CHAPTER 1

INTRODUCTION

1.1 Electromagnetic Problems and Classification

Electromagnetic (EM) problems are classified in terms of the equations describing them. The equations could be differential or integral or both. Most EM problems can be stated in terms of an operator equation

$$L\varphi = g \tag{1.1}$$

where L is an operator (differential, integral, or integro-differential), g is the known excitation or source, and φ is the unknown function to be determined. A typical example is an electrostatic problem involving Poisson's equation

$$-\nabla^2 V = \frac{\rho}{\varepsilon} \tag{1.2}$$

where $L = -\nabla^2$ is Laplacian operator, $g = \rho/\varepsilon$ is source term, and $\varphi = V$. In integral form, Poisson's equation is of the form

$$V = \int \frac{\rho dv}{4\pi\varepsilon r^2} \tag{1.3}$$

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where $L = \int \frac{dv}{4\pi r^2}$ is Laplacian operator, $g = V$ is source term, and $\varphi = \rho/\varepsilon$.

Electromagnetic problems involve linear, second-order differential equations. In general, a second-order partial differential equation (PDE) is given by

$$a\frac{\partial^2\varphi}{\partial x^2} + b\frac{\partial^2\varphi}{\partial x\partial y} + c\frac{\partial^2\varphi}{\partial y^2} + d\frac{\partial\varphi}{\partial x} + e\frac{\partial\varphi}{\partial y} + f\varphi = g \tag{1.4}$$

where the differential operator is

$$L = a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f. \tag{1.5}$$

The coefficients, a, b, and c, in general are functions of x and y; they may also depend on φ itself, in which case the PDE is said to be nonlinear. A PDE in which $g(x, y)$ equals zero is termed homogeneous; it is inhomogeneous if $g(x, y)$ is not equal to zero.

A PDE, in general, can have both boundary values and initial values. PDEs whose boundary conditions (BCs) are specified are called steady-state equations. If only initial values are specified, they are called transient equations.

Any linear second-order PDE can be classified as elliptic, hyperbolic, or parabolic depending on the coefficients a, b, and c. The terms hyperbolic, parabolic, and elliptic are derived from the fact that the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \tag{1.6}$$

represents a hyperbola, parabola, or ellipse if $b^2 - 4ac$ is positive, zero, or negative, respectively.

In each of these categories, there are PDEs that model certain physical phenomena. Such phenomena are not limited to electromagnetics but extend to almost all areas of science and engineering. Thus the mathematical model specified here arises in problems involving heat transfer, boundary-layer flow, vibrations, elasticity, electrostatic, wave propagation, and so on.

Elliptic PDEs are associated with steady-state phenomena, that is boundary-value problems. Typical examples of this type of PDE include Laplace's equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{1.7}$$

and Poisson's equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = g(x, y) \tag{1.8}$$

where in both cases $a = c = 1$, $b = 0$. An elliptic PDE usually models an interior problem, and hence the solution region is usually closed or bounded.

Hyperbolic PDEs arise in propagation problems. The solution region is usually open so that a solution advances outward indefinitely from initial conditions while always satisfying specified BCs. A typical example of hyperbolic PDE is the wave equation in one dimension

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} \tag{1.9}$$

where $a = v^2$, $b = 0$, $c = 1$. If the time dependence is suppressed, the equation is merely the steady-state solution.

Parabolic PDEs are generally associated with problems in which the quantity of interest varies slowly in comparison with the random motions which produce the variations. The most common parabolic PDE is the diffusion (or heat) equation in one dimension

$$\frac{\partial^2 \varphi}{\partial x^2} = k\frac{\partial \varphi}{\partial t} \tag{1.10}$$

where $a = 1, b = c = 0$.

In hyperbolic and parabolic PDEs, the solution region is usually open. The initial conditions and BCs typically associated with parabolic equations resemble those for hyperbolic problems except that only one initial condition at $t = 0$ is necessary since the parabolic equation (PE) is only first-order in time. Also, parabolic and hyperbolic equations are solved using similar techniques, whereas elliptic equations are usually more difficult and require different techniques.

The type of problem represented by $L\varphi = g$ is said to be deterministic, since the quantity of interest can be determined directly. Another type of problem where the quantity is found indirectly is called non-deterministic or eigenvalue. The standard eigenproblem is of the form $L\varphi = \lambda\varphi$. A more general version is the generalized eigenproblem having the form $L\varphi = M\lambda\varphi$, where M, like L, is a linear operator for EM problems. Here, only some particular values of λ called eigenvalues are permissible; associated with these values are the corresponding solutions called eigenfunctions. Eigenproblems are usually encountered in vibration and waveguide problems where the eigenvalues λ correspond to physical quantities such as resonance and cutoff frequencies, respectively.

Our problem consists of finding the unknown function φ of a PDE. In addition to the fact that φ satisfies $L\varphi = g$ within a prescribed solution region R, φ must satisfy certain conditions on S, the boundary of R. Usually these BCs are

$$\varphi(\mathbf{r}) = 0, \ \mathbf{r} \text{ on } S, \ \text{(Dirichlet type)} \tag{1.11}$$

$$\frac{\partial \varphi(\mathbf{r})}{\partial n} = 0, \ \mathbf{r} \text{ on } S, \ \text{(Neumann type).} \tag{1.12}$$

Here the normal derivative φ vanishes on S for Neumann type. Where a boundary has both, a mixed (Cauchy) boundary condition (CBC) is said to exist

$$\frac{\partial \varphi(\mathbf{r})}{\partial n} + h(\mathbf{r})\varphi(\mathbf{r}) = 0, \ \mathbf{r} \text{ on } S \tag{1.13}$$

where $h(\mathbf{r})$ is known function, $\frac{\partial \varphi(\mathbf{r})}{\partial n} = \mathbf{n}\,\nabla\varphi(\mathbf{r})$ is the directional derivative of φ along the outward normal to the boundary S, and $\mathbf{n}$ is a unit normal directed out of R. Note that the Neumann BC is a special case of the mixed condition with $h(\mathbf{r}) = 0$.

1.2 Maxwell Equations

The Maxwell equations are four differential equations which show classical properties of EM fields by using electric and magnetic fields. The equations are summarized in Table 1.1. Here, ρ_v is the electric volume charge density in C/m^3, $\mathbf{J}$ is the electric current density vector in A/m^2, $\mathbf{E}$ and $\mathbf{H}$ show the electric and magnetic field intensity vectors in V/m and A/m, respectively, and, $\mathbf{D}$ and $\mathbf{B}$ show the electric and magnetic flux density vectors in C/m^2 and Wb/m^2, respectively. The first two equations are related to the divergence of vectors, and the others are related to the curl operation of vectors. The relations between these vectors in simple medium are $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{J} = \sigma\mathbf{E}$, where ε, μ, and σ denote the permittivity (F/m), the permeability (H/m), and the electric conductivity (S/m) of the medium.

Table 1.1 The Maxwell equations using differential form.

Name	Differential Form	Name	Differential Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	Maxwell–Ampere's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$

1.3 Guided Waves and Transverse/Longitudinal Decomposition

Guided wave propagation problems can be solved by using simplified equations obtained from the longitudinal and transverse decomposition of Maxwell equations that yield the transverse electric (TE), transverse magnetic (TM), and transverse electromagnetic (TEM) representations under different polarizations, such as perpendicular/parallel polarizations or horizontal/vertical polarizations in applications.

The open form of Maxwell curl operations of electric and magnetic fields for rectangular coordinate system can be written as

$$\begin{aligned}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\mathbf{z}} \\ = -\mu\frac{\partial H_x}{\partial t}\hat{\mathbf{x}} - \mu\frac{\partial H_y}{\partial t}\hat{\mathbf{y}} - \mu\frac{\partial H_z}{\partial t}\hat{\mathbf{z}}\end{aligned} \tag{1.14}$$

$$\begin{aligned}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\hat{\mathbf{z}} \\ = \left(J_x + \varepsilon\frac{\partial E_x}{\partial t}\right)\hat{\mathbf{x}} + \left(J_y + \varepsilon\frac{\partial E_y}{\partial t}\right)\hat{\mathbf{y}} + \left(J_z + \varepsilon\frac{\partial E_z}{\partial t}\right)\hat{\mathbf{z}}.\end{aligned} \tag{1.15}$$

A rectangular waveguide is a classical three-dimensional (3D) guiding structure. If a rectangular waveguide is located longitudinally along z-axis, TE/TM cases are defined by assuming no electric/magnetic field component in the direction of propagation, therefore the governing equations are given in Table 1.2.

The boundary should also be taken into consideration to determine the polarization type [1]. First of all, let us define the plane of incidence as the plane containing the normal to the boundary surface and the direction of propagation of the wave. For

Table 1.2 The governing equations for rectangular waveguide along z-axis.

TE Polarization	TM Polarization
$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t}$	$\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} = \mu \frac{\partial H_x}{\partial t}$
$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t}$	$\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} = \mu \frac{\partial H_y}{\partial t}$
$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = \mu \frac{\partial H_z}{\partial t}$	$\frac{\partial E_y}{\partial x} = \frac{\partial E_x}{\partial y}$
$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \varepsilon \frac{\partial E_x}{\partial t}$	$\frac{\partial H_y}{\partial z} = -J_x - \varepsilon \frac{\partial E_x}{\partial t}$
$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \varepsilon \frac{\partial E_y}{\partial t}$	$\frac{\partial H_x}{\partial z} = J_y + \varepsilon \frac{\partial E_y}{\partial t}$
$\frac{\partial H_y}{\partial x} = \frac{\partial H_x}{\partial y}$	$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \varepsilon \frac{\partial E_z}{\partial t}$

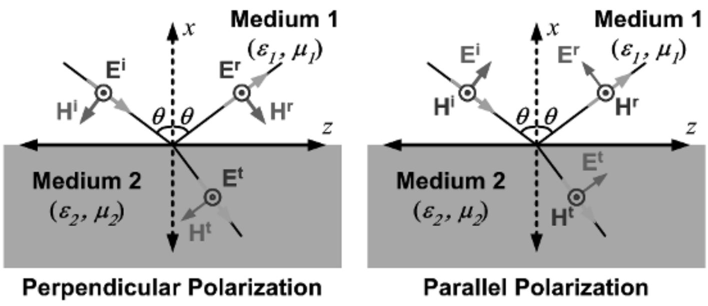

Figure 1.1 Perpendicular and parallel polarization on the zx-plane.

example, the plane of incidence is zx-plane in Fig. 1.1. Here, the electric field is either perpendicular to the plane of incidence for perpendicular polarization or parallel to the plane of incidence for parallel polarization. The governing equations are given in Table 1.3.

1.4 Two Dimensional Helmholtz's Equation

The wave equations for the electric and magnetic fields can be obtained by using Maxwell equations. If the electric and magnetic fields are to be time harmonic

Table 1.3 The governing equations for the plane of incidence on the zx-plane.

TE Polarization	TM Polarization
$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t}$	$\frac{\partial H_y}{\partial x} = J_z + \varepsilon \frac{\partial E_z}{\partial t}$
$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t}$	$\frac{\partial H_y}{\partial z} = -J_x - \varepsilon \frac{\partial E_x}{\partial t}$
$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \varepsilon \frac{\partial E_y}{\partial t}$	$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t}$

with the time dependence $\exp(-i\omega t)$, the wave equations in a linear, homogeneous, isotropic, source-free medium for each component of fields can be written as

$$\nabla^2 U - \mu\varepsilon \frac{\partial^2 U}{\partial t^2} = 0 \text{ (in time-domain)} \tag{1.16}$$

$$\nabla^2 U + k^2 U = 0 \text{ (in frequency-domain)} \tag{1.17}$$

where $k = \omega\sqrt{\mu\varepsilon}$ is the wavenumber, ω is the angular frequency, ∇^2 is the Laplace operator, and U shows the components of time harmonic either electric field or magnetic field. This is called homogeneous wave equation or Helmholtz's equation, that is the elliptic PDE. Assume an zx-plane as the two-dimensional (2D) environment, the Helmholtz's equation can be written as

$$\frac{\partial^2 U}{\partial z^2} + \frac{\partial^2 U}{\partial x^2} + k^2 U = 0. \tag{1.18}$$

1.5 Validation, Verification, and Calibration Procedure

Real-life engineering and EM problems can be handled via measurements or numerical simulations because only a limited number of problems with idealized geometries have mathematical exact solutions. The challenge in solving real-life engineering problems is therefore the reliability of the results. Reliability is achieved after a series of (model) validation, (data) verification, and (code) calibration (VV&C) tests [2].

Three fundamental building blocks of a simulation are the real-world problem entity being simulated, the conceptual model representation of that entity, and the computer implementation model. As illustrated in Fig. 1.2, engineers start with the definition of the real-life problem at hand. Electromagnetic problems, in general, are modeled with Maxwell equations and EM theory is well established by these equations. Maxwell equations are general and represent all linear EM problems. Once the geometry of the problem at hand (i.e., BCs) is given, they represent a unique

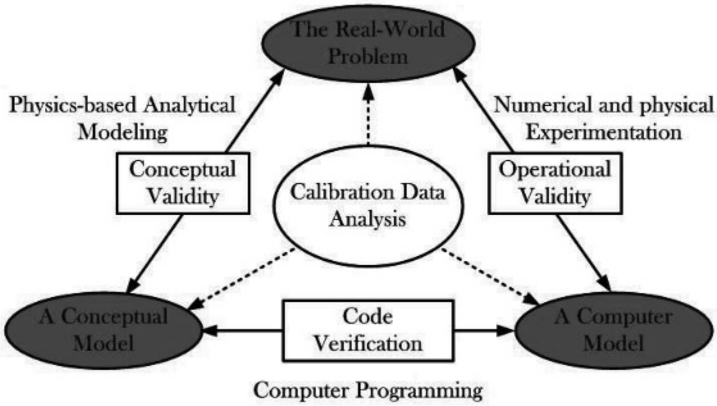

Figure 1.2 Validation, verification, and calibration procedure.

solution; the solution found by using Maxwell equations plus BC is the solution we are looking for. Unfortunately, there are only a few real-life problems which have mathematical exact solutions, therefore many different and approximate conceptual models can be used. It is the process of conceptual validity which shows that chosen conceptual model fits into the real-life problem the best under the specified initial and/or operational conditions. The next step is to develop a computer code for the chosen conceptual model. It is only after code verification via a computer programming process applied to show that the developed code represents the chosen conceptual model under given sets of conditions (accuracy, resolution, uncertainty, etc.). Finally, the solution for the real-life problem is obtained with confidence after numerical and/or physical experimentation; nothing but the operational validity process [3].

For the parabolic wave equation (PWE) chosen in this book, the VV&C procedure necessitates quantitatively and qualitatively answering these questions: (i) How precise is the PWE model? (ii) To what extent does the PWE correspond to the real-life problem? (iii) Under what/which conditions do different numerical methods yield reliable solutions? (iv) What is the accuracy of the numerical calculations?

1.6 Fourier Transform, DFT and FFT

The Fourier transform has been widely used in circuit analysis and synthesis, from filter design to signal processing, image reconstruction, etc. The reader should keep in mind that the time-domain and frequency-domain relations in electromagnetics are very similar to the relations between spatial and wavenumber domains. A simplest propagating (e.g., along z) plane wave is in the form of $\Phi\left(r,t\right) \propto e^{-i(\omega t - kz)}$ (where k and z are the wavenumber and position, respectively) and $\exp\left(-i\omega t\right)$ are also applicable to $\exp\left(ikz\right)$. Some characteristics are outlined as

- A rectangular time (frequency) window corresponds to a beam type (Sinc($\cdot$) function) variation in frequency (time)-domain.
- Similarly, a rectangular aperture (array) in spatial-domain corresponds to a beam type (Sinc($\cdot$) function) variation in wavenumber domain.
- The wider the antenna aperture the narrower the antenna beam; or, the narrower the pulse in time-domain the wider the frequency band.

Therefore, Fourier transform has also been used in electromagnetics from antenna analysis to imaging and non-destructive measurements, even in propagation problems. For example, the split-step parabolic equation (SSPE) method (which is nothing but the beam propagation method in optics) has been in use for several decades and is based on sequential Fourier transform operations between the spatial and wavenumber domains. Two- and three-dimensional propagation problems with non-flat realistic terrain profiles and inhomogeneous atmospheric variations above have been solved with this method successfully [3–5].

The principle of a transform in engineering is to find a different representation of a signal under investigation. The Fourier transform is the most important transform that is widely used in electrical engineering. The transformations between the time and the frequency-domains are based on the Fourier transform and its inverse Fourier transform. They are defined via

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{i2\pi ft} dt, \text{ and } s(t) = \int_{-\infty}^{\infty} S(f)\, e^{-i2\pi ft} df. \tag{1.19}$$

Here, $s(t)$, $S(f)$, and f are the time signal, the frequency signal, and the frequency, respectively. We, the physicists and engineers, sometimes prefer to write the transform in terms of angular frequency $\omega = 2\pi f$, as

$$S(\omega) = \int_{-\infty}^{\infty} s(t)\, e^{i\omega t} dt, \text{ and } s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{-i\omega t} d\omega \tag{1.20}$$

which, however, destroys the symmetry. To restore the symmetry of the transforms, the convention is to divide $1/(2\pi)$ term into two and use $1/\sqrt{2\pi}$ during both Fourier transform and inverse Fourier transform. The Fourier transform is valid for real or complex signals, and, in general, is a complex function of ω (or f).

The Fourier transform is valid for both periodic and non-periodic time signals that satisfy certain conditions. Almost all real-world signals easily satisfy these requirements. It should be noted that the Fourier series is a special case of the Fourier transform. Mathematically, Fourier transform is defined for continuous time signals and in order to go to the frequency-domain, the time signal must be observed from an infinite-extend time window. Under these conditions, the Fourier transform defined above yields frequency behavior of a time signal at every frequency, with zero frequency resolution. Some functions and their Fourier transform are listed in Table 1.4. To compute the Fourier transform numerically on a computer, discretization plus numerical integration are required. This is an approximation of the true

Table 1.4 Some functions and their Fourier transforms.

Time Domain	Fourier Domain
Rectangular window	Sinc function
Sinc function	Rectangular window
Constant function	Dirac delta function
Dirac delta function	Constant function
Dirac comb (Dirac train)	Dirac comb (Dirac train)
Cosine function	Two real-even delta function
Sine function	Two imaginary-odd delta functions
Exponential function $\{\exp(-i\omega t)\}$	One positive-real delta functions
Gaussian function	Gaussian function

(i.e., mathematical), analytically defined Fourier transform in a synthetic (digital) environment, and is called the discrete Fourier transform (DFT). There are three difficulties with the numerical computation of the Fourier transform.

- *Discretization* (introduces periodicity in both time and frequency-domains),
- *Numerical integration* (introduces approximation and numerical round off and truncation errors),
- *Finite time duration* (introduces maximum frequency and resolution limitations).

The DFT of a continuous time signal sampled over a record period of T, with a sampling rate of Δt can be given as

$$S(m\Delta f) = \frac{T}{N} \sum_{n=0}^{N-1} s(n\Delta t)\, e^{i2\pi m \Delta f n \Delta t} \tag{1.21}$$

where $\Delta f = 1/T$, and, is valid at frequencies up to $f_{max} = 1/(2\Delta t)$. A simple MATLAB *dft_sin.m* file computes (1.21) for a time record $s(t)$ of two sinusoids whose frequencies are user specified. The record length and sampling time interval are also supplied by the user and DFT of this record is calculated inside a simple integration loop.

Let us plot two sinusoids with 10 Hz/1 V and 50 Hz/0.25 V both in time and frequency-domains. Choose $f_{max} = 200$ Hz, $\Delta f = 2$ Hz, $T = 0.5$ s, and $\Delta t = 2.5$ ms. Note that these parameters may be chosen arbitrarily in DFT but frequency resolution and maximum frequency will be $\Delta f = 1/T$ and $f_{max} = 1/(2\Delta t)$, respectively. Results are shown in Fig. 1.3.

The DFT requires an excessive amount of computation time, particularly when the number of samples N is high. The fast Fourier transform (FFT) is an algorithm to speed up DFT computations. The FFT forces one further assumption that N is an

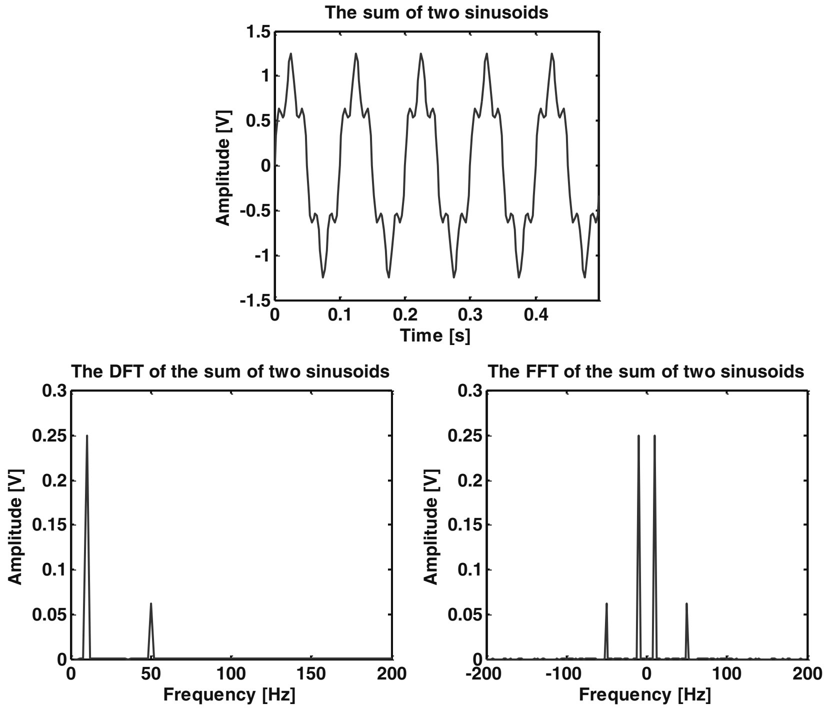

Figure 1.3 (Top) Time variations of two sinusoids, (bottom) frequency variations of two sinusoids obtained with (left) DFT and (right) FFT.

integer multiple of 2. This allows certain symmetries to occur reducing the number of calculations.

To write an FFT routine is not as simple as DFT routine, but there are many internet addresses where one can supply FFT subroutines (including source codes) in different programming languages, from Fortran to C++. Therefore, the reader does not need to go into details, rather include them in their codes by simply using *include* statements or *call* commands. In MATLAB, the calling command is *fft(s,N)* for the FFT and *ifft(S,N)* for the inverse FFT, where s and S are the recorded N-element time array and its Fourier transform, respectively. In order to do that one needs to replace the loop for DFT with a line code *Sf = fftshift(fft(st)*dt)*, that is included in *fft_sin.m* file. Note that one needs to scale the results in the frequency-domain (i.e., multiply the result by Δt) since MATLAB *fft(s,N)* command assumes $\Delta t = 1$; also, swap the first $N/2$ samples with the second half using the *fftshift(s)* command). Figure 1.3 also shows the FFT of the same signal.

As stated above, performing Fourier transform in a discrete environment introduces artificial effects. These are called aliasing effects, spectral leakage, and scalloping loss [3]. It should be kept in mind when dealing with DFT that

- Multiplication in the time-domain corresponds to a convolution in the frequency-domain.
- The Fourier transform of an impulse train in the time-domain is also an impulse train in the frequency-domain with the frequency samples separated by $T_0 = 1/f_0$.
- The narrower the distance between impulses (T_0) in the time-domain the wider the distance between impulses (f_0) in the frequency-domain (and vice versa).
- The sampling rate must be greater than twice the highest frequency of the time record, that is $\Delta t \geq 1/(2f_{max})$ (Nyquist sampling criterion).
- Since *time–bandwidth* product is constant, narrow transients in the time-domain possess wide bandwidths in the frequency-domain.
- In the limit, the frequency spectrum of an impulse is constant and covers the whole frequency-domain (that is why an impulse response of a system is enough to find out the response of any arbitrary input).

If the sampling rate in the time-domain is lower than the Nyquist rate, *aliasing* occurs [3]. Two signals are said to alias if the difference of their frequencies falls in the frequency range of interest, which is always generated in the process of sampling (aliasing is not always bad; it is called mixing or heterodyning in analog electronics and is commonly used in tuning radios and TV channels). It should be noted that although obeying Nyquist sampling criterion is sufficient to avoid aliasing, it does not give a high quality display in time-domain record. If a time signal sinusoid is not bin-centered in the frequency-domain then *spectral leakage* occurs. In addition, there is a reduction in coherent gain if the frequency of the sinusoid differs in value from the frequency samples, which is termed *scalloping loss*.

Fourier transform is used for *energy signal* which contains finite energy. This means $\int_{-\infty}^{\infty} |s(t)|^2\, dt$ is finite. Periodic functions do not satisfy this property. *Power signals* have finite power in one period (i.e., $\frac{1}{P}\int_{-P/2}^{P/2} |s(t)|^2\, dt$ is finite where P is the period of the signal). Power signals are represented in terms of Fourier series. A function $f(x)$ is periodic, with period P, if

$$f(x) = \sum_{n=1}^{\infty} f(x + nP)\,. \tag{1.22}$$

A periodic function $f(x)$ can be approximated by using Fourier series expansion as

$$f(x) \approx \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n x}{P}\right) + B_n \sin\left(\frac{2\pi n x}{P}\right) \tag{1.23}$$

where

$$A_0 = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\, dx \tag{1.24}$$

$$A_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\cos\left(\frac{2\pi nx}{P}\right)dx \tag{1.25}$$

$$B_n = \frac{2}{P}\int_{-P/2}^{P/2} f(x)\sin\left(\frac{2\pi nx}{P}\right)dx. \tag{1.26}$$

Non-periodic functions may also be approximated by Fourier series inside a finite region by assuming the finite region as the period of that function. In this case, it should be remembered that the Fourier series representation no longer represents the function outside the region. Equations (1.24)–(1.26) show that, one needs to multiply the function with sine and cosine functions along the whole period and then integrate in order to find out Fourier coefficients A_n and B_n.

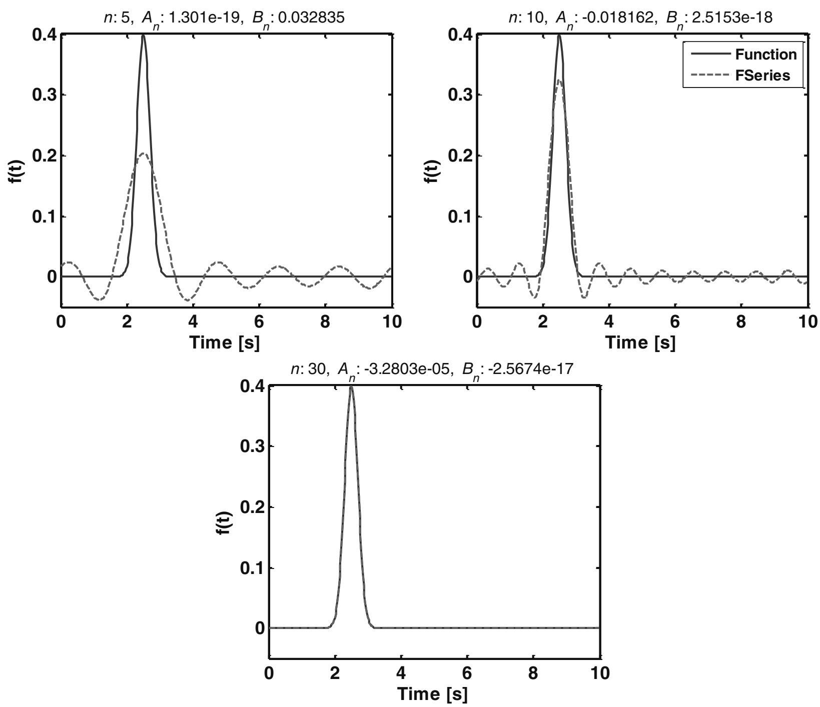

Figure 1.4 Gaussian function and its Fourier series approximation with 5, 10, 30 terms.

Fseries.m lists a simple MATLAB code for the Fourier series representation of a given function. A few examples plotted with this MATLAB code are given in Fig. 1.4.

The Gaussian function used in this example is

$$f(t) = \frac{\exp\left(-12.5(t-2.5)^2\right)}{\sqrt{2\pi}} \tag{1.27}$$

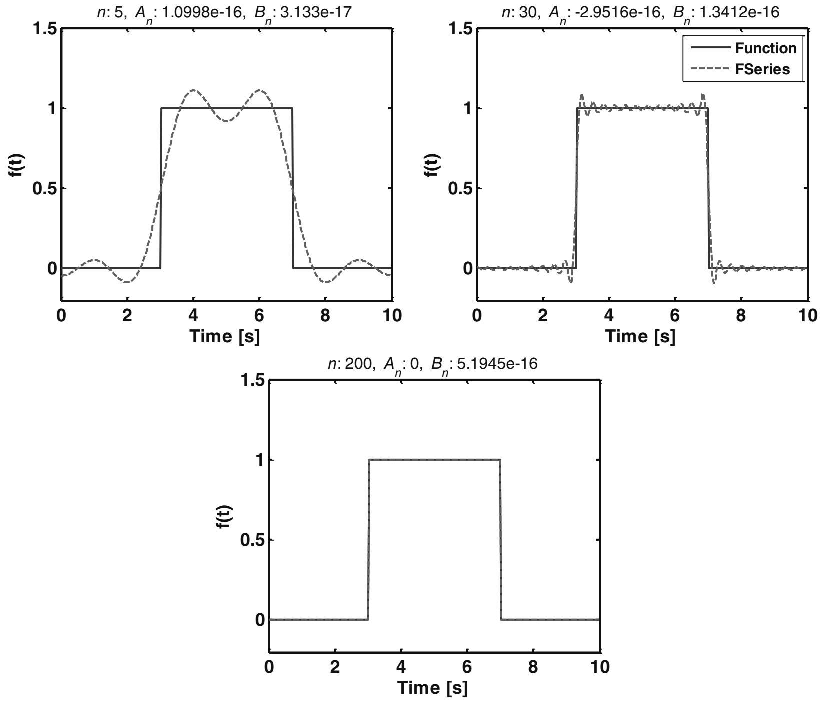

Figure 1.5 Rectangular pulse and its Fourier series approximation with 5, 30, 200 terms.

and the interval (i.e., the period) is $0 \leq t \leq 10$. First, only the first five terms are used and the result is plotted in Fig. 1.4. The solid and dashed lines in the figures correspond to the function and its Fourier series approximation, respectively. Second figure corresponds to the same scenario but with the first ten terms in the series summation. Finally, the last figure belongs to the same comparisons with the first thirty terms. The agreement in curves in Fig. 1.4 shows that, thirty terms are adequate for this function in this interval (period).

As shown above, any piecewise continuous function may be approximated by a series summation of sine and cosine functions. The number of terms required in the Fourier series representation depends on the smoothness of the function and the specified accuracy. The degree of smoothness of the function determines the number of terms in its Fourier representation. In addition, only sine or cosine terms contribute the function if it is odd or even symmetric. Understanding digital communication concepts, the frequency content of the rectangular pulse should be well analyzed. A symmetric rectangular pulse is defined as

$$\text{Rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & t > \left|\frac{T}{2}\right| \end{cases} \tag{1.28}$$

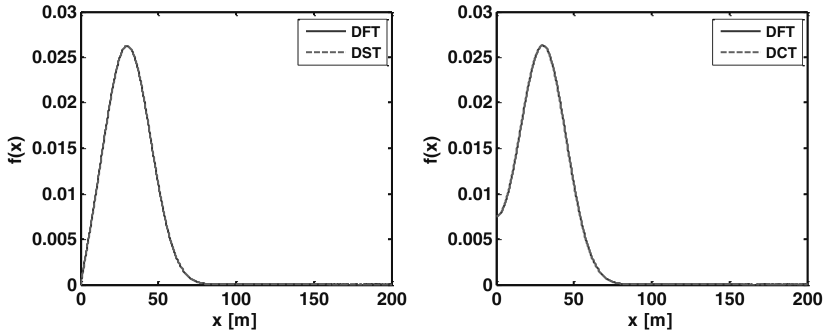

Figure 1.6 Comparison of DFT and DST/DCT analysis for the given Gaussian antenna profile: (left) DBC, (right) NBC.

and an infinite number of terms is required to fully represent this function with the Fourier series summation. The terms are called harmonics. It is interesting to visualize term by term contributions in the Fourier series representation. This is illustrated in Fig. 1.5 for a 4 s.-rectangular pulse between 3 s. and 7 s.

Fourier transform and Fourier series expansion are important procedures in engineering [6]. The DFT and FFT are discrete tools to analyze time-domain signals. One needs to know the problems caused because of the discretization and specify the parameters accordingly to avoid non-physical and non-mathematical results. Moreover, extra attention should be paid when using built-in commands in different computer languages (e.g., MATLAB). As pointed out above in the text, one needs to multiply the results of the FFT taken using MATLAB's *fft(s,N)* command with Δt in order to obtain correct amplitude values.

Since the DFT cannot handle the BCs automatically in propagation problems, the discrete sine transform (DST) and discrete cosine transform (DCT) can be used for various BCs on Earth. Using DFT, to satisfy the BC over perfect electric conductor (PEC) ground, the boundary is extended from [0, hmax] to [-hmax, hmax], and then, in accordance with the image theory, the odd and even symmetric field profiles are constructed for Dirichlet boundary condition (DBC) and Neumann boundary condition (NBC), respectively, to be able to apply the DFT. Another option, to avoid the height extension, is to reduce DFT to one-sided DST or DCT, for DBC and NBC, respectively. A MATLAB code *fft_dst_dct.m* compares DFT and DST/DCT of a Gaussian field profile. A 30 m height antenna with 0.1° beamwidth at 3 GHz is used in Fig. 1.6.

In addition, MATLAB functions for Fourier transforms are given below. Note that the initial and end values of s are zero for DST.

DFT of field	*fftshift(fft(ifftshift(s)))*	Inverse DFT of field	*fftshift(ifft(ifftshift(S)))*
DST of field	*dst(s(2:end-1))*	Inverse DST of field	*[0;idst(S);0]*
DCT of field	*dct(s)*	Inverse DCT of field	*idct(S)*