1

Universal Functions for Nonisothermal and Conjugate Heat Transfer

1.1 Formulation of Conjugate Heat Transfer Problem

As it follows from the above discussion, the domain of any conjugate problem consists at least of two subdomains according to the interaction components. Therefore, to formulate conjugate problem, it is necessary to specify two sets of equations: initial and boundary conditions governing the problem in each of subdomains in order to further conjugation of the corresponding solutions. In the case of heat transfer, such subdomains and sets of governing equations and boundary conditions are as follows:

- **Body domain:**
  Unsteady conduction equation

$$\frac{\partial T}{\partial t} = \alpha_w \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{q_{\mathrm{v}}}{\rho_w c_w}, \qquad \frac{\partial T}{\partial t} = \alpha_w \nabla^2 T + \frac{q_{\mathrm{v}}}{\rho_w c_w} \tag{1.1}$$

  or steady conduction equations:
  Laplace's and Poisson's equations (without and with heat source q_{v}) (Com. 1.1)

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0, \qquad \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = -\frac{q_{\mathrm{v}}}{\lambda_w} \tag{1.2}$$

  or simplified conduction equations for "thin body" and "thermally thin body" (Com. 1.1)

$$\alpha_w \frac{d^2 T}{dy^2} + \frac{q_{\mathrm{v}}}{\lambda_w} = 0, \quad \frac{1}{\alpha_w}\frac{\partial T_{\mathrm{av}}}{\partial t} - \frac{\partial^2 T_{\mathrm{av}}}{\partial x^2} + \frac{q_{w1} + q_{w2}}{\lambda_w \Delta} - \frac{q_{\mathrm{v.av}}}{\lambda_w} = 0 \tag{1.3}$$

Applications of Mathematical Heat Transfer and Fluid Flow Models in Engineering and Medicine,
First Edition. Abram S. Dorfman.

- **Fluid flow domain:**
  For laminar flow: Navier-Stokes and energy equations (S. 7.1)

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1.4}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \tag{1.5}$$

$$\rho\left(\frac{\partial \mathrm{v}}{\partial t}+u\frac{\partial \mathrm{v}}{\partial x}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial y}+w\frac{\partial \mathrm{v}}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 \mathrm{v}}{\partial x^2}+\frac{\partial^2 \mathrm{v}}{\partial y^2}+\frac{\partial^2 \mathrm{v}}{\partial z^2}\right) \tag{1.6}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+\mathrm{v}\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{1.7}$$

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\mathrm{v}\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\lambda\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\mu S$$

$$S=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2 \tag{1.8}$$

or simplified equations for high and low Reynolds and Peclet numbers:
Boundary layer equations (S. 7.4.4.1)

$$\frac{\partial u}{\partial x}+\frac{\partial \mathrm{v}}{\partial y}=0 \tag{1.9}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{dp}{dx}-\nu\frac{\partial^2 u}{\partial y^2}=0 \tag{1.10}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\mathrm{v}\frac{\partial T}{\partial y}-\alpha\frac{\partial^2 T}{\partial y^2}-\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2=0 \tag{1.11}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\qquad -\frac{1}{\rho}\frac{dp}{dx}=U\frac{dU}{dx} \tag{1.12}$$

Creeping flow equations (S. 7.4.1)

$$\nabla\cdot\mathbf{V}=0\qquad \nabla p=\mu\nabla^2\mathbf{V} \tag{1.13}$$

For turbulent flow: Reynolds averaged Navier-Stokes and energy equations in Einstein notations (S. 7.1.2.2)

$$\frac{\partial u_i}{\partial x_i}=0,\qquad \rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_j}\left[\left(\mu+\mu_{tb}\right)\frac{\partial u_i}{\partial x_j}\right] \tag{1.14}$$

$$\rho\frac{\partial T}{\partial t}+\rho u_j\frac{\partial T}{\partial x_j}=\frac{\partial}{\partial x_j}\left[\left(\frac{\mu}{\mathrm{Pr}}+\frac{\mu_{tb}}{\mathrm{Pr}_{tb}}\right)\frac{\partial T}{\partial x_j}\right]+\frac{\mu+\mu_{tb}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2 \tag{1.15}$$

or simplified boundary layer equations for high Reynolds and Peclet numbers (S. 8.3):

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\mathrm{v}\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{dp}{dx}-\frac{\partial}{\partial x}\left[\left(\nu+\nu_{tb}\right)\frac{\partial u}{\partial x}\right]=0 \tag{1.16}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+\mathrm{v}\frac{\partial T}{\partial y}-\frac{\partial}{\partial y}\left[\left(\alpha+\alpha_{tb}\right)\frac{\partial T}{\partial y}\right]-\frac{\nu+\nu_{tb}}{c_p}\left(\frac{\partial u}{\partial y}\right)^2=0 \tag{1.17}$$

and continuity (1.9) and Bernoulli (1.12) equations.

- **Conjugate conditions** (boundary conditions of the forth kind) (Com.1.1)

$$T^+=T^-,\qquad \lambda_w\frac{\partial T}{\partial y}\bigg|^+=-\lambda\frac{\partial T}{\partial y}\bigg|^- \tag{1.18}$$

Comment 1.1 Body domain: (i) in equations (1.2) ∇^2 is the Laplace operator (S. 7.1.2.1) and $\nabla^2 T$ is called Laplacian of T, (ii) the first equation (1.3) is obtained from two-dimensional second equation (1.2) for thin body ($\Delta/L \ll 1$), which thermal resistance is of the same order or greater than that of coolant ($\mathrm{B}i = h\Delta/\lambda_w \geq 1$), and due to that longitudinal derivative may be neglected, (iii) the second equation (1.3) is derived for thermally thin body (Bi ≤ 1) by integration of two-dimensional equation (1.1) in y-direction taken into account that transverse resistance of such body is small in comparison with that of the coolant.

Fluid flow domain: (i) the Navier-Stokes and energy equations are used in both three- and two-dimensional forms, whereas the boundary layer equations are basically employed in two-dimensional form that is appropriate for majority of applications, (ii) the creeping flow equations written in vector form (1.13) may be obtained from Navier-Stokes and energy equations after neglecting the inertia terms (left parts of equations (1.5)–(1.7) (S. 7.4.1), in this case, the continuity equation is the same equation (1.4), (iii) Navier-Stokes and energy equations are presented in instantaneous parameters, whereas the equations (1.14)–(1.17) for turbulent flow are written in averaged parameters using the same notations (S. 8.2), (iv) eddy-viscosity coefficient μ_{tb} in these equations is determined by one of turbulent models (S. 8.3) and the value of turbulent Prandtl number Pr_{tb} is usually taken to be equal or close to unit (S. 2.1.2.4).

Conjugate conditions are expressions providing continuity of the temperature fields at the interface in the form of equalities of temperatures and heat fluxes computed from both interface sides and marked by (+) and (−) for body and fluid, respectively (Exer. 1.1–1.7).

- Initial and boundary conditions for subdomains (S.7.2)

The equations just considered are used to solve the subdomain problems. The relevant initial and boundary conditions depend on the type and order of governing equation. For example, the conduction equation is of the first order in time and of the second order in space. Because of that, the solution of one-dimensional conduction problem depends on one initial and two boundary conditions given at two points. Solutions of more complicated two- or three-dimensional conduction problems also require satisfaction of one initial condition and of boundary conditions, but specified around the outline of the whole problem domain. The initial condition defines the temperature of the system as a function of coordinates at some instant $t = 0$, which is taken to be a beginning of the process, whereas the boundary conditions prescribe the values of some parameters on the boundaries of the system as the functions of time and position.

Besides the boundary conditions of the fourth kind (1.18), there are three usually employed boundary conditions that differ from each other by kind of variables assigned at some points or around boundaries of domain. The boundary condition of the first kind designates the temperature values, the second one specifies heat fluxes, and the third kind of boundary condition presents Newton's expression with known heat transfer coefficient and temperature head. For example, at some point of domain with coordinate x, three types of conditions have the form

$$T|_x = T_w, \qquad or \qquad q_w = -\lambda_w \left.\frac{\partial T}{\partial x}\right|_x, \qquad or \qquad q_w = h(T|_{w,x} - T_\infty) \tag{1.19}$$

The equations specifying the conjugate problem formulation considered above are partial differential equation of the second order. The appropriate boundary conditions for those equations depend on the type of a particular equation. It is said that such equation in its canonical form is of elliptic or hyperbolic type, depending whether it consist of a sum or a difference of two second order derivatives, whereas such canonical equation with only one second partial derivative is called a parabolic partial differential equation (Exer. 1.8). Thus, the two- and three-dimensional conduction equations, Laplace and Poisson equations as well as Navier-Stokes and energy equations for laminar flow and similar equations for turbulent flow are of elliptic type, whereas both sets of boundary layer equations for laminar and turbulent flows are the parabolic equations. The parabolic equation requires relatively simple boundary conditions specifying variable values only on a part of computational domain. For boundary layer equations, these conditions are: (i) the no-slip condition $u = \mathrm{v} = 0$ for dynamic equation and conjugate (1.18) or one of the regular (1.19) conditions for thermal equation on the body surface ($y = 0$) and (ii) asymptotic conditions $u \to U$, $T \to T_\infty$ far from the body on the outer edge of the boundary layer ($y \to \infty$) (S. 7.4.4).

In contrast to that, the elliptic equations require boundary conditions specifying the values of parameters around the entire computational domain, and formulation of such conditions is more complicated procedure. In this case, two problem formulations with different types of boundary conditions, known as Dirichlet and Neumann problems, are usually considered. The Dirichlet problem is stated using boundary conditions of the first kind by specifying the temperature on the boundaries of domain. The Neumann problem is composed similarly by employing the boundary conditions of the second kind by specifying the derivatives of the temperatures on the boundaries domain.

The Dirichlet problem is a well-posted problem, whereas the Neumann problem is an ill-posted problem. Physically it means that the solution behavior of an elliptic equation with Dirichlet boundary conditions is regular, whereas the solution of an elliptic equation under the ill-posted Neumann boundary conditions requires satisfaction on some additional conditions. For example, the solution of the Neumann heat transfer problem demands the thermal equilibrium that a system reaches when the total heat flux inside it is zero.

Comment 1.2 Formulation of boundary conditions for the Navier-Stokes equation is associated with additional difficulties arising due to fluid nature. To understand these complications, consider a flow through a channel or past a body immersed in a parallel fluid stream. Considering the Dirichlet problem, one should specify velocities along the boundaries of domain, which for a channel or tube include the walls or body surface plus entrance and exit sections of a channel or body. However, in such a case, only zero velocity boundary conditions on the surfaces of the immersed channel or body are known and a uniform stream velocity U_∞ at $x \to \pm\infty$ far away from the immersed object. The velocity profiles at the entrance and the exit sections required for the Dirichlet problem formulation are unknown because: (i) the profile at the entrance is established due to interaction of initial uniform stream with surrounding during the way from $x \to -\infty$ to channel or body entrance, and (ii) the velocity profile at the exit is formed as a result of processes inside the flow in the channel or around the body. Because of that in practice, the experimental data or relevant assumptions are used. The same situation holds for full energy equation since this equation is of elliptic type as well, and the initial uniform temperature profiles are deformed along with the velocity profiles (S. 7.2, Exam. 7.4 and 7.5, Exer. 1.9 and 1.10).

1.2 Methods of Conjugation

Physical analysis shows that any heat transfer conjugate problem is a question of thermal interaction of a body and a fluid with unknown temperature and heat flux distribution on the body/fluid interface. This becomes clear if one looks at what is known at the beginning of the conjugate problem solution. Indeed at the beginning, we know only: (i) set of equations governing heat transfer in a body and in a fluid separately, and (ii) conjugate conditions (1.18). However, the separate boundary conditions for each subdomain are unknown since data on interface may be obtained only as a result of conjugate problem solution. Thus, the situation is deficient: to solve a particular conjugate problem we need boundary conditions for each subdomain, which may be obtained only after solution of the same conjugate problem. There are several methods for resolving this challenging problem. Here, we consider two mostly used numerical methods and analytical approach based on employing so called universal functions [120]. Examples of other procedures for solving this problem are discussed in applications.

1.2.1 Numerical Methods

One relatively simple way to realize conjugation is to apply the iterations. The idea of such a approach is that each solution for the body or for the fluid produces a boundary condition for other component of the system. The process of interactions starts by assuming that one of the boundary conditions (1.19) exists on the interface. Then, one solves the problem for a body or for fluid applying this boundary condition and uses the result of solution as a boundary condition for solving the set of governing equations for other component and so on. If this iterative process converges, it might be continued until the desired accuracy is achieved. However, the rate of convergence of the iterations highly depends on the first guessing boundary condition, and there is no way to find an appropriate condition, except using the trial-and-error approach (Exer. 1.12).

Another known numerical conjugate procedure is grounded on the simultaneous solution of a large set of governing equations for both subdomains and conjugate conditions. Patankar [306] proposed a method and software for such a solution using one generalized expression for continuous computing of the velocities and temperatures fields through the whole problem domain that includes satisfying the conjugate boundary conditions. To make sure that one generalized equation provides the correct results in different part of the whole domain, the corresponding value of physical properties for each subdomain is employed. Thus, to ensure that the velocity is zero in the solid when the velocity field is calculated, one puts a very large value of viscosity coefficient for the grids points in the body domain, whereas for the grid points in the fluid domain, the real fluid viscosity coefficient is applied. When the temperature field is computed, the real values of heat transfer characteristics for the fluid and for the body are used, which gives the actual temperature field as a result of matching the temperature distribution in both computing subdomains (Exer. 1.13).

1.2.2 Using Universal Functions

Since the boundary conditions for subdomins are unknown, the required analytical solutions for body and fluid might be obtained only applying arbitrary nonisothermal boundary conditions. In other words, it is necessary to find a solution in the form that satisfies the governing equation at any (or arbitrary) boundary conditions (Exer. 1.14). Such solutions are given, in particular, in the form of universal functions, which are called universal because they satisfied a particular equation independent of boundary conditions [120]. In the next several paragraphs, we present two forms of universal functions used in this text for investigation of nonisothermal and conjugate heat transfer, including the performance of a conjugation procedure described in Section 2.2.2.

1.3 Integral Universal Function (Duhamel's Integral)

1.3.1 Duhamel's Integral Derivation

The Duhamel's integral presents a solution of some problems with varying variables in terms of known solutions of similar problems with the same variables considered as a constant parameters. This idea is based on two principles: (i) at a small interval of a variable, the function of interest may be approximately considered as a constant, and (ii) a solution of linear differential equation is presentable as a sum of other solutions of the same equation (superposition principle) (Exer. 1.15).

Letting the solution of the problem in question depend on some variable t according to function $F(t)$ and function $f(x,t)$ is a solution of similar problem for a different but constant t. For example, we consider the heat transfer from a body with time-dependent surface temperature $F(t)$, and function $f(x,t)$ is a known solution of the same problem with constant surface temperature. During a small interval Δt of variable t (Fig. 1.1) the given function F may be considered as an approximate constant. Then, on this small interval $\Delta F = F'(t)\Delta t$ an approximate solution of the problem in question is defined as a product $f(x,t)\Delta F$, where $f(x,t)$ is the approximate solution for constant t and ΔF is the interval. Consequently, for the first interval we have $f(x,t)F(0)$, where $F(0)$ stands for $\Delta F(0)$ at the beginning at $t = 0$ (Fig. 1.1).

For the next interval, the approximation starts at the time $t - \Delta t$ instead of t for the first one. Thus, the approximate solution is $f(x, t-\tau_1)\Delta F(\tau_1) = f(x, t-\tau_1)F'(\tau_1)\Delta\tau_1$, where $\tau_1 = \Delta t$ is the time lag in the second interval, and the small variation of function F is determined for time τ_1 when the second interval begins (Fig. 1.1). For the third interval, one gets similar solutions

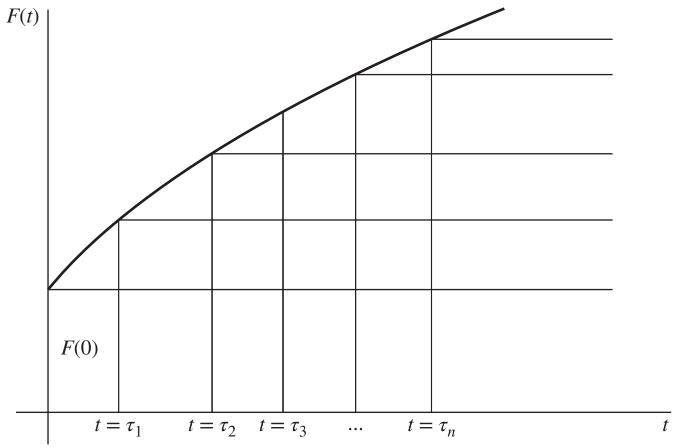

Figure 1.1 Duhamel's integral derivation: approximation of arbitrary dependence by step function

as $f(x, t-\tau_2)\Delta F(\tau_2) = f(x, t-\tau_2)F'(\tau_2)\Delta\tau_2$ with time lag τ_2 in the third interval, and so forth. The sum of those results gives the approximate solution of the considered problem, which in the limit at $\Delta t \to 0$ transforms in Duhamel's integral

$$T(x,t) = f(x,t)F(0) + \sum_{k=1}^{n} f(x, t-\tau_k)F'(\tau_k)\Delta\tau_k \tag{1.20}$$

$$T(x,t) = f(x,t)F(0) + \int_0^t f(x, t-\tau)F'(\tau)d\tau \tag{1.21}$$

Expression (1.21) presents the temperature of a body with given time-variable surface temperature $F(t)$ but in fact this is a general relation applicable to many other problems governed by linear equations. To use this integral in any particular case, it is enough to know desired function $F(t)$ and the relevant simple solution (Exer. 1.16 and 1.17).

In creating an universal function describing dependence between heat flux q_w and temperature head $\theta_w(x) = T_w - T_\infty$ in heat transfer, the role of function $F(t)$ plays temperature head $\theta_w(x)$, which is considered as a given, like a function $F(t)$. Since we are looking for relation defining the heat flux, it is clear that a heat transfer coefficient h from some solved problem should be taken instead of function $f(x,t)$. In such a case, the product $h\theta_w(x)$ determines the heat flux q_w as well as product $f(x,t)F(t)$ defines the body temperature in the example considered above. Then, the limit of a sum of products $h\theta_w(x)$, similar to the limit of the sum in equation (1.20), gives the desired universal function in the form of Duhamel's integral

$$q_w = h_1\theta_w(0) + \int_0^x h_\tau \frac{d\theta_w}{d\xi} d\xi \tag{1.22}$$

In this equation h_1 and h_τ are the heat transfer coefficients for the first and all other steps from a known solution of simple problem playing the same role as the functions $f(x,t)$ and $f(x, t-\tau)$ in equation (1.21) (Exer. 1.18).

The general expression (1.22) is applicable to any flow regime: laminar, turbulent, at zero, or non zero pressure gradient, if the heat transfer coefficients h_1 and h_τ from some relevant problem are known. It is common [201, 123] to consider as a known solution of a standard problem of heat transfer after temperature jump on a plate with an isothermal initial zone. In such a problem, the wall temperature remains constant (with isothermal heat transfer coefficient h_*) up to some point $x = \xi$ and then suddenly changes to another value resulting after temperature jump in heat transfer with coefficient h_ξ. The ratio of coefficients h_* and h_ξ defines the influence function $f(x, \xi) = h_\xi / h_*$ so called because it describes the effect of initial isothermal zone on heat transfer intensity after jump. The influence function is usually employed for putting $h_1 = h_*$ and $h_\tau = h_\xi$ which transforms equation (1.22) to the following standard form (Exer. 1.19)

$$q_w = h_* \left[\theta_w(0) + \int_0^x f(x, \xi) \frac{d\theta_w}{d\xi} d\xi \right] \tag{1.23}$$

This expression is an universal function because it determines the heat flux for arbitrary (for any) temperature head distribution $\theta_w(x)$ through integral of it derivative $d\theta_w / dx$.

1.3.2 Influence Function

Relation (1.23) is general as well as equation (1.22) and is also applicable to any flow regime, if the influence function for a specific case is known (Exer. 1.20). However, determining this function is another difficult task. For some simple cases, when the influence function depends on ratio ξ / x instead of each of those variables, an influence function was obtained using approximate methods. For the simplest case of laminar flow, Prandtl number close to one, and zero pressure gradient, the influence function was found by integral method (S. 7.6) in the form [123, 201] (Exam. 7.13)

$$f(\xi / x) = [1 - (\xi / x)^{3/4}]^{-1/3} \tag{1.24}$$

The more general result for the laminar gradient flow, the influence function was obtained for self-similar flows with power-law velocities $U = Cx^m$ (S. 7.5.2), but only for fluids with large or small Prandl numbers. It was shown that the same formula (1.24) is valid with exponents depending on velocity power m for large and small Prandtl numbers according to the first and second formula, respectively

$$f(\xi / x) = [1 - (\xi / x)^{3(m+1)/4}]^{-1/3} \qquad f(\xi / x) = [1 - (\xi / x)^{m+1}]^{-1/2} \tag{1.25}$$

The first formula was obtained assuming a linear velocity distribution in the thermal boundary layer [201, 230]. Therefore, it is applicable to fluids with large Prundtl numbers for which such assumption is close to reality because in this case the thermal boundary layer is thin relatively to velocity layer (S. 7.7). At the limit $\mathrm{Pr} \to \infty$, this approximate formula becomes exact. For another limiting case $\mathrm{Pr} \to 0$, the situation is opposite: the dynamic boundary layer thickness is so thin that velocity across the thermal boundary layer is practically equal to the external velocity $U(x)$ (S. 7.7). Since in this case, the velocity in the thermal layer is independent on y, the thermal boundary layer equation simplifies, and for self-similar flows, second formula (1.25) is derived [338].

For turbulent flows, the structure of known relations for influence function remains also the same. In particular, for this case at the same conditions as for laminar flow (Pr $\approx$ 1, zero pressure gradient), it was shown that formula (1.24) is valid with exponents 9/10 at (ξ/x) and $(-1/9)$ at brackets [123, 201]. There are also several relations for turbulent flows derived from experimental data. Those are presented also in the form (1.24), but with slightly different exponents: unity at (ξ/x) and at the brackets: (-0.114) [208], (-0.12) [201, 275] and (-0.2) or with giving practically the same results exponents 39/40 at (ξ/x) and $(-7/39)$ at the brackets (instead of unity and -0.2) [267].

The review shows that the known relations for influence function for laminar and turbulent flows pertain only for the simplest cases: limiting values of Prandtl numbers and basically at zero pressure gradients. Later we will see that employing the universal functions permits general expressions for influence function from which follow the particular results.

1.4 Differential Universal Function (Series of Derivatives)

This form of universal function may be obtained from the integral relation (1.23) using successive integration by parts. We start from simple case of zero pressure gradient. As we have seen above, for this simple case, the influence function depends on the ratio ξ/x, rather than of each variable separately, and integral formula (1.23) takes the form (Exer. 1.21)

$$q_w = h_* \left[\theta_w(0) + \int_0^x f(\xi/x) \frac{d\theta_w}{d\xi} d\xi \right] \tag{1.26}$$

Denoting $\zeta = \xi/x$ and applying for integration the following parts

$$u_1 = \frac{d\theta_w}{d\xi}, \; d\mathrm{v}_1 = xf(\zeta)d\zeta, \qquad \mathrm{v}_1 = x\left(\int_0^\zeta f(\gamma)d\gamma \; - 1 \right), \tag{1.27}$$

we get according to formula for integration by parts an expression (Exer. 1.22)

$$u\mathrm{v}\Big|_{\xi=0}^{\xi=x} - \int_0^x \mathrm{v}du = x\frac{d\theta_w}{d\xi}\left(\int_0^1 f(\zeta)d\zeta \; - 1 \right) + x\frac{d\theta_w}{d\xi}\Big|_{x=0}$$
$$- x\int_0^x \left(\int_0^\zeta f(\zeta)d\zeta - 1 \right) \frac{d^2\theta_w}{d\xi^2} d\xi \tag{1.28}$$

Substitution of this result in equation (1.26) leads to modified relation for heat flux

$$q_w = h_* \left[\theta_w(0) + \frac{x}{1!}\frac{d\theta_w}{dx}\Big|_{x=0} + g_1 x\frac{d\theta_w}{dx} - x\int_0^x \left(\int_0^\zeta f(\gamma)d\gamma - 1 \right) \frac{d^2\theta_w}{d\xi^2} d\xi \right] \tag{1.29}$$

$$\mathrm{g}_1 = \int_0^1 f(\zeta)d\zeta - 1$$

Here g_1 is a constant defined by the first integral in the right hand part of equation (1.28).

Comment 1.3 In this derivation we used several variables: x, ξ, γ, $\zeta = \xi/x$. Two of those, x and ζ, are working variables, whereas two others, ξ and γ, are so-called dummy variables, which play a subsidiary role. In this case, dummy variables are used for carry out integrals (to distinguish from upper limit of integral). Thus, γ is employed in integral (1.27) instead of ζ, and ξ is applied instead of x in the last integral in (1.28).

Relation (1.28) is further modified using the following parts for k integration

$$u_k = \frac{d^k\theta_w}{d\xi^k}, \quad \mathrm{v}_k = \int_0^\zeta \mathrm{v}_{k-1} d\gamma + \frac{(-1)^k x}{k!} \tag{1.30}$$

Putting here $k = 2$, we obtain the parts (1.31) for transforming the last integral in relation (1.29) via integration by parts that result in further modification of expression (1.29) for heat flux in a similar form (1.32) (Exer. 1.23)

$$u_2 = \frac{d^2\theta_w}{d\xi^2}, \quad \mathrm{v}_2 = x\int_0^\zeta d\gamma\left(\int_0^\zeta f_2(\gamma)d\gamma - 1\right) + \frac{x}{2!}, \quad f_2(\zeta) = \int_0^\zeta d\zeta\int_0^\zeta f(\zeta)d\zeta - \zeta + \frac{1}{2} \tag{1.31}$$

$$q_w = h_*\left[\theta_w(0) + \frac{x}{1!}\frac{d\theta_w}{dx}\bigg|_{x=0} + \frac{x^2}{2!}\frac{d^2\theta_w}{dx^2}\bigg|_{x=0} + g_1 x\frac{d\theta_w}{dx} + g_2 x^2\frac{d^2\theta_w}{dx^2} + x^2\int_0^x f_2(\zeta)\frac{d^3\theta_w}{d\xi^3}d\xi\right] \tag{1.32}$$

In the first equation $f_2(\zeta) = \mathrm{v}_2(\zeta)/x$, and in the second equation coefficient $g_2 = -f_2(1)$ is the value of this function at $\zeta = 1$. As it is seen, function $\mathrm{v}_2(\zeta)$ arises in the last integral in equation (1.32) and is defined by relations (1.31), where for simplicity the dummy variable γ is substituted by variable $\zeta = \xi/x$ (Exer. 1.24).

Repeating the integration by applying the parts indicated by equation (1.30), we finally arrive in the following series with coefficients g_k determined by relations (1.34)

$$q_w = h_*\left[\theta_w(0) + \frac{x}{1!}\frac{d\theta_w}{dx}\bigg|_{x=0} + \frac{x^2}{2!}\frac{d^2\theta_w}{dx^2}\bigg|_{x=0} + \ldots + \frac{x^k}{k!}\frac{d^k\theta_w}{dx^k}\bigg|_{x=0} + \ldots + g_1 x\frac{d\theta_w}{dx}\right.$$
$$\left. + g_2 x^2\frac{d^2\theta_w}{dx^2} + \ldots + g_k x^k\frac{d^k\theta_w}{dx^k} \ldots + (-1)^k x^k\int_0^x f_k(\zeta)\frac{d^{k+1}\theta_w}{d\xi^{k+1}}d\xi\right] \tag{1.33}$$

$$g_k = (-1)^{k+1} f_k(1), \quad f_k(\zeta) = \int_0^\zeta d\zeta\int_0^\zeta d\zeta \ldots \int_0^\zeta f(\zeta)d\zeta + \sum_{n=1}^{n=k}\frac{(-1)^n\zeta^{k-n}}{n!(k-n)!} \tag{1.34}$$

The k times repeated integral and the sum in the last equation may be presented as follows (Exer. 1.25 and 1.26)

$$\int_0^\zeta d\zeta\int_0^\zeta d\zeta \ldots \int_0^\zeta f(\zeta)d\zeta = \frac{1}{(k-1)!}\int_0^\zeta (1-\zeta)^{k-1} f(\zeta)d\zeta \tag{1.35}$$

$$\sum_{n=1}^{n=k} \frac{(-1)^n \zeta^{k-n}}{n!(k-n)!} = \frac{1}{k!}[(\zeta-1)^k - \zeta^k] \tag{1.36}$$

Then, the coefficients g_k according to relation (1.34) are defined as (Exer. 1.27)

$$g_k = \frac{(-1)^{k+1}}{k!}\left(k\int_0^1 (1-\zeta)^{k-1} f(\zeta)d\zeta - 1\right) \tag{1.37}$$

Analyzing equation (1.33) for heat flux, one sees that the first sum, starting with $\theta_w(0)$, represents an expansion of function $\theta_w(x)$ as a Taylor series at $x = 0$. Therefore, if at $k \to \infty$, the last integral in equation (1.33) (which is a remainder) goes to zero, this expression turns into infinite series determining the heat flux in terms of derivatives of temperature head distribution $\theta_w(x)$ in the following form (Exer. 1.28)

$$q_w = h_*\left(\theta_w + \sum_{k=1}^{\infty} g_k x^k \frac{d^k\theta_w}{dx^k}\right) \tag{1.38}$$

As shown in [101] and repeated in [119, p. 55], this relation is an exact particular result for zero pressure gradient obtained from more general exact solution of thermal boundary layer equation. At the same time, we just derive this relation from integral formula for heat flux (1.26) using the exact procedure of integration by parts. These two facts show that both relations—the integral (1.26) and differential formula (1.38)—are two equivalent exact expressions for heat flux in a flow with a zero pressure gradient. These expressions are universal because they both describe the dependence between heat flux and arbitrary (say any) surface temperature head in two forms: as integral consisting of the first derivative of temperature head or as a series of derivatives of it (Exer. 1.29).

1.5 General Forms of Universal Function

The more general expression for heat flux than relation (1.38), derived in [101] and mentioned above, is an exact solution of thermal boundary layer equation for self-similar flows with power-low external velocity distribution $U = Cx^m$ (S. 7.5.2). This solution is the same series (1.38) but written in Görtler variable Φ (S. 7.4.4.2) (see [119])

$$q_w = h_*\left(\theta_w + \sum_{n=1}^{\infty} g_k \Phi^k \frac{d^k\theta_w}{d\Phi^k}\right) \qquad \Phi = \frac{1}{\nu}\int_0^x U(\xi)d\xi \tag{1.39}$$

In the case of zero pressure gradient ($U = const$), series (1.39) transforms in (1.38) since for constant U variable $\Phi = \mathrm{Re}_x$, and due to that Φ is proportional to x (Exer. 1.30). Because series (1.38) and integral relation (1.26) are equivalent, it is clear that substituting the Görtler variable Φ for x in integral (1.26) yields expression also valid for self-similar flows with power-low external velocity distribution $U = Cx^m$ (Exer. 1.31 and 1.32)

$$q_w = h_*\left[\theta_w(0) + \int_0^{\Phi} f(\xi/\Phi)\frac{d\theta_w}{d\xi}d\xi\right] \tag{1.40}$$

The two equivalent universal functions (1.39) and (1.40) are solutions of boundary layer equations because, as it noted at the beginning of this section, expression (1.39) is obtained from thermal boundary layer equation. It is known, that solutions of that type are applicable to majority of practically important applications. Two facts ensure this conclusion: (i) the properties (kinematic viscosity and thermal diffusivity) of the essential technical fluids, such as air, water, oil, and liquid metals, are small, and due to that the corresponding Reynolds and Peclet numbers are large, and (ii) as it was shown by Prandtl, the Navier-Stokes equations simplifies to boundary layer equations for high Re and Pe because in this case, the viscosity effects are significant only in a thin layer adjacent to the body surface (S. 7.4.4.1).

To employ universal functions (1.39) and (1.40) for calculations, the series coefficients g_k and influence function $f(\xi/\Phi)$ for the integral are required. Examples considered above in Section 1.3.2 show that all known formulae for influence function have the same structure with different values of exponents. Proceeding from that fact of similarity, we use analogous expression for general form of influence function with unknown exponents C_1 and C_2

$$f(\xi/\Phi) = [1 - (\xi/\Phi)^{C_1}]^{-C_2} \tag{1.41}$$

In the next sections, the coefficients g_k and exponents C_1 and C_2 are calculated for laminar, turbulent flows, and several other regimes and conditions.

Exercises

It is assumed (see Preface) that to perform some exercises a reader gets additional knowledge from Part III using references indicated in text.

1.1 Explain how the conjugate problem differs from other heat transfer problems.

1.2 Why do conjugate heat transfer problems contain at least two subdomains? Name these subdomains.

1.3 What is the difference between Laplace and Poisson equations? In what cases and why are these equations simplified? Explain physically the difference between "thin" and "thermally thin" bodies.

1.4 Compare Navier-Stokes and full energy equations with simplified boundary layer and creeping equations. Explain what and why part of equation terms may be neglected? Are the simplified equations exact or approximate? Think: why the set of equations for creeping flow does not include the equation for turbulent flow?

1.5 Study Sections 8.2.3 to understand the Reynolds averaging and arising Reynolds stresses in averaged Navier-Stokes equations. Think: why are these stresses so much greater than molecular ones? Explain the differing between physical nature of coefficients μ and μ_{tb}, Compare Navier-Stokes (1.4)–(1.7) and Reynolds (1.14) equations to better understand the Einstein notations.

1.6 What is the essential difference between boundary conditions of the third and forth kinds?

1.7 Is the problem of atmosphere and ocean interaction a conjugate one? Explain your answer. What are the subdomains in such a problem? What equations are relevant?

1.8 Study some features of partial differential equation of second order using, for example, *Advanced Engineering Mathematics Course*, to understand what is a canonical equation form? How do canonical equations differ from each other? Learn or recall why the Navier-Stokes equation is difficult to solve. What is nonlinearity? How do nonlinear and linear equations differ?

1.9 Discuss with other students or colleagues the difficulties arising in formulating the boundary conditions for Navier-Stokes and full energy equations. Explain why it is easier to formulate boundary conditions for boundary layer equations.

1.10 Compare well- and ill-posted problems. What causes the additional difficulties in formulating Dirichlet problem for Navier-Stokes and energy equations?

1.11 Explain the term "deficient situation". Why do such situations occur in conjugate problem statements? How can this difficulty be resolved?

1.12 What is the basic idea of iteration method of conjugation? What is the method of trial-and-error? (see article "Trial and Error" on Google, on Wikipedia)

1.13 Explain how the problem of different physical properties of body and fluid is resolved in the conjugation by one equation for entire domain.

1.14 What is the universal function? Why is such a function required for an analytical solution of the conjugate problem?

1.15 The idea of Duhamel's integral is based on a superposition principle. Read about the superposition method to understand why it is applicable for linear, and is not appropriate for nonlinear equations (see Exer. 1.8).

1.16 In the development of Duhamel's integral, we used the first ordinate of each interval as a value of function $F(t)$. Think: will the result (1.20) be the same if we use the middle or final ordinate of each interval instead of the first one? Explain your answer.

1.17 What is the time lag? Remind or read about this term, for example, on Wikipedia. Think: what is the difference between time legs in the two equations (1.20) and (1.21)? Can the first one be transformed in the second?

1.18 Repeat the derivation of equation (1.22) from Duhamel's integral (1.21) to answer why: (i) function $\theta_w(x)$ and heat transfer coefficient represent in this case functions $F(t)$ and $f(x, t)$ in initial integral (1.21), and (ii) heat transfer coefficients h_1 and h_τ relate to functions $f(x, t)$ and $f(x, t - \tau)$.

1.19* Describe the problem of heat transfer after a temperature jump on the plate with an isothermal initial zone, which is usually used as a standard problem with a known solution. Draw a graph showing the temperature variation along the plate. Obtain equation (1.23).

1.20 What is an influence function? How does it relate to Duhamel's integral?

1.21 Discuss with your friend or colleague the benefits of universal function for conjugate heat transfer problems. Think about other examples where universal functions might be useful. What is the basic characteristic that distinguishes this type of relations from others?

1.22* Repeat the first integration by parts and derive equation (1.29) from universal function (1.26). Hint: note that the integration is performed by variable ξ, and because of that, x is considered as a constant parameter in process of this integration.

1.23* Obtain parts for third integration from general relation (1.30). Hint: begin from deriving the parts for first and second integrations using relations (1.30) to understand the procedure.

1.24 Extend the series (1.32) by adding two next terms without farther calculations. Compare your results with formulae (1.33) and (1.34). Hint: first, analyze the rules according to which the existing terms are constructed and then proceed using these rules.

1.25* Expression (1.35) for k-times repeated integral is obtained by series of integration by parts. Show that this is true for $k = 2$. Hint: take the integral with function $f(\zeta)$ as one part (u) and $d\zeta$ as another part (dv).

1.26 Check equation (1.36) for k = 2 and $k = 3$ to see that this equation is correct.

1.27 Derive equation (1.37) from relation (1.34) using expressions (1.35) and (1.36). Hint: take into account that $k!$ is a product of (k-1)! and k.

1.28 Recall or study the Taylor series to understand the procedure of transforming equation (1.33) into series (1.38).

1.29 At the end of this section (S. 1.4) it is stated that the integral with the first derivative of temperature head and a series with successive derivatives of it are universal functions, because they describe the dependence between heat flux and arbitrary (say any) surface temperature head. Explain why this statement is true or in other words, why a function describing a dependence of some arbitrary variable is universal? In what sense is such function universal?

1.30 Show that in the case of constant external flow $U = const$, the Görtler variable Φ becomes Reynolds number, and expression (1.39) transforms into relation (1.38).

1.31 Prove that in the case of constant external flow the expression (1.40) transforms into equation (1.26). Explain why it is possible to write $f(\xi/\Phi) = f(\xi/x)$ in this case. Hint: think about the connection between dummy and working variables explained in analyzing examples in Comment 1.2. Are the dummy variables ξ the same in both influence functions in equations (1.26) and (1.40)?

1.32 Recall why the arbitrary external velocity corresponds to arbitrary pressure gradient. What equation from a set of relations given in the beginning of this chapter tells us about this fact? What is the name of this equation? In what type of flow does such connection between velocity and pressure hold? Hint: see Section 7.1.2.5.

1.6 Coefficients g_k and Exponents C_1 and C_2 for Laminar Flow

In this section we discuss the basic features of coefficients g_k that determine the differential universal function for laminar flows and show how the exponents C for the general form (1.41) of influence function may be estimated using known coefficients g_k. Then, in the next sections, similar coefficients and exponents are evaluated for turbulent, compressible flows and for some other cases.

1.6.1 Features of Coefficients g_k of the Differential Universal Function

We discussed in the previous section how the exact solution of the thermal boundary layer equation is obtained for self-similar laminar flows with power-law external velocity distribution $U = cx^m$. Such distribution occurs on the wage with open angle $\pi\beta$ (Fig. 1.2) streamlined by potential flow. The exponent m in velocity distribution and open wage angle β are connected by relation $m = \beta/(2 - \beta)$. The first four coefficients g_k of series (1.39) calculated for this case are plotted in Figures 1.3 and 1.4 as functions of Prandl number for different external flow velocities (different β).

These data are obtained numerically (details in [119]). For limiting cases $\Pr \to 0$ and $\Pr \to \infty$, the corresponding simplified equations are solved analytically leading to following

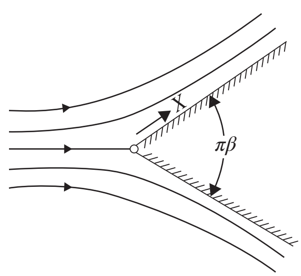

Figure 1.2 Flow past a wage. At the leading edge, the potential velocity is $U = cx^m$

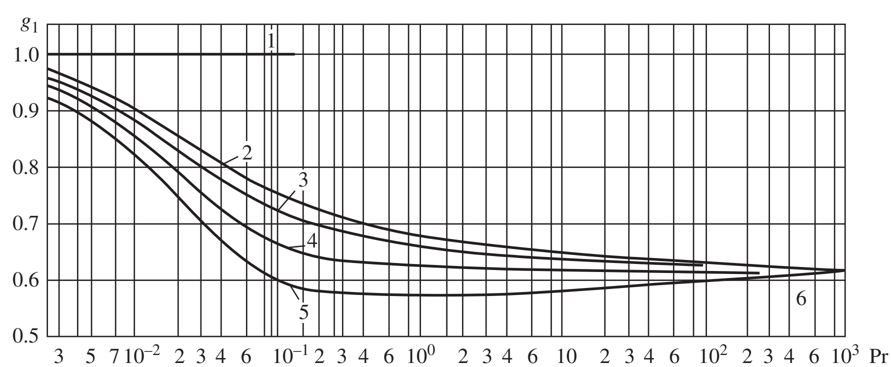

Figure 1.3 Coefficient $g_1(\Pr, \beta)$ of universal function (1.39) for laminar boundary layer. Asymptotes: $1 - \Pr = 0$, $6 - \Pr \to \infty$; β: 2-1 (stagnation point), 3 – 0.5 (favorable pressure gradient), 4-0 (zero pressure gradient), 5-(−0.16) (preseparation pressure gradient) (S. 7.5.2)

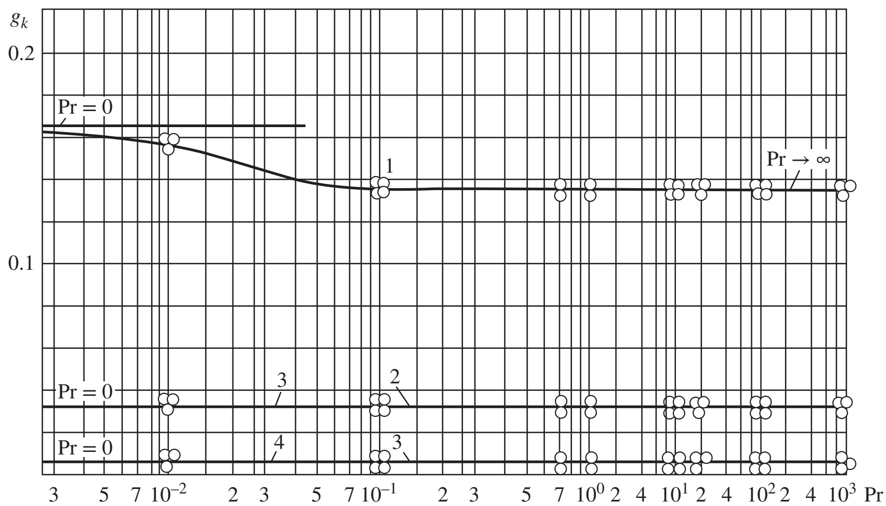

Figure 1.4 Coefficients g_k(Pr) of universal function (1.39) for laminar boundary layer. $1-(-g_2)$, 2-g_3, $3-(-g_4)$, ∘ – numerical integration

formulae and numerical results [119] ($\Gamma(j)$ -is gamma function, Exer. 1.33)

$$g_k = \frac{(-1)^{k+1}}{k!(2k-1)} \quad \text{Pr} \to 0, \quad g_k = \Gamma\left(\frac{2}{3}\right) \sum_{i=0}^{i=k} \frac{(-1)^{k+i}\Gamma\ [(4i/3)+1]}{(k-i)!\, i!\, \Gamma\ [(4i/3)+2/3]} \quad \text{Pr} \to \infty \tag{1.42}$$

$$g_1 = 1,\ g_2 = -1/6,\ g_3 = 1/30,\ g_4 = -1/168,\ g_5 = 1/1080,\ g_6 = -1/7920 \quad \text{Pr} = 0 \tag{1.43}$$

$$g_1 = 0.6123,\ g_2 = -0.1345,\ g_3 = 0.0298,\ g_4 = -0.0057 \qquad \text{Pr} \to \infty \tag{1.44}$$

Data from Figures 1.3 and 1.4 and limiting values (1.43) and (1.44) yield the following basic features of coefficients g_k of universal function (1.39): (i) coefficient g_1 depends on the external velocity (via β) and on the Prandtl number; this dependence is more significant for small Prandtl numbers (Pr < 0.5); for Pr → 0 and Pr → ∞, the values of g_1 for all β tend to the greatest $g_1 = 1$ and to the lowest $g_1 = 0.6123$ values, respectively, (ii) coefficient g_2 is practically independent of the external velocity and depends slightly only on the Prandtl number in the region of small Prandtl numbers; for Pr → 0 and Pr → ∞ the values of g_2 also tend to the greatest absolute value $|g_2| = 1/6$ and to the lowest absolute value $|g_2| = 0.1345$, respectively; (iii) coefficients g_3 and g_4 are independent of both the external velocity and the Prandtl number, so that numerically obtained values for whole diapason of Prandtl numbers practically coincide with the limiting values $g_3 = 1/30$ and 0.0298 and $g_4 = -1/168$ and -0.0057, (iv) therefore, for $k \geq 3$, coefficients g_k may be estimated using simple formula (1.42) for Pr → 0, (v) coefficients g_k for universal function (1.39) rapidly decrease with the number of terms, so that using first two or three coefficient usually gives acceptable results (Exer. 1.34).

Comment 1.4 The isothermal heat transfer coefficient h_* required for employing both universal functions (1.39) and (1.40) may be estimated by any of known methods reviewed, for example, in [369]. These methods are well tested during many years when the isothermal heat transfer coefficient was used for practical calculations, in particular, as a part of the boundary conditions of the third kind.

Analysis of just considered features of coefficients g_k shows that the exact for power-law external flows universal function (1.39) provides practically accurate approximate results for arbitrary external flow velocity distribution. The following facts specified this statement: (i) coefficient g_1 slightly depends on external velocity (say on β), whereas other coefficients are practically independent of the external velocity, (ii) if coefficients g_1 were also independent of β, the relation (1.39) would be an exact relation for arbitrary external velocity $U(x)$, (iii) in reality, according to Figure 1.3, the effect of β (i.e., external velocity) on the first coefficient reaches the maximum of $\pm 12\%$ from average value $g_1 \approx 0.675$ in vicinity of $\text{Pr} = 0.1$ and then decreases to zero in both limiting cases of Prandtl number at $\text{Pr} \to 0$ and $\text{Pr} \to \infty$.

Thus, universal function (1.39) in general case of arbitrary external flow (or pressure gradient) with average values of coefficients g_k provides the calculation results with inaccuracy less than $\pm 12\%$ which is comparable with accuracy of other existing approximate methods (S. 7.6) (see also [338]). Accuracy may be increased by estimating the value of β. In some cases, β is known, for example, it is clear that for the flow past plate $\beta = 0$ as well as for transverse flow past circular cylinder or other body with blunt nose $\beta = 1$ close to the stagnation point, whereas for the rest part of the surface of such body the value of β may be approximately considered as zero. In other cases, the parameter β may be estimated using some simple relation, for example, formula

$$\beta = 2(1 - \Phi/\text{Re}_x) \tag{1.45}$$

which results from assumption that average velocities of considering distribution $U(x)$ and power-law flow distribution $U = cx^{\beta/(2-\beta)}$ are equal (Exer. 1.35).

Comment 1.5 To understand why Görtler variable extends applicability of universal function (1.38) obtained for zero pressure gradients, consider Görtler variable in the form $\Phi = U_{av}x/\nu = \text{Re}_{(av)x}$ (Exer. 1.36). Here, $U_{av}(x)$ is the average external flow velocity for the interval from the leading edge ($x = 0$) to point with coordinate x. It follows from this presentation of Görtler variable that function Φ (1.39) takes into account the flow history. Physically, it means that the flow characteristics at point x are determined not only by local parameters at this point but also by those at other points along the whole considering interval $(0, x)$. Thus, Görtler variable takes into account, in particular, the variation of pressure gradients along the considering interval.

Comment 1.6 In general, the characteristics of some point of interest are governed by: (i) in the simplest case, local data only (at this point); for example, the coefficients of friction and heat transfer at some point in a flow past wage with given surface and free stream temperatures are specified only by local values of Reynolds and Nusselt numbers (S. 7.5.2), (ii) local and historical data (at this and behind points) as, for example, in the same problem for wage but with nonisothermal surface when to get characteristics of some point, the surface temperature at this and behind points is required in addition (S. 2.1.1), and (iii) local, historical and future data (at this, behind and advanced points) as, for example, in any Dirichlet problem for elliptical equation when for characteristics of any point, the information is necessary from whole boundary of domain (Exer. 1.37).

1.6.2 Estimation of Exponents C_1 and C_2 for Integral Universal Function

Relation (1.37) establishes the connection between coefficients g_k and influence function $f(\zeta)$ where for general case $\zeta = \xi/\Phi$. Substituting expression (1.41) into relation (1.37), expanding $(1-\zeta)^{k-1}$ via a binominal formula, and introducing a new variable $r = \zeta^{C_1}$ leads to an equation

$$g_k = \frac{(-1)^{k+1}}{k!}\left[\frac{k}{C_1}\sum_{m=0}^{k-1}(-1)^m \frac{(k-1)!}{m!(k-m-1)!}B\left(\frac{m+1}{C_1}, 1-C_2\right) - 1\right] \qquad (1.46)$$

determining the dependence between coefficients g_k and exponents C_1 and C_2 required for integral universal function (1.40). Here, $B(i,j)$ is beta function, which is specified through combination of gamma functions (Exer. 1.38)

$$B(i.j) = \int_0^1 r^{i-1}(1-r)^{j-1}dr = \Gamma(i)\Gamma(j)/\Gamma(i+j) \qquad (1.47)$$

It is easy to calculate coefficients g_k employing relation (1.46) and knowing exponents of some influence function. For example, since relation (1.46) is a result of exact solution for self-similar flows, this equation may be used to check the accuracy of influence function (1.24) and others of this type approximate equations. Substitution $C_1 = 3/4$ and $C_2 = 1/3$ in equation (1.46) gives: $g_1 = 0.612$, $g_2 = -0.131$, $g_3 = 0.03$, $g_4 = -0.0056$. Those are practically the same as (1.44) obtained from exact solution for the limiting case $\Pr \to \infty$. This tells us that function (1.24) is correct (Exer. 1.38). Similarly, equation (1.46) with exponents $C_1 = 1$ and $C_2 = 1/2$ gives coefficients g_k that are in agreement with the values (1.43) obtained from exact solution for $\Pr \to 0$ (details in [119]).

More complicated is the inverse problem of determining exponents in influence function (1.41) knowing coefficients g_k. Two facts cause the difficulties in solving this problem: (i) relation (1.46) is transcendental, and due to that could not be solved for exponent C_1 or C_2 (Exerc. 1.39), and (ii) there are only two unknowns C_1 and C_2, but countless known coefficients g_k. While the first difficulty is a technical question that may be resolved applying a graphic approach, or software based on trial and error or on other numerical method, the second problem is a fundamental complexity, known as an overdetermined system, when a number of equations exceeds the number of unknowns. In this case, such situation produces numerous results because each pair of coefficients g_k after substitution in equation (1.46) gives two equations defining unknown C_1 and C_2. However, as we have seen, this particular set of coefficients g_k consists of only two weighty coefficients g_1 and g_2, whereas the others are comparatively negligible. Due to that, it may be expected that exponents C_1 and C_2 obtained from system of two equations with coefficients g_1 and g_2 would be appropriate (Exer. 1.40 and 1.41).

This is confirmed by calculation results plotted in Figure 1.5 showing that all particular cases of laminar flow considered in Section 1.3.2 follow from data of this pattern. It is seen from Figure 1.5 that for whole interval of Prandtl number, exponents C_1 and C_2 vary slightly from 1 to 3/4 and from 1/2 to 1/3, respectively, as it should be according to relations (1.25).

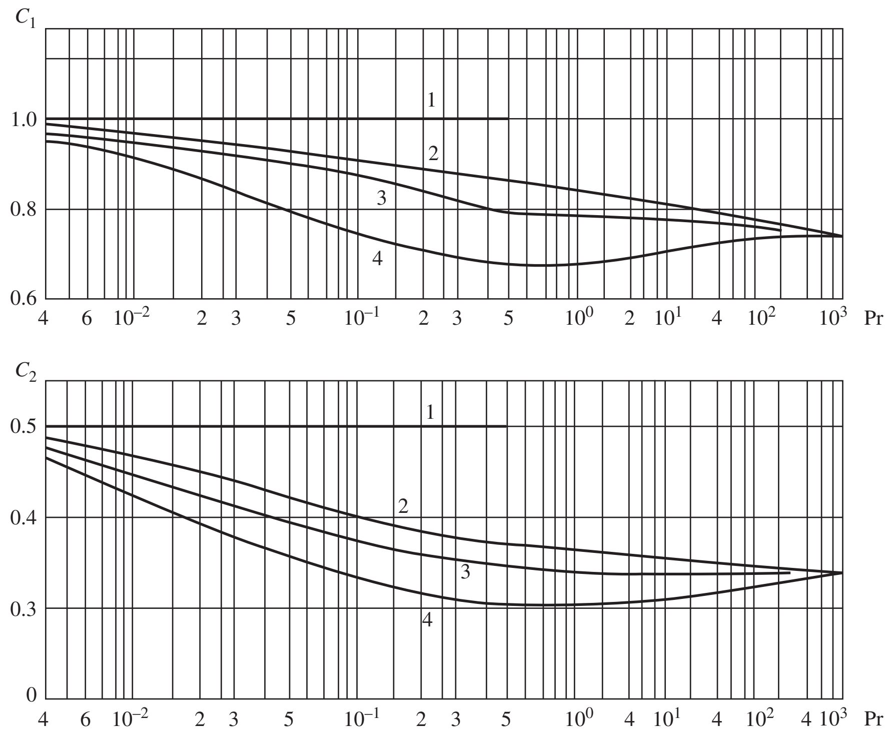

Figure 1.5 Exponents C_1 and C_2 for laminar boundary layer. 1 – Pr = 0, 2 – $\beta = 1$, 3 – 0, 4-(−0.16)

These data tells us as well as data of a slight dependence of g_k on pressure gradient (on β from Figs 1.3 and 1.4) identifies that the Görtler variable Φ takes into account the flow history (Com. 1.5). This property of variable Φ results in independence of the exponents in influence function (1.41) written in variables (ξ/Φ) on pressure gradients. That means that the values of C_1 and C_2 remains the same 3/4 and 1/3 at $\mathrm{Pr} \to \infty$ as well as 1 and 1/2, at $\mathrm{Pr} \to 0$ for arbitrary external velocity distribution, as those in the known simple influence functions (1.25) for the plate at zero pressure gradient ($m = 0$) presented in variables (ξ/x).

This is also in line with data from Figure 1.5 showing that C_1 and C_2 are independent on β in both limiting cases being the same for arbitrary external velocity as just indicated values 3/4 and 1/3 at $\mathrm{Pr} \to \infty$ and 1 and 1/2, at $\mathrm{Pr} \to 0$. Moreover, it is easy to check that the functions (1.25) for self-similar flows follow from relation (1.41) because in the case of $U = cx^m$, one gets: $\xi/\Phi = (\xi/x)^{m+1}$ (Exer. 1.42).

Now we have a full set of constants for using universal functions for general case of laminar flow. The same universal function (1.39) and (1.40) are applicable to other flow regimes and situations, however, with proper coefficients g_k and exponents C. Next sections present those constants for several other cases.

Comment 1.7 As we will see in applications, employing two forms of universal function provides accurate results of calculations. This is achieved by using the differential form with several first terms, when the series converges fast, and employing the integral form if the achieved by series accuracy is not satisfactory.

1.7 Universal Functions for Turbulent Flow

To get coefficients g_k and exponents C for the two forms of universal functions in the case of turbulent flow, the solution of thermal boundary layer equation for this case, analogous to that given in [101] for laminar flow, was obtained in [106] using the same Görtler variable (1.39).

Comment 1.8 This solution as well as some others considering in this and in the next chapters are presented more detailed in author's monograph [119]. Therefore, below we indicate in the brackets only the relevant page or section of this book without repeating citation [119]. For the case of turbulent flow solution such citation is [Chapter 4] instead of [119, Chapter 4].

The thermal boundary layer equation (1.17) for turbulent flow differs from similar equation (1.11) for laminar flow by last two terms containing additional dynamic ν_{tb} and thermal α_{tb} turbulent transfer coefficients. In contrast to analogous ν and α laminar coefficients, which are physical properties, the turbulent coefficients are complex functions depending on flow characteristics, and because of that a turbulence model is required for their estimation. Usually, to estimate turbulent thermal diffusivity, the turbulent Prandtl number $\mathrm{Pr}_{tb} = \nu_{tb}/\alpha_{tb}$ similar to physical Prandtl number is introduced. Then, coefficient ν_{tb} is defined, and turbulent diffusivity is found as ratio $\alpha_{tb} = \nu_{tb}/\mathrm{Pr}_{tb}$ using Pr_{tb} which usually is taken to be close or equal unity (S. 2.1.2.4, Exer. 1.43).

Solution of turbulent thermal boundary layer equation requires also the velocity profiles in boundary layer. In solution derived in [106], the Mellor-Gibson turbulence model [260] is used to calculate both transfer coefficients and velocity profiles. This model is one of the modern algebraic models based on dependences valid for equilibrium turbulent boundary layers, which are as well as the self-similar laminar boundary layers flows with constant dimensionless pressure gradient (S. 8.3.2 and S. 8.3.3)

$$\beta = \frac{\delta_1}{\tau_w}\frac{dp}{dx} \tag{1.48}$$

where δ_1 and τ_w are displacement thickness and skin friction stress (S. 7.5.1.1). As we have seen in Section 1.6.1, the parameter β determines the wage angle $\pi\beta$ and in the case of laminar self-similar flows is connected with the exponent in velocity power law $U = cx^m$. For this case, it is easy to check that the complex (1.48) is a constant (Exer. 1.44). In the case of the equilibrium turbulent boundary layer that can be demonstrated as well by more complicated analysis [422].

It is shown [106 or p. 99, Com. 1.8] that solution of thermal turbulent boundary layer equation for heat flux on arbitrary nonisothermal surface may be presented by the same two universal functions in slightly different Görtler variables (S. 8.3.6.3) with specific coefficients g_k of series (1.39) and exponents C of influence function in integral (1.40). In this case, these specific parameters depend on pressure gradient via β and on Prandtl number as in the case of laminar flow and in addition on Reynolds number. Calculations were performed for $\beta = -0.3$ (flow at stagnation point), $\beta = 0$ (zero pressure gradient flow), $\beta = 1$ and $\beta = 10$ (flows with weak and strong adverse pressure gradients) for the following Prandtl and Reynolds numbers:

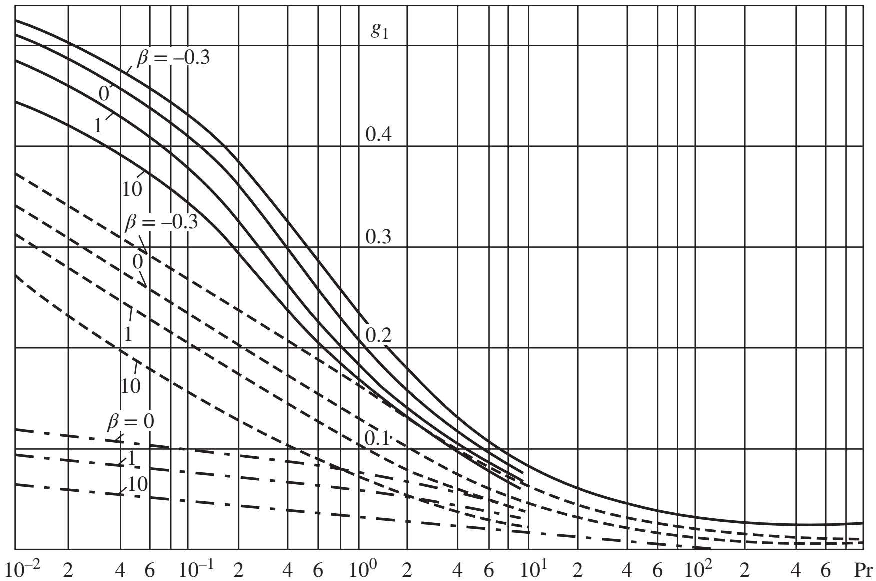

Figure 1.6 Coefficient g_1 for turbulent boundary layer: ___ $\mathrm{Re}_{\delta_1} = 10^3$,_ _ _ 10^5, _. _._ 10^9

Pr = 0.01, 0.1, 1, 10, 100, 1000 $\mathrm{Re}_{\delta_1} = 10^3, 10^5, 10^9$. Other details of turbulent boundary layer computing may be found in [106, p. 102]. The results are plotted in Figures 1.6 and 1.7 for coefficients g_k and in Figures 1.8 and 1.9 for exponents C as functions of Prandtl number for different values of β and Re_{δ_1}.

The following conclusions are formulated analyzing these data: (i) the values of the coefficients g_k rapidly decrease with increasing k as well as in the case of laminar flow; hence, one may use only a few of the first terms in series (1.39) to get satisfactory accuracy, (ii) as in the case of laminar flow, the coefficients g_k decrease with increasing Prandtl number (Exer. 1.45); however, in contrast to the case of laminar flow, where at large Prandtl numbers the value of coefficients become independent of Pr, but remain finite, in the case of turbulent flow they tends to zero with increasing Pr, so that starting with some value of Prandtl number (say $\approx 10^2$), the effect of nonisothermicity becomes negligible, (iii) for this case, coefficients g_k are smaller than the corresponding coefficients for laminar flow, and they decrease with increasing Reynolds number indicating that the nonisothermicity affects the heat transfer in turbulent flows relatively lesser (Exer. 1.46), (iv) coefficients g_1 and g_2 depend slightly on β, whereas the others are practically independent of β, this allows one to use the universal function (1.39) for turbulent flows with an arbitrary pressure gradient as for laminar flows estimating value of β by the same approach, (v) exponents C_1 and C_2 increase with decreasing Prandtl and Reynolds numbers, (vi) exponent C_2 increases with decreasing pressure gradient, whereas the exponent C_1 is practically independent of pressure gradient.

It follows from Figures 1.8 and 1.9 that $C_1 = 1$, $C_2 = 0.18$ for zero pressure gradient flow when Pr = 1 and $\mathrm{Re}_{\delta_1} = 10^3$, but under the same conditions and $\mathrm{Re}_{\delta_1} = 10^5$, the same figures give $C_1 = 0.84$, $C_2 = 0.1$. These computed results are close to exponents in well-known influence function considered in Section 1.3.2: the first exponents are almost the same as $C_1 = 1$

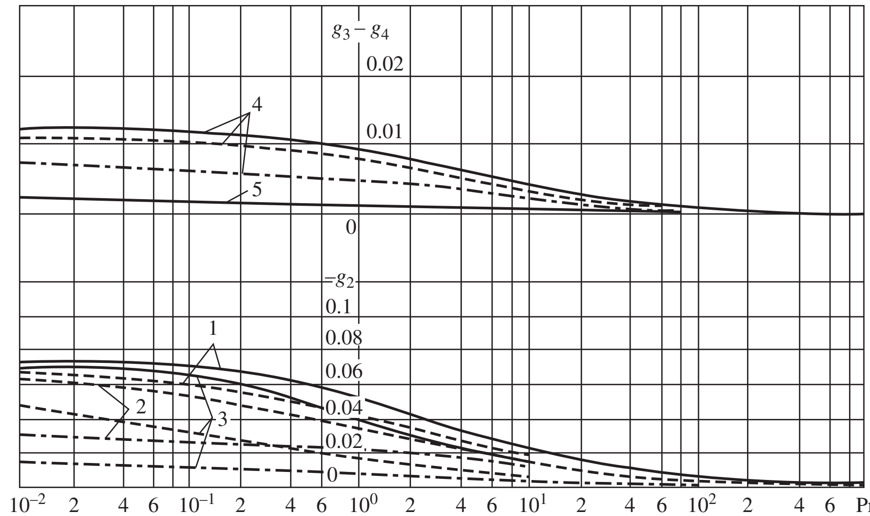

Figure 1.7 Coefficients g_k for turbulent boundary layer: g_2 : $1 - \beta = -0.3$, $2 - \beta = 0$, $3 - \beta = 1$, $4 - g_3$, $5 - g_4$, ___ $\mathrm{Re}_{\delta_1} = 10^3$, _ _ _ 10^5, _._._ 10^9

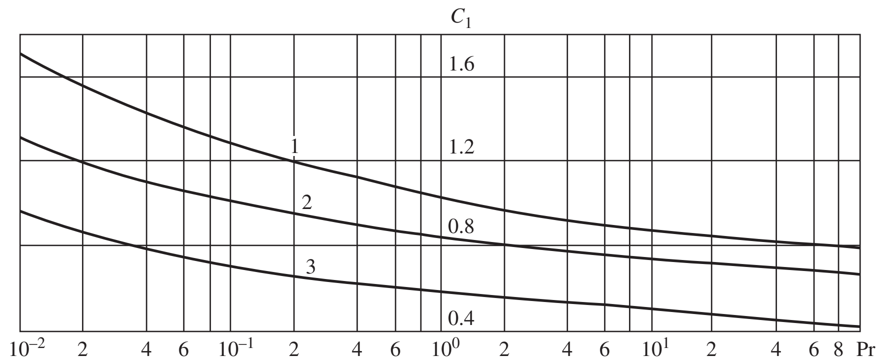

Figure 1.8 Exponent C_1 for turbulent boundary layer: $\mathrm{Re}_{\delta_1} = 10^3$, $2 - 10^5$, $3 - 10^9$

and $C_2 = 0.2$ derived theoretically and confirmed by measuring data at relatively low Reynolds number $\mathrm{Re}_x = 5 \cdot 10^5$ [201], and the second values are close to $C_1 = 9/10$ and $C_2 = 1/9$ obtained experimentally at greater Reynolds number $\mathrm{Re}_x = 10^8$ in [267] (Exer. 1.47).

Comment 1.9 In the case of turbulent flow, the isothermal heat transfer coefficient h_* is required also for applying both universal functions (1.39) and (1.40). For this purpose, a special investigation was provided for wide range of Reynolds and Prandtl numbers [107] using the same Mellor turbulent model. We present those results and comparison with available experimental data in applications in Section 2.1.2.3.

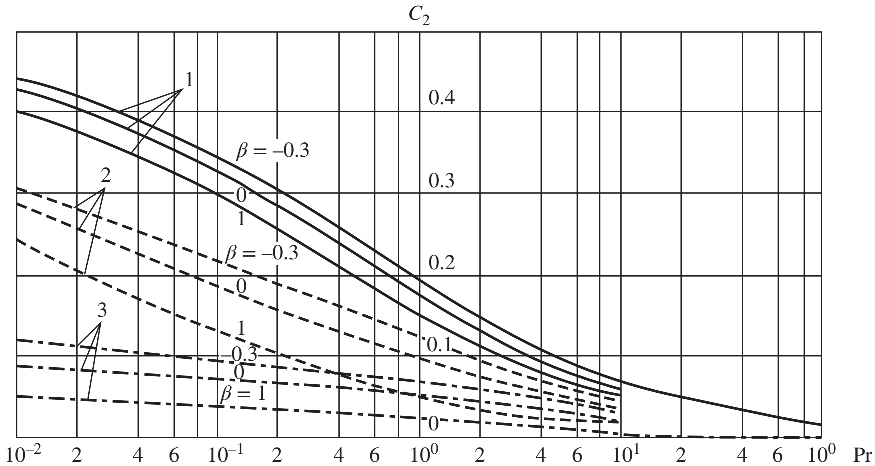

Figure 1.9 Exponent C_2 for turbulent boundary layer: $\mathrm{Re}_{\delta_1} = 10^3$, $2 - 10^5$, $3 - 10^9$

Exercises

1.33 Read about gamma function in some *Advanced Engineering Mathematics Course* and calculate limiting values (1.43) and (1.44) using formulae (1.42). Hint: for Gamma function estimation use some handbook or Mathcad.

1.34 Estimate a rate of convergence of series (1.39) by comparing the first four values of terms using limiting coefficients for Pr → 0 or Pr → ∞ and assuming that all products $\Phi^k(d^k\theta_w/d\Phi^k)$ are of the same order.

1.35 Derive relation (1.45) for estimating β. Hint: equal results of integration of two flows velocity distributions: given $U(x)$ and self-similar $U = cx^{\beta/(2-\beta)}$, which is used for estimating β.

1.36 Show that the Görtler variable Φ (1.39) is the average value of Reynolds number on interval $(0, x)$.

1.37 Think about other examples of problems and different types of information required to understand the role of local, historical, and future data in determining point characteristics of interest.

1.38 Read about beta function (see Exer. 1.33) and calculate coefficients g_k using relation (1.46) and exponents from influence function (1.24).

1.39 Recall or read about a term "transcendental expression" using, for example, Wikipedia to understand the difficulties in solving the inverse problem and methods to overcome those.

1.40* Think how the inverse problem may be solved graphically. Hint: (i) calculate two functions $g_1(C_1, C_2)$ and $g_2(C_1, C_2)$ using equation (1.46) and varying C_1 from 1 to 3/4 and

C_2 from $1/2$ to $1/3$, (ii) plot these functions and choose coefficients g_1 and g_2 from Figures 1.3 and 1.4 for the same value of Prandtl number, (iii) draw lines $g_1 = const.$ and $g_2 = const.$ on each of graphs to get two relations between exponents C_1 and C_2, (iv) find the intersection point of two curves plotted according to data for C_1 and C_2 from (iii).

1.41* Create a software for solving the inverse problem applying trial and error method. Hint: (i) choose coefficients g_1 and g_2 from Figures 1.3 and 1.4 for the same value of Prandtl number, (ii) take two equations (1.46) for g_1 and g_2 and guess a value of C_1 or C_2, (iii) by varying another than guessed exponent satisfy one of equation (1.46) to get the corresponding coefficient g, (iv) by varying another exponent satisfy another equation (1.46) to get the second coefficient g, (iv) continue the procedure until desired accuracy of both refined coefficients g_1 and g_2 will be achieved, (vi) if the process would be poor converged, change the value of the first guessing coefficient.

1.42 Show using the Görtler variable that for self-similar flows with velocity $U = cx^m$, the formula (1.25) is valid. Hint: calculate variable Φ for $U = cx^m$, and substitute ξ/Φ for ξ/x in equation (1.24).

1.43 What means physically $\mathrm{Pr}_{tb} = 1$? Recall, what is the Reynolds analogy? Explain physically why such an analogy exits. Hint: think about similarity of transport processes (S. 7.1.1).

1.44 Prove that for laminar boundary layer the parameter $\beta = (\delta_1/\tau_w)(dp/dx)$ is constant. Hint: use the external velocity distribution $U = Cx^{\beta/(2-\beta)}$, second equation (1.12) for dp/dx, and equations for δ_1 and τ_w from Section 7.5.1.

1.45 Explain physically why coefficients g_k decrease with a Reynolds or Prandtl number increasing. Hint: think about relation between dynamic and thermal boundary layer thicknesses (Exam. 7.8).

1.46 Explain why effect of nonisothermicty (and coefficients g_k) in turbulent flow is less than in laminar flow. Hint: think about different nature of transport process in both cases (S.8.2.1).

1.47 Calculate exponents C_1 and C_2 for several examples using laminar and turbulent data for g_1 and g_2 and graphical approach or software described in Exercises 1.41 and 1.42, respectively.

1.8 Universal Functions for Compressible Low

The results obtained above for incompressible flows are applicable to the compressible flows with variable density in the case of zero pressure gradient. This follows from the fact that boundary layer equations for compressible flow past plate in Dorodnizin or Illingworth-Stewartson variables (according Russian or English literature) has the same form as the boundary layer equations for incompressible flow in physical variables (Exer. 1.48). Therefore, the solution of thermal boundary layer equation for heat flux on a plate with arbitrary temperature distribution in compressible flow in such variables may be presented as universal function (1.38), but written in differences of enthalpy instead of temperature head

[111, p. 78]

$$q_w = \frac{q_{w*} C_x}{\sqrt{C}} \left(i_{0w} + \sum_{k=1}^{\infty} g_k x^k \frac{d^k i_{0w}}{dx^k} \right), \quad i_{0w} = \frac{J_w - J_{ad}}{J_\infty} = i_w - \frac{r}{2}(k-1)\mathrm{M}^2 \tag{1.49}$$

$$C = \sqrt{\frac{T_{w.av}}{T_\infty} \frac{T_\infty + T_S}{T_{w.av} + T_S}}$$

Here, q_{w*} is the heat flux on an isothermal surface with average temperature head of considering nonisothermal plate (like h_* in relation (1.38)), J_w and J_{ad} are wall and adiabatic wall enthalpy, i_{0w} is stagnation enthalpy difference (Exer. 1.49), C and C_x are coefficients of proportionality in approximation law for gas viscosity $\mu/\mu_\infty = C\ (T/T_\infty)$ calculated using average $T_{w.\mathrm{av}}$ or local T_w surface temperature, r is a recovery factor (S. 1.14), M is Mach number, and T_S is Sutherland gas constant. Equation (1.49) is an exact solution of a thermal boundary layer equation for the plate with an arbitrary temperature distribution. The well-known Chapman-Rubesin solution for polynomial surface temperature follows from this relation as a particular case.

The equivalent integral universal function for compressible flow past plate is obtained from equation (1.26) after similar substitution of stagnation enthalpy difference i_{0w} and product $q_{w*}C_x/\sqrt{C}$ before brackets for the temperature head and isothermal heat transfer coefficient h_*, respectively

$$q_w = \frac{q_{w*} C_x}{\sqrt{C}} \left[i_{0w}(0) + \int_0^x f(\xi/x) \frac{di_{0w}}{d\xi} d\xi \right] \tag{1.50}$$

1.9 Universal Functions for Power-Law Non-Newtonian Fluids

The term “non-Newtonian fluid” belongs to fluids which rheology (a science of flow of a matter) behavior is different from that of Newtonian fluids. In particular, the Newtonian fluids viscosity depends only on the temperature, whereas there are countless other fluids and materials (like polymers) whose viscosity is governed by more complex laws depending on spatial or/and on time deformation (Com. 5.8). The universal function was obtained for non-Newtonian and non-Fourier power laws fluids [351, p. 83] which characteristics obeys the following expressions of power types

$$\boldsymbol{\tau} = K_\tau \left(\frac{1}{2} I_2\right)^{\frac{n-1}{2}} \mathbf{e}, \quad \mathbf{q} = K_q \left(\frac{1}{2} I_2\right)^{\frac{s}{2}} \operatorname{grad} T, \quad I_2 = 4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial \mathrm{v}}{\partial y}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial \mathrm{v}}{\partial x}\right)^2 \tag{1.51}$$

Here, $\boldsymbol{\tau}$ and $\mathbf{e}$ are the stress and rate of deformation tensors, $\mathbf{q}$ is the heat flux vector, K_τ and K_q are constants and I_2 is a second invariant of rate of deformation tensor. Non-Newtonian power laws satisfactory describe the behavior of a group of substances, like suspensions, polymer solutions and melts, starch pastes, clay mortars.

Comment 1.10 Tensor is a general term defining the other quantities by order of its degree. A tensor order depends on array of numerical values (or indices) that determines the tensor. In this terminology, the scalar is a tensor of zero order, whereas the vector is the tensor of order one since three values (three coordinate) is necessary to define a vector. Similarly, the tensor of second order is an array of nine values.

Relations (1.51) simplifies for boundary layer flows for which they become

$$\tau = K_\tau \left(\frac{\partial u}{\partial y} \right)^n, \qquad q = K_q \left(\frac{\partial u}{\partial y} \right)^S \frac{\partial T}{\partial y} \tag{1.52}$$

The Newton and Fourier laws follow from (1.52) as well as from (1.51) at $n = 1$ and $s = n - 1 = 0$ (Exer. 1.50 and 1.51).

It is shown that universal functions (1.39) and (1.40) are valid also for power law non-Newtonian fluids, but only for such, for which the condition $s = n - 1$ is satisfied. This equality means that viscosity K_τ and heat conductivity K_q (called apparent parameters for non-Newtonian fluids, Exer. 1.52) are proportional to each other as for Newtonian fluids, and due to that, the analogous exact solution of thermal boundary layer equation may be obtained. [111, p. 84] In that, more general case, the Görtler variable and parameter β, defined the self-similar flows, are determined as follows (Exer. 1.53)

$$\Phi = \frac{\rho}{K_\tau} \int_0^x U^{2n-1}(\xi) d\xi \qquad \beta = \frac{(n+1)m}{(2n-1)m+1} \qquad m = \frac{Ux}{\int_0^x U(\xi) d\xi} - 1 \tag{1.53}$$

These equations indicate that for power-law fluids, the pressure gradient parameter β is depended not only on exponent m of the external velocity $U = cx^m$ as for Newtonian fluid, but also on exponent n in rheology law (1.51) or (1.52). Because of that, the pressure gradient is characterized not by β, which depends also on n, but by using directly the exponent m applying the third formula (1.53) instead of relation (1.45) for β in the case of Newtonian fluid (Exer. 1.54).

The coefficients g_k were calculated in the same way only for large Prandtl numbers Pr = 10, 100, 1000 typical for non-Newtonian fluids, exponents n from 0.2 to 1.8, and for $m = 0$ (zero pressure gradient), $m = 1/3$ (negative pressure gradient) and $m = 1$ (stagnation point flow). The results given in Figure 1.10 indicate that: (i) the dependences of coefficients g_k on Prandtl number and pressure gradient are similar to those for Newtonian fluids (Fig.1.3), (ii) for Pr > 10 and relatively small pressure gradients $m = 0$ and $m = 1/3$, coefficients g_k are practically independent on Pr as well as for Newtonian fluid, (iii) as the pressure gradient increases ($m = 1$), this dependence becomes more significant, but still remains slight, (iv) functions $g_1(n)$ and $g_2(n)$ (for $m = 1/3$, $m = 1$ and Pr = 100 they merge in one curve 2) present the effect of non-Newtonian behave of fluid (exponent n) on heat transfer intensity showing that nonisotermicity effect increases markedly with growing exponent n, (v) the greatest coefficient g_1 becomes larger from two to more than three times as the exponent n increases from 0.2 to 1.8.

In Figure 1.11 are plotted the results obtained for coefficient g_0 which defines the isothermal heat transfer coefficient h_* for non-Newtonian fluids according to formula for Nusselt number [111, p. 84]

$$\frac{\mathrm{Nu}_*}{\mathrm{Re}^{\frac{n}{n+1}}} = g_0 \left(\frac{C_f}{2} \mathrm{Re}^{\frac{1}{n+1}} \right)^{\frac{2n-1}{2n}} \left(\frac{\Phi}{\mathrm{Re}} \right)^{\frac{1}{2(n+1)}} \tag{1.54}$$

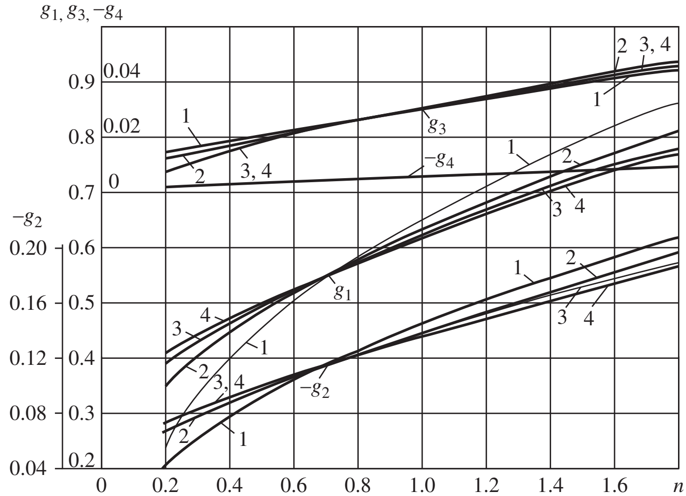

Figure 1.10 Coefficients g_k for non-Newtonian fluids: $s = n - 1$, $1 - m = 1$, Pr = 10, $2 - m = 1$, Pr = 100 and $m = 1/3$, Pr > 10, $3 - m = 1$, Pr > 1000, $4 - m = 0$, Pr > 10

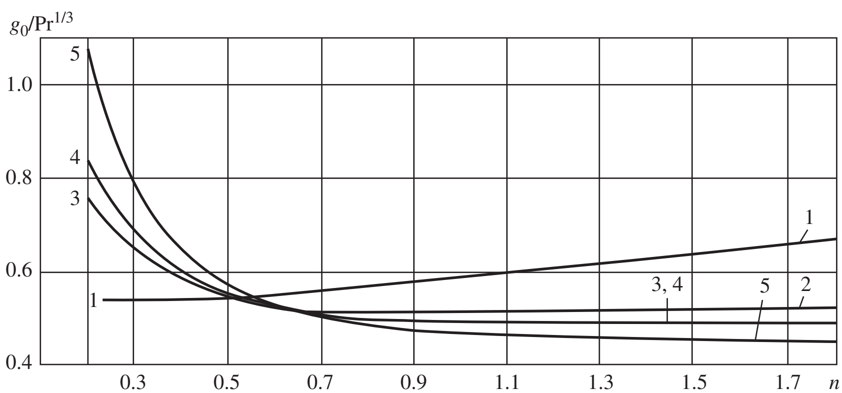

Figure 1.11 Quotient $g_0/\mathrm{Pr}^{1/3}$ of the coefficient g_0 and Prandtl number $\mathrm{Pr}^{1/3}$ for non-Newtonian fluids: $s = n - 1$, $1 - m = 0$, Pr > 10, $2 - m = 1/3$, Pr > 10, $3 - m = 1$. Pr > 1000, $4 - m = 1$, Pr = 100, $5 - m = 1$, Pr = 10

where Nu and Re are special numbers for non-Newtonian fluids given in nomenclature. The friction coefficient C_f containing in (1.54) may be estimated using special literature, for example [351]. Figure 1.11 presents the value $g_0/\mathrm{Pr}^{1/3}$ (instead of g_0) as a function of exponent n for different values of pressure gradient (different m) and Prandtl number. It is seen that for large Pr, this quotient slightly depends on Prandl number. This indicates that the heat transfer coefficient for an isothermal surface for power-law fluids is practically proportional to $\mathrm{Pr}^{1/3}$ as well as for Newtonian fluids (S. 7.5.1).

Although the coefficients C_1 and C_2 here are not given, they may be calculated by the same way using data for coefficients g_k (Exer. 1.55). For the general case of arbitrary exponents n and s (not connected by condition $s = n - 1$) for laws (1.51) only approximate similar solutions for universal function have been obtained [104].

1.10 Universal Functions for Moving Continuous Sheet

The systems in which a continuous material goes out of a slot and moves through surrounding coolant are used in a number of industrial processes, such as forming of synthetic films and fibers, the rolling of metals, glass production, and so on. Due to viscosity of the surrounding, on the surface of such moving sheet a boundary layer similar to that on a streamlined or a flying body forms as it schematically is shown on Figure 1.12. Despite both boundary layers are similar, at the middle of the last century it was shown [333] that the boundary layer on continuous sheet differs from the well-known boundary layer existing on streamlined bodies. As it is clear from Figure 1.12 in this case, the boundary layer grows in the direction of the motion, whereas on the moving or streamlined body, the boundary layer develops in an opposite to the moving direction.

It can be shown that in coordinate system attached to the moving surface, the boundary layer equations are unsteady (Exer. 1.56), and hence, they differ from the usual steady equations for a plate, but the boundary conditions in the moving frame are identical with those for streamlined plate. At the same time, in a frame attached to the slot, the problem of the moving sheet is steady, and both boundary layer equations coincide, however, in this case, the boundary conditions differ because the flow velocity on a sheet relatively to a slot is not zero.

The first calculations revealed that for the moving continuous sheet, the friction coefficient and the isothermal heat transfer coefficient at Pr = 0.7 are greater than those for streamlined plate by 34% and 20%, respectively [333, 395]. Exact solution for arbitrary nonisothermal surface in the same form of universal functions (1.38) and (1.26) for zero pressure gradient was obtained in [109] for stationary and blowing surrounding coolant with different ratio $\phi = U_\infty / U_w$, where U_w and U_∞ are velocities of surface and coolant. The coefficients g_k and the exponents C_1 and C_2 are given in Figure 1.13 and 1.14. The coefficient g_0 plotted in Figure 1.15

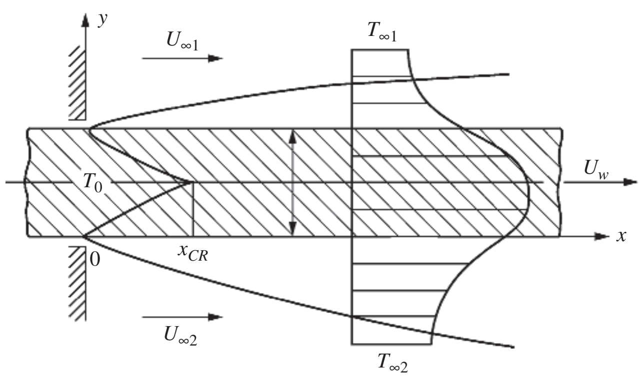

Figure 1.12 Schematic pattern of boundary layer on moving continuous sheet for symmetric ($U_1 = U_2, T_{\infty 1} = T_{\infty 2}$) and asymmetric flows

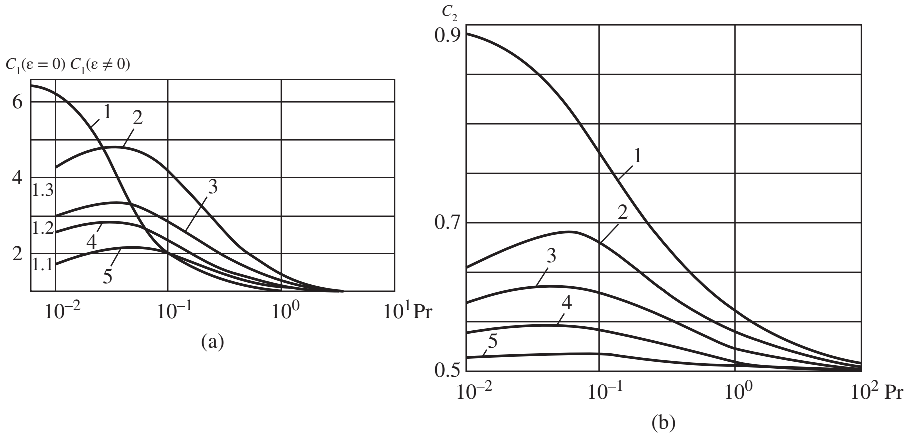

Figure 1.13 Coefficients $g_1(a)$ and $g_2(b)$ as a function of Prandtl number and a ratio $\phi = U_\infty/U_w$ for moving continuous sheet: $1 - \phi = 0$, $2 - 0.1$, $3 - 0.3$, $4 - 0.5$, $5 - 0.8$, 6-streamlined plate, 7(b)-g_3

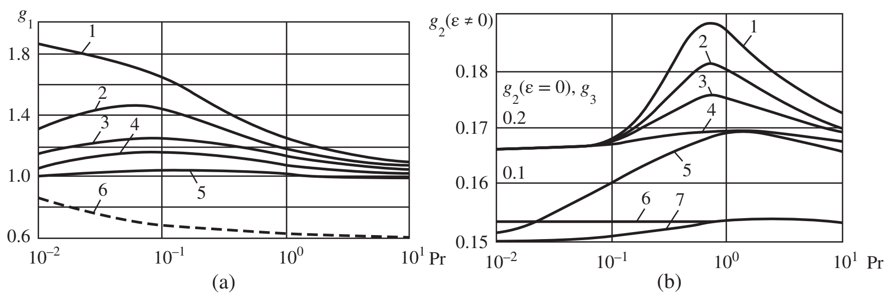

Figure 1.14 Exponent C_1 and C_2 as a functions of Prandtl number and ratio $\phi = U_\infty/U_w$ for a plate moving through surrounding. $1 - \phi = 0$, $2 - 0.1$, $3 - 0.3$, $4 - 0.5$, $5 - 0.8$

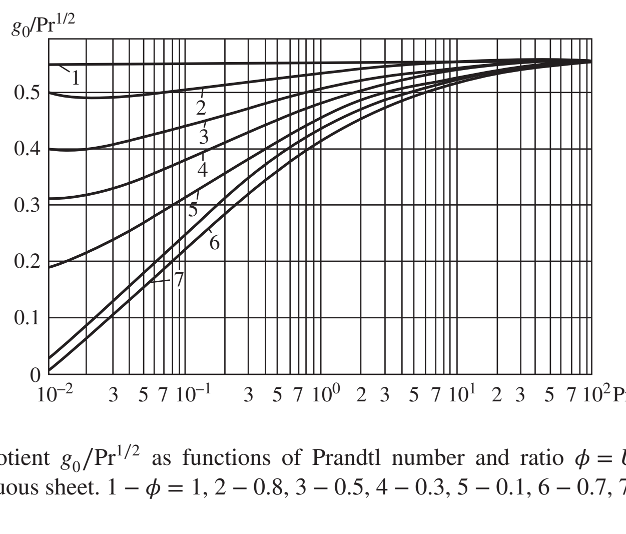

Figure 1.15 Quotient $g_0/\mathrm{Pr}^{1/2}$ as functions of Prandtl number and ratio $\phi = U_\infty/U_w$ for moving isothermal continuous sheet. $1 - \phi = 1$, $2 - 0.8$, $3 - 0.5$, $4 - 0.3$, $5 - 0.1$, $6 - 0.7$, $7 - (-0.05)$

for different ratio $\phi = U_\infty / U_w$ defines the isothermal heat transfer coefficient according to formula $h_* = g_0\sqrt{U_w/\nu x}$ (Exer. 1.57).

The coefficients g_k in this case are markedly greater than those for streamlined plate (Figs. 1.3 and 1.4), but decrease rapidly as well with coefficient number increasing. For instance, in the case of stationary surrounding, the first coefficient g_1 for the moving sheet is twice greater which indicates that the effect of nonsothermcity in this case is significantly larger than that for streamlined stationary or moving plate. As well as for usual case, the coefficients g_k for $k \geq 3$ are practically independent of the Prandtl number and blowing parameter ϕ and may be calculated by the same formula (1.42) for Pr → 0 as for universal functions (1.38).

1.11 Universal Functions for a Plate with Arbitrary Unsteady Temperature Distribution

An exact solution of unsteady thermal boundary layer equation for a plate with arbitrary unsteady temperature distribution $T_w(t,x)$ shows that in this case, a differential universal function has the form similar to series (1.38) [117, p. 75] (Exer. 1.58)

$$q_w = h_* \left(\theta_w + g_{10} x \frac{\partial \theta_w}{\partial x} + g_{01} \frac{x}{U} \frac{\partial \theta_w}{\partial t} + g_{20} x^2 \frac{\partial^2 \theta_w}{\partial x^2} + g_{02} \frac{x^2}{U^2} \frac{\partial^2 \theta_w}{\partial t^2} + g_{11} \frac{x^2}{U} \frac{\partial^2 \theta_w}{\partial x \partial t} + \dots \right) \tag{1.55}$$

Series (1.55) contains three types of terms with derivatives depending: on coordinate only, on time only, and on both coordinate and time with coefficients g_{k0}, g_{0i} and g_{ki}, respectively. Coefficients g_{k0} are the same as g_k in universal function (1.38) for steady temperature distribution. The two others depend on Prandtl number as well as g_k and on time (Exer. 1.59). For the case of zero pressure gradient and Pr = 1, the first four coefficients $g_{ki} (i \neq 0)$ are computed numerically. They are plotted on Figure 1.16, which shows that the coefficients g_{ki} are similar to coefficients g_k rapidly decreasing with growing number *ki*. Due to that, it is possible to obtain satisfactory accurate calculations using only first several terms of the series (1.55) as well as in the case of employing steady state universal function (1.38).

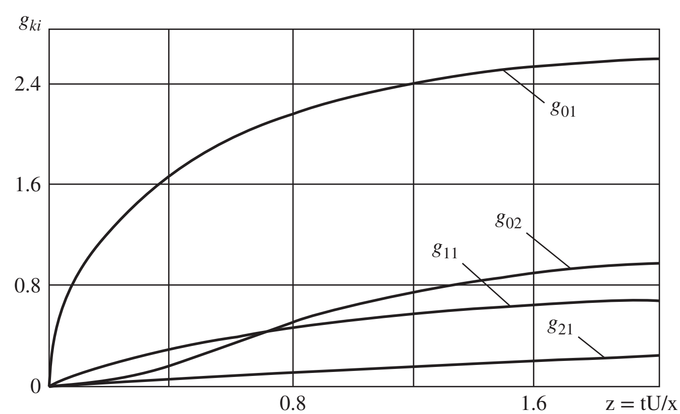

Figure 1.16 Coefficients $g_{ki} (i \neq 0)$ as functions of z for zero pressure gradient and Pr = 1

It follows from Figure 1.16 that the coefficients g_{ki} gradually grow with time and finally attain the values of $(g_{ki})_{t\to\infty}$ that coincide with those obtained by Sparrow in a similar problem without an initial conditions [370]. The ratio $g_{ki}(z)/(g_{ki})_{t\to\infty}$ is about 0.99 when dimensionless time $z = Ut/x$ (see Exer. 1.59) becomes $z > 2.4$. Hence, for $z > 2.4$, the coefficients g_{ki} are practically independent of time. Comparing these coefficients with coefficients g_k of series (1.38) for zero pressure gradient and Pr = 1 reveals that coefficients g_{ki} for large time are much greater. In particular, the first coefficient $g_{01} \approx 2.4$ is four times larger than corresponding value of steady state $g_1 \approx 0.6$ (Fig.1.3). This specifies that the nonisothermicity effect caused by time temperature gradient $\partial\theta_w/\partial t$ is four times greater than the effect of nonisothermicity produced at the same conditions by spatial temperature gradient $\partial\theta_w/\partial x$.

The following integral universal function corresponds to unsteady differential universal function (1.55)

$$q_w = h_* \left[\theta_w(t, 0) + \int_0^x f[(\xi/x,)\,0,\, z]\frac{\partial\theta_w}{\partial\xi}d\xi + \int_0^t f[0,\,(\eta/t),\, z]\frac{\partial\theta_w}{\partial\eta}d\eta \right.$$
$$\left. + \int_0^t d\eta \int_0^x f\left[(\xi/x),\,(\eta/t),\, z\right]\frac{\partial^2\theta_w}{\partial\xi\partial\eta}d\xi \right] \tag{1.56}$$

Applying the same technique of repeated integration by part as in the case of steady-state heat transfer described in Section 1.4, one can show that this expression is identical with the differential form in series (1.55).

1.12 Universal Functions for an Axisymmetric Body

According to Stepanov and Mangler (in conformity with Russian and English literature), the problem for an axisymmetric body is transformed to two-dimensional problem by using variables which in case of non-Newtonian power law fluids are [351]

$$\tilde{x} = \int_0^x R^{n+1}(\xi)d\xi, \quad \tilde{y} = Ry, \tag{1.57}$$

where R is a cross section radius (Exer. 1.60). It follows from this result that universal functions (1.39) and (1.40) are valid for flow past axisymmetric body in Görtler variable transformed according to relation (1.57). Such variable for the case of the power law non-Newtonian fluids is obtained after substituting $R^{n+1}dx$ for dx in the corresponding Görtler variable for non-Newtonian fluids (1.53). The heat flux q_w obtained for plane two-dimensional problems should be multiplied in this case by R^n to get

$$\Phi = \frac{\rho}{K_\tau}\int_0^x U^{2n-1}(\xi)R^{n+1}(\xi)d\xi, \qquad \tilde{q}_w = R^n q_w \tag{1.58}$$

In the case of Newtonian fluid ($n = 1$), the first expression (1.58) in conformity with equations (1.57) transforms in the Görtler variable Φ (1.39) modified by multiplying the integrand by R^2. The second equation (1.58) in this case reduces to $\tilde{q}_w = Rq_w$ where q_w is the heat flux obtained by universal function (1.39) for plane problem, which also is in line with the second equation (1.57) (Exer. 1.61).

Comment 1.11 Special variables play a significant role in simplifying the initial form of equations. So far, we consider those three variables: Görtler variable, which transforms the solution for zero pressure gradient to form applicable for flows with arbitrary pressure gradient, Dorodnizin-Illingworth-Stefartson variable transforming the equations for compressible fluid in an incompressible form, and the Stepanov-Mangler variable which reduces the axisymmetric equation to the equivalent two-dimensional equation. There are other specific variables transforming equations, in particular, as later we will see the Falkner-Skan, Blasius, and other similarity variables, which change the partial boundary layer equations in ordinary differential equations (S. 7.5.1 and 7.5.2), Prandtl-Mises variables converting boundary layer equation in the form, which is close to one-dimensional conduction equation (S. 7.4.4.2). These examples are taken from boundary layer theory. However, the equations transforming by applying new variables is a general method widely used in mathematics.

1.13 Inverse Universal Function

The inverse universal function is obtained as a result of solution of the inverse problem when the surface heat flux distribution is specified and the corresponding temperature head distribution should be found. We consider two such problems: first, the general inverse universal functions for surface with given arbitrary heat flux distribution is obtained and then, the specific inverse problem determining the universal function for recovery factor is solved.

1.13.1 Differential Inverse Universal Function [112]

Because we are seeking a temperature distribution, we solve equation (1.39) for temperature head to obtain

$$\theta_w + \sum_{k=1}^{\infty} g_k \Phi^k \frac{d^k \theta_w}{d\Phi^k} = \frac{q_w}{h_*} = \theta_{w*}(\Phi) \tag{1.59}$$

The right hand part of this equation is a known function of x and hence, of Görtler variable Φ since in the problem in question the heat flux distribution is given. Physically the ratio q_w/h_* determines the temperature head which would be established by given heat flux if the considering surface were isothermal. Therefore, we use for this function the notation $\theta_{w*}(\Phi)$ (Exer. 1.62). In fact, equation (1.59) is a differential equation defining unknown function $\theta_w(\Phi)$, which solution may be presented in terms of known function $\theta_{w*}(\Phi) = q_w/h_*$ as a series similar to universal function (1.39)

$$\theta_w = \theta_{w*} + \sum_{n=1}^{\infty} h_n \Phi^n \frac{d^n \theta_{w*}}{d\Phi^n}, \tag{1.60}$$

where h_n denotes coefficients similar to g_k. To prove this, we show that relation (1.60) satisfies equation (1.59) which we transform by replacing the right hand part to the left

$$\theta_w - \theta_{w*}(\Phi) + \sum_{k=1}^{\infty} g_k \Phi^k \frac{d^k \theta_w}{d\Phi^k} = 0 \tag{1.61}$$

Substitution of equation (1.60) into this equation leads to following equation (Exer. 1.63)

$$\sum_{n=1}^{\infty} h_n \Phi^n \frac{d^n\theta_{w*}}{d\Phi^n} + \sum_{n=1}^{\infty} g_k \Phi^k \frac{d^k\theta_{w*}}{d\Phi^k} + \sum_{n=1}^{\infty} g_k \Phi^k \frac{d^k}{d\Phi^k} \sum_{n=1}^{\infty} h_n \Phi^n \frac{d^n\theta_{w*}}{d\Phi^n} = 0 \tag{1.62}$$

After changing indices n to k and performing differentiation in the last term, we modify this equation by assembling terms containing the same groups $\Phi^k(\partial^k\theta_{w*}/\partial\Phi^k)$ for $k = 1, 2, 3 \ldots$. The expression obtained in such a way is a summation of the partial sums of groups $\Phi^k(\partial^k\theta_{w*}/\partial\Phi^k)$ for different numbers k, which according to (1.62) should be equal zero. Since those partial sums of different groups are independent of each other, the required condition of zero may be satisfied only if each sum of groups would be equal zero. The equalities obtained in such procedure by setting each of partial sums to zero contain known k coefficients g_k and $k - 1$ coefficients h_k. Therefore, they define coefficients h_k in sequence so that the first equality for $k = 1$ specifies h_1, the second one gives h_2 via known h_1, and so on, resulting in the following equations (Exer. 1.64)

$$\begin{gathered} h_1 + g_1(h_1 + 1) = 0, \quad h_2 + g_1(2h_2 + h_1) + g_2(2h_2 + 2h_1 + 1) = 0, \\ h_3 + g_1(3h_3 + h_2) + g_2(6h_3 + 4h_2 + h_1) + g_3(6h_3 + 6h_2 + 3h_1 + 1) = 0, \\ h_4 + g_1(4h_4 + h_3) + g_2(12h_4 + 6h_3 + h_2) + g_3(24h_4 + 18h_3 + 6h_2 + h_1) \\ +g_4(24h_4 + 24h_3 + 12h_2 + 4h_1 + 1) = 0 + \ldots \end{gathered} \tag{1.63}$$

Figures 1.17 and 1.18 present the values of the first four coefficients h_k for laminar and turbulent flows. For the limiting Prandtl numbers the coefficients h_k are

$$h_1 = -1/2, \quad h_2 = 3/16, \quad h_3 = -5/96, \quad h_4 = 35/1968 \qquad \mathrm{Pr} \to 0 \tag{1.64}$$

$$h_1 = -0.38, \quad h_2 = 0.135, \quad h_3 = -0.037, \quad h_4 = 0.00795 \qquad \mathrm{Pr} \to \infty \tag{1.65}$$

Like coefficients g_k, the first few coefficients h_k are a weak functions of β and the others are practically independent of β and Pr.

1.13.2 Integral Inverse Universal Function [112]

The integral inverse universal function is obtained in the same way by considering the universal function (1.40) as an equation for temperature head determination to get

$$\int_0^{\Phi} f(\xi/\Phi)\frac{d\theta_w}{d\xi}d\xi = \frac{q_w(\Phi)}{h_*(\Phi)} \qquad \text{or} \qquad \int_0^{\Phi} [1 - (\xi/\Phi)^{C_1}]^{-C_2} \frac{d\theta_w}{d\xi}d\xi = \frac{q_w(\Phi)}{h_*(\Phi)} \tag{1.66}$$

It is seen that in this case, the unknown function θ_w is located under sign of an integral with variable limit. To find such unknown function, one should solve the second equation (1.66), which is Volterra integral equations (Exer. 1.65). Because there is no standard approach for solving integral equations, the solution of such a problem is a hard task. In this specific case, applying new variables: $\Phi^{C_1} = z$, $\xi^{C_1} = \zeta$, and $q_w(\Phi)/h_*(\Phi)\, z^{C_2} = F(z)$ converts second equation (1.66) into Abel integral equation (1.67), of which the known solution for this

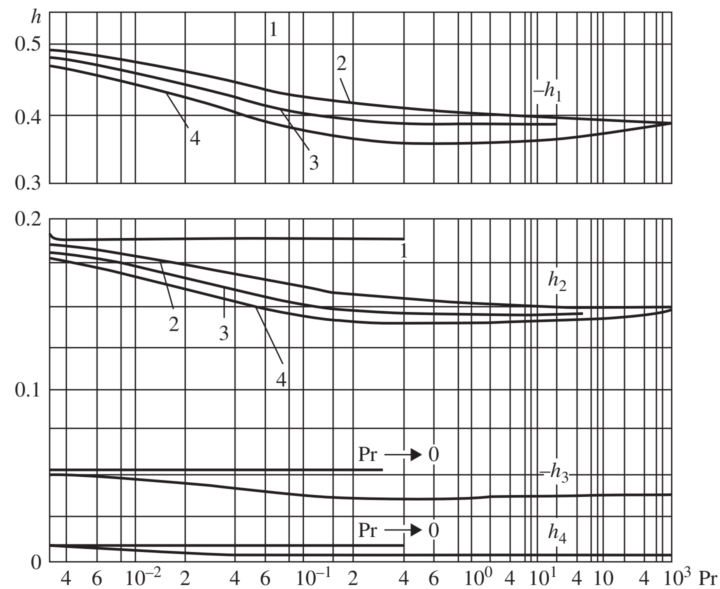

Figure 1.17 Coefficients h_k for laminar boundary layer. Asymptotes $1 - \mathrm{Pr} = 0,\ \beta : 2 - 1,\ 3 - 0,\ 4 - (-0.16)$

particular case is shown by second expression (1.67) (Exer. 1.66)

$$\int_0^z \frac{d\theta_w}{d\zeta} \frac{d\zeta}{(z-\zeta)^{C_2}} = F(z) \quad \frac{d\theta_w}{dz} = \frac{1}{(C_2-1)!(-C_2)!} \frac{d}{dz} \int_0^z \frac{F(\zeta) d\zeta}{(z-\zeta)^{1-C_2}} \tag{1.67}$$

Returning to variables Φ and ξ yields a relation for the temperature head (Exer. 1.67)

$$\theta_w = \frac{C_1}{\Gamma(1-C_2)\Gamma(C_2)} \int_0^\Phi \left[1 - \left(\frac{\xi}{\Phi}\right)^{C_1}\right]^{C_2-1} \left(\frac{\xi}{\Phi}\right)^{C_1(1-C_2)} \frac{q_w(\xi)}{h_*(\xi)\xi} d\xi \tag{1.68}$$

For the case of zero pressure gradient, this expression coincides with known relation

$$\theta_w = \frac{0.623}{\lambda} \mathrm{Re}_x^{-1/2}\, \mathrm{Pr}^{-1/3} \int_0^x \left[1 - \left(\frac{\xi}{x}\right)^{3/4}\right]^{-2/3} q_w(\xi) d\xi \tag{1.69}$$

obtained in [201] using another approach (Exer. 1.68).

1.14 Universal Function for Recovery Factor

The recovery factor determines a part of fluid mechanical energy that is recovered as thermal energy. This process is important for estimation of the wall adiabatic temperature (S. 7.3.5).

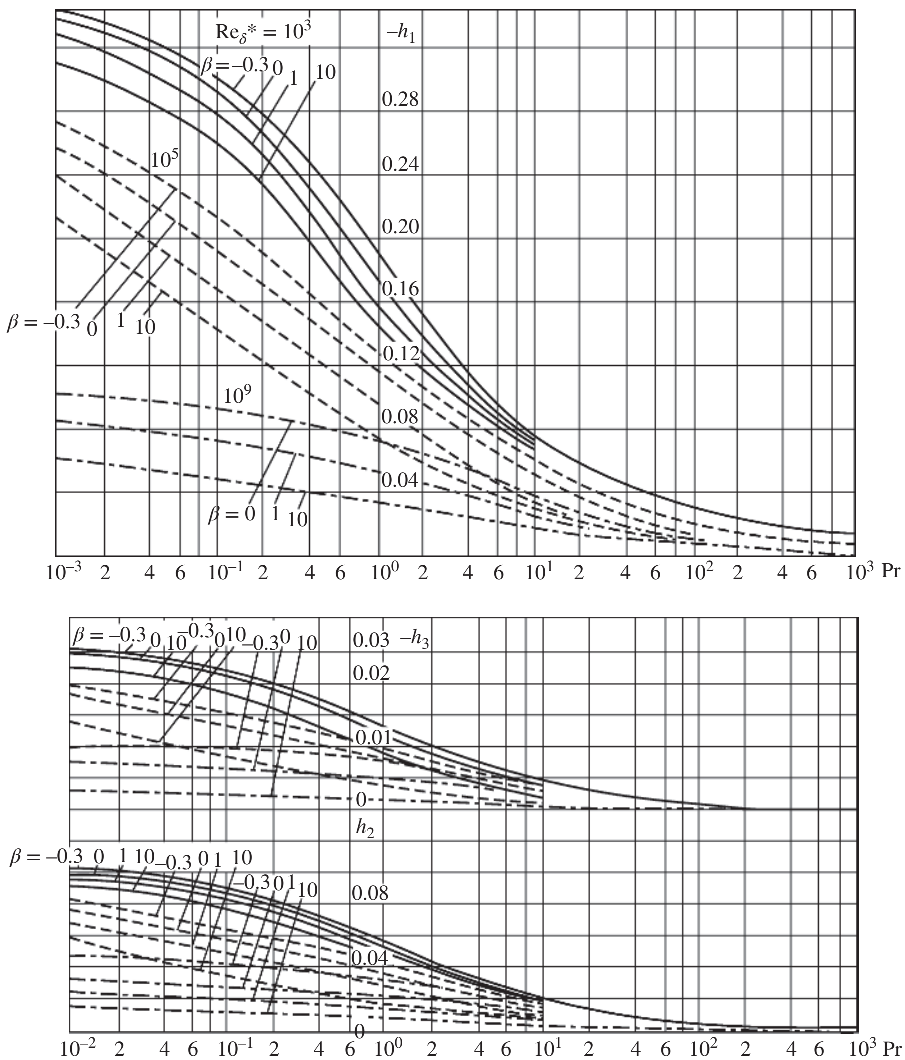

Figure 1.18 Coefficients h_k for turbulent boundary layer ___ $\mathrm{Re}_{\delta_1} = 10^3$,_ _ _ 10^5,_. _. 10^9

Actually, the recovery factor calculation is an inverse problem of the temperature head determining under condition of zero heat flux (Exer. 1.49) and significant mechanical energy dissipation. Solution of such a problem is similar to that considered in the previous section, but in this case in addition, the effect of mechanical energy dissipation should be taken into account.

It is shown [119, p. 57] that the exact solution of the laminar thermal boundary layer equation for the case with significant mechanical energy dissipation differs from universal function (1.39) by the term $g_d(U^2/c_p)$ where g_d is a special coefficient similar to coefficients g_k. Taking this into account, we solve equation (1.39) with such additional term (the first equation (1.70)) for temperature head θ_w in the same way as in Section 1.13.1 to obtain the second equation (1.70) instead of similar equation (1.59)

$$q_w = h_* \left(\theta_w + \sum_{n=1}^{n=k} g_k \Phi^k \frac{d^k \theta_w}{d\Phi^k} - g_d \frac{U^2}{c_p} \right), \quad \theta_w + \sum_{k=1}^{\infty} g_k \Phi^k \frac{d^k \theta_w}{d\Phi^k} = \frac{q_w}{h_*} + g_d \frac{U^2}{c_p} \tag{1.70}$$

Considering the last equation (1.70) as the differential equation defining temperature head as a sum $(q_w/h_* + g_d U^2/c_p)$ instead of θ_{w*}, we get the same solution (1.60) for the problem with significant mechanical dissipation in which $\theta_{w*} = q_w/h_*$ is substituted by the sum $(q_w/h_* + g_d U^2/c_p)$ regarding the effect of dissipation

$$\theta_w = q_w/h_* + g_d U^2/c_p + \sum_{n=1}^{\infty} h_n \Phi^n \frac{d^n (q_w/h_* + g_d U^2/c_p)}{d\Phi^n} \tag{1.71}$$

After putting here $q_w = 0$, this expression gives the universal function for recovery factor

$$r = \frac{T_{ad} - T_\infty}{U^2/2c_p} = \frac{T_w - T_{ad}}{U^2/2c_p} = 2g_d \left(1 + \sum_{k=1}^{\infty} h_k \frac{\Phi^k}{U^2} \frac{d^k U^2}{d\Phi^k} \right) \tag{1.72}$$

where T_{ad} is adiabatic wall (i.e., isolated) temperature, and T_w is the temperature of the usual non-isolated wall defined by equation (170) which wall has if it would be nonisolated. It follows from the last two relations that the adiabatic may be determined as a temperature of isolated wall or as a wall temperature cooled due to dissipation mechanical energy (Exer. 1.69 and 1.70).

Accordingly, integral universal function for recovery factor is obtained from relation (1.68) by the same substitution of the sum $(q_w/h_* + g_d U^2/c_p)$ for $\theta_{w*} = q_w/h_*$ in its integrand and following putting $q_w = 0$ (Exer. 1.71)

$$r = \frac{T_w - T_{ad}}{U^2/2c_p} = \frac{2g_d C_1}{\Gamma(1-C_2)\Gamma(C_2)U^2} \int_0^{\Phi} \left[1 - \left(\frac{\xi}{\Phi} \right)^{C_1} \right]^{C_2 - 1} \left(\frac{\xi}{\Phi} \right)^{C_1(1-C_2)} \frac{U^2(\xi)}{\xi} d\xi \tag{1.73}$$

Relations (1.72) and (1.73) present recovery factor for flows with arbitrary external flow $U(x)$. For zero pressure gradient with $U = const.$ both expressions (1.72) and (1.73) give the well-known result $r = 2g_d$ (Exer. 1.72).

The effect of dissipation, which is proportional to the square of velocity $U^2(x)$., is minor for incompressible fluids due to typical relatively small incompressible flows velocities. In such a case, the effect of dissipation becomes significant only for large Prandtl numbers when the last term in thermal boundary layer equation (1.11), which is proportional to Prandl number, is comparable with other terms (Exer.1.73). The large Prandtl numbers are common, in particular, for non-Newtonian fluids. Because of that, the coefficient g_d was calculated for non-Newtonian ($n = 0.6 - 1.2$) including Newtonian ($n = 1$) fluids only for large Prandtl numbers, Pr >1000, for zero pressure gradient ($m = 0$) and for stagnation point flow ($m = 1$). The same approach as for coefficients g_k calculation (S. 1.6.1) was used (Exer. 1.74). The results are plotted in Figure 1.19. It is seen from Figure 1.19 that the recovery factor and hence, the effect of dissipation increases with pressure gradient and decreases as the exponent n increases. This implies that the effect of dissipation for non-Newtonian fluids with $n > 1$ (dilatant fluids) is greater and limited for with $n < 1$ (pseudoplasic fluids) is smaller than that for Newtonian fluid with $n = 1$.

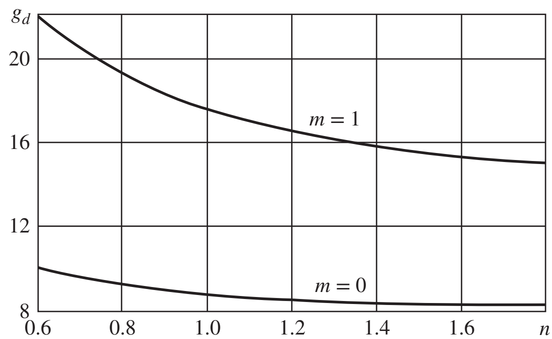

Figure 1.19 Coefficients g_d for Newtonian ($n = 1$) and non-Newtonian fluids ($s = n - 1$, S.1.6), Pr >1000

Comment 1.12 One merit of using the adiabatic wall temperature is that the substitution of the adiabatic temperature head defined as $\theta_{ad} = T_w - T_{ad}$ for usual temperature head $\theta_w = T_w - T_\infty$ in known relations yields the expressions valid for the case with heat dissipation. For example, equation (1.49) for compressible fluid written using adiabatic stagnation enthalpy difference i_{0w} has the same form as the relation (1.39) written in terms of usual temperature head θ_w for incompressible fluid. This is achieved due to applying adiabatic stagnation enthalpy difference according to the second equation (1.49) (Exer. 1.75). Applying adiabatic temperature is also important to understand some heat exchange processes. For instance, as it shown in Section 2.1.4.3, the fluid cools the wall until it reaches the adiabatic temperature, and then, the heat flux changes its direction so that the fluid heats the wall.

Exercises

1.48* The basic idea of transforming boundary layer equation for compressible fluid to the form of incompressible fluid is to take into account the variability of gas density using integral of density $\eta = \eta_0 \int_0^y \frac{\rho(\xi)}{\rho_0} d\xi$. That gives the transverse variable η, which converts the compressible boundary layer equation to the form of incompressible boundary layer equation. Note, that expression for η is of the same type as Görtler variable Φ defined by equation (1.39). Compare these relations, thinking of their similarity and dissimilarity. Are the considerations about different kinds of information and flow history outlined in comments 1.5 and 1.6 applicable to relation for η defined as integral of density? Think about dummy variables in this relation and in Görtler variable (1.39). Compare both these dummy variables denoted by the same letter ξ. Are they actually the same? If no, explain why do you think so, and how differ these dummy variables from each other?

1.49 Recall what is the adiabatic wall temperature (S.7.3.5) and enthalpy to understand what is the stagnation enthalpy difference i_{0w} and how is it calculated? Compare relations (1.49) and (1.50) for compressible fluids with analogous relations (1.38) and (1.26) for incompressible flows.

1.50 Show that for the boundary layer flows, the relations (1.52) follow from general expressions (1.51). Hint: compare magnitude order of terms as in derivation boundary layer equations (S. 7.4.4.1) and use Newton's law for shear stress.

1.51 Obtain Newton's and Fourier's laws from relations (1.52) for Newtonian fluids ($n = 1$) and prove that in this case the proportionality between viscosity and conductivity exists because of $s = n - 1$. Hint: note that in this case the deformation tensor **e** (see (1.51)) equals $\partial u/\partial y$.

1.52* Show the same from relations (1.52) for non-Newtonian fluid (at any n) for the case when $s = n - 1$. Hint: modify the relations (1.52) to the forms $\tau = K_\tau \left(\frac{\partial u}{\partial y}\right)^{n-1} \frac{\partial u}{\partial y}$ and $q = K_q \left(\frac{\partial u}{\partial y}\right)^{n-1} \frac{\partial T}{\partial y}$ which are similar to those obtained for Newtonian fluid in example 1.51. Compare the Newton's and Fourier's laws for Newtonian fluid with similar relations for non-Newtonian fluids to see the difference between viscosity μ and conductivity λ in the first case and factors $K\left(\frac{\partial u}{\partial y}\right)^{n-1}$ at velocity and temperature derivatives in the second case. Think: why those coefficients for non-Newtonian fluids are called "apparent" viscosity and conductivity (see Com. 5.8)?

1.53* Show that in the case of constant external flow $U = const$, the Görtler variable (1.53) becomes Reynolds number in a special form for non-Newtonian fluids (see nomenclature) as in similar case for Newtonian fluid (Exer. 1.30).

1.54 Derive the third formula (1.53) for exponent m using approach described in Exer. 1.35.

1.55* Calculate exponents C_1 and C_2 for several examples using coefficients g_k from Figure 1.10 by graphical approach or software described in Exer. 1.41 and 1.42, respectively; see also Exer. 1.47.

1.56 Think and explain why the problem of a moving continuous sheet is unsteady in the frame attached to the sheet, but is steady in unmoved frame located out of sheet. Hint: consider what the observer sees looking at the boundary layer when he or she: (i) moves along with the sheet and (ii) sit out of the moving sheet.

1.57* Compare coefficients g_0, g_1 and g_2 for different values of parameter $\phi = U_\infty/U_w$ on Figures 1.13 and 1.15. Think: why coolant blowing increases the first coefficient, but reduces two others showing that the cooling effect of isothermal surface grows, whereas the effect of nonisithermicity decreases as coolant velocity increases? What physically causes those opposite effects? Hint: analyze the mechanism of such two phenomena.

1.58 Continue series (1.55) for $i = 0$, $k > 0$, $k = 0$, $i > 0$, $k > 0$, $i > 0$ knowing that the general term is $g_{ki}(x^{k+i}/U^i)(\partial^{k+i}\theta_w/\partial x^k \partial t^i)$. Hint: first check the given terms.

1.59* Show that time and mixed derivatives are in fact derivatives with respect to dimensionless time $z = Ut/x$ so that, for example, the third, fifth and sixth terms in equation (1.55) actually are $\partial\theta_w/\partial z$, $\partial^2\theta_w/\partial z^2$, $\partial^2\theta_w/\partial x\partial z$.

1.60* Compare variable (1.57) with others of this type, Görtler variable (1.39) and Dorodnizin-Illingworth-Stewartson variable, considered in Exercise 1.48. Answer the same questions about variable (1.57): are the considerations about different kinds of information outlined in comment 1.3 applicable to this variable defined as integral of cross section radius? What variable represents the same dummy variable ξ in this integral?

1.61 Show that in the case of Newtonian fluid, equation (1.58) transforms in both Görtler variable for plane two-dimensional problem and for axisymmetric body that are relation (1.39) and modified relation (1.39), respectively.

1.62 Explain physically why the ratio q_w/h_* presents the temperature head on an isothermal surface and why the notation $\theta_{w*}(\Phi)$ is proper for this ratio.

1.63 Obtain expression (1.62) by substituting equation (1.60) into equation (1.61). Hint: the first sum in (1.62) represents the difference $\theta_w - \theta_{w\bullet}$, which is found from equation (1.60), the second and the third terms are obtained by substituting θ_w defined by the same equation (1.60) into derivatives in the last term of equation (1.61).

1.64* Derive relations (1.63) following directions from text and calculate coefficients h_k for limiting cases. Explain why the summation of the sums of groups with the same complexes $\Phi^k(\partial^k\theta_{w*}/\partial\Phi^k)$ equals zero it follows that each these sums equals zero.

1.65 Read the article about integral equations on Wikipedia (at least the beginning part of definitions) to understand the difference between Volterra and Fredholm integral equations. Read also on Wikipedia about young mathematician Abel who became famous despite he died at the age of 26.

1.66* Show that substitution of new variables indicated in the text transforms the second integral equation (1.66) into integral Abel equation (1.67). Hint: note that ζ is the integration variable, whereas z is considered as a parameter (see Exercise 1.22).

1.67 Obtain the inverse universal function (1.68) from solution of Abel equation (1.67) by returning to physical variables. Hint: take care of difference between functions of variables Φ and ξ. Note also, that factorial of noninteger numbers is determined by gamma function as $\gamma\,! = \Gamma(\gamma + 1)$.

1.68* Prove that expression (1.68) becomes (1.69) in the case of zero pressure gradient. Hint: as in previous example, be careful in using variable Φ, which in this case is proportional to x, and variable ξ. To calculate gamma function apply known formulae, for example, $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x$.

1.69* Show that the first and second expressions (1.72) for recovery factor follow from equations (1.71) and (1.70), respectively. Verify that that both definition of adiabatic temperature given in text follow from these equations.

1.70 Derive differential universal function (1.72) for recovery factor using relation (1.60) as described in the text. Comparing sums in functions (1.60) and (1.72), one sees that the second sum may be obtained directly from the first one by substituting U^2 for θ_{w*}. Explain physically why such substitution yields true result.

1.71 Obtain integral universal function (1.73) for recovery factor using relation (1.68). Holds the same physical explanation from the previous exercise in this case?

1.72 Show that in the case of zero pressure gradient, relations (1.72) and (1.73) yield well-known result for recovery factor $r = 2g_d$. Hint: in integrand (1.73) take a new variable $z = (\xi/x)^{3/4}$ and transform integral (1.73) to beta function defined by equation (1.47).

1.73 Prove that that only the last term in thermal boundary layer equation (1.11) is proportional to Prandl number. Explain physically why high Prandtl number leads to significant effect of mechanical energy dissipation.

1.74* What parameters should be changed to make relations (1.72) and (1.73) applicable for turbulent flows?

1.75 Derive the second equation (1.49) for adiabatic stagnation enthalpy difference i_{0w} at zero pressure gradient from relation (1.72) knowing that Mach number is $\mathrm{M} = U/U_{sd}$ where speed of sound is $U_{sd} = \sqrt{kRT_\infty}$.