d})^H \lambda$$

and setting the gradient to zero (see Section 1.4.2):

$$\frac{\partial L(\mathbf{x}, \lambda)}{\partial \mathbf{x}} = (\mathbf{A} \mathbf{x})^* + \frac{1}{2} \mathbf{C}^T \lambda^* = (\mathbf{A} \mathbf{x})^* + \frac{1}{2} (\mathbf{C}^H \lambda)^* = \mathbf{0}$$

whose solution is

$$\mathbf{x} = -\frac{1}{2} \mathbf{A}^{-1} \mathbf{C}^H \lambda$$

which, inserted into the constraints  $\mathbf{C} \mathbf{x} = \mathbf{d}$ , gives the solution for the multipliers:

$$-\frac{1}{2} \mathbf{C} \mathbf{A}^{-1} \mathbf{C}^H \lambda = \mathbf{d} \rightarrow \lambda = -2(\mathbf{C} \mathbf{A}^{-1} \mathbf{C}^H)^{-1} \mathbf{d}$$

Back-substituting into the problem leads to the general solution for the constrained optimization problem:

$$\mathbf{x}_{opt} = \mathbf{A}^{-1} \mathbf{C}^H (\mathbf{C} \mathbf{A}^{-1} \mathbf{C}^H)^{-1} \mathbf{d}.$$

## Appendix A: Arithmetic vs. Geometric Mean

Let  $x_1, \dots, x_N$  be a set of positive values; the arithmetic mean is always larger than the geometric mean:

$$\frac{\sum_{k=1}^N x_k}{N} \geq \left( \prod_{k=1}^N x_k \right)^{1/N}$$

the equality is for  $x_1 = x_2 = \dots = x_N$ .



The inequality holds for any weighted average with weighting  $\lambda_k \geq 0$ :

$$\sum_{k=1}^N \lambda_k x_k \geq \prod_{k=1}^N x_k^{\lambda_k},$$

the equality is still for  $x_1 = x_2 = \dots = x_N$ . In some cases it is convenient (or stems from the problem itself, e.g., in information theory related problems) to adopt logarithms:

$$\log \left( \sum_{k=1}^N \lambda_k x_k \right) \geq \sum_{k=1}^N \lambda_k \log x_k$$

By extension, the following inequalities hold:

$$\sqrt{\frac{x_1^2 + \dots + x_N^2}{N}} \geq \frac{x_1 + \dots + x_N}{N} \geq \sqrt[N]{x_1 \dots x_N} \geq \frac{N}{\frac{1}{x_1} + \dots + \frac{1}{x_N}}$$

