BIG IDEA 1

Thinking in Cubes

The new brain science shows that five different pathways are involved when people think about mathematical ideas, and two of these are visual. When we make mathematics visual for students, we help them learn and hold ideas in powerful ways in their brains, as the introduction to this book explains. Similarly, we now know that movement really helps with mathematical ideas and is important for brain development. When students move with mathematics, it means that the mathematical ideas are held in the sensory-motor portions of the brain, which helps students understand the ideas powerfully. We see evidence of people holding mathematical ideas in these parts of their brains when they gesture to illustrate an idea; when people talk about circles, for example, they often draw a circle in the air. This big idea gives students opportunities to touch and feel mathematical ideas, and that is meaningful to students of any age.

In our Visualize activity, students will build with cubes and develop connections between two- and three-dimensional representations of solids. They will be asked to think about the outside and inside of cubes, which is important geometric thinking. As they physically model and also draw, they will build significant brain connections.

In the Play activity, students will construct cities of cubes that match views that we give them, again using brain pathways that will develop mathematical thinking. Students also will build their own cities, which will be engaging and exciting for them, enhancing the learning potential of the activities. As students think visually and also bring in numerical thinking, their brains will develop pathways between the areas that are used for these different types of thinking.

In our Investigate activity, students will again have the opportunity to feel cubes and consider their size physically and with numbers, encouraging brain connections. They also get to work with some constraints that will guide their thinking and learning. Students will be asked to investigate the volume of rectangular solids by packing little boxes into larger boxes of their own design. Any time that students are asked to bring their own ideas into mathematics, such as when they make their own designs, they are working with agency, which will help them enjoy mathematics and also see it as an active subject that they should think deeply about. When students work with agency, their work is closer to that of a mathematician, and inviting students to combine their own ideas with formal mathematical ideas is a really worthwhile goal. The activities that make up this big idea provide plenty of opportunities for students to combine their own thinking with major mathematical ideas and principles.

Jo Boaler

Solids, Inside and Out

Snapshot

Students build connections between two- and three-dimensional representations of solids by using views of a rectangular solid to construct a model with cubes. Students investigate what the inside looks like and compare results.

> Connection to CCSS
> 5.MD.3
> 5.MD.4

Agenda

Activity	Time	Description/Prompt	Materials
Launch	5 min	Show students the two-dimensional views of a rectangular solid constructed out of 60 cubes. Challenge students to build this solid.	Rectangular Solids Sheet, to display for the class
Explore	30 min	Partnerships try to build a rectangular solid from 60 cubes so that it matches the views provided. Students then consider what the inside looks like and figure out how to construct and draw a model of the cubes that cannot be seen.	• Rectangular Solids Sheets, one per partnership • Snap or multilink cubes for each partnership: 15 each of red, green, yellow, and blue • Drawing Solids Sheet, one per partnership
Discuss	20 min	Students compare their results and discuss how they used the views to construct the solid. Students discuss the differences between their models of the inside.	

(*Continued*)

Activity	Time	Description/Prompt	Materials
Extend	30–60 min	Partnerships construct their own rectangular solid puzzles and swap with other groups to solve.	• Drawing Solids Sheet, at least one per partnership • Cubes • Colors • Baskets, trays, or bags for students' puzzles

To the Teacher

Students often struggle when moving between two-dimensional representations of solids objects, like the ones shown in Figure 1.1, and three-dimensional representations. Two-dimensional drawings of solids force us to imagine the parts we cannot see, and students need experiences with mentally rotating and imagining these invisible parts. Similarly, we often encounter multiple views of three-dimensional objects aimed at helping us see what one view does not show. These offer a different challenge: constructing the three-dimensional whole from parts. In this visual activity, students are asked to move repeatedly between two-dimensional and three-dimensional representations to build connections and a deeper sense of what it means to be solid.

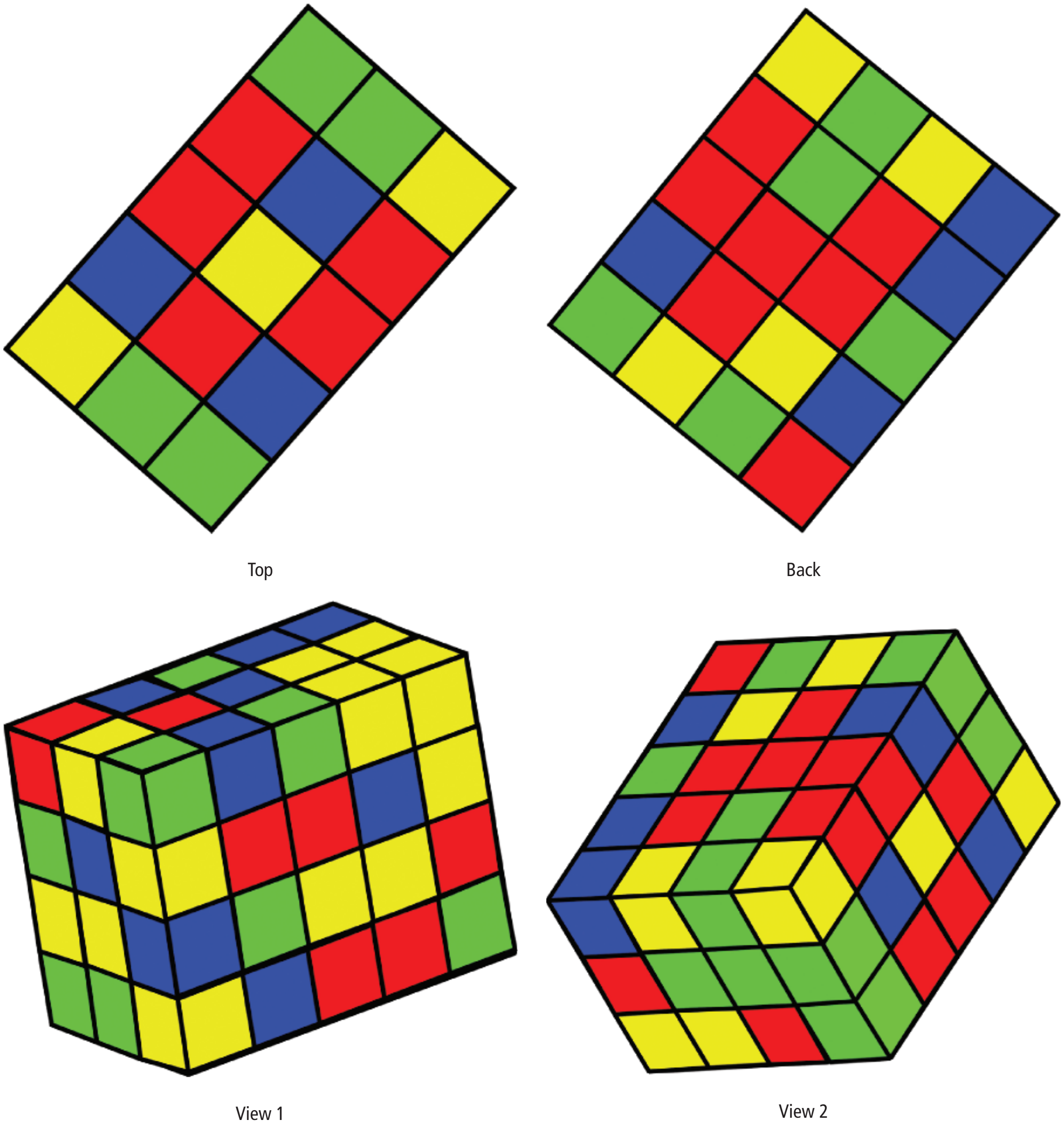


Figure 1.1 Four Views as Shown on the Rectangular Solids Sheet

The extension activity pushes these connections even further by asking students to create and draw their own solids and then try to build solids designed by others. This extension could easily take another full day or more, depending on how excited students are. They may want to construct multiple puzzles and solve many as well. We encourage you to follow students' interests and allow as much time as students want for constructing puzzles. The repeated engagement with building, drawing, and mentally manipulating the solids will help develop important ways of thinking about solids and volume for future work.

Activity

Launch

Launch this lesson by telling students that we are going to be exploring ways to fill space. In the past, they have likely spent a lot of time thinking about how to cover two-dimensional shapes with squares to find the area, but now we are going to think about how to fill three-dimensional solids with cubes. Show students the views provided on the Rectangular Solid Sheet shown in Figure 1.1. Tell students that all of these images are of the same rectangular solid, and each image is called a *view*. Tell students that the solid is made of 60 blocks: 15 are red, 15 are green, 15 are yellow, and 15 are blue. Their challenge today is to build this solid. Partners will each get a copy of the views and enough blocks to construct the solid. If you are not able to copy the Rectangular Solid Sheet in color for students, we suggest you display the images in color using a projector and provide students with the blank version, which they can color in themselves. Coloring in the individual cubes may also help students attend their arrangement on each face and support students with building the solid.

Explore

Have students work in partnerships. Provide each partnership with 15 red, 15 green, 15 yellow, and 15 blue snap or multilink cubes, and a copy of the Rectangular Solid Sheet. Partnerships use the views to try to construct the rectangular solid so that it matches the views.

After students have built the solid and both partners agree that it matches the views and uses all 60 cubes, then challenge students to figure out what cubes are on the inside. Ask, What cubes are we not able see? Can you make a model of what is inside your solid? Provide partners with additional cubes and a copy of the Drawing Solids Sheet. Ask them to draw an image of what is inside their solid once they have built their model.

Discuss

Bring students together with their models of the rectangular solid, models of the inside, and drawings of the inside. Pair partnerships up to share their two models and drawings, as in a turn and talk. Then bring all students together to discuss the following questions:

- How did you figure out how to build your solid? What did you have to think about?
- How did you use the images to help you?
- What was hard about building the rectangular solid so that it matched the views you were given?
- What did you notice when you compared your models to those others made?

Then discuss the models of the inside of the solid:

- What was inside your solid? How did you figure it out?
- How are the insides of our solids similar? Different? Why?

Extend

Invite students to create their own rectangular solid puzzle for another partnership to solve. Provide students with additional copies of the Drawing Solids Sheet, colors, and a container for their puzzle. Students should first use any number of their cubes to build a rectangular solid. They then use their solid to construct multiple views of it on a copy of the Drawing Solids Sheet. Push students to think carefully about which views they will show so that it is challenging but possible to build the solid with the views they provide. Once they are confident that their views are accurate, ask students to deconstruct their solid and place their views and cubes into a container (basket, tray, or bag). You may want to ask students to name their puzzle or put their names on it. Then students can trade puzzles with another group and try to build their solid. After students have spent some time exploring one another's puzzles, you may want to bring them back together to discuss questions like these:

- What makes a good puzzle?
- What made solving a rectangular solid puzzle easier or harder?
- What strategies for solving these puzzles have you and your partner come up with?

Look-Fors

- **Are students creating a solid?** Students may be tempted to construct only what they see in the images and presume that there is nothing inside the solid at all. Challenge students who have leftover cubes by asking: Is your solid constructed with 60 cubes? Why not? Does it match the views you were given? How can you make your solid meet all of the constraints of the problem?

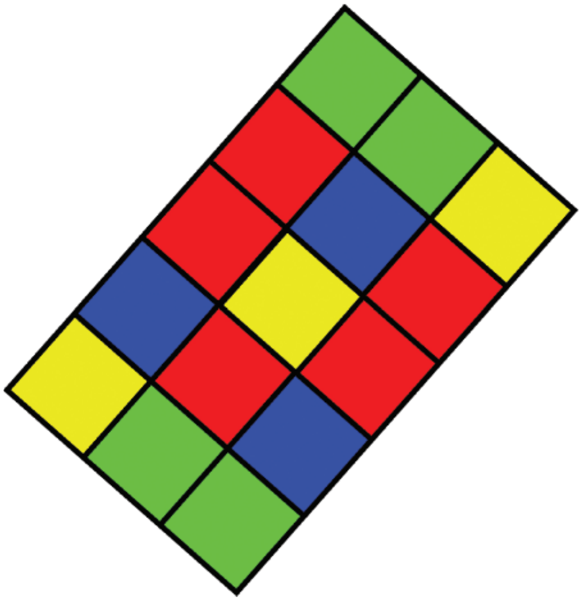
- **How are students mentally rotating the images?** Students may struggle to mentally manipulate the images provided to figure out how the pieces fit together. Often students don't realize that rotating the paper can help them see the images differently. Some students may be overwhelmed by the images and not know how to get started. Consider prompting students by asking: Is there one image that makes sense to you? How could you get started with that one image and then see how it fits with the others? What could you do with your paper to help you see how the pieces fit together?
- **How are students interpreting the images?** Drawing a three-dimensional object on two-dimensional paper is a challenge. If students have not yet had experience with isometric dot paper (see appendix), you may want to ask them to try drawing a single cube first, just to explore how the dot paper works. You might also ask them to think about the views they were provided and how they could use those as models.

Reflect

What surprised you about building rectangular solids from cubes and views?

Rectangular Solids Sheet

Use 15 yellow, 15 red, 15 blue, and 15 green cubes to build the rectangular solid.



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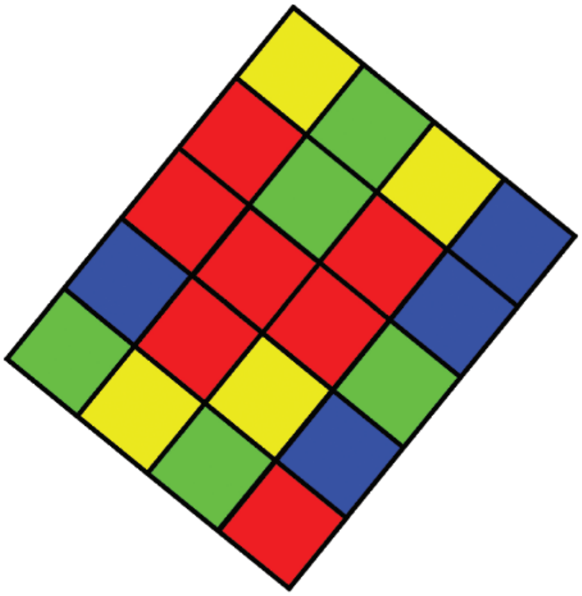

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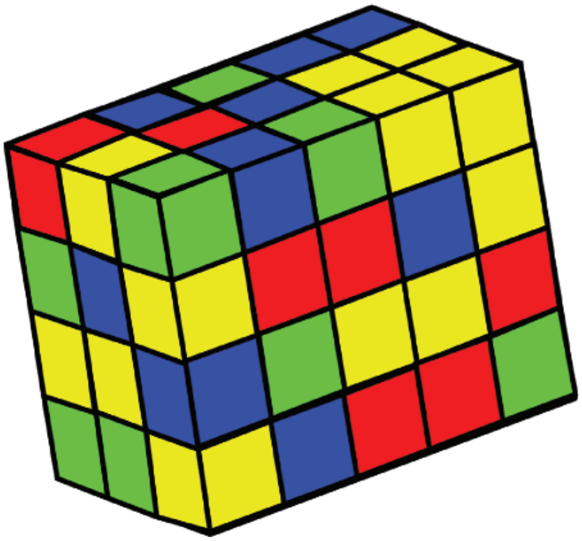
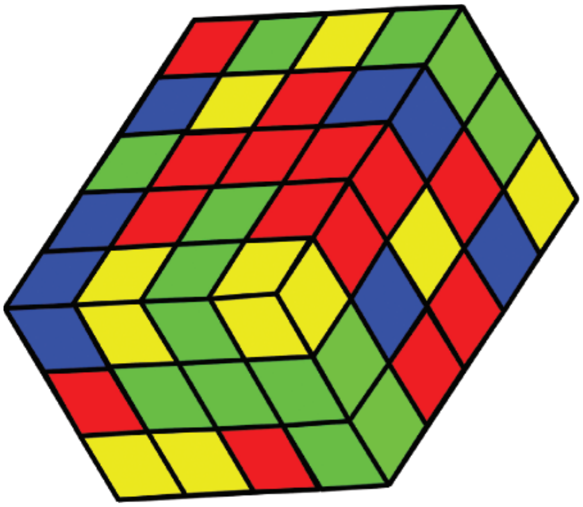
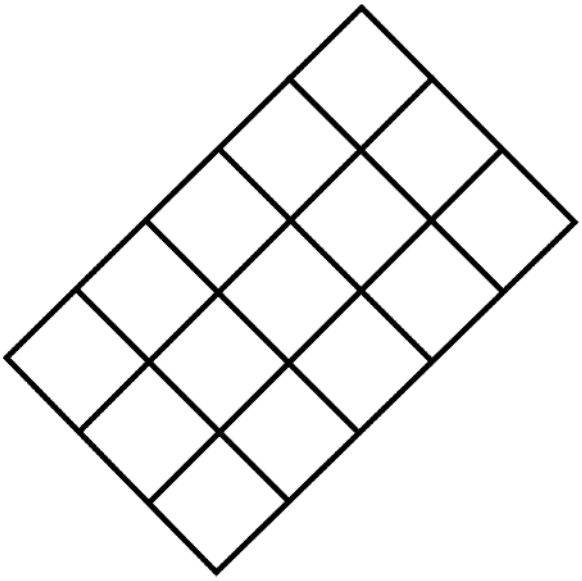


View 1



View 2

Drawing Solids Sheet



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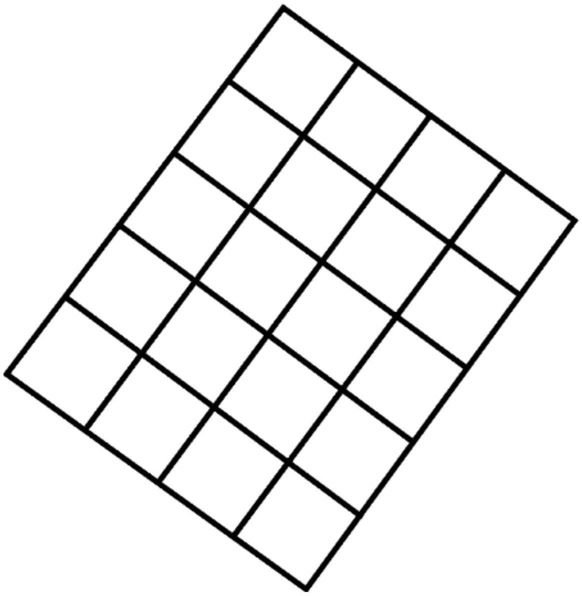

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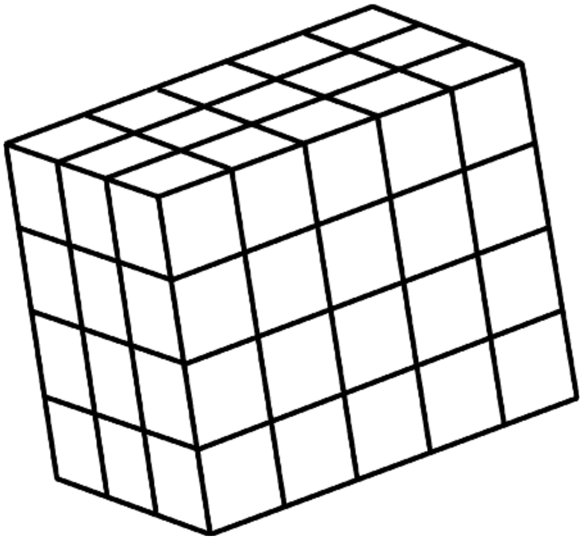
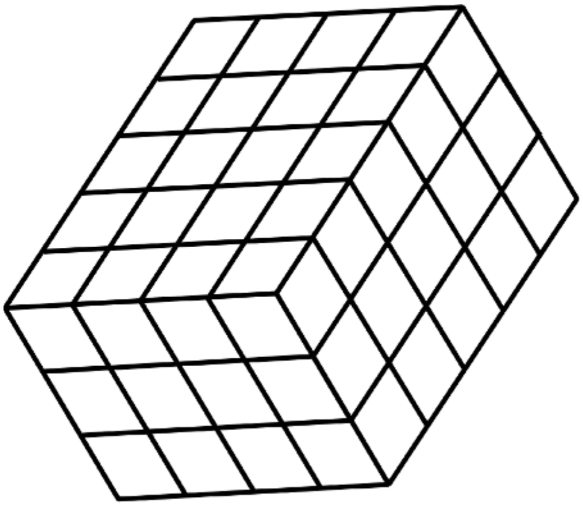


View 1



View 2

City of Cubes

Snapshot

Students use multiple views of block towers to construct cities of cubes that match those views. Students learn that the number of cubes used to build the city is its volume, and develop their own City of Cubes puzzles, which support students in learning how to record complex three-dimensional figures on paper and interpret those representations.

> Connection to CCSS
> 5.MD.3
> 5.MD.4,
> 5.MD.5c

Agenda

Activity	Time	Description/Prompt	Materials
Launch	5–10 min	Show students examples of architectural images and discuss what they communicate. Show students similar views from a City of Cubes puzzle and make sure students understand what is shown.	• Architectural Drawing sheets • Views from the first City of Cubes Puzzle sheet
Play	30+ min	Students work in partners to try to build the City of Cube puzzles from different sets of views. Students explore whether more than one city is possible and how many cubes it would take to build those cities.	• All three City of Cubes Puzzle sheets, copied for partners to choose • Snap, multilink, or unifix cubes in multiple colors, for all partnerships • Isometric dot and grid paper (see appendix)
Discuss	15 min	Discuss the strategies students used for solving the puzzles and the multiple possible solutions they developed.	• Student-built City of Cubes puzzles

(*Continued*)

Activity	Time	Description/Prompt	Materials
Play	30+ min	Partners work together to design their own City of Cubes puzzles. Then partnerships swap puzzles, solve each other's, share solutions, and offer feedback for improving the puzzles.	• Snap, multilink, or unifix cubes in multiple colors, for all partnerships • Isometric dot and grid paper (see appendix) • Colors
Discuss	15 min	Discuss the challenges of creating these puzzles and what students learned from the cycles of solving, sharing, and giving and receiving feedback.	

To the Teacher

The City of Cubes views we have provided use multiple colors to support students in making sense of the images. Each tower uses a single color of cubes. You may not have the same color cubes as indicated in our drawings. Provide students with whatever colors you have available and tell students that they can represent the towers in whatever colors they choose. Changing the colors, though, will likely make moving between the views and their own work a bit more difficult. You may want to encourage students to jot down on the views which colors they are using for each tower to help them track their own work.

In the second half of this lesson, we give students the opportunity to create their own puzzles. Recording a three-dimensional figure on paper is quite different from interpreting a set of drawings to build a figure. Students need opportunities to engage with both activities, and each will challenge students in different ways. Remind students how important it is to feel challenged and to work on mathematical ideas that allow them to struggle. We anticipate that this lesson will take more than one day. After the lesson, you may also want to incorporate students' puzzles into stations or games for indoor recess.

Activity

Launch

Launch the lesson by showing students some images of architectural drawings that architects produce to communicate what the buildings they design might look like (Figure 1.2). They often create floor plans that show what each floor looks like from above, and they make views of the outside so that you can see how buildings look in relationship to each other.

Front and Side View

Source: Shutterstock.com/paparoma

Aerial View

Source: Shutterstock.com/bioraven

Figure 1.2 Architectural Drawings

Tell students that today they are going to use some images of cube cities to try to build what is shown. Show students an example of the views from the first City of Cubes puzzle. Show students the aerial view and be sure they understand it as how the city might look from a helicopter looking straight down. Show students side views and be sure they understand these as what they would see standing on the outside looking at the city. You may want to give students a moment to turn and talk to a partner about what they notice in these images, then collect some observations. Ask, Can you build this city using these views?

Play

Invite partners to choose one of the three City of Cubes puzzles to try to build first. Students should have access to colored cubes to build each city. For each city they build, students try to figure out the following:

- How many cubes could it take to build this city?
- Is there more than one way to build a city that matches the views given?
- If there is more than one way, what is the fewest number of cubes it could take to build it? What is the greatest number of cubes you could use to build it?

Students should document their findings, either by saving the models they construct (if you have enough cubes) or by devising a way of recording on paper the possibilities they create. Provide isometric dot paper or grid paper (see appendix) for those who would like these tools.

Discuss

When students have had a chance to fully explore at least one of these puzzles, gather students together with their evidence (models or drawings) to discuss the following questions:

- How did you tackle building the cities? What strategies were useful? How did you use the colors?
- How many cubes did it take to build each city? How do you know?
- Which cities can be built in more than one way? What is the fewest number of cubes it could take to build the city? What is the greatest? How do you know?

During this discussion, name the number of cubes needed to construct the city as its *volume.* At the close of the discussion, challenge students to create their own City of Cubes puzzles.

Play

Partners work to create their own City of Cubes puzzle(s). Encourage students to build their cities and then use their buildings to create images on grid and/or dot paper (see appendix) that they could give to others as a puzzle. Remind students how colors help them make sense of the images, and ask students to use color in their puzzle drawings. Students should consider what views they want to provide and whether they want their puzzle to have multiple solutions.

When partnerships have finished creating a puzzle, they can swap with other partners and try their puzzles. For each puzzle students try, ask them to consider the following:

- What could the volume be?
- Are there multiple possible cities that match the views you were given?
- Do you have suggestions for the puzzle creators for how to make the puzzle clearer or more interesting?

After students swap, they can return the puzzle to its creators, share solutions, and provide some feedback on how to make the puzzle clearer or more interesting. Swapping, solving, and sharing rounds can be repeated as long as students are engaged.

Discuss

After the class has had a chance to create, swap, and share results from their puzzles, gather students together for one final discussion:

- What was challenging about making your own? What was challenging about creating the drawings? How did you address these challenges?
- What makes a city hard to build? What images make it easier to build?
- What strategies did you use?
- How did you figure out the volume of the city?
- What feedback did you receive that was most useful for improving your puzzle? Why was it helpful?

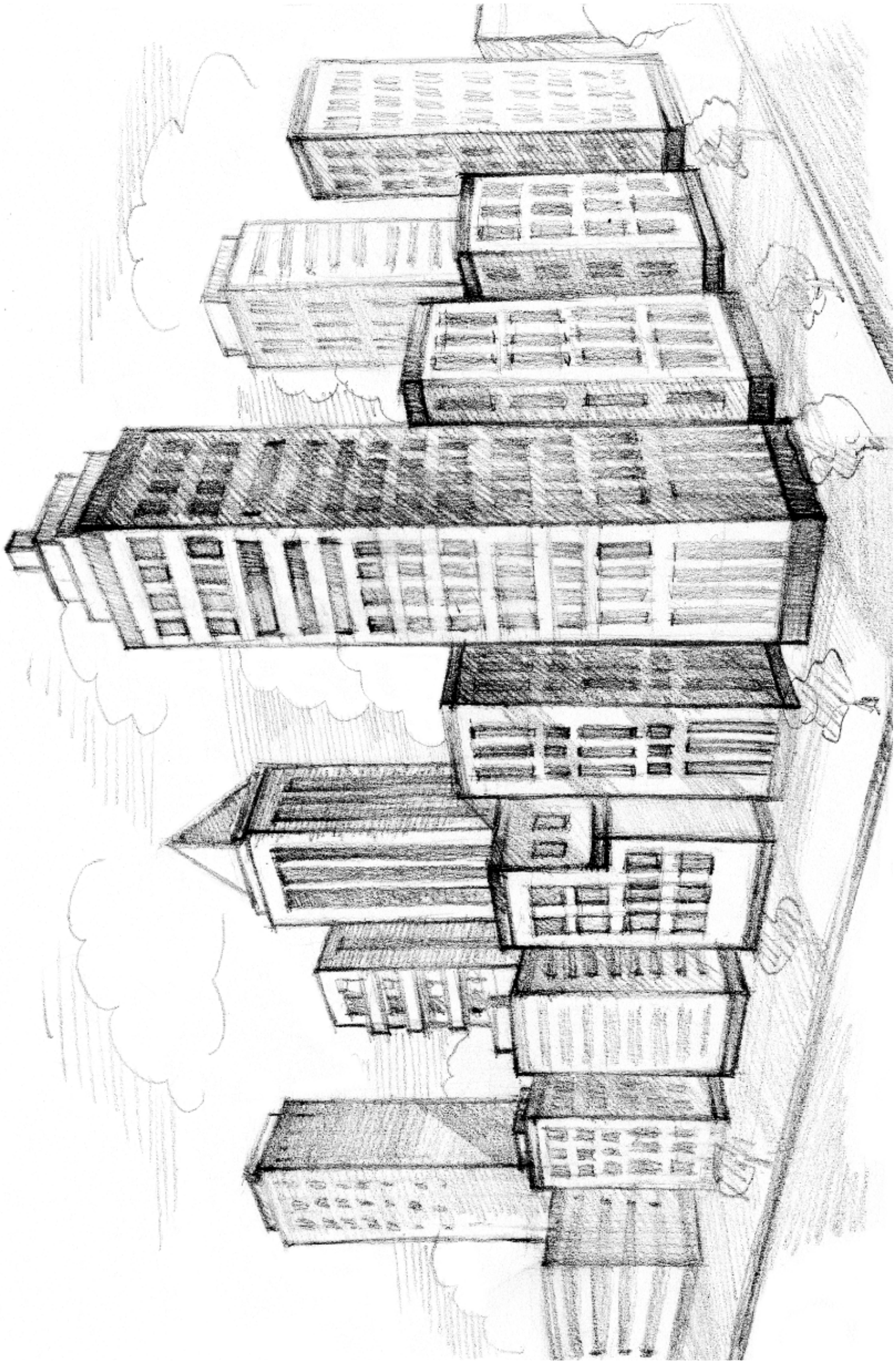
Look-Fors

- **How are students interpreting the two-dimensional representations of three-dimensional space?** Students may struggle with perceiving the depth in the images and interpreting different towers as in front or behind one another. Some students may struggle with the aerial views. Remind them how important struggle is for brain growth. You may want to prompt students to put their model on the floor and hover over it to understand the aerial view and compare their work to the puzzle. For students struggling with the side views, you may want to compare these to the architectural drawings and help students notice what they can and cannot see, and how their minds fill in the towers hidden behind. Then have them move back to the puzzle drawings and try to interpret these with the same thinking.
- **How are students recording their work?** For students who are considering how to record the various cities they built to match a puzzle, ask them what makes their cities different. The differences will likely center on how many cubes are in each tower and the positions of the towers. You might ask: How could you capture these differences? Which view(s) could help you show what makes them different? How could labels help? For students creating their own puzzles, guide them to use the view provided in the puzzles as models for how they might record their own cities. Ask, How did these views show the city? What views would help another person build your city? Students might also benefit from encouragement to focus on a single view and talk about what they see before they try to draw. Finally, you might also encourage students to go see how other partnerships are recording their thinking and learn from each other.
- **Are students seeing more than one way to construct the city?** Students might see the first way they construct as the only way the puzzle could work. Once they have built one way that they can prove matches the views they were given, encourage students to ask themselves, What could we change about this city and still have it match the views? Encourage students to try altering the city and checking it against the views to figure out what kinds of changes work.

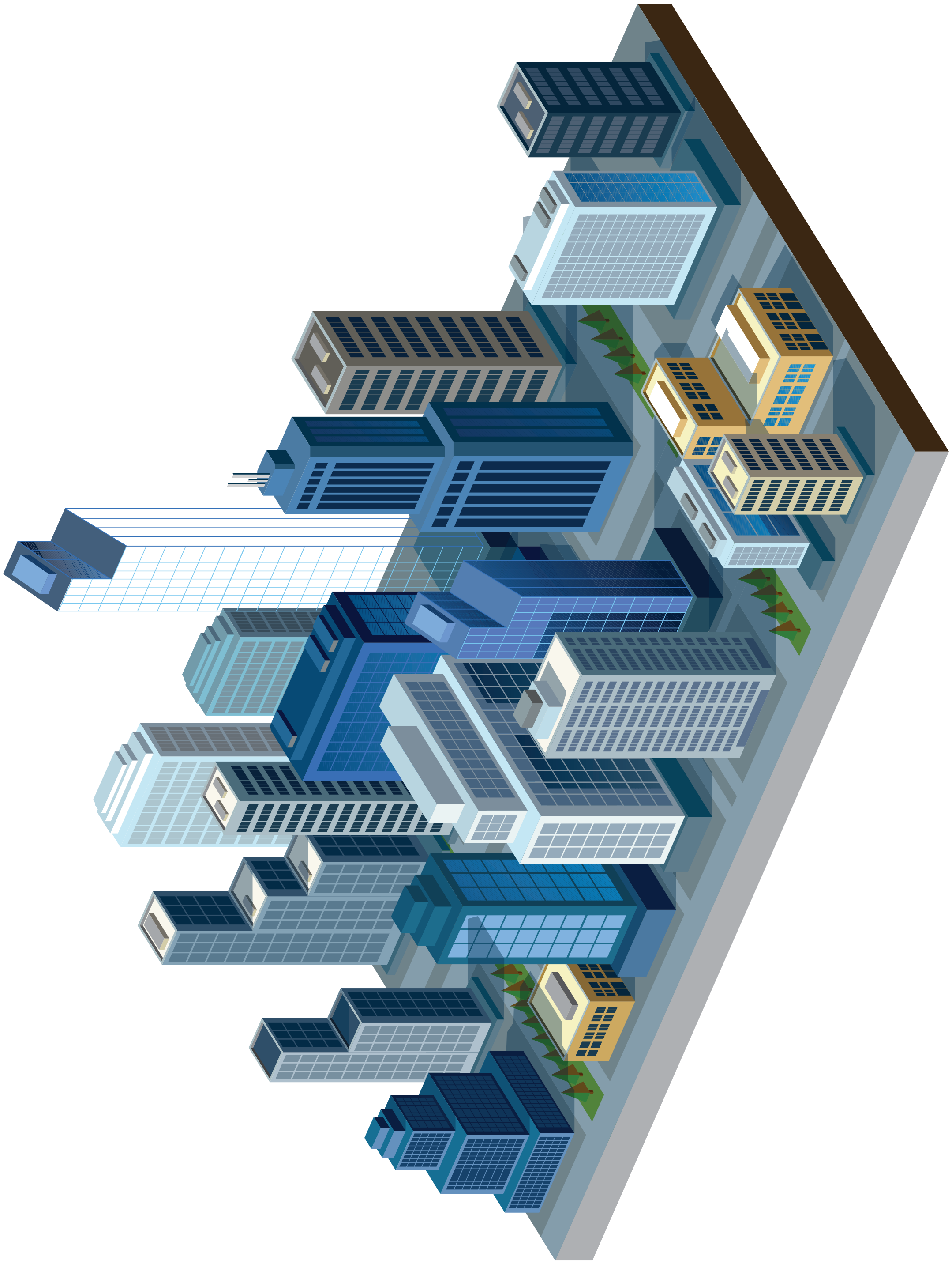
Reflect

What is volume?

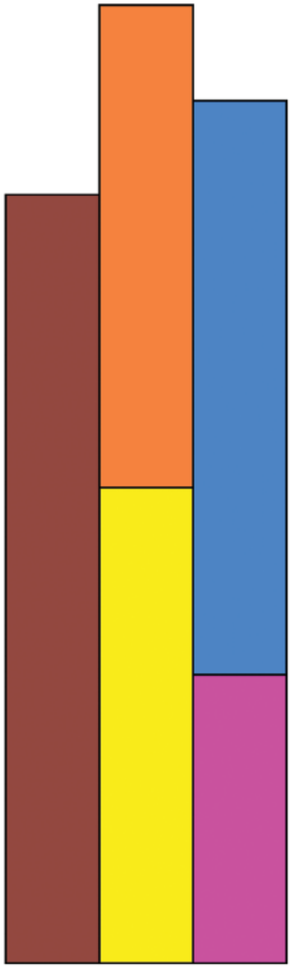
Architectural Drawing—Front and Side View



Architectural Drawing—Aerial View



City of Cubes Puzzle



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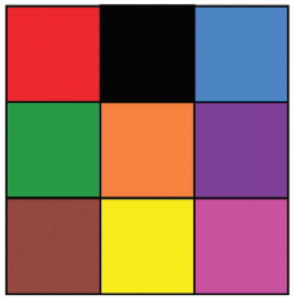
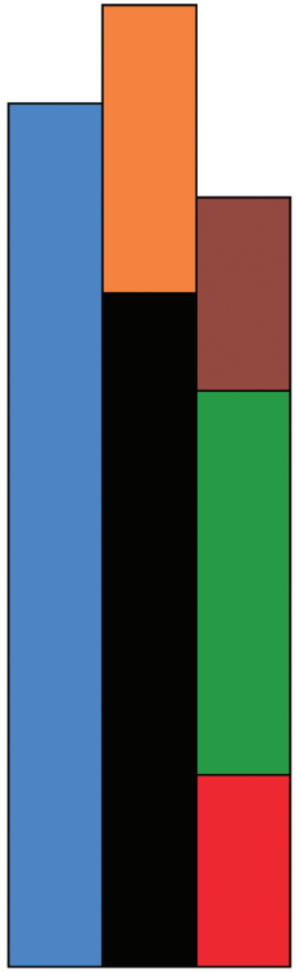


Aerial View



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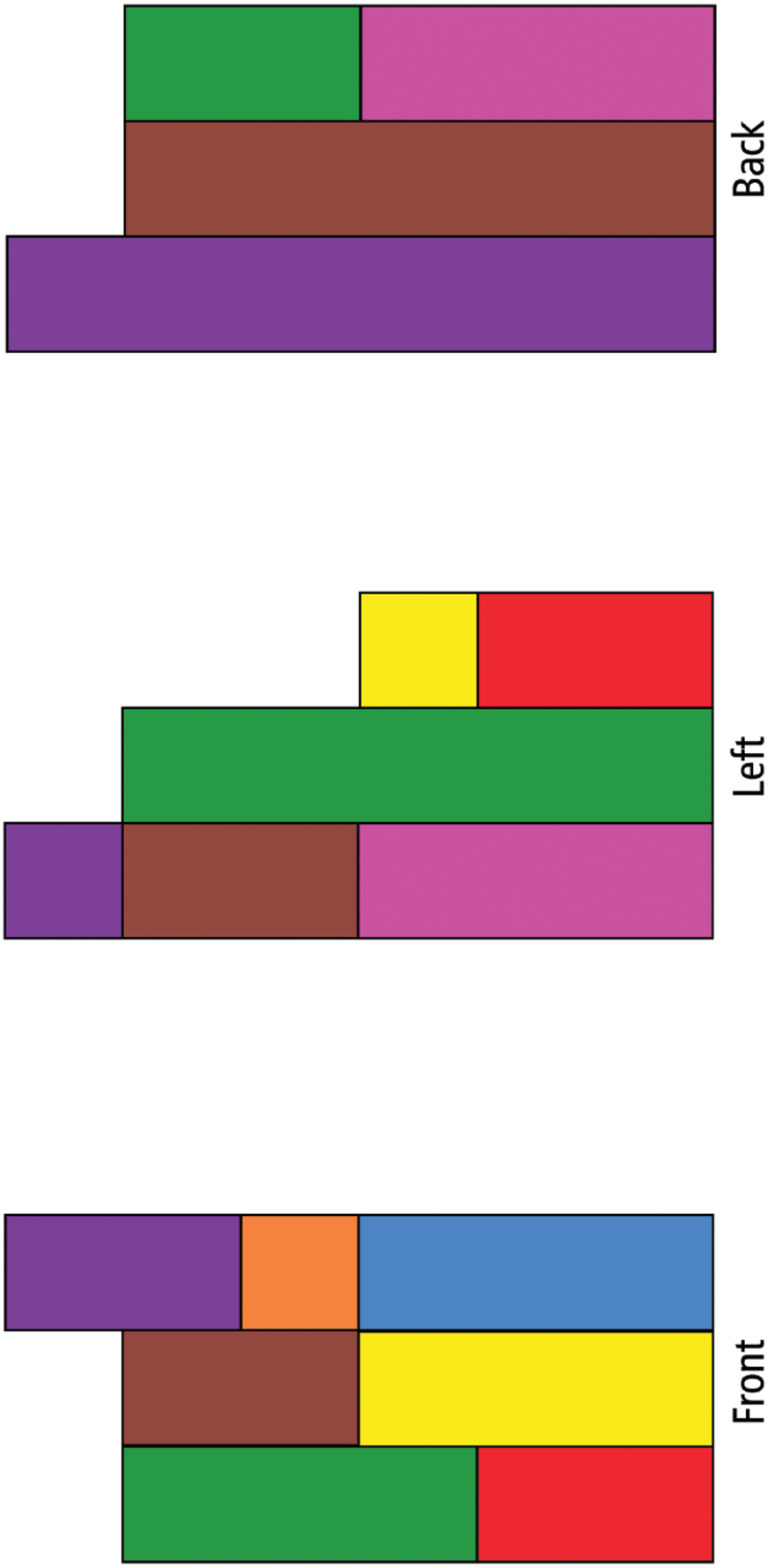
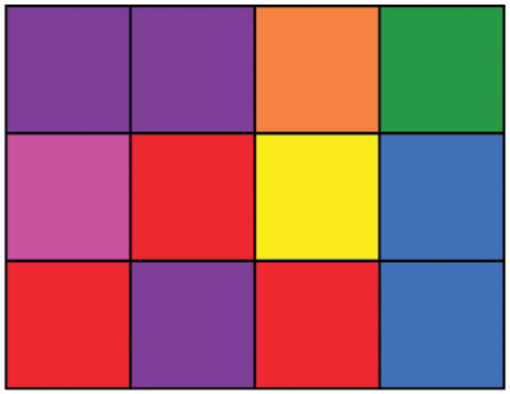
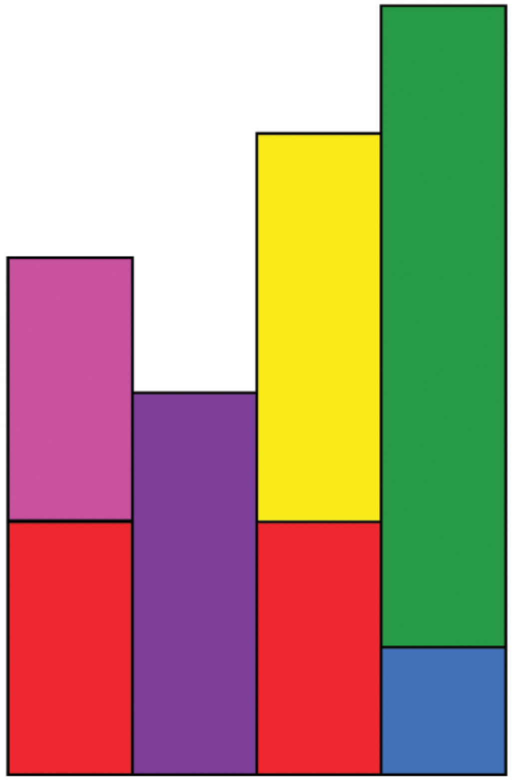
City of Cubes Puzzle



City of Cubes Puzzle

Each color represents a building.
Builds of the same color are the same height.





A Box of Boxes

Snapshot

Students investigate the volume of rectangular solids by packing little boxes into larger boxes of their own design, while trying to minimize the volume of empty space.

Connection to CCSS
5.MD.3
5.MD.4
5.MD.5

Agenda

Activity	Time	Description/Prompt	Materials
Launch	10 min	Show students some small boxes and discuss what happens when you pack them into a larger box to mail. Show students images of four small boxes and challenge them to design a box to hold them all.	• Examples of boxes for packing • Little Boxes sheet, to show class
Explore	30 min	Partners work to design boxes to hold a set of small boxes that they can construct from cubes: 3 × 1 × 1, 2 × 2 × 2, 3 × 2 × 2, and 3 × 2 × 1. Partners try to create a box with the smallest volume of empty space.	• Snap or multilink cubes, 30 per partnership • Little Boxes sheet, for each partnership • Isometric dot or grid paper (see appendix), for partners to record (See Play lesson) • Optional: colors
Discuss	15 min	Discuss the solutions students discovered and how to minimize empty space. Record students' solutions in a table to focus attention on dimensions and volume.	Chart and markers

Activity	Time	Description/Prompt	Materials
Explore	30+ min	Partners investigate minimizing empty space by changing the parameter of the problem. They can remove a little box, add a new little box, or create an entirely new set of little boxes. They try to design packing boxes for their sets that minimize empty space. Some students may investigate the opposite: what set of boxes requires a lot of empty space.	• Snap or multilink cubes, at least 30 per partnership • Little Boxes sheet, for each partnership • Box of Boxes Recording Sheet, at least one per partnership
Discuss	15+ min	Students share their findings from their various investigations, and the class discusses how thinking about volume helps in designing boxes.	Students' work

To the Teacher

The idea of packing boxes inside of a larger box may be familiar to some students, but it can be difficult to discuss with clarity. We encourage you to launch the lesson by showing some smaller boxes, and perhaps a larger packing box, so that students can clearly see the idea. You may want to ask students when they have seen this before—such as getting a package in the mail or packing a lunchbox—to help them make connections. You may want to show students what happens when a box is too large for its contents to help motivate them to create a box with the smallest volume of empty space.

The second half of the activity includes four avenues for investigation: removing a little box, adding a new little box, creating a set of boxes with no empty space, and creating a set of boxes that requires a lot of empty space. Given the range of choices, the class may find that this investigation stretches across multiple days. Students may want to spend one day exploring one of these questions and then try a new question the following day. The longer that students investigate, the more connections and patterns they will discover that support them in reasoning about volume and lead them toward multiplicative thinking.

Activity

Launch

Launch the investigation by showing the class a collection of boxes. You might use some boxes from around your classroom or school, such as boxes for tissues, paper clips, games, snacks, file folders, pencils, or markers. Make sure your collection includes boxes of different sizes. Ask students to imagine that you need to pack these different boxes into a larger box to send in the mail. When you mail a package, you want to make sure there is as little empty space as possible, to prevent the objects inside from rattling around, to conserve paper, and to protect the box from getting squashed. How would you choose the right size box? What is the smallest box you could fit all of these little boxes into?

Tell students that for today's investigation, they will be trying to find the smallest box that can hold a collection of smaller boxes. Show students the images of the four boxes they will be trying to fit together into a larger box: $3 \times 1 \times 1$, $2 \times 2 \times 2$, $3 \times 2 \times 2$, and $3 \times 2 \times 1$. Tell students that we name rectangular solids like these by their dimensions and that we say that a box measures "3 by 2 by 1" cubes. Point out these dimensions in the drawings and be sure students understand the language used.

The goal is for students to design a box that will hold all four and have the least amount of empty space when packed. The box must be a rectangular solid, as typical packing boxes are.

Explore

Provide students with snap or multilink cubes, a copy of the Little Boxes sheet, and isometric dot or grid paper (see appendix) to record their thinking. You may also want to provide colors for students to color-code the different little boxes in their drawings. Students work in partnerships to design a box that can hold the following smaller boxes: $3 \times 1 \times 1$, $2 \times 2 \times 2$, $3 \times 2 \times 2$, and $3 \times 2 \times 1$. The large box must be a rectangular solid. As students work, they should investigate the following questions:

- What is the smallest box you can make that will hold all of these boxes?
- What are the dimensions of your box? What is its volume? How much empty space will be left in your box?

Partners record their different solutions on isometric dot or grid paper (see appendix) as they try to find the smallest box with the least amount of empty space. For each solution, students record the dimensions of the packing box, the volume, and the empty space. This will help them compare their different solutions.

Discuss

When students have had a chance to investigate and come up with some possible solutions, gather them together to discuss the questions here. Record students' solutions on a chart showing the dimensions of each box, its volume, and the empty space remaining (Figure 1.3).

- What boxes did you create?
- How did you find the volume of the boxes you created?
- How did you figure out the amount of empty space in each of your packing boxes? What box leaves the least amount of empty space?

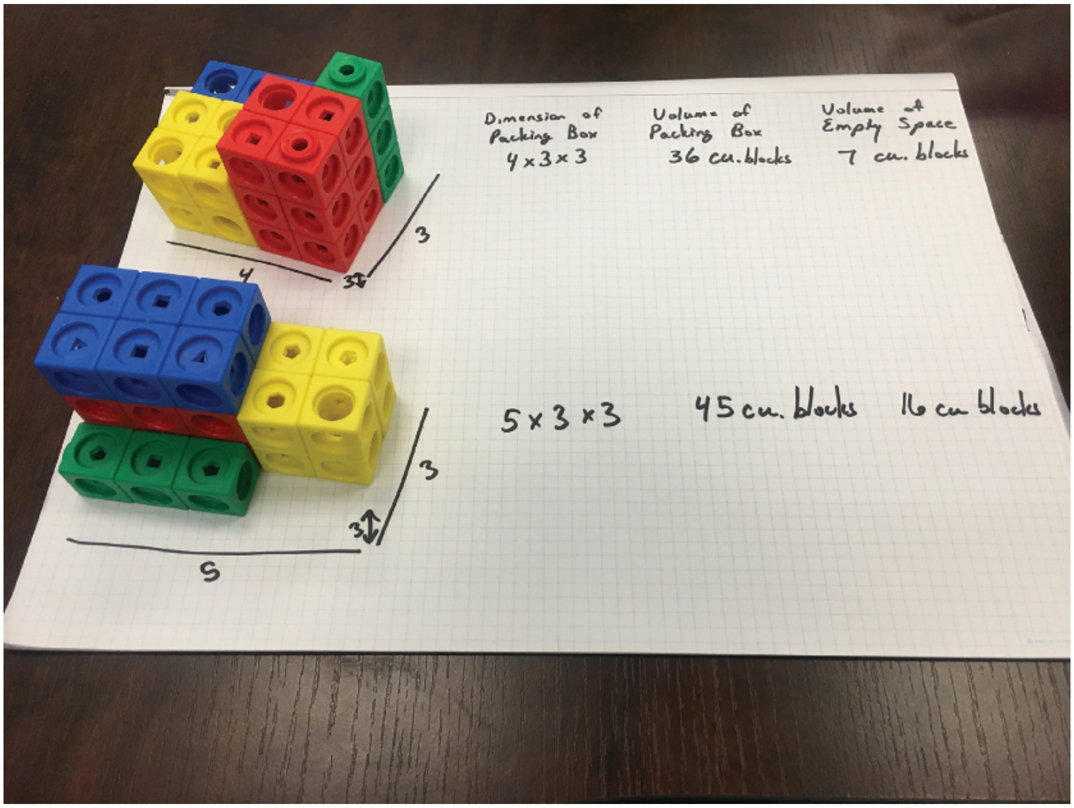


Figure 1.3 Two Pieces of Student Work Entered into a Table for Classroom Display

- What strategies did you come up with for minimizing the empty space?
- How can we be sure that we, as a class, have found the smallest packing box possible?

Ask students to look at the chart you have created out of the class's solutions. Ask, What patterns do you notice in the chart? Take advantage of connections students make between the dimensions of the packing box and its volume and the relationship between the packing box volume and the empty space volume (that is, the bigger the box, the more empty space it leaves).

Explore

After the discussion, challenge students to investigate what happens to the smallest packing box when they change the parameters of the problem. Students can choose to explore one or more of the following questions. For each question partners explore, they record the dimensions of the little boxes they are working with, the dimensions of the packing box, its volume, and the volume of empty space. A recording sheet is provided, though students may want to devise their own recording method.

- What happens if you remove one of the four small boxes? Now what is the smallest box you can make to hold the three remaining boxes?
- What happens if you add a new box? What size box would you want to add? Now what is the smallest box you can make to hold all five boxes?
- Can you find a set of boxes that leaves no empty space?
- Can you find a set of boxes that must leave a lot of empty space?

Discuss

Students will have investigated different constraints for packing boxes: packing fewer boxes, packing more boxes, packing with no empty space, and packing with lots of empty space. You'll want to discuss all of these pathways for investigation, but if one of them was more intriguing for students, be sure to give it extra time in your discussion. You may want to have students share their recording sheets on a document camera to show their solutions and how they refined their ideas through multiple trials.

- What interesting things did you discover when trying to pack boxes?
- What kinds of small boxes are challenging to pack? Why?

- What happens to the box you can make when you change what's inside?
- How did you refine your strategies for finding the smallest box?
- Did anyone find a set of boxes that leaves no empty space? Why does that work for these boxes?
- Did anyone find a set of boxes that you think must leave a lot of empty space? Why do you think this set requires a lot of empty space?
- How does thinking about volume help you design a packing box?

Look-Fors

- **Are students constructing an outer box of cubes around their little boxes?** Students might try to use the cubes to build a box around the smaller cubes, encasing them in additional cubes. This will create misleading solutions, because the cubes themselves have volume—they are actually little boxes. You may want to bring students back to the example cardboard boxes you shared in the launch to notice that packing boxes are thin. They might benefit from actually making the walls of the larger box out of paper (rather than cubes) to see that the box can be measured in cubes based on what is inside.
- **How are students finding the volume of empty space?** Some students may be able to imagine the missing cubes from the packing box they construct, but other students will need to see them. You might encourage students to fill in the missing space with a different color of cubes to help them see. They can then remove them to count the cubes representing empty space.
- **How are students finding the volume of the packing box?** Encourage students to build with cubes and count if that helps them. But you'll also want to push students to notice patterns that help them count faster and ultimately see volume as multiplicative. For instance, if students count the cubes on the bottom of their box (or in any layer), you might ask, Is there a way you could use that information so that you don't have to count all of the cubes? Pushing students to develop new ways will encourage brain growth.

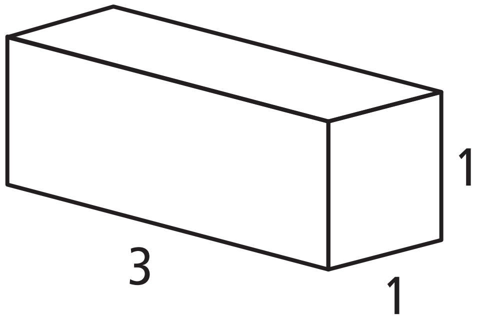
Reflect

How do you find the volume of a rectangular solid?

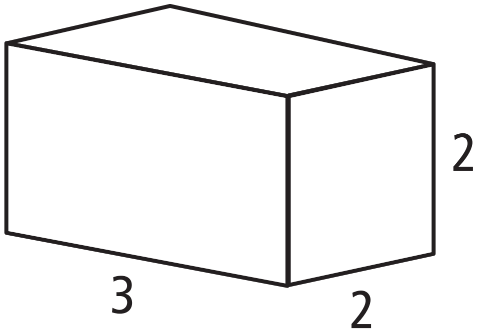
Little Boxes

You have four shapes as shown below. What is the smallest size box that will fit these four shapes inside it?

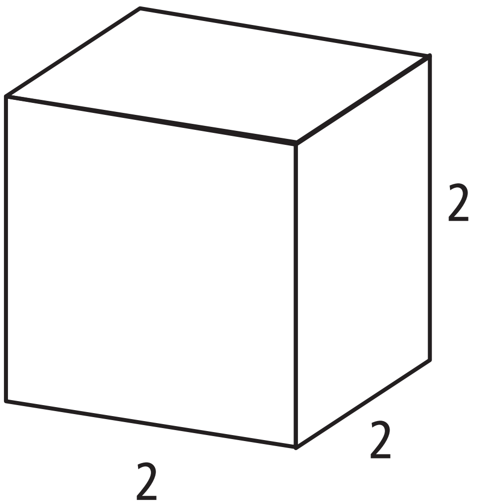
The box must be a rectangular prism with length, width, and height. Is there more than one way to pack these shapes?



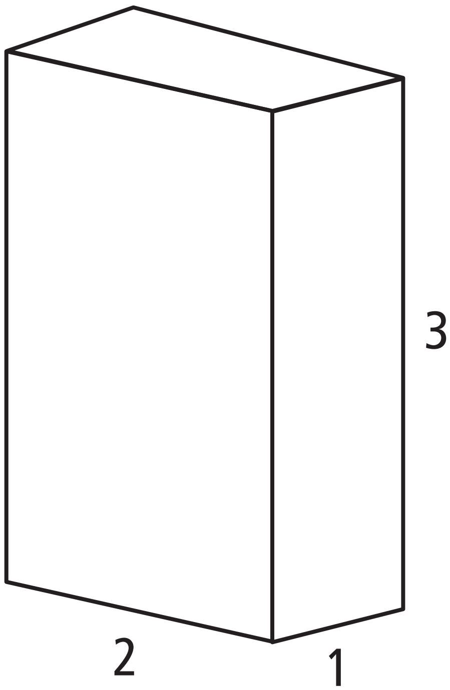
$3 \times 1 \times 1$



$3 \times 2 \times 2$



$2 \times 2 \times 2$



$2 \times 1 \times 3$

Box of Boxes Recording Sheet

Dimensions of Little Boxes	Dimensions of Packing Box	Volume of Packing Box	Volume of Empty Space