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Fundamental Concepts

The last couple of decades have witnessed a phenomenal growth in both the extent and nature of real-life scientific problems which encounter nonlinear data. More specifically, such data as being referred to here do not have support on the real line but are mapped on a circle, sphere, torus, cylinder, disc, or their multivariate generalizations. Examples abound from Astrophysics, Bioinformatics, Defense Sciences, Environmental and Ecological Sciences, Geological Sciences, Health Sciences, Molecular Biophysics, and in fact from virtually all areas of applied research. Uncertainty in the physical world thus demands the development of stochastic models, more specifically of probability distributions, for the underlying populations.

In this chapter, we briefly discuss several characteristics and special features which arise in the construction of such distributions as mentioned above. We expose some interesting properties which differ between these distributions and those for the distributions defined on the $\mathbb{R}$ or $\mathbb{R}^k$. Also discussed are the challenges faced and short comings in generalizing the methods presented for the univariate to the multivariate case. Briefly noted at the end are the distinguishing features which call for special considerations needed for developing statistical inference procedures for these distributions.

1.1 Preliminary Notions

The simplest non-Euclidean manifold we start with is that of the circle S , which can be considered to be obtained from the Euclidean manifold $\mathbb{R}^2$ by the embedding approach $\mathbb{R}^2 \rightarrow S$. Thus, a point on $(x_1, x_2) \in S$ is defined as (i) $(x_1, x_2) | x_1^2 + x_2^2 = 1$. We will be dealing with points on the circumference of the unit circle. Thus three more representations of this point are available: (ii) the angle θ , $\theta \in [0, 2\pi)$, (iii) the vector representation in terms of the trigonometric functions, $(\cos \theta, \sin \theta)$, and (iv) a point in the complex plane $\mathbb{C}$, $e^{i\theta}$, where $i = \sqrt{-1}$. It is usually more

convenient to work in terms of θ as compared to the other representations for the case of the circle, torus, and cylinder. However, we will see in the later chapters that the Euclidean coordinates defining the related polar transformations may be quite useful for the cases of hyper-sphere and hyper-disc. The non-Euclidean manifolds, e.g. circle, torus, cylinder, sphere, etc. to be discussed in this book, will be referred to in general, simply as (corresponding) manifolds.

Random variables (rvs) defined on $\mathbb{R}$ will be referred to as linear rvs while those on circle, torus, etc. as circular, toroidal rvs, etc. The support of the probability density function (pdf) $f(\theta)$ of the rv Θ may be taken as $[0, 2\pi)$ or equivalently $[-\pi, \pi)$. Their graphical representations on the real line may differ and tend to be misleading, but their equivalence is obvious from that on the circle. For example, a symmetric pdf $f(\theta)$ with mode at 0 will be U-shaped on $[0, 2\pi)$, but bell-shaped on $[-\pi, \pi)$ when plotted on the real line. Similarly, $f(\theta)$ plots that may appear asymmetric on the real line will emerge as symmetric when plotted on the circle if they are truly so.

The pdf $f(x)$ of a linear rv X may not exist or be available in any analytical form. For example, for X with support on entire $\mathbb{R}$, a pdf is not meaningful. Also, for the stable family of distributions for X , the pdfs of its members, save only three, cannot be presented analytically. In both these cases, however, the pdfs of the “corresponding” (by wrapping) Θ do exist and can be explicitly presented as will be elaborated below.

Just as differential equations, ordinary or partial, are used to study various linear pdfs, the same can be one for circular or toroidal pdfs too (see, e.g. SenGupta et al., 2026). However, we will not pursue that approach here, mainly because it does not usually lead to interesting new distributions on manifolds, but rather reveals some growth behavior of those in general.

1.2 Characteristic Functions and Trigonometric Moments

We briefly discuss here some important features of the characteristic function and (trigonometric) moments of distributions on circle and torus. Such properties corresponding to other manifolds, e.g. sphere, will be taken up in details in Chapter 8.

1.2.1 Characteristic Functions

Characteristic function (c.f.) provides an important tool for the derivation of the pdf of the associated rv Θ . This is so because of its elegant representation in terms of the trigonometric moments of Θ , and in turn the representation of the pdf of Θ in terms of these trigonometric moments for its Fourier series representation.

An important property of the c.f. of Θ , $\phi_p(\Theta)$, is given by

Result 1.1 $\phi_p(\Theta)$ is defined only for $p \in \mathbb{Z}$, the set of all integers.

The result follows from the fact that $\theta \equiv \theta + p \cdot 2\pi$, $p \in \mathbb{Z}$, $\forall \theta \in [0, 2\pi)$.

Thus, the c.f. of Θ can be written as

$$\phi_p(\Theta) = E(e^{ip\Theta}) = E\{\cos(p\Theta)\} + iE\{\sin(p\Theta)\} \equiv \alpha_p + i\beta_p.$$

By virtue of the result that any strictly periodic function of Θ can be represented by its Fourier series, the pdf of Θ may thus be written as

$$f(\theta) = \frac{1}{2\pi} \left[1 + 2 \sum_{p=1}^{\infty} \{ \alpha_p \cos(p\theta) + \beta_p \sin(p\theta) \} \right].$$

Note that for a symmetric $f(\theta)$, $\beta_p = 0$, $\forall p = 1, 2, 3, \dots$

Let us obtain a circular rv Θ from a linear rv X by wrapping, that is, $\Theta = X \pmod{2\pi}$. A very useful and fundamental property relating the c.f. of the generated rv Θ to that of its generating rv X is given by

Result 1.2 $\phi_p(\Theta) = \phi_{t=p}(X)$, $p \in \mathbb{Z}$, $t \in \mathbb{R}$.

For a symmetric pdf of X , the pdf of the corresponding wrapped rv Θ is then given by

$$\textbf{Result 1.3} \quad f(\theta) = \phi_p(\Theta) = \frac{1}{2\pi} \left\{ 1 + 2 \sum_{p=1}^{\infty} \phi_{t=p}(X) \cos(p\theta) \right\}.$$

This general approach can yield a wide variety of circular pdfs starting from the c.f.s of linear rvs. The striking fact is that Result 1.3 yields the pdf of Θ in the above simple analytic form even though the pdf of the generating rv X may not be so available. Two important examples of this fact are the wrapped stable and wrapped Linnik families of circular rvs.

1.2.2 Trigonometric Moments

Unlike that for a linear rv X , usual p th power moments $E(\Theta^p)$ are not sensible measures for Θ . Instead, one defines $\gamma_p(\Theta) \equiv (\alpha_p, \beta_p)$ as the trigonometric moment of order p , $p = 1, 2, 3, \dots$. It is worthwhile to note that, unlike the case of linear pdfs, the (trigonometric) moment γ_p exists for all orders for all circular distributions. However, the descriptive measures need not do so. As an example the population mean direction defined by $\mu = \arctan^*(\beta_1/\alpha_1)$, where “*” denotes the quadrant specific value, does not exist for the circular uniform distribution since $\alpha_1 = \beta_1 = 0$ for it. A measure of circular dispersion is given by $E[1 - \cos(\Theta - \mu)]$.

Bivariate or joint trigonometric moments of order (p, q) of two circular rvs Θ and Φ are defined similarly by using the product moments of the trigonometric functions as $(E\{\cos(p\Theta) \cdot \cos(q\Phi)\}, E\{\cos(p\Theta) \cdot \sin(q\Phi)\}, E\{\sin(p\Theta) \cdot \cos(q\Phi)\}, E\{\sin(p\Theta) \cdot \sin(q\Phi)\})$, $p, q = 1, 2, \dots$. Any measure of association between the two circular rvs should involve an association measure of their corresponding vectorial representations, $\mathbf{X}(\Theta) = (\cos \Theta, \sin \Theta)'$, $\mathbf{Y}(\Phi) = (\cos \Phi, \sin \Phi)'$, that is, involve all the four components of the joint trigonometric moment. Such a measure was obtained through the canonical correlation between $\mathbf{X}(\Theta)$ and $\mathbf{Y}(\Phi)$ by Jupp and Mardia (1980) and shown to involve the usual product moment correlations of the four pairs of the trigonometric rvs $(\cos \Theta, \cos \Phi)$, $(\cos \Theta, \sin \Phi)$, $(\sin \Theta, \cos \Phi)$, $(\sin \Theta, \sin \Phi)$.

1.3 Derivations of Probability Density Functions

Pdfs of rvs on manifolds possess some unique and interesting features. We discuss a few of those below.

1.3.1 Periodicity, Truncation, and Wrapping

Since Θ is an angle, its pdf $f(\theta)$ must be strictly periodic, that is, $f(\theta) = f(\theta + k \cdot 2\pi)$, $k \in \mathbb{Z}$, $\forall \theta \in [0, 2\pi)$. This requirement is in addition to the usual requirements of non-negativity and integrability to 1 as needed to define a pdf for a linear rv X . Thus, the representation of its pdf in terms of the trigonometric functions and Fourier series as mentioned above follow naturally. This is also true for the joint pdf of two circular rvs Θ and Φ , that is, for the pdf on the torus, for which a bivariate (doubly periodic) Fourier series always exists and can be explicitly written down. Here also the approach of using c.f.s as in the univariate case is rewarding and leads to explicit forms of pdfs on the torus for which the pdfs need not even exist in analytical forms, e.g. bivariate stable and Linnik families of toroidal distributions. Further details will be provided in Chapter 9.

We noted that mere truncation of a linear pdf on $(0, 2\pi)$ will not yield periodicity and also fails to utilize the remaining part of the pdf. However, the method of wrapping, which wraps the truncated pdf as given below, satisfies both these requirements. Let the linear rv X be wrapped as $\Theta = X \pmod{2\pi}$. Let X have the pdf $g(x)$. Then the corresponding wrapped pdf $f(\theta)$ of Θ is given by

$$f(\theta) = \sum_{k=-\infty}^{\infty} g(\theta + k \cdot 2\pi), \quad \theta \in [0, 2\pi).$$

Choice of various distributions for X leads to the corresponding variety of circular wrapped distributions.

Unit distributions may appear to be quite attractive for deriving circular distributions therefrom, since the corresponding rv X supported on $[0,1]$ may be easily transformed to have support on $[0, 2\pi)$. However, here also the periodicity needs to be enforced. Such constructions may nevertheless be fruitful, e.g. see circular beta distribution in Chapter 4.

Another powerful method of constructing distributions on manifolds from those for the Euclidean space is that of projection. For the univariate case, consider the transformation from the real line to the unit circle given by $x = \tan(\theta/2)$. This can be equivalently viewed as the corresponding Möbius transformation from the line to the complex plane. Thus, given the pdf $g(x)$ of X one can derive the pdf of Θ and vice versa. Minh and Farnum (2003) have shown that the linear Cauchy and t -distributions are obtained by this method from the circular uniform and Cartwright distributions. This method also generalizes (stereographic and inverse stereographic projections) to derive distributions on the sphere manifold S^2 from those on $\mathbb{R}^2$. Interestingly, this method faces difficulties in deriving toroidal distributions.

1.3.2 Marginal Distributions and Independence

In the generalizations of univariate to multivariate distributions, two properties are often preferred: (i) marginals and joint distributions should be in the similar families and (ii) joint distributions should include the case of independence of the variables. For example, the bivariate normal distribution possesses both these properties, with the nullity of the parameter ρ , the correlation coefficient in it, yielding independence. However, this is not the case with its counterpart, the bivariate von Mises (BvM) distribution, defined on the torus, as will be seen in Chapter 9. The marginals of the BvM distribution are not univariate von Mises (vM) distributions (unless these are independent) but are of no known standard forms. Independence in BvM distribution however results from the nullity of a matrix of its parameters as will be shown there also. A generalization of vM distribution to the toroidal case which possesses both the properties (i) and (ii) is not yet available.

Toroidal distributions derived from their c.f.s, as in their Fourier series representations, in general will possess property (i). Property (ii) may not be obvious from these representations, but follows from their corresponding c.f.s.

In many cases, c.f. may not be readily available and it may be nontrivial to check whether independence is included in the pdf. An example of this fact is the possibly asymmetric pdf derived by a series of conformal mappings for the hyper-disc given in Chapter 11.

1.4 Statistical Inference and Applications

Special methods, unique to the distributions on manifolds, are often needed for developing related statistical inference procedures.

1.4.1 Estimation and Testing of Hypotheses

Even for the circular case, the normalizing constants can be quite complicated involving series of special functions, and routine maximum-likelihood (ML) methods may be slow in convergence. Here the trigonometric moments method (TMM) are frequently found to be of great help by using the representations in terms of the corresponding Fourier series. In fact, these TMM estimates may well serve as initial values when iterative techniques are called for in the ML method.

For testing of hypotheses, tackling of the nuisance parameter(s) can be a challenging problem. This is more so when the nuisance parameter is the concentration parameter, as for the case of even the vM distribution. Since no invariance or similarity property exists for such a parameter, exact or unconditional tests are often not available to mitigate this nuisance parameter.

1.4.2 Testing Goodness-of-Fit

Indiscriminate use of a distribution for modeling the underlying data can be often grossly misleading with devastating consequences. Thus, the suitability of the distribution used is of prime importance. This is achieved through the Goodness-of-Fit (GoF) tests. However, the GoF tests used for the linear case are not appropriate for the manifolds considered here. This is so due to the requirement of invariance with respect to choice of origin for the data. Simple modifications of the test statistics for linear data have been successfully incorporated to yield the corresponding tests for the circular case, e.g. Watson's U^2 test as a modification of the vM functional test. C.f.-based GoF tests can be used for certain manifolds, e.g. the torus where the c.f. is conveniently available. However, to date there does not seem to exist GoF tests for pdfs on several other manifolds, e.g. the pdfs on the cylinder, hyper-disc, etc. This is an important open problem.

1.4.3 Applications

With the emergence of new technologies, interesting datasets on various manifolds are in the need of proper analysis. In the context of Big Data, data on manifolds is a member of this group due to its rich variety. Thus, for supervised and unsupervised learning with these datasets, derivations of distributions on manifolds have become imperative, e.g. for model-based analysis in Statistical Machine Learning. Herein lies a rich and fertile arena for challenging and useful research with a panorama of datasets in our modern era.

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