BIG IDEA 1

Moving Shapes

In this book, we focus on a set of big ideas that extend across the eighth-grade curriculum, bringing in a greater focus on geometrical thinking. Geometry has been a neglected part of the eighth-grade curriculum for some time.

Ginsberg, Cooke, Leinwand, Noell, and Pollock (2005) investigated US students' geometrical experiences, looking at the international tests TIMSS and PISA, and found that US students spend 50% less time on geometry than students in other countries. Not surprisingly given this lack of attention, students' achievement in these areas was also significantly lower than students in other countries (Driscoll, DiMatteo, Nikula, & Egan, 2007).

Many teachers and students associate geometry with rules, remembering their high school years reproducing two-column proofs. This is the unfortunate outcome of a misguided approach to mathematics, when important ideas are lost as mathematical thinking is reduced to a set of rules. What is more critical to geometry is reasoning and adaptability. In this big idea, we introduce the ideas of congruence and similarity. Rather than just learning definitions for these, students look at cases and consider deeply the question, How do we know if two shapes are congruent or similar? Definitions play a part, but the most important act is reasoning; students should be encouraged to consider such questions as, What do we know now about this shape? What else do we need to know? Can I move or adapt my shape to give me more information? Can I convince someone else that my shapes are similar or congruent? What would I use to convince them? A great starting discussion for this sequence of lessons would be the question, What does it mean to be the same? Transformational geometry, congruence, and similarity are key ideas. We have chosen to focus our attention on triangles, the building blocks of geometric shapes and the coordinate plane, an important visual space for algebra.

In the Visualize activity, students are asked to consider the question, How do we know when two figures are the same? We ask students to study triangles where their vertices are provided. As students plot the points and connect the vertices with segments, they are asked to determine which triangles are congruent. We have created triangles that are congruent but may not appear so because they have been rotated and flipped. Others combine to make a triangle. Students can hone their detective skills by investigating each set. Students explore the key ideas visually.

In the Play activity, students are asked to transform shapes, rotating and reflecting them. We think that students will enjoy working out how one shape turns into another, developing patterns that explain the transformations. This is the work of computer animation, which has been important to the cartoon filmmaking industry for many years. Students will be given the opportunity to create their own puzzle transformations, which they can share with each other.

The Investigate activity provides students the experience of continuous transformations that are repeated over and over again. Students will be invited to design their own shape and think about what happens when they repeat the same transformation on the shape. In doing so, they will become pattern creators, which we hope they will find exciting. The work will help them understand what happens when transformations happen continuously, and the patterns that can result.

Jo Boaler

References

Driscoll, M. J., DiMatteo, R. W., Nikula, J., & Egan, M. (2007). *Fostering geometric thinking: A guide for teachers, grades 5–10.* Portsmouth, NH: Heinemann.

Ginsburg, A., Cooke, G., Leinwand, S., Noell, J., & Pollock, E. (2005). *Reassessing U.S. international mathematics performance: New findings from the 2003 TIMSS and PISA.* Washington, DC: American Institutes for Research and Department of Education.

What Does It Mean to Be the Same?

Snapshot

Using a set of coordinate pairs that describe triangles, students explore what makes two figures the "same" and develop a shared definition of *congruence*.

Connection to CCSS
8.G.1, 8.G.2

Agenda

Activity	Time	Description/Prompt	Materials
Launch	10 min	Show students the sets of coordinate pairs on the Point-by-Point Triangle sheet and ask them how they might figure out which of these triangles are the same. Discuss some initial ideas.	Point-by-Point Triangle sheet, to display
Explore	30 min	Partners map the triangle set onto a coordinate plane and explore which triangles they think are the "same." Partners develop a working definition of *same* for geometric figures and gather evidence to support which shapes are the same and which are not.	• Point-by-Point Triangle sheet, per partnership • Coordinate Plane sheet, per partnership • Make available: patty paper, rulers, and angle rulers or protractors
Discuss	15 min	Discuss which triangles students found to be the same and different, and the evidence they have to support these claims. Partners share the different working definitions they developed, and the class comes to agreement on a shared definition, which is then labeled as *congruence* on a class chart.	Chart paper and markers

To the Teacher

The core idea of this activity is congruence. We introduce students to geometric transformations by posing the question, How do we know when two figures are the same? The conventional definition says that two figures are congruent if you can obtain one from the other through a series of translations (slides), rotations (turns), or reflections (flips). That is, if you can slide, flip, or turn a shape and then lay it on top of another, such that the sides and angles align, then the two shapes are congruent. This excludes shapes that must be dilated to align; shapes that must be shrunk or expanded to align with one another are not congruent. We will return to dilations in Big Idea 2, which focuses on similarity.

This activity is designed to provoke discussion about what it means for two shapes to be the "same" and to provide an opportunity for the class to develop a definition of congruence. As part of gathering evidence for two triangles being the "same," we invite students to consider the corresponding points or vertices, or the related parts of two triangles being compared. The concept of corresponding sides and vertices of geometric figures reappears throughout geometry and is useful for decomposing the triangles in this activity to determine congruence. This may trigger the need to have names for the different parts of the triangles. We have given letter labels to the coordinate pairs that locate the vertices, and you can encourage students to use these to describe corresponding vertices. Students may not know how to describe the sides; if they are searching for ways to name these, you can tell them that it is a convention in mathematics to name sides by the two vertices that form the endpoints. For example, side AB ($\overline{AB}$) is between points A and B. It is not necessary for students to use formal language, but if they are struggling to describe their observations with precision, your providing language and teaching conventions can be useful.

Activity

Launch

Launch the activity by showing the Point-by-Point Triangle sheet. Tell students that the coordinate pairs in this table make triangles and that today their task is to figure out which of these triangles are the same. Ask, How could you do that? Give students a chance to turn and talk to a partner about a plan.

Invite students to share some initial ideas, but keep the conversation brief so that students still have plenty to think about. Point out that they will need to make a convincing argument for any shapes they believe are the same. If students raise questions about the meaning of *same,* you might tell them that deciding what it means to be the same is one of the goals for today's work and that they should think with their partner about what their definition of *same* will be.

Explore

Provide partners with the Point-by-Point Triangle sheet and the Coordinate Plane sheet. Make available patty paper, scissors, and angle rulers or protractors. Partners work together to map the triangles onto the plane and explore the following questions:

- Which shapes are the same?
- How could you prove it?
- If you find two shapes that are the same, which points (vertices) correspond?
- Which shapes are not the same? What is your evidence?
- What does it mean for two shapes to be the same?

As you talk with students, press them to develop a precise working definition of sameness that the class can discuss.

Discuss

Gather the class and discuss the following questions:

- Which shapes are the same?
- How can you convince us that two shapes are the same?
- Which shapes are not the same? What is your evidence?
- What does it mean for two shapes to be the same?

When you discuss these questions, press students to explain the kinds of evidence that are necessary to prove sameness or disprove it. When you discuss this final question, build a chart of criteria for sameness and name this idea as *congruence*. We expect that this discussion may generate some debate, and we encourage you to embrace disagreements while continuing to ask for reasoning.

Look-Fors

- **Are partners accurately locating points on the coordinate plane?** Students may need to be reminded of the convention that coordinate pairs are given as (x, y) rather than (y, x). Transposing these is a common mistake. Interestingly, if students transpose all of the points consistently, they will still be able to explore congruence and make arguments about which shapes are the same and which are not. However, if students transpose some points, but not all, sometimes locating them as (x, y) and sometimes as (y, x), they will likely find no congruent figures.
- **Do partners have a clear and shared definition of *same*?** We expect that each group of students will develop their own definition of what is necessary for two figures to be considered the same and that these definitions will vary across the class. However, if students within a group are each using different criteria for sameness or their criteria are vague, they will not be prepared to make a coherent argument to justify which figures are the same. If you encounter students using vague or informal ways of determining sameness (such as, "They look the same"), ask, What is the rule for two figures being the same? Press them to develop a list of features, a test, or a definition that others could use. Students might also find it helpful to consider the evidence for difference. You might ask, How do you know that these two triangles are not the same? What are you noticing? Discussing contrasting cases can support students in naming the geometric features that could be relevant for defining congruence.

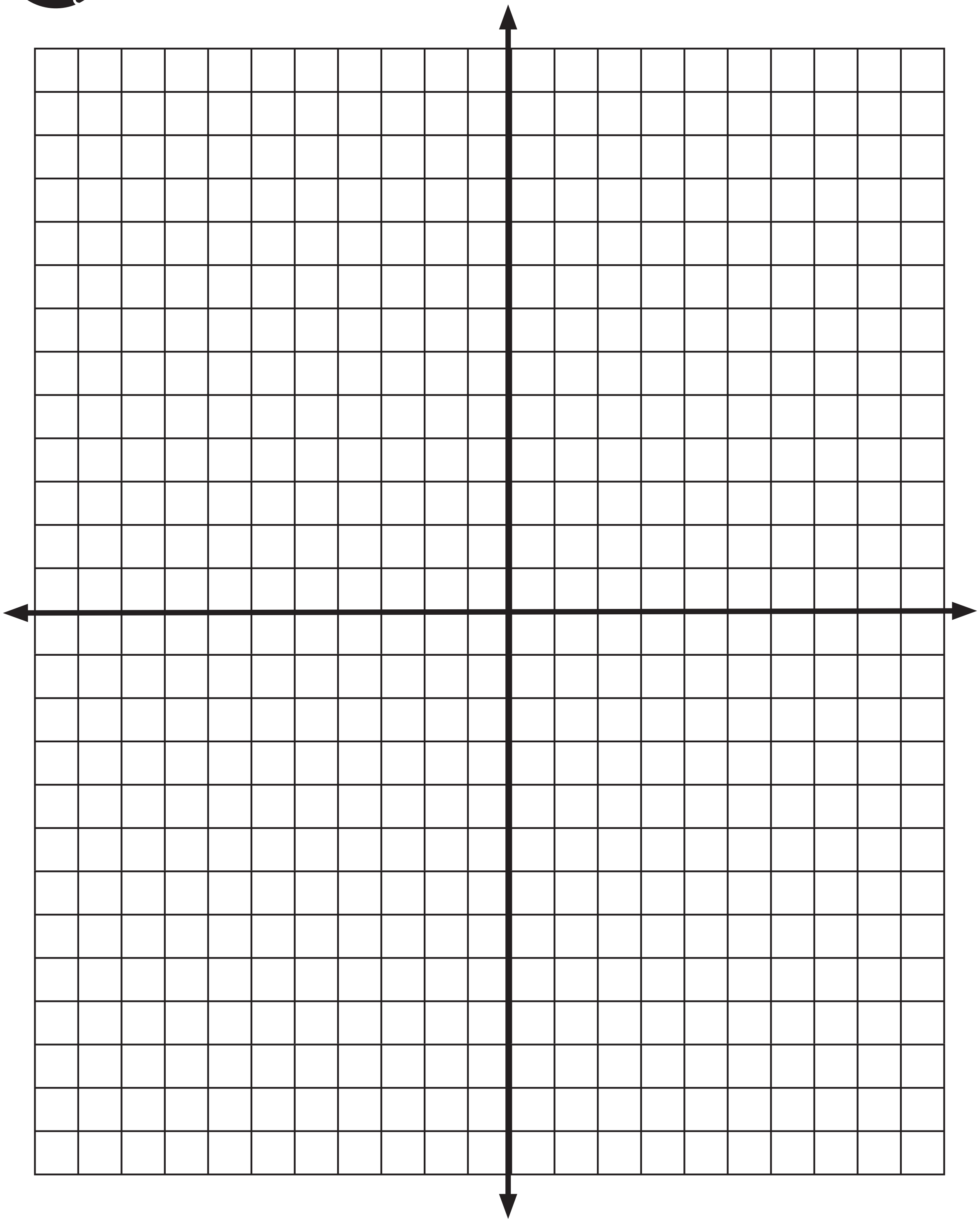
Reflect

How can you tell if two figures are congruent?

Point-by-Point Triangle

Triangle	Vertices
ΔABC	A(8, 8), B(8, 12), C(6, 8)
ΔDEF	D(3, 6), E(–5, 10), F(–7, 6)
ΔGHI	G(–4, 0), H(–4, 4), I(–1, 4)
ΔJKL	J(5, 1), K(8, 1), L(8, –3)
ΔMNO	M(–10, –11), N(–6, –3), O(–10, –3)
ΔPQR	P(0, –4), Q(0, –7), R(4, –7)
ΔSTU	S(1, –12), T(9, –6), U(9, –12)

Coordinate Plane



Pixel Puzzles

Snapshot

Students use pixilated designs on grids to analyze transformations and develop language for describing translations, rotations, and reflections. Students play with solving and designing transformation puzzles.

Connection to CCSS
8.G.1, 8.G.2, 8.G.3

Agenda

Activity	Time	Description/Prompt	Materials
Launch	5–10 min	Show students the figures on the Before and After sheet and invite them to describe how the figure changed. Discuss words for the type of transformation and the direction or magnitude of the transformation.	Before and After sheet, to display
Play	20 min	Partners work together to develop ways of describing the transformations shown on the Name the Change sheet.	• Name the Change sheet, per partnership • Make available: patty paper
Discuss	15–20 min	Discuss the ways that students described the transformations they analyzed. Develop a class chart that includes precise descriptions of the three types of transformations, connects these to the language students used, and includes words for describing direction or magnitude.	Chart paper and markers

Activity	Time	Description/Prompt	Materials
Play	30+ min	Partners play with a set of pixel puzzles on the Make the Change sheet, which challenge students to perform and undo transformations on patterns of their own design. When finished, students create their own puzzles on the DIY Transformations sheet.	• Make the Change sheet, per partnership • Patty paper, grid paper (see appendix), square tiles, and colors, per partnership • DIY Transformations sheet, as needed
Discuss	15 min	Discuss the strategies that students developed for understanding and performing different transformations. Discuss the challenges of reversing a transformation and the mistakes that emerged throughout the activity.	

To the Teacher

This activity is adapted from designs in Lou Kroner's *Slides, Flips, and Turns* (1994), which is sadly out of print. Although we often explore transformations with polygons, Kroner made use of patterns on grids, which provide a framework for comparing corresponding parts of the figures. The grids do not make it easy, though, and we think that students will find these pixilated designs challenging to transform. You may find that students are excited to make and test their own puzzles well beyond what we have offered here. With grid paper (see appendix), students can extend their play by making puzzles for themselves or each other.

The central idea in this activity is developing shared ways of describing transformations and understanding how these transformations look. In the first part of the lesson, we invite students to develop their own ways to describe changes, and we expect that you might hear a wide range of language for changes and directions, from "turn left" and "flip up" to "rotate counterclockwise 90 degrees." In the first discussion, make connections between the different ways that students are describing the same change. We encourage you to record these and their meanings in a chart. In the second part of the activity, students will likely run into questions about what it means to flip a shape vertically or horizontally, or what 90-degree, 180-degree, or 270-degree turns look like. Press students to reason through these questions, support one another in thinking about what it could mean, and use their experience in the first part of the activity as a reference. It is worth noting that with rotations, more than one way of describing the same transformation exists, as this may come up in discussions. For example, rotating a figure 90 degrees counterclockwise is equivalent to rotating the same figure 270 degrees clockwise.

Activity

Launch

Launch the activity by showing students the Before and After sheet on the document camera. Ask, What happened to the shape? How could we describe how it changed from before to after? Give students a chance to turn and talk to a partner. Take some examples of ways students describe the change. Be sure to draw attention to language that describes the nature of the change and language that describes direction or magnitude. There is no need to formalize the language students use at this point; you simply want to open the door to different ways that transformations might be described. Tell students that today they are going to play with describing and performing transformations using patterns on a grid.

Play

Provide partners with a Name the Change sheet and access to patty paper. Partners play with the strings of pixel puzzles, in which each row shows a series of transformations, to answer the following questions:

- What happened to the figure to go from one image to the next across the page?
- How can you describe it clearly? Can you find more than one way?

Students record in the boxes between figures their ways of naming the transformations so that they can discuss clear and precise ways of describing changes as a class.

Discuss

Discuss the following questions as a class:

- How did you describe the changes from shape to shape? (Embrace debate as you discuss each puzzle.)

- Are all slides (translations) the same? Why or why not? How can you describe the different types?
- Are all turns (rotations) the same? Why or why not? How can you describe the different types?
- Are all flips (reflections) the same? Why or why not? How can you describe the different types?
- When is there more than one way to describe the change?

Make a chart to formalize the names for translations (slides), rotations (turns), and reflections (flips), and show how these terms are connected to other language students have used to describe the same transformations. Include in your chart directional language that students come up with (left, right, up, down, horizontal, vertical, clockwise, counterclockwise, etc.) to moderate the types of transformations they saw.

Play

Provide partners with the Make the Change sheet, patty paper, grid paper (see appendix), square tiles, and colors. Ask students to first design a shape for the middle grid in each puzzle. Then partners determine what their figure would look like performing the two transformations moving forward (to the right) and how to reverse the transformations to find the original image (to the left).

When students finish, invite them to create their own patterns and rules to explore on the DIY Transformations sheet. Partners might each create a puzzle and swap, or work together to construct a puzzle for themselves or others to solve. Students might include some transformation directions in the labels and some pictures.

Discuss

Discuss as a class the following questions:

- What did you notice? What did you discover? (Discuss some of the individual parts to the puzzles.)
- What strategies did you use to slide, flip, or turn your designs? Which were most helpful, and why?
- What was hard about transforming these shapes? What did you struggle with the most? Why?
- What mistakes did you make? How did you notice your mistake? What did you have to do differently to address your mistake?

Be sure to discuss the challenges of reversing a transformation and how students thought through these parts of the puzzle. If students made puzzles for one another to solve, be sure to discuss how they had to think to design a puzzle, rather than just solve one.

Look-Fors

- **Are students moving their papers or using strategies for movement?** Surprisingly, we've found that many students want to perform transformations in their minds alone and devalue physical strategies for transformations or testing their ideas. If students struggle to visualize a rotation or a reflection, ask, What could you do to see how that transformation would look? What tools or strategies could you use? Encourage students to rotate their papers or to get up and look at their papers from a new angle. Students can make use of patty paper to perform reflections, or literally flip their papers over and hold them up to the light, so that they can see what a horizontal or vertical flip looks like. These are valid mathematical strategies for developing spatial reasoning, and students who engage in them will build skills for visualizing transformations.
- **Are students attending to corresponding parts?** As students record their transformations, they will need to pay attention to how each cell in the grid corresponds to a new cell in the adjacent grid. For instance, the top left cell may correspond to the top right cell after a flip or turn. You may notice errors in the figure, with cells moved inconsistently or with parts of a figure forgotten. As students map each figure onto a new grid, you might ask them, How do you know you have made the new figure accurately? What strategies are you using to check your thinking?
- **Are students reasoning about direction and magnitude?** With transformations, the description of the nature of the change isn't complete without some additional information about direction or magnitude. With a translation, students can simply describe it as up, down, left, or right, and in these puzzles, the direction is clear. But rotations and reflections are particularly challenging because they rely on language that isn't used every day, and students will have to reason about what it means to reflect vertically or horizontally and what it means to rotate a specified number of degrees and in which direction. Across this activity, press students to describe transformations as precisely as they can and to justify how they made sense out of a given transformation. Ask, How do you know to change the figure in this way? What does it mean to perform

this transformation? Ask students to talk about all the parts of a transformation—not just flipping, but flipping how?

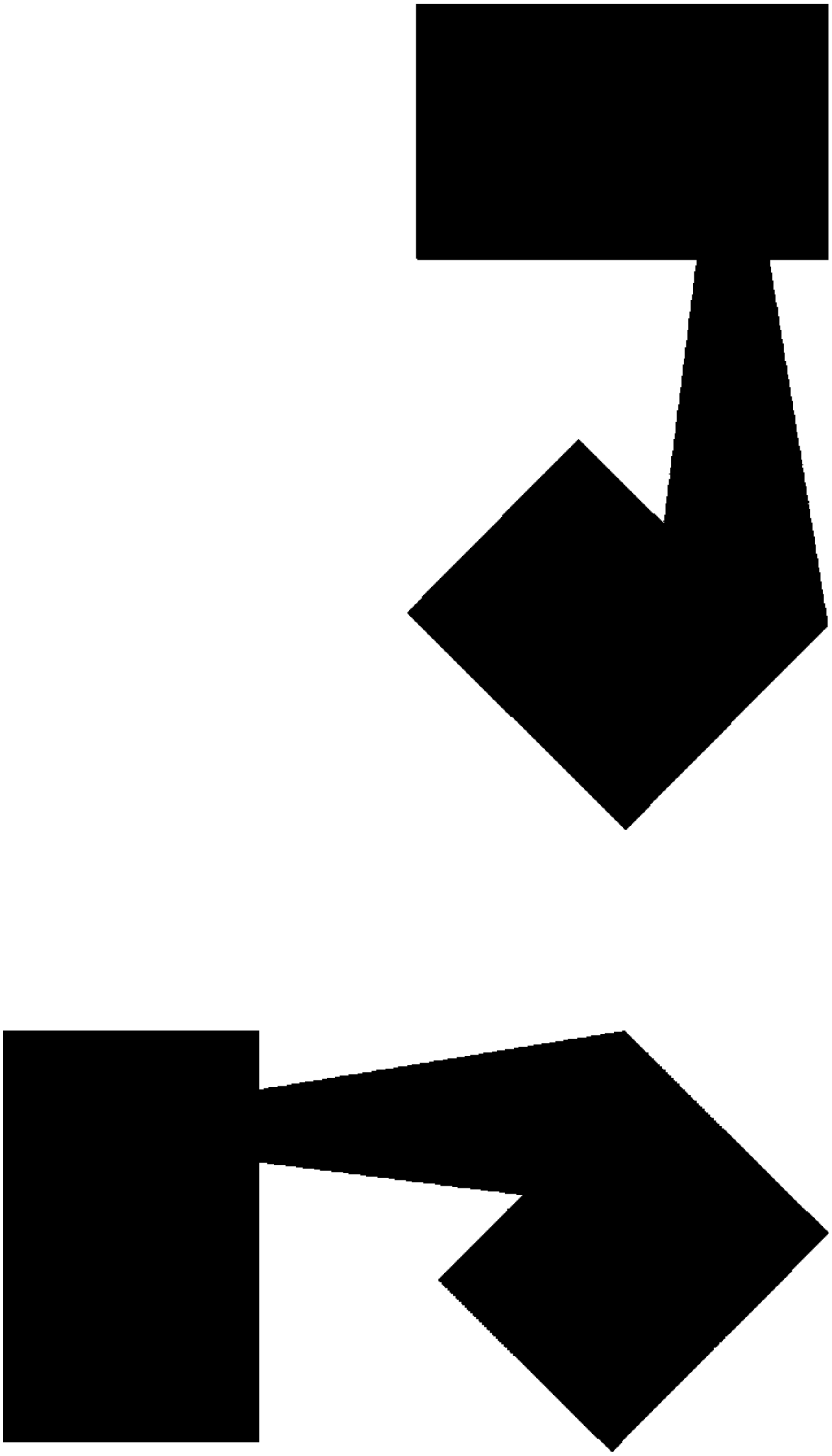
Reflect

Which transformation did you find trickiest to visualize? Why? What strategies did you use to help yourself?

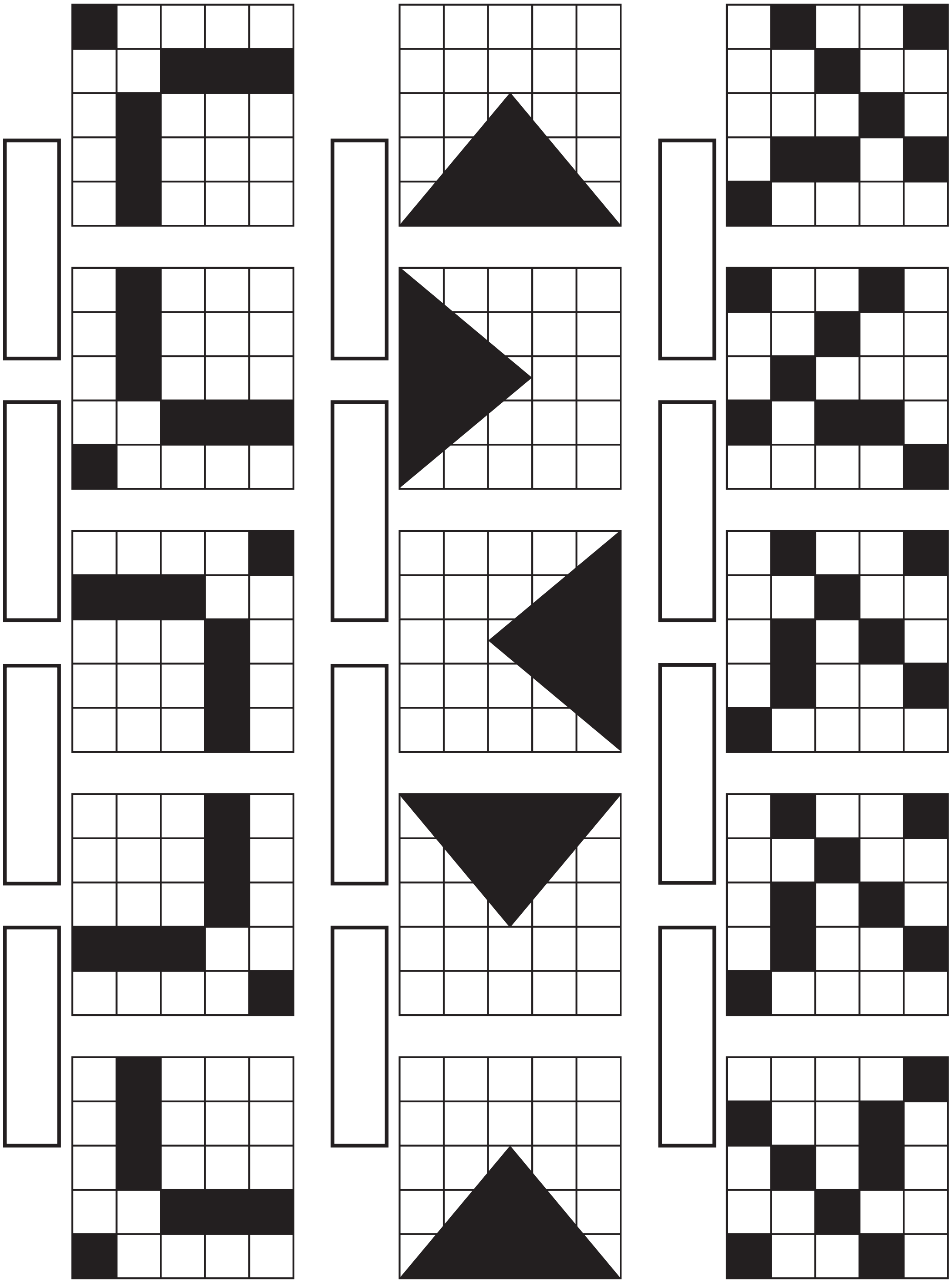
Reference

Kroner, L. R. (1994). *Slides, flips, and turns*. Parsippany, NJ: Dale Seymour.

Before and After



Name the Change



Make the Change

Rotate -90°

Rotate 180°

Rotate -270°

Reflect horizontal

Reflect vertical

Rotate 270°

Slide

Rotate -90°

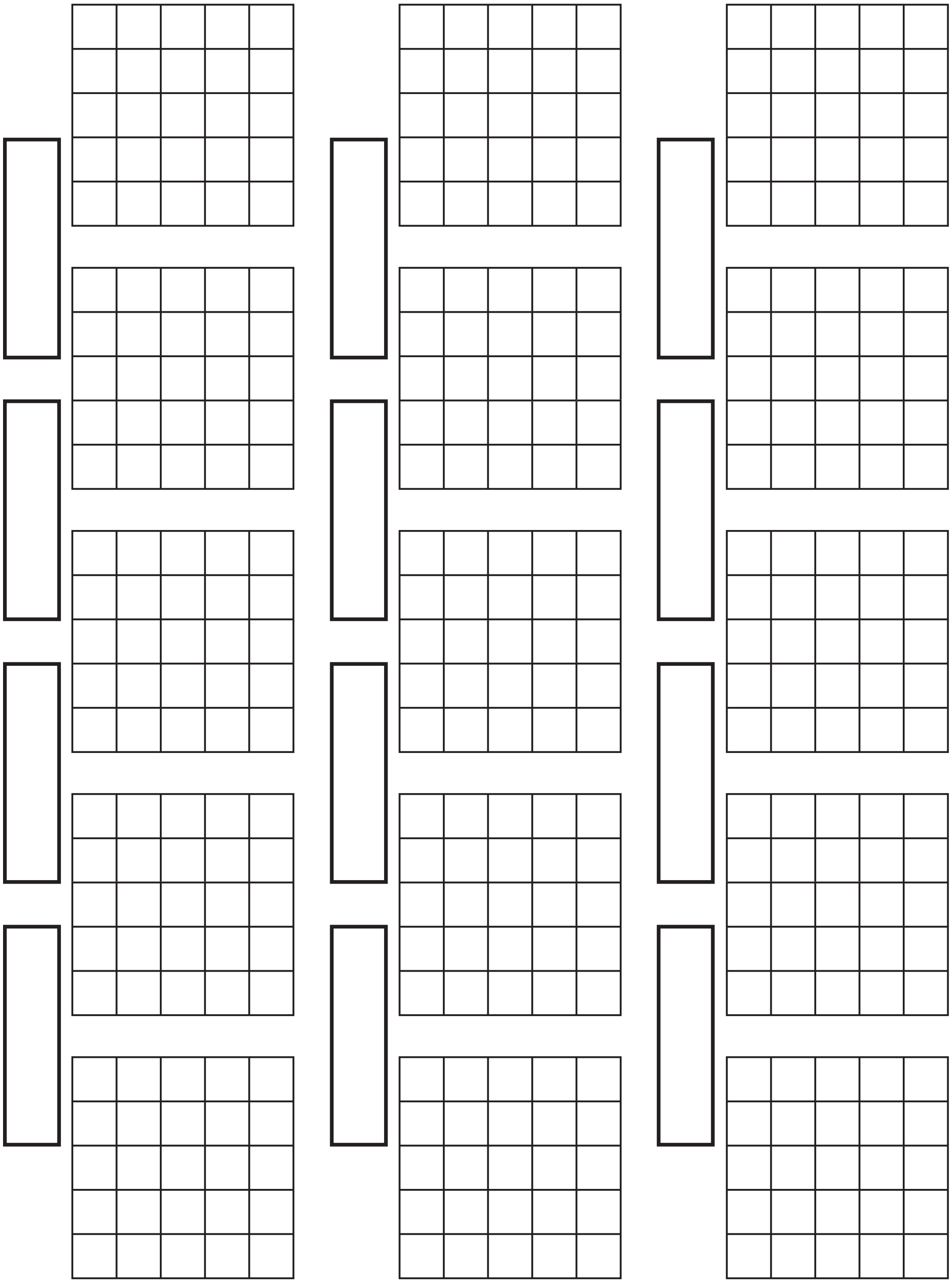
Rotate -180°

Rotate 90°

Reflect horizontal

Slide

DIY Transformations



Slide It, Flip It, Turn It

Snapshot

Students investigate the patterns created by applying a transformation rule repeatedly to a geometric shape of their own design. Pooling their findings, the class looks for patterns in the kinds of rules that create different effects.

Connection to CCSS
8.G.1, 8.G.2, 8.G.3

Agenda

Activity	Time	Description/Prompt	Materials
Launch	5–10 min	Show students the Chain Pattern and discuss the different ways that students see and could extend it. Tell students that the rule for this pattern was a translation (or slide) of a certain number of units and this simple rule created repeating forms.	Chain Pattern sheet, to display
Explore	40+ min	Partners design a shape on dot paper and a rule for transforming it. Applying the rule repeatedly to the shape, students investigate the patterns created and why these might occur. Partners investigate different kinds of rules with different shapes to develop ideas about what transformations generate different kinds of patterns.	Make available: isometric and square dot paper (see appendix), rulers, compasses, patty paper, protractors or angle rulers, and colors
Discuss	15+ min	Post students' patterns in a display space and do a gallery walk. Discuss what kinds of rules generate different sorts of patterns, such as spirals or circles.	
Extend	Ongoing	Students hunt for transformation patterns in the world and bring in examples.	

To the Teacher

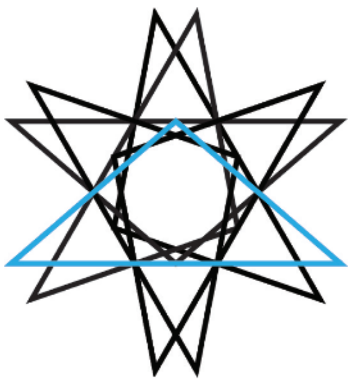
In this activity, we invite students to explore how a single transformation rule can create patterns when used repeatedly. We begin by looking at the Chain Pattern, which shows a series of overlapping triangles that was created with a rule that involved a translation to the right (or left, depending on how you look at it) a certain number of units. These now-overlapping triangles create new forms, such as rhombuses, and the pattern can continue forever to the left and right in a line.

An isosceles triangle, in blue, is duplicated and transformed using a slide to the right.

More complex patterns are possible using rotations, reflections, and translations in combination. For instance, look at the pattern here. In this case, the rule for transforming the same shape was a slide to the right with a –30-degree rotation.

An isosceles triangle, in blue, duplicated with a slide and a –30-degree rotation.

Both of these previous patterns could extend forever, but some patterns begin to overlap in such a way that nothing new is created. For example, the pattern here shows the same triangle being rotated 60 degrees around a fixed point. After five repetitions, the triangles overlap, and the pattern ceases to change.



An isosceles triangle, in blue, is duplicated five times. Each duplication is rotated another 60 degrees.

Combining different transformations and exploring the results with different shapes is the focus of this investigation. Encourage students to try the same rule with different shapes, or different rules with the same shape, to see what results are attributable to the rule and which may be particular to a type of shape being transformed.

Some students may find this type of exploration particularly compelling given its connections to digital animation. As animators design movement, they must consider the algorithms, or rules, for repeated changes. When repeated transformations are seen in quick succession, they are perceived as smooth movement—for example, a car driving across the screen or a person turning their head. There are many intriguing online resources that explore the mathematics of animation, and we encourage you to support students in exploring these connections.

Activity

Launch

Launch the activity by showing the Chain Pattern on the document camera. Ask, How do you see this pattern? What would you draw to continue the pattern? Give students a chance to turn and talk to a partner. Discuss how students see the elements of the pattern. Invite them to come up to the document camera to point out how they see it and to show how they might draw it.

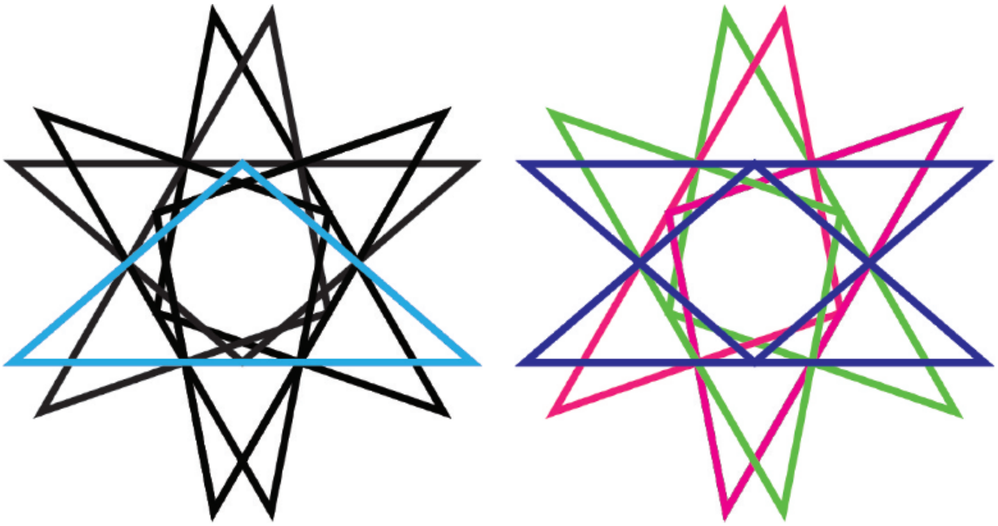
Point out that this pattern is made from transforming a shape. In this case, we've taken a triangle and repeatedly translated, or slid, it to the right some number of units. This rule created overlapping and repeating forms. Tell students that today they are going to experiment with transforming shapes to make patterns.

Explore

Make available materials for constructing and investigating transformation patterns, including isometric and square dot paper (see appendix), rulers, compasses, patty paper, protractors or angle rulers, and colors. Partners work together to draw a simple geometric figure on dot paper and create a rule for transforming it. The rule can be simple (one step) or compound (multiple steps). Be sure the rule is clear and includes both the type of transformation(s) and direction and magnitude language so that someone else could follow the rule. Students apply their rule repeatedly to the shape they have designed, drawing it on the same grid, as with the Chain Pattern example. Partners investigate the following questions:

- What happens? Why?
- What patterns are created when you follow your rule? Why?
- What interesting patterns can you make?

Encourage students to investigate another rule on the same or a different shape (or the same rule with a different shape) and continue to explore these questions. For every pattern created, ask students to label it with the rule used. Experiment with patterns by using color. How do different colors change what you see in the patterns you've made?



The same pattern with different colors

Discuss

Ask groups to post all of their patterns, labeled with the rule used, in a display space. Invite students to do a gallery walk and look for patterns across the patterns.

As a class, discuss the following questions:

- What kinds of rules make similar patterns?
- What kinds of rules make patterns that appear to end?
- What kinds of rules make patterns that could continue to grow forever?
- What rules created surprising patterns?
- How do different shapes change the appearance of the pattern?

You may want to work as a class to sort the patterns into groups based on common features, such as those that are circular, spiral, or linear. Sorting could support students in then determining what kinds of rules lead to these kinds of repeating patterns.

Extend

Students can extend their thinking about transformation patterns by investigating the question, Where in the world can you find patterns that were created with slides, flips, and turns? Encourage students to hunt in the school building, in architecture, in animation, in graphic design, art, tile work, cultural artifacts, and books. This could be an ongoing investigation, during which students are on the lookout for patterns and bring in examples or photos from their community, their reading, or the internet.

Look-Fors

- **Are students' rules specific?** The rules that students create need to be specific enough so that they (or another person) can follow them with precision. The rule description needs to include as many as four kinds of information: the type of transformation (slide, flip, or turn), the direction (up, down, left, right, horizontal, vertical, clockwise, or counterclockwise), the magnitude (the number of units or degrees), and the order (if students have more than one step, they need to specify which transformation is first, second, and so on). Ask students questions about their rules that support them in specifying each of these aspects of their rule, such as, What direction will you slide it? How far will you rotate it? Which comes first, the rotation or the reflection?
- **Are students' rules and shapes manageable?** Another issue that could emerge is that students may create a large shape or a rule that generates movement across a large distance, causing their pattern to skitter off the paper before they have a chance to see what happens. In these instances, ask, What happened? Why? Support students in pinpointing the issue. Then students need to decide how to handle it. They can either retain the shape and the rule, and tape multiple sheets of dot paper together to make a much larger surface, or revise the shape and/or the rule to make it more manageable. In either case, be sure to invite these students to share what they found with the class to add to everyone's learning about the results of transformations.

- **How are students dealing with challenges of rotation?** Rotating a figure is challenging, and the selection of the angle of rotation makes a difference in how complex that challenge is. On square dot paper, rotations of 90 degrees are more straightforward; on isometric dot paper, rotations of 60 degrees are supported by the grid. However, students may find that they want to explore smaller increments of rotation, such as 45 degrees or 30 degrees, and these will create more interesting patterns. As students are choosing an angle of rotation, ask, How will you use the dot paper to support you in rotating precisely? What other tools or strategies could you use? Students might, for example, use an angle ruler or protractor and patty paper to rotate through any angle.

Reflect

What did you discover that surprised you? What are you wondering now?

Chain Pattern