1

Introduction and General Matrix Methods

1.1 Brief Introduction

The study of the convergence of infinite series is an ancient art. In ancient times, people were more concerned with orthodox examinations of convergence of infinite series. Series that did not converge were of no interest to them until the advent of L. Euler (1707–1783), who took up a serious study of "divergent series"; that is, series that did not converge. Euler was followed by a galaxy of great mathematicians, such as C.F. Gauss (1777–1855), A.L. Cauchy (1789–1857), and N.H. Abel (1802–1829). The interest in the study of divergent series temporarily declined in the second half of the nineteenth century. It was rekindled at a later date by E. Cesàro, who introduced the idea of $(C, 1)$ convergence in 1890. Since then, many other mathematicians have been contributing to the study of divergent series. Divergent series have been the motivating factor for the introduction of summability theory.

Summability theory has many uses in analysis and applied mathematics. An engineer or physicist who works with Fourier series, Fourier transforms, or analytic continuation can find summability theory very useful for his/her research.

Throughout this chapter, we assume that all indices and summation indices run from 0 to ∞, unless otherwise specified. We denote sequences by $\{x_k\}$ or (x_k), depending on convenience.

Consider the sequence

$$\{s_n\} = \{1, 0, 1, 0, \dots\},$$

which is known to diverge. However, let

$$t_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1},$$

$$\text{i.e.,} \quad t_n = \begin{cases} \frac{k+1}{2k+1}, & \text{if } n = 2k; \\ \frac{k+1}{2k+2}, & \text{if } n = 2k+1, \end{cases}$$

An Introductory Course in Summability Theory, First Edition. Ants Aasma, Hemen Dutta, and P.N. Natarajan.

proving that

$$t_n \to 1/2, \ n \to \infty.$$

In this case, we say that the sequence $\{s_n\}$ converges to $1/2$ in the sense of Cesàro or $\{s_n\}$ is $(C, 1)$ summable to $1/2$. Similarly, consider the infinite series

$$\sum_n a_n = 1 - 1 + 1 - 1 + \cdots.$$

The associated sequence $\{s_n\}$ of partial sums is $\{1, 0, 1, 0, \dots\}$, which is $(C, 1)$-summable to $1/2$. In this case, we say that the series $\Sigma_n a_n = 1 - 1 + 1 - 1 + \cdots$ is $(C, 1)$-summable to $1/2$.

With this brief introduction, we recall the following concepts and results.

1.2 General Matrix Methods

Definition 1.1 Given an infinite matrix $A = (a_{nk})$, and a sequence $x = \{x_k\}$, by the A-transform of $x = \{x_k\}$, we mean the sequence

$$A(x) = \{(Ax)_n\},$$
$$(Ax)_n = \Sigma_k a_{nk} x_k,$$

where we suppose that the series on the right converges. If $\lim_{n\to\infty} (Ax)_n = t$, we say that the sequence $x = \{x_k\}$ is summable A or A-summable to t. If $\lim_{n\to\infty} (Ax)_n = t$ whenever $\lim_{k\to\infty} x_k = s$, then A is said to be preserving convergence for convergent sequences, or sequence-to-sequence conservative (for brevity, Sq-Sq conservative). If A is sequence-to-sequence conservative with $s = t$, we say that A is sequence-to-sequence regular (shortly, Sq-Sq regular). If $\lim_{n\to\infty} (Ax)_n = t$, whenever, $\Sigma_k x_k = s$, then A is said to preserve the convergence of series, or series-to-sequence conservative (i.e., Sr-Sq conservative). If A is series-to-sequence conservative with $s = t$, we say that A is series-to-sequence regular (shortly, Sr-Sq regular).

In this chapter and in Chapters 2 and 3, for conservative and regular, we mean only Sq–Sq conservativity and Sq-Sq regularity.

If X, Y are sequence spaces, we write

$$A \in (X, Y),$$

if $\{(Ax)_n\}$ is defined and $\{(Ax)_n\} \in Y$, whenever, $x = \{x_k\} \in X$. With this notation, if A is conservative, we can write $A \in (c, c)$, where c denotes the set of all convergent sequences. If A is regular, we write

$$A \in (c, c; P),$$

P denoting the "preservation of limit."

Definition 1.2 A method $A = (a_{nk})$ is said to be lower triangular (or simply, triangular) if $a_{nk} = 0$ for $k > n$, and normal if A is lower triangular if $a_{nn} \neq 0$ for every n.

Example 1.1 Let A be the Zweier method; that is, $A = Z_{1/2}$, defined by the lower triangular method $A = (a_{nk})$ where (see [2], p. 14) $a_{00} = 1/2$ and

$$a_{nk} = \begin{cases} \frac{1}{2}, & \text{if } k = n-1 \text{ and } k = n; \\ 0, & \text{if } k < n-1 \end{cases}$$

for $n \geq 1$. The method $A = Z_{1/2}$ is regular. The transformation $(Ax)_n$ for $n \geq 1$ can be presented as

$$(Ax)_n = \frac{x_{n-1} + x_n}{2}.$$

Then,

$$\lim_n (Ax)_n = \lim_k x_k$$

for every $x = \{x_k\} \in c$; that is, $Z_{1/2} \in (c, c; P)$.

We now prove a landmark theorem in summability theory due to Silverman–Toeplitz, which characterizes a regular matrix in terms of the entries of the matrix (see [3–5]).

Theorem 1.1 (Silverman-Toeplitz) $A = (a_{nk})$ is regular, that is, $A \in (c, c; P)$, if and only if

$$\sup_{n \geq 0} \sum_k |a_{nk}| < \infty; \tag{1.1}$$

$$\lim_{n \to \infty} a_{nk} := \delta_k; \tag{1.2}$$

and

$$\lim_{n \to \infty} \sum_k a_{nk} = \delta \tag{1.3}$$

with $\delta_k \equiv 0$ and $\delta \equiv 1$.

Proof: Sufficiency. Assume that conditions (1.1)–(1.3) with $\delta_k \equiv 0$ and $\delta \equiv 1$ hold. Let $x = \{x_k\} \in c$ with $\lim_{k \to \infty} x_k = s$. Since $\{x_k\}$ converges, it is bounded; that is, $x_k = O(1),\ k \to \infty$, or, equivalently, $|x_k| \leq M$, $M > 0$ for all k.

Now

$$\sum_k |a_{nk} x_k| \leq M \sum_k |a_{nk}| < \infty,$$

in view of (1.1), and so

$$(Ax)_n = \sum_k a_{nk} x_k$$

is defined. Now

$$(Ax)_n = \sum_k a_{nk}(x_k - s) + s\sum_k a_{nk}. \tag{1.4}$$

Since $\lim_{k\to\infty} x_k = s$, given an $\epsilon > 0$, there exists an $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, such that

$$|x_k - s| < \frac{\epsilon}{2L}, \quad k > N, \tag{1.5}$$

where $L > 0$ is such that

$$|x_n - s| \leq L, \quad \sum_k |a_{nk}| \leq L, \tag{1.6}$$

and hence

$$\sum_k a_{nk}(x_k - s) = \sum_{k=0}^{N} a_{nk}(x_k - s) + \sum_{k=N+1}^{\infty} a_{nk}(x_k - s),$$

$$\left|\sum_k a_{nk}(x_k - s)\right| \leq \sum_{k=0}^{N} |a_{nk}||x_k - s| + \sum_{k=N+1}^{\infty} |a_{nk}||x_k - s|.$$

Using (1.5) and (1.6), we obtain

$$\sum_{k=N+1}^{\infty} |a_{nk}||x_k - s| \leq \frac{\epsilon}{2L}\sum_k |a_{nk}| \leq \frac{\epsilon}{2L}L = \frac{\epsilon}{2}.$$

By (1.2), there exists a positive integer n_0 such that

$$|a_{nk}| < \frac{\epsilon}{2L(N+1)}, \quad k = 0, 1, \ldots, N, \text{ for } n > n_0.$$

This implies that

$$\sum_{k=0}^{N} |a_{nk}||x_k - s| < L(N+1)\frac{\epsilon}{2L(N+1)} = \frac{\epsilon}{2}, \quad \text{for } n > n_0.$$

Consequently, for every $\epsilon > 0$, we have

$$\left|\sum_k a_{nk}(x_k - s)\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for } n > n_0.$$

Thus,

$$\lim_{n\to\infty} \sum_k a_{nk}(x_k - s) = 0. \tag{1.7}$$

Taking the limit as $n \to \infty$ in (1.4), we have, by (1.7), that

$$\lim_{n\to\infty} (Ax)_n = s,$$

since $\delta = 1$. Hence, A is regular, completing the proof of the sufficiency part.

Necessity. Let A be regular. For every fixed k, consider the sequence $x = \{x_n\}$, where

$$x_n = \begin{cases} 1, & n = k; \\ 0, & \text{otherwise}. \end{cases}$$

For this sequence x, $(Ax)_n = a_{nk}$. Since $\lim_{n\to\infty} x_n = 0$ and A is regular, it follows that $\delta_k \equiv 0$. Again consider the sequence $x = \{x_n\}$, where $x_n = 1$ for all n. Note that $\lim_{n\to\infty} x_n = 1$. For this sequence x, $(Ax)_n = \Sigma_k a_{nk}$. Since $\lim_{n\to\infty} x_n = 1$ and A is regular, we have $\delta = 1$. It remains to prove (1.1). First, we prove that $\Sigma_k |a_{nk}|$ converges. Suppose not. Then, there exists an $N \in \mathbb{N}$ such that

$$\sum_k |a_{Nk}| \text{ diverges.}$$

In fact, $\Sigma_k |a_{Nk}|$ diverges to ∞. So we can find a strictly increasing sequence $k(j)$ of positive integers such that

$$\sum_{k=k(j-1)}^{k(j)-1} |a_{Nk}| > 1, \quad j = 1, 2, \ldots. \tag{1.8}$$

Define the sequence $x = \{x_k\}$ by

$$x_k = \begin{cases} \frac{|a_{Nk}|}{j a_{Nk}}, & \text{if } a_{Nk} \neq 0 \text{ and } k(j-1) \leq k < k(j),\ j = 1, 2, \ldots; \\ 0, & \text{if } k = 0 \text{ or } a_{Nk} = 0. \end{cases}$$

Note that $\lim_{k\to\infty} x_k = 0$ and $\Sigma_k a_{nk} x_k$ converges. In particular, $\Sigma_k a_{Nk} x_k$ converges. However,

$$\sum_k a_{Nk} x_k = \sum_{j=1}^{\infty} \sum_{k=k(j-1)}^{k(j)-1} \frac{|a_{Nk}|}{j} = \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=k(j-1)}^{k(j)-1} |a_{Nk}| > \sum_{j=1}^{\infty} \frac{1}{j}.$$

This leads to a contradiction since $\sum_{j=1}^{\infty} \frac{1}{j}$ diverges. Thus,

$$\sum_k |a_{nk}| \text{converges for every } n \in \mathbb{N}.$$

To prove that (1.1) holds, we assume that

$$\sup_{n\geq 0} \sum_k |a_{nk}| = \infty$$

and arrive at a contradiction.

We construct two strictly increasing sequences $\{m(j)\}$ and $\{n(j)\}$ of positive integers in the following manner.

Let $m(0) = 0$. Since $\Sigma_k |a_{m(0),k}| < \infty$, choose $n(0)$ such that

$$\sum_{k=n(0)+1}^{\infty} |a_{m(0),k}| < 1.$$

Having chosen the positive integers $m(0), m(1), \ldots, m(j-1)$ and $n(0), n(1), \ldots,$ $n(j-1)$, choose positive integers $m(j) > m(j-1)$ and $n(j) > n(j-1)$ such that

$$\sum_{k} |a_{m(j),k}| > j^2 + 2j + 2; \tag{1.9}$$

$$\sum_{k=0}^{n(j-1)} |a_{m(j),k}| < 1; \tag{1.10}$$

and

$$\sum_{k=n(j)+1}^{\infty} |a_{m(j),k}| < 1. \tag{1.11}$$

Now define the sequence $x = \{x_k\}$, where

$$x_k = \begin{cases} \frac{|a_{m(j),k}|}{j a_{m(j),k}}, & \text{if } n(j-1) < k \leq n(j), a_{m(j),k} \neq 0, j = 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\lim_{k\to\infty} x_k = 0$. Since A is regular, $\lim_{n\to\infty} (Ax)_n = 0$. However, using (1.9)–(1.11), we have

$$\begin{aligned} |(Ax)_{m(j)}| &= \left| \sum_k a_{m(j),k} x_k \right| \\ &= \left| \sum_{k=0}^{n(j-1)} a_{m(j),k} x_k + \sum_{k=n(j-1)+1}^{n(j)} a_{m(j),k} x_k + \sum_{k=n(j)+1}^{\infty} a_{m(j),k} x_k \right| \\ &\geq \left| \sum_{k=n(j-1)+1}^{n(j)} a_{m(j),k} x_k \right| - \sum_{k=0}^{n(j-1)} |a_{m(j),k} x_k| - \sum_{k=n(j)+1}^{\infty} |a_{m(j),k} x_k| \\ &> \frac{1}{j} \sum_{k=n(j-1)+1}^{n(j)} |a_{m(j),k}| - 1 - 1 \\ &= \frac{1}{j} \left[\sum_k |a_{m(j),k}| - \sum_{k=0}^{n(j-1)} |a_{m(j),k}| - \sum_{k=n(j)+1}^{\infty} |a_{m(j),k}| \right] - 2 \\ &> \frac{1}{j} [(j^2 + 2j + 2) - 1 - 1] - 2 = j + 2 - 2 = j, j = 1, 2, \ldots. \end{aligned}$$

Thus, $\{(Ax)_{m(j)}\}$ diverges, which contradicts the fact that $\{(Ax)_n\}$ converges. Consequently, (1.1) holds. This completes the proof of the theorem.

Example 1.2 Let A be the Cesàro method $(C,1)$; that is, $A = (C,1)$. This method is defined by the lower triangular matrix $A = (a_{nk})$, where $a_{nk} = 1/(n+1)$ for all $k \leq n$. It is easy to see that all of the conditions of Theorem 1 are satisfied. Hence, $(C,1) \in (c,c;P)$.

Example 1.3 Let $A_{-1,1}$ be the method defined by the lower triangular matrix (a_{nk}), where $a_{00} = 1$ and

$$a_{nk} = \begin{cases} -1, & \text{if } k = n-1; \\ 1, & \text{if } k = n; \\ 0, & \text{if } k < n-1 \end{cases}$$

for $n \geq 1$. It is easy to see that, in this case, $\delta_k \equiv 0$, $\delta = 0 \neq 1$ and condition (1.1) holds. Therefore, $A_{-1,1}$ does not belong to $(c,c;P)$. However, $A_{-1,1} \in (c,c)$ and $A_{-1,1} \in (c_0,c_0)$, where c_0 denotes the set of all sequences converging to 0 (see Exercises 1.1 and 1.4).

Let m (or ℓ_∞) denote the set of all bounded sequences. For $x = \{x_k\} \in \ell_\infty$, define

$$\| x \| = \sup_{k \geq 0} |x_k|. \tag{1.12}$$

Then, it is easy to see that m is a Banach space and c is a closed subspace of m with respect to the norm defined by (1.12).

Definition 1.3 The matrix $A = (a_{nk})$ is called a Schur matrix if $A \in (m,c)$; that is, $\{(Ax)_n\} \in c$, whenever, $x = \{x_k\} \in m$.

The following result gives a characterization of a Schur matrix in terms of the entries of the matrix (see [3–5]).

Theorem 1.2 (Schur) $A = (a_{nk})$ is a Schur matrix if and only if (1.2) holds and

$$\sum_k |a_{nk}| \text{ converges uniformly in } n. \tag{1.13}$$

Proof: Sufficiency. Assume that (1.2) and (1.13) hold. Then, (1.13) implies that the series $\Sigma_k |a_{nk}|$ converge, n belongs to N. By (1.2) and (1.13), we obtain that

$$\sup_{n \geq 0} \sum_k |a_{nk}| = M < \infty.$$

Thus, for each r, we have

$$\lim_{n \to \infty} \sum_{k=0}^{r} |a_{nk}| \leq M.$$

Hence,

$$\sum_{k=0}^{r} |\delta_k| \leq M \text{ for every } r,$$

and so

$$\sum_k |\delta_k| < \infty.$$

Thus, if $x = \{x_k\} \in m$, it follows that $\Sigma_k a_{nk} x_k$ converges absolutely and uniformly in n. Consequently,

$$\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} \sum_k a_{nk} x_k = \sum_k \delta_k x_k,$$

proving that $\{(Ax)_n\} \in c$; that is, $A \in (m, c)$, proving the sufficiency part.

Necessity. Let $A = (a_{nk}) \in (m, c)$. Then, $A \in (c, c)$ and so (1.2) holds. Again, since $A \in (c, c)$, we get that (1.1) holds; that is,

$$\sup_{n\geq 0} \sum_k |a_{nk}| < \infty.$$

As in the sufficiency part of the present theorem, it follows that $\Sigma_k |\delta_k| < \infty$. We write

$$b_{nk} = a_{nk} - \delta_k.$$

Then, $\{\Sigma_k b_{nk} x_k\}$ converges for all $x = \{x_k\} \in m$. We now claim that

$$\sum_k |b_{nk}| \to 0, n \to \infty. \tag{1.14}$$

Suppose not. Then,

$$\overline{\lim_{n\to\infty}} \sum_k |b_{nk}| = c > 0.$$

So,

$$\sum_k |b_{mk}| \to c,\ m \to \infty$$

through some subsequence of positive integers. We also note that

$$\lim_{m\to\infty} b_{mk} = 0 \text{ for all } k \in \mathbb{N}.$$

We can now find a positive integer $m(1)$ such that

$$\left| \sum_k |b_{m(1),k}| - c \right| < \frac{c}{10}$$

and

$$|b_{m(1),0}| + |b_{m(1),1}| < \frac{c}{10}.$$

Since $\Sigma_k |b_{m(1),k}| < \infty$, we can choose $k(2) > 1$ such that

$$\sum_{k=k(2)+1}^{\infty} |b_{m(1),k}| < \frac{c}{10}.$$

It now follows that

$$\left|\sum_{k=2}^{k(2)} |b_{m(1),k}| - c\right| = \left|\left(\sum_{k} |b_{m(1),k}| - c\right) - (|b_{m(1),0}| + |b_{m(1),1}|) - \sum_{k=k(2)+1}^{\infty} |b_{m(1),k}|\right|$$
$$< \frac{c}{10} + \frac{c}{10} + \frac{c}{10} = \frac{3c}{10}.$$

Now choose a positive integer $m(2) > m(1)$ such that

$$\left|\sum_{k} |b_{m(2),k}| - c\right| < \frac{c}{10}$$

and

$$\sum_{k=0}^{k(2)} |b_{m(2),k}| < \frac{c}{10}.$$

Then, choose a positive integer $k(3) > k(2)$ such that

$$\sum_{k=k(3)+1}^{\infty} |b_{m(2),k}| < \frac{c}{10}.$$

It now follows that

$$\left|\sum_{k=k(2)+1}^{k(3)} |b_{m(2),k}| - c\right| < \frac{3c}{10}.$$

Continuing this way, we find $m(1) < m(2) < \cdots$ and $1 = k(1) < k(2) < k(3) < \cdots$ so that

$$\sum_{k=0}^{k(r)} |b_{m(r),k}| < \frac{c}{10}; \tag{1.15}$$

$$\sum_{k=k(r+1)+1}^{\infty} |b_{m(r),k}| < \frac{c}{10}; \tag{1.16}$$

and

$$\left|\sum_{k=k(r)+1}^{k(r+1)} |b_{m(r),k}| - c\right| < \frac{3c}{10}. \tag{1.17}$$

We now define a sequence $x = \{x_k\}$ as follows: $x_0 = x_1 = 0$ and

$$x_k = (-1)^r \operatorname{sgn}\ b_{m(r),k},$$

if $k(r) < k \leq k(r+1)$, $r = 1, 2, \ldots$. Note that $x = \{x_k\} \in m$ and $\|x\| = 1$. Now

$$\begin{aligned}\left|\sum_k b_{m(r),k}x_k - (-1)^r c\right| &= \left|\sum_{k=0}^{k(r)} b_{m(r),k}x_k + \sum_{k=k(r)+1}^{k(r+1)} b_{m(r),k}x_k \right.\\ &\quad \left. + \sum_{k=k(r+1)+1}^{\infty} b_{m(r),k}x_k - (-1)^r c\right|\\ &= \left|\left\{\sum_{k=k(r)+1}^{k(r+1)} |b_{m(r),k}| - c\right\}(-1)^r \right.\\ &\quad \left. + \sum_{k=0}^{k(r)} b_{m(r),k}x_k + \sum_{k=k(r+1)+1}^{\infty} b_{m(r),k}x_k\right|\\ &< \frac{3c}{10} + \frac{c}{10} + \frac{c}{10} = \frac{c}{2},\end{aligned}$$

using (1.15), (1.16) and (1.17).

Consequently, $\{\Sigma_k b_{nk}x_k\}$ is not a Cauchy sequence and so it is not convergent, which is a contradiction. Thus, (1.14) holds. So, given $\epsilon > 0$, there exists a positive integer n_0 such that

$$\sum_k |b_{nk}| < \epsilon, \quad n > n_0. \tag{1.18}$$

Since $\Sigma_k |b_{nk}| < \infty$ for $0 \leq n \leq n_0$, we can find a positive integer M such that

$$\sum_{k=M}^{\infty} |b_{nk}| < \epsilon, \quad 0 \leq n \leq n_0. \tag{1.19}$$

In view of (1.18) and (1.19), we have

$$\sum_{k=M}^{\infty} |b_{nk}| < \epsilon \quad \text{for all } n,$$

that is, $\Sigma_k |b_{nk}|$ converges uniformly in n. Since $\Sigma_k |\delta_k| < \infty$, it follows that $\Sigma_k |a_{nk}|$ converges uniformly in n, proving the necessity part. The proof of the theorem is now complete.

Example 1.4 Let $A = (a_{nk})$ be defined by the lower triangular matrix

$$a_{nk} := \frac{1}{(n+1)(k+1)}. \tag{1.20}$$

Then, $\delta_k = 0$ and

$$\sum_k |a_{nk}| = \frac{1}{n+1}\sum_{k=0}^{n} \frac{1}{k+1} = \frac{1}{n+1} O(\ln(n+1)) \to 0 \ \text{ if } \ n \to \infty;$$

that is, condition (1.13) is fulfilled. Hence, $A \in (m, c)$ by Theorem 1.2.

Using Theorems 1.1 and 1.2, we can deduce the following important result.

Theorem 1.3 (Steinhaus) An infinite matrix cannot be both regular and a Schur matrix. In other words, given a regular matrix, there exists a bounded, divergent sequence which is not A-summable.

Proof: Let A be a regular and a Schur matrix. Then, (1.2) and (1.3) hold with $\delta_k \equiv 0$ and $\delta \equiv 1$. Using (1.13), we get

$$\lim_{n\to\infty} \sum_k a_{nk} = \sum_k \left(\lim_{n\to\infty} a_{nk} \right) = 0$$

by (1.2), which contradicts (1.3). This establishes our claim.

For the proof of the following results, we need some additional notations. Let

$$cs := \left\{ x = (x_k) \ : \ (X_n) \in c; \ X_n := \sum_{k=0}^{n} x_k \right\},$$

$$cs_0 := \left\{ x = (x_k) \mid (X_n) \in c_0; \ X_n := \sum_{k=0}^{n} x_k \right\},$$

$$l := \left\{ x = (x_k) \ : \ \sum_k |x_k| < \infty \right\},$$

$$bv := \{ x = (x_k) : \ (\Delta x_k) \in l \},$$

where

$$\Delta x_k := (\Delta^1 x)_k = x_k - x_{k+1},$$

and

$$bv_0 := bv \cap c_0.$$

It is easy to see that the set of sequences cs is equivalent to the set of all convergent series.

Theorem 1.4 (Hahn) Let $A = (a_{nk})$ be a matrix method. Then, $A \in (l, c)$ if and only if condition (1.2) holds and

$$a_{nk} = O(1). \tag{1.21}$$

Proof: For every fixed k, let e^k be the sequence in which 1 occurs in the $(k+1)^{\text{th}}$ place and 0 elsewhere. As $e^k \in l$, then condition (1.2) is necessary. It is easy to see that we can consider a matrix A as a continuous linear operator from l to c with the norm $\|A\| = \sup_{n,k} |a_{nk}|$. The proof now follows from the Banach–Steinhaus theorem.

Example 1.5 It is easy to see that the methods $Z_{1/2}$, $(C,1)$, $A_{-1,1}$ and the methods A, defined by (1.20), considered in Examples 1.1–1.4, belong (l,c).

Theorem 1.5 Let $A=(a_{nk})$ be a matrix method. Then, $A \in (cs,c)$ if and only if condition (1.2) holds and

$$\sum_k |\Delta_k a_{nk}| = O(1). \tag{1.22}$$

Moreover,

$$\lim_n A_n x = \delta_0 \lim Sx + \sum_k \Delta\delta_k (X_k - \lim Sx) \tag{1.23}$$

for every $x := (x_k) \in cs$.

Proof: First, we find conditions for the existence of the transform Ax for every $x \in cs$. Define

$$y_n^m = \sum_{k=0}^{m} a_{nk} x_k \tag{1.24}$$

for $x := (x_k) \in cs$. Using the Abel's transform (see, e.g., [1], p. 18)

$$\sum_{k=0}^{m} \epsilon_k x_k = \sum_{k=0}^{m-1} \Delta\epsilon_k X_k + \epsilon_m X_m,$$

where

$$X_k := \sum_{l=0}^{k} x_l, \tag{1.25}$$

we can write

$$y_n^m = \sum_{k=0}^{m-1} \Delta a_{nk} X_k + a_{nm} X_m.$$

This implies that Ax exists for every $x \in cs$ if and only

$$y_n := \lim_m y_n^m \tag{1.26}$$

has a finite limit for every $(X_k) \in c$ and $n \in \mathbb{N}$, since, for every $(X_k) \in c$ there exists an $(x_l) \in cs$, such that (1.25) holds. Hence, for every $n \in \mathbb{N}$, the limit y_n in (1.26) exists for every $(X_k) \in c$ if and only if the matrix $D^n := (d^n_{mk}) \in (c,c)$, where

$$d^n_{mk} = \begin{cases} \Delta_k a_{nk}, & \text{if } k < m; \\ a_{nm}, & \text{if } k = m; \\ 0, & \text{if } k > m. \end{cases}$$

As

$$\lim_m d^n_{mk} = \Delta_k a_{nk} \text{ and } \sum_{k=0}^{m} d^n_{mk} = a_{n0},$$

$D^n \in (c, c)$ if and only if

$$\sum_{k=0}^{m-1} |\Delta_k a_{nk}| + |a_{nm}| = O_n(1). \tag{1.27}$$

As

$$|a_{nm}| \leq \sum_{k=0}^{m-1} |\Delta_k a_{nk}| + |a_{n0}|,$$

(1.27) is equivalent to the condition

$$\sum_k |\Delta_k a_{nk}| = O_n(1). \tag{1.28}$$

Moreover, using (1.38) (see Exercise 1.1), from the existence of the limits in (1.26) we obtain

$$y_n - a_{n0} \lim_k X_k = \sum_k (\Delta_k a_{nk})(X_k - \lim_k X_k). \tag{1.29}$$

As $(X_k - \lim_k X_k) \in c_0$, then, using Exercise 1.3, we conclude that transform (1.29) and the finite limit $\lim_n (y_n - a_{n0} \lim_k X_k)$ exists if and only if

$$\text{there exists the finite limit } \lim_n \Delta_k a_{nk} := d_k \tag{1.30}$$

and condition (1.22) holds. We note also that condition (1.28) follows from (1.22). In addition,

$$\lim_n (y_n - a_{n0} \lim_k X_k) = \sum_k d_k (X_k - \lim_k X_k). \tag{1.31}$$

Now (1.31) implies that, for the existence of the finite limit $\lim_n y_n$ it is necessary that

$$\text{there exists the finite limit } \delta_0. \tag{1.32}$$

Therefore, using (1.30), we obtain that, for every $k \geq 1$ the existence of finite limits δ_k; that is, condition (1.2) is necessary. From the other side, condition (1.30) follows from (1.2).

Finally, the validity of (1.23) follows from (1.31) and (1.32).

Example 1.6 Let A be the Zygmund method of order 1; that is, $A = Z^1$, defined by the lower triangular matrix (a_{nk}), where

$$a_{nk} = 1 - \frac{k}{n+1}.$$

Then, $\delta_k \equiv 1$ and $\Delta a_{nk} = 1/(n+1)$ for every $k \le n$. Hence,

$$\sum_k |\Delta_k a_{nk}| = 1;$$

that is, condition (1.22) holds. Thus, $Z^1 \in (cs, c)$ by Theorem 1.5. Moreover, Z^1 is Sr–Sq regular (see Exercise 1.5).

Theorem 1.6 Let $A = (a_{nk})$ be a matrix method. Then, $A \in (bv, c)$ if and only if conditions (1.2), (1.3) hold and

$$\sum_{k=0}^{m} a_{nk} = O(1). \tag{1.33}$$

Moreover, for (bv_0, c) condition (1.3) is redundant.

Proof: As $e \in bv$, then it is necessary for $A \in (bv, c)$ that

$$\text{all series } \sum_{k=l}^{\infty} a_{nk} \text{ converge.} \tag{1.34}$$

Hence,

$$\sum_{k=0}^{m} a_{nk} = O_n(1).$$

Let

$$x_k - \lim x_k := v_k$$

and y_n^m be defined by (1.24) for every $x = (x_k) \in bv$. As $(v_k) \in c_0$ and

$$\sum_k |v_k - v_{k-1}| < \infty \ (v_{-1} = 0)$$

(i.e., $(v_k - v_{k-1}) \in l$), then

$$y_n^m - \lim_k x_k \sum_{k=0}^{m} a_{nk} = \sum_{k=0}^{m} a_{nk} v_k = -\sum_{k=0}^{m} a_{nk} \sum_{l=k+1}^{\infty} (v_l - v_{l-1})$$

$$= -\sum_{l=1}^{m} \left(\sum_{k=0}^{l-1} a_{nk} \right) (v_l - v_{l-1}) - \sum_{l=m+1}^{\infty} \left(\sum_{k=0}^{m} a_{nk} \right) (v_l - v_{l-1}).$$

Hence, for $m \to \infty$, we obtain

$$y_n - \lim_k x_k \sum_k a_{nk} = -\sum_{l=1}^{\infty} \left(\sum_{k=0}^{l-1} a_{nk} \right) (v_l - v_{l-1}), \tag{1.35}$$

where $\lim_m y_n^m = y_n$. Thus, transformation (1.35) exists if condition (1.34) holds. So we can conclude from (1.35) that conditions (1.2) and (1.33) are necessary and sufficient for the existence of the finite limit $\lim_n \left(y_n - \lim_k x_k \sum_k a_{nk}\right)$ by Theorem 1.4, since the existence of the finite limits

$$\lim_n \sum_{k=0}^{l-1} a_{nk}$$

is equivalent to (1.2). As $e \in bv$, then condition (1.3) is necessary. Therefore, from the existence of the finite limit $\lim_n \left(y_n - \lim_k x_k \sum_k a_{nk}\right)$ follows $(y_n) \in c$ for every $x \in bv$.

It is easy to see that for (bv_0, c) the existence of the finite limit δ is redundant.

Example 1.7 As $bv \subset c \subset m$, then (see Examples 1.1–1.4) the methods $Z_{1/2}$, $(C, 1)$, $A_{-1,1}$ and the method A, defined by (1.20), belong (bv, c).

In addition to Theorems 1.1–1.6 we also need the following results.

Theorem 1.7 Let $A = (a_{nk})$ be a matrix method. Then, $A \in (c_0, cs)$ if and only if

$$\text{all series } \mathfrak{A}_k := \sum_n a_{nk} \text{ are convergent,} \tag{1.36}$$

$$\sum_k \left| \sum_{n=0}^{l} a_{nk} \right| = O(1). \tag{1.37}$$

Theorem 1.8 Let $A = (a_{nk})$ be a matrix method. Then, $A \in (c_0, cs_0)$ if and only if $\mathfrak{A}_k \equiv 0$ and condition (1.37) holds.

Theorem 1.9 Let $A = (a_{nk})$ be a matrix method. Then, $A \in (m, bv) = (c, bv) = (c_0, bv)$ if and only if

$$\left| \sum_{n\in L} \sum_{k\in K} (a_{nk} - a_{n-1,k}) \right| = O(1),$$

where K and L are arbitrary finite subsets of $\mathbf{N}$.

As the proofs of Theorems 1.7–1.9 are rather complicated, we advise the interested reader to consult proofs of these results from [7] and [6]. We also note that the proofs of Theorems 1.1–1.6 can be found in monographs [1–3]. We now present some examples.

Example 1.8 For the method $A_{-1,1}$, $\mathfrak{A}_k \equiv 0$ and

$$\sum_k \left| \sum_{n=0}^{l} a_{nk} \right| = 1;$$

that is, condition (1.37) holds. Thus, by Theorem 1.8, $A_{-1,1} \in (c_0, cs_0)$.

Example 1.9 Let a lower triangular method $A = (a_{nk})$ be defined by

$$a_{nk} := \frac{1}{(n+1)^2(k+1)^2}.$$

Then, clearly the series $\mathfrak{A}_k$ converges to some non-zero number for every k, and

$$\sum_{k=0} \left| \sum_{n=0}^{l} a_{nk} \right| = \sum_{k=0}^{l} \frac{1}{(k+1)^2} \sum_{n=k}^{l} \frac{1}{(n+1)^2} = O(1);$$

that is, condition (1.37) holds. Thus, by Theorem 1.7, $A \in (c_0, cs)$, but A does not belong to (c_0, cs_0).

Example 1.10 The method A, defined by (1.20), belongs $(m, bv) = (c, bv) = (c_0, bv)$. Indeed,

$$\left| \sum_{n\in L} \sum_{k\in K} (a_{nk} - a_{n-1,k}) \right| = \sum_{n\in L} \frac{1}{n(n+1)} \sum_{k\in K} \frac{1}{k+1}$$

$$\leq \sum_{n\in L} \frac{1}{n(n+1)} \sum_{k=0}^{n} \frac{1}{k+1} = O(1) \sum_{n\in L} \frac{\ln(n+1)}{n(n+1)} = O(1);$$

that is, condition of Theorem 1.9 is fulfilled.

1.3 Excercise

Exercise 1.1 Prove that $A = (a_{nk})$ is conservative, that is, $A \in (c, c)$ if and only if (1.1) holds and the finite limits δ_k and δ exist.
In such a case, prove that

$$\lim_{n\to\infty} (Ax)_n = s\delta + \sum_k (x_k - s)\delta_k, \tag{1.38}$$

$\lim_{k\to\infty} x_k = s.$

Hint. Use Theorem 1.1.

Exercise 1.2 Try to prove the Steinhaus theorem without using Theorem 1.2, that is, given a regular matrix, construct a bounded, divergent sequence $x = \{x_k\}$ such that $\{(Ax)_n\}$ diverges.

Exercise 1.3 Prove that $A = (a_{nk}) \in (c_0, c)$ if and only if conditions (1.1) and (1.2) hold.

Hint. Use Theorem 1.1.

Exercise 1.4 Let $A = (a_{nk})$ be a matrix method. Prove that, $A \in (c_0, c_0)$ if and only if conditions (1.1) and (1.2) with $\delta_k \equiv 0$ hold.

Exercise 1.5 Prove that $A = (a_{nk})$ is Sr–Sq regular if and only if condition (1.22) holds and $\delta_k \equiv 1$.

Hint. Use the proof of Theorem 1.5.

Exercise 1.6 Prove that method $A = (a_{nk}) \in (m, m) = (c, m) = (c_0, m)$ if and only if condition (1.1) is satisfied.

Hint. For the proof of the necessity, see the proof of Theorem 1.1.

Exercise 1.7 Prove that $A = (a_{nk}) \in (c, c_0)$ if and only if (1.2), (1.3) are satisfied and

$$\lim_n \sum_k a_{nk} = 0.$$

Exercise 1.8 Prove that a method $A = (a_{nk}) \in (m, c)$ if and only if conditions (1.1), (1.2) are satisfied and

$$\lim_n \sum_k |a_{nk} - \delta_k| = 0.$$

Prove that in this case

$$\lim_n A_n x = \sum_k \delta_k x_k$$

for every $x := (x_k) \in m$.

Hint. We note that this result is a modification of Theorem 1.2.

Exercise 1.9 Prove that $A = (a_{nk}) \in (m, c_0)$ if and only if

$$\lim_n \sum_k |a_{nk}| = 0.$$

Hint. Use Theorem 1.2.

Exercise 1.10 Prove that $A = (a_{nk}) \in (l, m)$ if and only if condition (1.21) holds.

Hint. A matrix A can be considered as a continuous linear operator from l to m. To find the norm of A, use the principle of uniform boundedness.

Exercise 1.11 ([Knopp–Lorentz theorem]) Prove that $A = (a_{nk}) \in (l, l)$ if and only if

$$\sum_n |a_{nk}| = O(1).$$

Hint. The proof is similar to the proof of Theorem 1.4. See also hint of Exercise 1.10.

Exercise 1.12 Prove that $A = (a_{nk}) \in (l, bv)$ if and only if

$$\sum_n |a_{nk} - a_{n-1,k}| = O(1).$$

Hint. Let

$$Y_n := \sum_{k=0}^{n} y_k$$

If $(y_k) \in l$, then $(Y_n) \in bv$, and vice versa, if $(Y_n) \in bv$, then $(y_k) \in l$. Denoting $Y_n := A_n x$ for every $x = (x_k) \in l$, we can say that $(Y_n) \in bv$ for every $x \in l$ if and only $(y_k) \in l$ for every $x \in l$, where

$$y_n = Y_n - Y_{n-1} = \sum_k (a_{nk} - a_{n-1,k}) x_k.$$

To find conditions for the existence of Ax, use Theorem 1.4. Then, use Exercise 1.10.

Exercise 1.13 Prove that $A = (a_{nk}) \in (bv, bv)$ if and only if

$$\text{series } \sum_k a_{nk} \text{ are convergent,} \tag{1.39}$$

$$\sum_n \left| \sum_{k=0}^{l} (a_{nk} - a_{n-1,k}) \right| = O(1).$$

Moreover, for (bv_0, bv) condition (1.39) is redundant.

Hint. As $e \in bv$, then for finding conditions for the existence of Ax, use Theorem 1.4. Further use Exercise 1.10.

Exercise 1.14 Let A be defined by (1.20). Does $A \in (c, c)$, $A \in (cs, c)$, $A \in (m, c)$, $A \in (l, l)$, $A \in (l, m)$, $A \in (l, bv)$, and $A \in (c_0, cs)$? Is A Sq–Sq or Sr–Sq regular? Why?

Exercise 1.15 Does $A_{-1,1} \in (m, c)$, $A_{-1,1} \in (l, l)$, $A_{-1,1} \in (l, m)$, $A_{-1,1} \in (l, bv)$, and $A_{-1,1} \in (m, bv)$? Why?

Exercise 1.16 Does $Z_{1/2} \in (m, c)$, $Z_{1/2} \in (c_0, c_0)$, $Z_{1/2} \in (l, l)$, $Z_{1/2} \in (l, m)$, $Z_{1/2} \in (l, bv)$, $Z_{1/2} \in (c_0, cs)$, and $Z_{1/2} \in (m, bv)$? Why?

Exercise 1.17 Does $Z^1 \in (m, c)$, $ZZ^1 \in (c_0, c_0)$, $Z^1 \in (l, l)$, $Z^1 \in (l, m)$, $Z^1 \in (l, bv)$, and $Z^1 \in (m, bv)$? Why?

Exercise 1.18 Prove that the method $A = (a_{nk})$, defined by the lower triangular matrix with

$$a_{nk} := \frac{2k}{(n+1)^2},$$

is Sq–Sq regular.

Exercise 1.19 Prove that m is a Banach space with respect to the norm defined by (1.1).

Exercise 1.20 Prove that c, c_0 are closed subspaces of m under the norm defined by (1.1).

Exercise 1.21 If $A = (a_{nk})$, $B = (b_{nk}) \in (c, c)$, prove that $A + B$, $AB \in (c, c)$, where AB denotes the usual matrix product.

Exercise 1.22 Is A regular, if $(Ax)_n = 2x_n - x_{n+1}$ for all n? Why?

Exercise 1.23 Prove that $A = (a_{nk})$ is a Schur matrix if and only if A sums all sequences of 0's and 1's.

Exercise 1.24 ([Mazur–Orlicz theorem]) If a conservative matrix sums a bounded, divergent sequence, prove that it sums an unbounded one too.

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