BIG IDEA 1

Connecting 2D and 3D Worlds

Researchers have recently shown that when we work on a mathematics problem, five areas of the brain are involved, and two of them are visual pathways (Menon, 2015) (as I explained in the introduction to this book). Our brains are helped when we work visually; and when we connect visuals with numbers, important brain connections occur. In addition, other brain areas are involved when we touch, move things, and interact physically with mathematical ideas. The area of research concerning physical touch and movement is known as "embodied cognition," and researchers in this field point out the importance of students' holding mathematical ideas in the motor and perceptual areas of the brain (Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012), which comes about when they learn mathematical ideas through touch and movement. In this big idea, students are learning about the features of 3D shapes. This is an area that is particularly important to experience physically, as students will not develop a complete understanding of the mathematical features of shapes if they only see them in two-dimensional pictures in textbooks.

When we taught 83 middle school students a few summers ago, we invited them to build larger cubes out of sugar cubes. A year later, one of the students told me that he still remembers the meaning of "1 cubed" by recalling the feel and the features of the small cube he held in his hands. He told me that it was continuing to help him as he learned geometry. The three activities in this big idea give students a physical opportunity to hold objects in their hands and experience their different features.

In the Visualize activity, students are invited to slice 3D objects to make different two-dimensional shapes. Students will enjoy working with clay, and they will learn about the three-dimensional objects they hold in their hands as they touch and feel the dimensions. As they form the shapes and then sketch them, they will be able to make connections between areas of the brain that deal with physical touch and with drawing.

In the Play activity, students will get the opportunity to explore further with the objects they build out of clay. This activity also includes an element we design into tasks wherever possible—giving students choice. This is something that will increase their interest, which, in turn, will increase their learning and achievement.

Our Investigate activity poses questions that we hope will enable students to struggle, as that causes positive brain activity, and to extend their ideas to high levels. The questions have an openness that is rare in textbooks but that is important, as this openness enables students to develop their own ideas and to develop mathematical thinking in response to ill-defined problems of the type they will meet in the world outside school. We pose the following questions:

- How could you slice this solid so that the face that is made has the same area as the base?
- How could you slice it so that the shape has an area bigger than the base?

Both questions will give students opportunities to think deeply, to wonder about relationships, and to connect ideas.

Jo Boaler

References

Menon, V. (2015). Salience network. In A. W. Toga (Ed.), *Brain mapping: An encyclopedic reference* (Vol. 2, pp. 597–611). San Diego, CA: Academic Press.

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences*, *21*(2), 287–323. doi:10.1080/10508406.2011.611445

Seeing Slices

Snapshot

Students visualize and explore the two-dimensional shapes that can be made by slicing a rectangular solid.

Connection to CCSS
7.G.3

Agenda

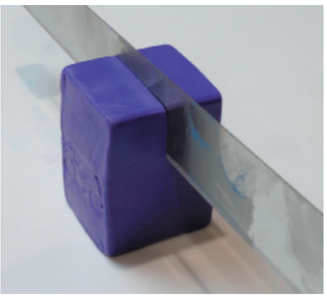
Activity	Time	Description/ Prompt	Materials
Launch	10–15 min	Show students a rectangular solid made of clay, and then slice that solid on an angle. Without separating the pieces, ask student to predict what shape the face of the slice is. Discuss students' predictions, then reveal and discuss the shape of the face.	• Rectangular solid made of clay • Cutting tool (dental floss or a wire cutter)
Explore	25–30 min	Small groups explore the question, What different two-dimensional shapes can you make by slicing a rectangular prism? Groups create a rectangular solid from clay and a net to match. Then groups use a cutting tool to slice and re-form the solid repeatedly to explore the shapes of the sliced faces. Students sketch the solid, how it was sliced, and the resulting face shape.	• Clay, enough for each group to form a rectangular solid • Cutting tool (dental floss or a wire cutter), for each group • Regular and isometric dot paper (see appendix), multiple sheets per group
Discuss	15 min	Discuss the two-dimensional shapes students created by slicing their rectangular solids. Discuss patterns for creating rectangles, other quadrilaterals, or triangles. Ask, What shapes cannot be made? Why?	Optional: chart and markers

To the Teacher

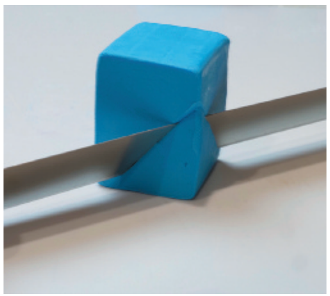
For this activity and the others in this big idea, we want students to have the chance to physically interact with three-dimensional figures and have the opportunity to slice them. To accomplish this, we recommend the use of clay to make the solids and dental floss for cutting. If you have an art department that has wire cutters for clay, those are even better. You may also try plastic knives, if you have those available. We like clay because it is stiff, and if you use a thin cutting tool, clay will maintain the shape of the slice better than play dough or other softer modeling substances. We recommend that you try working with the clay yourself to see what strategies work best with your specific clay for forming crisp solids and slicing them so that the solid does not warp. This will make it easier for you to model your cutting technique for the class during the launch. You may want to provide tips to the class for forming solids and cutting smoothly based on your experience. One additional option instead of clay is kinetic sand. It is much more expensive, which is why we have not focused on it here, but it molds beautifully, with crisp edges, and cuts well.

Students may notice that slicing a prism makes two different kinds of shapes: two new three-dimensional solids and a new two-dimensional face. In this activity, we're going to focus on the two-dimensional face made by slicing. However, you'll want to be precise with your language to avoid confusion. For instance, if you simply asked, What shape is made by slicing the rectangular solid? students may try to name the new *solids* created, rather than the shape of the new *face* created.

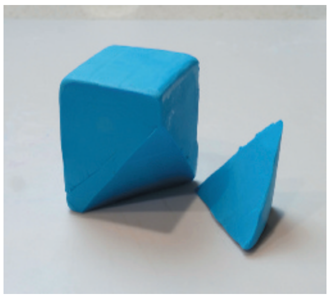
In the closing discussion, we ask the class to consider patterns for slicing the rectangular solid to get different kinds of face shapes. An exhaustive list is not necessary, but we hope that students will discover that to create a rectangle, they can slice parallel or perpendicular to one of the rectangular faces. To make a triangular slice, you must slice through only three faces, which looks like slicing a corner off the solid. Encourage students to look for as many patterns as they can.



A vertical cut perpendicular to the base.



A diagonal slice passing through three faces.

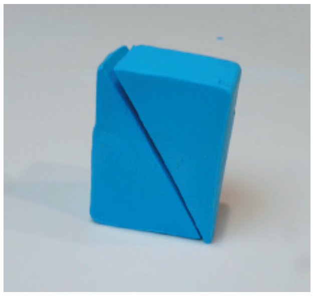


The shape of the cut through three faces is a triangle.

Activity

Launch

Launch the activity by showing students a rectangular prism made out of clay. Your rectangular prism should be large enough that students can see it well, and the faces, edges, and vertices should be as crisp as possible. Show students your cutting tool (dental floss or wire cutter), and then use it to cut the clay into two parts on an angle, so that your cut is not parallel to one face.



The square prism sliced diagonally.

Without separating the two pieces, ask the class, What shape do you think the face of the slice is? Give students a chance to turn and talk to a partner about how they are visualizing the face made by slicing the solid. Invite students to come up and draw their hypotheses on the board or document camera. Ask students to explain why they believe that their shape is the one made by the slice. For each predication, ask the class whether they agree or disagree and why. The class may not come to agreement on a prediction, but do invest some time in debating predictions and how students are visualizing the slice.

Separate the two pieces of the rectangular solid to reveal the shape of the sliced face. Trace the shape of the slice onto a piece of paper on the document camera or onto the board. Discuss what you found and its relationship with students' ideas. Ask, How accurate were our predictions? Does the shape surprise you? Why or why not?

Pose the question for exploration: What different two-dimensional shapes can you make by slicing a rectangular prism?

Explore

Provide each group with clay, a cutting tool, and isometric and regular dot paper (see appendix). Ask each group to make a rectangular prism and construct a net for that prism on the regular dot paper as a record of their solid. Students explore the question, What different two-dimensional shapes can you make by slicing a rectangular prism?

Students cut the solid they have formed so as to find different shapes that can be made by slicing. For each slice, the group makes a sketch of how they cut the solid and traces the sliced face to record the new shape created. Students may find that isometric dot paper is useful for sketching the solid and their cut lines, while the regular dot paper is more useful for tracing the shape of the sliced face. Students can use their net to help them re-form the same solid as they cut it again and again.

Discuss

Ask, What shapes were you able to make by slicing your rectangular prisms? Invite groups to share some examples of how they cut and the shapes they made.

Discuss the following questions:

- How do you cut the rectangular prism to get different sorts of shapes?
- What are the rules for cutting to get a rectangle? Or a quadrilateral? Or a triangle? (For each kind of shape students created, discuss the ways they must slice to produce that shape. You may want to record these patterns on a chart.)
- What shapes are not possible when slicing a rectangular solid? Why?

Look-Fors

- **Are students interpreting the shape of the face accurately?** As described in the To the Teacher section, students may struggle to interpret which shape they are exploring—the two new solids created by slicing, the new two-dimensional face made by slicing, or even the changed two-dimensional faces of the former rectangular solid. In the launch, we encourage you to be as clear as possible, but if you see students interpreting the task differently, you may want to get your rectangular solid model and point out the feature under investigation. Further, even if students understand that it is the face we are thinking closely about, their eyes may play tricks on them. Students may observe a cut face that is a quadrilateral and see it as a rectangle or parallelogram, even when it is not. Ask probing questions such as, How do you know it is a rectangle (or parallelogram)? How can we be sure? What happens when you trace it onto the dot paper? Be sure students are using more than their eyes to name the shapes they create.

- **How are students sketching their solids?** Making two-dimensional representations of three-dimensional solids is challenging, because it can be difficult for the eye to translate what it sees in the three-dimensional space onto a flat surface. Isometric dot paper (see appendix) can help students capture the depth of a shape, but even the use of this tool requires students to take something they understand and represent it somewhat differently than it is in the real world. For instance, students will know that the faces of their solid are rectangular, but to draw the solid with depth, students will need to represent some faces as parallelograms. If students are struggling with the difference between what they know about a shape and how to draw it, encourage them to look closely at the solid from one angle and describe what the faces *look* like from that angle—not what they actually *are*, but what they look like. Students may need to first challenge what they *see* to be able to draw it.
- **Are students noticing patterns?** Students may focus on cutting, sketching, and re-forming their solids, but looking for patterns is a goal of this exploration. As you circulate to talk to groups, deliberately ask students, What patterns are you noticing in the shapes you can make by slicing your solid? Students may not have thought about patterns yet, but your question will prompt them to reflect on the many shapes they have created to look for patterns. Note also whether students seem to be cutting repeatedly, but in the same ways, which will constrain what patterns they can see and what shapes they find. If you notice that they are, for instance, always cutting parallel to one face of the solid, you might ask, What happens if you cut on an angle?

Reflect

When might you need to slice a solid? Why might the shape of the sliced face matter?

Playing with Clay

Snapshot

Students play with slicing a variety of three-dimensional figures to see what shapes the sliced faces can make and how these compare with slicing rectangular solids.

Connection to CCSS
7.G.3

Agenda

Activity	Time	Description/ Prompt	Materials
Launch	5 min	Remind students of the two-dimensional shapes they made by slicing rectangular prisms. Show students some of the figures in the solid set. Ask, If we sliced a different solid, how might that change the shapes the slice could make?	Solid set
Explore	20 min	Each group selects a solid to explore, and forms one like it using clay. Groups construct a net for their solid as a record. Students form hypotheses about the two-dimensional shapes that could be made by slicing the solid in different ways, and record their predictions.	• Slice Hypotheses sheet, one per group • Clay, for each group • Isometric and regular dot paper (see appendix), for each group • Solids, one per group
Discuss	10 min	Discuss the groups' hypotheses about the shapes that could be made by slicing their solids. Support students in describing how they visualized the slicing to make their predictions.	

Activity	Time	Description/ Prompt	Materials
Play	30+ min	Groups test their hypotheses by slicing their solid with a cutting tool repeatedly. For each slice, students record a sketch of how they sliced the figure and the new two-dimensional shape formed by the slice. Groups construct a chart for their solid that shows their hypotheses and their findings, using drawings and diagrams to make it as visual as possible.	• Slice Hypotheses sheet, one per group • Clay and a cutting tool, for each group • Isometric and regular dot paper (see appendix), for each group • Solids, one per group • Chart, markers, and tape, for each group
Discuss	15–20 min	Post groups' charts and do a brief gallery walk to look for patterns and surprises. Discuss what students found by slicing, how these findings compare to their hypotheses, and the patterns they notice looking across the different solids represented on charts. Discuss what shapes were not made and whether they might still be possible or are impossible.	

To the Teacher

In this activity, we continue to use clay and cutting tools to form and slice solids. See the To the Teacher section in the Visualize activity for more details about these materials. In addition to these, this activity requires a set of solids. You need only one set to make this activity work, as each group will explore only one figure. There are many types of solid sets available, ranging from those that are truly solid and made of wood to hollow figures that can be filled with water or sand, and even some with removable inserts that can unfold into nets. Any kind will work for this activity. However, if you do have available hollow solids, students may be able to use them as molds, making this activity more efficient. This is particularly effective if instead of clay you use kinetic sand, which molds easily and forms crisp edges.

Activity

Launch

Launch the activity by reminding students of the slicing work they did in the Visualize activity and the kinds of two-dimensional shapes they made by slicing rectangular prisms. Show students several three-dimensional figures from your solid set, such as cylinders, different kinds of prisms, pyramids, and spheres. Pose the question, If we sliced a different solid, how might that change the shapes the slice could make?

Explore

Provide each group with clay, isometric and regular dot paper (see appendix), and a Slice Hypotheses sheet. Invite each group to choose one solid from the solid set to explore. Ask each group to use the figure they chose to help them make a solid like it with clay. Groups then construct a net for that solid as a record, using regular dot paper.

Before students begin to play with slicing their solids, ask them to make a prediction, a formal hypothesis, of the shapes that could be made by slicing the solid. Encourage students to visualize different kinds of slices and sketch multiple predictions on the Slice Hypotheses sheet.

Discuss

Discuss the following questions:

- What are your hypotheses? What slice shapes might be made with your solid?
- How did you make your hypotheses? What were you seeing in your mind to help you predict the shape that would be formed by slicing?

Invite students to share their sketches or to hold up their solids to gesture the ways they imagine slicing it and what the results might be. Focus students' discussion on trying to visualize what is produced by slicing in different ways.

Play

Groups explore the question, What two-dimensional shapes can be made by slicing our solid? Provide each group with a cutting tool and invite them to cut the solid to find the different shapes that the sliced face can make. For each slice, students sketch on isometric dot paper a picture of how they cut the solid and trace the face (on isometric or regular dot paper) to record the slice shape. Students can use their net to help them re-form the solid and cut again.

Provide each group with chart, markers, and tape, and ask them to construct a display with their shape, their Slice Hypotheses sheet, and all the shapes they were able to make by slicing. Encourage students to make these displays as visual as possible to help others see the slicing they were trying.

Discuss

Have groups post their charts and do a short gallery walk. As students walk, ask them to look for patterns or surprises. Then gather the class to discuss the following questions:

- What shapes were you able to make by slicing your solids? How were they different than the shapes we made when slicing the rectangular solid? Invite students to share some examples of how they cut and the shapes they made.
- What patterns do you notice in the shapes of the slices that different groups made with different solids?
- What shapes were not possible with different solids? Why?
- Compare your hypotheses to what you found. How were they different? The same? Did you discover anything that surprised you?

Look-Fors

- **Are students making precise solids?** It can be physically difficult to work with clay and create solids with crisp edges and flat faces. However, the more precise the solid, the better the conclusions students can draw from slicing it. Although you don't want students spending inordinate amounts of time forming a meticulous solid, it makes sense to point out moments when greater precision will ultimately help students explore. You may want to notice aloud how the solid the group has created differs from the example from the solid set, or call attention to the challenges students face in trying to draw a net of a wobbly solid. Students may be supported with some tools with straight edges or flat faces to help them, such as rulers and the table top.
- **How are students dealing with ambiguity when slicing?** When students slice their solid, it is likely to warp somewhat. This makes exploring the shape of the sliced face a challenge. Ask students how they are dealing with the bends or crushed corners after they slice. Some students may discover that

they can reshape the face to re-create it. Ask, How do you know that this face is the shape of the slice, and not changed? What can you do to increase your confidence in your findings? As a test, students might try putting the two portions of the solid back together to see whether they meet accurately.

- **Are they slicing in new ways?** As in the Visualize activity, students may end up slicing the solid in similar ways repeatedly. This repeated slicing could be part of an important exploration of pattern. For instance, students exploring a pyramid might slice it parallel to the base repeatedly as they notice that they get similar figures of decreasing size as they move to the top vertex. However, this could also lead students to conclude that only a narrow set of slice shapes can be formed. Ask students questions about how they are deciding how to slice, the patterns they are noticing, and whether they can devise any creative slicing techniques to generate new shapes.

Reflect

What two-dimensional shapes have you not yet seen made by slicing? What kind of solid might you need to slice to get those shapes? How might you need to slice it?

Slice Hypotheses Sheet

I think if we slice it like this . . .	It will make a slice like this . . .

The Area of Slices

Snapshot

Students investigate the area of faces made by slicing different types of geometric solids.

Connection to CCSS
7.G.3

Agenda

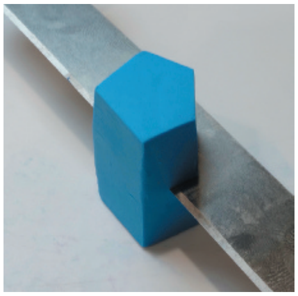
Activity	Time	Description/ Prompt	Materials
Launch	5–10 min	Remind students of their explorations of the shapes made by slicing solids. Pose the questions, How could you slice this solid so that the face that is made has the same area as the base? How could you slice it so that the shape has an area bigger than the area of the base?	Solid, either from solid set or made from clay
Explore	40+ min	Groups choose solids to explore, and develop conjectures about how to slice solids to yield shapes with the same area as the base, a larger area than the base, and the largest possible area. Students frame their conjectures as if-then statements and use grid paper to gather evidence about area.	• Make available: solid set • Clay and a cutting tool, for each group • Grid paper (see appendix), multiple sheets for each group
Discuss	20 min	Record students' conjectures on three charts, one for each question under investigation. Use the discussion to push students to be more precise about the conditions under which a conjecture might be true.	Charts and markers

Activity	Time	Description/ Prompt	Materials
Extend	20+ min	If the class is uncertain whether some conjectures are always true or true only under some conditions, send students back to construct new figures to test and refine these ideas. Discuss students' findings and revise your charts accordingly.	• Make available: solid set • Clay and a cutting tool, for each group • Grid paper (see appendix), multiple sheets for each group

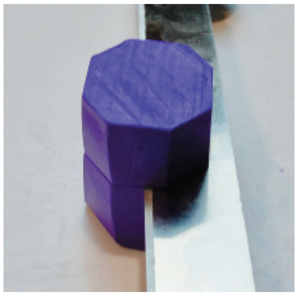
To the Teacher

This activity builds on students' explorations of slicing solids in the previous two activities in this big idea. Here we incorporate area as an element of the investigation and invite students to make conjectures about how to slice solids to get an area that is the same as or larger than the base and then the largest possible area. Before students can grapple with this idea, the class must be clear about which faces of a solid can be called a *base*. For prisms, there are two faces that could be called the base; these are the two opposite faces joined by rectangles. For a triangular prism, these bases are triangles; for cylinders, they are the circles. For pyramids and cones, the base is opposite the vertex where the greatest number of faces meet. Spheres have no base. If students are unfamiliar with identifying bases, you'll want to include in your launch of the activity some discussion of what constitutes a base for these different sorts of figures.

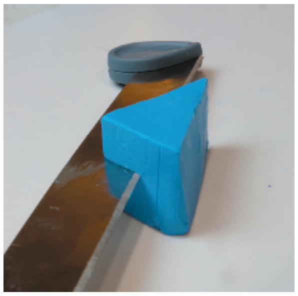
When students begin to develop conjectures about the ways to slice three-dimensional figures to get the desired area, prisms are a nice entry point because they can lead to some more generalizable conjectures, rather than conjectures specific to each solid. For instance, when prisms are sliced parallel to the base, the slice is congruent to the base.



A pentagonal prism is sliced parallel to the base.

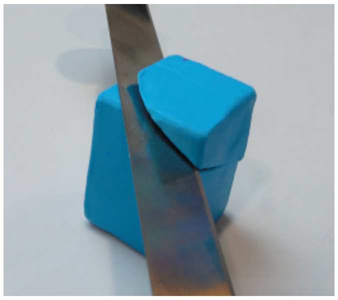


An octagonal prism is sliced parallel to the base.

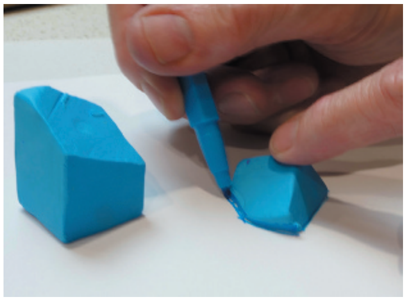


A triangular prism is sliced parallel to the base.

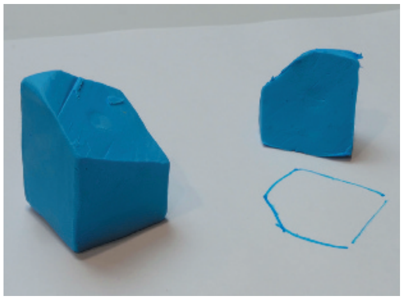
When prisms are sliced at an angle (that is, not parallel to the base) and through the side faces, the area of the slice is larger than the area of the base. Students will have likely built some intuition about this in the Visualize and Play activities.



A square prism is sliced at an angle through five faces.



The shape of the cut face is traced.



The shape of the cut face is a pentagon.

These are fairly straightforward conjectures that can be supported by investigation of multiple solids. But it is important to note that these conjectures only apply to prisms, and if students extend their investigation to include pyramids and cones, they will find different patterns. Identifying the largest possible area of a slice is also trickier and depends more on the individual shape, if students push toward precision.

Activity

Launch

Launch this activity by reminding students of the work they have done visualizing the shapes made by slicing solids, and that they have found that many different shapes are possible no matter what the solid is. Tell students that today they are going to investigate not just the shapes that can be made but the size of these slices.

Hold up a solid from your solid set, or a clay solid you have made. Pose the questions, How could you slice this solid so that the face that is made has the same area as the base? How could you slice it so that the shape has an area bigger than the base? How do you slice the solid for the greatest possible area of the slice? Give students a chance to turn and talk to begin to think about these questions and visualize slicing the solids. Be sure the class is clear about which face(s) in different types of solids are considered the base.

Explore

Provide groups with clay, a cutting tool, regular or isometric dot paper (see appendix), and grid paper (see appendix). Invite students to choose a solid from the class's geometric solid set to make for this investigation. They can make more than one kind of three-dimensional figure as they try to develop conjectures to address the following questions:

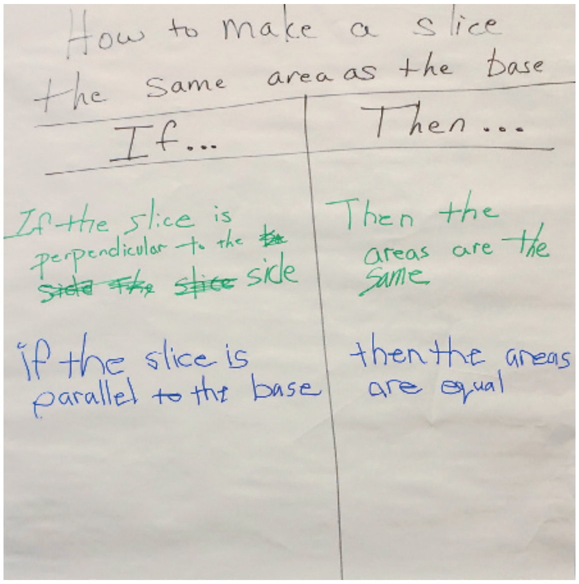
- How would you slice a solid so that the slice has the same area as one of its bases?
- How would you slice a solid so that the slice has a bigger area than the base?
- How would you slice a solid for the greatest area possible?

Encourage student to frame their conjectures in the form of "If . . ., then . . ." For instance: "If you slice a prism parallel to its base, then the slice is the same shape and area as the base."

Students will need to check the area of the slices they make, using grid paper (see appendix). The areas that students find will likely not be precise because of the movement of the clay, but the results should provide students with enough evidence to make conjectures.

Discuss

Set up three posters for students' conjectures: "How to make a slice with the same area as a base," "How to make a slice with an area larger than a base," and "How to make a slice with the largest possible area." Set up each as a T-chart, with If . . . and Then . . . headers.



Student conjectures written as if . . . then . . . statements.

Moving poster by poster, collect students' conjectures and ask the class whether they have evidence that the conjecture is always true, sometimes true, or not true.

Highlight any conjectures about which the class seems unsure. Push students to defend the conditions under which a conjecture might be true. For instance, a conjecture that might be true for all prisms may not be true for any pyramid. As the discussion enables students to become more precise about these conditions, modify the If . . . column to show when a conjecture is true. Use the discussion to come to agreement about students' conjectures.

Extend

You can send the class back to make additional solids to test any conjectures about which the class was unsure of the conditions where it might hold true. Reconvene to discuss these conjectures and what students found.

Look-Fors

- **How are students using the grid paper to find the area of sliced faces?** The shapes of the sliced faces are likely to be somewhat imprecise, and finding area will involve some strategy. Students may count the squares on the grid paper, but they will need to deal with the partial squares to estimate the total area of the face. Partial units pose a challenge because when sides cut across multiple squares, students will need a method for approximating how these pieces can be assembled into whole units. Students can be more precise if they make the edges of the face neater and place the face on grid paper intentionally so that at least one edge aligns with the grid. If students become frustrated trying to estimate area, you'll want to zero in on which issue is at the root of their struggle—the shape of the face and its position on the paper, or developing methods for counting partial squares. Ask questions such as, What makes estimating the area hard? What could you try to make this easier?
- **Are students testing their ideas across shapes to generalize?** As students test individual slices on specific three-dimensional figures, they will generate several findings about these cases. Encourage students to think beyond these case-by-case findings and consider what the findings might indicate about general conclusions. You might ask, What patterns are you noticing about slicing and area? When do you seem to get an area the same as the base? When do you seem to get an area larger than the base? Stepping into generalizations can be a difficult move because students need to use the evidence they have to infer what would happen in cases they haven't yet tried. You might support students by asking them to think about what the pattern *might* be, and then asking how they could test this pattern to see whether it holds true.

- **How are students thinking about conditions for their conjectures?** On the other end of the generalization spectrum, students may be inclined to see a small amount of evidence and make sweeping generalizations about how to slice all solids. Ask students questions to encourage them to think about the conditions under which their conjectures hold and when they might not. You might ask, Will this work for all solids? Are there any solids where this pattern might not hold true? You might encourage students to go look at the solid set with one of their conjectures in mind and try to visualize whether it will work across all shapes. Students may also need support in developing more precise language about their conditions. For instance, students may need to use language such as *parallel, perpendicular, edges, faces,* and *vertices* in addition to the names of all the solids to help them name precisely how to slice those solids to yield the results they want.

Reflect

What other questions do you have about slicing solids? How might you investigate them?