CHAPTER 1

MAGNETIC CIRCUITS

OUR KNOWLEDGE OF MAGNETIC phenomena is literally as old as science itself. According to the Greek philosopher Aristotle, the attractive power of magnets was known even at that time. However, it was not until the sixteenth century that experimental work on magnetism began in earnest. Notable, among scientists active at that time, were the works of Gilbert, who discovered the earth's magnetism, Volta, who developed the voltaic cell, and Oersted, who related the magnetic field to the flow of current. However, it is on the works of Biot and Savart, Ampere, and finally Faraday that the modern theory of magnetism is based. In their experiments, the force on a current-carrying wire due to the flow of current in another wire was carefully measured and forms the experimental basis for the entire theory of magnetism.

1.1 BIOT–SAVART LAW

Using modern notation, the experiments of these pioneers can be expressed compactly in a single vector equation. With reference to Figure 1.1, let O_1 and O_2 be two very thin closed conducting current loops in which steady (DC) currents flow. The coordinates along the loop O_1 can be designated by x_1, y_1, z_1 and the coordinates along the second loop O_2 as x_2, y_2, z_2. The arc lengths along the loops O_1 and O_2 are denoted as vector quantities $\boldsymbol{dl}_1$ and $\boldsymbol{dl}_2$, respectively. From the experiments of Biot, Savart, and Ampere, the differential force in Newtons expressed as a vector exerted on a small piece of loop 2 carrying current I_2 due to the current I_1 in a small piece of loop 1 can be expressed, in modern notation and units as

$$d\boldsymbol{F}_{21} = \left(\frac{\mu_0}{4\pi}\right)\frac{I_2 \boldsymbol{dl}_2 \times [I_1 \boldsymbol{dl}_1 \times \boldsymbol{u}_{r12}]}{R^2} \quad \text{newtons(N)} \tag{1.1}$$

where

$$R = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and $\boldsymbol{u}_{r12}$ is a unit vector pointing from $\boldsymbol{dl}_1$ to $\boldsymbol{dl}_2$. Essentially, this force acts to align the two differential elements (i.e., make $\boldsymbol{dl}_1$ and $\boldsymbol{dl}_2$ collinear). This expression can

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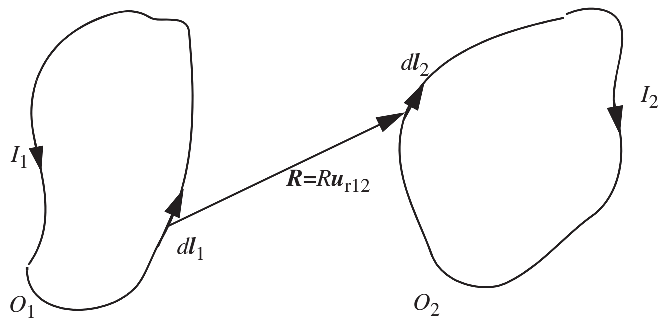

Figure 1.1 Illustration of Biot-Savart's law.

be integrated around coil 1 to find the total force exerted on the differential element of coil 2 as

$$d\boldsymbol{F}_{21} = \frac{\mu_0}{4\pi} \oint_{O_1} \frac{I_2 d\boldsymbol{l}_2 \times (I_1 d\boldsymbol{l}_1 \times \boldsymbol{u}_{r12})}{R^2} \tag{1.2}$$

To find the total force on wire 2, one can simply integrate a second time to form what is called Biot–Savart law or, alternatively, Ampere's law of force,

$$\boldsymbol{F}_{21} = \frac{\mu_0}{4\pi} \oint_{O_2} \oint_{O_1} \frac{I_2 d\boldsymbol{l}_2 \times (I_1 d\boldsymbol{l}_1 \times \boldsymbol{u}_{r12})}{R^2} \quad N \tag{1.3}$$

When the force $\boldsymbol{F}_{21}$ is measured in newtons and the currents are in amperes with the tests made in a vacuum the proportionality constant μ_0 is equal to $4\pi \times 10^{-7}$ newtons per ampere squared (eventually defined as henries per meter). Thus, the proportionality constant μ_0, is called the *permeability of free space*. It can be shown that reciprocity holds, that is, $\boldsymbol{F}_{12} = -\boldsymbol{F}_{21}$

1.2 THE MAGNETIC FIELD *B*

One of the great philosophical contributions of mathematics to science was the use of so-called "fields" to explain the action at a distance, a concept justly troubling to these early researchers. Upon examination of equation (1.3), one can define an incremental magnetic field vector $d\boldsymbol{B}_{21}$ at point 2 due to a current element at point 1 as

$$d\boldsymbol{B}_{21} = \frac{\mu_0}{4\pi} \frac{I_1 d\boldsymbol{l}_1 \times \boldsymbol{u}_{r12}}{R^2} \tag{1.4}$$

The magnetic field resulting from the entire circuit 1 is then

$$\boldsymbol{B}_{21} = \frac{\mu_0}{4\pi} \oint_{O_1} \frac{I_1 d\boldsymbol{l}_1 \times \boldsymbol{u}_{r12}}{R^2} \tag{1.5}$$

whereupon, the force equation, equation 1.3, becomes the much simpler "*BIl*" form,

$$\boldsymbol{F}_{21} = \oint_{O_2} I_2 d\boldsymbol{l}_2 \times \boldsymbol{B}_{21} \tag{1.6}$$

In contrast to equation (1.3), this formulation evaluates the force on a current loop in terms of the interaction of this current with a *magnetic field* $\boldsymbol{B}$. It is important to remember that the basic unit of magnetic field is *newtons per ampere-meter* which will lead to the interpretation of magnetic flux lines as "lines of force." Note that there need be no restriction on the value of $\boldsymbol{u}_{r12}$ and R in equation (1.5). That is, these quantities need not be concerned with the actual distance between two current elements on two circuits. In this case, $\boldsymbol{B}$ is well defined everywhere in space and thereby constitutes what is called a *vector field*.

One of the advantages of the field formulation is that when $\boldsymbol{B}$ is known, the relation permits one to evaluate what would be the force exerted on a current-carrying conductor placed anywhere in the $\boldsymbol{B}$ field without consideration as to what are the system of currents actually giving rise to this field.

An alternative expression for the vector $\boldsymbol{B}$ can be obtained if the current loops cannot be considered to have negligibly small cross-sectional areas, that is

$$\boldsymbol{B}_{21} = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J} \times \boldsymbol{u}_{r12}}{R^2} dV \tag{1.7}$$

where V is the volume and the vector $\boldsymbol{J}$ is the volumetric current density in *amperes per meter*^{2}.

1.3 EXAMPLE—COMPUTATION OF FLUX DENSITY *B*

The computation of flux density within an electrical machine forms the basic principle behind the machine design process. Consider here the simple example in which a short segment of wire of length L carries a current I as shown in Figure 1.2. Since the

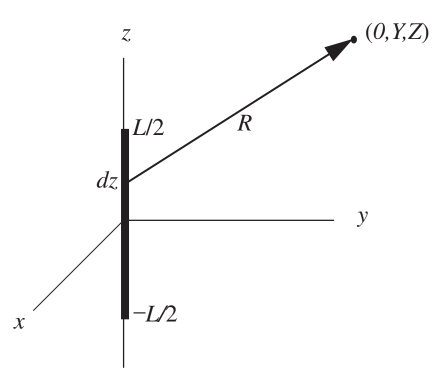

Figure 1.2 Magnetic field of a short wire.

current flows in the z-axis, equation (1.5) becomes

$$d\boldsymbol{B}_{21} = \frac{\mu_0}{4\pi}\frac{Idz\boldsymbol{u}_z \times \boldsymbol{u}_{r12}}{R^2} \tag{1.8}$$

Since the cross product cannot result in a z component and is also normal to the unit vector $\boldsymbol{u}_{r12}$, the flux density must also be normal to the plane containing the $\boldsymbol{u}_z$ as well as the vector $\boldsymbol{u}_{r12}$.

The magnitude of the total field at any point on the $x = 0$ plane is

$$B_{21} = \frac{\mu_0 I}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{|\boldsymbol{u}_z \times \boldsymbol{u}_{r12}|dz}{R^2} \tag{1.9}$$

On the $x = 0$ plane, the unit vector $\boldsymbol{u}_{r12}$ is given by

$$\boldsymbol{u}_{r12} = \frac{Y}{\sqrt{Y^2 + (Z - z)^2}}\boldsymbol{u}_y + \frac{Z}{\sqrt{Y^2 + (Z - z)^2}}\boldsymbol{u}_z \tag{1.10}$$

where Y and Z designate a specific point on the $x = 0$ plane. After taking the cross products, equation (1.9) becomes

$$\boldsymbol{B}(0, Y, Z) = \frac{\mu_0 I}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{YdZ}{\sqrt[3]{Y^2 + (Z - z)^2}}\boldsymbol{u}_x \tag{1.11}$$

Upon integrating,

$$\boldsymbol{B}(0, Y, Z) = \frac{\mu_0 I}{4\pi Y}\left[\frac{Z + \frac{L}{2}}{\sqrt{Y^2 + \left(Z + \frac{L}{2}\right)^2}} - \frac{Z - \frac{L}{2}}{\sqrt{Y^2 + \left(Z - \frac{L}{2}\right)^2}}\right]\boldsymbol{u}_x \tag{1.12}$$

Of general interest is the magnetic field on the plane perpendicular to the wire and at the center line of the conductor, where $Z = 0$. Here,

$$\boldsymbol{B}(0, Y, 0) = \frac{\mu_0 I}{4\pi Y}\left[\frac{L}{\sqrt{Y^2 + \left(\frac{L}{2}\right)^2}}\right]\boldsymbol{u}_x \tag{1.13}$$

For a current of infinite length, $L \to \infty$ and equation (1.13) becomes

$$\boldsymbol{B}(0, Y, 0) = \frac{\mu_0 I}{2\pi Y}\boldsymbol{u}_x \tag{1.14}$$

and becomes independent of Z. When the wire is infinitely long, the general result is clearly, from symmetry,

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi R} \quad \text{Wb/m}^2 \tag{1.15}$$

where R is the radial distance from the wire and the direction of $\boldsymbol{B}$ is normal to the plane containing both the wire and the length R.

1.4 THE MAGNETIC VECTOR POTENTIAL *A*

The expression for magnetic field can be further simplified by introducing the concept of the magnetic vector potential. It can be easily shown that the following expression is an identity:

$$\frac{\boldsymbol{u}_{r12}}{R^2} = -\nabla\left(\frac{1}{R}\right) \tag{1.16}$$

where ∇ is the gradient operator defined by

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{u}_x + \frac{\partial}{\partial y}\boldsymbol{u}_y + \frac{\partial}{\partial z}\boldsymbol{u}_z \tag{1.17}$$

Using equation (1.16), equation (1.7) can be written as

$$\boldsymbol{B}_{21} = \frac{\mu_0}{4\pi}\int_V \boldsymbol{J} \times \nabla\left(-\frac{1}{R}\right) dV \tag{1.18}$$

The vector differential operator affects only the variables at the point at which $\boldsymbol{B}_{21}$ is evaluated while the integral is taken over the region for which the current density $\boldsymbol{J}$ is defined. However, another identity states that if f is any scalar function of x, y, and z, and $\boldsymbol{v}$ is any vector

$$\nabla \times (f\boldsymbol{v}) = f\nabla \times \boldsymbol{v} + \nabla f \times \boldsymbol{v} \tag{1.19}$$

where $\nabla\times$ denotes the curl operator.

Then, letting f be $(1/R)$ and setting $\boldsymbol{v}$ to $\boldsymbol{J}$

$$\nabla \times \left(\frac{\boldsymbol{J}}{R}\right) = \left(\frac{1}{R}\right)\nabla \times \boldsymbol{J} + \nabla\left(\frac{1}{R}\right) \times \boldsymbol{J} \tag{1.20}$$

In Cartesian coordinates, the curl of any vector $\boldsymbol{F}$ is expressed as the determinant of the matrix

$$\nabla \times \boldsymbol{F} = \det\begin{bmatrix} \boldsymbol{u}_x & \boldsymbol{u}_y & \boldsymbol{u}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} \tag{1.21}$$

Upon defining the concept of the electric field (later in Section 1.7) it will become evident that the first term on the right-hand side of equation (1.20) is zero by virtue of equations (1.41) and (1.42). (This result assumes that the current distribution is not time-dependent or that the frequency is sufficiently low as is typically the case with electrical machinery.) Thus, equation (1.18) becomes

$$\boldsymbol{B}_{21} = \nabla \times \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}}{R} dV \tag{1.22}$$

The curl operation can be brought outside of the integral since the two operations are independent; that is, the integral is taken over the volume containing $\boldsymbol{J}$ while the differential operator operates at the point defining $\boldsymbol{B}_{21}$. The field $\boldsymbol{B}_{21}$ has now been defined by the curl of a function that can be designated as $\boldsymbol{A}$,

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.23}$$

which can be formally defined as

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}}{R} dV \quad \text{Wb/m} \tag{1.24}$$

The use of the subscript "21" on $\boldsymbol{B}$ has now been dropped for simplicity. The quantity $\boldsymbol{A}$ is called the *magnetic vector potential* and must be formally evaluated by decomposing the integrand into components along the three coordinates. That is, for the x component of $\boldsymbol{A}$,

$$A_x = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}_x}{R} dV \tag{1.25}$$

and so forth for A_y and A_z. Note that the vector potential in the x direction is caused only by the current distribution in the x direction. Hence, the problem of computing the magnetic field $\boldsymbol{B}$ has been reduced to solving three decoupled scalar integrals.

The circuit equivalent to equation (1.24) is

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_L \frac{\boldsymbol{I}}{R} dl \tag{1.26}$$

and is generally more useful when currents flow through wires having negligible cross section.

1.5 EXAMPLE—CALCULATION OF MAGNETIC FIELD FROM THE MAGNETIC VECTOR POTENTIAL

Consider that it is necessary to determine the vector potential and the resulting flux density at a distance Y from the center of a current element of length L and on a line perpendicular to its midpoint as shown in Figure 1.3.

Since the current is directed solely in the z direction, the magnetic vector potential will have only a z component. By symmetry, equation (1.26) reduces to

$$A_z = \frac{\mu_0}{4\pi} \int_0^{\frac{L}{2}} \frac{I}{R} dz \tag{1.27}$$

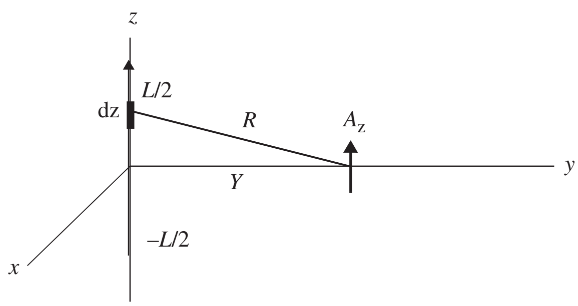

Figure 1.3 Magnetic vector potential of a current element.

so that

$$A_z = \frac{\mu_0 I}{4\pi} \int_0^{\frac{L}{2}} \frac{1}{\sqrt{Y^2 + z^2}} dz \tag{1.28}$$

which becomes ultimately

$$A_z = \frac{\mu_0 I}{4\pi} \ln\left(\frac{L}{2Y} + \sqrt{1 + \left(\frac{L}{2Y}\right)^2}\right) \tag{1.29}$$

where ln denotes the natural logarithm. The solution for magnetic vector potential can now be used to obtain the magnetic flux density at the same point. From equation (1.23),

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{1.30}$$

and from equation (1.21), if $\boldsymbol{A}$ has only a z-directed component,

$$\boldsymbol{B} = \frac{\partial A_z}{\partial y}\boldsymbol{u}_x - \frac{\partial A_z}{\partial x}\boldsymbol{u}_y \tag{1.31}$$

Since A_z does not vary with x, the second term is zero and

$$B_x = \frac{\partial A_z}{\partial Y} = \frac{\mu_0 I}{2\pi Y}\left(\frac{\frac{L}{2}}{\sqrt{\left(\frac{L}{2}\right)^2 + Y^2}}\right) \tag{1.32}$$

which is the same as equation (1.13).

1.6 CONCEPT OF MAGNETIC FLUX

It has been determined that the magnetic field $\boldsymbol{B}$ can be expressed in terms of the curl of an auxiliary vector potential function $\boldsymbol{A}$. However, again from vector calculus, the divergence of the curl of any function is always zero, that is

$$\nabla \bullet (\nabla \times \boldsymbol{A}) = 0$$

where, in the Cartesian system, the divergence operator is defined as

$$\nabla \bullet \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{1.33}$$

From the definition of $\boldsymbol{A}$, it follows that the divergence of $\boldsymbol{B}$ must be identically zero.

$$\nabla \bullet \boldsymbol{B} = 0 \tag{1.34}$$

If equation (1.34) is integrated over a volume

$$\int_V \nabla \bullet \boldsymbol{B} dV = 0 \tag{1.35}$$

whereupon, from Gauss' law, one obtains the result that

$$\int_V \nabla \bullet \boldsymbol{B} dV = \oint_S \boldsymbol{B} \bullet d\boldsymbol{S} = 0 \tag{1.36}$$

in which the surface S encloses the volume V.

In many cases, it is advantageous to think of a vector field as the "flow" of a quantity and in the case of the magnetic field, as suggested from equation (1.36), it is useful to now think of $\boldsymbol{B}$ as a density of flow of "something" per unit area. In the SI system of units, it has been agreed to term this "something" as *magnetic flux* with unit *webers*. Consequently, $\boldsymbol{B}$ can now be considered to have units *webers per square meter* and when the webers per unit area are integrated over a closed surface, the total amount of magnetic flux enclosed is identically zero. The modern SI unit for $\boldsymbol{B}$ is the *tesla* which is identically equal to a weber per square meter. However, both these terms will be used interchangeably throughout this book.

For an arbitrary surface S, bounded by a closed contour O as shown in Figure 1.1, the total magnetic flux Φ passing through the surface S is expressed by

$$\Phi = \int_S \boldsymbol{B} \bullet d\boldsymbol{S} \quad \text{webers} \tag{1.37}$$

The flux which passes through the surface S is said to link the contour O and is generally referred to as the *flux linkage* of the contour. The flux which links a contour O may also be expressed in terms of the vector potential $\boldsymbol{A}$. Since $\boldsymbol{B}$ is the curl of $\boldsymbol{A}$, one can write

$$\Phi = \int_S \boldsymbol{B} \bullet d\boldsymbol{S} = \int_S \nabla \times \boldsymbol{A} \bullet d\boldsymbol{S}$$

This expression can be transformed to a contour integral by using Stokes' theorem, in which case,

$$\Phi = \int_S \nabla \times \boldsymbol{A} \bullet d\boldsymbol{S} = \oint \boldsymbol{A} \bullet d\boldsymbol{l} \tag{1.38}$$

This expression is sometimes more convenient to evaluate than equation (1.37) and will be particularly useful when finite element analysis is investigated later in this text.

1.7 THE ELECTRIC FIELD *E*

In a manner similar to the magnetic field discussion above, the force impressed on one electric charge by another located some distance away can be described by an *electric field* acting directly on the charge. The force is usually expressed as the force on a unit "test" charge as

$$\frac{\boldsymbol{F}_{21}}{q_2} = \frac{1}{4\pi\varepsilon}\int_V \frac{\boldsymbol{u}_{r12}\rho_c(V)}{R^2}dV \tag{1.39}$$

where ρ_c is the charge density in coulombs per meter^3, R is the distance from the differential charge $\rho_c dV$ to the point at which $\boldsymbol{E}$ is evaluated, $\boldsymbol{u}_{r12}$ is the corresponding unit vector, ε is the permittivity of the material. The unit vector $\boldsymbol{u}_{r12}$ again points from the location of the charge, point 1, to the point at which $\boldsymbol{E}$ is to be evaluated, point 2.

Whereas the force exists only on the test charge q_2, a field can again be said to exist everywhere in space given by the vector

$$\boldsymbol{E}_{21}(x, y, z) = \frac{1}{4\pi\varepsilon}\int_V \frac{\boldsymbol{u}_{r12}\rho_c(V)}{R^2}dV \tag{1.40}$$

The electric field is usually given in fundamental units of newtons per coulomb. The free space value of permittivity is $\varepsilon_0 = (1/36\pi) \times 10^{-9}$ coulombs^2/Nm^2.

Finally, when the electric field exists inside a conducting material, the presence of the field establishes a current according to Ohm's law, that is,

$$\boldsymbol{J} = \sigma\boldsymbol{E} \tag{1.41}$$

From the form of the definition of $\boldsymbol{E}$, equation (1.40), and by using equation (1.16) and the vector identity $\nabla \times \nabla(1/R) = 0$, it can be readily shown that

$$\nabla \times \boldsymbol{E} = 0 \tag{1.42}$$

if ρ_c is not time-dependent. Equation (1.42) remains valid for DC current flow in a conductor since the charge at each point in the wire is always the same.

From Stokes' theorem, equation (1.42) has the property that

$$\int_S (\nabla \times \boldsymbol{E}) \bullet d\boldsymbol{S} = \oint_O \boldsymbol{E} \bullet d\boldsymbol{l} = 0 \tag{1.43}$$

where O bounds the surface area S. Equation (1.43) essentially implies that the line integral of $\boldsymbol{E}$ between any two points is independent of the path resulting in the electrical field being said to be conservative. That is, no energy is lost or gained in moving a charged particle around a closed path in an electromagnetic field produced by static (non-moving charges) or by steady currents. In a practical sense, this statement implies that if a conductor is placed in a steady field, no current will flow in the conductor in the steady state.

Upon examining equation (1.43), it is apparent that the electric field has the properties of a "gradient," that it is expressed in terms of "some quantity" per meter. This quantity is formally defined as a *volt* in which case, the electric field is defined to have unit volts per meter for which a volt has fundamental units of newton-meter per coulomb. The unit of permittivity ε_0 in terms of the voltage as a unit is coulomb per volt meter.

1.8 AMPERE'S LAW

Ampere's law forms the fundamental basis upon which all machine design begins. While often presented as a separate law to that of Biot and Savart, its basis is, in actuality, embedded in the definition of the magnetic field $\boldsymbol{B}$. Upon taking the curl of $\boldsymbol{B}$ as defined by equation (1.7), and replacing the curl-curl operator by the equivalent expression, the gradient of the divergence minus the Laplacian, it is possible to obtain

$$\nabla \times \boldsymbol{B} = \nabla \times \nabla \times \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}}{R} dV \tag{1.44}$$

$$= \frac{\mu_0}{4\pi} \int_V \left[\nabla\nabla \bullet \frac{\boldsymbol{J}}{R} - \boldsymbol{J}\nabla^2 \left(\frac{1}{R} \right) \right] dV \tag{1.45}$$

The differential operators have been taken behind the integral since these operators are taken with respect to the point at which $\boldsymbol{B}$ is desired whereas the integral is taken over the region where the current density $\boldsymbol{J}$ exits.

The first term in equation (1.45) can be written alternatively as

$$\int_V \nabla\nabla \bullet \frac{\boldsymbol{J}}{R} dV = \nabla \int_V \nabla \bullet \frac{\boldsymbol{J}}{R} dV \tag{1.46}$$

where the gradient operator has been brought out from under the integral sign since the gradient and integral operations can again be interchanged. However from Gauss' theorem, this integral can be replaced by the expression

$$\nabla \int_V \nabla \bullet \frac{\boldsymbol{J}}{R} dV = -\nabla \oint_S \frac{\boldsymbol{J}}{R} \bullet d\boldsymbol{S} \tag{1.47}$$

The minus sign appears in this expression since the divergence operator is taken with respect to the point defining $\boldsymbol{B}$ whereas the integral is taken over the volume defining the current density $\boldsymbol{J}$. However, the surface S describes the outer surface of the conductor over which a net current clearly is not escaping. Thus the dot product $\boldsymbol{J} \bullet d\boldsymbol{S}$ is zero on this surface and the first term in equation (1.45) is, therefore, zero.

The expression for the curl of $\boldsymbol{B}$ reduces to

$$\nabla \times \boldsymbol{B} = -\frac{\mu_0}{4\pi} \int_V \boldsymbol{J}\nabla^2 \left(\frac{1}{R} \right) dV \tag{1.48}$$

where $\nabla^2 = (\nabla \bullet \nabla)$ is the Laplacian operator. That is,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.49}$$

where the set (x, y, z) denotes the point at which $\boldsymbol{B}$ is defined. Also, R is the distance from the point where $\boldsymbol{B}$ is evaluated at the differential element dV locating $\boldsymbol{J}$. In a similar manner, one can define the set (x', y', z') as denoting the point at which the differential volume dV is defined and the corresponding Laplacian operator

$$\nabla'^2 = \frac{\partial^2}{\partial (x')^2} + \frac{\partial^2}{\partial (y')^2} + \frac{\partial^2}{\partial (z')^2} \tag{1.50}$$

By formal differentiation, it can be shown that the expression $\nabla^2(1/R)$ is identically zero everywhere except at the point of singularity, namely where $R \to 0$. At the point of singularity, the points (x,y,z) and (x',y',z') coincide so that $\nabla^2(1/R) = \nabla'^2(1/R)$ where the prime indicates differentiation with respect to the prime variables. Since ∇^2 is equivalently written as $\nabla \bullet \nabla$, equation (1.48) can also be expressed as

$$\nabla \times \boldsymbol{B} = -\lim_{R\to 0} \frac{\mu_0}{4\pi} \int_V \boldsymbol{J} \nabla' \bullet \nabla' \left(\frac{1}{R}\right) dV \tag{1.51}$$

which, by Gauss' theorem becomes

$$\nabla \times \boldsymbol{B} = -\lim_{R\to 0} \frac{\mu_0}{4\pi} \oint_S \boldsymbol{J} \nabla' \left(\frac{1}{R}\right) d\boldsymbol{S} \tag{1.52}$$

The differential surface area in spherical coordinates is $\boldsymbol{u}_{\rm n}(R^2 \sin\theta d\phi d\theta)$, where $\boldsymbol{u}_{\rm n}$ is the unit normal to the surface $d\boldsymbol{S}$. However, the gradient of $1/R$ is equal to $-\boldsymbol{u}_{\rm n}/R^2$ so that the integral becomes

$$\nabla \times \boldsymbol{B} = \lim_{R\to 0} \frac{\mu_0}{4\pi} \oint_S \boldsymbol{J} \sin\theta d\phi d\theta \tag{1.53}$$

Since the radius of the small sphere approaches zero, the current density vector $\boldsymbol{J}$ can be removed from the integrand since it becomes a constant. The remaining integral can now be evaluated as simply 4π. Equation (1.44) finally reduces to *Ampere's law*

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{1.54}$$

The integral form of Ampere's law can be obtained by integrating equation (1.54) over an arbitrary finite open surface which includes the region where the current density $\boldsymbol{J}$ is flowing, whereupon,

$$\int_S \nabla \times \boldsymbol{B} \cdot d\boldsymbol{S} = \mu_0 \int_S \boldsymbol{J} \bullet d\boldsymbol{S} \tag{1.55}$$

The right-hand side of equation (1.55) is clearly proportional to the current *I* flowing through the surface *S*. With the use of Stokes' theorem, the left hand side can be altered to the form

$$\int_S (\nabla \times \boldsymbol{B}) \bullet d\boldsymbol{S} = \oint_O \boldsymbol{B} \bullet d\boldsymbol{l} \tag{1.56}$$

where the path *O* corresponds to the outer edge of the surface *S*. Thus, the integral form of Ampere's law is

$$\int_O \boldsymbol{B} \bullet d\boldsymbol{l} = \mu_0 I \tag{1.57}$$

1.9 MAGNETIC FIELD INTENSITY *H*

Thus far, the behavior of the magnetic field in a vacuum has been considered. When dealing with material bodies, orbiting electrons of each individual atom can be considered as a current loop. With no external magnetic field, the orbiting atoms are randomly positioned so that they do not produce, themselves, a magnetic field (except for permanent magnets). The presence of the magnetic field influences the orbits of the individual atoms creating what is called a *magnetic dipole moment* $\boldsymbol{m}$. The dipole moment is defined as equal to the product of the area of the circular loop defined by the orbiting electron times the magnitude of the circulating current and with a direction perpendicular to the plane of the loop in the direction of a right-hand screw. That is, if the current loop is located in the *x*,*y* plane and orbiting in a counterclockwise direction as shown in Figure 1.4, the magnetic dipole moment is defined by

$$\boldsymbol{m} = \pi r^2 I \boldsymbol{u}_z \quad \text{A-m}^2$$

The vector potential for this small electric circuit is

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi} \oint_O \frac{d\boldsymbol{l}}{R_1} \tag{1.58}$$

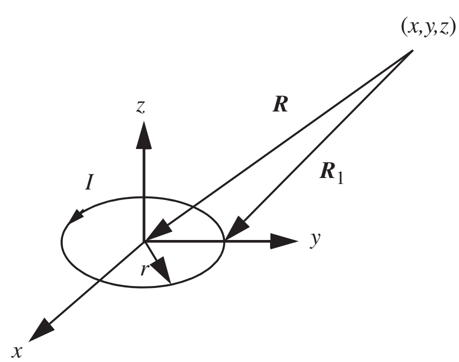

Figure 1.4 The magnetic dipole.

If R^2 is much greater than r^2, then

$$\frac{1}{R_1} \approx \frac{1}{R}\left(1 + \frac{rx}{R^2}\cos\phi' + \frac{ry}{R^2}\sin\phi'\right) \tag{1.59}$$

Equation (1.58) then integrates to yield

$$\boldsymbol{A} = \frac{\mu_0 I r^2}{4R^3}(-y\boldsymbol{u}_x + x\boldsymbol{u}_y) \tag{1.60}$$

However,

$$\boldsymbol{u}_z \times \boldsymbol{R} = \boldsymbol{u}_z \times (x\boldsymbol{u}_x + y\boldsymbol{u}_y + z\boldsymbol{u}_z) = -y\boldsymbol{u}_x + x\boldsymbol{u}_y$$

so that the vector potential can be expressed in vector form as

$$\boldsymbol{A} = \frac{\mu_0 m}{4\pi}\boldsymbol{u}_z \times \frac{\boldsymbol{u}_r}{R^2} \tag{1.61}$$

In general, if the magnetization axis direction is arbitrary,

$$\boldsymbol{A} = \frac{\mu_0 m}{4\pi}\boldsymbol{u}_m \times \frac{\boldsymbol{u}_r}{R^2} \tag{1.62}$$

or, alternatively, from equation (1.16), as

$$\boldsymbol{A} = -\frac{\mu_0 m}{4\pi}\boldsymbol{u}_m \times \nabla\left(\frac{1}{R}\right) \tag{1.63}$$

A mathematical representation of the overall magnetic dipole moment of a finite body can be obtained by multiplying $m\boldsymbol{u}_m$ by the number of atoms per unit volume N_a to obtain the *magnetic polarization vector* $\boldsymbol{M}$

$$\boldsymbol{M} = N_a \boldsymbol{m} = N_a m \boldsymbol{u}_m \quad \mathrm{A/m} \tag{1.64}$$

so that it is possible to write, in place of (1.63),

$$\boldsymbol{A}(x, y, z) = -\frac{\mu_0}{4\pi}\int_V \boldsymbol{M}(x', y', z') \times \nabla\left(\frac{1}{R}\right) dV' \tag{1.65}$$

where $\boldsymbol{M}$ is considered here as varying within the body (i.e., a function of (x', y', z') and R represents the distance between the external point (x,y,z) and the internal point (x', y', z'). Now,

$$\boldsymbol{M} \times \nabla\left(\frac{1}{R}\right) = -\boldsymbol{M} \times \nabla'\left(\frac{1}{R}\right) \tag{1.66}$$

and

$$\boldsymbol{M}(x', y', z') \times \nabla'\left(\frac{1}{R}\right) = \left(\frac{1}{R}\right)\nabla' \times \boldsymbol{M}(x', y', z') - \nabla' \times \frac{\boldsymbol{M}(x', y', z')}{R} \tag{1.67}$$

Equation (1.65) can now be written as

$$\boldsymbol{A} = -\frac{\mu_0}{4\pi}\int_V \left(\frac{1}{R}\right)\nabla' \times \boldsymbol{M}(x', y', z')dV' + \frac{\mu_0}{4\pi}\int_V \nabla' \times \frac{\boldsymbol{M}(x', y', z')}{R}dV' \tag{1.68}$$

It can be shown as a homework problem that a corollary to Stokes' theorem is the fact that

$$\int_V \nabla' \times \frac{\boldsymbol{M}(x', y', z')}{R} dV' = -\oint_S \frac{\boldsymbol{M} \times \boldsymbol{u}_\mathrm{n}}{R} dS \tag{1.69}$$

for any vector field $\boldsymbol{M}$ wherein $\boldsymbol{u}_\mathrm{n}$ is the unit normal to the surface dS. Thus, finally,

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \oint_S \frac{M \times \boldsymbol{u}_\mathrm{n}}{R} dS + \frac{\mu_0}{4\pi} \int_V \left(\frac{1}{R}\right) \nabla' \times \boldsymbol{M}(x', y', z') dV' \tag{1.70}$$

If one compares this expression of the vector potential with that for true currents, it is apparent that one can interpret the term $\boldsymbol{M} \times \boldsymbol{u}_\mathrm{n}$ as an equivalent surface polarization current $\boldsymbol{K}_m$. Similarly, the curl of the magnetization vector $\nabla \times \boldsymbol{M}$ is an equivalent volumetric polarization current $\boldsymbol{J}_m$. The expression for vector potential becomes

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \left(\oint_S \frac{\boldsymbol{K}_m}{R} dS + \int_V \frac{\boldsymbol{J}_m}{R} dV \right) \tag{1.71}$$

Note that this equation again generates three "decoupled" equations involving only x, y, and z components of $\boldsymbol{A}$, $\boldsymbol{K}_m$, and $\boldsymbol{J}_m$, respectively.

Consider now the magnetic flux density at any point within a material body having both true current $\boldsymbol{J}$ and polarization current $\boldsymbol{J}_m$. From equation (1.22),

$$\boldsymbol{B} = \nabla \times \left(\frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J}_m + \boldsymbol{J}}{R} dV \right) \tag{1.72}$$

In Section 1.8, it was shown that $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$. In an analogous manner it is evident that with polarization currents,

$$\nabla \times \boldsymbol{B} = \mu_0 (\boldsymbol{J} + \boldsymbol{J}_m) \tag{1.73}$$

Since the volumetric polarization current $\boldsymbol{J}_m$ is equal to the curl of the magnetization $\boldsymbol{M}$, this expression can be written as

$$\nabla \times \left(\frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \right) = \boldsymbol{J} \tag{1.74}$$

in which the vector on the left-hand side of the equation $\boldsymbol{B}/\mu_0 - \boldsymbol{M}$ has as its source, only the true currents $\boldsymbol{J}$. It is, therefore, useful to define a new quantity, the *magnetic field intensity* $\boldsymbol{H}$ as

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M} \quad \mathrm{A/m} \tag{1.75}$$

from which it can be established that

$$\mu = \frac{\mu_0}{1 - \mu_0 \left(\frac{\boldsymbol{M}}{\boldsymbol{B}} \right)} \tag{1.76}$$

in which case

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{1.77}$$

or, alternatively,

$$\boldsymbol{B} = \mu_r \mu_0 \boldsymbol{H} \tag{1.78}$$

where $\mu_r = \mu/\mu_0$ is defined as the *relative permeability*. The differential form for Ampere's law is finally obtained, namely

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \tag{1.79}$$

Since $\nabla \bullet \boldsymbol{H} = -\nabla \bullet \boldsymbol{M}$, the divergence of the magnetic field intensity is not zero as is the case for the divergence of $\boldsymbol{B}$. The magnetic field intensity is also sometimes called the *magnetic potential gradient*.

Starting with equation (1.79), and applying Stokes' theorem, results in the usual integral form of Ampere's law,

$$\int_S (\nabla \times \boldsymbol{H}) \bullet d\boldsymbol{S} = \oint_O \boldsymbol{H} \bullet dl = \int_S \boldsymbol{J} \bullet d\boldsymbol{S} = I \tag{1.80}$$

where I is the total current enclosed by the path defined by O. If current I is confined to a conductor and flows N times through the loop O, I is replaced by NI in equation (1.80).

1.10 BOUNDARY CONDITIONS FOR *B* AND *H*

In the derivation of the differential form for Ampere's law, points within the material were specified and not points on the boundary, where an additional polarization current component, $\boldsymbol{K}_m$ exists. Hence, for points on the boundary, the results obtained must be modified to take account of this current which results from a discontinuity in the magnetization vector $\boldsymbol{M}$. Consider now an interface between two material bodies with permeability μ different from μ_0 as illustrated in Figure 1.5.

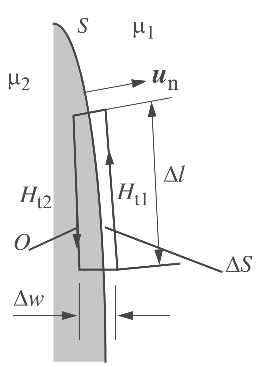

Figure 1.5 Determination of boundary condition for the tangential component of ***H***.

Let H_{t1} and H_{t2} be the components of $\boldsymbol{H}$ tangent to the interface surface in the material body and air, respectively. From equation (1.79), $\nabla \times \boldsymbol{H} = \boldsymbol{J}$ and if the path O is chosen as shown in Figure 1.5, the integral form for Ampere's law gives

$$\int_{\nabla S} (\nabla \times \boldsymbol{H}) \bullet d\boldsymbol{S} = \oint_{\Delta O} \boldsymbol{H} \bullet d\boldsymbol{l} = \int_{\Delta S} \boldsymbol{J} \bullet d\boldsymbol{S} \tag{1.81}$$

where ΔO is the outer contour of the surface ΔS. Since no physical current is enclosed, these expressions are equal to zero. If one shrinks Δw to a negligibly small value, then $H_{t1}\Delta l - H_{t2}\Delta l = 0$ so that

$$H_{t1} = H_{t2} \tag{1.82}$$

which states, in effect, that the tangential components of $\boldsymbol{H}$ must be continuous across a boundary not carrying a true surface current $\boldsymbol{K}$. It is clear that if the boundary supports a physical current surface current $\boldsymbol{K}$, then equation (1.82) must be replaced with

$$H_{t1} = H_{t2} = K \tag{1.83}$$

where positive current is taken in the direction made by a right-hand screw when rotated in the direction defined by the path O. In vector form, the equivalent expression is written as

$$\boldsymbol{u}_{\mathrm{n}} \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{K} \tag{1.84}$$

Although true surface currents are essentially impossible, equation (1.83) is often used to approximate physical situations in an electrical machine design.

As a corollary to equation (1.82), it is apparent that the tangential components of B are discontinuous across a boundary separating materials with different permeabilities, that is

$$\frac{B_{t1}}{\mu_1} = \frac{B_{t2}}{\mu_2} \tag{1.85}$$

when no surface currents flow on the boundary.

The behavior of the normal components of B and H can also be determined as shown in Figure 1.6.

In this case, from equation (1.36),

$$\oint_S \boldsymbol{B} \bullet d\boldsymbol{S} = 0$$

If this expression is applied to the pill box shape of Figure 1.6, then

$$B_{n1}\Delta S - B_{n2}\Delta S = (B_{n1\mathrm{s}} + B_{n2\mathrm{s}})\,(\pi \Delta r \Delta w)$$

where B_{n1}, B_{n2} pertains to the top and bottom of the pill box and $B_{n1\mathrm{s}}$ and $B_{n2\mathrm{s}}$ pertains to the sides of the pillbox in materials 1 and 2, respectively. If the sides of the pillbox are made arbitrarily small, then $B_{n1}\Delta S = B_{n2}\Delta S$, or finally across any boundary,

$$B_{n1} = B_{n2} \tag{1.86}$$

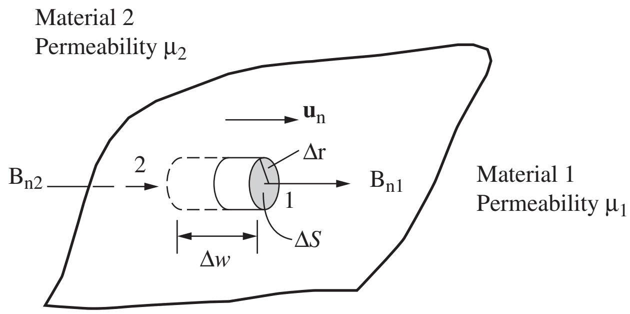

Figure 1.6 Determination of the boundary condition for the normal component of ***B***.

In vector form, this expression is equivalent to

$$\boldsymbol{u}_n \bullet (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0 \tag{1.87}$$

The corresponding boundary condition for ***H*** is clearly

$$\mu_1 H_{n1} = \mu_2 H_{n2} \tag{1.88}$$

It is observed that the normal component of ***B*** is continuous but not so for ***H***.

1.11 FARADAY'S LAW

It was Michael Faraday and Joseph Henry who jointly discovered that the electric field becomes non-conservative when the line integral of ***E*** is evaluated in a magnetic field which is non-steady, that is, when the magnetic field linking the line integral path varies with time. In this case, equation (1.43) must be modified to form

$$\oint_O E \bullet d\boldsymbol{l} = -\frac{d}{dt}\int_S \boldsymbol{B} \bullet d\boldsymbol{S} = -\frac{d\Phi}{dt} \tag{1.89}$$

where O bounds surface S. In a practical sense, this expression shows that an additional electric field is produced by a time-changing magnetic field and consequently, a voltage is produced in a closed short-circuited coil when placed in this field. The strength of this voltage is proportional to the time rate of change of flux linking the coil and, in turn, induces a current in the conducting loop. The negative sign indicates that the voltage is directed in such a manner so as to produce a current which produces a consequent magnetic field which reduces the net flux linking the loop.

The differential form of equation (1.89) may be obtained by using Stokes' theorem to replace the line integral by a surface integral so that

$$\oint_O E \bullet dl = \int_S \nabla \times \boldsymbol{E} \bullet d\boldsymbol{S} = -\frac{d}{dt}\int_S \boldsymbol{B} \bullet d\boldsymbol{S} \tag{1.90}$$

or

$$\int_S \left(\nabla \times \boldsymbol{E} + \frac{d\boldsymbol{B}}{dt} \right) \bullet d\boldsymbol{S} = 0 \tag{1.91}$$

from which,

$$\nabla \times \boldsymbol{E} = -\frac{d\boldsymbol{B}}{dt} \tag{1.92}$$

The current which flows in the conducting loop creates also an electric field within a conducting material which is proportional to the current, the basis for Ohm's law. Expressed at a point rather than averaged over a finite body, Ohm's law at a point is

$$\boldsymbol{J} = \sigma \boldsymbol{E} \tag{1.93}$$

where, σ is the conductivity of the material in amperes per volt-meter (*ohm-meters*)$^{-1}$.

1.12 INDUCED ELECTRIC FIELD DUE TO MOTION

Since a changing magnetic field linking a conducting coil can also be produced by simply moving it physically through a stationary, non-uniform field, equation (1.89) is equally valid for this condition as well. Movement of the coil through the field, however, produces an accompanying phenomenon, the Lorentz force which states that moving charges in a magnetic field experience a force proportional to the velocity of the charge and the strength of the magnetic field according to the vector equation

$$\boldsymbol{F} = q_c \boldsymbol{v} \times \boldsymbol{B} \tag{1.94}$$

The force is seen acting in a direction perpendicular to both $\boldsymbol{v}$ and $\boldsymbol{B}$. Note that this is just an equivalent form of the Biot–Savart law since an ampere flowing through a meter of wire is essentially the same as 10^{-6} coulombs of electrons traveling at 10^6 m/s. This force is proportional to the electric field since

$$\frac{\boldsymbol{F}}{q_c} = \boldsymbol{E} = \boldsymbol{v} \times \boldsymbol{B} \tag{1.95}$$

The corresponding field, in turn, induces a voltage in the coil resulting in current flow according to Ohm's law, equation (1.93). This induced voltage is typically called the *electromotive force* (a misnomer since the actual units of the quantity are fundamentally newtons/coulomb). Note from Figure 1.7 that this voltage is in such a direction so as to produce a current which resists any change in the flux linking the coil. The degree to which this is accomplished, depends on the resistance of the coil. If superconducting, flux linking the coil will not change at all. The interrelationship between the force on a moving coil and the resulting current (or vice versa) is the key component in the principle of *electromechanical energy conversion*.

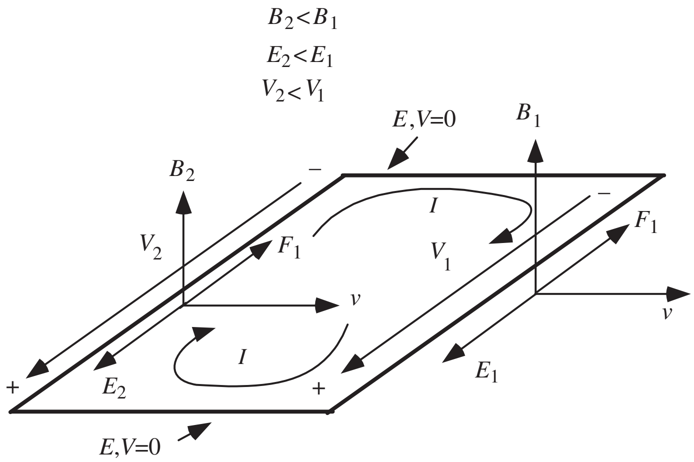

Figure 1.7 Induced voltage in a coil moving in a direction so as to increase the flux linking the coil. Force assumed to be impressed on a negative charge (electron).

1.13 PERMEANCE, RELUCTANCE, AND THE MAGNETIC CIRCUIT

The solution to the general magnetostatic boundary-value problem involving conduction currents in the presence of magnetic material is very difficult to obtain analytically. Fortunately, applications involving electric machine design allow for good approximate solutions to be obtained. The analysis procedure parallels that of DC circuits which are composed of series and parallel resistors. Consider, for example, the field in the region of a toroid with a rectangular cross section wound with N turns with the coil current I as illustrated in Figure 1.8.

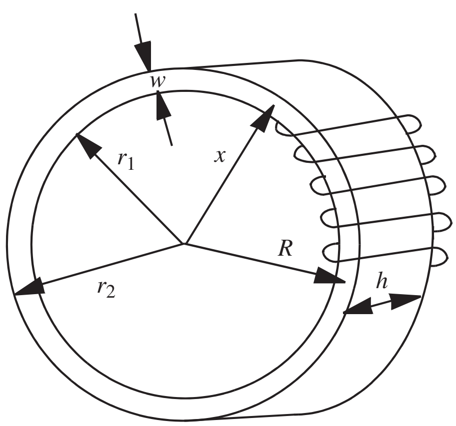

Figure 1.8 Flux distribution of a toroid with a rectangular cross section.

Due to symmetry, the magnetic field intensity has only a circumferential component. At any point in the core x meters from its center, the magnetic potential gradient is

$$H = \frac{NI}{2\pi x} \tag{1.96}$$

where N is the number of turns enclosed by the path for H. The flux density at the distance x is therefore

$$B = \mu H = \frac{\mu NI}{2\pi x} \tag{1.97}$$

However, the flux density at any point is equal to $d\Phi/dA$. The total flux over a cross-sectional area ($h\,dx$) is

$$d\Phi = BdA = Bh\,dx = \mu \frac{NIh}{2\pi}\frac{dx}{x} \tag{1.98}$$

The total flux over the area A is given by

$$\begin{aligned}
\Phi &= \int_{r_1}^{r_2} \mu \frac{NIh}{2\pi}\frac{dx}{x} \\
&= \mu \frac{NIh}{2\pi} \ln(r_2/r_1) \\
&= \mu \frac{NIh}{2\pi} \ln\left(\frac{R + w/2}{R - w/2}\right) \\
&= \mu \frac{NIh}{2\pi} \ln\left(\frac{1 + \frac{w}{2R}}{1 - \frac{w}{2R}}\right) \\
&= \mu \frac{NIh}{2\pi} \ln\left(1 + \frac{w}{R} + \frac{w^2}{2R^2} + \frac{w^3}{4R^3} + \cdots\right)
\end{aligned} \tag{1.99}$$

so that finally,

$$\Phi = \mu \frac{NI}{2\pi} h \frac{w}{R} \quad \text{if } w/R \ll 1 \tag{1.100}$$

When $w/R = 0.2$, then the natural log of the expansion $1 + w/R + w^2/2R^2 + \cdots$ is equal to w/R to within 0.3%. Therefore, for cases where the core width is small in comparison with the mean radius R, one can assume the flux density to be uniform. Thus

$$\Phi = \frac{\mu NIhw}{2\pi R} \tag{1.101}$$

Considering the core width w to be small compared with R enables one to assume that H is constant at all points of the toroid and equal to its value at the center. In this case, from equation (1.97)

$$\frac{\mu NI}{2\pi R} = B \tag{1.102}$$

and

$$hw = A \tag{1.103}$$

Therefore,

$$\Phi = \frac{\mu NI}{2\pi R}A \quad (= BA) \tag{1.104}$$

Note that a rectangular toroid has purposely been chosen for this example. The case of a circular toroid reduces in the same manner but the exact solution involves Bessel functions. It is useful noting that in this book, in deference to convention, the symbol A will be used for both vector potential and cross-sectional area. Similarly, S will be used for surface area and per unit slip of an induction motor, hopefully without much confusion.

It can be observed that equation (1.104) may be written as

$$\Phi = \left(\frac{\mu A}{2\pi R}\right) NI \tag{1.105}$$

where $A = hw$. Note that the coefficient of NI is a constant depending upon the geometry of the magnetic circuit and its permeability. Defining this constant as the permeance

$$\mathcal{P} = \frac{\mu A}{2\pi R} \tag{1.106}$$

Since $2\pi R$ is the length of the magnetic path, it may be replaced for purposes of generalization by l. Therefore, in general, when the flux is uniformly distributed over a constant cross-sectional area,

$$\mathcal{P} = \frac{\mu A}{l} \tag{1.107}$$

The permeance $\mathcal{P}$ is given in units of *webers per ampere-turn* or *henries*.

It is also useful to define

$$\mathcal{F}_{12} = \int_1^2 H \bullet d\mathbf{l} \tag{1.108}$$

The quantity $\mathcal{F}_{12}$ is said to express the magnetomotive force (MMF) acting between points 1 and 2 which has the SI unit *amperes*. The units used in this text will be *ampere-turns* as a reminder of the important influence of the number of winding turns on this quantity. When the closed path O encloses a circuit of N turns carrying I amperes, it is clear from Ampere's law that

$$\mathcal{F} = NI \tag{1.109}$$

Therefore, in general, the flux in a magnetic circuit can be expressed as

$$\Phi = \mathcal{P}\mathcal{F} \tag{1.110}$$

In almost all practical cases, the permeance is not so easy to find. Therein lies the art and science of electrical machine design. When the flux density varies over the cross-sectional area, the differential form of equation (1.110) is often useful. In this case

$$d\Phi = \mathcal{F}d\mathcal{P} \tag{1.111}$$

where

$$d\mathcal{P} = \frac{\mu dA}{l}$$

The total permeance is found by taking

$$\mathcal{P} = \int_0^A \mu \frac{dA}{l} \tag{1.112}$$

where l is frequently a function of A.

In the case of the example rectangular toroid, $dA = hdx$ and $l = 2\pi x$. Equation (1.112) becomes

$$\mathcal{P} = \int_{r_1}^{r_2} \mu \frac{hdx}{(2\pi x)}$$

whereupon,

$$\mathcal{P} = \frac{\mu h}{2\pi} \ln\left(\frac{r_2}{r_1}\right)$$

directly.

The reciprocal of permeance also has great utility in magnetic circuit analysis. Formally, by definition, reluctance is

$$\mathcal{R} = 1/\mathcal{P} \tag{1.113}$$

and if the cross-sectional area A and the permeability are constants, independent of the length of the circuit,

$$\mathcal{R} = \frac{l}{\mu A} \tag{1.114}$$

The quantity reluctance carries units of ampere-turns per weber. In the MKS unit system, this corresponds to inverse henries (H^{-1}). In SI units, reluctance has not been formally given a unique name such as *siemens* for Ω^{-1} so that it is generally described in terms of its basic units. In this text, inverse henries is adopted as the preferred unit. The reluctance is often found to be a more useful quantity in the analysis of electrical machines than permeance.

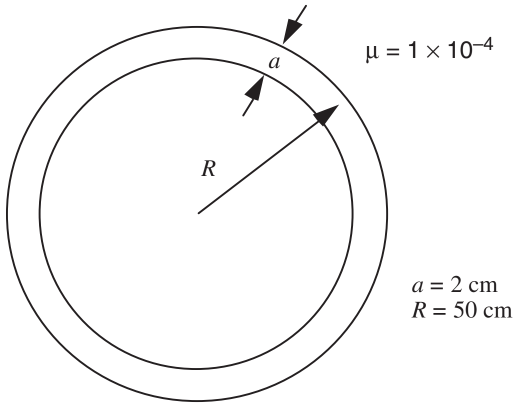

Figure 1.9 Square toroid for Example.

1.14 EXAMPLE—SQUARE TOROID

1. Find the reluctance of a toroid of square cross section and radius R to the center of the core. The core is designed such that $\mu = 1 \cdot 10^{-4}$, $R = 50$ cm, and $a = 2$ cm. Refer to Figure 1.9. The reluctance $\mathcal{R}$ is given by

$$\begin{aligned}\mathcal{R} &= \frac{l}{\mu A} \\ &= \frac{2\pi(0.5)}{1 \cdot 10^{-4}(0.02)^2} \\ &= 7.85 \cdot 10^7 \quad H^{-1}\end{aligned}$$

2. The toroid has 1570 ampere-turns uniformly wound around its core. What is the value of the flux inside the toroid?

$$\begin{aligned}\Phi &= \mathcal{F}/\mathcal{R} \\ &= \frac{1570}{7.85 \cdot 10^7} = 2 \cdot 10^{-5} \quad \text{Wb}\end{aligned}$$

3. What is the flux density in the core of the toroid?

$$B = \frac{\Phi}{A} = \frac{2 \cdot 10^{-5}}{(0.02)^2} = 0.05 \quad \text{Wb/m}^2$$

1.15 MULTIPLE CIRCUIT PATHS

When the same flux is set up in a magnetic circuit made of the same material but with parts of different cross sections or when the parts of the circuit are of materials of different permeabilities but of the same or different cross sections, the flux can be found by finding the reluctance of the different parts, adding them to give the total reluctance of the circuit, and finally dividing the magnetomotive force by the total

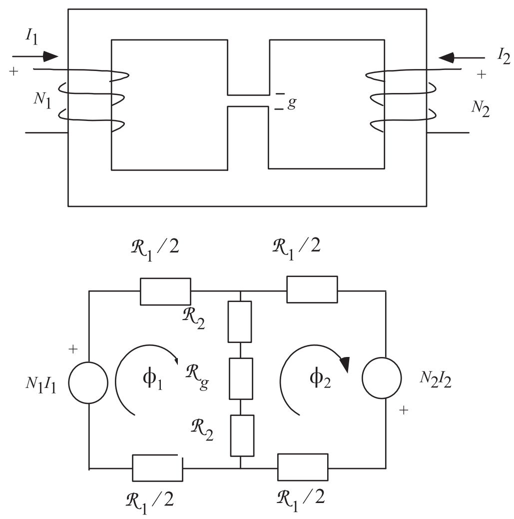

Figure 1.10 A type of iron core transformer and its equivalent magnetic circuit.

reluctance. As in the case of the resistances of an electric circuit, the reluctances in a magnetic circuit when in series are added to give the total reluctance of the circuit.

When the magnetic circuits are in parallel, their total permeance is equal to the sum of the permeances of the parallel circuits. The total reluctance is simply the reciprocal of the total permeance. When the circuit is more complicated, the usual application of Kirchhoff's laws will generally yield an answer. For example, consider the iron-core transformer with an air gap in the center leg as shown in Figure 1.10. (a) The problem is to compute the flux linkage for the coil with N_2 turns when a current I_2 flows in the other coil. The equivalent magnet circuit is illustrated in Figure 1.10b. The magnetomotive force N_1I_1 is applied to the circuit and it acts in series with the reluctance $\mathcal{R}_1$ of the left leg. The reluctance can be split into two portions of magnitude $\mathcal{R}_1/2$. Following the usual DC circuit analysis approach, the following two equations may be obtained:

$$\begin{aligned} N_1I_1 &= \Phi_1(\mathcal{R}_1 + 2\mathcal{R}_2 + \mathcal{R}_g) - \Phi_2(2\mathcal{R}_2 + \mathcal{R}_g) \\ N_2I_2 &= \Phi_2(\mathcal{R}_1 + 2\mathcal{R}_2 + \mathcal{R}_g) - \Phi_1(2\mathcal{R}_2 + \mathcal{R}_g) \end{aligned} \tag{1.115}$$

Solving for Φ_2 yields

$$\Phi_2 = [N_1I_1(2\mathcal{R}_2 + \mathcal{R}_g) + N_2I_2(\mathcal{R}_1 + 2\mathcal{R}_2 + \mathcal{R}_g)]/[\mathcal{R}_1(\mathcal{R}_1 + 4\mathcal{R}_2 + 2\mathcal{R}_g)] \tag{1.116}$$

1.16 GENERAL EXPRESSION FOR RELUCTANCE

Assume now a body of homogeneous permeability but of an arbitrary shape. If flux lines can be approximated, flux tubes containing a specified number of flux lines can be identified which take the general shape of the sketch in Figure 1.11.

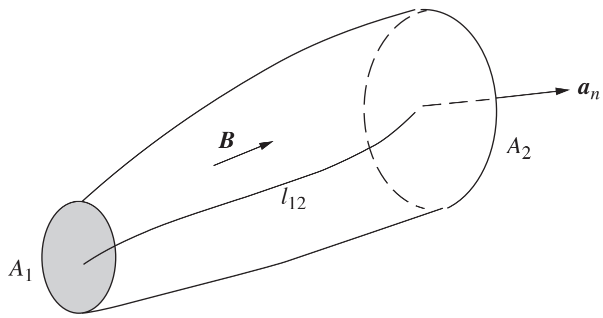

Figure 1.11 An arbitrary flux tube.

The difference of magnetic potential between the two faces A_1 and A_2 can be evaluated by again taking

$$\int_{l_{12}} H \bullet d\boldsymbol{l} = \mathcal{F}_1 - \mathcal{F}_2 \tag{1.117}$$

where l_{12} is the path from A_1 to A_2 along the side or within the flux tube. The flux which is confined within the flux tube is

$$\int_{A} B \bullet d\boldsymbol{S} = \Phi \tag{1.118}$$

where A is the area A_1, A_2, or any cross-sectional area across the flux tube. By definition, the reluctance between the two surfaces A_1 and A_2 is

$$\mathcal{R} = \frac{\displaystyle\int_{l_{12}} H \bullet d\boldsymbol{l}}{\mu \displaystyle\int_{A} H \bullet d\boldsymbol{S}} \tag{1.119}$$

Although this expression is rather simple in form, the values of the integrals cannot be established easily since the location of the flux lines must be known before the integrals can be carried out. Evaluation of the reluctance can be made more accurate if the cross section of the flux tubes are considered as curvilinear squares or rectangles. That is, it is assumed that the corners of the cross-sectional area of a flux tube are always 90 degrees but the sides of the rectangles are allowed to be curved lines. This behavior of the cross-sectional area is a natural consequence of the fact that the lines of constant magnetic potential must be at right angles to the lines of magnetic flux. Plots of the magnetic field using "curvilinear squares" is a traditional method which can yield remarkably accurate results when care is taken to always maintain a curvilinear (right angle) relationship between the potential and flux lines when sketching the field plot.

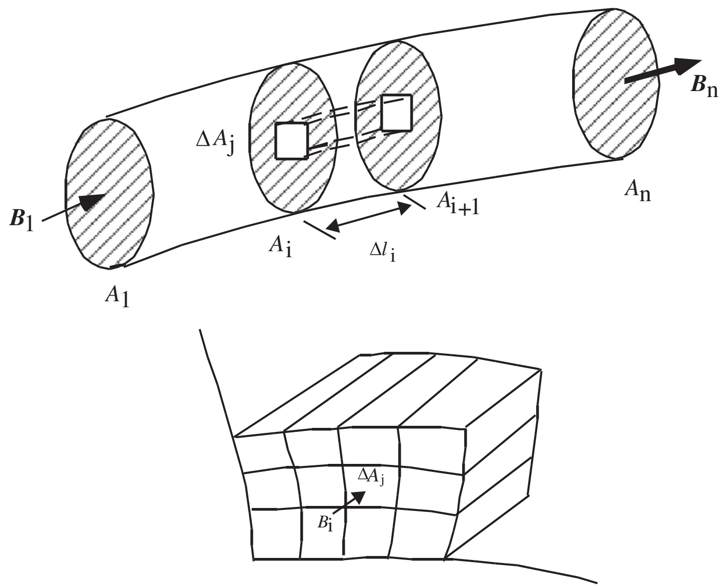

Figure 1.12 Orthogonal curvilinear squares used to portray a magnetic flux tube.

Consider the more accurate flux plot of Figure 1.12, where two equipotential surfaces A_i and A_{i+1} are identified. Let the potential difference between these surfaces be $\Delta\mathcal{F}_s$.

The region between A_i and A_{i+1} can be decomposed into a number of more elementary flow tubes of length Δl_i and cross section ΔA_{ij}. The reluctance of an arbitrary elementary flow tube is

$$\mathcal{R}_i = \frac{\Delta\mathcal{F}_i}{\Delta\Phi_{ij}} = \frac{\Delta l_i}{\mu\Delta A_{ij}} \tag{1.120}$$

Since permeances add directly in parallel, the total permeance between surfaces A_i and A_{i+1} is

$$\Delta\mathcal{P}_i = \sum_j \frac{\mu\Delta A_{ij}}{\Delta l_i} \tag{1.121}$$

The corresponding reluctance is

$$\Delta\mathcal{R}_i = \frac{1}{\displaystyle\sum_j \frac{\mu\Delta A_{ij}}{\Delta l_i}} \tag{1.122}$$

The total reluctance is obtained by adding up all such reluctances over the total length of the flux tube. The result is

$$\mathcal{R} = \sum_i \frac{1}{\sum_j \frac{\mu \Delta A_{ij}}{\Delta l_i}} \tag{1.123}$$

It is important to note that the equation remains a function of the geometry of the magnetic structure only. For more details on the method of curvilinear squares for flux plotting, the reader is referred to any good basic text on electromagnetic fields.

1.17 INDUCTANCE

In most practical cases the magnetic flux links a number of circuit loops N or "turns" in which case one defines the flux linkage λ as

$$\lambda = N\Phi \tag{1.124}$$

The inductance of a coil is defined as "the number of flux linkages in weber turns per ampere of current flowing in the coil." Flux linkages per ampere is formally defined as a *henry.* Interpreted in mathematical form

$$L = \frac{\lambda}{I} = \frac{N\Phi}{I} \quad \text{weber-turns/ampere or henries} \tag{1.125}$$

where

L is the inductance in henries

N is the number of turns of the coil

Φ is the flux in webers linking the turns

I is the current in the turns in amperes

If the current I and flux Φ correspond to the same circuit, then the resulting inductance is termed self-inductance. When the current I and flux Φ correspond to different circuits, a mutual inductance can be defined.

The self-inductance can be written in several other useful forms:

$$L = \frac{N^2\Phi}{\mathcal{F}} \tag{1.126}$$

$$= N^2\mathcal{P} \tag{1.127}$$

and if the cross-sectional area and μ are constant,

$$L = \mu\frac{N^2A}{l} \tag{1.128}$$

Note that the inductance is proportional to the square of the number of turns. Since the inductance has been formally defined in henries, the permeability μ formally takes on the alternate unit of *henries per meter*.

In the above expressions for inductance, it was assumed that the magnetic path or magnetic circuit is defined. When the path of the magnetic flux is not defined as in a solenoid, formulas for inductances are derived from field theory, flux plotting, experimentation, or numerical solution of Laplace's or Poisson's equation.

1.18 EXAMPLE—INTERNAL INDUCTANCE OF A WIRE SEGMENT

It was shown in Example 1.1 that the flux density of an infinitely long wire was given by

$$B = \frac{\mu_0 I}{2\pi R} \tag{1.129}$$

Consider a more detailed view of the wire as shown in Fig. 1.13 showing now a circular inner portion of the wire with radius r. If it is assumed that the current density J in A/m^2 is directed along the wire length and is uniform over the wire cross section, then by Ampere's law, the integral along a circular path C with radius r within the wire will yield

$$\oint_C \boldsymbol{H} \bullet d\boldsymbol{l} = H_\phi 2\pi r = J\pi r^2 \quad 0 < r < R_w, \tag{1.130}$$

where R_w is the outer radius of the wire. Since the wire is assumed to be non-magnetic,

$$B_\phi = \mu_0 H_\phi = \frac{\mu_0 J r}{2} \tag{1.131}$$

Since J is constant, the current in the wire is $I = J\left(\pi R_w^2\right)$ and thus

$$B_\phi = \mu_0 I \frac{r}{2\pi R_\mathrm{w}^2} \tag{1.132}$$

where R_w is the radius of the wire.

Consider now a circular inner portion of the wire with radius r. The flux in an annular portion of length l and thickness dr will be

$$d\Phi(r) = B_\phi l dr \tag{1.133}$$

$$= \frac{\mu_0 I l}{2\pi} \frac{r}{R_\mathrm{w}^2} dr \tag{1.134}$$

This flux links the only the current $I\frac{r^2}{R_\mathrm{w}^2}$ so that the corresponding flux linkages

$$d\lambda = \frac{\mu_0 I l}{2\pi} \frac{r^3}{R_\mathrm{w}^4} dr \tag{1.135}$$

and the total flux linkages become

$$\lambda = \frac{\mu_0}{2\pi} I l \int_0^{R_\mathrm{w}} \frac{r^3}{R_\mathrm{w}^4} dr \tag{1.136}$$

$$= \frac{\mu_0}{8\pi} I l \tag{1.137}$$

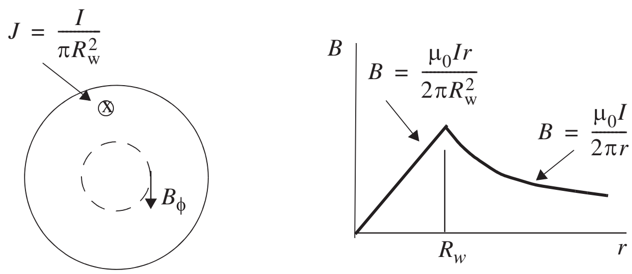

Figure 1.13 Calculation of flux density and inductance within a circular conductor.

Thus, the internal inductance of a wire of length L is

$$L_{\text{internal}} = \frac{\lambda}{I} = \frac{\mu_0}{8\pi}L \tag{1.138}$$

1.19 MAGNETIC FIELD ENERGY

Utilization of the magnetic field energy is often a convenient method for determining the inductance. The energy stored in a magnetic field can be expressed as

$$W_m = \frac{1}{2}\int_V (\boldsymbol{B} \bullet \boldsymbol{H})\, dV \tag{1.139}$$

since, also,

$$W_m = \frac{1}{2}LI^2 \tag{1.140}$$

then, when $\boldsymbol{B}$ and $\boldsymbol{H}$ arise from the same current,

$$L_{\text{self}} = \frac{1}{I^2}\int_V (\boldsymbol{B} \bullet \boldsymbol{H})\, dV \tag{1.141}$$

Alternative forms of equation (1.141) are useful. In machine analysis, it is frequently possible to assume that the field intensity and flux density are only radially directed in the air gap and that they vary only circumferentially. Furthermore, since $H = B/\mu$, if $\mu \rightarrow \infty$, H can be assumed as zero in the iron. Alternatively, the relatively small MMF drop in the iron can be corrected by appropriately increasing the MMF in an equivalent gap g_e. If θ denotes the angular measure in the circumferential direction, r the radial direction, and l the axial direction, after performing integration in the radial and axial directions of a typical cylindrically shaped geometry, equation (1.141) can be written as

$$L_{\text{self}} = \frac{grl}{I^2}\int_0^{2\pi} B_g(I,\theta)\, H_g(I,\theta)\, d\theta \tag{1.142}$$

Since H can be assumed constant in the gap then

$$H_g g = \mathcal{F}_g \tag{1.143}$$

where $\mathcal{F}_g$ represents the MMF acting across the air gap. One can now write this equation as either

$$L_{\text{self}} = \frac{rl}{I} \int_0^{2\pi} B_g(I,\theta) \frac{\mathcal{F}_g(I,\theta)}{I} d\theta \tag{1.144}$$

or

$$L_{\text{self}} = \mu_0 \frac{rl}{g} \int_0^{2\pi} \left[\frac{\mathcal{F}_g(I,\theta)}{I} \right]^2 d\theta \tag{1.145}$$

Since the gap g is comprised of air, the MMF must vary linearly with the current so that it can be expressed as the product of current times a second function which only depends on θ. Thus, the self-inductance can be obtained from either

$$L_{\text{self}} = \frac{rl}{I} \int_0^{2\pi} B_g(I,\theta) N(\theta) d\theta \tag{1.146}$$

or the expression,

$$L_{\text{self}} = \mu_0 \frac{rl}{g} \int_0^{2\pi} N(\theta)^2 d\theta \tag{1.147}$$

The new quantity $N(\theta) = \mathcal{F}_g(I,\theta)/I$ is called the *winding function* and is frequently employed in the circuit analysis of AC machines.

The field representation of stored energy can also be used to calculate mutual inductance. When $\boldsymbol{B}$ and $\boldsymbol{H}$ arise from currents in two different circuits,

$$W_m = \frac{1}{2} \int_V (\boldsymbol{B}_1 + \boldsymbol{B}_2) \bullet (\boldsymbol{H}_1 + \boldsymbol{H}_2) dV \tag{1.148}$$

However, it is also true from circuit theory that

$$W_m = \frac{1}{2} L_1 I_1^2 + L_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2 \tag{1.149}$$

Comparing equations (1.148) and (1.149), the terms involving the mutual inductance can be equated whereupon,

$$L_{12} I_1 I_2 = \frac{1}{2} \int_V (\boldsymbol{B}_1 \bullet \boldsymbol{H}_2 + \boldsymbol{B}_2 \bullet \boldsymbol{H}_1) dV \tag{1.150}$$

When the ***B***s and ***H***s are collinear, the product exists only in the gap and is only a function of θ, equation (1.150) clearly reduces to

$$L_{12} = \mu_0 \frac{rl}{g} \int_0^{2\pi} N_1(\theta) N_2(\theta)\, d\theta \tag{1.151}$$

where $(B_1/\mu_0)g = H_1 g = \mathcal{F}_1$, $(B_2/\mu_0)g = H_2 g = \mathcal{F}_2$, and

$$N_1(\theta) = \frac{\mathcal{F}_1(\theta)}{I_1}; \quad N_2(\theta) = \frac{\mathcal{F}_2(\theta)}{I_2} \tag{1.152}$$

When the flux density produced by one of the two windings is known, the following expression for the mutual inductance L_{12} is also convenient.

$$L_{12} = \frac{rl}{I_1} \int_0^{2\pi} B_1((\theta) N_2(\theta))d\theta \tag{1.153}$$

1.20 THE PROBLEM OF UNITS

One of the facts of life concerning the electromagnetic design of an electric machine is inconsistency regarding physical units. This inconsistency is a consequence of the long history associated with this discipline. In practice, three unit systems are used based on the SI or MKS system (Europe originally and now worldwide), the CGS unit system (small transformers, permanent magnet or PM machines, and small sub-fractional horsepower (HP) machines) and the English unit system (fractional HP machines and above, during early work in the United States). The English unit system is a throwback to the use of inches and pounds whereas the CGS system came into use via the physicists. Clearly, the SI system is the unit system of choice today. However, in view of the tremendous work done in the past incorporating the other units, it is important to be equally familiar with all three sets of units.

In this text, a particular relationship will be derived using SI units and then the result converted, if desired, to the different units. As an example, consider the *constituent equation* for magnetic materials.

$$B(\mathrm{Wb/m^2}) = \mu_r \mu_0\, (\mathrm{H/m})\, H\, (\mathrm{A/m}) \tag{1.154}$$

Multiplying this equation by 10^4 and substituting explicitly for μ_0,

$$10^4 B(\mathrm{Wb/m^2}) = \mu_r [4\pi \cdot 10^{-3} H\, (\mathrm{A/m})] \tag{1.155}$$

If one defines

$$B\,(\mathrm{G}) = 10^4 B(\mathrm{Wb/m^2}) \tag{1.156}$$

and

$$H\,(\mathrm{Oe}) = 4\pi \cdot 10^{-3} H\, (\mathrm{A/m}) \tag{1.157}$$

then in CGS units

$$B\,(\mathrm{G}) = \mu_r H\,(\mathrm{Oe}) \tag{1.158}$$

where the new units of B and H are the *gauss* and the *oersted*, respectively.

Consider now the possibility of converting this equation to English units. Multiplying equation (1.154) by 10^8, the result is

$$10^8 B(\mathrm{Wb/m^2}) = 10^8 \mu_r \mu_0, H\,(\mathrm{A/m}) \tag{1.159}$$

One can now define a new unit of flux called the *maxwell* or *line* such that

$$1 \text{ weber} = 10^8 \text{ lines or maxwells} \tag{1.160}$$

Making this substitution in equation (1.159), and explicitly substituting for B and μ_o,

$$B\left(\frac{\text{lines}}{\mathrm{m^2}}\right) = \mu_r\,(40\pi)\,H\,(\mathrm{A/m}) \tag{1.161}$$

Now,

$$B\left(\frac{\text{lines}}{\mathrm{m^2}}\right) = B\left(\frac{\text{lines}}{\text{in.}^2}\right)\left(\frac{1\text{ in.}}{2.54\text{ cm}}\right)^2\left(\frac{100\text{ cm}}{\text{m}}\right)^2 \tag{1.162}$$

and

$$H\,(\mathrm{A/m}) = H\,(\text{A/in.})\left(\frac{1\text{ in.}}{2.54\text{ cm}}\right)\left(\frac{100\text{ cm}}{\text{m}}\right) \tag{1.163}$$

so that in the English system,

$$\begin{aligned} B\left(\frac{\text{lines}}{\text{in.}^2}\right) &= \mu_r 40\pi\left(\frac{2.54}{100}\right)^2\left(\frac{100}{2.54}\right)H\left(\frac{\text{A}}{\text{in.}}\right) \\ B\left(\frac{\text{lines}}{\text{in.}^2}\right) &= \mu_r\left(\frac{4\pi}{10}\right)2.54\,H\left(\frac{\text{A}}{\text{in.}}\right) \end{aligned} \tag{1.164}$$

The quantity $(4\pi/10)2.54 = 3.192$ is sometimes called the "free space" permeability in the English system.

A similar development can be carried out for the magnetic circuit equation, equation (1.110), that is

$$\Phi(\mathrm{Wb}) = \mathcal{P}\,(\mathrm{H})\,\mathcal{F}(\text{A-t}) \tag{1.165}$$

Multiplying both sides by 10^8 and making use of equations (1.160) and (1.107)

$$\begin{aligned} \Phi\,(\text{lines}) &= (10^8)(4\pi\cdot 10^{-7})\mu_r \frac{A(\mathrm{m^2})}{l\,(\mathrm{m})}\mathcal{F}\,(\text{A-t}) \\ &= 40\pi\mu_r \mathcal{F}\,(\text{A-t})\,\frac{A(\mathrm{m^2})}{l\,(\mathrm{m})} \end{aligned} \tag{1.166}$$

Now

$$A(\mathrm{m^2}) = A(\mathrm{cm^2})\left(\frac{1\text{ m}}{100\text{ cm}}\right)^2 \tag{1.167}$$

$$l(\mathrm{m}) = l(\mathrm{cm})\left(\frac{1\text{ m}}{100\text{ cm}}\right) \tag{1.168}$$

TABLE 1.1 Comparison of magnetic circuit equations with various systems of units

Constit. Eq.	MKS (SI) units $B = \mu_0\mu_r H$	CGS units $B = \mu_r H$	English units $B = \mu_0\mu_r H$
Magnetic Ohm's law	$\Phi = \mu_0 \frac{\mu_r A}{l}\mathcal{F}$	$\Phi = \frac{\mu_r A}{l}\mathcal{F}$	$\Phi = \mu_0 \frac{\mu_r A}{l}\mathcal{F}$
Faraday's law	$v = N\frac{d\Phi}{dt}$	$v = N\frac{d\Phi}{dt} \cdot 10^{-8}$	$v = N\frac{d\Phi}{dt} \cdot 10^{-8}$
Free space permeability	$\mu_0 = 4\pi \cdot 10^{-7}$	$\mu_0 = 1$	$\mu_0 = \frac{4\pi}{10} \cdot 2.54$ $= 3.192$
	B in Wb/m^2	B in gauss	B in lines/in.^{2}
	H in A-t/m	H in Oe	H in A-t/in.
	Φ in Wb	Φ in lines (Maxwells)	Φ in lines
	$\mathcal{F}$ in A-t/m	$\mathcal{F}$ in Gb	$\mathcal{F}$ in A-t/m

so that equation (1.166) becomes

$$\Phi\,(\text{lines}) = 0.4\pi\mu_r \frac{A(\text{cm}^2)}{l\,(\text{cm})}\mathcal{F}\,(\text{A-t}) \tag{1.169}$$

In the CGS system the *gilbert* is defined as

$$\mathcal{F}\,(\text{gilberts}) = 0.4\pi\mathcal{F}\,(\text{A-t})$$

so that in the CGS system, the magnetic circuit equation, equation (1.169), becomes

$$\Phi\,(\text{maxwells}) = \mu_r \frac{A(\text{cm}^2)}{l(\text{cm})}\mathcal{F}\,(\text{gilberts}) \tag{1.170}$$

In the English system, it is necessary to convert lengths to inches:

$$\Phi\,(\text{maxwells}) = \frac{(0.4\pi)\,\mu_r A(\text{in.}^2)}{(2.54)\,l\,(\text{in.})}\mathcal{F}\,(\text{A-t}) \tag{1.171}$$

The key equations in magnetic circuit analysis in the three systems are summarized in Table 1.1. The reader should get to know these equations well. In order to help sort out the various conversion factors between the three systems, "flow charts" for the important variables are provided in Figure 1.14.

1.21 MAGNETIC PATHS WHOLLY IN IRON

The analogies between the electric circuit and the magnetic circuit seem to indicate, upon first consideration, that the Ohm's law type of relationship among MMF, flux, and reluctance or permeance ought to provide a straightforward method for solving magnetic circuit problems. However, the problem is much more difficult than the simple examples thus far considered. The direct application of the method is made difficult in practice by the relatively large flux leakage encountered in magnetic circuit

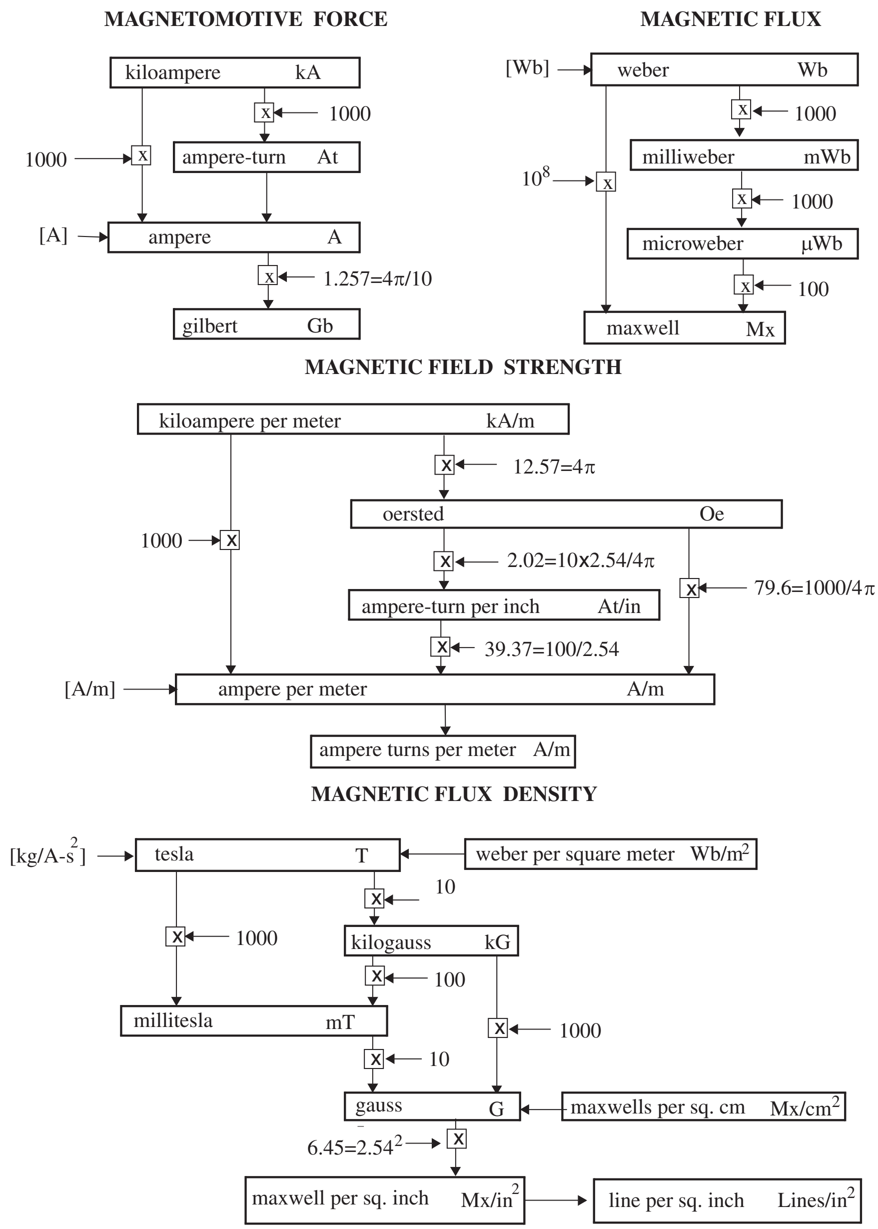

Figure 1.14 Conversion factors for magnetic field quantities.

problems and by the dependence of the reluctance of a ferromagnetic material upon the flux density; that is, the problem is nonlinear.

In general, solutions of magnetic circuit problems are solved to resolve two key questions: (a) the determination of the MMF required to produce a desired flux or flux density in a specified part of a structure, (b) the determination of the flux or flux density produced at specified places in a magnetic structure brought about by MMFs impressed at various places throughout the structure. Strictly speaking, the magnetic circuit method of analysis does not yield flux densities except as averages of total fluxes over the cross-sectional areas of the circuit so that the exact determination of the flux distribution becomes a field problem.

When the problem is to determine the MMF required to produce a desired total flux or flux density, that is, the calculation as in (a), the procedure is direct, provided that the leakage flux is neglected or estimated. In each portion of a series magnetic path having a cross-sectional area A, the average value of flux density B is equal to the ratio of the total flux Φ to the area A. The value of magnetizing force H required to establish this value of B is determined as a curve of B plotted as a function of H for the particular material. This value of H is then multiplied by the length of that portion of the path for which B is assumed constant, to give the magnetic potential difference $\mathcal{F}_{ab}$ between the ends of that portion of the path, that is

$$\mathcal{F}_{ab} = Hl_{ab} \tag{1.172}$$

where the distance a to b is the length of the path of uniform material and cross-sectional area. If the path includes portions of different kinds of ferromagnetic material, the value of H for each material is multiplied only by the length of the path in that material to give the magnetic potential difference for that portion of the path. The sum of the magnetic potential differences for all such portions of paths a–b, b–c, c–d, etc. taken around the series circuit gives the total MMF required, that is

$$\mathcal{F} = \mathcal{F}_{ab} + \mathcal{F}_{bc} + \mathcal{F}_{cd} + \cdots + \mathcal{F}_{na} \tag{1.173}$$

If the construction of the circuit is such that the average flux density differs markedly from the extremes of flux density on the cross section, more elaborate magnetic circuits must be employed.

When the problem is to determine the total flux or flux density produced by MMFs impressed at various places, that is, the calculation as in (b) above, the procedure is not so straightforward even if leakage fluxes are neglected. In certain simple combinations of paths, graphical methods are applicable. These are illustrated in the examples to follow. In complicated combinations of paths, a successive-approximation method leads rapidly to a solution. For such problems, the MMF required to produce an assumed value of flux Φ_1, is first calculated. If the calculated MMF does not approach the assumed impressed value within limits, a second trial value Φ_2 is chosen, greater or less by the amount required to equal the magnetic potential drop produced by the assumed flux Φ_1. After a few iterations, a solution is easily obtained. A Newton–Raphson iteration procedure using a digital computer is convenient for this purpose.

1.22 MAGNETIC MATERIALS

Historically, electric motors have been constructed from magnetic steels usually in the form of thin laminations, electrical conductors (either copper or aluminum), insulation for the conductors and slots, high tensile strength steel for shafts, and steel or copper alloys for bearings. The laminations used in most general-purpose motors have been "common iron" or low carbon steel. Although low in cost, this material typically produces machines of only modest efficiency. More recently, high efficiency motors often feature higher quality silicon steels at a correspondingly higher cost. The percent of silicon in the steel has a beneficial effect in reducing losses in the steel but at the same time tends to reduce the saturation flux density. The percent of silicon in motor steels typically range from 1% to 4%. The corresponding 60 Hz AC losses range from 0.6 watt per pound of core for 3.25% silicon steel to 1.0 watt per pound for 1% silicon steel at a peak flux density of 15,000 gauss (1.5 tesla). Nickel alloys, such as permalloy, have low losses but are very expensive and have low saturation flux density. The cobalt alloys such as Supermendur (49% iron, 49% cobalt, and 2% vanadium) have peak flux densities over 2 teslas, but are also very expensive ($7–$8 per pound) and have higher losses.

When the magnetic structure is assembled by means of stacking laminations punched from thin sheet material, the volume occupied by the stacked laminations does not truly represent the volume of iron that supports the magnetic flux. A region whose permeability is that of air exists between the laminations because of the presence of irregularities in the laminations or due to a thin coat of insulating varnish applied to avoid circulating current flow between laminations (eddy currents). In order to allow for this effect, the effective cross-sectional area of iron is equal to the cross-sectional area of the stack, times a factor called the *stacking factor*. The stacking factor, defined as the ratio of the cross-sectional area of the iron to the cross-sectional area of the stack, ranges between about 0.95 and 0.90 for lamination thickness between 0.025 in. and 0.014 in. (25 and 14 mils), respectively. For thinner laminations, for example, 1 mil to 5 mil thick, the stacking factor can be in the range of 0.4–0.75. Thinner laminations than 14 mils are generally not used unless iron loss is a severe problem. This choice typically occurs when the machine operates at high frequencies, for example an aircraft generator.

A new group of alloys has been developed, grouped under the generic title of amorphous metal alloys. These materials represent a new state of matter for electromagnetic materials, the so-called amorphous or non-crystalline state. Ordinary window glass is a typical example of an amorphous material. Some of these new amorphous alloys have magnetic properties which surpass the properties of conventional alloys. Thus, they appear to be a potentially useful new class of soft magnetic material. These alloys contain about 80% ferritic elements such as iron, nickel, and cobalt, and 20% glasseous elements such as silicon, phosphorous, boron, and carbon. A good example of an amorphous alloy having 80% iron and 20% boron by atomic weight is Fe80B20 (*Metglas* from Metglas Inc.). Major advantages of amorphous metal include low cost (roughly $0.30 per pound vs. $0.50 for silicon steel), very low core loss (one fifth that of the best silicon steels), low annealing temperature, and high tensile strength. Unfortunately, this new material has not yet been successfully used in a large scale because the high tensile strength also makes the material difficult to punch. Also, amorphous materials are presently only available in thicknesses of 1–2 mils (0.001″ to 0.002″) which results in a poor stacking factor and creates problems during assembly.

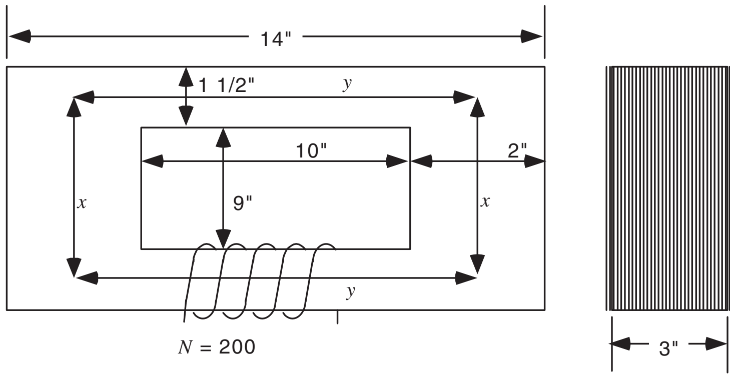

Figure 1.15 Core type transformer structure with two different cross-sectional areas.

1.23 EXAMPLE—TRANSFORMER STRUCTURE

The magnetic structure shown in in Figure 1.15 is similar to that of a core type transformer. The core is made of 29 gauge (14 mils) fully processed steel. The *B–H* curve for this material is shown in Figure 1.16. The sheets are stacked into a 3-inch stack. The stacking factor is 0.91. The exciting winding has 200 turns. Compute the current required in the exciting winding to produce a maximum core flux density of 1.2 tesla. Leakage flux is to be neglected.

Solution.

The cross-sectional area of the iron portion of legs x is $2 \times 3 \times 0.91 = 5.46$ in.^{2} and of the legs y is $1.5 \times 3 \times 0.91 = 4.1$ in.^{2}

- The maximum flux density will clearly occur in the y member having the smallest cross section. If $B_y = 1.2$ tesla then in the x leg $B_x = 1.2 \times 4.1/5.46 = 0.9$ tesla.
- From Figure 1.16, the magnetizing force is 2.9 Oe or 5.8 A-t/in. for legs y and 1.4 Oe or 2.85 A-t/in for legs x.
- The mean length of the flux paths in Figure 1.15 is estimated as 21 in. for the two x legs and 24 in. for the two y legs.
- The sum of the MMFs acting on the two y legs is $5.8 \times 24 = 140$ A-t and for the x legs $2.85 \times 21 = 60$ A-t. The total ampere-turns for the entire magnetic circuit is therefore $140 + 60 = 200$.
- The excitation current required to produce a flux density of 1.2 T in the transformer core is thus $200/200 = 1.0$ A.
- The flux in the core is clearly $\Phi = B_y \times A_y = 1.2 \times 3 \times 1.5 \times 0.91 \times (0.0254)^2 = 3.17$ mWb.
- The saturated inductance is then $L = N\Phi/i = 200 \times 3.17 \times 10^{-3}/1.0 = 0.63$ H.

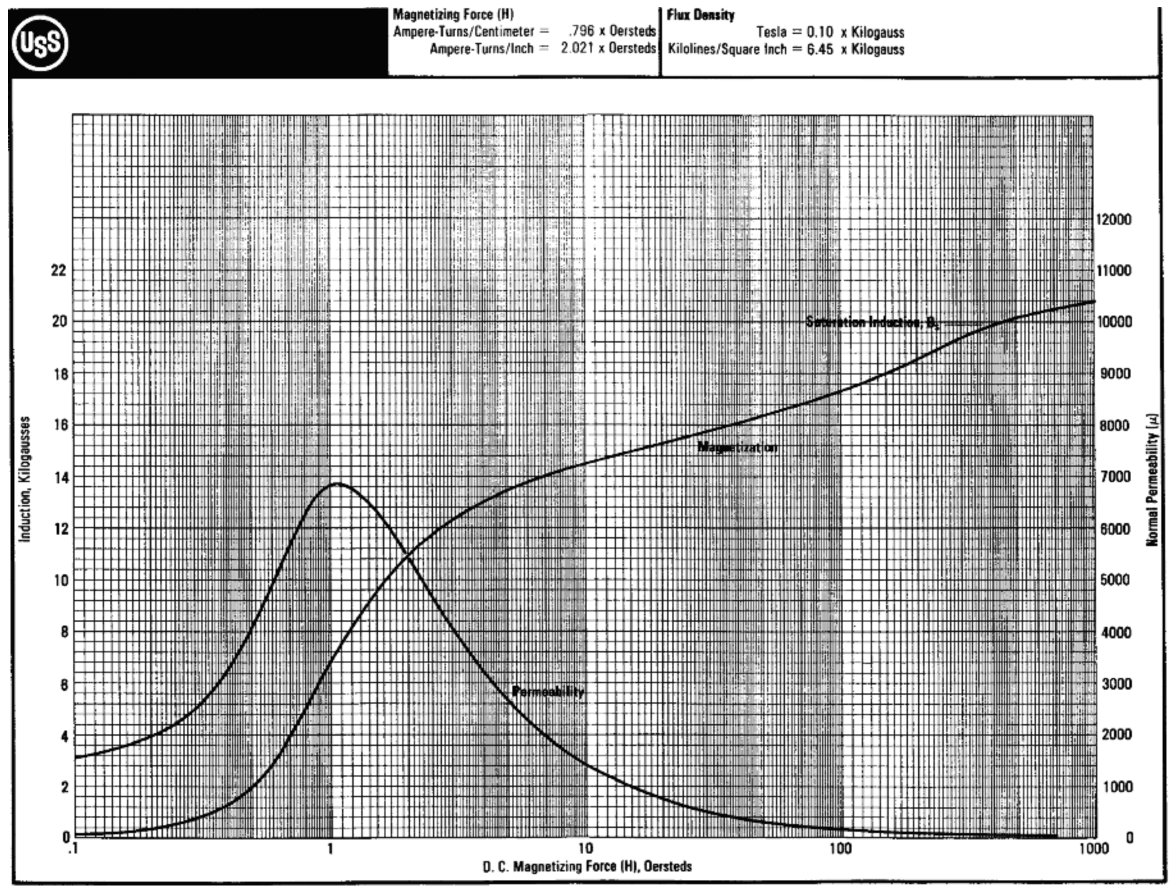

Figure 1.16 *B-H* curves for 29 Gage M27 fully processed steel.

When a specified MMF acts on a core, the inverse problem of calculating the fluxes is not simple. Assume, for example, that the results of the previous calculations are not known and that the core is excited with 200 A-t. In this case, it is necessary to estimate the probable magnetic potential difference between the ends of each core portion.

- Since the cross-sectional area of the y legs is much smaller than that of the x legs, the flux density is much larger in the y legs and would consume the major part of the MMF drop. As a first approximation, all the MMF is assumed to drop along the y legs. The resulting potential gradient is therefore 200/24 = 8.3 A-t/in. or 4.1 Oe.
- From Figure 1.16, the flux density in the y legs is then about 1.3 T. By proportionality, the flux density in the x legs is (4.1/5.46)1.3 = 0.97 T. Again from Figure 1.16, the x legs require an MMF of 1.6 × 2.021 × 21 = 68 A-t. The MMF required by the entire circuit is 200 + 68 = 268, which is, of course, too much MMF to satisfy Ampere's law.
- As a second approximation, the MMF drops in the x and y legs can be estimated by taking ratios. For the y legs assume $\mathcal{F}_y = (200/268)200 = 149$ A-t so that $H_y =$ 6.2 A/in. or 3.1 Oe. The second iteration yields $B_y = 1.22$ T resulting in a flux density in the x leg of $B_x = (4.1/5.46)1.22 = 0.91$.
- The corresponding field intensity in the x leg becomes 1.45 Oe or 2.9 A-t/in.
- The MMF drop in the x legs of 2.9 × 21 = 61 A-t.
- The total MMF drop around the circuit is now estimated to be 149 + 61 = 210 A-t which is now just slightly greater than the correct value of 200 A-t. As a third iteration, it is now possible to assume that $\mathcal{F}_y = (149/210)149 = 106$ A-t.

Note that the method oscillates about the correct solution, but nonetheless converges rapidly if implemented on a digital computer since the B–H curve is a simple monotonically increasing function. The iteration method for the y leg MMF can be made less oscillatory by changing the new estimate by only a fraction of the error from the last iteration by using the algorithm,

$$\mathcal{F}_i = \mathcal{F}_{i-1} + \kappa_{\mathrm{a}}(\mathcal{F}_{i(\mathrm{est})} - \mathcal{F}_{i-1}) \tag{1.174}$$

The quantity κ_{a} is an acceleration factor which can be taken as roughly 0.5 and $\mathcal{F}_{i(\mathrm{est})}$ is the estimated MMF for the ith iteration using ratios as described above.

For simpler problems, one can also resort to a graphical method. The procedure is to first determine the relationship between the total flux and the total MMF for each of the two nonlinear portions of the circuit. The curves of Φ_x as a function of $\mathcal{F}_x$ and Φ_y as a function of $\mathcal{F}_y$ is plotted in Figure 1.17 in such a manner that the abscissa for the x leg runs from left to right and for the y leg from right to left. The plot for the y legs, turned end for end, is called a negative magnetization curve and its origin is put at the point where $\mathcal{F}$ equals 200 on the plot for the x legs. The point 200 is chosen because it is equal to the applied MMF. Since the same total flux is present in both legs x and y, the solution for the impressed value of 200 A-t. is the intersection of the two curves.

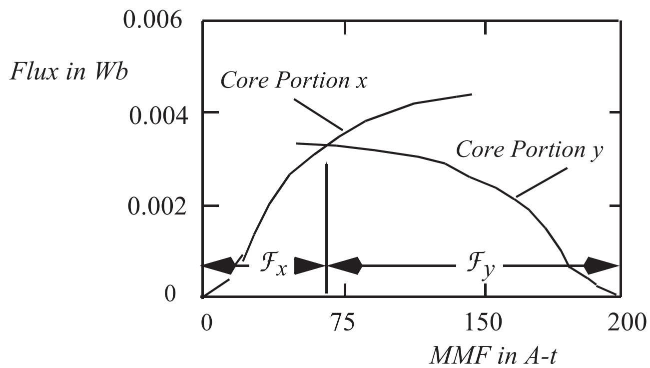

Figure 1.17 Graphical solution of Example.

1.24 MAGNETIC CIRCUITS WITH AIR GAPS

Because electrical machines involve magnetic circuits in relative rotation, an air gap must exist between the stator and rotor. In addition, other air gaps frequently occur because of limitations inherent in the construction. Air gaps are often introduced into iron-core inductors in order to make the inductance of the element essentially independent of the current in the coil throughout its working range, but at the same time to make the inductance larger than if the inductor had the same coil and only an air core.

When an air gap is inserted in a magnetic circuit, the flux spreads out, or fringes, around the gap as shown by the sketch of Figure 1.18 and the flux density in the gap assumes a non-uniform distribution. The flux that terminates near the edges of the gap is called the *fringing flux*. Because of the spreading of the flux, the apparent reluctance of the gap is not that of an air space of the same dimensions as the gap. Since the permeability of iron is several thousand times that of air the reluctance of

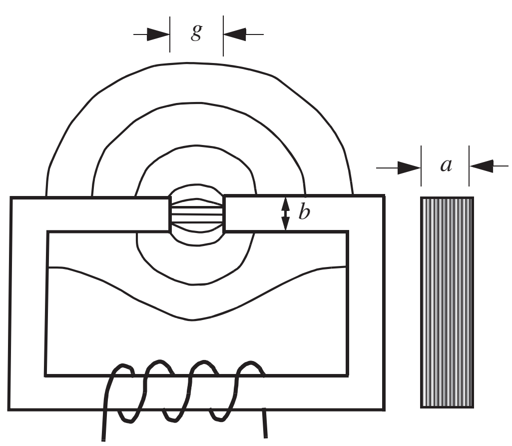

Figure 1.18 Magnetic circuit showing fringing flux.

even a short air gap is usually large compared to that of the iron portion making the magnetic potential between the stator and roor teeth relatively large. Relatively large magnetic potentials may also exist between iron parts not immediately near the gap. For example, in a synchronous machine, the main flux that traverses the gap fringes at the pole tips and because of the large reluctance of the air gap, considerable flux goes directly from rotor pole to rotor pole, constituting rotor leakage flux. This flux is often as much as 25% of the flux in the core of the field pole and contributes considerably to the saturation of the pole body.

When the air gap is short compared with its cross-sectional dimensions and has parallel faces, the fringing effect can be incorporated into the analysis by the use of simple correction factors. If the cross-sectional dimensions of the core are the same on both sides of the gap, the equivalent gap is assumed to have a length g equal to the actual air gap, but to have an equivalent cross-sectional area

$$A = (a + g)(b + g) \tag{1.175}$$

where a and b are the cross-sectional dimensions of the core faces. If one of the faces of the gap has a cross-sectional dimension much larger than the corresponding dimensions of the other, a correction of $2g$ should be used. Experience has shown that these rules give satisfactory results if the correction applied does not exceed about 1/5 of the physical cross-section.

If the total MMF applied is known a successive approximation solution can again be used. The first approximation can be obtained by considering that all the ampere-turns are required to overcome the reluctance of the air gap. A direct graphical method can also be used. The required solution is again obtained by superposing a plot of Φ_s as a function of $\mathcal{F}_s$ with a plot of Φ_a as a function of $\mathcal{F}_a$, where s denotes the steel portion and a the air portion of the flux path. The construction is shown in Figure 1.19, where $\mathcal{F}_t$ denotes the total impressed MMF. Note that the ordinate intersection of the air gap line is readily determined since

$$\Phi_a = \frac{\mathcal{F}_a}{\mathcal{R}_a} = \frac{\mu_0 A}{g}\mathcal{F}_a \tag{1.176}$$

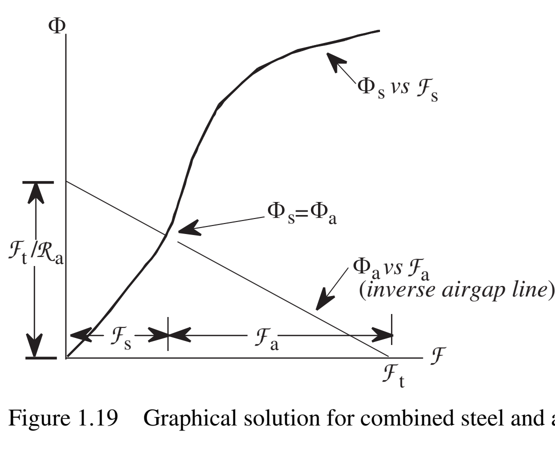

Figure 1.19 Graphical solution for combined steel and air magnetic circuit.

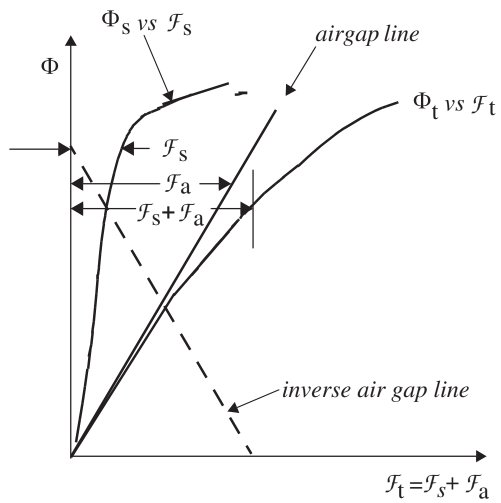

Figure 1.20 Illustrating the concept of the air gap line.

The intersection of the negative air gap line with the saturation characteristic of the steel for all values of $\mathcal{F}_t$ will generate the flux versus MMF characteristics for the overall device, that is, the "sat curve." The net saturation curve can be readily visualized as the sum of the iron and air saturation curves at each value of flux. Figure 1.20 shows such a construction.

1.25 EXAMPLE—MAGNETIC STRUCTURE WITH SATURATION

A magnetic structure similar to Figure 1.18 is made of 29 gage sheet steel laminations 0.014 in. thick stacked 2 in. thick. Dimension b is 2.5 in. The air gap length g is 0.10 in.

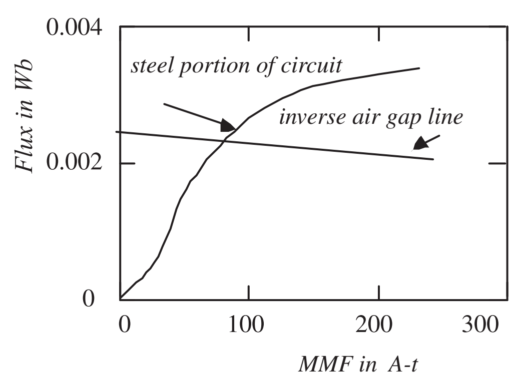

Figure 1.21 Graphical solution of Example.

The mean length of the steel part of the circuit is 30 in. Find the resultant flux if the applied MMF is 1400 A-t.

The equivalent air gap area, using a $2g$ correction is $(2.0 \times 0.91 + 0.2)$ $(2.5 + 0.2) = 5.45$ in.^{2} The negative air gap line intersects the abscissa at $\mathcal{F} = 1400$ A-t. The intersection on the ordinate is found by solving

$$\Phi_{\mathrm{a}} = (\mu_0)\frac{A}{l}\mathcal{F}_{\mathrm{a}} \tag{1.177}$$

or

$$\Phi_{\mathrm{a}} = 4\pi 10^{-7}\left(\frac{5.45}{0.1}\right)\frac{1}{39.37}(1400) = 2.44\,\mathrm{mWb} \tag{1.178}$$

The saturation curve for the iron is found by neglecting the MMF drop of the gap. A plot of the steel saturation curve and the negative air gap line is shown in Figure 1.21. The point of intersection of the two curves is read as 2.2 mWb.

1.26 EXAMPLE—CALCULATION FOR SERIES–PARALLEL IRON PATHS

A type of core construction used frequently for certain transformers in which a relatively small magnetic coupling between primary and secondary coils is desired is shown in Figure 1.22. This type of geometry will also be used in the subsequent analysis of a machine for calculating the flux entering the core through both the tooth and the slot. Because of the nonlinearity of the core material the fraction of the total flux bypassed through the leg y varies with the amount of magnetic saturation.

For illustration, assume a flux of 3.8 mWb is set up in the leg x by a coil wound around this leg. The ampere-turns required are to be calculated. The magnetic material is again fully processed steel, 14 mil thick. The stacking factor is 0.91. The paths *axb* and *azb* are assumed to have a mean length of 21 in. The mean length of core in the center leg is 7.9 in.

- In order to solve this problem, the cross-sectional area of the core and air gap is first calculated. For the core, the area is $2 \times 2 \times 0.91 = 3.64$ in.^{2} For the air gap, including fringing, $2.1 \times 2.1 = 4.41$ in.^{2}

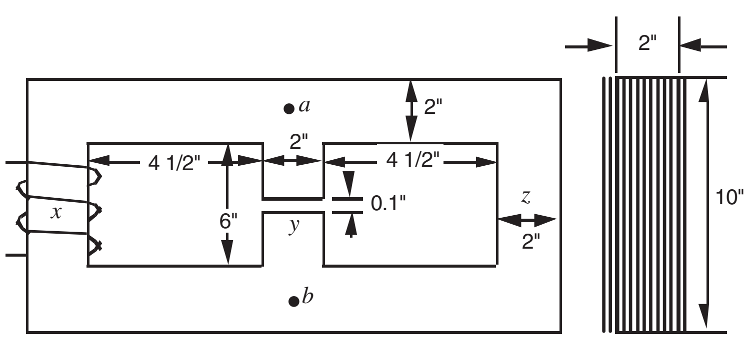

Figure 1.22 Magnetic circuit with series-parallel paths.

- The flux density in the x leg is calculated to be $0.00380/(3.64 \times 0.0254^2) = 1.62$ tesla. The MMF required to set up this flux density in the path axb is calculated as $830 \times 21 = 1743$ A-t. Figure 1.16 as $83 \times 21 = 1743$ A-t.

The total flux of 3.8 mWb in the path axb divides between the z and y legs in such a manner that the MMF from a to b is the same using either path azb or ayb. The division of flux must be calculated by assuming a tentative flux distribution and then correcting. The flux in the air gap or y leg is assumed so that the MMF from a to b necessary to produce this flux can be calculated. Since this MMF also acts on leg z, the flux in the z leg is now calculated and added to the flux assumed to exist in the y leg. The result is compared to the assumed total value of flux and if the result differs, the amount of flux assumed in the air gap is corrected accordingly. The procedure continues iteratively until convergence occurs.

- For example, assume that 0.80 mWb of flux exists in the y leg of the transformer. The corresponding value of flux density in the y leg iron is $0.00080/(3.64 \times 0.0254^2) = 0.34$ tesla. Since this is a relatively small value of flux density, the MMF drop in the iron can be neglected relative to the drop in the air.
- The MMF consumed in the air gap is, from equation (1.110), the result (0.0008) $0.1/[0.0254 \times (2 + 0.1)^2 \mu_0] = 568$ A-t. This MMF drop acts on the z leg producing a magnetizing force of $568/21 = 27$ A-t/in. which from the data of Figure 1.16 establishes a flux density of 1.47 tesla.
- The corresponding flux in the z leg is then $(1.47)(3.64)(0.0254^2)$ or 3.5 mWb. The total flux in the two legs is 4.3 mWb which is somewhat more than the specified value of 3.8 mWb.
- If necessary, a second trial is now made using the value $0.8 \times 3.8/4.3 = 0.71$ as the value of flux assumed to flow in the y leg. The procedure proceeds iteratively to about 0.47 mWb in the y leg and 3.33 mWb in the z leg.
- The MMF drop in the y leg is now computed to be 333 A-t resulting in a total MMF drop around the path $xaybx$ corresponding to the ampere-turns required to produce 3.8 mWb in the x leg, namely $\mathcal{F} = 1743 + 333 = 2076$ A-t.

1.27 MULTIPLE WINDING MAGNETIC CIRCUITS

In many cases, the magnetic circuit includes the effect of two or more sources of MMF. This was, for example, the case when the transformer of Section 1.15 was examined. This is also typically the case in many electrical machines which have not only an excitation component but also a separate load component typically on different members of the machine. When the iron is allowed to saturate the problem is now not so simple.

The basic issue can be demonstrated by the simple electromagnet shown in Figure 1.23. This device can be represented by the equivalent magnetic circuit shown in Figure 1.24.

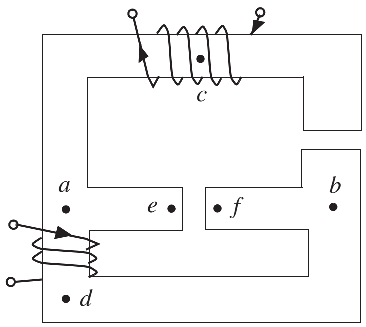

Figure 1.23 Electromagnetic with two sources of excitation.

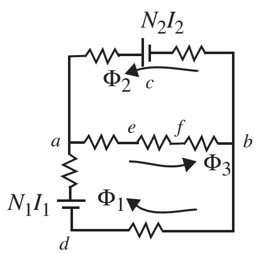

Figure 1.24 Equivalent magnetic circuit of Figure 1.23.

In this case, the MMF drop from point a to point b can be written in terms of three equations, namely

$$\mathcal{F}_{ab}(\Phi_1) = N_1I_1 - \mathcal{R}_{adb}(\Phi_1)\Phi_1 \tag{1.179}$$

$$\mathcal{F}_{ab}(\Phi_2) = N_2I_2 - \mathcal{R}_{acb}(\Phi_2)\Phi_2 \tag{1.180}$$

$$\mathcal{F}_{ab}(\Phi_3) = \mathcal{R}_{aefb}(\Phi_3)\Phi_3 \tag{1.181}$$

where

$$\Phi_1 + \Phi_2 = \Phi_3 \tag{1.182}$$

Three MMF versus flux curves can be constructed as shown in Figure 1.25.

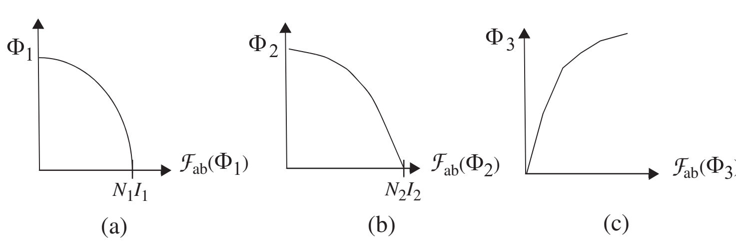

Figure 1.25 MMF vs. flux curves for the three magnetic paths: (a) lower member, (b) upper member, (c) middle member.

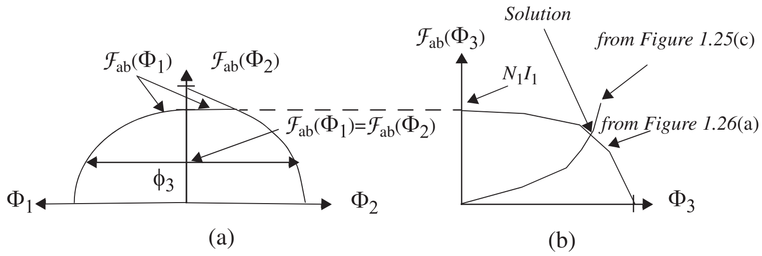

Figure 1.26 Graphical solution of the dual excitation problem.

Clearly, $\mathcal{F}_{ab}(\Phi_1) = \mathcal{F}_{ab}(\Phi_2)$. If both Φ_1 and Φ_2 are plotted versus $\mathcal{F}_{ab}$, then the flux Φ_3 can be determined for every value of $\mathcal{F}_{ab}$. The resulting construction is shown in Figure 1.26a. The actual solution is the point where the MMF as determined by this curve matches the MMF as determined by the function $\mathcal{F}_{ab}(\Phi_3)$ as computed in Figure 1.25c.

1.28 MAGNETIC CIRCUITS APPLIED TO ELECTRICAL MACHINES

Although the methods and simplifying assumptions outlined in the preceding sections yield results of reasonable accuracy for simple geometries, electrical machines present a considerably more complicated problem. Nonetheless, the basic principles which have been discussed form the basis for analysis and design of any machine structure. For illustration, Figure 1.27 shows the magnetic circuit of a typical two-pole DC machine. The center-slotted member is the rotor which carries the rotor winding in the slots. It is usually assembled from laminated steel punchings having 1–3% silicon. The outer member is the field structure, the cylindrical portion being the yoke or frame which frequently is of cast iron or cast steel. The protruding

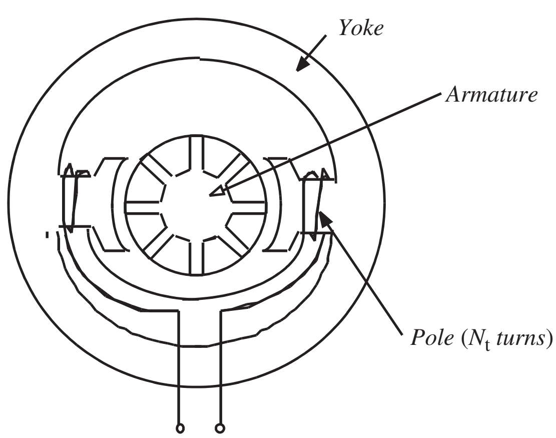

Figure 1.27 Magnetic circuit of a DC machine.

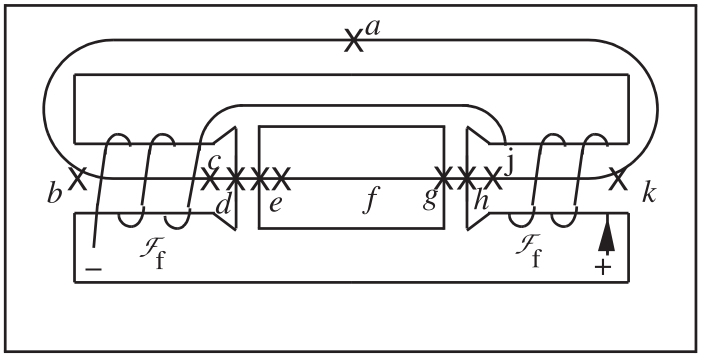

Figure 1.28 Simplified magnetic equivalent circuit corresponding to Figure 1.27.

portions of the field structure of *salient poles* are the poles which are usually made of laminated steel.

With a DC voltage applied to the field windings, the steady-state value of field current is determined entirely by the resistance of the circuit. For the polarity shown, the field winding produces an MMF in the direction to establish a flux from left to right through the field poles, air gaps, and armature. The flux path is then completed through the frame. The magnetic circuit is redrawn schematically in Figure 1.28. In order to determine the portions of the magnetic circuit whose properties predominate, a graph of the relative MMF of various points around the circuit is sketched in Figure 1.29. The MMFs are given with respect to an arbitrary point at the center of the yoke, point *a*. The magnetic potential drop from *a* to *b* is shown as a negatively sloped line in Figure 1.29. From *b* to *c*, similar conditions hold but since the material is different a slightly different slope is shown. The air gap *d*–*e* offers a very large

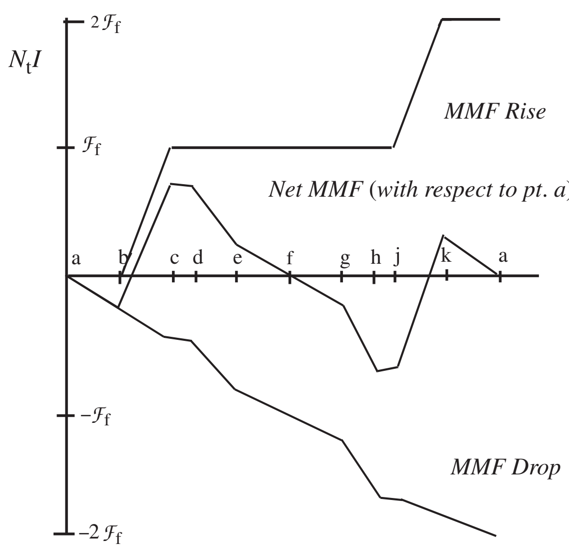

Figure 1.29 Magnetic potential drops and rises in a two-pole DC machine.

MMF drop relative to the iron. If the plot is completed for the remainder of the path, returning to the point *a*, the total reluctance drop is found to equal twice the MMF drop from *a* to *f*.

For the drop from *a* to *f* to exist, an equal and opposite rise in MMF must exist somewhere in the circuit. The required potential rise is established by the MMF of the field windings, the magnitude of the contribution being the same as the MMF drop from *a* to *f* for each of the two field windings. The upper curve of Figure 1.29 shows the rise in potential given by the windings. The next MMF curve shows the sum of the rises and drops at each point in the circuit and thus gives the actual MMF at each point with respect to point *a*.

Several important points can be deduced from Figure 1.29. First, note that the points *a* and *f* are at the same potential and the MMFs from *f* to *a* and *a* to *f* are mirror images of each other. Therefore, it can be deduced that all calculations can be made on a "per pole" basis. Second, the resultant curve shows that all points on the yoke are nearly at the same potential. Therefore, the leakage flux from *b* to *k* through the air will be relatively small. Third, the same is not true for the tips of the field poles. The potential difference through the pole body increases until at the pole tips the MMF is a maximum. In particular, the difference in potential from the top of one pole to the tip of the other is $2N_tI$. Since the distance between the tops of adjacent pole shoes is relatively small, the leakage flux from pole to pole is likely to be appreciable. Indeed, even for a good design, this leakage flux is generally 10 to 20% of the useful flux.

1.29 EFFECT OF EXCITATION COIL PLACEMENT

Up to this point, little attention has been paid to the exact placement of the core coil making up an inductor. In practice, the location of the exciting coil has a considerable effect on the overall losses as well as the exact value of the inductance obtained. Consider again the simple air-gapped core of Figure 1.18. Figure 1.30 shows three cases in which the coil is placed (a) on the limb farthest from the gap, (b) on the limb with the gap and (c) on the upper and lower limbs. In each case, the MMF is plotted from point *a* to point *f* as identified in the figure. The potential is plotted with respect to point *a*. While the remainder of the flux path is not plotted (from *f* back to *a* in the lower portion of the core), it is identical (mirror image) to the upper half. In case (a) where the coil is on the side away from the gap, the difference in MMF potential between the upper half of the core and the lower half is large over the distance *c* to *d* resulting in a large flux passing from the top limb to the bottom limb which closes through the left-hand limb. Since the useful flux is presumably the flux in the air gap region, this additional flux could be considered as leakage flux.

In case (b) of Figure 1.30, MMF is large only in the region near the air gap (*d* to *e*) resulting in "leakage" flux concentrated in this region. Clearly, if the purpose of the inductor design is to create a specified amount of flux in the air gap region, then this design would wind up being the smallest and lightest since most of the iron path does not have to support these additional leakage flux lines. Unfortunately, this option is often not a good choice since heating of the copper conductors around the

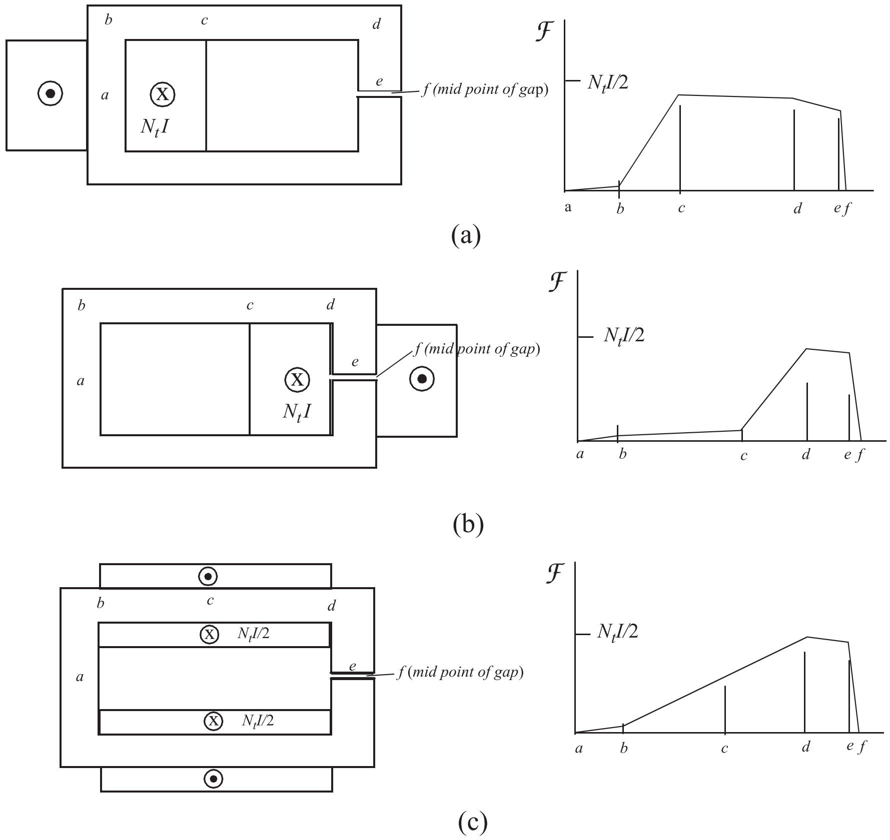

Figure 1.30 Three different methods of winding a simple gapped core. MMF potential plotted with respect to point *a*.

air gap will occur due to eddy current effects. The issue of losses will be taken up in more detail in Chapter 5.

With case (c), a compromise can be reached concerning these additional flux lines. Since the MMF in this case increases linearly over the entire length of the upper and lower limbs, the overall average difference in potential decreases resulting in leakage flux lines somewhere between cases (a) and (b).

Although a poor choice for inductor design, case (b) clearly does produce the desirable effect of creating the maximum number of flux lines in the air gap. This result is also a valid and important observation in the design of electrical machines where the process of creating a maximum amount of flux lines in the air gap for a given amount of ampere-turns is of critical importance. The design of case (b) teaches that it is important to design machines in which the copper exciting the main magnetic circuit be as near to the air gap as possible. Hence, many shallow slots are preferable to fewer deep slots containing the same amount of ampere-turns. Also, magnets far away from the air gap (buried magnets) are poorer in creating torque than a machine with magnets simply fixed on the rotor surface, that is, in the air gap. Much more will be presented concerning these interesting aspects of machine design in future chapters.

1.30 CONCLUSION

This chapter has served as a brief, but intensive, review of electromagnetic fields as applied to electric machine design. Although the math appears formidable, fortunately, with reasonable approximations, most calculations required in the design process can be carried out without an advanced mathematical treatment. Generally, the simple concepts presented in Section 1.13 provide a good starting point for the design process.

REFERENCE

[1] R. Plonsey and R. E. Collin, *Principles and Applications of Electromagnetic Fields*, McGraw-Hill Book Company, Inc., 1961.