CHAPTER 1

Conditions of Uniform Convergence

1.1 Pointwise, Absolute, and Uniform Convergence. Convergence on a Set and Subset

Example 1. A function $f(x, y)$, defined on $X \times Y$, has a limit for any fixed $x \in X$ as y approaches y_0, that is, $f(x, y)$ converges pointwise to a limit function $\varphi(x)$ as y approaches y_0, but the convergence of $f(x, y)$ to $\varphi(x)$ is nonuniform on X.

Solution
Let us consider $f(x, y) = \frac{xy}{x^2+y^2}$ defined on $[0, 1] \times (0, 1]$ and choose $y_0 = 0$. If $x = 0$, then $f(0, y) = 0$ and consequently $\lim_{y\to 0} f(0, y) = \lim_{y\to 0} 0 = 0$. If $x \neq 0$, then $\lim_{y\to 0} f(x, y) = \lim_{y\to 0} \frac{xy}{x^2+y^2} = 0$. Therefore, the limit function is defined for any $x \in [0, 1]$ and it is zero: $\varphi(x) = \lim_{y\to 0} f(x, y) = 0$. However, the convergence to $\varphi(x)$ is not uniform on $X = [0, 1]$. Indeed for $\forall y \in Y = (0, 1]$, there exists $x_y = y \in (0, 1]$ such that

$$|f(x_y, y) - \varphi(x_y)| = \frac{y^2}{2y^2} = \frac{1}{2} \underset{y\to 0}{\nrightarrow} 0,$$

that is, for $\varepsilon_0 = \frac{1}{2}$ whatever radius δ is chosen, there exists the point $x_y = y \in (0, 1]$ such that $|f(x_y, y) - \varphi(x_y)| = \frac{y^2}{2y^2} \geq \varepsilon_0$ although $|y| < \delta$. It means that the convergence is not uniform.

Remark 1. In the case of $Y = \mathbb{N}$, a similar example can be formulated as follows: a sequence of functions $f_n(x)$ converges (pointwise) on a set X, but this convergence is nonuniform. One of the counterexamples is $f_n(x) = x^n$, $X = (-1, 1)$. Since $|x| < 1$, one gets $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0 = f(x)$, $\forall x \in X$. To show that this convergence is nonuniform, let us pick up $x_n = \left(1 - \frac{1}{n}\right) \in X$, for $\forall n \in \mathbb{N}$, $n \geq 2$; and for these points, we obtain

$$|f_n(x_n) - f(x_n)| = \left(1 - \frac{1}{n}\right)^n \underset{n\to\infty}{\rightarrow} e^{-1} \neq 0.$$

Counterexamples on Uniform Convergence: Sequences, Series, Functions, and Integrals, First Edition.
Andrei Bourchtein and Ludmila Bourchtein.
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Companion website: www.wiley.com/go/bourchtein/counterexamples_on_uniform_convergence

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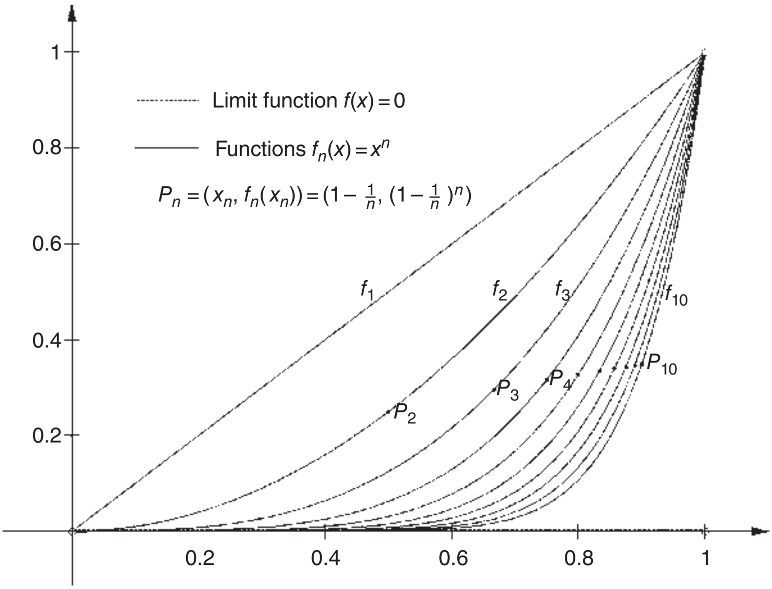

Figure 1.1 Examples 1, 4, and 28, sequence $f_n(x) = x^n$.

In other words, for $\varepsilon_0 = \frac{1}{4}$ whatever natural number N is chosen, for some (in this case, actually, for any) $n \in \mathbb{N}$, $n \geq 2$, there exists the point $x_n = 1 - \frac{1}{n}$ such that $|f_n(x_n) - f(x_n)| = \left(1 - \frac{1}{n}\right)^n \geq \frac{1}{4}$, that is, the convergence is not uniform. (In the last inequality, we have used the fact that the sequence $\left(1 - \frac{1}{n}\right)^n$ is increasing.)

Remark 2. A similar formulation can be made in the case of series: a series of functions converges (pointwise) on a set, but this convergence is nonuniform. The respective counterexample can be given with the series $\sum_{n=0}^{\infty} x^n$, $x \in X = (-1, 1)$. It is well known that the geometric series is convergent for $|x| < 1$ and $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = f(x)$. To analyze the character of this convergence, first let us find the partial sums $f_n(x) = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ and the corresponding remainders $r_n(x) = f(x) - f_n(x) = \frac{x^{n+1}}{1-x}$. Choosing now $x_n = 1 - \frac{1}{n+1}$, $\forall n \in \mathbb{N}$, we obtain

$$r_n(x_n) = \frac{\left(1 - \frac{1}{n+1}\right)^{n+1}}{1 - 1 + \frac{1}{n+1}} = (n+1)\left(1 - \frac{1}{n+1}\right)^{n+1} \underset{n\to\infty}{\to} \infty.$$

Therefore, the convergence is nonuniform.

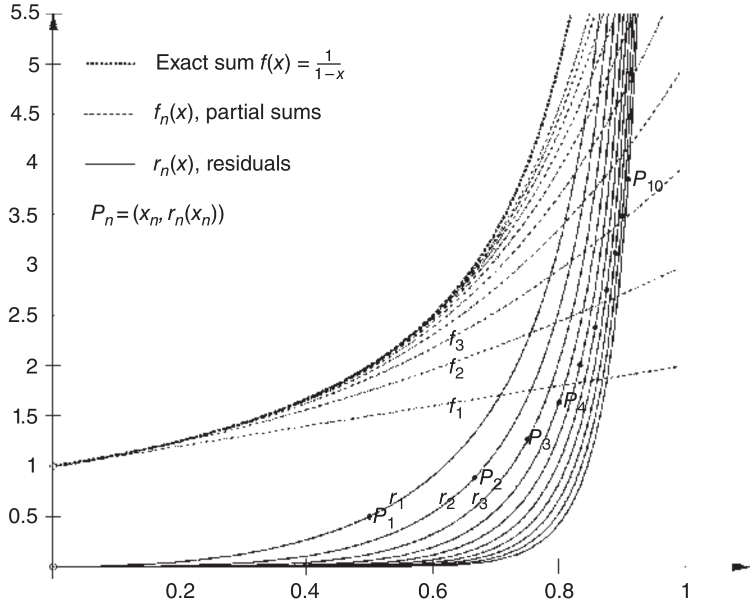

Figure 1.2 Examples 1 and 4, series $\sum_{n=0}^{\infty} x^n$.

Example 2. A series of functions converges on X and a general term of the series converges to zero uniformly on X, but the series converges nonuniformly on X.

Solution

Let us consider the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ on $X = [0, 1)$. This series converges for $\forall x \in X$, because $0 \le \frac{x^n}{n} \le x^n$, $\forall n$, and the geometric series $\sum_{n=1}^{\infty} x^n$ is convergent for $|x| < 1$. We can even find the sum of the series if we recall that the function $\ln\,(1+x)$ has expansion in Taylor's series $\ln\,(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ convergent on $(-1, 1]$. Then, replacing x by $-x$, we obtain $\ln\,(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ with convergence on $[-1, 1)$ and, in particular, on $X = [0, 1)$. Further, the general term $u_n(x) = \frac{x^n}{n}$ converges to 0 uniformly on $X = [0, 1)$, because $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and evaluation $|u_n(x)| = \frac{|x|^n}{n} < \frac{1}{n}$ holds for $\forall x \in X$. Hence, the conditions of the statement are satisfied. However, the series is not convergent uniformly on X that can be shown by verifying the Cauchy criterion of the uniform convergence. In fact, for $\forall x \in X$ and for $\forall n, p \in \mathbb{N}$ we have the following evaluation:

$$\left| \sum_{k=n+1}^{n+p} \frac{x^k}{k} \right| = \frac{x^{n+1}}{n+1} + \cdots + \frac{x^{n+p}}{n+p} > p\frac{x^{n+p}}{n+p}.$$

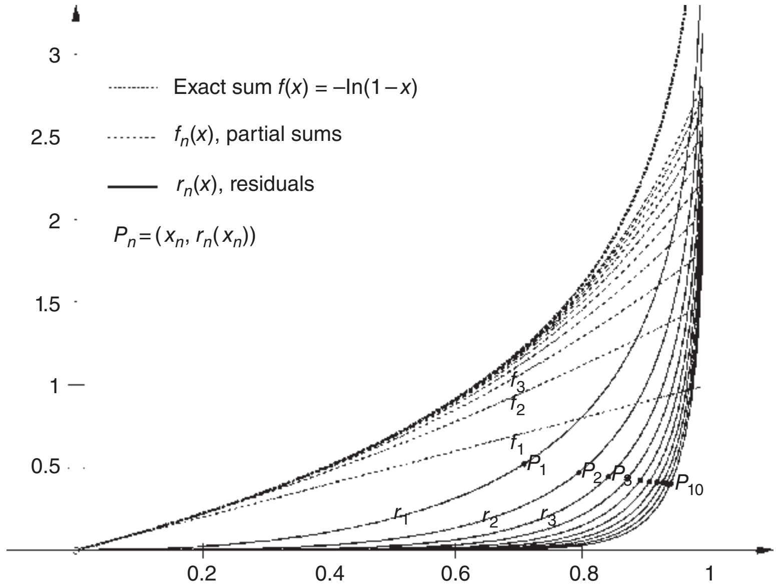

Figure 1.3 Examples 2, 26, 27, and 30, series $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Now, for $\forall n \in \mathbb{N}$, choosing $p_n = n$ and $x_n = \frac{1}{\sqrt[n]{2}} \in X$, we get

$$\left| \sum_{k=n+1}^{n+p_n} \frac{x_n^k}{k} \right| > n \frac{\left(1/\sqrt[n]{2}\right)^{2n}}{2n} = \frac{1}{8} \underset{n\to\infty}{\not\to} 0,$$

which means that the Cauchy criterion is not satisfied and, therefore, the series does not converge uniformly on X.

Remark 1. The series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$ considered on $X = (-1, 1)$ provides a similar counterexample. First, it converges for $\forall x \in X$, because $0 \le \left| \frac{x^n}{\sqrt{n}} \right| \le |x|^n$, $\forall n$, and the geometric series $\sum_{n=1}^{\infty} |x|^n$ is convergent for $|x| < 1$. Second, the inequality $|u_n(x)| = \left| \frac{x^n}{\sqrt{n}} \right| < \frac{1}{\sqrt{n}}$ holds for $\forall x \in X$; since $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, it implies the uniform convergence of $\frac{x^n}{\sqrt{n}}$ to 0 on X. Hence, the statement conditions hold. To analyze the nature of the convergence of the series, let us evaluate the sum $\sum_{k=n+1}^{n+p} \frac{x^k}{\sqrt{k}}$ for $\forall n \in \mathbb{N}$, $p_n = n$, and $x_n = \frac{1}{\sqrt[n]{3}} \in X$:

$$\left| \sum_{k=n+1}^{n+p_n} \frac{x_n^k}{\sqrt{k}} \right| > n \frac{\left(1/\sqrt[n]{3}\right)^{2n}}{\sqrt{2n}} = \frac{1}{9\sqrt{2}} \sqrt{n} \underset{n\to\infty}{\to} \infty.$$

This means that the Cauchy criterion is not satisfied and, therefore, the series does not converge uniformly on X.

Remark 2. The converse general statement is true: if a series $\sum u_n(x)$ converges uniformly on X, then its general term converges to zero uniformly on X.

Example 3. A sequence of functions converges on X and there exists its subsequence that converges uniformly on X, but the original sequence does not converge uniformly on X.

Solution

Let us consider the sequence $f_n(x) = \begin{cases} \frac{x}{n}, n = 2k-1 \\ \frac{1}{n}, n = 2k \end{cases}$, $\forall k \in \mathbb{N}$, $X = \mathbb{R}$.
For any fixed $x \in \mathbb{R}$, we have two partial limits: if $n = 2k-1$, then $\lim\limits_{k\to\infty} f_{2k-1}(x) = \lim\limits_{k\to\infty} \frac{x}{2k-1} = 0$; and if $n = 2k$, then $\lim\limits_{k\to\infty} f_{2k}(x) = \lim\limits_{k\to\infty} \frac{1}{2k} = 0$. Therefore, this sequence converges to 0 on $\mathbb{R}$: $\lim\limits_{n\to\infty} f_n(x) = 0 = f(x)$. Note also that the subsequence $f_{2k}(x)$ converges uniformly on $\mathbb{R}$, since the same evaluation $|f_{2k}(x) - f(x)| < \varepsilon$ holds simultaneously for all $x \in \mathbb{R}$. Or equivalently, for $\forall \varepsilon > 0$ there exists $K_\varepsilon = \left[\frac{1}{2\varepsilon}\right]$ such that for $\forall k > K_\varepsilon$ and simultaneously for all $x \in \mathbb{R}$ it follows that $|f_{2k}(x) - f(x)| < \varepsilon$. That is, the definition of the uniform convergence is satisfied for $f_{2k}(x)$. Nevertheless, the sequence $f_n(x)$ does not converge uniformly on $\mathbb{R}$. Indeed, whatever large index N we choose, there exists the index $n_N = 2N-1 > N$ and the real point $x_N = 2N-1$ such that

$$|f_{2N-1}(x_N) - f(x_N)| = \frac{2N-1}{2N-1} = 1 \underset{N\to\infty}{\nrightarrow} 0.$$

Hence, the convergence of $f_n(x)$ is not uniform on $\mathbb{R}$.

Example 4. A function $f(x, y)$ defined on $(a, b) \times Y$ converges to a limit function $\varphi(x)$ as y approaches y_0, and this convergence is uniform on any interval $[c, d] \subset (a, b)$, but the convergence is nonuniform on (a, b).

Solution

We can employ here the same functions used in Example 1. First, we consider $f(x, y) = \frac{xy}{x^2+y^2}$ defined on $(0, 1) \times (0, 1)$ and choose $y_0 = 0$. As in Example 1, the limit function is zero: $\varphi(x) = \lim\limits_{y\to 0} f(x, y) = 0$, $\forall x \in (0, 1)$. However, the convergence to $\varphi(x)$ is not uniform on $(0, 1)$, because for $\forall y \in (0, 1)$ there exists $x_y = y$ such that

$$|f(x_y, y) - \varphi(x_y)| = \frac{y^2}{2y^2} = \frac{1}{2} \underset{y\to 0}{\nrightarrow} 0.$$

On the other hand, for any interval $[c, d] \subset (0, 1)$, the convergence is uniform. Indeed, for all $x \in [c, d]$ and for any $y > 0$, it follows that

$$|f(x, y) - \varphi(x)| = \frac{xy}{x^2 + y^2} \leq \frac{xy}{x^2} \leq \frac{1}{c}y.$$

Therefore, for any $\varepsilon > 0$, there exists $\delta_\varepsilon = c\varepsilon > 0$ (which is the same for all points in $[c, d]$) such that if $0 < y < \delta$, then $|f(x, y) - \varphi(x)| \leq \frac{1}{c}y < \varepsilon$ for all $x \in [c, d]$ simultaneously. It means that the convergence is uniform on $[c, d]$.

Remark 1. The sequence $f_n(x) = x^n$ on $x \in (-1, 1)$ from Example 1 provides the following counterexample: a sequence of functions $f_n(x)$, defined on (a, b), converges uniformly on any interval $[c, d] \subset (a, b)$, but the convergence is nonuniform on (a, b). The fact that the convergence to the limit function $f(x) = 0$ is not uniform on $(-1, 1)$ was already proved in Example 1. Let us show that the convergence is uniform on any $[c, d] \subset (-1, 1)$. Since we can always construct the interval $[-q, q]$, where $q = \max\{|c|, |d|\}$, such that $[c, d] \subset [-q, q] \subset (-1, 1)$, it is sufficient to prove the uniform convergence on $[-q, q]$. For this interval, we get $|f_n(x) - f(x)| = |x^n| \leq q^n$, for all $x \in [-q, q]$ at the same time. Since $\lim_{n\to\infty} q^n = 0$, that is, for any $\varepsilon > 0$ (it is sufficient to consider $\varepsilon < 1$), there exists $N_\varepsilon = \left[\frac{\ln \varepsilon}{\ln q}\right]$ such that $q^n < \varepsilon$ if $n > N_\varepsilon$, we can conclude that for any $\varepsilon > 0$ there exists exactly the same $N_\varepsilon = \left[\frac{\ln \varepsilon}{\ln q}\right]$ such that when $n > N_\varepsilon$, then $|x^n| \leq q^n < \varepsilon$ for all $x \in [-q, q]$ simultaneously. The last sentence is the definition of the uniform convergence on $[-q, q]$, and consequently, on $[c, d]$.

Remark 2. Finally, the series of Example 1 $\sum_{n=0}^{\infty} x^n$, $x \in (-1, 1)$ is an example of the situation when a series of functions converges uniformly on any interval $[c, d] \subset (a, b)$, but the convergence is nonuniform on (a, b). It was already shown in Example 1 that the convergence of the given series is not uniform on $(-1, 1)$. Let us consider an interval $[-q, q] \subset (-1, 1)$, $q > 0$ and show that the convergence is uniform on such an interval (this will imply the uniform convergence on any interval $[c, d] \subset (-1, 1)$). Since $|x^n| \leq q^n$ for any $x \in [-q, q]$ and the numerical series $\sum_{n=0}^{\infty} q^n$ is convergent (the geometrical series with $|q| < 1$), according to the Weierstrass test the series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-q, q]$.

Remark 3. The nearly converse situation also takes place, as it is shown in Example 5.

Example 5. A sequence $f_n(x)$ converges on X, but this convergence is nonuniform on a closed interval $[a, b] \subset X$.

Solution
One of the counterexamples is $f_n(x) = nxe^{-n^2x^2}$ on $X = \mathbb{R}$. It is easy to show that this sequence approaches $f(x) \equiv 0$ on $\mathbb{R}$. In fact, for $x = 0$ one has $f_n(0) = 0, \forall n$ and, consequently, $\lim\limits_{n\to\infty} f_n(0) = 0$. For $x \neq 0$, one can use the change of variable $t = nx$ and apply l'Hospital's rule:

$$\lim_{n\to\infty} f_n(x) = \lim_{t\to\pm\infty} \frac{t}{e^{t^2}} = \lim_{t\to\pm\infty} \frac{1}{2te^{t^2}} = 0.$$

Consider now $[a, b] \subset \mathbb{R}$ such that $a \leq 0 < b$. Choosing $N > \frac{1}{b}$ and $x_n = \frac{1}{n}$, one obtains the following evaluation for $\forall n > N$:

$$|f_n(x_n) - f(x_n)| = n|x_n|e^{-n^2x_n^2} = e^{-1} \underset{k\to\infty}{\nrightarrow} 0,$$

which means that the convergence of $f_n(x)$ to 0 is nonuniform on such a closed interval.

Remark 1. A similar example for a series goes as follows: a series $\sum u_n(x)$ converges on a set X, but this series does not converge uniformly on a closed subinterval $[a, b] \subset X$. The series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ provides an example. First, we show that it is convergent on $\mathbb{R}$. If $x_k = k\pi$, $\forall k \in \mathbb{Z}$, then $\sum_{n=1}^{\infty} \frac{\sin nx_k}{n} = \sum_{n=1}^{\infty} 0 = 0$. For $x \neq k\pi$, we can apply Dirichlet's theorem. For the partial sums $B_n(x) = \sum_{k=1}^{n} \sin kx$, the following evaluation holds:

$$\begin{aligned}
|B_n(x)| &= \left| \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} 2\sin kx \sin\frac{x}{2} \right| \\
&= \left| \frac{1}{2\sin\frac{x}{2}} \sum_{k=1}^{n} \left(\cos\left(k - \frac{1}{2}\right)x - \cos\left(k + \frac{1}{2}\right)x \right) \right| \\
&= \frac{1}{2\left|\sin\frac{x}{2}\right|} \left| \cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x \right| \\
&= \frac{1}{2\left|\sin\frac{x}{2}\right|} \left| 2\sin\frac{n+1}{2}x \cdot \sin\frac{n}{2}x \right| \leq \frac{1}{\left|\sin\frac{x}{2}\right|}
\end{aligned}$$

(note that the division by $\sin\frac{x}{2}$ is possible, since $x \neq k\pi$). Therefore, the sums $B_n(x)$ are bounded for any fixed $x \neq k\pi$. Besides, the numerical sequence $c_n = \frac{1}{n}$ is decreasing and approaches 0 as $n \to \infty$. Hence, all the conditions of Dirichlet's theorem are satisfied, and therefore the series is convergent on $\mathbb{R}$.

Let us show that this convergence is nonuniform on $(0, 2\pi)$ (and consequently on $[0, 2\pi]$ or any other interval containing $(0, 2\pi)$). To this end, we evaluate the sum $\sum_{k=n+1}^{n+p} u_k(x_n) = \sum_{k=n+1}^{n+p} \frac{\sin kx_n}{k}$ in the Cauchy criterion of uniform convergence. Choosing in this sum $p_n = n$ and $x_n = \frac{\pi}{6n}$, noting that $\frac{\pi}{6} < kx_n \leq \frac{\pi}{3}$

for any k such that $n < k \le n + p_n = 2n$, and recalling that $\sin t$ is positive and strictly increasing on $\left(0, \frac{\pi}{2}\right)$, we obtain

$$\left|\sum_{k=n+1}^{n+p_n} u_k(x_n)\right| = \left|\sum_{k=n+1}^{2n} \frac{\sin kx_n}{k}\right|$$
$$= \frac{\sin\left(\frac{\pi}{6} + \frac{\pi}{6n}\right)}{n+1} + \frac{\sin\left(\frac{\pi}{6} + \frac{2\pi}{6n}\right)}{n+2} + \cdots + \frac{\sin\frac{\pi}{3}}{2n}$$
$$> \frac{\sin\frac{\pi}{6}}{n+1} + \frac{\sin\frac{\pi}{6}}{n+2} + \cdots + \frac{\sin\frac{\pi}{6}}{2n}$$
$$= \frac{1}{2}\left(\frac{1}{n+1} + \cdots + \frac{1}{2n}\right) > \frac{1}{2}\frac{n}{2n} = \frac{1}{4}.$$

Hence, there exists $\varepsilon_0 = \frac{1}{4}$ such that for $\forall n$ there are $p_n = n$ and $x_n = \frac{\pi}{6n} \in (0, 2\pi)$ such that $\left|\sum_{k=n+1}^{n+p_n} u_k(x_n)\right| > \varepsilon_0$. This means that the Cauchy criterion is not satisfied on $(0, 2\pi)$ and, consequently, the series does not converge uniformly on this interval.

At the same time, the application of Dirichlet's theorem of uniform convergence reveals that the series converges uniformly on the interval $[a, 2\pi - a]$ for any $a \in (0, \pi)$. Indeed, since $\sin\frac{x}{2} > 0$, $\forall x \in [a, 2\pi - a]$, we can apply the same evaluations as above for the partial sums $B_n(x) = \sum_{k=1}^{n} \sin kx$ to obtain

$$|B_n(x)| \le \frac{1}{|\sin x/2|} = \frac{1}{\sin x/2} \le \frac{1}{\sin a/2}, \quad \forall x \in [a, 2\pi - a],$$

that is, the sums $B_n(x)$ are uniformly bounded on $[a, 2\pi - a]$. Since $c_n = \frac{1}{n} \underset{n\to\infty}{\to} 0$ and c_n is strictly decreasing, all the conditions of Dirichlet's theorem of uniform convergence are satisfied. Hence, the series converges uniformly on any interval $[a, 2\pi - a]$, $a \in (0, \pi)$.

Remark 2. For functions depending on a parameter, the corresponding formulation is as follows: a function $f(x,y)$ defined on $X \times Y$ has a limit $\lim_{y\to y_0} f(x,y) = \varphi(x)$ for $\forall x \in X$, but $f(x,y)$ converges to $\varphi(x)$ nonuniformly on a subinterval $[a,b] \subset X$. The function $f(x,y) = \frac{x^2y^2}{x^4+y^4}$ considered on $\mathbb{R} \times (0, +\infty)$ with the limit point $y_0 = 0$ provides the counterexample. This function converges to $\varphi(x) \equiv 0$ on $\mathbb{R}$ as $y \to 0$: for $x = 0$, one has $f(0,y) = 0$, $\forall y \in (0, +\infty)$ which implies $\lim_{y\to 0} f(0,y) = 0$; and for $x \ne 0$, one obtains by the arithmetic rules of the limits $\lim_{y\to 0} \frac{x^2y^2}{x^4+y^4} = \frac{0}{x^4} = 0$. Choose now $[a,b] \subset \mathbb{R}$ such that $a \le 0 < b$ and evaluate the difference $|f(x,y) - \varphi(x)|$ for $\forall y \in (0,b)$ and $x_y = y \in [a,b]$:

$$|f(x_y, y) - \varphi(x_y)| = \frac{y^4}{2y^4} = \frac{1}{2} \underset{y\to 0}{\not\to} 0.$$

This result shows that the convergence is nonuniform on a chosen closed interval.

Remark 3. A strengthened versions of these statements are presented in Example 6.

Example 6. A sequence $f_n(x)$ converges on a set X, but it does not converge uniformly on any subinterval of X.

Solution

To construct a counterexample, let us place all the rational numbers of the interval $[0, 1]$ in a specific order of a numerical sequence r_n, $n = 1, 2, \cdots$ (this can be done, since the set of all the rational numbers of any interval is countable). Define now the functions $f_n(x)$ on $[0, 1]$ as follows: $f_n(x) = \begin{cases} 1, x = r_1, r_2, \cdots, r_n \\ 0, \text{otherwise} \end{cases}$. This sequence is monotone in n for any fixed $x \in [0, 1]$ (since $f_n(r_{n+1}) = 0 < 1 = f_{n+1}(r_{n+1})$ and $f_n(x) = f_{n+1}(x)$, $\forall x \neq r_{n+1}$) and bounded (since $0 \leq f_n(x) \leq 1$, $\forall n \in \mathbb{N}$, $\forall x \in [0, 1]$). Therefore, this sequence is convergent at any fixed $x \in [0, 1]$ and $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \in \mathbb{I} \end{cases} = D(x)$. The convergence is nonuniform on $[0, 1]$ since for $\forall n$ there exist $x_n = r_{n+1}$ such that

$$|f_n(x_n) - f(x_n)| = |f_n(r_{n+1}) - f(r_{n+1})| = 1 \underset{n\to\infty}{\nrightarrow} 0.$$

Let us show that the convergence is also nonuniform on any interval $[a, b] \subset [0, 1]$, which will imply that the convergence is nonuniform on any interval in $[0, 1]$. In fact, since any interval contains infinitely many rational points, in $[a, b]$ there are infinitely many points of the sequence $r_1, r_2, r_3, \cdots$, which form a subsequence $r_{n_1}, r_{n_2}, r_{n_3}, \cdots, r_{n_k} \in [a, b]$, $\forall k \in \mathbb{N}$. Then for any $k \in \mathbb{N}$, there exist $n_k > k$ and $x_k = r_{n_{k+1}} \in [a, b]$ such that

$$|f_{n_k}(x_k) - f(x_k)| = |f_{n_k}(r_{n_{k+1}}) - f(r_{n_{k+1}})| = 1 \underset{k\to\infty}{\nrightarrow} 0,$$

which means that the convergence is nonuniform on $[a, b]$.

Remark 1. The corresponding example for a series that converges on X, but does not converge uniformly on any subinterval of X, can be easily constructed using the sequence of the given counterexample as partial sums of the series. For instance, the series $\sum u_n(x)$ with the terms $u_n(x) = \begin{cases} 1, x = r_n \\ 0, x \neq r_n \end{cases}$ defined on $[0, 1]$ has the partial sums $f_n(x)$ of the above counterexample and, consequently, this series converges on $[0, 1]$, but does not converge uniformly on any subinterval of $[0, 1]$.

Remark 2. Another example of this type, albeit for a sequence of continuous on X functions $f_n(x)$, is given in Example 26 of Chapter 2.

Example 7. A series $\sum u_n(x)$ converges uniformly on an interval, but it does not converge absolutely on the same interval.

Solution

The series $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$ converges uniformly on the interval $[-a,a]$, $\forall a>0$. In fact, for any fixed $x\in\mathbb{R}$, this is an alternating series, which converges by Leibniz's test: $\lim\limits_{n\to\infty}\frac{x^2+n}{n^2}=0$ and $\frac{x^2+n}{n^2}$ is strictly decreasing in n for any fixed $x\in\mathbb{R}$. For alternating series, the remainder can be evaluated through its first term: $|r_n(x)|\le|u_{n+1}(x)|=\frac{x^2+n+1}{(n+1)^2}$. Therefore, for all $x\in[-a,a]$, we get

$$|r_n(x)|\le\frac{x^2+n+1}{(n+1)^2}\le\frac{a^2}{(n+1)^2}+\frac{1}{n+1}\underset{n\to\infty}{\to}0,$$

which implies the uniform convergence of the series on $[-a,a]$. However, the series of the absolute values $\sum_{n=1}^{\infty}|u_n(x)|=\sum_{n=1}^{\infty}\frac{x^2+n}{n^2}$ diverges for any x: for $x=0$, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is harmonic and divergent; for $\forall x\ne 0$, the series $\sum_{n=1}^{\infty}\left(\frac{x^2}{n^2}+\frac{1}{n}\right)$ is the sum of the two series—$\sum_{n=1}^{\infty}\frac{x^2}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$—where the former is convergent (p-series with $p=2$) and the latter is divergent (the harmonic series), which implies the divergence of the sum. Hence, the given series is convergent on $\mathbb{R}$, uniformly convergent on $[-a,a]$, $\forall a>0$, but it does not possess absolute convergence at any point.

Remark. The converse situation is considered in Example 8.

Example 8. A series $\sum u_n(x)$ converges absolutely on an interval, but it does not converge uniformly on the same interval.

Solution

The series $\sum_{n=0}^{\infty}(-1)^n x^n$ converges absolutely on the interval $X=[0,1)$, because for $\forall x\in[0,1)$ the series $\sum_{n=0}^{\infty}|u_n(x)|=\sum_{n=0}^{\infty}x^n$ is the geometric series with the nonnegative ratio less than 1. However, the convergence is not uniform on $[0,1)$, because for $\forall n$ we can choose $x_n=1-\frac{1}{n+1}\in[0,1)$ that gives the following evaluation:

$$|r_n(x_n)|=\left|\frac{(-1)^{n+1}x_n^{n+1}}{1+x_n}\right|=\frac{\left(1-\frac{1}{n+1}\right)^{n+1}}{1+1-\frac{1}{n+1}}\underset{n\to\infty}{\to}\frac{1}{2}e^{-1}\ne 0.$$

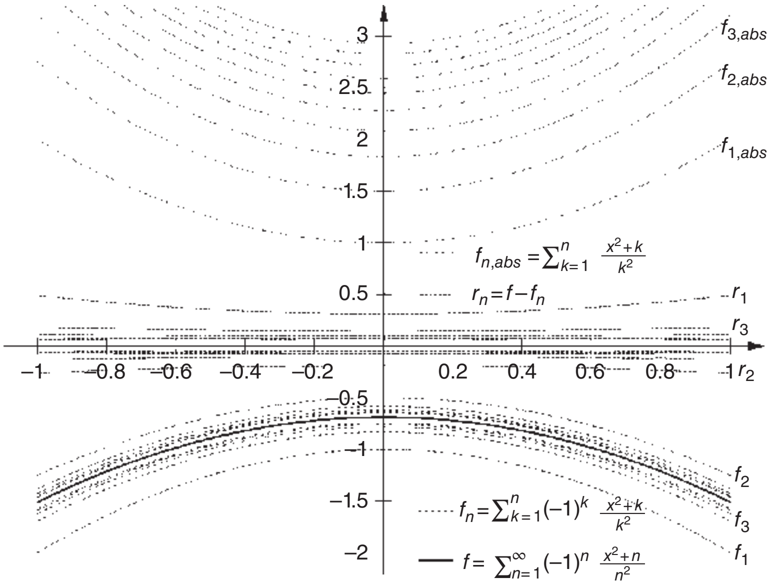

Figure 1.4 Example 7, series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$.

Example 9. A series $\sum u_n(x)$ converges absolutely and uniformly on $[a, b]$, but the series $\sum |u_n(x)|$ does not converge uniformly on $[a, b]$.

Solution

Let us consider the series $\sum_{n=0}^{\infty} u_n(x) = \sum_{n=0}^{\infty} (-1)^n (1-x)x^n$ on $[0, 1]$. If $x = 1$, then $u_n(1) = 0$ and the series converges at this point. If $x \neq 1$, then for each fixed x we have the geometric series with the ratio $q = -x$, and since $|q| = |x| < 1$, the series is convergent. The series of the absolute values $\sum_{n=0}^{\infty} |u_n(x)| = \sum_{n=0}^{\infty} (1-x)x^n$ is convergent on $[0, 1]$ for the very same reasons.

Let us analyze the uniform convergence of the original series. Since this series is alternating, we have the following evaluation for its remainder $r_n(x)$ for any n:

$$|r_n(x)| \leq |u_{n+1}(x)| = (1-x)x^{n+1}, \forall x \in [0, 1].$$

Note that the continuous function $h(x) = (1-x)x^{n+1}$ is positive on $(0, 1)$ and at the end points $h(0) = h(1) = 0$. Therefore, $h(x)$ achieves its global maximum in some interior point of $[0, 1]$. Solving the critical point equation

$$h'(x) = (n+1)x^n - (n+2)x^{n+1} = x^n[(n+1) - (n+2)x] = 0,$$

we find the only critical point $x_n = \frac{n+1}{n+2}$ on $(0, 1)$, which is the global maximum point of $h(x)$ on $[0, 1]$. Thus, for $\forall x \in [0, 1]$,

$$|r_n(x)| \le (1-x)x^{n+1} \le (1-x_n)x_n^{n+1}$$
$$= \frac{1}{n+2}\left(1 - \frac{1}{n+2}\right)^{n+1} \underset{n\to\infty}{\to} 0 \cdot e^{-1} = 0,$$

that is, the convergences is uniform on $[0, 1]$.

Finally, let us show that the series $\sum_{n=0}^{\infty} |u_n(x)| = \sum_{n=0}^{\infty} (1-x)x^n$ does not converge uniformly on $[0, 1]$. It is sufficient to show that the convergence is nonuniform on $[0, 1)$, so let us evaluate the remainder $\tilde{r}_n(x)$ for $x \in [0, 1)$:

$$\tilde{r}_n(x) = \sum_{k=n+1}^{\infty} |u_k(x)| = \frac{(1-x)x^{n+1}}{1-x} = x^{n+1}.$$

Choosing now the points $x_n = \frac{1}{\sqrt[n+1]{2}} \in [0, 1)$ for each n, we obtain $\tilde{r}_n(x_n) = \left(\frac{1}{\sqrt[n+1]{2}}\right)^{n+1} = \frac{1}{2} \underset{n\to\infty}{\nrightarrow} 0$, which shows that the convergence is nonuniform on $[0, 1)$ and, consequently, on $[0, 1]$.

Remark. The converse general statement is true: if a series $\sum_{n=0}^{\infty} |u_n(x)|$ converges uniformly on $[a, b]$, then the series $\sum_{n=0}^{\infty} u_n(x)$ converges absolutely and uniformly on $[a, b]$.

Example 10. A series $\sum u_n(x)$ converges absolutely and uniformly on X, but there is no bound of the general term $u_n(x)$ on X in the form $|u_n(x)| \le a_n$, $\forall n$ such that the series $\sum a_n$ converges.

Solution

One of the counterexamples is the series $\sum_{n=1}^{\infty} u_n(x)$ with the general term $u_n(x) = \begin{cases} 0, x \in [0, 2^{-n-1}] \cup [2^{-n}, 1] \\ \frac{1}{n}\sin^2(2^{n+1}\pi x), x \in (2^{-n-1}, 2^{-n}) \end{cases}$ defined on $X = [0, 1]$. Note that $u_n(x) \ge 0$ for $\forall x \in [0, 1]$, so the convergence and absolute convergence is the same thing for this series. At the points $x = 0$, $x = 2^{-n}$, $\forall n \in \mathbb{N}$, and for $\forall x \in \left[\frac{1}{2}, 1\right]$, we get $u_n(x) = 0$, $\forall n \in \mathbb{N}$ and the series converges to zero. If $x \in \left(0, \frac{1}{2}\right)$, $x \ne 2^{-n}$, $n \ge 2$, then each of such points lies in only one of the intervals $(2^{-n-1}, 2^{-n})$, because these intervals have no common points: $(2^{-n-1}, 2^{-n}) \cap (2^{-n-2}, 2^{-n-1}) = \emptyset$, $\forall n \in \mathbb{N}$. Therefore, there is only one n_x such that $x \in (2^{-n_x-1}, 2^{-n_x})$. Then $u_{n_x}(x) = \frac{1}{n_x}\sin^2(2^{n_x+1}\pi x)$ and $u_n(x) = 0$, $\forall n \ne n_x$ and, consequently, $\sum_{n=1}^{\infty} u_n(x) = u_{n_x}(x)$, which shows the (absolute) convergence of this series on $[0, 1]$.

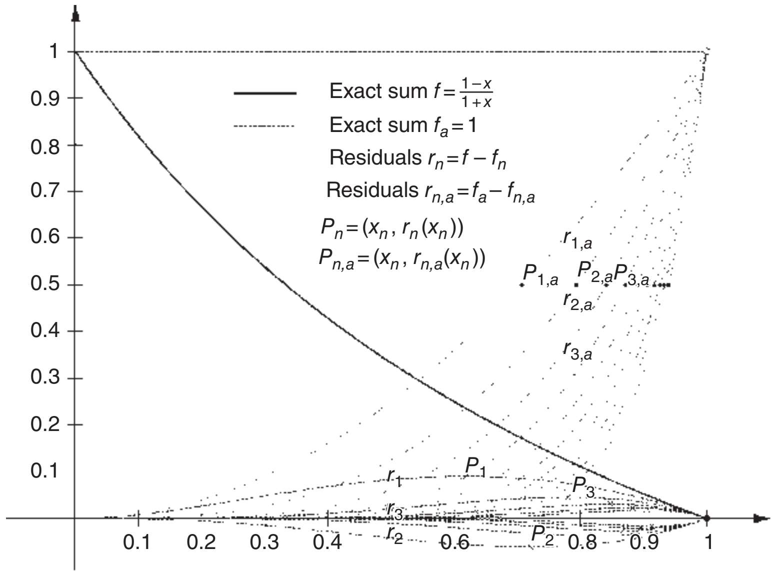

Figure 1.5 Examples 9 and 10 (second counterexample), series $\sum_{n=0}^{\infty}(-1)^n(1-x)x^n$.

Applying the Cauchy criterion and employing similar reasoning, we can also prove that the convergence is uniform. Indeed, since for any fixed $x \in [0,1]$ at most only one term in the entire series is nonzero and this term satisfies the inequality $|u_{n_x}(x)| = \left|\frac{1}{n_x}\sin^2(2^{n_x+1}\pi x)\right| \le \frac{1}{n_x}$, we obtain the following evaluation $\left|\sum_{k=n+1}^{n+p} u_k(x)\right| \le \frac{1}{n+1} < \frac{1}{n}$, which holds for $\forall n,p \in \mathbb{N}$ and simultaneously for $\forall x \in [0,1]$. Hence, for $\forall \varepsilon > 0$, there exists $N_\varepsilon = \left[\frac{1}{\varepsilon}\right]$ such that for $\forall n > N_\varepsilon$, $\forall p \in \mathbb{N}$ and simultaneously for all $x \in [0,1]$, it follows that $\left|\sum_{k=n+1}^{n+p} u_k(x)\right| < \frac{1}{n} < \varepsilon$, that is, the series converges uniformly on $[0,1]$ according to the Cauchy criterion of the uniform convergence.

Nevertheless, the functions $u_n(x)$ do not admit majoration on $[0,1]$ by the constants a_n such that the series $\sum_{n=1}^{\infty} a_n$ converges. Indeed, for $\forall n \in \mathbb{N}$, the inequality $|u_n(x)| \le \frac{1}{n}$ is exact (in the sense that $\frac{1}{n}$ is the lowest upper bound for $|u_n(x)|$) on $[0,1]$, because there exists the point $x_n = 3 \cdot 2^{-n-2} \in (2^{-n-1}, 2^{-n})$ such that $u_n(x_n) = \frac{1}{n}\sin^2\left(\frac{3}{2}\pi\right) = \frac{1}{n}$, and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Another interesting counterexample is the series of Example 9: $\sum_{n=0}^{\infty} u_n(x) = \sum_{n=0}^{\infty}(-1)^n(1-x)x^n$ on $[0,1]$. It was shown in Example 9 that this series converges absolutely and uniformly on $[0,1]$. For each fixed n, the function $|u_n(x)| > 0$, $\forall x \in (0,1)$, and $|u_n(0)| = |u_n(1)| = 0$. Therefore, the continuous function $|u_n(x)|$ achieves its global maximum in an interior point of $[0,1]$,

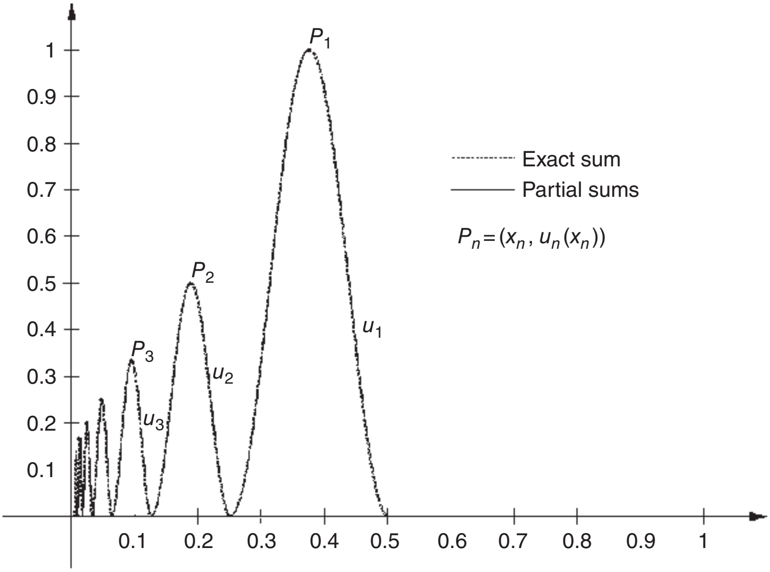

Figure 1.6 Example 10, series $\sum_{n=1}^{\infty} u_n(x)$, $u_n(x) = \begin{cases} 0, x \in [0, 2^{-n-1}] \cup [2^{-n}, 1] \\ \frac{1}{n}\sin^2(2^{n+1}\pi x), x \in (2^{-n-1}, 2^{-n}) \end{cases}$.

which can be found by solving the critical point equation:

$$|u_n(x)|' = (x^n - x^{n+1})' = nx^{n-1} - (n+1)x^n$$
$$= (n+1)x^{n-1}\left(\frac{n}{n+1} - x\right) = 0.$$

The unique solution on $(0, 1)$ is $x_n = \frac{n}{n+1}$ and, consequently,

$$|u_n(x)| \le \max_{[0,1]} |u_n(x)| = |u_n(x_n)| = \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n.$$

Since $\lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)^n = e^{-1}$ and the series $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges, according to the comparison test, the series $\sum_{n=0}^{\infty} \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n$ also diverges. Note that for each n, the majorant term $a_n = \frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^n$ is exact (i.e., the minimum possible) for $|u_n(x)|$ on $[0, 1]$. Therefore, there is no convergent majorant series $\sum_{n=0}^{\infty} a_n$ such that $|u_n(x)| \le a_n$.

Remark. The converse general statement is true and represents the famous Weierstrass M-test.

1.2 Uniform Convergence of Sequences and Series of Squares and Products

Example 11. A sequence $f_n(x)$ converges uniformly on X to a function $f(x)$, but $f_n^2(x)$ does not converge uniformly on X to $f^2(x)$.

Solution

The sequence $f_n(x) = \ln\frac{nx}{n+1}$ converges on $X = (0, +\infty)$ to $f(x) = \ln x$, because $\lim\limits_{n\to\infty} \ln\frac{nx}{n+1} = \ln\left(\lim\limits_{n\to\infty}\frac{n}{n+1}\right)x = \ln x$. The following evaluation shows that this convergence is uniform:

$$|f_n(x) - f(x)| = \left|\ln\frac{nx}{n+1} - \ln x\right| = \left|\ln\frac{n}{n+1}\right| < \varepsilon.$$

So for $\forall\varepsilon > 0$, there exists $N_\varepsilon = \left[\frac{1}{e^\varepsilon - 1}\right]$, which depends only on ε, such that for $\forall n > N_\varepsilon$ and simultaneously for all $x \in X$ we have $|f_n(x) - f(x)| < \varepsilon$.

Due to arithmetic properties of the limits, the sequence $f_n^2(x)$ also converges to $f^2(x) = \ln^2 x$ for any fixed $x \in X$ (it can also be shown directly: $\lim\limits_{n\to\infty}\ln^2\frac{nx}{n+1} = \left(\ln\left(\lim\limits_{n\to\infty}\frac{n}{n+1}\right)x\right)^2 = \ln^2 x$). However, this convergence is not uniform. In fact, for each $x \in X$, we get

$$\begin{aligned}|f_n^2(x) - f^2(x)| &= \left|\ln^2\frac{nx}{n+1} - \ln^2 x\right| \\ &= \left|\ln\frac{nx}{n+1} - \ln x\right| \cdot \left|\ln\frac{nx}{n+1} + \ln x\right| \\ &= \ln\frac{n+1}{n} \cdot \left|\ln\frac{nx^2}{n+1}\right|.\end{aligned}$$

Choosing now $x_n = \frac{1}{ne^n} \in X$, we obtain

$$\begin{aligned}|f_n^2(x_n) - f^2(x_n)| &= \ln\frac{n+1}{n} \cdot \left|\ln\frac{n}{(n+1)n^2e^{2n}}\right| \\ &= \ln\frac{n+1}{n} \cdot \ln(n(n+1)e^{2n}) \\ &= \ln\frac{n+1}{n} \cdot \ln n(n+1) + 2n\ln\frac{n+1}{n} > 1\end{aligned}$$

for sufficiently large n (since the first term is positive and the limit of the second equals two: $\lim\limits_{n\to\infty} 2n\ln\frac{n+1}{n} = \lim\limits_{n\to\infty} 2\ln\left(1 + \frac{1}{n}\right)^n = 2\ln e = 2$, that is, $2n\ln\frac{n+1}{n} > 1$ for large n). Therefore, the convergence is not uniform: for $\varepsilon_0 = 1$ whatever N is chosen, it can be found that $\tilde{n} > N$ and corresponding $x_{\tilde{n}} \in X$ such that $|f_{\tilde{n}}^2(x_{\tilde{n}}) - f^2(x_{\tilde{n}})| > \varepsilon_0 = 1$.

Remark 1. Naturally, the following example can also be constructed: sequences $f_n(x)$ and $g_n(x)$ converge uniformly on X to $f(x)$ and $g(x)$, respectively, but $f_n(x)g_n(x)$ does not converge uniformly on X to $f(x)g(x)$. In the case $f_n(x) = g_n(x)$, we have the original example with the square of function. For different sequences, we can use the same $f_n(x) = \ln\frac{nx}{n+1}$ and slightly different $g_n(x) = \ln\frac{nx}{2n+5}$. The sequence $g_n(x)$ converges to $g(x) = \ln\frac{x}{2}$, and this convergence is uniform on $X = (0, +\infty)$ due to the evaluation

$$|g_n(x) - g(x)| = \left|\ln\frac{nx}{2n+5} - \ln\frac{x}{2}\right| = \left|\ln\frac{2n}{2n+5}\right| = \ln\left(1+\frac{5}{2n}\right) \underset{n\to\infty}{\to} 0.$$

Consequently, $f_n(x)g_n(x)$ converges to $f(x)g(x) = \ln x \ln\frac{x}{2}$ for each fixed $x \in X$ due to arithmetic rules of the limits. However, this convergence is not uniform on X, as it is shown below: for each $x \in X$, we have

$$\begin{aligned}
|f_n(x)g_n(x) - f(x)g(x)| &= \left|\ln\frac{nx}{n+1}\ln\frac{nx}{2n+5} - \ln x\ln\frac{x}{2}\right| \\
&= \left|\ln\frac{nx}{n+1}\ln\frac{nx}{2n+5} - \ln\frac{nx}{n+1}\ln\frac{x}{2} \right. \\
&\quad \left. + \ln\frac{nx}{n+1}\ln\frac{x}{2} - \ln x\ln\frac{x}{2}\right| \\
&= \left|\ln\frac{nx}{n+1}\ln\frac{2n}{2n+5} + \ln\frac{x}{2}\ln\frac{n}{n+1}\right|,
\end{aligned}$$

and for the special choice of the points $x_n = \frac{1}{ne^n} \in X$, we obtain

$$\begin{aligned}
|f_n(x_n)g_n(x_n) - f(x_n)g(x_n)| &= \left|\ln\frac{1}{(n+1)e^n}\ln\frac{2n}{2n+5} + \ln\frac{1}{2ne^n}\ln\frac{n}{n+1}\right| \\
&= \left|(n+\ln(n+1))\ln\left(1+\frac{5}{2n}\right) + (n+\ln 2n)\ln\left(1+\frac{1}{n}\right)\right| \\
&= \frac{5}{2}\cdot\frac{2n}{5}\ln\left(1+\frac{5}{2n}\right) + \frac{\ln(n+1)}{2n/5}\frac{2n}{5}\ln\left(1+\frac{5}{2n}\right) \\
&\quad + n\ln\left(1+\frac{1}{n}\right) + \frac{\ln 2n}{n}n\ln\left(1+\frac{1}{n}\right) \underset{n\to\infty}{\to} \frac{5}{2}\cdot 1 \\
&\quad + 0\cdot 1 + 1 + 0\cdot 1 = \frac{7}{2}.
\end{aligned}$$

In the evaluation of the last limit, we have used the following auxiliary limits:

$$\lim_{n\to\infty}\frac{n}{\alpha}\ln\left(1+\frac{\alpha}{n}\right) = \lim_{n\to\infty}\ln\left(1+\frac{\alpha}{n}\right)^{n/\alpha} = \ln e = 1, \quad \forall\alpha \neq 0$$

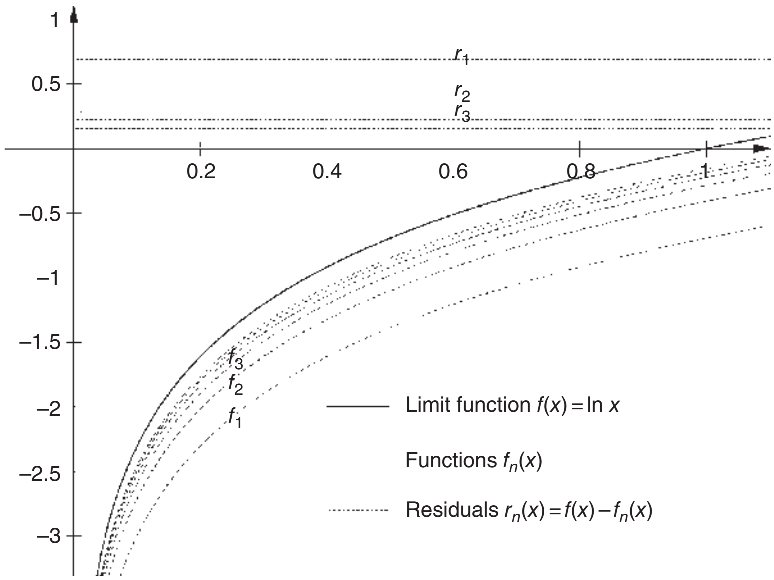

Figure 1.7 Examples 11 and 17, sequence $f_n(x) = \ln \frac{nx}{n+1}$.

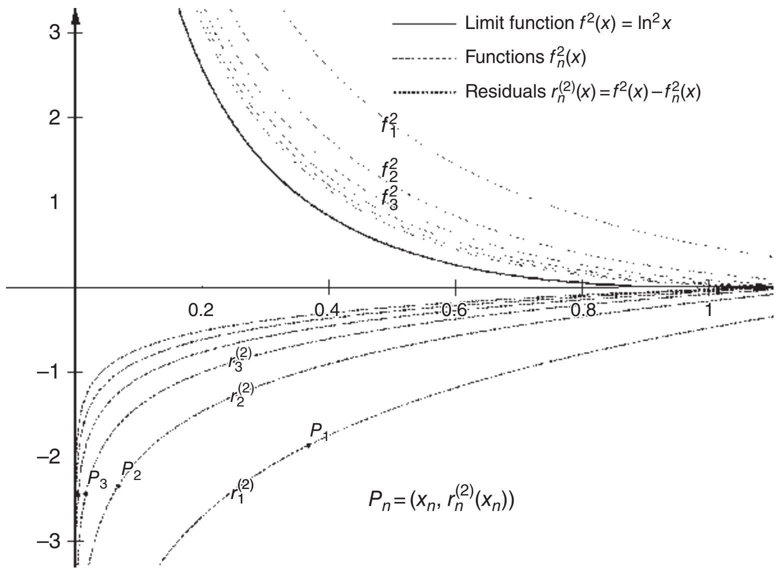

Figure 1.8 Example 11, sequence of squares $f_n^2(x) = \ln^2 \frac{nx}{n+1}$.

according to the second remarkable limit, and

$$\lim_{t\to\infty}\frac{\ln(\alpha t+\beta)}{t}=\lim_{t\to\infty}\frac{\alpha/(\alpha t+\beta)}{1}=0,\qquad \forall\alpha>0,\forall\beta$$

due to l'Hospital's rule. Therefore, for $\varepsilon_0=1$ whatever N is chosen, there is $\tilde{n}>N$ and corresponding $x_{\tilde{n}}\in X$ such that $|f_{\tilde{n}}(x_{\tilde{n}})g_{\tilde{n}}(x_{\tilde{n}})-f(x_{\tilde{n}})g(x_{\tilde{n}})|>\varepsilon_0=1$, that is, the convergence is nonuniform.

Remark 2. The following general statement is true for the sum and difference: if $f_n(x)$ and $g_n(x)$ converge uniformly on X to $f(x)$ and $g(x)$, respectively, then $f_n(x)\pm g_n(x)$ converges uniformly on X to $f(x)\pm g(x)$.

Remark 3. The following general statement is true for the product: if $f_n(x)$ and $g_n(x)$ converge uniformly on X to $f(x)$ and $g(x)$, respectively, and $f(x)$ and $g(x)$ are bounded on X, then $f_n(x)\cdot g_n(x)$ converges uniformly on X to $f(x)\cdot g(x)$. (Note the requirement of boundedness of the limit functions in this formulation.)

Example 12. Sequences $f_n(x)$ and $g_n(x)$ converge nonuniformly on X to $f(x)$ and $g(x)$, respectively, but $f_n(x)\cdot g_n(x)$ converges to $f(x)\cdot g(x)$ uniformly on X.

Solution

Consider the sequences $f_n(x)=\frac{1}{n\sqrt{x}}$ and $g_n(x)=nxe^{-nx}$ on $X=(0,+\infty)$. Both sequences converge to 0 for any fixed $x\in X$:

$$\lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{1}{n\sqrt{x}}=0=f(x);$$

$$\lim_{n\to\infty}g_n(x)=\lim_{n\to\infty}nxe^{-nx}=\lim_{t\to+\infty}\frac{t}{e^t}=\lim_{t\to+\infty}\frac{1}{e^t}=0=g(x).$$

Therefore, $\lim\limits_{n\to\infty}f_n(x)\cdot g_n(x)=0$.

Let us investigate the nature of the convergence of these sequences. For $f_n(x)$, choosing $x_n=\frac{1}{n^2}\in X$, we obtain

$$|f_n(x_n)-f(x_n)|=\frac{1}{n\sqrt{x_n}}=\frac{n}{n}=1\underset{n\to\infty}{\nrightarrow}0,$$

that is, this convergence is nonuniform on X. Similarly, choosing $x_n=\frac{1}{n}\in X$, we can show the nonuniform convergence of $g_n(x)$ on X:

$$|g_n(x_n)-g(x_n)|=nx_ne^{-nx_n}=1\cdot e^{-1}\underset{n\to\infty}{\nrightarrow}0.$$

Finally, for $f_n(x) \cdot g_n(x)$, we have $|f_n(x)g_n(x) - f(x)g(x)| = \sqrt{x}e^{-nx}$. The derivative of the right-hand side is

$$(\sqrt{x}e^{-nx})_x = \left(\frac{1}{2\sqrt{x}} - n\sqrt{x}\right)e^{-nx} = \frac{n}{\sqrt{x}}e^{-nx}\left(\frac{1}{2n} - x\right).$$

Therefore, the point $x_n = \frac{1}{2n} \in X$ is the only local (and global) maximum of $\sqrt{x}e^{-nx}$ on X. Consequently,

$$|f_n(x)g_n(x) - f(x)g(x)| \le \max_{(0,+\infty)} \sqrt{x}e^{-nx} = \sqrt{x_n}e^{-nx_n} = \frac{1}{\sqrt{2n}}e^{-1/2} \underset{n\to\infty}{\to} 0,$$

that is, $f_n(x) \cdot g_n(x)$ converges uniformly on X to 0.

Example 13. A sequence $f_n^2(x)$ converges uniformly on X, but $f_n(x)$ diverges on X.

Solution

Consider the sequence $f_n(x) = (-1)^n \frac{n+1}{n} x$ on $X = (0, 1]$. The sequence of squares $f_n^2(x) = \frac{(n+1)^2}{n^2} x^2$ converges uniformly on $(0, 1]$ to x^2, because

$$|f_n^2(x) - x^2| = \left|\frac{(n+1)^2}{n^2}x^2 - x^2\right| = x^2\frac{2n+1}{n^2} \le \frac{2}{n} + \frac{1}{n^2} \underset{n\to\infty}{\to} 0.$$

However, there is no limit of $f_n(x)$ for any fixed $x \in (0, 1]$, since two partial limits give different results: $f_{2n}(x) = \frac{2n+1}{2n}x \underset{n\to\infty}{\to} x$ and $f_{2n+1}(x) = -\frac{2n+2}{2n+1}x \underset{n\to\infty}{\to} -x$.

Remark 1. The same sequence can be used to exemplify the following situation: a sequence $|f_n(x)|$ converges uniformly on X, but $f_n(x)$ diverges on X. Indeed, although $f_n(x) = (-1)^n \frac{n+1}{n} x$ does not converge on $X = (0, 1]$, the sequence $|f_n(x)| = \frac{n+1}{n}x$ converges uniformly to x on $X = (0, 1]$:

$$||f_n(x)| - x| = \left|\frac{n+1}{n}x - x\right| = \frac{1}{n}|x| \le \frac{1}{n} \underset{n\to\infty}{\to} 0.$$

Remark 2. Note that the inequality $||f_n(x)| - |f(x)|| \le |f_n(x) - f(x)|$ ensures the validity of the converse general statement: if $f_n(x)$ converges uniformly on X to $f(x)$, then $|f_n(x)|$ converges uniformly on X to $|f(x)|$.

Remark 3. The following example also takes place: a sequence $f_n^2(x)$ converges uniformly on X and $f_n(x)$ converges on X, but the convergence of $f_n(x)$ is nonuniform. Consider the sequence $f_n(x)$ on $[0, 1]$ similar to that analyzed in Example 6: $f_n(x) = \begin{cases} 1, x = r_1, r_2, \cdots, r_n \\ -1, \text{otherwise} \end{cases}$, where r_n is the sequence of all the rational points in $[0,1]$ ordered in some way. This sequence is monotone in n for any fixed $x \in [0,1]$ (since $f_n(r_{n+1}) = -1 < 1 = f_{n+1}(r_{n+1})$ and $f_n(x) = f_{n+1}(x)$, $\forall x \neq r_{n+1}$) and bounded (since $-1 \leq f_n(x) \leq 1$, $\forall n \in \mathbb{N}$, $\forall x \in [0,1]$). Therefore, this sequence is convergent at any fixed $x \in [0,1]$ and $f(x) = \lim\limits_{n\to\infty} f_n(x) = \begin{cases} 1, x \in \mathbb{Q} \\ -1, x \in \mathbb{I} \end{cases}$. The sequence of the squares consists of the same constant function $f_n^2(x) = 1$, $\forall n \in \mathbb{N}$ and, therefore, it converges uniformly on $[0,1]$ to $f^2(x) = 1$. At the same time, using the same reasoning as in Example 6, one can show that the convergence of $f_n(x)$ is nonuniform on $[0,1]$ and on any subinterval of $[0,1]$.

Example 14. A sequence $f_n(x) \cdot g_n(x)$ converges uniformly on X to 0, but neither $f_n(x)$ nor $g_n(x)$ converges to 0 on X.

Solution

The sequences $f_n(x) = nx + (-1)^n nx$ and $g_n(x) = nx - (-1)^n nx$ are divergent for every fixed $x \in X = (0, +\infty)$:

$$f_{2n}(x) = 4nx \underset{n\to\infty}{\to} +\infty, f_{2n+1}(x) = 0 \underset{n\to\infty}{\to} 0;$$
$$g_{2n}(x) = 0 \underset{n\to\infty}{\to} 0, g_{2n+1}(x) = (4n+2)x \underset{n\to\infty}{\to} +\infty.$$

However, $f_n(x) \cdot g_n(x) = n^2x^2 - n^2x^2 = 0$ converges uniformly to 0 on X.

Another interesting counterexample includes the sequences $f_n(x) = \begin{cases} \frac{nx}{n+1}, x \in \mathbb{Q} \cap X \\ \frac{x}{n}, x \in \mathbb{I} \cap X \end{cases}$ and $g_n(x) = \begin{cases} \frac{x}{n}, x \in \mathbb{Q} \cap X \\ \frac{nx}{n+1}, x \in \mathbb{I} \cap X \end{cases}$ defined on $X = (0,1]$. Both sequences converge on X to nonzero functions $f(x) = \lim\limits_{n\to\infty} f_n(x) = \begin{cases} x, x \in \mathbb{Q} \cap X \\ 0, x \in \mathbb{I} \cap X \end{cases}$ and $g(x) = \lim\limits_{n\to\infty} g_n(x) = \begin{cases} 0, x \in \mathbb{Q} \cap X \\ x, x \in \mathbb{I} \cap X \end{cases}$, respectively. At the same time, $\lim\limits_{n\to\infty} f_n(x)g_n(x) = \lim\limits_{n\to\infty} \frac{x^2}{n+1} = 0$, for $\forall x \in (0,1]$ and this convergence is uniform on X, since the estimate

$$|f_n(x)g_n(x) - 0| = \frac{x^2}{n+1} \leq \frac{1}{n+1} \underset{n\to\infty}{\to} 0$$

holds simultaneously for all $x \in (0,1]$.

Example 15. A sequence $f_n(x)$ converges uniformly on X to a function $f(x)$, $f_n(x) \neq 0, f(x) \neq 0$, $\forall x \in X$, but $\frac{1}{f_n(x)}$ does not converge uniformly on X to $\frac{1}{f(x)}$.

Solution

The sequence $f_n(x) = \frac{nx}{n+2}$ converges uniformly on $X = (0,1)$ to $f(x) = x$:

$$|f_n(x) - f(x)| = \left| \frac{nx}{n+2} - x \right| = \frac{2}{n+2}|x| < \frac{2}{n+2} < \varepsilon$$

and the last inequality holds for $\forall \varepsilon > 0$ and simultaneously for all $x \in X$ if we choose $\forall n > N_\varepsilon = \left[\frac{2}{\varepsilon}\right]$. On the other hand,

$$\left|\frac{1}{f_n(x)} - \frac{1}{f(x)}\right| = \left|\frac{n+2}{nx} - \frac{1}{x}\right| = \frac{2}{n}\frac{1}{|x|},$$

and for $x_n = \frac{1}{n} \in X$, it follows that $\left|\frac{1}{f_n(x_n)} - \frac{1}{f(x_n)}\right| = n\frac{2}{n} = 2$, which means that the convergence is nonuniform.

Example 16. A sequence $f_n(x)$ is bounded uniformly on $\mathbb{R}$ and converges uniformly on $[-a, a]$, $\forall a > 0$, to a function $f(x)$, but the numerical sequence $\sup\limits_{x\in\mathbb{R}} f_n(x)$ does not converge to $\sup\limits_{x\in\mathbb{R}} f(x)$.

Solution

Consider the sequence $f_n(x) = e^{-(x-n)^2}$, which is defined and uniformly bounded on $\mathbb{R}$: $0 < e^{-(x-n)^2} \le 1$, $\forall n$, $\forall x \in \mathbb{R}$. This sequence converges to zero on $\mathbb{R}$, since for any fixed $x \in \mathbb{R}$ one has $(x-n)^2 \underset{n\to\infty}{\to} +\infty$ and, consequently, $\lim\limits_{n\to\infty} e^{-(x-n)^2} = \lim\limits_{t\to+\infty} e^{-t} = 0$. Hence, $f(x) \equiv 0$ on $\mathbb{R}$ and, consequently, $\sup\limits_{x\in\mathbb{R}} f(x) = 0$. On the other hand, $\sup\limits_{x\in\mathbb{R}} f_n(x) = 1$, $\forall n \in \mathbb{N}$, since $f_n(x) \le 1$ and $f_n(n) = 1$. This means that $\sup\limits_{x\in\mathbb{R}} f_n(x) = 1$ does not converge to $\sup\limits_{x\in\mathbb{R}} f(x) = 0$. It just remains to prove the uniform convergence of $f_n(x)$ on $[-a, a]$, $\forall a > 0$. For any fixed $a > 0$, there exists the natural number $N_a > a$. Then for $\forall n > N_a$, one gets $(x-n)^2 \ge (a-n)^2$ for each $x \in [-a, a]$. Therefore,

$$|f_n(x) - f(x)| = e^{-(x-n)^2} \le e^{-(a-n)^2}$$

for all $x \in [-a, a]$. Since $\exp\left(-(a-n)^2\right) \underset{n\to\infty}{\to} 0$, the last inequality guarantees the uniformity of the convergence on $[-a, a]$, where $a > 0$ is arbitrary. Note, however, that the convergence of $f_n(x)$ is not uniform on $\mathbb{R}$, which is evident if one chooses $x_n = n$ leading to

$$|f_n(x_n) - f(x_n)| = e^{-(n-n)^2} = 1.$$

Remark 1. Two other interesting counterexamples are $f_n(x) = \arctan\frac{x}{n}$ and $f_n(x) = \frac{2nx}{n^2+x^2}$. For instance, for the first function the reasoning can be as follows. First, note that the sequence is uniformly bounded on $\mathbb{R}$ ($\left|\arctan\frac{x}{n}\right| < \frac{\pi}{2}$, $\forall n \in \mathbb{N}$ and $\forall x \in \mathbb{R}$). Second, it converges to zero for any fixed $x \in \mathbb{R}$ ($\lim\limits_{n\to\infty} \arctan\frac{x}{n} = 0$). Further, this convergence is uniform on $[-a, a]$, $\forall a > 0$ due to the evaluation

$$|f_n(x) - f(x)| = \left|\arctan\frac{x}{n}\right| = \arctan\frac{|x|}{n} \le \arctan\frac{a}{n},$$

that holds for all $x \in [-a, a]$ (here we used the properties that $\arctan t$ is an odd and a strictly increasing function). Since $\lim\limits_{n\to\infty} \arctan \frac{a}{n} = 0$, the last evaluation implies the uniform convergence. Hence, all the conditions of the example are satisfied, but still the sequence $\sup\limits_{x\in\mathbb{R}} f_n(x)$ does not converge to $\sup\limits_{x\in\mathbb{R}} f(x)$, because $\sup\limits_{x\in\mathbb{R}} \arctan \frac{x}{n} = \frac{\pi}{2}$ for any n, while $\sup\limits_{x\in\mathbb{R}} f(x) = \sup\limits_{x\in\mathbb{R}} 0 = 0$. Note, that just like in the first counterexample, the convergence of $f_n(x)$ is not uniform on $\mathbb{R}$: for any n one can choose $x_n = n$ to obtain

$$|f_n(x_n) - f(x_n)| = \arctan \frac{n}{n} = \arctan 1 = \frac{\pi}{4} \neq 0.$$

Remark 2. The following general statement is true: if a sequence $f_n(x)$ is bounded uniformly on $\mathbb{R}$ and converges uniformly on $\mathbb{R}$ to a function $f(x)$, then the numerical sequence $\sup\limits_{x\in\mathbb{R}} f_n(x)$ converges to $\sup\limits_{x\in\mathbb{R}} f(x)$. (Note the requirement of uniform convergence on $\mathbb{R}$ to the limit function in this formulation.)

Remark 3. The condition of uniform convergence of a sequence $f_n(x)$ to a function $f(x)$ on X is equivalent to the condition $\lim\limits_{n\to\infty} \sup\limits_{x\in X} |f_n(x) - f(x)| = 0$.

Example 17. Suppose each function $f_n(x)$ maps X on Y and function $g(y)$ is continuous on Y; the sequence $f_n(x)$ converges uniformly on X, but the sequence $g_n(x) = g(f_n(x))$ does not converge uniformly on X.

Solution

Let us consider the sequence $f_n(x) = \ln \frac{nx}{n+1}$ on $X = (0, +\infty)$ and function $g(y) = e^y$. Each of the functions $f_n(x)$ maps $(0, +\infty)$ on the entire real line and the function $g(y)$ is continuous on $\mathbb{R}$. In Example 11, it was shown that the sequence $f_n(x)$ converges uniformly on X to the function $f(x) = \ln x$. The corresponding sequence $g_n(x) = g(f_n(x)) = \frac{nx}{n+1}$ converges for any fixed $x \in (0, +\infty)$:

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \frac{nx}{n+1} = x = g(f(x)),$$

but this convergence is not uniform. In fact,

$$|g_n(x) - g(f(x))| = \left| \frac{nx}{n+1} - x \right| = \frac{1}{n+1} x$$

and choosing $x_n = (n+1) \in X$, one gets

$$|g_n(x_n) - g(f(x_n))| = \frac{x_n}{n+1} = 1.$$

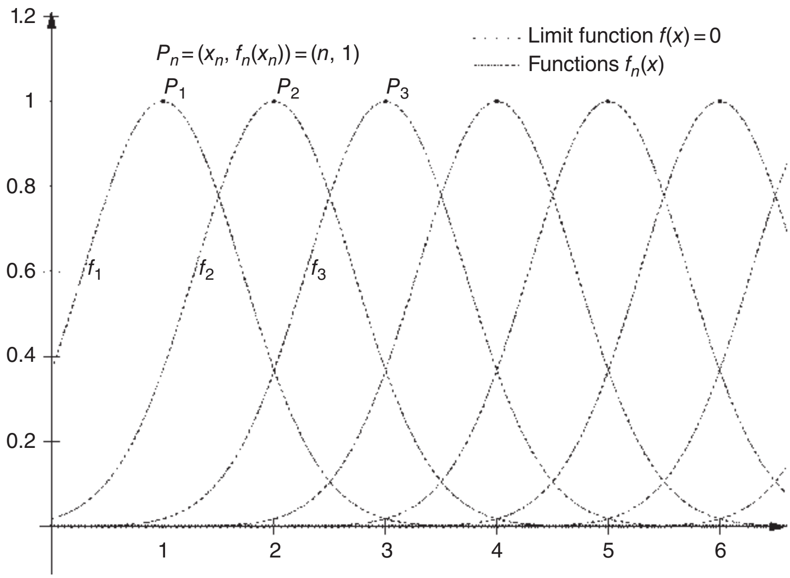

Figure 1.9 Example 16, sequence $f_n(x) = e^{-(x-n)^2}$.

Remark. The following example also takes place: suppose functions $f_n(x)$ map X on Y and function $g(y)$ is continuous on Y; the sequence $f_n(x)$ converges nonuniformly on X, but the sequence $g_n(x) = g(f_n(x))$ converges uniformly on X. In the trivial case, one can use an arbitrary nonuniformly convergent sequence $f_n(x)$ and the constant function $g(y) \equiv 1$. For a nonconstant function $g(y)$, one can use the above sequence $f_n(x) = \frac{nx}{n+1}$ defined on $X = (0, +\infty)$ and the function $g(y) = \ln y$.

Example 18. A series $\sum u_n^2(x)$ converges uniformly on X, but the series $\sum u_n(x)$ does not converge uniformly on X.

Solution

In Remark 1 to Example 5, it was shown that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges nonuniformly on $\mathbb{R}$. However, the series $\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^2}$ converges uniformly on $\mathbb{R}$ according to the Weierstrass test: $\frac{\sin^2 nx}{n^2} \le \frac{1}{n^2}$, $\forall x \in \mathbb{R}$ and the majorant series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example 19. A series $\sum u_n^2(x)$ converges uniformly on X, but the series $\sum u_n(x)$ does not converge (even pointwise) on X.

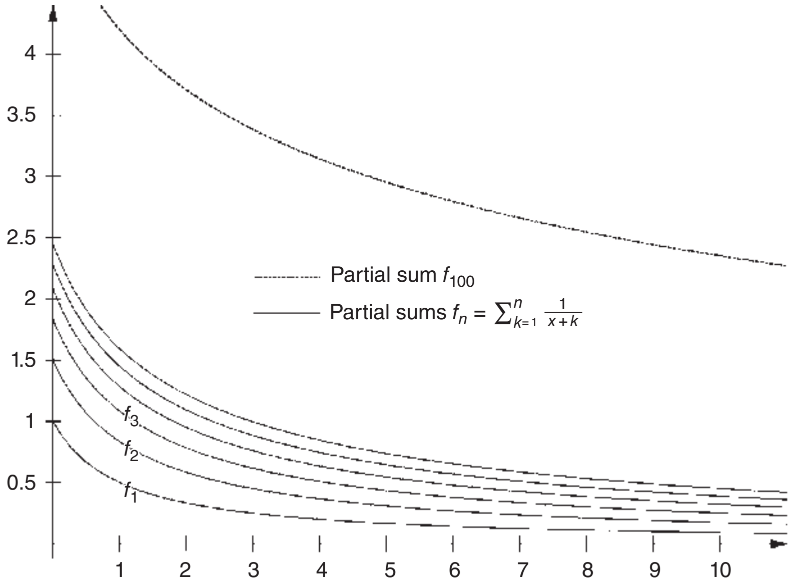

Figure 1.10 Examples 19, 20, and 21, series $\sum_{n=1}^{\infty} \frac{1}{x+n}$.

Solution

Consider the series $\sum_{n=1}^{\infty} \frac{1}{x+n}$ on $X = [0, +\infty)$. This series is divergent at each point $x \in X$, since $\lim_{n\to\infty} \frac{1/n}{1/(x+n)} = 1$ and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. At the same time, the series $\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$ converges uniformly on $X = [0, +\infty)$ due to the Weierstrass test: $\frac{1}{(x+n)^2} \le \frac{1}{n^2}$, $\forall x \in X$ and the majorant series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Remark. If $\sum u_n(x)$ diverges, it may happen that $\sum_{n=1}^{\infty} u_n^2(x)$ also diverges. For instance, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{x}+\sqrt{n}}$ diverges at each point $x \in X = [0, 1]$ according to the comparison test: $\lim_{n\to\infty} \frac{1/\sqrt{n}}{1/(\sqrt{x}+\sqrt{n})} = 1$ and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent. Due to the very same arguments, the series of squares also diverges on $X = [0, 1]$: $\lim_{n\to\infty} \frac{1/n}{1/(\sqrt{x}+\sqrt{n})^2} = 1$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Example 20. A series $\sum u_n(x)v_n(x)$ converges uniformly on X, but at least one of the series $\sum u_n(x)$ or $\sum v_n(x)$ does not converge uniformly on X.

Solution

Consider $u_n(x) = \frac{1}{x+n}$ and $v_n(x) = \frac{1}{x^2+n^2}$ on $X = [0, +\infty)$. The series $\sum_{n=1}^{\infty} u_n(x)v_n(x)$ converges uniformly on X due to the Weierstrass test:

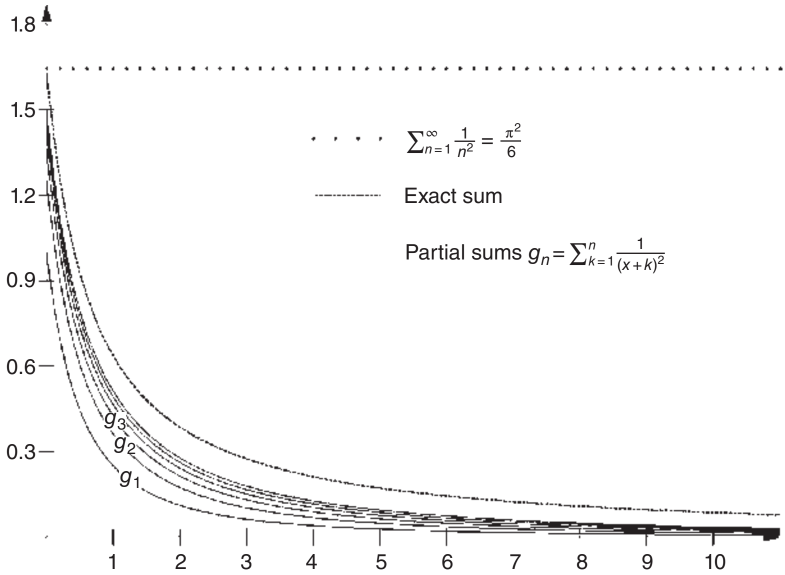

Figure 1.11 Example 19, series of squares $\sum_{n=1}^{\infty} \frac{1}{(x+n)^2}$.

$$|u_n(x)v_n(x)| = \frac{1}{x+n}\frac{1}{x^2+n^2} \le \frac{1}{n^3}, \quad \forall x \in [0, +\infty)$$

and $\sum \frac{1}{n^3}$ is a convergent series. The same reasoning shows the uniform convergence of the series $\sum_{n=1}^{\infty} v_n(x)$ on X:

$$|v_n(x)| = \frac{1}{x^2+n^2} \le \frac{1}{n^2}, \quad \forall x \in [0, +\infty)$$

and $\sum \frac{1}{n^2}$ is a convergent series. However, the series $\sum_{n=1}^{\infty} u_n(x)$ diverges on X, since $\lim\limits_{n\to\infty} \frac{1/n}{1/(x+n)} = 1$ and the series $\sum \frac{1}{n}$ is divergent.

Example 21. A series $\sum u_n(x)v_n(x)$ converges uniformly on X, but neither $\sum u_n(x)$ nor $\sum v_n(x)$ converges (even pointwise) on X.

Solution
Consider $u_n(x) = \frac{1}{x+n}$ and $v_n(x) = \frac{1}{\sqrt{x}+\sqrt{n}}$ on $X = [0, +\infty)$. The series $\sum_{n=1}^{\infty} u_n(x)v_n(x)$ converges uniformly on X due to the Weierstrass test:

$$|u_n(x)v_n(x)| = \frac{1}{x+n}\frac{1}{\sqrt{x}+\sqrt{n}} \le \frac{1}{n^{3/2}}, \quad \forall x \in [0, +\infty)$$

and $\sum \frac{1}{n^{3/2}}$ is a convergent series. However, both $\sum_{n=1}^{\infty} u_n(x)$ and $\sum_{n=1}^{\infty} v_n(x)$ diverge on X according to the comparison test: $\lim\limits_{n\to\infty} \frac{1/n}{1/(x+n)} = 1$ and the series $\sum \frac{1}{n}$ diverges; $\lim\limits_{n\to\infty} \frac{1/\sqrt{n}}{1/(\sqrt{x}+\sqrt{n})} = 1$ and the series $\sum \frac{1}{\sqrt{n}}$ diverges.

Example 22. Series $\sum u_n(x)$ and $\sum v_n(x)$ converge nonuniformly on X, but $\sum u_n(x)v_n(x)$ converges uniformly on X.

Solution
The series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges nonuniformly on $\mathbb{R}$ (see Remark 1 to Example 5), and so does the series $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ (applying for the latter series the same reasoning as for the former in Remark 1 to Example 5). However, the series $\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^{3/2}}$ converges uniformly on $\mathbb{R}$ according to the Weierstrass test: $\frac{\sin^2 nx}{n^{3/2}} \le \frac{1}{n^{3/2}}$, $\forall x \in \mathbb{R}$ and the majorant series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

Example 23. A series $\sum u_n(x)$ converges uniformly on X, but $\sum u_n^2(x)$ does not converge uniformly on X.

Solution
The uniform convergence of the series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt[3]{n}}$ on $X = (0,1)$ can be proved by applying Abel's theorem. In fact, the series $\sum (-1)^n \frac{1}{\sqrt[3]{n}}$ converges by Leibniz's test of alternating series (and this convergence is uniform on X, since the series does not depend on x), and the sequence x^n is monotone in n for each $x \in (0,1)$ and is uniformly bounded, since $x^n < 1$, $\forall x \in (0,1)$, $\forall n \in \mathbb{N}$.

On the other hand, the convergence of the series $\sum_{n=1}^{\infty} u_n^2(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^{2/3}}$ is nonuniform on $X = (0,1)$. In fact, this series converges on $X = (0,1)$, since $u_n^2(x) = \frac{x^{2n}}{n^{2/3}} \le x^{2n}$, $\forall n \in \mathbb{N}$, and the series $\sum x^{2n}$ is convergent by being a geometric series with the ratio in $(0,1)$. At the same time, applying the Cauchy criterion of the uniform convergence with $p_n = n$ and $x_n = \left(1 - \frac{1}{4n}\right) \in X$, one obtains

$$\left|\sum_{k=n+1}^{n+p_n} u_k^2(x_n)\right| = \sum_{k=n+1}^{n+p_n} \frac{x_n^{2k}}{k^{2/3}} > n\frac{x_n^{4n}}{(2n)^{2/3}} = \frac{n^{1/3}}{4^{1/3}}\left(1 - \frac{1}{4n}\right)^{4n} \underset{n\to\infty}{\to} +\infty,$$

that is, the series $\sum u_n^2(x)$ converges nonuniformly on $X = (0,1)$.

Remark 1. If, in the given counterexample, one changes the set X to $(0,1]$, then $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt[3]{n}}$ converges uniformly on $X = (0,1]$ (due to the

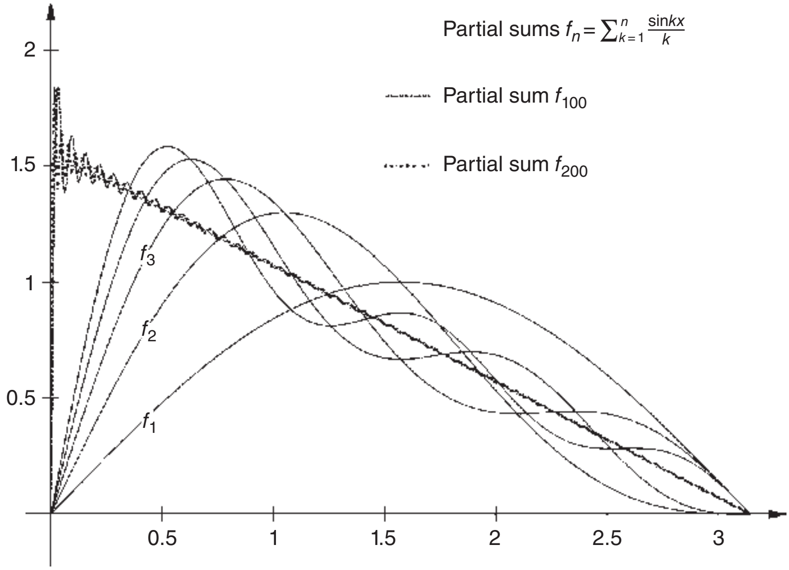

Figure 1.12 Examples 22, 5, and 18, series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

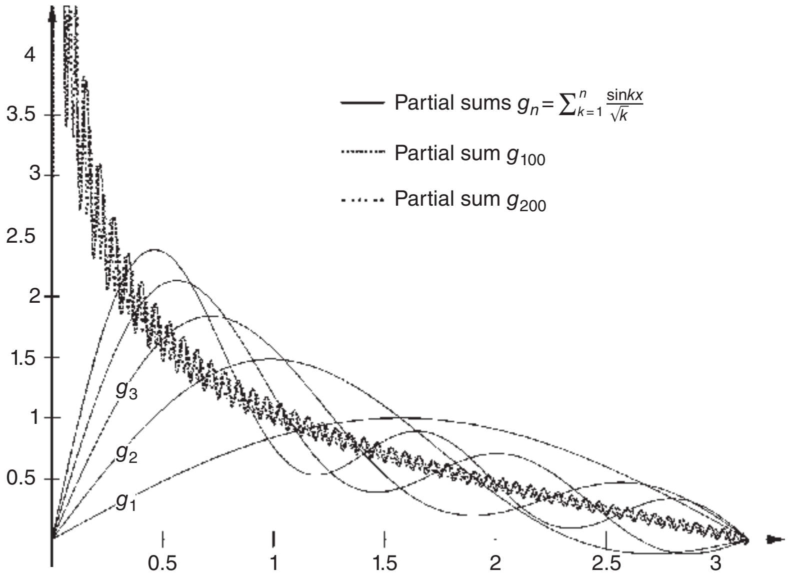

Figure 1.13 Examples 22 and 24, series $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$.

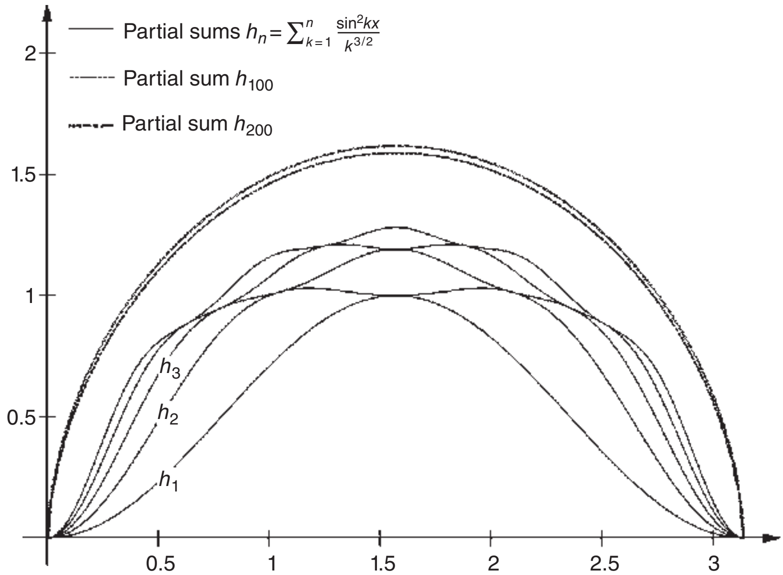

Figure 1.14 Example 22, series of products $\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n^{3/2}}$.

same reasoning as before), but $\sum_{n=1}^{\infty} u_n^2(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^{2/3}}$ diverges at $x = 1$ since $\sum_{n=1}^{\infty} u_n^2(1) = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a divergent p-series.

Remark 2. Evidently, the following more general example can also be constructed: both series $\sum u_n(x)$ and $\sum v_n(x)$ converge uniformly on X, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X. In the particular case $u_n(x) = v_n(x)$, the counterexample is already provided above. Let us consider the case when $u_n(x) \neq v_n(x)$. For instance, using the same arguments as before, one can prove that both $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt[3]{n}}$ and $\sum_{n=1}^{\infty} v_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$ converge uniformly on $X = (0, 1)$, but the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^{5/6}}$ converges nonuniformly on $X = (0, 1)$. For the last series, its convergence follows from the evaluation $\frac{x^{2n}}{n^{5/6}} \leq x^{2n}$, $\forall n \in \mathbb{N}$ and the convergence of the geometric series $\sum_{n=1}^{\infty} x^{2n}$ for $\forall x \in (0, 1)$, while the nonuniformity can be shown by the Cauchy criterion, choosing as above $p_n = n$ and $x_n = \left(1 - \frac{1}{4n}\right) \in X$:

$$\left| \sum_{k=n+1}^{n+p_n} u_k(x_n)v_k(x_n) \right| = \sum_{k=n+1}^{2n} \frac{x_n^{2k}}{k^{5/6}} > n\frac{x_n^{4n}}{(2n)^{5/6}} = \frac{n^{1/6}}{2^{5/6}}\left(1 - \frac{1}{4n}\right)^{4n} \underset{n\to+\infty}{\to} \infty.$$

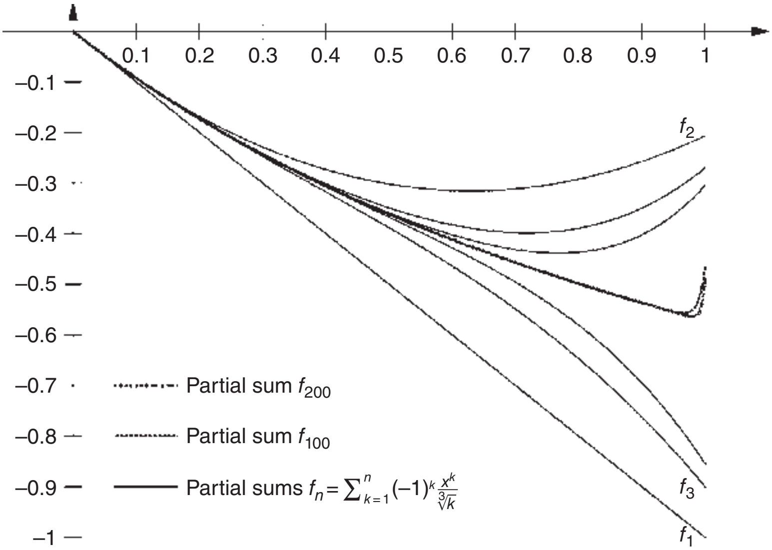

Figure 1.15 Example 23, series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt[3]{n}}$.

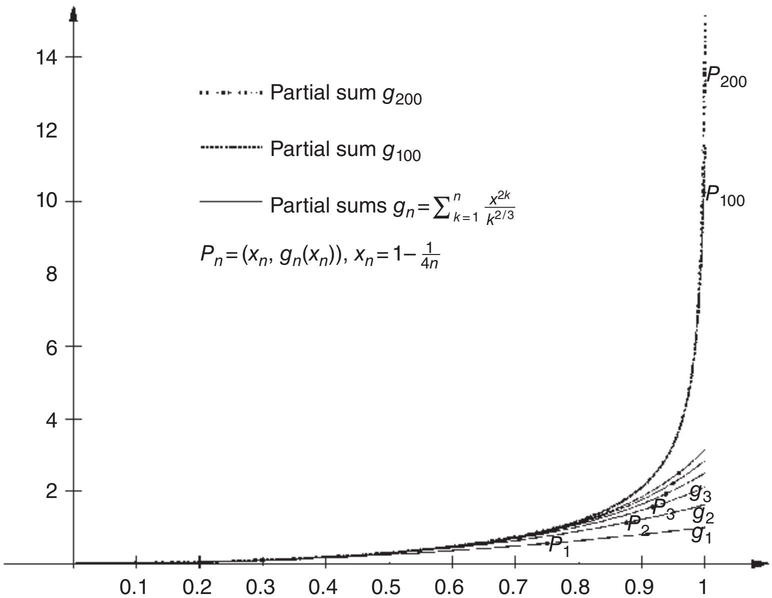

Figure 1.16 Example 23, series of squares $\sum_{n=1}^{\infty} \frac{x^{2n}}{n^{2/3}}$.

Example 24. A series $\sum u_n(x)$ converges uniformly on X, but $\sum u_n^2(x)$ does not converge (even pointwise) on X.

Solution

The series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ is uniformly convergent on $X = [a, \pi - a]$, $\forall a \in \left(0, \frac{\pi}{2}\right)$, which can be shown using the same considerations as for $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ in Remark 1 to Example 5. Let us prove that the series $\sum_{n=1}^{\infty} u_n^2(x) = \sum_{n=1}^{\infty} \frac{\sin^2 nx}{n}$ is divergent on X. Note that the series $\sum_{n=1}^{\infty} \frac{\cos 2nx}{n}$ is uniformly convergent on X just like the series in Remark 1 to Example 5. The series of squares can be rewritten in the form $\sum_{n=1}^{\infty} u_n^2(x) = \sum_{n=1}^{\infty} \frac{1-\cos 2nx}{2n}$, that is, the general term $u_n^2(x)$ is the difference of the general term of the divergent harmonic series and uniformly convergent series. This implies the divergence of the series $\sum_{n=1}^{\infty} \frac{\sin^2 nx}{n}$ at each point of X.

Remark. Naturally, the following more general situation also takes place: both series $\sum u_n(x)$ and $\sum v_n(x)$ converge uniformly on X, but the series $\sum u_n(x)v_n(x)$ does not converge (even pointwise) on X. In the particular case $u_n(x) = v_n(x)$, the counterexample is given above. Let us consider the case when $u_n(x) \neq v_n(x)$. For instance, using the same arguments as before, one can prove that both $\sum_{n=1}^{\infty} \frac{\cos nx}{\sqrt[3]{n}}$ and $\sum_{n=1}^{\infty} \frac{\cos nx}{\sqrt[4]{n}}$ converge uniformly on $X = [a, \pi - a]$, $\forall a \in \left(0, \frac{\pi}{2}\right)$, but the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{\cos^2 nx}{n^{7/12}}$ diverges at each point of X, because its general term can be represented in the form $\frac{\cos^2 nx}{n^{7/12}} = \frac{1}{2n^{7/12}} + \frac{\cos 2nx}{2n^{7/12}}$, where the first summand is a general term of the divergent p-series, while the second is a general term of the uniformly convergent series.

Example 25. Both series $\sum u_n(x)$ and $\sum v_n(x)$ are nonnegative for $\forall x \in X$, $\lim\limits_{n\to\infty} \frac{u_n(x)}{v_n(x)} = 1$ and one of these series converges uniformly on X, but another series does not converge uniformly on X.

Solution

Consider the two nonnegative series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{x^2}{n^4+x^4}$ and $\sum_{n=1}^{\infty} v_n(x) = \sum_{n=1}^{\infty} \frac{x^2}{n^4+x^2}$ on $X = \mathbb{R}$. The limit of the general terms equals 1: $\lim\limits_{n\to\infty} \frac{u_n(x)}{v_n(x)} = \lim\limits_{n\to\infty} \frac{n^4+x^2}{n^4+x^4} = 1$ for $\forall x \in \mathbb{R}$. Also, the series $\sum_{n=1}^{\infty} \frac{x^2}{n^4+x^4}$ converges uniformly on $X = \mathbb{R}$ by the Weierstrass test, since $u_n(x) = \frac{1}{2n^2} \frac{2n^2x^2}{n^4+x^4} \leq \frac{1}{2n^2}$, $\forall x \in \mathbb{R}$ and the series $\sum \frac{1}{2n^2}$ converges. Therefore, all the statement conditions hold. At the same time, the second series $\sum_{n=1}^{\infty} \frac{x^2}{n^4+x^2}$ converges on $X = \mathbb{R}$ since $\frac{x^2}{n^4+x^2} \leq \frac{x^2}{n^4}$, $\forall x \in \mathbb{R}$ and the series $\sum \frac{1}{n^4}$ converges. However, the convergence of the second

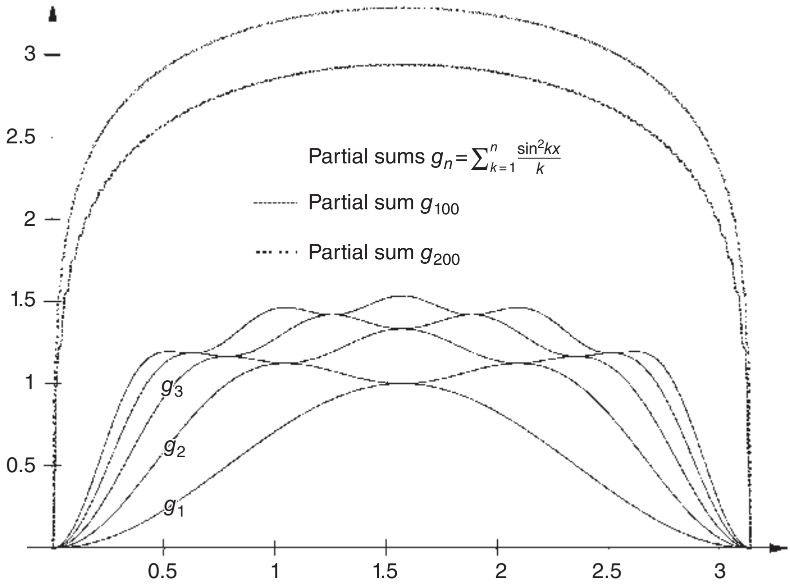

Figure 1.17 Example 24, series of squares $\sum_{n=1}^{\infty}\frac{\sin^2 nx}{n}$.

series is nonuniform since its general term does not converge to 0 uniformly: for $x_n = n^2$, one gets $v_n(x_n) = \frac{x_n^2}{n^4+x_n^2} = \frac{1}{2} \underset{n\to\infty}{\nrightarrow} 0$.

Remark. For numerical series and for the pointwise convergence of series of functions, the corresponding general statement is true and represents a particular case of the Comparison test for nonnegative series: if both series $\sum u_n(x)$ and $\sum v_n(x)$ are nonnegative for $\forall x \in X$, $\lim\limits_{n\to\infty} \frac{u_n(x)}{v_n(x)} = const > 0$ and one of these series converges on X, then another series also converges on X.

1.3 Dirichlet's and Abel's Theorems

Remark to Examples 26–29. In the following four examples, the conditions of Dirichlet's theorem, which provides sufficient conditions for the uniform convergence of the series $\sum u_n(x)v_n(x)$, are analyzed. It is shown that none of the three conditions stated in the theorem can be dropped. At the same time, these conditions are not necessary: all of them can be violated for an uniformly convergent series.

Example 26. The partial sums of $\sum u_n(x)$ are bounded for $\forall x \in X$, and the sequence $v_n(x)$ is monotone in n for each fixed $x \in X$ and converges uniformly on X to 0, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X.

Solution

Let $u_n(x) = x^n$ and $v_n(x) = \frac{1}{n}$ be defined on $X = (0, 1)$. The series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} x^n$ converges on X, since this is a geometric series with the ratio in $(0, 1)$. Therefore, the partial sums of this series are bounded for each fixed $x \in X$. However, the boundedness is not uniform as is seen from the evaluation of the partial sums at the points $x_n = 1 - \frac{1}{n}$, $n > 1$ lying in X:

$$\sum_{k=1}^{n} x_n^k = \frac{x_n}{1-x_n}(1-x_n^n) = \frac{1-1/n}{1/n}\left(1-\left(1-\frac{1}{n}\right)^n\right)$$
$$= (n-1)\left(1-\left(1-\frac{1}{n}\right)^n\right) \underset{n\to\infty}{\to} +\infty,$$

that is, the first condition in Dirichlet's theorem is weakened. The remaining two conditions hold: $v_n(x) = \frac{1}{n}$ is monotone and $v_n(x) = \frac{1}{n} \underset{n\to\infty}{\to} 0$ (and the last convergence is uniform, because v_n does not depend on x).

The series of the products $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges on $(0, 1)$, since $0 < \frac{x^n}{n} \le x^n, \forall n \in \mathbb{N}$ and the geometric series converges for $\forall x \in (0, 1)$. However, this convergence is nonuniform, which can be shown by the Cauchy criterion: choosing $p_n = n$ and $\tilde{x}_n = 1 - \frac{1}{2n} \in X, \forall n \in \mathbb{N}$, one obtains:

$$\left|\sum_{k=n+1}^{n+p_n} u_k(\tilde{x}_n)v_k(\tilde{x}_n)\right| = \sum_{k=n+1}^{2n} \frac{\tilde{x}_n^k}{k} > n\frac{\tilde{x}_n^{2n}}{2n} = \frac{1}{2}\left(1-\frac{1}{2n}\right)^{2n} \underset{n\to+\infty}{\to} \frac{1}{2}e^{-1} \neq 0.$$

Example 27. The partial sums of $\sum u_n(x)$ are uniformly bounded on X, and the sequence $v_n(x)$ converges uniformly on X to 0, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X.

Solution

Let $u_n(x) = \frac{(-1)^n}{\sqrt{n}}$ and $v_n(x) = (-1)^n \frac{x^n}{\sqrt{n}}$ be defined on $X = (0, 1)$. The series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by Leibniz's test and, consequently, its partial sums are bounded (and this boundedness is uniform on X, because the general term does not depend on x). The sequence $v_n(x) = (-1)^n \frac{x^n}{\sqrt{n}}$ converges uniformly on X to 0 due to the following evaluation: $\left|(-1)^n \frac{x^n}{\sqrt{n}}\right| \le \frac{1}{\sqrt{n}}$, $\forall x \in (0, 1)$ and $\frac{1}{\sqrt{n}} \underset{n\to+\infty}{\to} 0$. Thus, both conditions of the statement hold, but the series of the products $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges nonuniformly on $(0, 1)$ as was shown in Example 26. Note that the sequence $v_n(x) = (-1)^n \frac{x^n}{\sqrt{n}}$

is not monotone in n, that is, the condition of the monotonicity of $v_n(x)$ in Dirihlet's theorem is violated.

Example 28. The partial sums of $\sum u_n(x)$ are uniformly bounded on X, and the sequence $v_n(x)$ is monotone in n for each fixed $x \in X$ and converges on X to 0, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X.

Solution

Consider $u_n(x) = (-1)^n$ and $v_n(x) = x^n$ on $X = (0, 1)$. The partial sums $\sum_{k=1}^{n} u_k(x)$ are uniformly bounded on X: $\left|\sum_{k=1}^{n} u_k(x)\right| = \left|\sum_{k=1}^{n} (-1)^k\right| \le 1$, for $\forall n \in \mathbb{N}$ and $\forall x \in (0, 1)$. The sequence $v_n(x) = x^n$ is decreasing in n and $v_n(x) = x^n \underset{n\to+\infty}{\to} 0$ for each fixed $x \in (0, 1)$. Thus, the conditions of the statement are satisfied. However, the series of the products converges nonuniformly. In fact, $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} (-1)^n x^n$ is a convergent geometric series on $(0, 1)$ (the ratio $-x \in (-1, 0)$), but the evaluation of its residual shows that this convergence is nonuniform: choosing $x_n = 1 - \frac{1}{n+1} \in (0, 1)$, one obtains

$$\left|\sum_{k=n+1}^{\infty} (-1)^k x_n^k\right| = \left|\frac{(-1)^{n+1}x_n^{n+1}}{1 + x_n}\right|$$
$$= \frac{1}{2 - \frac{1}{n+1}}\left(1 - \frac{1}{n+1}\right)^{n+1} \underset{n\to+\infty}{\to} \frac{1}{2}e^{-1} \ne 0.$$

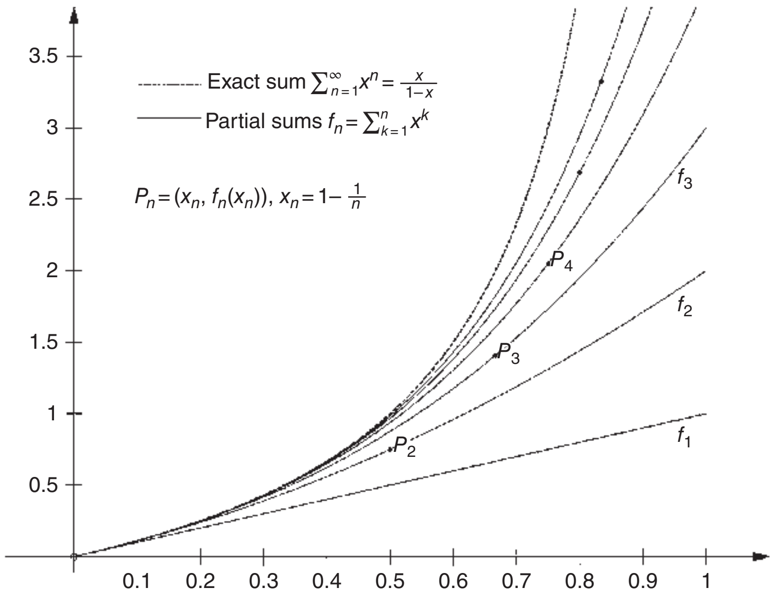

Figure 1.18 Examples 26, 29, and 33, series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} x^n$.

Note that the third condition of Dirichlet's theorem (the uniform convergence of $v_n(x)$) is weakened in the above statement, and the chosen sequence $v_n(x) = x^n$ converges nonuniformly to 0 on $X = (0,1)$, since for $x_n = 1 - \frac{1}{n} \in (0,1), \forall n \in \mathbb{N}$ it follows $|v_n(x_n)| = \left(1 - \frac{1}{n}\right)^n \underset{n\to+\infty}{\to} e^{-1} \neq 0$.

Example 29. The partial sums of $\sum u_n(x)$ are not uniformly bounded on X, and the sequence $v_n(x)$ is not monotone in n and does not converge uniformly on X to 0, but still the series $\sum u_n(x)v_n(x)$ converges uniformly on X.

Solution

Consider $u_n(x) = x^n$ and $v_n(x) = \frac{(-1)^n}{xn^2}$ on $X = (0,1)$. Let us check the conditions of the statement. First, the partial sums of $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} x^n$ are not uniformly bounded on X (see for details Example 26). Second, the sequence $v_n(x)$ is not monotone in n (it is alternating for each fixed $x \in (0,1)$). Finally, $v_n(x) = \frac{(-1)^n}{xn^2}$ converges to 0 for each fixed $x \in (0,1)$, but this convergence is not uniform, because choosing $x_n = \frac{1}{n^2}, \forall n \in \mathbb{N}$ one obtains $\left|\frac{(-1)^n}{x_n n^2}\right| = 1 \underset{n\to+\infty}{\nrightarrow} 0$. In this way, all the conditions in Dirichlet's theorem are violated. Nevertheless, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n-1}}{n^2}$ converges uniformly on (0, 1)

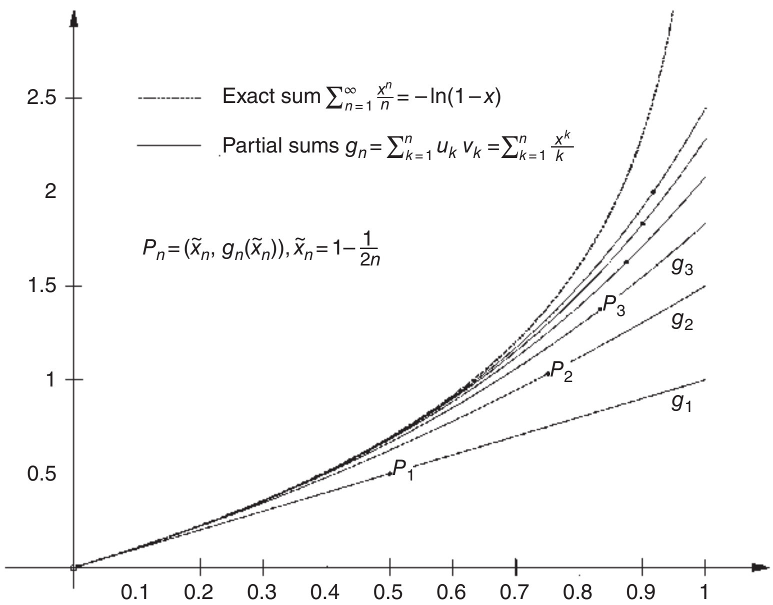

Figure 1.19 Examples 26, 27, 30, 31, and 32, series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$.

according to the Weierstrass test: $\left|(-1)^n \frac{x^{n-1}}{n^2}\right| \le \frac{1}{n^2}$, for $\forall n \in \mathbb{N}$ and $\forall x \in (0,1)$, and the majorant series $\sum \frac{1}{n^2}$ converges.

Remark. The functions $u_n(x) = \frac{x}{n}$ and $v_n(x) = (-1)^n x$ considered on $X = (0, 10]$ exhibit even "wilder" behavior. In fact, the partial sums of the series $\sum_{n=1}^{\infty} \frac{x}{n}$ are not bounded at any point $x \in (0, 10]$ since this series is positive and divergent at each $x \in (0, 10]$. The sequence $(-1)^n x$ is not monotone in n and diverges at each $x \in (0, 10]$. Nevertheless, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n}$ converges uniformly on $(0, 10]$ since the following evaluation of the residual (resulting from Leibniz's test for alternating series)

$$|r_n(x)| = \left|\sum_{k=n+1}^{\infty} (-1)^k \frac{x^2}{k}\right| \le \left|(-1)^{n+1} \frac{x^2}{n+1}\right| \le \frac{100}{n+1} \underset{n\to+\infty}{\to} 0$$

is true for all $x \in (0, 10]$ simultaneously.

Remark to Examples 30–33. In the next four examples, we analyze the sufficient conditions of Abel's theorem for the uniform convergence of the series $\sum u_n(x)v_n(x)$. The situation here is quite similar to that for Dirichlet's

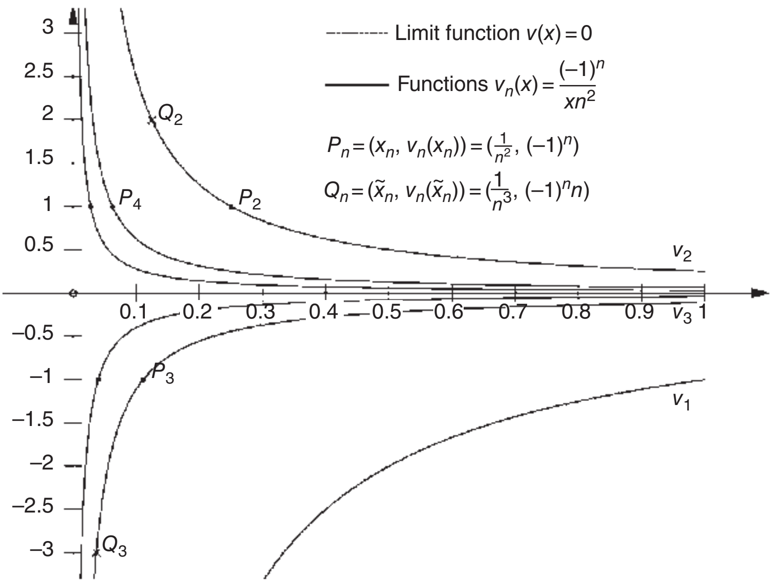

Figure 1.20 Examples 29 and 33, sequence $v_n(x) = \frac{(-1)^n}{xn^2}$.

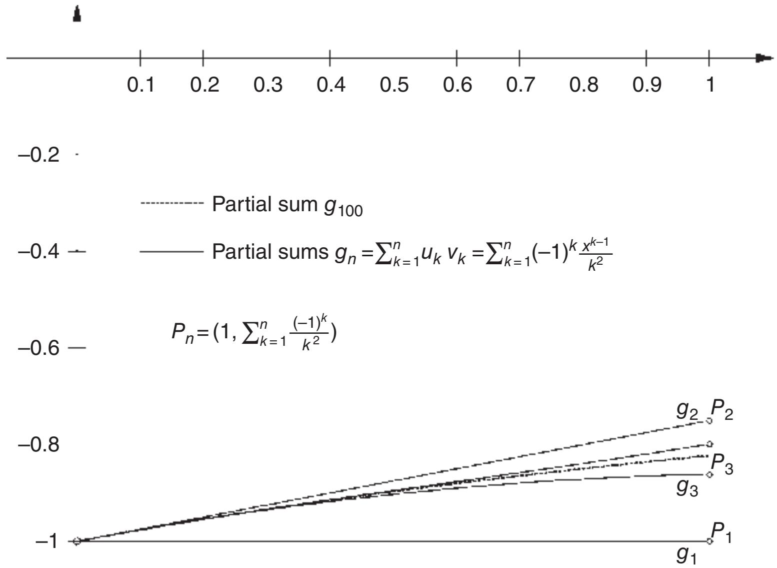

Figure 1.21 Examples 29 and 33, series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n-1}}{n^2}$.

theorem: none of the three conditions can be dropped, but, at the same time, all of them can be violated for an uniformly convergent series.

Example 30. A series $\sum u_n(x)$ converges on X, and a sequence $v_n(x)$ is monotone in n for each fixed $x \in X$ and uniformly bounded on X, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X.

Solution

For $u_n(x) = x^{n-1}$ and $v_n(x) = \frac{x}{n}$ on $X = (0, 1)$, the series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} x^{n-1}$ is convergent at each point $x \in (0, 1)$, since this is a geometric series with the ratio in $(0, 1)$, and the sequence $v_n(x) = \frac{x}{n}$ is monotone in n and uniformly bounded on X: $\left|\frac{x}{n}\right| \le \frac{1}{n} \le 1$, $\forall n \in \mathbb{N}$, $\forall x \in X$. Thus, all the statement conditions are satisfied. However, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges nonuniformly on $(0, 1)$ (see Example 26 for details). Note, that in the statement conditions, the first condition of Abel's theorem (the uniform convergence of $\sum u_n(x)$) is weakened, and the chosen series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} x^{n-1}$ converges nonuniformly on $X = (0, 1)$, which can be seen from the evaluation of the series residual for $x_n = 1 - \frac{1}{n} \in X$, $\forall n > 1$:

$$\left|\sum_{k=n+1}^{\infty} x_n^{k-1}\right| = \frac{x_n^n}{1 - x_n} = n \cdot \left(1 - \frac{1}{n}\right)^n \underset{n\to+\infty}{\to} \infty.$$

Example 31. A series $\sum u_n(x)$ converges uniformly on X, and a sequence $v_n(x)$ is uniformly bounded on X, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X.

Solution
Consider $u_n(x) = \frac{(-1)^n}{n}$ and $v_n(x) = (-1)^n x^n$ on $X = (0, 1)$. The series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by Leibniz's test, and this convergence is uniform since $u_n(x)$ does not depend on x. The uniform boundedness of $v_n(x) = (-1)^n x^n$ is also easily verified: $|(-1)^n x^n| \le 1$, for $\forall n \in \mathbb{N}$, $\forall x \in (0, 1)$. Thus, all the statement conditions hold. However, as was shown in Example 26, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges nonuniformly on $(0, 1)$. Note that the condition of monotonicity of $v_n(x)$ in Abel's theorem is omitted in this example and, consequently, the choice of the nonmonotone sequence $v_n(x) = (-1)^n x^n$ resulted in nonuniform convergence of the series of the products.

Example 32. A series $\sum u_n(x)$ converges uniformly on X, and a sequence $v_n(x)$ is monotone in n for each fixed $x \in X$, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X.

Solution
For $u_n(x) = \frac{x^{n-1}}{n^2}$ and $v_n(x) = nx$ on $X = (0, 1)$, all the statement conditions are satisfied. In fact, the sequence $v_n(x) = nx$ is monotone in n for each fixed $x \in (0, 1)$, and the series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2}$ converges uniformly on $(0, 1)$ according to the Weierstrass test: $\left|\frac{x^{n-1}}{n^2}\right| \le \frac{1}{n^2}$, $\forall x \in (0, 1)$ and the series $\sum \frac{1}{n^2}$ is a convergent p-series. However, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges nonuniformly on $(0, 1)$ (see details in Example 26). Note that the condition of uniform boundedness of $v_n(x)$ in Abel's theorem is omitted in the statement. The chosen sequence $v_n(x) = nx$ is not bounded for any $x \in (0, 1)$, and this led to nonuniform convergence of the series $\sum u_n(x)v_n(x)$.

Remark. The following strengthened version of this example can also be constructed: a series $\sum u_n(x)$ converges uniformly on X, and a sequence $v_n(x)$ is monotone and bounded in n for each fixed $x \in X$, but the series $\sum u_n(x)v_n(x)$ does not converge uniformly on X. The counterexample can be provided by $u_n(x) = \frac{x^2}{(1+x)^n}$ and $v_n(x) = \frac{n^2}{(3n^2+2)x}$ on $X = (0, +\infty)$. The series $\sum_{n=0}^{\infty} u_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x)^n}$ converges on X as a geometric series with the ratio $\frac{1}{1+x} \in (0, 1)$, $\forall x \in X$. To show the uniformity of this convergence on X, let us consider the residual

$$r_n(x) = \sum_{k=n+1}^{\infty} \frac{x^2}{(1+x)^k} = \frac{\frac{x^2}{(1+x)^{n+1}}}{1 - \frac{1}{1+x}} = \frac{x}{(1+x)^n}$$

and solve the critical point equation for each $n > 1$ fixed:

$$r_n'(x) = \frac{1-(n-1)x}{(1+x)^{n+1}} = 0$$

that gives $x_n = \frac{1}{n-1}$. Since the derivative $r_n'(x)$ is positive at the left of x_n and negative at the right, the critical point x_n is the maximum on X and, consequently, for each fixed n one gets the following evaluation of the residual:

$$r_n(x) \le \max_{(0,+\infty)} |r_n(x)| = r_n(x_n) = \frac{1}{n-1}\left(1+\frac{1}{n-1}\right)^{-n} \underset{n\to\infty}{\to} 0 \cdot e^{-1} = 0,$$

that is, the series $\sum_{n=0}^{\infty} u_n(x)$ converges uniformly on X.

As for the sequence $v_n(x) = \frac{n^2}{(3n^2+2)x}$, for each fixed $x \in X$, its terms are monotonic ($v_{n+1}(x) > v_n(x)$) and bounded ($|v_n(x)| = \frac{n^2}{(3n^2+2)x} < \frac{1}{3x}$). Thus, all the conditions of the statement are satisfied.

Nevertheless, the series $\sum_{n=0}^{\infty} u_n(x)v_n(x) = \sum_{n=0}^{\infty} \frac{n^2}{3n^2+2}\frac{x}{(1+x)^n}$ converges nonuniformly on X. Indeed, the convergence on X follows from the inequality $0 < \frac{n^2}{3n^2+2}\frac{x}{(1+x)^n} < \frac{1}{3}\frac{x}{(1+x)^n}$ and the convergence of the geometric series $\sum \frac{x}{(1+x)^n}$ for each fixed $x \in X$. Applying now the Cauchy criterion with $p_n = n$ and $x_n = \frac{1}{n}$, one obtains

$$\left|\sum_{k=n+1}^{n+p_n} u_k(x_n)v_k(x_n)\right| = \sum_{k=n+1}^{2n} \frac{k^2}{3k^2+2}\frac{x_n}{(1+x_n)^k} > \frac{n}{4}\frac{x_n}{(1+x_n)^{2n}}$$
$$= \frac{n}{4}\frac{1}{n}\left(1+\frac{1}{n}\right)^{-2n} \underset{n\to\infty}{\to} \frac{1}{4}e^{-2} \neq 0,$$

which means that the convergence is nonuniform on $X = (0,+\infty)$. Note that although $v_n(x)$ is bounded for each fixed $x \in X$, it is not uniformly bounded on X, since for $x_n = \frac{1}{3n^2+2} \in X$ one gets $v_n(x_n) = n^2 \underset{n\to\infty}{\to} +\infty$.

Example 33. A series $\sum u_n(x)$ does not converge uniformly on X, and a sequence $v_n(x)$ is not monotone in n and is not uniformly bounded on X, but still the series $\sum u_n(x)v_n(x)$ converges uniformly on X.

Solution

Consider $u_n(x) = x^n$ and $v_n(x) = \frac{(-1)^n}{xn^2}$ on $X = (0,1)$. Let us check the conditions of the statement. First, using the same reasoning as in Example 30, one can prove that the series $\sum_{n=1}^{\infty} x^n$ converges nonuniformly on $(0,1)$. Then, the sequence $v_n(x)$ is not monotone in n (it is alternating for each fixed $x \in (0,1)$). Finally, $v_n(x) = \frac{(-1)^n}{xn^2}$ converges to 0 for each fixed $x \in (0,1)$, but this sequence does not bounded uniformly on $(0,1)$, since for $x_n = \frac{1}{n^3} \in (0,1)$ one has $|v_n(x_n)| = n \underset{n\to+\infty}{\to} \infty$. Thus, all the conditions of Abel's theorem are violated. Nevertheless, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^{n-1}}{n^2}$ converges uniformly on $(0,1)$ according to the Weierstrass test as was shown in Example 29.

Remark. The conditions of Abel's theorem are violated even stronger for the functions $u_n(x) = \frac{x}{n}$ and $v_n(x) = (-1)^n\sqrt{nx}$ considered on $X = (0, 2]$. In fact, the series $\sum_{n=1}^{\infty} \frac{x}{n}$ diverges at each $x \in (0, 2]$. The sequence $(-1)^n\sqrt{nx}$ is not monotone in n and is unbounded at each $x \in (0, 2]$ because $|v_n(x)| = |(-1)^n\sqrt{nx}| = \sqrt{nx} \underset{n\to+\infty}{\to} +\infty$, $\forall x \in (0, 2]$. Nevertheless, the series $\sum_{n=1}^{\infty} u_n(x)v_n(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{\sqrt{n}}$ converges uniformly on $(0, 2]$ as is seen from the evaluation of the residual (following from Leibniz's test for alternating series)

$$|r_n(x)| = \left| \sum_{k=n+1}^{\infty} (-1)^k \frac{x^2}{\sqrt{k}} \right| \le \left| (-1)^{n+1} \frac{x^2}{\sqrt{n+1}} \right| \le \frac{4}{\sqrt{n+1}} \underset{n\to+\infty}{\to} 0,$$

which is satisfied for all $x \in (0, 2]$ simultaneously.

Exercises

1 Show that Example 1 can be illustrated by the sequence $f_n(x) = \frac{nx}{n^2x^2+1}$ on $X = [0, 1]$.

2 Use the points $x_n = \frac{1}{\sqrt[n]{2}}$, $\forall n \in \mathbb{N}$ to prove that the sequence $f_n(x) = x^n$ of Example 1 converges nonuniformly on $X = (-1, 1)$.

3 Show that the series $\sum_{n=1}^{\infty} 2^n \sin \frac{1}{5^n x}$ converges on $X = (0, \infty)$, but the convergence is not uniform.

4 Check if the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[3]{n}}$ on $X = (-1, 1)$ can be used for Example 2.

5 Use the sequence $f_n(x) = \frac{x+n+(-1)^n x}{n^2}$ on $X = \mathbb{R}$ to illustrate the statement in Example 3.

6 Construct a counterexample to the following false statement: "if $f(x, y)$ defined on $[a, b] \times Y$ converges to a limit function $\varphi(x)$ as y approaches y_0, and this convergence is uniform on any interval $[c, b]$, $\forall c \in (a, b)$, then the convergence is also uniform on $[a, b]$." Compare with the statement in Example 4. Formulate similar false statements for sequences and series and disprove them by counterexamples. (Hint: for the functions depending on a parameter, try $f(x, y) = \frac{2xy^2}{x^2+y^4}$ on $[0, 1] \times (0, 1]$ with the limit point $y_0 = 0$; for the sequences—$f_n(x) = \frac{nx}{n^2x^2+1}$ on $[0, 1]$; and for the series—$\sum u_n(x) = \sum \frac{(1-x)^n}{n}$ on $(0, 1]$.)

7 Verify that

a) the function $f(x,y) = \frac{x^2}{y^2} e^{-x/y}$ on $X \times Y = [0, +\infty) \times (0, 1]$ with the limit point $y_0 = 0$

b) the function $f(x,y) = \begin{cases} \frac{x}{y} \sin \frac{y}{x}, x \neq 0 \\ 1, x = 0 \end{cases}$ on $X \times Y = \mathbb{R} \times (0, 1]$ with the limit point $y_0 = 0$

c) the sequence $f_n(x) = \frac{2n^2 x}{1+n^4x^2}$ on $X = \mathbb{R}$

d) the series $\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ on $X = \mathbb{R}$

provide counterexamples for Example 5.

8 Verify that the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n^2}{n^3}$ on $X = [-1, 1]$ is one more counterexample to the statement of Example 7.

9 Use the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$, $X = (-1, 1)$ for Example 8.

10 Show the feasibility of Example 9 by using counterexamples with

a) the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^2}{(1+x^2)^n}$, $X = \mathbb{R}$

b) the series $\sum_{n=0}^{\infty} (-1)^n x(1-x)^n$, $X = [0, 1]$.

11 Use the series $\sum_{n=1}^{\infty} u_n(x)$ with

a) $u_n(x) = \begin{cases} 0, x \in [0, 3^{-n-1}] \cup [3^{-n}, 1] \\ \frac{1}{\sqrt{n}} \cos^2\left(\frac{3^{n+1}}{2}\pi x\right), x \in (3^{-n-1}, 3^{-n}) \end{cases}$ on $X = [0, 1]$

b) $u_n(x) = (-1)^{n-1} \frac{x^2}{(1+x^2)^n}$ on $X = \mathbb{R}$

to show the feasibility of Example 10.

12 Verify that the sequence $f_n(x) = \frac{x^2 n^2}{2n^2+5}$ on $X = (0, 1]$ specifies Example 15.

13 Check the statement of Example 16 for the sequence $f_n(x) = \frac{2nx}{n^2+x^2}$ on $X = \mathbb{R}$.

14 Verify whether the sequences $f_n(x)$, $g_n(x)$ and $f_n(x) \cdot g_n(x)$ are convergent or divergent on X. In the case of the convergence, analyze its character:

a) $f_n(x) = \ln \frac{(n^2+1)x}{n^2}$, $g_n(x) = \ln \frac{3n+2}{n} x^2$ on $X = (0, +\infty)$

b) $f_n(x) = g_n(x) = \ln \frac{n^2x^2}{n^2+1}$ on $X = (0, +\infty)$

c) $f_n(x) = \frac{x}{n}$, $g_n(x) = \frac{\sin nx}{nx}$ on $X = (0, +\infty)$

d) $f_n(x) = \frac{x}{n}$, $g_n(x) = \ln \frac{(n+1)x}{n}$ on $X = (0, +\infty)$

e) $f_n(x) = \frac{x}{n}$, $g_n(x) = \frac{\sin nx}{nx}$ on $X = (0, 1]$

f) $f_n(x) = g_n(x) = (-1)^n \frac{5n^2+2}{3n^2+1} x$ on $X = (0, 1]$.

Formulate false statements for which these sequences represent counterexamples.

15 Show that the sequences $f_n(x) = \begin{cases} \frac{n^2+1}{n^2}x^2, x \in \mathbb{Q} \\ \frac{x^2}{n^2}, x \in \mathbb{I} \end{cases}$ and $g_n(x) = \begin{cases} \frac{1}{n^2}x^2, x \in \mathbb{Q} \\ \frac{n^2+1}{n^2}x^2, x \in \mathbb{I} \end{cases}$ defined on $X = [0,1]$ provide a counterexample to Example 14.

16 Verify whether the series $\sum u_n(x)$, $\sum v_n(x)$ and $\sum u_n(x) \cdot v_n(x)$ are convergent or divergent on X. In the case of the convergence, analyze its character:

a) $u_n(x) = v_n(x) = \frac{\sin nx}{\sqrt{n}}$ on $X = \mathbb{R}$
b) $u_n(x) = v_n(x) = \frac{\sin nx}{n^{2/3}}$ on $X = \mathbb{R}$
c) $u_n(x) = v_n(x) = \frac{1}{x^{2/3}+n^{2/3}}$ on $X = [0, +\infty)$
d) $u_n(x) = \frac{1}{x+n}$, $v_n(x) = \frac{1}{x+\ln^2 n}$ on $X = (0, +\infty)$
e) $u_n(x) = v_n(x) = \frac{\cos nx}{\sqrt{n}}$ on $X = [a, \pi - a]$, $\forall a \in \left(0, \frac{\pi}{2}\right)$
f) $u_n(x) = \frac{\sin nx}{\sqrt{n}}$, $v_n(x) = \frac{\sin nx}{\sqrt[4]{n}}$ on $X = [a, \pi - a]$, $\forall a \in \left(0, \frac{\pi}{2}\right)$
g) $u_n(x) = v_n(x) = (-1)^n \frac{x^n}{\sqrt{n}}$ on $X = (0,1)$
h) $u_n(x) = (-1)^n \frac{x^n}{\ln n}$, $v_n(x) = (-1)^n \frac{x^n}{\sqrt{n}}$, $n \geq 2$ on $X = (0,1)$
i) $u_n(x) = \frac{\sin nx}{n^{2/3}}$, $v_n(x) = \frac{\sin nx}{n}$ on $X = \mathbb{R}$
j) $u_n(x) = v_n(x) = \frac{x^n}{n}$ on $X = [0,1)$
k) $u_n(x) = \frac{x^n}{n}$, $v_n(x) = \frac{x^n}{n^{1/3}}$ on $X = [0,1)$
l) $u_n(x) = v_n(x) = (-1)^n \frac{x^n}{\sqrt[3]{n}}$ on $X = [0,1)$
m) $u_n(x) = v_n(x) = (-1)^n \frac{x}{\sqrt{n}}$ on $X = (0,1)$.

Formulate false statements for which these series represent counterexamples.

17 Show that the series $\sum_{n=1}^{\infty} \frac{2x^3}{n^6+x^6}$ and $\sum_{n=1}^{\infty} \frac{2x^3}{n^6+x^3}$ on $X = (0, +\infty)$ exemplify the statement in Example 25.

18 For given $u_n(x)$ and $v_n(x)$ on the specified set X, verify the conditions of Dirichlet's theorem and investigate the character of the convergence of the series $\sum u_n(x) \cdot v_n(x)$:

a) $u_n(x) = x^{n-1}$, $v_n(x) = \frac{x}{n}$ on $X = (-1,1)$
b) $u_n(x) = x^{n-1}$, $v_n(x) = (-1)^n \frac{x}{n}$ on $X = [0,1)$
c) $u_n(x) = x^{n+1}$, $v_n(x) = (-1)^n \frac{1}{n^{3/2}x}$ on $X = (0,1)$

d) $u_n(x) = (-1)^n x^{n-1}$, $v_n(x) = (-1)^n \frac{x}{n}$ on $X = [0, 1)$
e) $u_n(x) = \frac{1}{\sqrt{n}}$, $v_n(x) = \sin nx$ on $X = \left[\frac{\pi}{10}, \frac{19\pi}{10}\right]$
f) $u_n(x) = (-1)^n$, $v_n(x) = \left(\frac{x^2}{1+x^2}\right)^n$ on $X = \mathbb{R}$
g) $u_n(x) = x^2$, $v_n(x) = \frac{(-1)^n}{(1+x^2)^n}$ on $X = (0, +\infty)$
h) $u_n(x) = \frac{x}{n}$, $v_n(x) = \frac{\sin nx}{x}$ on $X = \left[\frac{\pi}{6}, \frac{11\pi}{6}\right]$
i) $u_n(x) = x^{2n}$, $v_n(x) = \frac{(-1)^n}{(1+x^2)^n}$ on $X = (-1, 1)$.

Formulate false statements for which these functions and series represent counterexamples.

19 For given $u_n(x)$ and $v_n(x)$ on the specified set X, verify the conditions of Abel's theorem and investigate the character of the convergence of the series $\sum u_n(x) \cdot v_n(x)$:
a) $u_n(x) = (-1)^n$, $v_n(x) = \left(\frac{x^2}{1+x^2}\right)^n$ on $X = \mathbb{R}$
b) $u_n(x) = x^{n-1}$, $v_n(x) = \frac{x}{n}$ on $X = (-1, 1)$
c) $u_n(x) = \frac{x^2}{n}$, $v_n(x) = \frac{\sin nx}{nx^2}$ on $X = (0, +\infty)$
d) $u_n(x) = \frac{1}{nx}$, $v_n(x) = \sqrt{nx} \sin nx$ on $X = \left[\frac{\pi}{10}, \frac{19\pi}{10}\right]$
e) $u_n(x) = \frac{(-1)^n}{n}$, $v_n(x) = (-1)^n \sin nx$ on $X = \mathbb{R}$
f) $u_n(x) = (-1)^n \frac{x^2}{(1+x^2)^n}$, $v_n(x) = \frac{2n+1}{(n+1)x^2}$ on $X = (0, +\infty)$
g) $u_n(x) = \frac{\sin nx}{\sqrt{n}}$, $v_n(x) = \frac{1}{\sqrt{n}}$ on $X = \mathbb{R}$
h) $u_n(x) = \frac{\sin nx}{n^2}$, $v_n(x) = n^{3/2}$ on $X = \mathbb{R}$
i) $u_n(x) = \frac{x^n}{n\sqrt{n}}$, $v_n(x) = x\sqrt{n}$ on $X = (0, 1)$
j) $u_n(x) = \frac{\sin nx}{n}$, $v_n(x) = (-1)^n \sqrt{n}$ on $X = \left(0, \frac{\pi}{2}\right)$.

Formulate false statements for which these functions and series represent counterexamples.

Further Reading

S. Abbott. *Understanding Analysis*, Springer, New York, 2002.

D. Bressoud, *A Radical Approach to Real Analysis*, MAA, Washington, DC, 2007.

T.J.I. Bromwich, *An Introduction to the Theory of Infinite Series*, AMS, Providence, RI, 2005.

B.M. Budak and S.V. Fomin, *Multiple Integrals, Field Theory and Series*, Mir Publisher, Moscow, 1978.

G.M. Fichtengolz, *Differential- und Integralrechnung, Vol.1–3,* V.E.B. Deutscher Verlag Wiss., Berlin, 1968.

V.A. Ilyin and E.G. Poznyak, *Fundamentals of Mathematical Analysis, Vol.1,2,* Mir Publisher, Moscow, 1982.

K. Knopp, *Theory and Applications of Infinite Series*, Dover Publication, Mineola, NY, 1990.

C.H.C. Little, K.L. Teo and B. Brunt, *Real Analysis via Sequences and Series*, Springer, New York, 2015.

W. Rudin, *Principles of Mathematical Analysis,* McGraw-Hill, New York, 1976.

V.A. Zorich, *Mathematical Analysis I, II*, Springer, Berlin, 2004.