BIG IDEA 1

Partitioning Shapes into Equal Parts

If you have picked up this book, I am going to assume that you want to introduce your students to a mathematics that is beautiful and creative and that encourages them to explore and be curious. After many years of working in mathematics education, researching and teaching different groups of students, I have come to realize that there are two completely different versions of mathematics. For many people in the world, mathematics is a subject of rules and procedures. But the mathematics we will explore together with your students is one that is open, creative, and beautiful. When your students meet this mathematics, I predict that they will be engaged and inspired, and they will learn the mathematical ideas deeply.

In this big idea, we help students understand two concepts that they will use for many years in mathematics and that are often misunderstood by students. The first is the meaning of equality; the second is the meaning of a fraction.

In students' mathematical journeys, they will meet the idea of "equals" often and in many different ways; in this activity, we introduce the students to the idea of equality between shapes. Researchers have identified that many problems occur for students when they develop the idea that *equals* means "do something," a kind of instruction or operation (Powell, 2012). This comes about when students see expressions such as 4 + 5 = ? They think the equals sign is telling them to add the numbers. But the equals sign is not an instruction; it is a relational symbol, showing that the numbers or shapes on either side of the symbol are in an equal relationship. This is an important and big idea for students. In the Visualize activity, students will discuss the meaning of equals as they consider different shapes.

As students compare shapes, they will also get an opportunity to combat a common misconception in fraction learning. Many students believe—incorrectly—that for fractions to be equal, they have to have the same shape. This misconception comes about because students are so often shown fractions that have been partitioned into shapes that are equal in area, as well as congruent. But when we think about equal fractions, we only need to think about the area; the shape of the fraction is not relevant. In this activity, students will see that different shapes fill a half of an object, as they have the same area even when the shape is different. As students play with spatial puzzles, and think about their qualities, they will learn that shapes, like numbers, can be understood flexibly.

In the Visualize activity, students consider the question, What does it mean for two or more shapes to be equal? They also think about ways to partition shapes into equal parts. Students will look at squares and cut squares to decide and then explain how they know that parts are equal. They will begin to use the language of fractions, and they will be naming shapes as well and justifying how they know whether they are equal or not.

The Play activity builds off the Visualize activity as students work to partition squares into fourths in as many different ways as possible, folding and cutting. Students are encouraged to find equal areas that are not congruent. Students should play with the shapes, rotating and flipping them, realizing that they can transform shapes and make new ones.

In our Investigate activity, students partition a rectangle into rows and columns where the pieces of the partition are equal. This activity starts to build toward a visual representation of multiplication and should be connected to area. Students investigate how to partition rectangles into rows and columns of square units, which should be challenging, giving students lots of opportunity for brain strengthening. It will be helpful and interesting to collect the different strategies they use.

Jo Boaler

Reference

Powell S. R. (2012). Equations and the equal sign in elementary mathematics textbooks. *Elementary School Journal, 112*, 627–648. http://doi.org/10.1086/665009.

Equal or Same?

Snapshot

Students explore squares that have been partitioned into different shapes to develop ideas about what it means for parts of a whole to be equal.

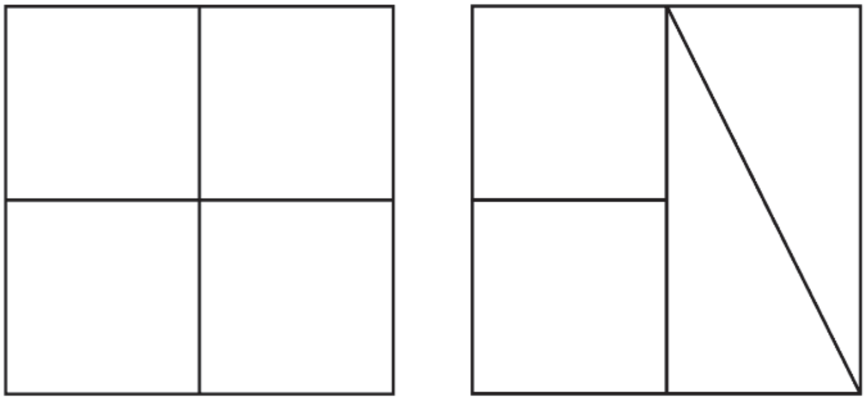
Connection to CCSS
2.G.3

Agenda

Activity	Time	Description/Prompt	Materials
Launch	5–10 min	Show students the Partitioned Square image and ask, Are the parts equal? How do you know? Surface different ideas about what it means to be equal.	Partitioned Square sheet, to display
Explore	20–25 min	Provide partners with the Partitioned Squares sheet. Partners explore the two squares shown to figure out how each is partitioned and whether the parts are equal. Partners develop ways of describing the parts of each square.	• Partitioned Squares sheet, per partnership • Make available: colors and scissors
Discuss	10–15 min	Discuss each square in turn, asking, Are the parts equal, and how do we know? As a class, develop ways of describing the parts. Ask, Do the parts have to be the same shape to be equal? Come to agreement that shapes can look different and still be equal portions of the whole.	Partitioned Squares sheet, to display
Extend	15–20 min	Partners explore how they could alter the square that does not have equal parts so that the parts become equal.	• Partitioned Squares sheet, per partnership • Ruler, per partnership

To the Teacher

One of the central ideas that students have been developing in first grade and now in second grade is what it means for two or more things to be equal. While students spend time grappling with this idea within equations, where the two sides of the equal sign must have the same value, they must also be thinking similarly about space. In this activity, we invite students to think about the ways that we can partition shapes into equal parts and what it means for those parts to be equal. Most often students encounter shapes that have been partitioned into shapes that are both equal (quarters, for example) *and* congruent (such as squares).



Left: Four congruent squares. *Right:* Four shapes with equal area where the squares are congruent and the triangles are congruent.

If students only see shapes partitioned as the figure on the left, they may conflate equivalence and congruence, assuming that shapes must look the same to be equal. But as the figure on the right shows and students will explore in this task, equal parts do not have to have the same shape. Instead, they must be the same portion of the whole or cover the same area. While area is not a concept that second graders work on explicitly, students can develop intuitive ideas about the ways that shapes are partitioned. For instance, it is enough to justify that the figure has been cut into two equal parts or halves and then that each half has been partitioned in half again, even though not in the same way. To press students to think about what it means for parts to be equal, we have provided two squares partitioned into noncongruent shapes. In one shape, the parts are all equal; in the other, not all parts are equal.

Activity

Launch

Launch the activity by showing the Partitioned Square image on the document camera. Tell students that this is a square that has been partitioned, or cut into parts. Ask, Are the parts equal? How do you know? Give students a chance to turn and talk to a partner about their ideas and reasoning. Invite students to share their thinking. Students do not need to come to agreement; in fact, it is more interesting when students begin to share ideas and disagreement surfaces. Focus the conversation on what it means to be equal: Do equal pieces have to have the same shape? This will be the focal question for this activity.

Explore

Provide partners with the Partitioned Squares sheet. Make available colors and scissors for students who want to color-code, fold, or cut the shapes to compare the parts. Partners explore the following questions about each square:

- How is the square partitioned?
- Are the parts equal? How do you know?
- How could you name each part?

Students may use fraction language, particularly the word *half*, when comparing, discussing, or naming the parts. Encourage students to explain what they mean when they use this language and how it relates to the way the square was partitioned.

Discuss

Use the following questions to discuss each square in turn:

- Are the parts equal? How do you know?
- What did you do to try to decide if the pieces are equal?
- How could you name each part?

Using students' ideas from discussing each of the two squares, ask, What does it mean for parts to be equal? Through this discussion, come to agreement that equal parts do not have to have the same shape; they just need to be equal size (or area) or be the same portion of the whole.

Extend

Provide students with the Partitioned Squares sheet and a ruler. Partners explore the following questions:

- How could you change the square that does not have equal parts so that it does?
- What line or lines would you move, and how?

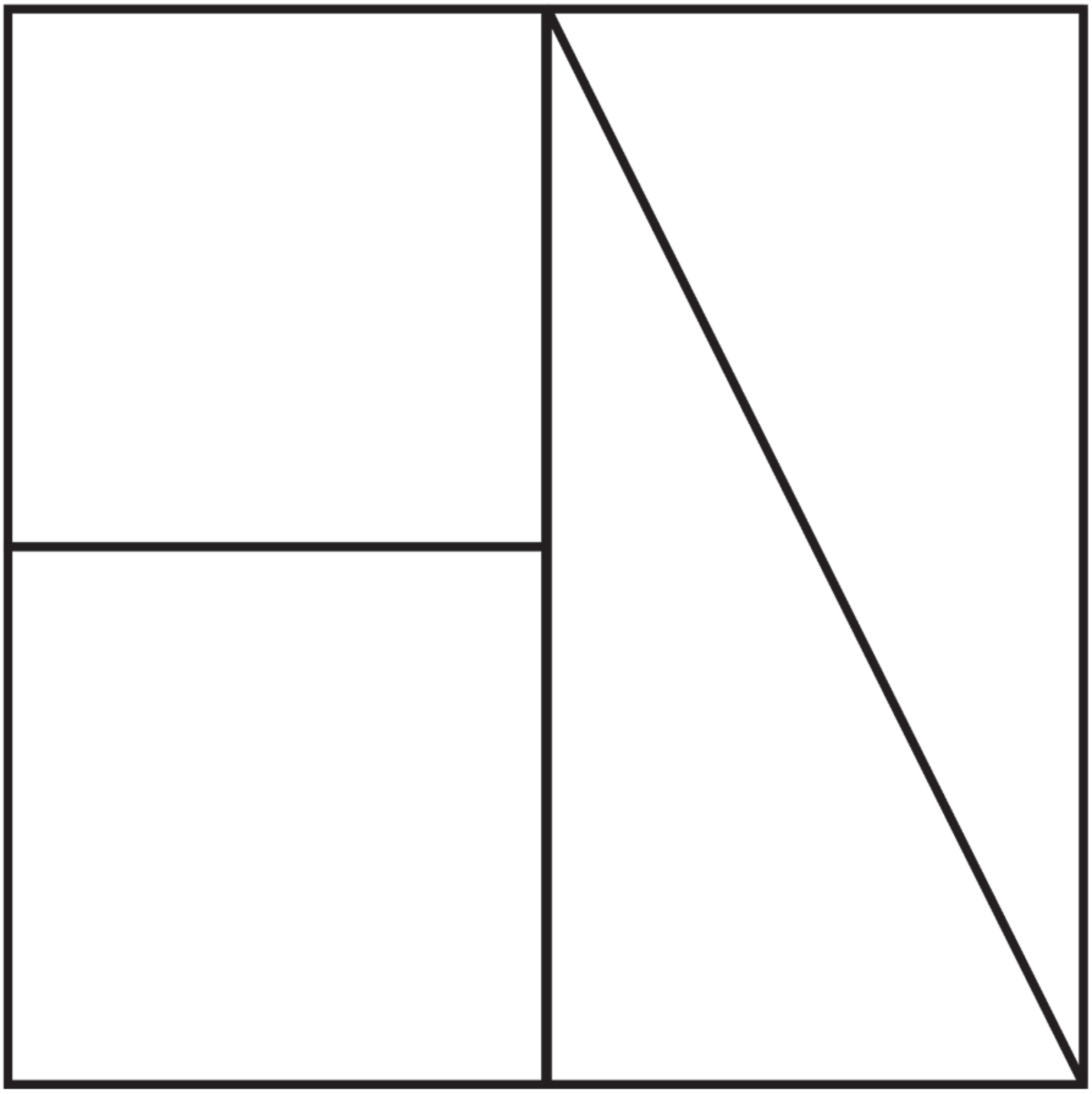
Look-Fors

- **Are students thinking about how the shape is partitioned and using ideas about half?** Before students ask whether the parts are equal, it can help if they think about how the shape was cut. What was the process? Where were the lines placed? Thinking about these questions can support students in thinking in terms of cutting in *half*, a concept that is directly tied to equal parts. If students are struggling to make sense of whether the parts are equal or if they disagree, ask them to go back to the figure and describe how they think the shape was cut, and in what order the lines were drawn. They may be supported by cutting out and folding the shape to see this process. If students use fraction language, press them to explain how they know that the figure was partitioned into halves—for example, how do they know that these were two *equal* parts rather than just two parts?
- **Are students manipulating the shapes through cutting or folding? Are they flipping, turning, or stacking pieces?** Reasoning about these two squares can be supported by physically manipulating each to see how the partitions were made and which pieces might be equal as a result. Some of the pieces are congruent. If students cut out the parts, they could rotate, flip, and stack them to see that they are congruent. This can be evidence of equal parts. However, if students compare noncongruent shapes, they will need to reason about the portion of the whole each part represents. Once students have cut or folded, encourage them to engage with the question of how to name the parts. If two parts have the same name, that, too, can be evidence of equivalence.

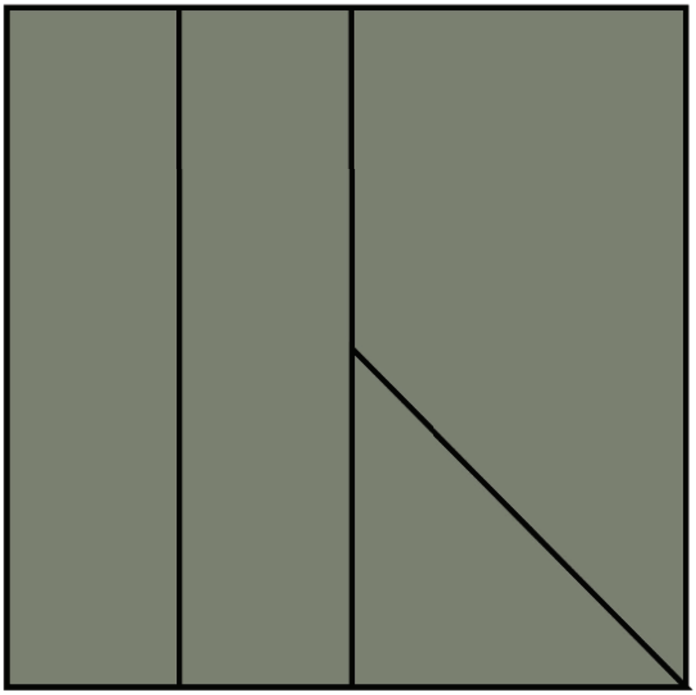
Reflect

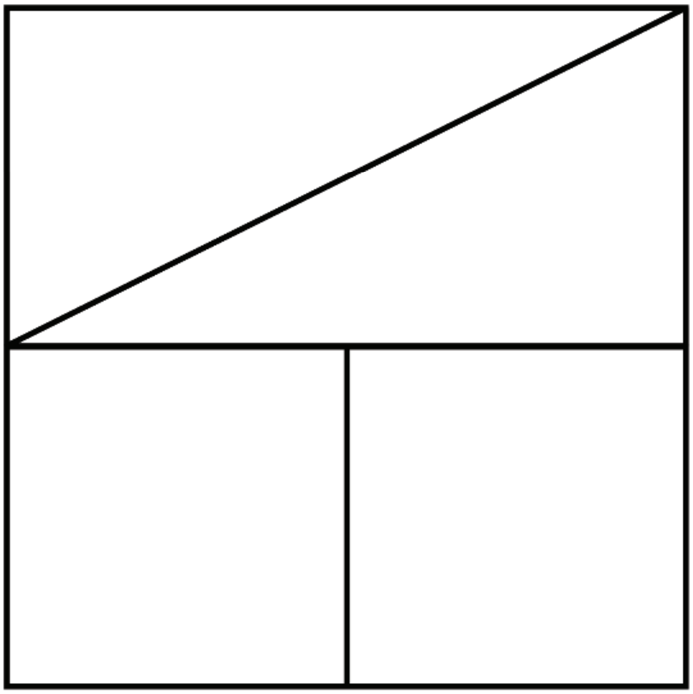
How do you know when the parts of a shape are equal or not equal?

Partitioned Square



Partitioned Squares





Four Fourths

Snapshot

Students generate as many ways as possible to partition a square into four equal, but not congruent, parts. Then they play with the wholes they can make from four fourths that are not squares, exploring the relationship between part and whole.

Connection to CCSS
2.G.3

Agenda

Activity	Time	Description/Prompt	Materials
Launch	5–10 min	Show students the Partitioned Square image from the Visualize activity and remind them that they found that parts can be equal without being the same shape. Show students a square piece of paper and ask, How could you cut or fold this square into fourths where the fourths are not all the same shape?	• Partitioned Square sheet, from the Visualize activity, to display • Square paper, such as origami paper
Play	25–30 min	Using equal-size pieces of square paper, partners generate as many ways as they can to partition the whole into four noncongruent parts. Then partners use the collection of fourths to construct new wholes that are not squares. Partners must decide how they know they have constructed a whole that is equivalent to the original square.	• Multiple equal pieces of square paper, such as origami paper, sticky notes, patty paper, or die-cut squares, per partnership • Scissors and tape, per partnership
Discuss	15 min	Discuss the ways that students developed to partition the whole square into four equal parts, and press students to justify how they knew these parts were equal. Then display the new wholes that partners created from their fourth parts. Discuss how students knew they had created a shape equivalent to the original square.	Space to display students' new wholes

To the Teacher

This activity builds directly on the work students did in the Visualize activity to distinguish between equal parts and congruent shapes. Here, we invite students to play with creating their own ways to partition squares into four equal, but not congruent, shapes, using the figure they explored in the Visualize activity as an example. For this activity to work, all students will need access to multiple copies of equal-size squares of paper. You can use origami paper or patty paper, or even sticky notes, though the stickiness can get in the way. Alternatively, if you have access to a die-cut machine, you can punch endless copies of the same square. Whatever resource you choose, you should have only one size square available to all students. This square becomes your class's *whole*.

In the second part of the exploration, students are invited to play with constructing new wholes using the fourth pieces they have created by partitioning the square paper. If all the parts they have created are fourths, then any shape they construct using four fourths is equivalent to a whole, or the original square piece of paper. This relationship between the part and the whole is the focus of this activity, and, because looks can be deceiving, students may struggle to make sense of how the new shapes they build out of fourths can be equivalent to the square when they appear so different. This is where discussion of students' reasoning is so critical; what they see may not be as helpful as what they know.

One key geometric idea embedded in this activity is transformations, or rotating and flipping shapes. Students typically do not encounter these as formal geometric concepts until much later in elementary school, but those concepts are built on a foundation of experience with moving pieces to see what happens. While some students get lots of experience with imagining what a shape will look like flipped or turned as they build with blocks or solve jigsaw puzzles, those who do not play in this way at home may find it much more challenging to envision the results of transformations. Although spatial relations are not the explicit focus of this lesson, in this activity students will get much-needed experience with the relationships involved in performing transformations.

Activity

Launch

Launch the activity by showing on the document camera the Partitioned Square image from the Visualize activity. Remind students that this square is partitioned into four equal parts, but those parts do not look the same. These parts are equal, but they do not all have the same shape. Show students a piece of square paper, such as origami paper. Ask, How could you cut or fold this square into fourths where the fourths are not all the same shape? Give students a chance to turn and talk, and invite students to share some of their ideas.

Play

Provide partners with multiple copies of the same square whole, such as origami paper, sticky notes, patty paper, or die-cut squares. Partners explore the following questions:

- How can you cut or fold each square into fourths where the fourths are not all the same shape?
- How do you know they are all fourths?

Partners work together to come up with as many different ways as they can to partition the square into four noncongruent fourths.

Provide partners with scissors and tape. Partners can, if they have not already, cut their squares into fourths so that they have individual pieces. Using their collection of fourths, partners explore the following questions:

- What new whole can you create that is equal to the original whole but not the same shape?
- How do you know it is one whole?

Students can tape their pieces together to make new wholes using the parts they created. As students work, ask questions that focus on how they know each piece is a fourth and how they know the new whole is equivalent to the original, namely because it is composed of four fourths.

Discuss

As a class, discuss the ways they approached partitioning the squares using the following questions:

- What strategies did you use or develop to partition the squares into fourths where the parts are not all the same shape?
- How did you know the parts were equal?

- What different ways of partitioning the square into fourths did you come up with?
- Do you think we found all the ways of partitioning this square into fourths that are not the same shape? Why or why not?

Display on a bulletin board or wall the many ways that students reconstructed a whole using their fourth parts. Discuss the following questions:

- What new wholes did you create?
- How did you know that the figures you created were one whole and not more or less?
- How did you transform the original whole into the new one? How could we draw or represent that change?

Look-Fors

- **Are students attending to making four equal parts, rather than simply four parts?** The first, most important task for students is to partition the whole square into four equal parts. This is a place to lean into the mathematical practice of attending to precision. While we don't ever expect perfection from students, precision is often needed to ensure that students have clarity on the concepts they are working with. In this case, there is a meaningful difference between four parts and four equal parts. Ask students, How do you know your parts are equal? If students have a conceptual reason, such as folding in half and then half again, then they are being precise, even if their folding is not perfectly even.
- **Are students drawing on the idea of fourth as half of a half?** We focus on fourths in this and the previous activity because one way of conceiving of a fourth is as half of a half. Halves are far easier to create with some degree of precision than thirds. However, you might see students trying to draw or fold each part as an individual region, which is unlikely to result in four equal parts. You might ask, What do you know about what a fourth is? How did we know when the four parts were equal (or not) in the Visualize activity? How can the meaning of fourth help you create fourths? Support students in connecting the meaning of fourth to the acts needed to create them.
- **Do students see that all of their fourths are equivalent?** Students might be convinced that the pieces within a whole—those cut from the same sheet

of square paper—are equivalent, because of how they constructed them. They may find it conceptually harder to see the fourths constructed from different (but same-size) pieces of square paper as equivalent. After students have had a chance to create and justify several different ways of partitioning squares, point to fourths that come from different squares and ask, Are these parts equal? Why or why not? Press students to think about the relationship between the part and the whole. If the whole is the same, then all fourths of that whole should also be equivalent.

- **Do students see that any four fourths can make a whole?** The final conceptual hurdle in this activity is to reconstruct a whole using the fourths as parts. Students may interpret this as simply using the pieces as building blocks to make a shape, regardless of size. Ask questions that focus students on the meaning of the term *whole* in this task, such as, What was the whole that you started with? Can you make another shape that is equal to that whole? How would you know whether your new shape was equal to the original square? The goal is for students to think of the whole as four fourths, such that any four fourth pieces can be used in any way to create a new whole.

Reflect

How do you know when two wholes are equal?

Rows and Columns

Snapshot

Students investigate how to partition rectangles into rows and columns of square units and what observations they can then make about partitioned rectangles.

Connection to CCSS
2.G.2., 2.OA.4

Agenda

Activity	Time	Description/Prompt	Materials
Launch	5–10 min	Show students first an empty rectangle and then the same rectangle partitioned into rows and columns of square units. Use students' observations of the different representations to point out that there is more information in the partitioned rectangle. Provide some language for how we describe these partitioned rectangles. Pose the question, How do you partition a rectangle into rows and columns of square units?	Rectangles sheet, to display
Explore	30+ min	Provide students with rectangles and materials for partitioning. Partners investigate how to partition rectangles into rows and columns of square units, what makes this challenging, and what strategies can be useful.	• Rectangles A, B, and C, per partnership, with access to additional copies • Make available materials for partitioning: spaghetti, pipe cleaners, rulers, square tiles, cubes, or scissors

Activity	Time	Description/Prompt	Materials
Discuss	15 min	Discuss the strategies that students developed for partitioning rectangles into rows and columns and how students knew whether these resulted in equal square units. Select one rectangle to focus on, and compare the different ways that students partitioned the same rectangle. Discuss why the number of rows, columns, and units may be different.	

To the Teacher

In this investigation, we turn from partitioning shapes into fractional parts to partitioning rectangles into rows and columns of square units. Here the square units are more central than the portion they represent. These are related, but somewhat different, ideas. Seeing rectangles as being composed of rows and columns of square units is a conceptual leap that opens the door to connecting shapes to measurement and multiplicative reasoning. Once a rectangle is partitioned into rows and columns of square units, we can name its dimensions, compare the dimensions of two different shapes, count the area, and begin to reason about the equal groups of squares in each row or column.

Students are often presented with rectangles on grids or otherwise partitioned for them. This is because partitioning into equal units is quite challenging. Ultimately, we want students to be able to use partitioned rectangles and draw rectangles using the support of partitioned spaces such as grid paper. But first, it can be useful for students to try to partition rectangles themselves so they can see what is involved and think through the challenges that arise. We suggest offering students a buffet of tools that they might try as they attempt to partition. Tools such as uncooked spaghetti or pipe cleaners can be laid down as moveable lines. Students may want to use rulers as lines or as tools for drawing. Some students may want to cut out the rectangle and fold it, as they may have done with shapes in previous activities. Alternatively, square objects, such as square tiles or cubes, could be arranged on top of the rectangle as the square units themselves. Each of the rectangles we have provided has dimensions that are a whole number of inches, so if you have 1″ square tiles, these should fit perfectly.

Partitioning these rectangles into equal and square units is an experiment. Students will and should make many mistakes, try many different materials and discard several, try multiple strategies, and find some pathways frustrating or dissatisfying. Do not swoop in to save students from this struggle. In the next big idea and beyond, students will see rectangles prepartitioned and will not have to do this work for themselves. This is their opportunity to develop an understanding of the mathematical work that goes into that partitioning.

Activity

Launch

Launch the activity by showing students on the document camera the blank rectangle at the top of the Rectangles sheet. Cover the second, partitioned rectangle shown below it. Ask, What can we say about this rectangle? Invite students to share their observations. They might point out that opposite sides are the same length or that there are two shorter sides and two longer sides, for instance.

Then show the image below the first, which depicts the same rectangle partitioned into rows and columns of square units. Tell students that it is the same rectangle. Ask, Now what can we say about this rectangle? Give students a chance to turn and talk to a partner, then discuss how the rectangle is partitioned into *rows* and *columns* of square units. Tell students that we can say that this rectangle is "3 units tall and 5 units wide" or that it is "three by five." Point out that there are many more observations we can make about rectangles when they are partitioned into rows and columns. Ask, How do you partition a rectangle into rows and columns of square units?

Explore

Provide students with access to Rectangles A, B, and C, and materials for exploring partitioning, such as spaghetti, pipe cleaners, rulers, square tiles, cubes, and scissors. Partners work together to explore the following questions:

- How can you partition a rectangle into rows and columns of square units?
- What materials or tools could help you?
- How do you know when the parts you have created are equal? How do you know when you have created square units?
- What makes partitioning into rows and columns challenging? What strategies help?

- What can you say about each rectangle once you have partitioned it? What observations can you make?
- Can you partition the same rectangle in different ways?

Encourage students to record their partitioning on the rectangle by drawing what they have done and to annotate the figure with their observations. If students use materials such as spaghetti or pipe cleaners, they may want to tape these down as a record of their partitioning.

Discuss

Discuss the following questions:

- What did you notice when you tried to partition these rectangles?
- What made partitioning into square units challenging?
- What strategies did you develop? Which worked, and which did not?
- How did you know when you did (or did not) have equal parts?

Choose a rectangle to discuss in detail. Invite students to share how they partitioned the rectangle and what they noticed about the rectangle once it was partitioned. Compare the different ways of partitioning the same rectangle by discussing the following questions:

- What different things do we learn about this rectangle when we see it partitioned in different ways?
- Why are the numbers of rows and columns different?

Be sure students notice that when the number of rows and columns goes up, the size of the units goes down. Throughout the discussion, be sure to use the language of arrays, such as *rows and columns*, *square units*, *dimensions*, and *partition*.

Look-Fors

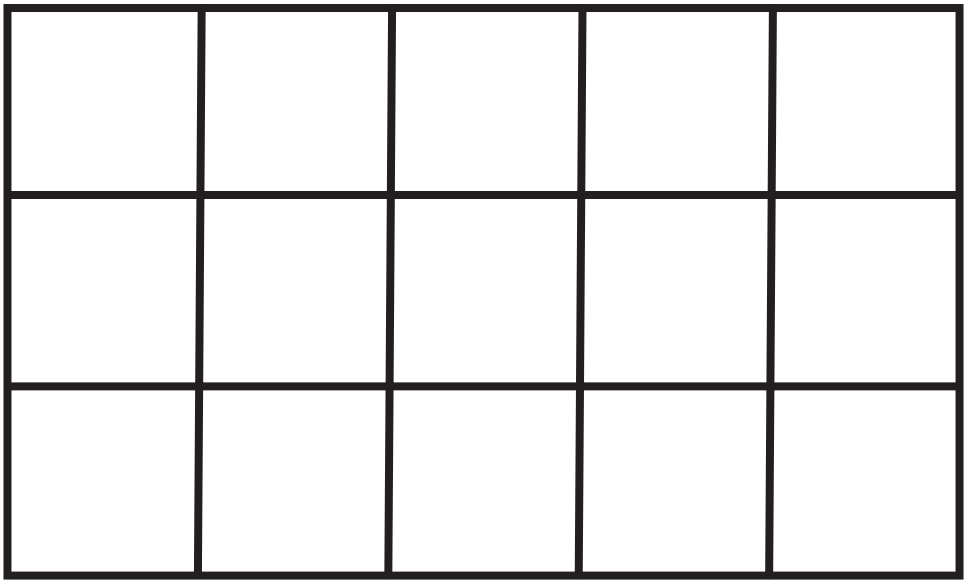
- **Are students frustrated by their efforts to partition?** Spaghetti rolls. Pipe cleaners get bent. Cubes do not always fit. Folding and refolding can lead to a mess of lines. Students may become frustrated by their efforts to partition a rectangle into equal square units. This is an opportunity to embrace struggle, promote perseverance, and point out what we learn from failures. Each time students try something that does not work, they have learned something valuable. Ask, What did you just learn? Focus students' attention on what they are learning from each attempt and what they can then try next. You might say, "So the spaghetti is rolling away, making it hard to get square units. What else could you try?"

- **Are students focused on creating both equal and square units?** Students could decide, for example, to fold a rectangle into four rows and four columns. This is something they are likely to be able to do with some precision and create equal units. However, these equal units may not be squares. In previous activities, students explored multiple equal shapes, but in this activity, we have shifted to equal-size square units. If students have not made that same shift, ask questions that focus their attention on all of the parameters of the task. Ask, Are your units equal? How do you know? Are your units square? How do you know? You can then point out the value in the work students have done, and what they now need to explore.
- **How are students making adjustments when their units do not fit or are not equal?** One challenge students will face during their experimentation is creating equal square units that do not fit into or fully fill the rectangle. For instance, students might partition one dimension into four rows, and then proceed to make these into squares, only to discover at the end that a little sliver is left in each row. Similarly, students might try to cover a rectangle with cubes and find that they overflow the boundaries. The key questions are, What does this tell you? and What adjustments can you make? This might mean creating more or fewer rows or columns, to make the square units larger or smaller. It might mean trying a different material for the same reason, or a different-size square tile or cube. Before students toss out their attempt, slow them down to reflect on what they learn from it. This promotes proportional reasoning about the space as students consider the consequences of making more or fewer rows, larger or smaller squares.

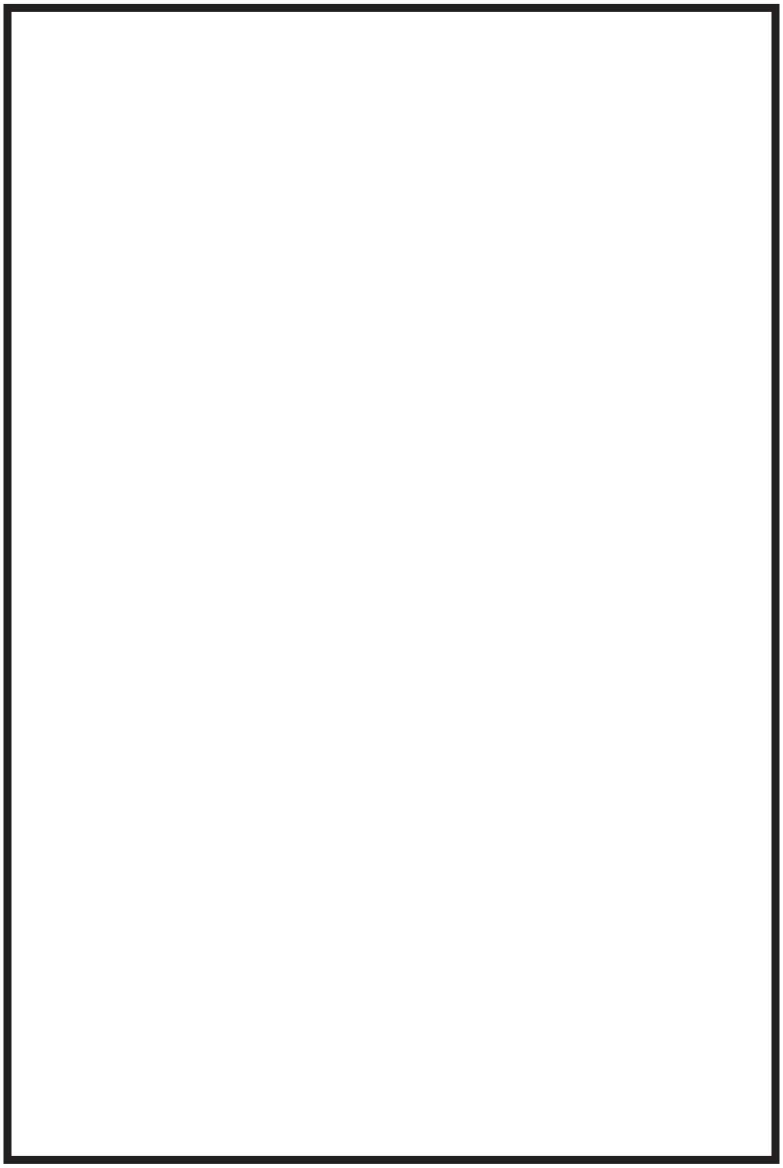
Reflect

When would partitioning a rectangle into rows and columns be useful?

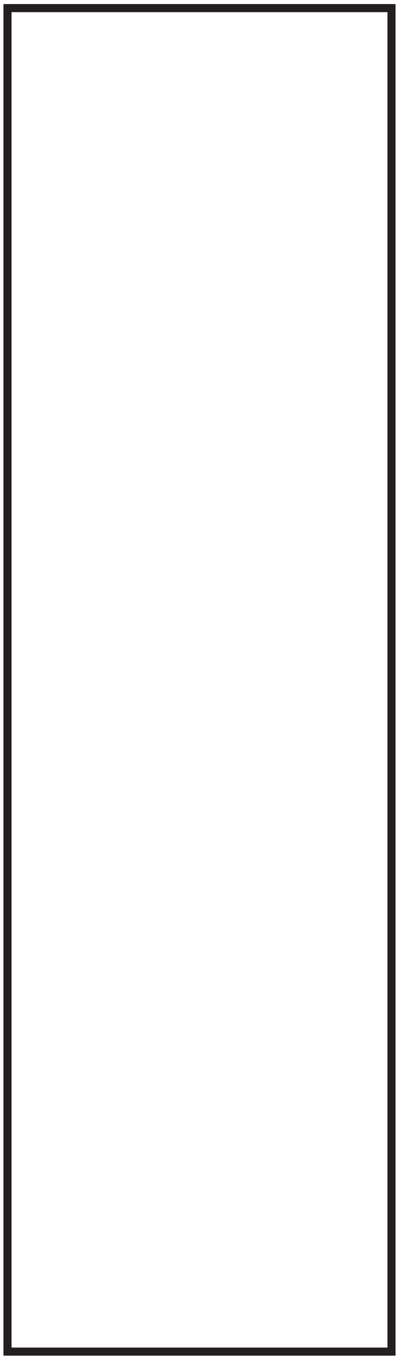
Rectangles



Rectangle A



Rectangle B



Rectangle C