Chapter 1

Interpolation

In this chapter, we will introduce interpolation theory, the first of two key topics that will form the foundation of everything that follows in this book. We will find that this concept leads quite naturally to finite difference methods and, when combined with the second key topic, the method of weighted residuals, provides the necessary mathematical concepts needed for all other numerical methods we present. So, let us get started.

1.1 Purpose

Interpolation is a method of constructing new data points between known values or for creating a function that fits exactly a known set of discrete data points defined within a specific range. Interpolation has many applications in science and engineering. In this book, it will be used to form the basis for numerical differentiation, numerical quadrature, numerical integration, and as a key part of several numerical methods used to solve differential and partial differential equations.

1.2 Definitions

We begin by introducing some interpolation notation. Consider a region

$$(a =)\ x_0 < x_1 < x_2 ... < x_n\ (= b) \tag{1.1}$$

as illustrated in Fig. 1.1

Figure 1.1: Discretized line spanning a to b.

Next assume there exists a function $f(x)$ that is a known function of x. We will use this function momentarily.

Numerical Methods for Solving Partial Differential Equations, First Edition. George F. Pinder.

Now also consider a function $P_n(x)$ that has the following properties:

1. $P_n(x)$ is a polynomial of degree n, that is.

$$P_n(x) = \sum_{i=0}^{n} a_i x^i \tag{1.2}$$

   where the coefficients a_i are known constants and x^i indicates the variable x to the ith power.

2. $P_n(x_i) = f(x_i)$ where x_i are particular values of x as seen in Fig. 1.1. In other words at the locations x_i the values of $f(x)$ and $P_n(x)$ are identical.

According to our definition of interpolation, this $P_n(x)$ is an interpolating polynomial of degree n. Note that by convention, i has a lower bound value of 0 rather than 1. To make the above clear, consider the following example.

1.3 Example

Consider the sine function shown in Fig. 1.2. and the information presented in Table 1.1.

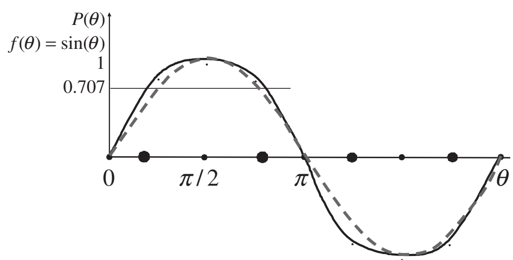

Figure 1.2: Sin (θ) curve with measured points indicated by small black dots (dashed curve) and interpolated values indicated by large black dots (solid curve).

Let the second-degree polynomial $P_2(\theta)$ be given by

$$P_2(\theta) = a_0 + a_1\theta + a_2\theta^2. \tag{1.3}$$

θ	$f(\theta)$	$P_2(\theta)$	$E(\theta) \equiv f(\theta) - P_2(\theta)$
0	0	0	0
$\pi/4$	0.707	0.753	-0.046
$\pi/2$	1	1	0
$3\pi/4$	0.707	0.751	-0.044
π	0	0	0
$5\pi/4$	-0.707	-1.18	0.37
$6\pi/4$	-1	-2.89	1.89

Table 1.1: Comparison of exact and calculated approximations for the sin curve.

Now create the special version of Eq. (1.3) that satisfies the three known values of $P_2(\theta)$, namely those at $\theta = 0,\ \pi/2$, and π. Then, substituting values of $P_2(\theta)$ from Table 1.1 we obtain

$$\begin{aligned} P_2(0) &= a_0 + a_1 0 + a_2 0^2 = 0 \\ P_2(\pi/2) &= a_0 + a_1(\pi/2) + a_2(\pi/2)^2 = 1 \\ P_2(\pi) &= a_0 + a_1(\pi) + a_2(\pi)^2 = 0 \end{aligned} \tag{1.4}$$

from which we can generate the set of equations

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & \pi/2 & (\pi/2)^2 \\ 1 & \pi & (\pi)^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solving for the coefficients a_0, a_1, and a_2 we obtain

$$P_2(\theta) = 0 + 1.27(\theta) - 0.40\left(\theta^2\right). \tag{1.5}$$

From this expression, we can obtain values of $P(\theta)$ for any θ. In Table 1.1, we calculate the values of $P(\theta)$ for various values of θ. Notice the difference in the error of the interpolation at $\theta = \pi/4$ and $\theta = (5\pi)/4$. Why did this happen? It is due to the fact that the value $\theta = 5\pi/4$ lies outside of the range of θ used to define $P(\theta)$.

The question now arises as to whether polynomials can be used to represent functions other than the sine. To answer this question we turn to the Weirstraus approximation theorem.

1.4 Weirstraus Approximation Theorem

The Weirstraus approximation theorem basically tells us that it is possible to calculate a polynomial approximation of any desired accuracy, provided you employ a suitably large number of terms in the polynomial. It states:

Theorem 1 *If $f(x)$ is continuous on a finite interval $[a,b]$, then, given any $\varepsilon > 0$, there exists an n and a polynomial $P_n(x)$ of degree n such that $|f(x) - P_n(x)| < \varepsilon$ for all x in $[a,b]$.*

Although this theorem indicates that a polynomial can be found to represent any function, it does not mean that the coefficients of all polynomials can be calculated. In some cases, especially for large n, the coefficient matrix for the polynomial coefficients can be ill-conditioned (almost singular) and the coefficient values unobtainable. Fortunately, in our work, we will be using polynomials that do not exhibit this pathology.

1.5 Lagrange Interpolation

Let us now focus on one special kind of interpolation that we will use extensively in subsequent material. Consider the approximation of a function $f(x)$ denoted as $\hat{f}(x)$ written as follows:

$$f(x) = \hat{f}(x) + E(x) \tag{1.6}$$

where we will call $E(x)$ the error of the approximation; in other words $E(x)$ is a measure of how well $\hat{f}(x)$ represents $f(x)$.

We now define the form of $\hat{f}(x)$ in a very special way, that is

$$\hat{f}(x) = \sum_{j=0}^{n} \ell_j^n(x) f(x_j) \tag{1.7}$$

where $\ell_j^n(x)$ is an as yet undefined polynomial of degree n. Next, substitute Eq. (1.7) into Eq. (1.6) to obtain

$$f(x) = \sum_{j=0}^{n} \ell_j^n(x) f(x_j) + E(x). \tag{1.8}$$

To determine the functional form of the polynomial $\ell_j^n(x)$ we will require that $E(x)$, the error in the approximation, vanishes at selected locations along x, namely at x_i, $i = 0, ...n$. We will call these locations nodes and they are indicated in Fig. 1.1 by the black dots. Recall that we required $f(x)$ to equal $f(x_j)$ in our general definition of a polynomial in Section 1.2. We can write this requirement formally, in terms of the errors $E(x_i)$, as

$$E(x_i) = 0 \quad i = 0, ..., n \tag{1.9}$$

where we have $n+1$ nodes. Now combine Eq. (1.9) with Eq. (1.8) to give

$$f(x_i) = \sum_{j=0}^{n} \ell_j^n(x_i) f(x_j) \quad i = 0, ..., n. \tag{1.10}$$

Equation (1.10) is Eq. (1.8) written for the specific nodal locations x_i where, by definition, the error must vanish. Note that the index i identifies the location, that is $f(x_i)$ and $\ell_j^{(n)}(x_i)$ where the polynomial is being evaluated and the index j indicates the term in the polynomial, that is $\ell_j^{(n)}(x_i) f(x_j)$. Let us now expand Eq. (1.10) as

$$f(x_1) = \ell_0^n(x_1) f(x_0) + \ell_1^n(x_1) f(x_1) + \cdots + \ell^n(x_1) f(x_n). \tag{1.11}$$

The form of Eq. (1.11) suggests that the polynomials $\ell_j^n(x_i)$ must have special properties. In order to satisfy the requirement that

$$E(x_i) = 0, \quad i = 0, ..., n. \tag{1.12}$$

Indeed, at the location $x_{1,}$ for example, Eq. (1.11) must yield the following:

$$\ell_1^n(x_1) = 1 \tag{1.13}$$
$$\ell_0^n(x_1) = \ell_2^n(x_1) ... = \ell_n^n(x_1) = 0. \tag{1.14}$$

In fact we can generalize this statement to any nodal location x_i, that is

$$\ell_j(x_i) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad i, j = 0, ..., n. \tag{1.15}$$

In other words, our polynomial must be unity at the nodal location for which it is defined, that is where the indices i and j are the same, and zero at all other nodes. Writing this in shorthand notation we get that

$$\ell_j^n(x_i) = \delta_{ij} \quad j = 0, ...n, \quad i = 0, ..., n \tag{1.16}$$

where δ_{ij} is the Kronecker delta. The Kronecker delta is defined such that

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.17}$$

Now let us assume the special case of $n = 1$; that is we are considering a linear polynomial. Let us focus on the jth polynomial, that is

$$\ell_j^1(x) = a_j + b_j x. \tag{1.18}$$

In light of Eq. (1.17), we can say that the following is true for $\ell_j^1(x)$ evaluated at node x_0

$$\ell_0^1(x_0) = a_0 + b_0 x_0 = 1 \tag{1.19}$$

$$\ell_0^1(x_1) = a_0 + b_0 x_1 = 0. \tag{1.20}$$

Writing this system of equations in matrix form we get

$$\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.21}$$

which we can solve to obtain

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \frac{1}{x_1 - x_0} \begin{bmatrix} x_1 & -x_0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{x_1}{x_1 - x_0} \\ \frac{-1}{x_1 - x_0} \end{bmatrix}. \tag{1.22}$$

Substitution of Eq. (1.22) into Eq. (1.18) yields

$$\ell_0^1(x) = a_0 + b_0 x = \frac{x_1}{x_1 - x_0} - \frac{1}{x_1 - x_0} x = \frac{x_1 - x}{x_1 - x_0}. \tag{1.23}$$

One can similarly obtain $\ell_1^1(x) = \frac{x - x_0}{x_1 - x_0}$. The functions $\ell_0^1(x)$ and $\ell_1^1(x)$ are **linear Lagrange polynomials** and are presented in Fig. 1.3

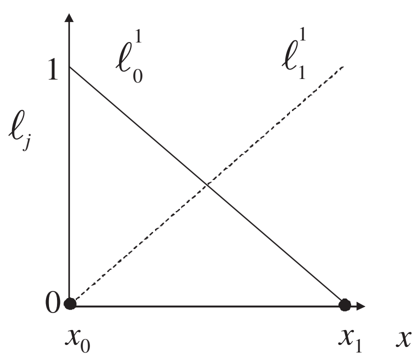

Figure 1.3: Linear Lagrange polynomials, one defined for the node located at x_0 (solid line) and one for the node located at x_1 (dashed line).

Let us check to see if these functions satisfy the requirements stated in Eqs. (1.19) and (1.20):

$$\text{when } x = x_0; \quad \ell_0^1(x_0) = \frac{x_1 - x_0}{x_1 - x_0} = 1 \tag{1.24}$$

$$\text{when } x = x_1; \quad \ell_0^1(x_1) = \frac{x_1 - x_1}{x_1 - x_0} = 0. \tag{1.25}$$

It appears the Lagrange polynomials, as defined above, work for the linear case.

The general form of the nth degree Lagrange polynomial is

$$\ell_j^n(x) = \Pi_{\substack{i=0\\i\neq j}}^{n} \frac{x - x_i}{x_j - x_i} \tag{1.26}$$

where the operator $\Pi_{\substack{i=0\\i\neq j}}^{n}$ says that for a specific value of j, each term denoted by the subscript i, $i = 0, 1, ..., n$ will be multiplied together except for the special case of $i = j$, which would result in a value of zero in the denominator. For example, $\Pi_{i=1}^{3} x_i = (x_1)(x_2)(x_3)$. Thus, for our linear case we obtain

$$\ell_0^1(x) = \Pi_{\substack{i=0\\i\neq j}}^{1} \frac{x - x_i}{x_0 - x_i} = \frac{x - x_1}{x_0 - x_1} \tag{1.27}$$

which is the same as we obtained in Eq. (1.23) after multiplying the numerator and denominator by (-1).

The strategy that was used above for the linear polynomial can be extended to define the quadratic. In this case, as we will see below, there are three unknown coefficients and therefore one needs three equations. The equations are obtained by imposing the constraints defined in Eq. (1.16) on the quadratic polynomial. Alternatively, since we have stated the general polynomial form in Eq. (1.16), we can write directly, by selecting $n = 2$ the relationship (where j is now equal to 0):

$$\ell_0^2(x) = \Pi_{\substack{i=0\\i\neq 0}}^{2} \frac{x - x_i}{x_0 - x_i} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}. \tag{1.28}$$

The shape of this function is shown in Fig. 1.4. By selecting other values of j, that is $j = 1$ or $j = 2$ two additional quadratics will be generated for location x_1 and x_2.

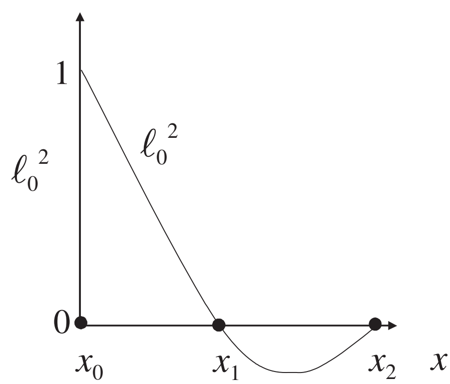

Figure 1.4: Quadratic Lagrange polynomial identified with node x_0.

1.5.1 Example

Let us use the linear function to represent the function $\ln(x)$. The linear Lagrange polynomials are presented in Fig. 1.3. There will be two terms, one associated with each node, that is $x = x_0$ and $x = x_1$, as seen in Fig. 1.3. Thus we have that $x \in [1, 2]$

$$\hat{f}(x) = \sum_{j=0}^{1} \ell_j^1(x) f(x_j) \qquad x \in [1,2]. \tag{1.29}$$

From this equation we see that we need $f(x_0)$ and $f(x_1)$. To get this information we need to evaluate $\ln(x)$ at $x = 1.0$ and $x = 2.0$. The following equation shows how this is used:

$$\begin{aligned}
\hat{f}(x) &= \ell_0^1(x) f(x_0) + \ell_1^1(x) f(x_1) \\
&= \ell_0^1(x) f(1.0) + \ell_1^1(x) f(2) \\
&= \ell_0^1(x)(0) + \ell_1^1(x)(0.693) \\
&= \frac{x - x_0}{x_1 - x_0}(0.693) \\
&= \frac{x-1}{2-1}(0.693) \\
&= -0.693 + 0.693x.
\end{aligned} \tag{1.30}$$

A comparison of the function $f(x)$ and the approximation $\hat{f}(x)$ is presented in Table 1.2. Note that at the node points $x = 1.0$ and $x = 2.0$ the solution is exact, as required by our definition of the approximating polynomial.

It is helpful to examine the information provided in Fig. 1.5. The interpolant $\hat{f}$ is given in the top pane. It is a straight line since it is made up of the weighted sum or two straight lines as

x	$f(x)$	$\hat{f}(x)$	$E(x)$
1.00	0.00	0.00	0.00
1.25	0.22	0.173	0.047
1.50	0.40	0.347	0.053
1.75	0.56	0.520	0.04
2.0	0.693	0.693	0.00

Table 1.2: Values of the function $f(x) = \ln(x)$, the approximation to $f(x)$, that is $\hat{f}(x)$ and the error $E(x) = f(x) - \hat{f}(x)$.

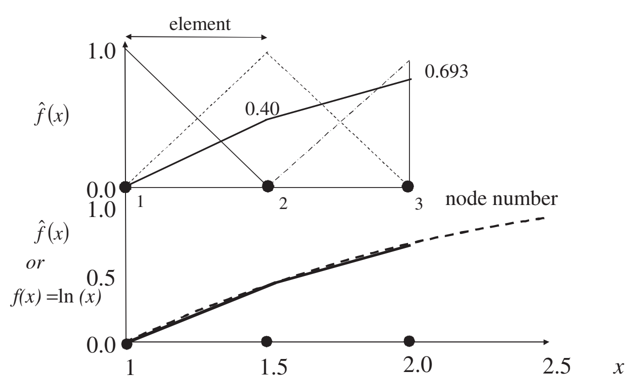

Figure 1.5: Representation of the $\ln(x)$ function using linear Lagrange polynomials.

can be seen from Eq. (1.30). The approximation is the sum of the linear Lagrange polynomial at $x = 1.0$ multiplied by the value of $\ln(1)$ and the linear Lagrange polynomial defined at $x = 2$ multiplied by $\ln(2)$. The weighted sum of linear polynomials always generates a linear polynomial approximation. The lower pane in this figure shows the comparison between the function $f(x) = \ln(x)$ and its approximation $\hat{f}(x)$.

1.6 Compare $P_2(\theta)$ and $\hat{f}(\theta)$

In this section we want to examine the relationship between the use of a quadratic polynomial, and an approximation based upon quadratic Lagrange polynomials, to interpolate. We start by writing the approximation of the $\sin(\theta)$ function, $\hat{f}(x)$, using the quadratic Lagrange polynomials, that is

$$\hat{f}(\theta) = \sum_{j=0}^{2} \ell_j^2(\theta) f_j(\theta) \tag{1.31}$$

where, using Eq. (1.28), we obtain three polynomials, one for each node in Fig. 1.4

$$\ell_0^2(\theta) = \frac{(\theta - \theta_1)(\theta - \theta_2)}{(\theta_0 - \theta_1)(\theta_0 - \theta_2)} \tag{1.32}$$

$$\ell_1^2(\theta) = \frac{(\theta - \theta_0)(\theta - \theta_2)}{(\theta_1 - \theta_0)(\theta_1 - \theta_2)} \tag{1.33}$$

$$\ell_2^2(\theta) = \frac{(\theta - \theta_0)(\theta - \theta_1)}{(\theta_2 - \theta_0)(\theta_2 - \theta_1).} \tag{1.34}$$

Now we multiply each function by the appropriate coefficient value $f(\theta)$ and get

$$\hat{f}(\theta) = \ell_0^2(\theta)(0) + \ell_1^2(\theta)(1) + \ell_2^2(\theta)(0). \tag{1.35}$$

Next we substitute the definitions of $\ell_1^2(\theta)$

$$\ell_1^2(\theta) = \frac{(\theta - 0)(\theta - \pi)}{(\pi/2 - 0)(\pi/2 - \pi)} = \boxed{1.27\theta - 0.406\theta^2}. \tag{1.36}$$

Comparison of this relationship with Eq. (1.5) shows that

$$P_2(\theta) = \hat{f}(\theta). \tag{1.37}$$

In essence, no matter how you manipulate quadratic polynomials, whether or not they are Lagrange polynomials, you will not change the approximation. The reason we use Lagrange polynomials and the $\hat{f}(\theta)$ machinery will become more evident later.

The value of ℓ_1^2 in Fig. 1.6 is the approximation to the $\sin(\theta)$ function.

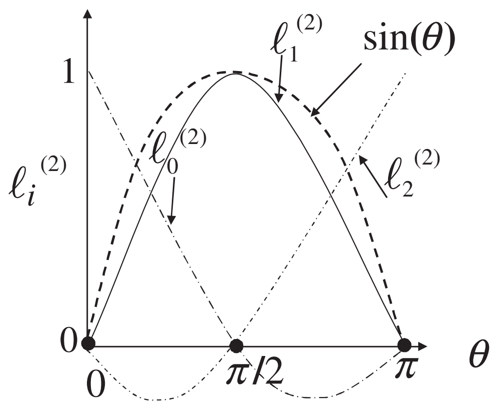

Figure 1.6: Quadratic polynomial approximation of $\sin(\theta)$.

1.7 Error of Approximation

The general idea in this section is to determine, in the absence of the function being approximated being available, how well the polynomial will approximate it. The argument is rather convoluted in that we need to build a set of concepts and then bring them all together at the end, so please be patient.

Step 1 (Define a function $F(x)$)

We start by defining

$$F(x) = \Pi_{i=0}^{n}(x - x_i). \tag{1.38}$$

For example for the special case of $n = 2$ we have the function

$$F(x) = \Pi_{i=0}^{2}(x - x_i) = (x - x_0)(x - x_1)(x - x_2) \tag{1.39}$$

which is given in Fig. 1.7.

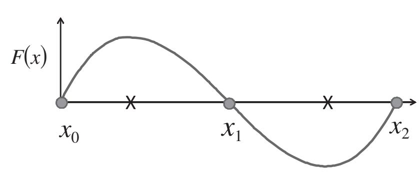

Figure 1.7: Function $F(x)$ with zeros at x_0, x_1, and x_2. The locations indicated by an x along the axis are where $\frac{dF(x)}{dx} = 0$.

It is important in this development to note that

$$\boxed{F(x) = \Pi_{i=0}^{n}(x - x_i) = 0 \quad for \;\; x = x_i, \;\; i = 0, 1, ..., n} \tag{1.40}$$

because when the term $(x_i - x_i)$ arises, it is zero and that eliminates the series, that is, function $F(x)$ vanishes at the nodes.

Because the approximation must equal the function at the nodal points x_i, we have

$$\hat{f}(x_i) = f(x_i) \qquad i = 0, 1, ..., n. \tag{1.41}$$

Alternatively, we can write

$$\boxed{f(x_i) - \hat{f}(x_i) = 0 \qquad i = 0, 1, ..., n.} \tag{1.42}$$

Step 2 (Define a function $g(x)$)

Now we change course and define the following function:

$$g(x) = f(x) - \hat{f}(x) - AF(x) \tag{1.43}$$

where A is a constant. Why we do this has no answer at this point, but its relevance becomes evident shortly. Note from Eq. (1.40) and the Fig. 1.7 that there are $n+1$ points x_i where $F(x)$ is zero. Therefore, at these points, according to Eq. (1.40-1.42)

$$g(x_i) = 0 \quad i = 0, 1, ..., n. \tag{1.44}$$

We now digress once again in Step 3 which follows:

Step 3 (Introduction of the concept of ξ)

Let us choose A such that $g(x) = 0$ at some **arbitrary** point $x_p \in [x_0, x_n]$. Now $g(x)$ is zero at at least $n+2$ points, that is at x_0, x_1,..., $x_p, x_{n-1,}$ x_n (see Fig. 1.8). Since $g(x)$ is smooth, it must have a minimum or maximum between each pair of zeroes at which points the derivatives of $g(x)$ vanish. These are indicated by the letter x in Fig. 1.8. Then $\frac{d(g)}{dx}$ has at least $(n+2) - 1 = n+1$ zeros in the interval $[x_0, x_n]$. Similarly at the points of inflection $\frac{d^2g(x)}{dx^2}$ has at least $(n+2) - 2 = n$ zeros. Using similar logic to look at even higher derivatives, we finally arrive at the observation that $\frac{d^{n+1}g(x)}{dx^{n+1}}$ has $(n+2) - (n+1) = 1$ zero.

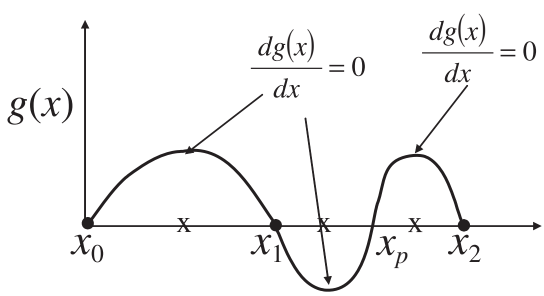

Figure 1.8: Function $g(x)$ as presented in Eq. 1.43.

Let $x = \xi$ represent the location of this zero, that is

$$\boxed{\left.\frac{d^{n+1}g(x)}{dx^{n+1}}\right|_{x=\xi} = 0.} \tag{1.45}$$

Since $\hat{f}(x)$ is a polynomial of degree n, $\frac{d^{n+1}\hat{f}(x)}{dx^{n+1}} = 0$. This observation will be used a little later.

Step 4 (Calculation of $\frac{d^{n+1}F(x)}{dx^{n+1}}$)

It is not obvious, but true, that by differentiating

$$F(x) = \Pi_{i=0}^{n}(x - x_i) \tag{1.46}$$

we obtain

$$\boxed{\frac{d^{n+1}F}{dx^{n+1}} = (n+1)!} \tag{1.47}$$

To show this to be the case in at least one situation we provide the following example. This is not a proof, but provides some degree of confidence.

Example Consider the example of $n = 1$

$$F(x) = (x - x_0)(x - x_1) \tag{1.48}$$

$$\frac{dF(x)}{dx} = (x - x_0) + (x - x_1) \tag{1.49}$$

$$\frac{d^2F(x)}{dx^2} = 1 + 1 = 2 = 2! \tag{1.50}$$

Step 5 (Calculation of A)

Let us now differentiate $g(x)$ $n+1$ times (see Eq. (1.43)) to give

$$\frac{d^{n+1}g(x)}{dx^{n+1}} = \frac{d^{n+1}f(x)}{dx^{n+1}} - \frac{d^{n+1}\hat{f}(x)}{dx^{n+1}} - A\frac{d^{n+1}F(x)}{dx^{n+1}}. \tag{1.51}$$

Consider what is happening at $x = \xi$ in Eq. (1.51) (we now add letters to refer to the terms):

$$\boxed{\underbrace{\left.\frac{d^{n+1}g(x)}{dx^{n+1}}\right|_{x=\xi}}_{D} = \left.\frac{d^{n+1}f(x)}{dx^{n+1}}\right|_{x=\xi} - \underbrace{\left.\frac{d^{n+1}\hat{f}(x)}{dx^{n+1}}\right|_{x=\xi}}_{B} - \underbrace{A\left.\frac{d^{n+1}F(x)}{dx^{n+1}}\right|_{x=\xi}}_{C}.} \tag{1.52}$$

We now address each of the terms in Eq. (1.52) one at a time. Term D is zero because, from Eq. (1.45)

$$\left.\frac{d^{n+1}g(x)}{dx^{n+1}}\right|_{x=\xi} = 0. \tag{1.53}$$

Term B is zero because we are taking the $n+1$ derivative of an nth degree polynomial.

$$\left.\frac{d^{n+1}\hat{f}(x)}{dx^{n+1}}\right|_{x=\xi} = 0 \tag{1.54}$$

From Eq. (1.47) we know that

$$\left.\frac{d^{n+1}F\left(x\right)}{dx^{n+1}}\right|_{x=\xi}=\left(n+1\right)! \tag{1.55}$$

We now combine this information with Eq. (1.52) to give

$$\left.\frac{d^{n+1}f}{dx^{n+1}}\right|_{x=\xi}=A\left.\frac{d^{n+1}F}{dx^{n+1}}\right|_{x=\xi}=A\left(n+1\right)! \tag{1.56}$$

or, solving for A

$$\boxed{A=\frac{1}{(n+1)!}\left.\frac{d^{n+1}f(x)}{dx^{n+1}}\right|_{x=\xi}.} \tag{1.57}$$

Remember that A is chosen in such a way that $g(x_p)=0$.

Step 6 (Calculation of $E\left(x\right)$)

Now we know by the definition of x_p, that at $x=x_p$, $g\left(x\right)=0$. Thus we have

$$g\left(x_p\right)=f\left(x_p\right)-\hat{f}\left(x_p\right)-AF\left(x_p\right)=0. \tag{1.58}$$

Rearranging Eq. (1.58) and substituting for A we obtain

$$f\left(x_p\right)-\hat{f}\left(x_p\right)=\frac{1}{\left(n+1\right)!}\left.\frac{d^{n+1}f\left(x\right)}{dx^{n+1}}\right|_{x=\xi}\left(F\left(x_p\right)\right). \tag{1.59}$$

Finally, because x_p was selected arbitrarily, we can replace it with x to obtain:

$$f\left(x\right)-\hat{f}\left(x\right)=\frac{1}{\left(n+1\right)!}\left.\frac{d^{n+1}f\left(x\right)}{dx^{n+1}}\right|_{x=\xi}\left(F\left(x\right)\right) \tag{1.60}$$

so we can write, using the definition of $F\left(x\right)$ from Eq. (1.46)

$$\boxed{E\left(x\right)=f\left(x\right)-\hat{f}\left(x\right)=\frac{1}{(n+1)!}\left.\frac{d^{n+1}f(x)}{dx^{n+1}}\right|_{x=\xi}\left(\Pi_{i=0}^{n}\left(x-x_i\right)\right)\qquad\xi\in\left[x_0,x_n\right].} \tag{1.61}$$

Take a closer look at Eq. (1.61). We see that the error in the interpolation is inversely proportional to a function of the number of nodes, that is the term $\frac{1}{(n+1)!}$. It is directly proportional to the value of the derivative, that is term $\left.\frac{d^{n+1}f(x)}{dx^{n+1}}\right|_{x=\xi}$, as one examines the values along x. Finally it is directly proportional to a function of the size of the distance between the location x and the nodal locations x_i, $i=0,...,n$; such that the smaller the spacing between nodes, the smaller the error. For a given domain length, this is related to the value of n. So a large n yields a small distance between nodes and they work together to yield a smaller error.

Example 1

To see how one might use the above concept of error by revisiting the case presented earlier of

$$f\left(x\right)=\ln\left(x\right)\qquad x\in\left[1,2\right]. \tag{1.62}$$

From Eq. (1.30) on page 7 we have the approximation of ln(x) using a piecewise linear Lagrange polynomial given as

$$\hat{f}(x) = -0.693 + 0.693x \tag{1.63}$$

and for $x = 1.25$ we have the following computed values and computed error

$$\hat{f}(1.25) = 0.173 \tag{1.64}$$
$$f(1.25) = 0.22 \tag{1.65}$$
$$E(1.25) = 0.047. \tag{1.66}$$

Now let us calculate the *theoretical* error using Eq. (1.61). Substituting the values for $x = 1.25$ we have

$$E(x) = f(x) - \hat{f}(x) = \frac{1}{(n+1)!}\frac{d^{n+1}f(x)}{dx^{n+1}}\Big|_{x=\xi}\left(\Pi_{i=0}^{n}(x - x_i)\right) \qquad \xi \in [x_0, x_n] \tag{1.67}$$

which upon substitution of $f(x) =$ln(x) and $n = 1$ gives

$$E(x) = f(x) - \hat{f}(x) = \frac{1}{(1+1)!}\frac{d^{1+1}\ln(x)}{dx^{1+1}}\Big|_{x=\xi}\left(\Pi_{i=0}^{1}(x - x_i)\right) \qquad \xi \in [1, 2] \tag{1.68}$$

which yields

$$E(1.25) = \frac{1}{2!}\frac{d^2\ln(x)}{dx^2}\Big|_{x=\xi}(x - x_0)(x - x_1) \tag{1.69}$$

$$\begin{aligned} E(1.25) &= \frac{1}{2!}\frac{d^2\ln(x)}{dx^2}\Big|_{x=\xi}(1.25 - 1)(1.25 - 2) \\ &= \frac{1}{2!}\left(-\frac{1}{\xi^2}\right)(1.25 - 1)(1.25 - 2) \\ &= \frac{0.094}{\xi^2}. \end{aligned} \tag{1.70}$$

So what do we do with this? Well we recognize that

$$1 \le \xi \le 2 \tag{1.71}$$

so, using the limiting (upper and lower bound) values of ξ we have

$$\frac{0.094}{(2)^2} \le E(1.25) \le 0.094. \tag{1.72}$$

or

$$0.023 \le E(1.25) \le 0.094.$$

Note that the actual error according to Eq. (1.64) is $E(1.25) = 0.047$, which is within the bounds indicated by Eq. (1.72).

Example 2

Let us consider the quadratic polynomial approximation to the ln(x) $\quad x \in [1, 2]$. We have

$$\hat{f}(x) = -1.21 + 1.33x - 0.214x^2 \tag{1.73}$$

which evaluated at $x = 1.25$ gives

$$\hat{f}\,(1.25) = 0.213 \tag{1.74}$$
$$f\,(1.25) = 0.22 \tag{1.75}$$
$$E\,(1.25) = 0.006. \tag{1.76}$$

From the error expression we have

$$E\,(x) = f\,(x) - \hat{f}\,(x) = \frac{1}{(n+1)!}\frac{d^{n+1}f\,(x)}{dx^{n+1}}\bigg|_{x=\xi}\left(\Pi_{i=0}^{n}\,(x-x_i)\right) \qquad \xi \in [x_0, x_n] \tag{1.77}$$

which, upon substitution of the function $f\,(x) =$ln(x) and using $n = 2$, yields

$$E\,(x) = f\,(x) - \hat{f}\,(x) = \frac{1}{(2+1)!}\frac{d^{2+1}\ln(x)}{dx^{2+1}}\bigg|_{x=\xi}\left(\Pi_{i=0}^{2}\,(x-x_i)\right) \qquad \xi \in [1, 2] \tag{1.78}$$

or, simplifying,

$$E\,(x) = f\,(x) - \hat{f}\,(x) = \frac{1}{3!}\frac{d^{3}ln\,(x)}{dx^{3}}\bigg|_{x=\xi}\left((x-x_0)\,(x-x_1)\,(x-x_2)\right). \tag{1.79}$$

Now we select the point of interest, that is $x = 1.25$, and substitute it into Eq. (1.79) to get

$$\begin{aligned} E\,(1.25) &= \frac{1}{3!}\left(\frac{2}{\xi^3}\right)(1.25-1)\,(1.25-1.5)\,(1.25-2.0) \\ &= \frac{2}{(3)\,(2)\,\xi^3}\,(0.0469) = \frac{0.0156}{\xi^3}. \end{aligned} \tag{1.80}$$

Now we need to determine a choice of ξ. If we use the upper and lower bounds of the interval $x \in [1, 2]$ and therefore substitute $\xi = 1$ and $\xi = 2$ in Eq. (1.80) we obtain

$$0.00195 \leq E\,(1.25) \leq 0.0156. \tag{1.81}$$

Since the true error is $E\,(1.25) = 0.006$, Eq. (1.81) shows that it lies within the computed interval.

1.8 Multiple Elements

To this point we have been dealing with one interval. We will now introduce some new notation to consider multiple intervals. We define the interval over which one polynomial is defined as an *element*. That element may have any number of nodes depending upon the degree of the polynomial. The higher the degree of Lagrange polynomial, the larger the number of nodes we would use per element. We have thus far considered as high as a quadratic Lagrange polynomial which required three nodes per element.

In numerical methods we find the use of one element rather uninteresting. Rather we want to concatenate several elements, and use a low degree Lagrange polynomial for each element. In this section we extend our earlier work to consider multiple elements.

Firstly, consider a two element approximation using linear Lagrange polynomials in each element. Such an arrangement is found in Fig. 1.9. Notice that there is a node in common

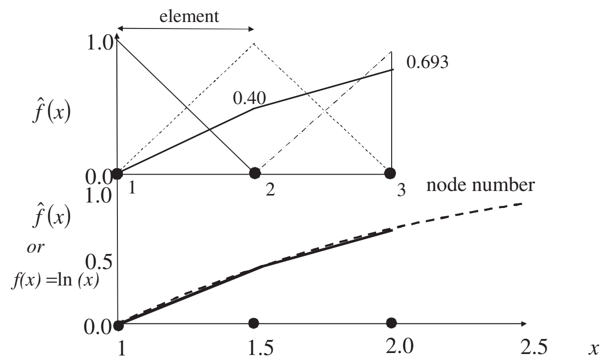

Figure 1.9: Two linear elements and their approximation (solid lines) of the logarithmic function (dashed line).

at the beginning and end of all internal elements. The elements on the ends of the domain of interest share only one internal node. To introduce and illustrate the multiple-element concept, we will use once again the function $f(x) = \ln(x)$ as our example. The values of $f(x) = \ln(x)$ at each node, that is $\ln(x_j)$ where $j = 1, 2, 3$ are used to define the linear approximations

$$\hat{f}(x) = \widehat{\ln}(x) = \sum_{j=1}^{2} \ell_j(x) \ln(x_j) \quad x \in [1, 1.5] \tag{1.82}$$

$$\hat{f}(x) = \widehat{\ln}(x) = \sum_{j=2}^{3} \ell_j(x) \ln(x_j) \quad x \in [1.5, 2.0]\,. \tag{1.83}$$

Note that the indices in the summation now refer to node numbers, for example x_1 denotes the value of x and node 1 and that to simplify notation from this point forward, **we will drop the superscript on ℓ_j^1 for the linear Lagrange polynomial**.

These two equations describe the piecewise linear approximations between the nodes in the upper panel of Fig. 1.9. In panel two, we see the relationship between this approximation and the function being approximated, that is $\ln(x)$. To proceed we need to find a convenient way to relate information at the local level, for example that is that associated with the element, to that of the global system, for example that is associated with the assemblage of elements in which the original problem is defined. To achieve this goal consider the information provided in Fig. (1.10) and Tables 1.3 and 1.4. Figure 1.10 shows the relationship between the global and local coordinate systems. In the local coordinate system, each element sees the world from the same perspective. In other words, an observer at a point in the local system sees only what is happening on the element on which he/she resides. He/she does not see beyond the nodes defining each end of the element. The same observer when associating his/her position with respect to the global system sees the entire domain of interest inclusive of all elements.

It is helpful to distinguish more clearly between the two types of coordinates. In this spirit let us define the local coordinate as χ. On the left is always node 0 and coordinate χ_0 and on the right is always node 1 and coordinate χ_1. Thus $0 \leq \chi \leq 1.$ for each and every element. Similarly the Lagrange polynomials are always represented as a function of χ, that is we have $\ell_0(\chi)$ and $\ell_1(\chi)$ no matter which element we are in.

Element	Local Node	Global Node
1	0	1
1	1	2
2	0	2
2	1	3

Table 1.3: Relationship between local and global node numbering.

Element	χ_0	χ_1
1	$x = 1$	$x = 1.5$
2	$x = 1.5$	$x = 2$

Table 1.4: Relationship between local and global coordinates

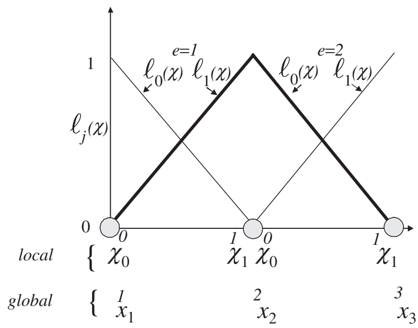

Figure 1.10: Two linear elements in the global (x) and local (χ) coordinate systems.

From the global perspective you see in Fig. 1.10 that the node numbers are increasing from left to right as are the coordinate values. Table 1.3 presents the relationship between nodal numbering at the local or element scale and numbering at the global scale. Similarly, Table 1.4 shows the relationship between the global and nodal coordinates.

Let us see how we can derive a relationship between the two coordinate systems. We know that in element 1 when $\chi = \chi_0$, $x = x_1$. We also know that at $\chi = \chi_1$, $x = x_2$. It is clear that x is a linear function of χ, so we will write

$$x = a + b\chi. \tag{1.84}$$

From our earlier observations we have

$$x_1 = a + b\chi_0 \tag{1.85}$$
$$x_2 = a + b\chi_1 \tag{1.86}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & \chi_0 \\ 1 & \chi_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{1.87}$$

Solving for a and b we have

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & \chi_0 \\ 1 & \chi_1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{\chi_1 - \chi_0} \begin{bmatrix} \chi_1 & -\chi_0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.88}$$

Substitution of Eq. (1.88) into Eq. 1.84 yields

$$\begin{aligned} x &= \frac{\chi_1 x_1 - \chi_0 x_2}{\chi_1 - \chi_0} + \frac{x_2 - x_1}{\chi_1 - \chi_0}\chi \\ &= \frac{\chi_1 - \chi}{\chi_1 - \chi_0} x_1 + \frac{\chi - \chi_0}{\chi_1 - \chi_0} x_2. \end{aligned} \tag{1.89}$$

We have seen this structure before; it is an expansion using Lagrange polynomials, in this case defined in the coordinate system χ. In other words, we can write

$$\begin{aligned} x &= \ell_0(\chi)\, x_1 + \ell_1(\chi)\, x_2 \\ &= \sum_{j=0}^{1} \ell_j(\chi)\, x(\chi_j) \end{aligned} \tag{1.90}$$

where

$$\ell_0(\chi) = \frac{\chi_1 - \chi}{\chi_1 - \chi_0} \tag{1.91}$$

and

$$\ell_1(\chi) = \frac{\chi - \chi_0}{\chi_1 - \chi_0}. \tag{1.92}$$

Thus we can see that we can move between the coordinate systems; that is we can determine a value of x given a value of χ if we know the nodal locations $x(\chi_0)$ and $x(\chi_1)$ which are x_1, and x_2, respectively in our example. If we were to rewrite Eq. (1.84) as

$$\chi = a + bx \tag{1.93}$$

we could show that for element 2 $(e = 2)$ in Fig. 1.10

$$\chi = \ell_2(x)\, \chi\Big|_{x=x_2} + \ell_3(x)\, \chi\Big|_{x=x_3} \tag{1.94}$$

Equation (1.94) states that, given a value of x, we can determine the value of χ.

1.8.1 Example

Suppose we want to find the value of x at location $\chi = 0.5$ in element 2 in Fig. 1.10. Using Eq. (1.90) we have

$$\begin{aligned} x &= \ell_0(\chi)\, x_2 + \ell_1(\chi)\, x_3 \\ x|_{\chi=0.5} &= \ell_0(0.5)\, x_2 + \ell_1(0.5)\, x_3 \\ &= (0.5)\, x_2 + (0.5)\, x_3 \\ &= \frac{x_2 + x_3}{2}. \end{aligned} \tag{1.95}$$

If we assume $x_3 = 2$ and $x_2 = 1.5$ as is the case in our approximation of $\ln(x)$ as shown in Fig. 1.9 we obtain

$$
\begin{aligned}
x|_{\chi=0.5} &= \frac{x_2 + x_3}{2} \\
&= \frac{1.5 + 2}{2} \\
&= \frac{3.5}{2}.
\end{aligned}
\tag{1.96}
$$

For multiple elements we have what is shown in Fig. 1.11.

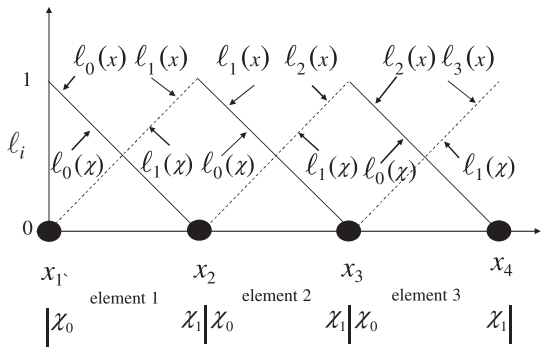

Figure 1.11: Global and local basis functions (Lagrange Polynomials) for four-node problem.

Let us now consider the approximations of our logarithm in each element in Fig. 1.9. In terms of the local coordinate χ it will be of the form:

$$
\hat{f}(\chi) = \sum_{j=0}^{1} \ell_j(\chi) f(\chi_j) = \ell_0(\chi) f(\chi_0) + \ell_1(\chi) f(\chi_1) \quad x \in [1, 1.5] \quad for\ e = 1 \tag{1.97}
$$

$$
\hat{f}(\chi) = \sum_{j=0}^{1} \ell_j(\chi) f(\chi_j) = \ell_0(\chi) f(\chi_0) + \ell_1(\chi) f(\chi_1) \quad x \in [1.5, 2] \quad for\ e = 2 \tag{1.98}
$$

To obtain our approximation to the logarithm, we substitute appropriate values for $f(\chi_j)$. For example, in Fig. 1.9 $f(x_1) = f(x)\Big|_{x=1} = f(\chi)\Big|_{\chi=0} = ln(1) = 0$ for element 1. Substituting the appropriate values observed at the nodes we get

$$
\hat{f}(\chi) = \ell_0(\chi)(0) + \ell_1(\chi)(0.40) \quad x \in [1, 1.5] \quad for\ e = \ 1 \tag{1.99}
$$

$$
\hat{f}\chi = \ell_0(\chi)(0.40) + \ell_1\chi(0.69) \quad x \in [1.5, 2.0] \quad for\ e = \ 2. \tag{1.100}
$$

If we now substitute in the definition of the linear Lagrange polynomials we obtain for element 1

$$
\begin{aligned}
\hat{f}(\chi) &= \left(\frac{\chi_1 - \chi}{\chi_1 - \chi_0}\right)(0) + \left(\frac{\chi_0 - \chi}{\chi_0 - \chi_1}\right)(0.40) \\
\hat{f}(\chi) &= \left(\frac{1.0 - \chi}{1.0}\right)(0) + \left(\frac{0 - \chi}{-1.0}\right)(0.40) \\
\hat{f}(\chi) &= 0.0 + 0.40\chi \quad x \in [1, 1.5]
\end{aligned}
\tag{1.101}
$$

and for element 2

$$\hat{f}(\chi) = \left(\frac{\chi_1 - \chi}{\chi_1 - \chi_0}\right)(0.40) + \left(\frac{\chi_0 - \chi}{\chi_0 - \chi_1}\right)(0.693)$$
$$\hat{f}(\chi) = \left(\frac{1.0 - \chi}{1.0}\right)(0.40) + \left(\frac{0 - \chi}{-1.0}\right)(0.693)$$
$$\hat{f}(\chi) = 0.29\chi + 0.40 \qquad x \in [1.5, 2.0] \tag{1.102}$$

which gives us an interpolation of the logarithm using the local coordinate system.

Now let us calculate the value of $\ln(1.25)$. We obtain for element 1

$$\hat{f}(\chi = 0.5) = 0.40(0.5) = 0.2 \tag{1.103}$$

since $\hat{f}(\chi)$ has been obtained. For the location 1.75 we need to use the approximation in element 2. Thus we have

$$\hat{f}(\chi = 0.5) = 0.29\,(0.5) + (0.4) = 0.546. \tag{1.104}$$

Tabulating the results, including the quadratic approximation we obtained in Eq. (1.73), we get

	$f(1.25)$	$\hat{f}(1.25)$	$E(1.25)$	$f(1.75)$	$\hat{f}(1.75)$	$E(1.75)$
One linear element	0.22	0.173	0.047	0.559	0.52	0.04
Two linear elements	0.22	0.20	0.02	0.559	0.546	0.013
One quadratic element	0.22	0.22	−0.006	0.559	0.560	0.001

Note for future reference that Eqs. (1.97) and (1.98) could be written in matrix notation for an arbitrary element e as

$$\left[\hat{f}(\chi)\right]^e = \begin{bmatrix} \ell_0(\chi) & \ell_1(\chi) \end{bmatrix}^e \begin{bmatrix} f(\chi_0) \\ f(\chi_1) \end{bmatrix}^e . \tag{1.105}$$

1.9 Hermite Polynomials

In this section we investigate another form of polynomial that allows us to directly interpolate not only the function $f(x)$ but also its derivative $\frac{df(x)}{dx}$; this is the Hermite polynomial. The Hermite polynomial is one of a suite of polynomials that when concatenated make up a numeric function that has a prescribed degree of smoothness. The point of departure for establishing the form of these piecewise Hermite polynomial functions is the following expression:

$$f(x) = \sum_{j=0}^{1}\left(h_j^0(x)\,f(x_j) + h_j^1(x)\left.\frac{df}{dx}\right|_{x_j}\right) + E(x)$$
$$= \hat{f}(x) + E(x) \tag{1.106}$$

where the functions $h_j^0(x)$ and $h_j^1(x)$ are the Hermite polynomials. The superscripts here are identified with the two forms of the Hermite *function*, one, h_j^0 associated with the value of the function at the node x_j, that is,. $f(x_j)$, and the other $h_j^1(x)$ associated with the *derivative* at the node, that is $\left.\frac{df}{dx}\right|_{x_j}$. There are four functions, two associated with each node in the element, and they are shown in Fig. 1.12. To determine the form of the functions $h_j^0(x)$ and $h_j^1(x)$ we require that, as in the case of the Lagrange polynomials, the approximating function exactly represent at the nodes the function being approximated. However, in addition to these constraints we require that the approximation of the derivatives of the function being approximated also be exact at the nodes. These sets of requirement, two at each of two nodes, yields:

$$\hat{f}(x_0) = f(x_0) \tag{1.107}$$

$$\hat{f}(x_1) = f(x_1) \tag{1.108}$$

$$\left.\frac{d\hat{f}}{dx}\right|_{x_0} = \left.\frac{df}{dx}\right|_{x_0} \tag{1.109}$$

$$\left.\frac{d\hat{f}}{dx}\right|_{x_1} = \left.\frac{df}{dx}\right|_{x_1}. \tag{1.110}$$

We now expand Eq. (1.106) for each of the above conditions; that is we expand the approximation and then impose the requirements of Eqs. (1.107)-(1.110). The first line of the two associated with each approximate value, for example $\hat{f}(x_0)$, is the expansion and the second is the value the expansion must represent.

$$\begin{aligned}\hat{f}(x_0) &= h_0^0(x_0)\, f(x_0)^* + h_1^0(x_0)\, f(x_1) + h_0^1(x_0)\left.\frac{df}{dx}\right|_{x_0} + h_1^1(x_0)\left.\frac{df}{dx}\right|_{x_1} \\ &= f(x_0)^* \end{aligned} \tag{1.111}$$

$$\begin{aligned}\hat{f}(x_1) &= h_0^0(x_1)\, f(x_0) + h_1^0(x_1)\, f(x_1)^* + h_0^1(x_1)\left.\frac{df}{dx}\right|_{x_0} + h_1^1(x_1)\left.\frac{df}{dx}\right|_{x_1} \\ &= f(x_1)^* \end{aligned} \tag{1.112}$$

$$\begin{aligned}\left.\frac{d\hat{f}}{dx}\right|_{x_0} &= \left.\frac{dh_0^0}{dx}\right|_{x_0} f(x_0) + \left.\frac{dh_1^0}{dx}\right|_{x_0} f(x_1) + \left.\frac{dh_0^1}{dx}\right|_{x_0}\left.\frac{df}{dx}\right|_{x_0}^* + \left.\frac{dh_1^1}{dx}\right|_{x_0}\left.\frac{df}{dx}\right|_{x_1} \\ &= \frac{df(x_0)}{dx}^* \end{aligned} \tag{1.113}$$

$$\begin{aligned}\left.\frac{d\hat{f}}{dx}\right|_{x_1} &= \left.\frac{dh_0^0}{dx}\right|_{x_1} f(x_0) + \left.\frac{dh_1^0}{dx}\right|_{x_1} f(x_1) + \left.\frac{dh_0^1}{dx}\right|_{x_1}\left.\frac{df}{dx}\right|_{x_0} + \left.\frac{dh_1^1}{dx}\right|_{x_1}\left.\frac{df}{dx}\right|_{x_1}^* \\ &= \left.\frac{df}{dx}\right|_{x_1}^*. \end{aligned} \tag{1.114}$$

In general, for Eqs. (1.111) through (1.114) to be satisfied, for any one of the four equations the starred terms in each expression should be equal. For this to be true, the information appearing in the following tables is required. The first line of Table 1.5, for example, should be read as follows: In order to have $\hat{f}(x_0) = f(x_0)$, we should have $h_0^0 = 1$. In addition, as seen in Table 1.6 $h_0^1(x_0)$ and $h_1^1(x_0)$ must also be zero at both x_0 and x_1 and $h_1^0(x_0) = 0$ where the locations x_0 and x_1 are the two nodal locations in the element.

From the first row in Tables 1.5 and 1.6 we see that there are four conditions to be imposed on $h_0^0(x)$ (read horizontally across in each of these tables). For example, $h_0^0(x)$ must take on a value of 1 at x_0 and 0 at x_1 and $\frac{dh_0^0(x)}{dx}$ must be 0 at both x_0 and x_1. Since a cubic polynomial is completely defined by four conditions (which is the number we have available

	$\hat{f}(x_0) = f(x_0)$	$\hat{f}(x_1) = f(x_1)$
h_0^0	1	0
h_1^0	0	1
h_0^1	0	0
h_1^1	0	0

Table 1.5: Necessary conditions to be imposed on the Hermite polynomials to assure that the approximation and function are the same at the nodes.

	$\left.\frac{d\hat{f}}{dx}\right\|_{x_0} = \left.\frac{df}{dx}\right\|_{x_0}$	$\left.\frac{d\hat{f}}{dx}\right\|_{x_1} = \left.\frac{df}{dx}\right\|_{x_1}$
$\frac{dh_0^0}{dx}$	0	0
$\frac{dh_1^0}{dx}$	0	0
$\frac{dh_0^1}{dx}$	1	0
$\frac{dh_1^1}{dx}$	0	1

Table 1.6: Necessary conditions to be imposed on the Hermite polynomials to assure that the approximation of the derivative and and the derivative are the same at the nodes.

for each of the functions $h_0^0(x)$, $h_1^0(x)$, $h_0^1(x)$, and $h_1^1(x)$), let us assume that each is a cubic polynomial of the form, for example:

$$h_0^0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3. \tag{1.115}$$

The coefficients a_0, b_0, c_0, and d_0 can be determined using the information in the above tables. For example one equation for the case of $h_0^0(x)$ we can obtain using the first element of the first row in Table 1.5; that is $h_0^0(x)$ must be 1 at x_0. We get:

$$h_0^0(x_0) = a_0 + b_0 x_0 + c_0 x_0^2 + d_0 x_0^3 = 1. \tag{1.116}$$

We know that the derivative of Eq. (1.115) at x_0 must equal zero from the first row in Table 1.6. Collecting this information for each element in row 1 of these tables we obtain the following set of equations for the coefficients a_0, b_0, c_0 and d_0

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 0 & 1 & 2x_0 & 3x_0^2 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 1 & 2x_1 & 3x_1^2 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} h_0^0(x_0) = 1 \\ \left.\frac{dh_0^0}{dx}\right|_{x_0} = 0 \\ h_0^0(x_1) = 0 \\ \left.\frac{dh_0^0(x_1)}{dx}\right|_{x_1} = 0. \end{matrix} \tag{1.117}$$

In reading Eq. (1.117) the information appearing to the right of the matrix equation identifies the conditions giving rise to each row of the equation.

Solving for a_j, b_j, c_j, and d_j and substituting in cubic polynomial we obtain

$$h_j^0(x) = \left(\ell_j^{(1)}(x)\right)^2 \left(1 - 2(x - x_j) \left.\frac{d\ell_j^{(1)}}{dx}\right|_{x_j}\right) \tag{1.118}$$

where the Lagrange polynomial approximations are used in the definition. One can make similar arguments to obtain

$$h_j^1(x) = \left(\ell_j^1(x)^2\right)(x - x_j). \tag{1.119}$$

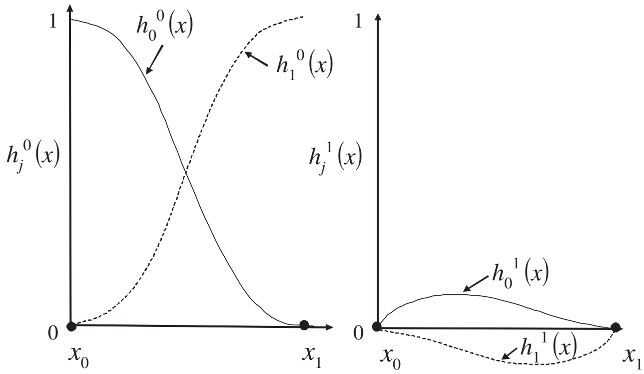

Figure 1.12: Hermite polynomials.

As noted earlier, the resulting functions are provided in Fig. 1.12. Note that the curves in this figure satisfy the constraints provided in Tables 1.5 and 1.6.

Example

Consider the function

$$f(x) = \text{exp}(x) \qquad x \in [0.5, 1] \tag{1.120}$$

which upon differentiation yields

$$\frac{df(x)}{dx} = \text{exp}(x) \qquad x \in [0.5, 1]\,. \tag{1.121}$$

The approximation $\hat{f}(x)$ is given by introducing the values in Eq. (1.120) and (1.121) into Eq. (1.106); that is

$$\hat{f}(x) = h_0^0(x)\,\text{exp}(0.5) + h_0^1(x)\,\text{exp}(0.5) + h_1^0(x)\,\text{exp}(1.0) + h_1^1(x)\,\text{exp}(1.0) \tag{1.122}$$

A visual representation of the resulting approximation is provided in Fig. 1.13

1.10 Error in Approximation by Hermites

We are not going to develop the error of the approximation using Hermites, but simply provide it for the our case (cubic with two nodes) below, viz.

$$E(x) = \frac{1}{(4)!}\left[\Pi_{i=0}^{1}(x - x_i)\right]^2 \left.\frac{d^4 f}{dx^4}\right|_{x=\xi}. \tag{1.123}$$

Example of Hermite Error Approximation

$$\begin{aligned} E(0.6) &= \frac{1}{4!}\left[(x - x_0)(x - x_1)\right]^2 \text{exp}(x)\,|_{x=\xi} \quad x \in [0.5, 1.0] \\ &= \frac{1}{4!}\left[(0.6 - 0.5)(0.6 - 1.0)\right]^2 \text{exp}(x)\,|_{x=\xi} \\ &= 6.67 \times 10^{-5} \text{exp}(x)\,|_{x=\xi}. \end{aligned} \tag{1.124}$$

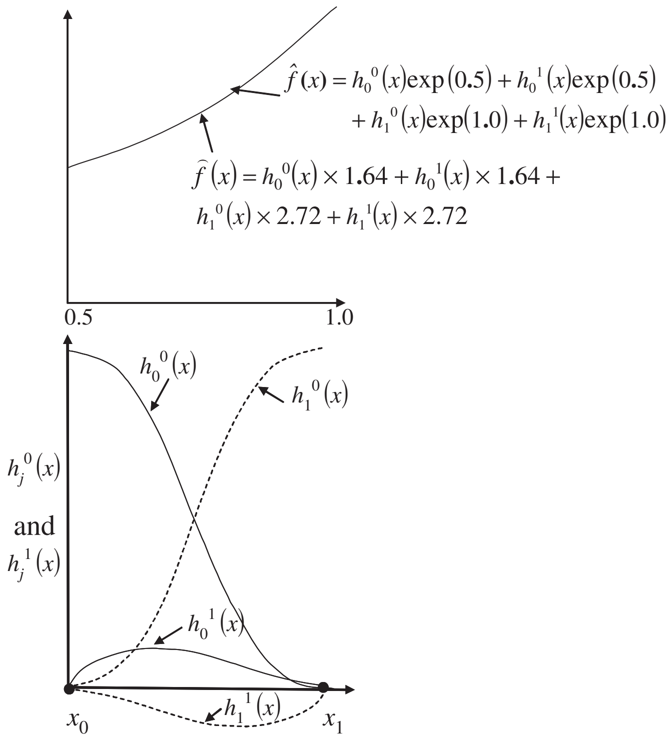

Figure 1.13: Hermite approximation of the exponential function.

Since

$$0.5 \leq \xi \leq 1.0 \tag{1.125}$$

then using these limits the error bound is

$$1.1 \times 10^{-4} \leq E(0.6) \leq 1.81 \times 10^{-4}. \tag{1.126}$$

The measured error is

$$E(0.6) = 1.82212 - 1.82198 = 1.4 \times 10^{-4} \tag{1.127}$$

which lies within the error bounds calculated.

1.11 Chapter Summary

Polynomial approximation theory, the subject of this chapter, is a fundamental theoretical underpinning for numerical methods. In this chapter we introduce this topic in the context of using polynomial approximation theory to approximate functions. The Lagrangian polynomials and the Hermite polynomials are considered along with the error associated with using them in approximating functions. The concept of discretizing a domain, say a length along x, into a set of concatenated linear segments called elements is introduced.

1.12 Problems

1. Determine (derive) the form of $\ell_1(x)$ (see Fig. 1.14). The result should be an algebraic equation that describes the function $\ell_1(x)$ for any interval (x_0, x_1). The strategy is to write the general form of the linear equation and then impose the conditions required of a linear approximating function.

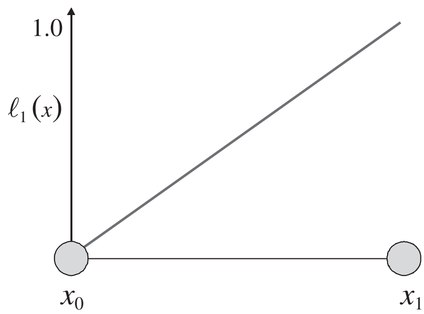

Figure 1.14: Linear basis function.

2. Using Eq. (1.128) below write the form of the second degree or quadratic Lagrange polynomial $\ell_1^2(x)$ shown below? How does it satisfy the requirements of a Lagrange polynomial?

$$\ell_j^{(n)}(x) = \Pi_{\substack{i=0 \\ i \neq j}}^{n} \frac{x - x_i}{x_j - x_i}. \tag{1.128}$$

The shape of this function is shown in Fig. 1.15 below.

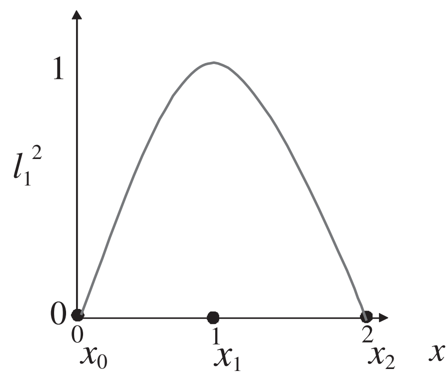

Figure 1.15: Quadratic Lagrange basis function defined over the interval $x_0 - x_2$.

3. $P_n(\theta)$ is a polynomial of degree n, that is,

$$P_n(\theta) = \sum_{i=0}^{n} a_i \theta^i \tag{1.129}$$

$P_n(\theta_i) = f(\theta_i)$ where θ_i are particular values of θ. Assume $f(x)$ =sin(θ) (see Fig. 1.16) and that $n = 3$ (the polynomial is a cubic). Determine the polynomial represented by 1.129 using points $\theta_i = 0, \pi/4, 3\pi/4, \pi$. Compare these errors to those presented in the example in the notes in Section 1.3. Sin (θ) curve with measured points indicated by small black dots and interpolated values indicated by large black dots.

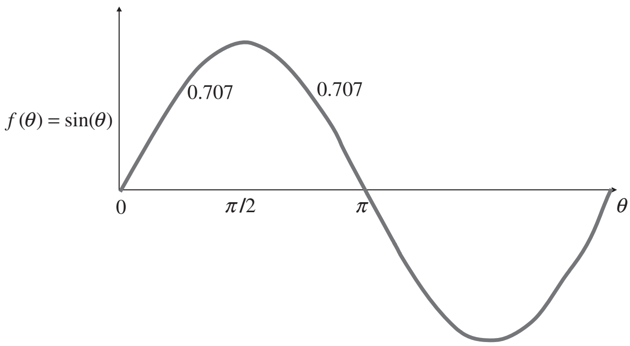

Figure 1.16: Plot of $\sin(\theta)$ over the range $\theta = 0$ to $\theta = 2\pi$.

4. Consider the function

$$f(x) = e^x \qquad x \in [0, 1]. \tag{1.130}$$

Use a piecewise linear polynomial to determine $e^{0.5}$. Now calculate the error of your approximation, that is, determine the difference between the exact value of $e^{0.5}$ and your estimate. Using the following equation for the theoretical error

$$E(x) = f(x) - \hat{f}(x) = \frac{1}{(n+1)!} \frac{d^{n+1} f(x)}{dx^{n+1}}\bigg|_{x=\xi} \left(\Pi_{i=0}^{n} (x - x_i)\right) \qquad \xi \in [x_0, x_n] \tag{1.131}$$

Show that the error of your estimate is consistent with the theoretical estimate.

5. Consider the situation presented in Fig.1.17. You know the value of x at the location at 1.5 in inches and you want to determine the same location in centimeters. Using the transformations given by

$$x = a + b\chi \tag{1.132}$$

where x is the distance in inches and χ is the distance in centimeters determine the location X in centimeters.

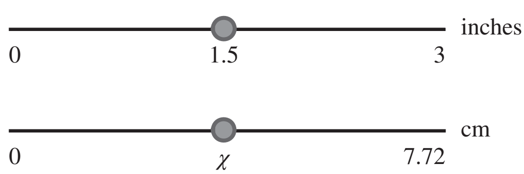

Figure 1.17: Representation of a line in terms of inches and centimeters.

6. Please do the following:

   (a) Using the information in Fig 1.18, draw the linear Lagrange functions $\ell_i(x)$ and $\ell_i(\chi)$. In the local coordinate system use a solid line for $\ell_0(\chi)$ and a dashed line for $\ell_1(\chi)$. Label the horizontal and vertical axes with appropriate values. Give each of the symbolic x_i and χ_i actual values, (in the sense of values of x_0 and χ_0 etc.) and actual numerical values (in the sense of 1.0 etc.) in the spaces provided. You will have five values of x_i of your choice and the same number of values of χ, but there will be duplicate values of χ at most of the nodes.

(b) What is the value of $h = x_{i+1} - x_i$?

(c) What is the value of $\chi_1 - \chi_0$?

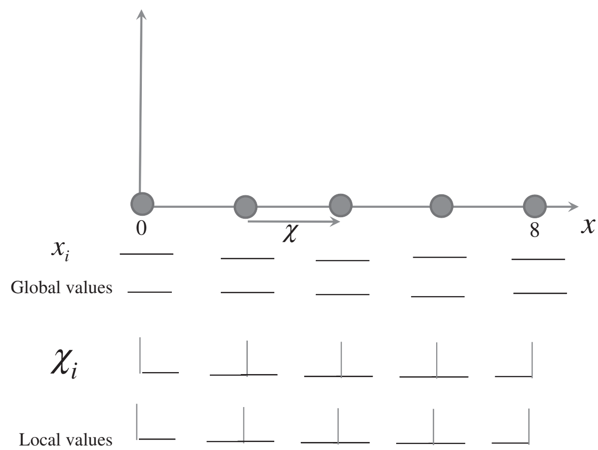

Figure 1.18: Representation of five points in global (x) and local (χ) coordinate systems.

7. Given the formula for the Lagrange polynomial, that is

$$\ell_j^n(x) = \Pi_{\substack{i=0\\i\neq j}}^{n} \frac{x - x_i}{x_j - x_i} \tag{1.133}$$

write the form of the cubic Lagrange polynomial. Then use this expression to write the approximation of the $\sin(\theta)$ segment from 0 to π. Use four equally spaced values of π to obtain this approximation (four π values). Now evaluate your polynomial at the points indicated in Table 1.1, to compute the error, and note how the errors compare with those already recorded in this table.

8. From the equation

$$x = \sum_{j=0}^{1} \ell_j(\chi)\, x(\chi_j) \qquad \chi \in (0,1), \quad x \in (1,2) \tag{1.134}$$

determine the value of x at $\chi = 0.75$.

9. State the four requirements (constraints) necessary to obtain a Hermite interpolating polynomial h_0^0 as used in the expression

$$\hat{u}(x) = u_0 h_0^0 + u_1 h_1^0 + \left.\frac{du}{dx}\right|_0 h_0^1 + \left.\frac{du}{dx}\right|_1 h_1^1 \tag{1.135}$$

assuming you start with the standard cubic polynomial.

10. Use a linear polynomial to approximate $\sin\theta, 0 \leq \theta \leq \pi$. shown in Fig. 1.19. Determine whether the error at $\pi/2$ falls within the range predicted by the error formula.

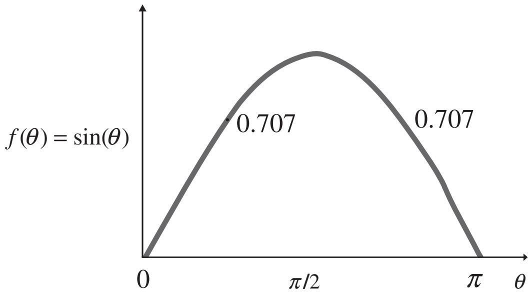

Figure 1.19: The $\sin(\theta)$ function from 0 to π.

11. Use quartic (fourth degree) Lagrange polynomials to approximate sin θ using nodes at $0, \pi/2, \pi, (3/2)\pi$ and π. Determine the value of $\sin\theta$ at $(5/4)\pi$. Compare the estimate to the value obtained using the quadratic Lagrange polynomial. Is it better; why (or why not)?

12. Given the $\sin(\theta)$ function shown above and linear polynomials written in terms of the local coordinate system χ, determine the value of the $\sin(\theta)$ at $\chi = 0.5$.

13. The goal here is to determine if one can use any two points along the line from $x = 0$ to $x = 2$ in Fig. 1.20 to estimate the value at $x = 1.0$. Please solve this problem by following these steps:

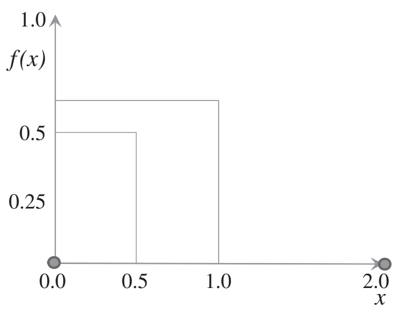

Figure 1.20: Problem definition.

(a) Assume $f(x)$ can be approximated by

$$\hat{f}(x) = \sum_{j=0}^{1} \ell_j(x) f_j \tag{1.136}$$

where $\ell_j(x)$ is the linear Lagrange polynomial and f_j is the value of $f(x)$ at $x = x_j$.

(b) Expand this expression to represent that needed for this problem.

(c) Evaluate the resulting expression at the point $x = 0.0$ and $x = 0.5$. Assume $\hat{f}(0.0) = 0.25$ and $\hat{f}(0.5) = 0.5$. You should have two equations in the two unknown values of f_1 and f_2 where the subscripts are the two nodal identifiers, that is nodes 1 and 2.

(d) Solve for the two unknown values of f_1 and f_2..

(e) Use these values to estimate the value of $f(x)$, where $x = 1.0$.

Hint: Remember that the linear Lagrange polynomials are 1 at the node for which they are defined and 0 elsewhere, so in this case they are two straight lines crossing at $x = 1$ and $\ell(1) = 0.5$.

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