Chapter
1

René Descartes explaining his work to Queen Christina of Sweden. In 1637 Descartes published his invention of the coordinate system described in this chapter.

Functions and Their Graphs

Functions lie at the heart of modern mathematics. This chapter begins by introducing the notion of a function, along with its domain and range.

By combining algebra and geometry, analytic geometry provides a tremendously powerful tool for visualizing functions. Thus we discuss the coordinate plane, which can be thought of as a two-dimensional analogue of the real line. Although functions are algebraic objects, often we can understand a function better by viewing its graph in the coordinate plane.

The third section of this chapter shows how algebraic transformations of a function change its domain, range, and graph.

The fourth section of this chapter deals with the composition of functions, which allows us to write complicated functions in terms of simpler functions. This idea has applications throughout wide areas of mathematics.

Inverse functions and their graphs become the center of attention in the last two sections of this chapter. Inverse functions will be key tools later in our treatment of roots, logarithms, and inverse trigonometric functions.

1.1 Functions

Learning Objectives

By the end of this section you should be able to

- evaluate functions defined by formulas;
- determine when two functions are equal;
- determine the domain of a function;
- determine the range of a function;
- use functions defined by tables.

Definition and Examples

Although we do not need to do so in this book, functions can be defined more generally to deal with objects other than real numbers.

Function and domain

A **function** associates every number in some set of real numbers, called the **domain** of the function, with exactly one real number.

We usually denote functions by letters such as f, g, and h. If f is a function and x is a number in the domain of f, then the number that f associates with x is denoted by $f(x)$ and is called the value of f at x.

Often (although not always) a function is defined by a single formula, as in the next example.

Example 1

Suppose a function f is defined by the formula

$$f(x) = x^2$$

for every real number x. Evaluate each of the following:

(a) $f(3)$

(b) $f(-\frac{1}{2})$

(c) $f(1+t)$

(d) $f(\frac{x-5}{\pi})$

The use of informal language when discussing functions is acceptable if the meaning is clear. For example, a textbook or your instructor might refer to "the function x^2" or "the function $f(x) = x^2$". Both these phrases are shorthand for the more formally correct "the function f defined by $f(x) = x^2$".

solution Here the domain of f is the set of real numbers, and f is the function that associates every real number with its square. To evaluate f at any number, we simply square that number, as shown by the solutions below.

(a) $f(3) = 3^2 = 9$

(b) $f(-\frac{1}{2}) = (-\frac{1}{2})^2 = \frac{1}{4}$

(c) $f(1+t) = (1+t)^2 = 1 + 2t + t^2$

(d) $f(\frac{x-5}{\pi}) = (\frac{x-5}{\pi})^2 = \frac{x^2-10x+25}{\pi^2}$

A function need not be defined by a single algebraic expression, as shown by the following example.

Example 2

The U.S. 2016 federal income tax for a single person with taxable income x dollars (this is the net income after allowable deductions and exemptions) is $g(x)$ dollars, where g is the function defined by federal law as follows:

$$g(x) = \begin{cases} 0.1x & \text{if } 0 \le x \le 9275 \\ 0.15x - 463.75 & \text{if } 9275 < x \le 37650 \\ 0.25x - 4228.75 & \text{if } 37650 < x \le 91150 \\ 0.28x - 6963.25 & \text{if } 91150 < x \le 190150 \\ 0.33x - 16470.75 & \text{if } 190150 < x \le 413350 \\ 0.35x - 24737.75 & \text{if } 413350 < x \le 415050 \\ 0.396x - 43830.05 & \text{if } 415050 < x. \end{cases}$$

This function changes each year, depending upon inflation and changes in the law.

What was the 2016 federal income tax for a single person whose taxable income that year was

(a) $25,000?

(b) $50,000?

solution

(a) Because 25000 is between 9275 and 37650, use the second line of the definition of g:

$$\begin{aligned} g(25000) &= 0.15 \times 25000 - 463.75 \\ &= 3286.25. \end{aligned}$$

Thus the 2016 federal income tax for a single person with $25,000 taxable income that year was $3,286.25.

(b) Because 50000 is between 37650 and 91150, use the third line of the definition of g:

$$\begin{aligned} g(50000) &= 0.25 \times 50000 - 4228.75 \\ &= 8271.25. \end{aligned}$$

Thus the 2016 federal income tax for a single person with $50,000 taxable income that year was $8,271.25.

The next example shows that using the flexibility offered by functions can be quicker than dealing with single algebraic expressions.

Example 3

Give an example of a function h whose domain is the set of real numbers and such that $h(1) = 10$, $h(3) = 2$, and $h(9) = 26$.

solution The function h could be defined as follows:

$$h(x) = \begin{cases} 10 & \text{if } x = 1 \\ 2 & \text{if } x = 3 \\ 26 & \text{if } x = 9 \\ 0 & \text{if } x \text{ is a real number other than 1, 3, or 9.} \end{cases}$$

The function h defined by

$$h(x) = x^2 - 8x + 17$$

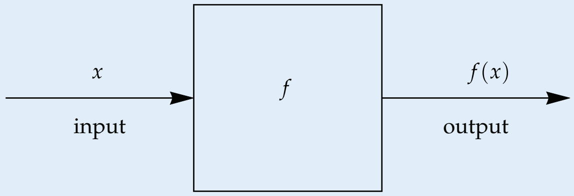
for all real numbers x provides another correct solution (as you can verify). However, finding this algebraic expression requires serious effort.

You might sometimes find it useful to think of a function f as a machine:

This machine might work using a formula, or it might work more mysteriously, in which case it is sometimes called a "black box".

Functions as machines

A function f can be visualized as a machine that takes an input x and produces an output $f(x)$.



For example, suppose f is the function whose domain is the interval $[-4, 6]$, with f defined by the formula $f(x) = x^2$ for every x in the interval $[-4, 6]$. Then giving input 3 to this machine produces output 9. The same input must always produce the same output; thus inputting 3 to this machine at a later time must again produce the output 9. Although each input has just one output, a given output may arise from more than one input. For example, the inputs -3 and 3 both produce the output 9 for this function.

When thinking of a function as the machine pictured above, the domain of the function is the set of numbers that the machine accepts as allowable inputs.

What if the number 8 is input to the machine described in the paragraph above? Because 8 is not in the domain of this function f, the machine does not produce an output for this input; the machine should produce an error message stating that 8 is not an allowable input.

Here is what it means for two functions to be equal:

Equality of functions

Two functions are **equal** if and only if they have the same domain and the same value at every number in that domain.

Example 4

Suppose f is the function whose domain is the set of real numbers, with f defined on this domain by

$$f(x) = x^2.$$

Suppose g is the function whose domain is the set of positive numbers, with g defined on this domain by

$$g(x) = x^2.$$

Are f and g equal functions?

Two functions with different domains are not equal as functions, even if they are defined by the same formula.

solution Note that, for example, $f(-3) = 9$, but the expression $g(-3)$ makes no sense because $g(x)$ has not been defined when x is negative. Because f and g have different domains, these two functions are not equal.

The next example shows that considering only the formula defining a function can be deceptive.

Example 5

Suppose f and g are functions whose domain is the set consisting of the two numbers $\{1, 2\}$, with f and g defined on this domain by the formulas

$$f(x) = x^2 \quad \text{and} \quad g(x) = 3x - 2.$$

Are f and g equal functions?

solution Here f and g have the same domain—the set $\{1, 2\}$. Thus it is at least possible that f equals g. Because f and g have different formulas, the natural inclination is to assume f is not equal to g. However,

$$f(1) = 1^2 = 1 \quad \text{and} \quad g(1) = 3 \cdot 1 - 2 = 1$$

and

$$f(2) = 2^2 = 4 \quad \text{and} \quad g(2) = 3 \cdot 2 - 2 = 4.$$

Thus

$$f(1) = g(1) \quad \text{and} \quad f(2) = g(2).$$

Because f and g have the same value at every number in their domain $\{1, 2\}$, the functions f and g are equal.

Although the variable x is commonly used to denote the input for a function, other symbols can also be used.

Notation

The variable used for inputs when defining a function is irrelevant.

Example 6

Suppose f and g are functions whose domain is the set of real numbers, with f and g defined on this domain by the formulas

$$f(x) = 3 + x^2 \quad \text{and} \quad g(t) = 3 + t^2.$$

Are f and g equal functions?

The symbols x and t here are simply placeholders to indicate that f and g associate with any number 3 plus the square of that number.

solution Because f and g have the same domain and the same value at every number in that domain, f and g are equal functions.

Mathematical notation is sometime ambiguous, with proper interpretation depending upon the context. For example, consider the expression

$$y(x + 2).$$

If y and x denote numbers, then the expression above equals $yx + 2y$. However, if y is a function, then the parentheses above do not indicate multiplication but instead indicate that the function y is evaluated at $x + 2$.

If f is a function, then the parentheses in

$$\frac{f(2x)}{f(x)}$$

do not indicate multiplication. Thus if f is a function, then neither f nor x can be canceled from the expression above.

The Domain of a Function

Although the domain of a function is a formal part of characterizing the function, often we are loose about the domain of a function. Usually the domain is clear from the context or from a formula defining a function. Use the following informal rule when the domain is not specified.

Domain not specified

If a function is defined by a formula, with no domain specified, then the domain is assumed to be the set of all real numbers for which the formula makes sense and produces a real number.

The next three examples illustrate this rule.

Example 7

Find the domain of the function f defined by

$$f(x) = (3x - 1)^2.$$

solution No domain has been specified, but the formula above makes sense for all real numbers x. Thus unless the context indicates otherwise, we assume the domain for this function is the set of real numbers.

The following example shows that avoiding division by 0 can determine the domain of a function.

Example 8

Here h is a function and $h(t)$ denotes the value of the function h at a number t.

Find the domain of the function h defined by

$$h(t) = \frac{t^2 + 3t + 7}{t - 4}.$$

solution No domain has been specified, but the formula above does not make sense when $t = 4$, which would lead to division by 0. Thus unless the context indicates otherwise, we assume the domain for this function is the set of real numbers that do not equal 4, which could be written as $(-\infty, 4) \cup (4, \infty)$.

The following example illustrates the requirement of the informal rule that the formula must produce a real number.

Example 9

Find the domain of the function g defined by

$$g(x) = \sqrt{|x| - 5}.$$

solution No domain has been specified, but the formula above produces a real number only for numbers x with absolute value greater than or equal to 5. Thus unless the context indicates otherwise, we assume the domain for this function is $(-\infty, -5] \cup [5, \infty)$.

The Range of a Function

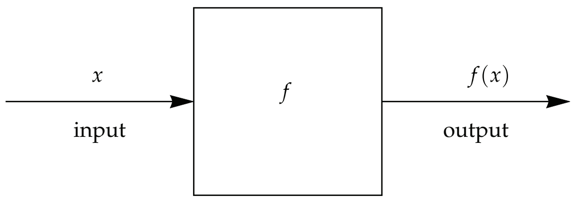
Another important set associated with a function, along with the domain, is the range. The range of a function is the set of all values taken on by the function.

Range

The **range** of a function f is the set of all numbers y such that $f(x) = y$ for at least one x in the domain of f.

Some books use the word **image** *instead of* **range**.

In other words, if we think of a function f as the machine below, then the range of f is the set of numbers that the machine produces as outputs (and the domain is the set of allowable inputs).



The set of inputs acceptable by this machine is the domain of f.
The set of outputs is the range of f.

Example 10

Suppose the domain of f is the interval $[2, 5]$, with f defined on this interval by the equation $f(x) = 3x + 1$.

(a) Is 10 in the range of f?

(b) Is 19 in the range of f?

(c) What is the range of f?

solution

(a) We need to determine whether $f(x) = 10$ for some x in the interval $[2, 5]$, which is the domain of f. In other words, we need to determine whether the equation

$$3x + 1 = 10$$

has a solution x in the interval $[2, 5]$. The only solution to the equation above is $x = 3$, which is in the interval $[2, 5]$. Thus 10 is in the range of f.

(b) We need to determine whether $f(x) = 19$ for some x in the interval $[2, 5]$, which is the domain of f. In other words, we need to determine whether the equation

$$3x + 1 = 19$$

has a solution x in the interval $[2, 5]$. The only solution to the equation above is $x = 6$, which is not in the interval $[2, 5]$. Thus 19 is not in the range of f.

(c) Suppose y is a real number. To decide whether y is in the range of f, we need to determine whether $f(x) = y$ for some x in the interval $[2, 5]$. In other words, we need to determine whether the equation

In the next section, we will see a graphical interpretation of the range of a function.

$$3x + 1 = y$$

has a solution x in the interval $[2, 5]$. The solution to the equation above is $x = \frac{y-1}{3}$. Thus we need to determine whether

$$2 \le \frac{y-1}{3} \le 5.$$

Multiplying all three parts of the inequality above by 3 and then adding 1 to all three parts gives the inequality $7 \le y \le 16$. Thus the range of f is the interval $[7, 16]$.

For a number y to be in the range of a function f, there is no requirement that the equation $f(x) = y$ have just one solution x in the domain of f. The requirement is that there be at least one solution. The next example shows that multiple solutions can easily arise.

Example 11

Suppose the domain of g is the interval $[1, 20]$, with g defined on this interval by the equation $g(x) = |x - 5|$. Is 2 in the range of g?

solution We need to determine whether the equation

$$|x - 5| = 2$$

This equation implies that $x - 5 = 2$ or $x - 5 = -2$.

has at least one solution x in the interval $[1, 20]$. The equation above has two solutions, $x = 7$ and $x = 3$, both of which are in the domain of g. We have $g(7) = g(3) = 2$. Thus 2 is in the range of g.

Functions via Tables

If the domain of a function consists of only finitely many numbers, then all the values of the function can be listed in a table.

Example 12

The common format shown here uses a table with two columns. The left column shows each number in the domain of the function, and the right column shows the corresponding value of the function.

Describe the function f whose domain consists of the three numbers $\{2, 7, 13\}$ and whose values are given by the following table:

x	$f(x)$
2	3
7	$\sqrt{2}$
13	-4

solution For this function we have

$$f(2) = 3, \quad f(7) = \sqrt{2}, \quad \text{and} \quad f(13) = -4.$$

The equations above give a complete description of the function f.

Thinking about why the result below holds should be a good review of the concepts of domain and range.

All the values of a function can be listed in a table only when the function has only finitely many numbers in its domain.

Domain and range from a table

Suppose all the values of the function are listed in a table using the format from the example above. Then

- the domain of the function is the set of numbers that appear in the left column of the table;
- the range of the function is the set of numbers that appear in the right column of the table.

When describing a function by a table, there should be no repetitions in the left column, which shows the numbers in the domain of the function. However, repetitions can occur in the right column, which shows the numbers in the range of the function, as in the following example.

Example 13

Suppose f is the function completely determined by the table below.

x	$f(x)$
1	6
2	6
3	-7
5	6

For this function we have $f(1) = f(2) = f(5) = 6$ *and* $f(3) = -7$.

(a) What is the domain of f?

(b) What is the range of f?

solution

(a) The left column of the table contains the numbers 1, 2, 3, and 5. Thus the domain of f is the set $\{1, 2, 3, 5\}$.

(b) The right column of the table contains only two distinct numbers, 6 and -7. Thus the range of f is the set $\{6, -7\}$.

Exercises

For Exercises 1–12, assume

$$f(x) = \frac{x+2}{x^2+1}$$

for every real number x. Evaluate and simplify each of the following expressions.

1 $f(0)$

2 $f(1)$

3 $f(-1)$

4 $f(-2)$

5 $f(2a)$

6 $f(\frac{b}{3})$

7 $f(2a+1)$

8 $f(3a-1)$

9 $f(x^2+1)$

10 $f(2x^2+3)$

11 $f(\frac{a}{b}-1)$

12 $f(\frac{2a}{b}+3)$

For Exercises 13–18, assume

$$g(x) = \frac{x-1}{x+2}.$$

13 Find a number b such that $g(b) = 4$.

14 Find a number b such that $g(b) = 3$.

15 Simplify the expression

$$\frac{g(x)-g(2)}{x-2}.$$

16 Simplify the expression

$$\frac{g(x)-g(3)}{x-3}.$$

17 Simplify the expression

$$\frac{g(a+t)-g(a)}{t}.$$

18 Simplify the expression

$$\frac{g(x+b)-g(x-b)}{2b}.$$

For Exercises 19–26, assume f is the function defined by

$$f(t) = \begin{cases} 2t+9 & \text{if } t < 0 \\ 3t-10 & \text{if } t \geq 0. \end{cases}$$

19 Evaluate $f(1)$.

20 Evaluate $f(2)$.

21 Evaluate $f(-3)$.

22 Evaluate $f(-4)$.

23 Evaluate $f(|x|+1)$.

24 Evaluate $f(|x-5|+2)$.

25 Find two different values of t such that $f(t) = 0$.

26 Find two different values of t such that $f(t) = 4$.

27 Using the tax function given in Example 2, find the 2016 federal income tax for a single person whose taxable income that year was \$60,000.

28 Using the tax function given in Example 2, find the 2016 federal income tax for a single person whose taxable income that year was \$120,000.

For Exercises 29–32, find a number b such that the function f equals the function g.

29 The function f has domain the set of positive numbers and is defined by $f(x) = 5x^2 - 7$; the function g has domain (b, ∞) and is defined by $g(x) = 5x^2 - 7$.

30 The function f has domain the set of numbers with absolute value less than 4 and is defined by $f(x) = \frac{3}{x+5}$; the function g has domain the interval $(-b, b)$ and is defined by $g(x) = \frac{3}{x+5}$.

31 Both f and g have domain $\{3, 5\}$, with f defined on this domain by the formula $f(x) = x^2 - 3$ and g defined on this domain by the formula $g(x) = \frac{18}{x} + b(x-3)$.

32 Both f and g have domain $\{-3, 4\}$, with f defined on this domain by the formula $f(x) = 3x + 5$ and g defined on this domain by the formula $g(x) = 15 + \frac{8}{x} + b(x-4)$.

For Exercises 33–38, a formula has been given defining a function f but no domain has been specified. Find the domain of each function f, assuming that the domain is the set of real numbers for which the formula makes sense and produces a real number.

33 $f(x) = \frac{2x+1}{3x-4}$

34 $f(x) = \frac{4x-9}{7x+5}$

35 $f(x) = \frac{\sqrt{x-5}}{x-7}$

36 $f(x) = \frac{\sqrt{2x+3}}{x-6}$

37 $f(x) = \sqrt{|x-6|-1}$

38 $f(x) = \sqrt{|x+5|-3}$

39 Suppose the domain of F is the interval $[3, 7]$, with F defined on this domain by the equation $F(x) = -2x + 5$. Find the range of F.

40 Suppose the domain of G is the interval $[-2, 3]$, with G defined on this domain by the equation $G(x) = -4x - 6$. Find the range of G.

For Exercises 41–46, find the range of h if h is defined by

$$h(t) = |t| + 1$$

and the domain of h is the indicated set.

41 $(1, 4]$

42 $[-8, -3)$

43 $[-3, 5]$

44 $[-8, 2]$

45 $(0, \infty)$

46 $(-\infty, 0)$

For Exercises 47–54, assume f and g are functions completely defined by the following tables:

x	$f(x)$
3	13
4	-5
6	$\frac{3}{5}$
7.3	-5

x	$g(x)$
3	3
8	$\sqrt{7}$
8.4	$\sqrt{7}$
12.1	$-\frac{2}{7}$

47 Evaluate $f(6)$.

48 Evaluate $g(8)$.

49 What is the domain of f?

50 What is the domain of g?

51 What is the range of f?

52 What is the range of g?

53 Find two different values of x such that $f(x) = -5$.

54 Find two different values of x such that $g(x) = \sqrt{7}$.

55 Find all functions (displayed as tables) whose domain is the set $\{2, 9\}$ and whose range is the set $\{4, 6\}$.

56 Find all functions (displayed as tables) whose domain is the set $\{5, 8\}$ and whose range is the set $\{1, 3\}$.

57 Find all functions (displayed as tables) whose domain is $\{1, 2, 4\}$ and whose range is $\{-2, 1, \sqrt{3}\}$.

58 Find all functions (displayed as tables) whose domain is $\{-1, 0, \pi\}$ and whose range is $\{-3, \sqrt{2}, 5\}$.

59 Find all functions (displayed as tables) whose domain is $\{3, 5, 9\}$ and whose range is $\{2, 4\}$.

60 Find all functions (displayed as tables) whose domain is $\{0, 2, 8\}$ and whose range is $\{6, 9\}$.

Problems

Some problems require considerably more thought than the exercises.

61 Suppose the only information you know about a function f is that the domain of f is the set of real numbers and

$$f(1) = 1,\ f(2) = 4,\ f(3) = 9,\ \text{ and } f(4) = 16.$$

What can you say about the value of $f(5)$?
[*Hint:* The answer to this problem is not "25". The shortest correct answer is just one word.]

62 Suppose g and h are functions whose domain is the set of real numbers, with g and h defined on this domain by

$$g(y) = \frac{4y}{y^2 + 5} \quad \text{and} \quad h(r) = \frac{4r}{r^2 + 5}.$$

Are g and h equal functions? Explain.

63 Give an example of a function whose domain is $\{2, 5, 7\}$ and whose range is $\{-2, 3, 4\}$.

64 Give an example of a function whose domain is $\{3, 4, 7, 9\}$ and whose range is $\{-1, 0, 3\}$.

65 Find two different functions whose domain is $\{3, 8\}$ and whose range is $\{-4, 1\}$.

66 Explain why there does not exist a function whose domain is $\{-1, 0, 3\}$ and whose range is $\{3, 4, 7, 9\}$.

67 Give an example of a function f whose domain is the set of real numbers and such that the values of $f(-1)$, $f(0)$, and $f(2)$ are given by the following table:

x	$f(x)$
-1	$\sqrt{2}$
0	19
2	-5

68 Give an example of a function F whose domain is the interval $[0, 2]$ and such that $F(0) = 0$ and $F(2) = 2$ but 1 is not in the range of F.

69 Give an example of two different functions f and g, both of which have the set of real numbers as their domain, such that $f(x) = g(x)$ for every rational number x.

70 Give an example of a function whose domain equals the set of real numbers and whose range equals the set $\{-1, 0, 1\}$.

71 Give an example of a function whose domain equals the set of real numbers and whose range equals the set of integers.

72 Give an example of a function whose domain is the interval $[0, 1]$ and whose range is the interval $(0, 1)$.

73 Give an example of a function whose domain is the interval $(0, 1)$ and whose range is the interval $[0, 1]$.

74 Give an example of a function whose domain is the set of positive integers and whose range is the set of positive even integers.

75 Give an example of a function whose domain is the set of positive even integers and whose range is the set of positive odd integers.

76 Give an example of a function whose domain is the set of integers and whose range is the set of positive integers.

77 Give an example of a function whose domain is the set of positive integers and whose range is the set of integers.

Worked-Out Solutions to Odd-Numbered Exercises

Do not read these worked-out solutions before attempting to do the exercises yourself. Otherwise you may mimic the techniques shown here without understanding the ideas.

Best way to learn: Carefully read the section of the textbook, then do all the odd-numbered exercises and check your answers here. If you get stuck on an exercise, then look at the worked-out solution here.

For Exercises 1–12, assume

$$f(x) = \frac{x + 2}{x^2 + 1}$$

for every real number x. Evaluate and simplify each of the following expressions.

1 $f(0)$

solution $f(0) = \frac{0+2}{0^2+1} = \frac{2}{1} = 2$

3 $f(-1)$

solution $f(-1) = \frac{-1+2}{(-1)^2+1} = \frac{1}{1+1} = \frac{1}{2}$

5 $f(2a)$

solution $f(2a) = \frac{2a+2}{(2a)^2+1} = \frac{2a+2}{4a^2+1}$

7 $f(2a+1)$

solution

$$f(2a+1) = \frac{(2a+1)+2}{(2a+1)^2+1} = \frac{2a+3}{4a^2+4a+2}$$

9 $f(x^2+1)$

solution

$$f(x^2+1) = \frac{(x^2+1)+2}{(x^2+1)^2+1} = \frac{x^2+3}{x^4+2x^2+2}$$

11 $f(\frac{a}{b}-1)$

solution We have

$$f(\frac{a}{b}-1) = \frac{(\frac{a}{b}-1)+2}{(\frac{a}{b}-1)^2+1} = \frac{\frac{a}{b}+1}{\frac{a^2}{b^2}-2\frac{a}{b}+2}$$
$$= \frac{ab+b^2}{a^2-2ab+2b^2},$$

where the last expression was obtained by multiplying the numerator and denominator of the previous expression by b^2.

For Exercises 13–18, assume

$$g(x) = \frac{x-1}{x+2}.$$

13 Find a number b such that $g(b) = 4$.

solution We want to find a number b such that

$$\frac{b-1}{b+2} = 4.$$

Multiply both sides of the equation above by $b+2$, getting

$$b-1 = 4b+8.$$

Now solve this equation for b, getting $b = -3$.

15 Simplify the expression

$$\frac{g(x)-g(2)}{x-2}.$$

solution We begin by evaluating the numerator:

$$g(x)-g(2) = \frac{x-1}{x+2} - \frac{1}{4}$$
$$= \frac{4(x-1)-(x+2)}{4(x+2)}$$
$$= \frac{4x-4-x-2}{4(x+2)}$$
$$= \frac{3x-6}{4(x+2)}$$
$$= \frac{3(x-2)}{4(x+2)}.$$

Thus

$$\frac{g(x)-g(2)}{x-2} = \frac{3(x-2)}{4(x+2)} \cdot \frac{1}{x-2}$$
$$= \frac{3}{4(x+2)}.$$

17 Simplify the expression

$$\frac{g(a+t)-g(a)}{t}.$$

solution We begin by computing the numerator:

$$g(a+t)-g(a)$$
$$= \frac{(a+t)-1}{(a+t)+2} - \frac{a-1}{a+2}$$
$$= \frac{(a+t-1)(a+2)-(a-1)(a+t+2)}{(a+t+2)(a+2)}$$
$$= \frac{3t}{(a+t+2)(a+2)}.$$

Thus

$$\frac{g(a+t)-g(a)}{t} = \frac{3}{(a+t+2)(a+2)}.$$

For Exercises 19–26, assume f is the function defined by

$$f(t) = \begin{cases} 2t+9 & \text{if } t<0 \\ 3t-10 & \text{if } t \geq 0. \end{cases}$$

19 Evaluate $f(1)$.

solution Because $1 \geq 0$, we have

$$f(1) = 3 \cdot 1 - 10 = -7.$$

21 Evaluate $f(-3)$.

solution Because $-3 < 0$, we have

$$f(-3) = 2(-3)+9 = 3.$$

23 Evaluate $f(|x|+1)$.

solution Because $|x|+1 \geq 1 > 0$, we have

$$f(|x|+1) = 3(|x|+1)-10 = 3|x|-7.$$

25 Find two different values of t such that $f(t) = 0$.

solution If $t < 0$, then $f(t) = 2t+9$. We want to find t such that $f(t) = 0$, which means that we need to solve the equation $2t+9 = 0$ and hope that the solution satisfies $t < 0$. Subtracting 9 from both sides of $2t+9 = 0$ and then dividing both sides by 2 gives $t = -\frac{9}{2}$. This value of t satisfies the inequality $t < 0$, and we do indeed have $f(-\frac{9}{2}) = 0$.

If $t \geq 0$, then $f(t) = 3t - 10$. We want to find t such that $f(t) = 0$, which means that we need to solve the equation $3t - 10 = 0$ and hope that the solution satisfies $t \geq 0$. Adding 10 to both sides of $3t - 10 = 0$ and then dividing both sides by 3 gives $t = \frac{10}{3}$. This value of t satisfies the inequality $t \geq 0$, and we do indeed have $f(\frac{10}{3}) = 0$.

27 Using the tax function given in Example 2, find the 2016 federal income tax for a single person whose taxable income that year was \$60,000.

solution Because 60000 is between 37650 and 91150, use the third line of the definition of g in Example 2:

$$\begin{aligned} g(60000) &= 0.25 \times 60000 - 4228.75 \\ &= 10771.25. \end{aligned}$$

Thus the 2016 federal income tax for a single person with \$60,000 taxable income that year was \$10,771.25.

For Exercises 29–32, find a number b such that the function f equals the function g.

29 The function f has domain the set of positive numbers and is defined by $f(x) = 5x^2 - 7$; the function g has domain (b, ∞) and is defined by $g(x) = 5x^2 - 7$.

solution For two functions to be equal, they must at least have the same domain. Because the domain of f is the set of positive numbers, which equals the interval $(0, \infty)$, we must have $b = 0$.

31 Both f and g have domain $\{3, 5\}$, with f defined on this domain by the formula $f(x) = x^2 - 3$ and g defined on this domain by the formula $g(x) = \frac{18}{x} + b(x - 3)$.

solution Note that

$$f(3) = 3^2 - 3 = 6 \quad \text{and} \quad f(5) = 5^2 - 3 = 22.$$

Also,

$$g(3) = \frac{18}{3} + b(3 - 3) = 6 \quad \text{and} \quad g(5) = \frac{18}{5} + 2b.$$

Thus regardless of the choice of b, we have $f(3) = g(3)$. To make the function f equal the function g, we must also have $f(5) = g(5)$, which means that we must have

$$22 = \frac{18}{5} + 2b.$$

Solving this equation for b, we get $b = \frac{46}{5}$.

For Exercises 33–38, a formula has been given defining a function f but no domain has been specified. Find the domain of each function f, assuming that the domain is the set of real numbers for which the formula makes sense and produces a real number.

33 $f(x) = \dfrac{2x + 1}{3x - 4}$

solution The formula above does not make sense when $3x - 4 = 0$, which would lead to division by 0. The equation $3x - 4 = 0$ is equivalent to $x = \frac{4}{3}$. Thus the domain of f is the set of real numbers not equal to $\frac{4}{3}$. In other words, the domain of f equals $(-\infty, \frac{4}{3}) \cup (\frac{4}{3}, \infty)$.

35 $f(x) = \dfrac{\sqrt{x - 5}}{x - 7}$

solution The formula above does not make sense when $x < 5$ because we cannot take the square root of a negative number. The formula above also does not make sense when $x = 7$, which would lead to division by 0. Thus the domain of f is the set of real numbers greater than or equal to 5 and not equal to 7. In other words, the domain of f equals $[5, 7) \cup (7, \infty)$.

37 $f(x) = \sqrt{|x - 6| - 1}$

solution Because we cannot take the square root of a negative number, we must have $|x - 6| - 1 \geq 0$. This inequality is equivalent to $|x - 6| \geq 1$, which means that $x - 6 \geq 1$ or $x - 6 \leq -1$. Adding 6 to both sides of these inequalities, we see that the formula above makes sense only when $x \geq 7$ or $x \leq 5$. In other words, the domain of f equals $(-\infty, 5] \cup [7, \infty)$.

39 Suppose the domain of F is the interval $[3, 7]$, with F defined on this domain by the equation $F(x) = -2x + 5$. Find the range of F.

solution Suppose y is a real number. To decide whether y is in the range of F, we need to determine whether $F(x) = y$ for some x in the interval $[3, 7]$. In other words, we need to determine whether the equation

$$-2x + 5 = y$$

has a solution x in the interval $[3, 7]$. The solution to the equation above is $x = \frac{5-y}{2}$. Thus we need to determine whether

$$3 \leq \frac{5 - y}{2} \leq 7.$$

Multiplying all three parts of the inequality above by -2 (reversing directions of the inequality) and then adding 5 to all three parts gives the inequality $-1 \geq y \geq -9$. Thus the range of F is the interval $[-9, -1]$.

For Exercises 41–46, find the range of h if h is defined by

$$\boldsymbol{h(t) = |t| + 1}$$

and the domain of h is the indicated set.

41 $(1, 4]$

solution For each number t in the interval $(1, 4]$, we have $h(t) = t + 1$. Thus the range of h is obtained by adding 1 to each number in the interval $(1, 4]$. This implies that the range of h is the interval $(2, 5]$.

43 $[-3, 5]$

solution For each number t in the interval $[-3, 0)$, we have $h(t) = -t + 1$, and for each number t in the interval $[0, 5]$ we have $h(t) = t + 1$. Thus the range of h consists of the numbers obtained by multiplying each number in the interval $[-3, 0)$ by -1 and then adding 1 (this produces the interval $(1, 4]$), along with the numbers obtained by adding 1 to each number in the interval $[0, 5]$ (this produces the interval $[1, 6]$). This implies that the range of h is the interval $[1, 6]$.

45 $(0, \infty)$

solution For each positive number t we have $h(t) = t + 1$. Thus the range of h is the set obtained by adding 1 to each positive number. Hence the range of h is the interval $(1, \infty)$.

For Exercises 47–54, assume f and g are functions completely defined by the following tables:

x	$f(x)$
3	13
4	-5
6	$\frac{3}{5}$
7.3	-5

x	$g(x)$
3	3
8	$\sqrt{7}$
8.4	$\sqrt{7}$
12.1	$-\frac{2}{7}$

47 Evaluate $f(6)$.

solution Looking at the table, we see that $f(6) = \frac{3}{5}$.

49 What is the domain of f?

solution The domain of f is the set of numbers in the first column of the table defining f. Thus the domain of f is the set $\{3, 4, 6, 7.3\}$.

51 What is the range of f?

solution The range of f is the set of numbers that appear in the second column of the table defining f. Numbers that appear more than once in the second column need to be listed only once when finding the range. Thus the range of f is the set $\{13, -5, \frac{3}{5}\}$.

53 Find two different values of x such that $f(x) = -5$.

solution Looking at the table, we see that $f(4) = -5$ and $f(7.3) = -5$.

55 Find all functions (displayed as tables) whose domain is the set $\{2, 9\}$ and whose range is the set $\{4, 6\}$.

solution Because we seek functions f whose domain is the set $\{2, 9\}$, the first column of the table for any such function must have 2 appear once and must have 9 appear once. In other words, the table must start like this:

x	$f(x)$
2	
9	

or this

x	$f(x)$
9	
2	

The order of the rows in a table that defines a function does not matter. For convenience, we choose the first possibility above.

Because the range must be the set $\{4, 6\}$, the second column must contain 4 and 6. There are only two slots in which to put these numbers in the first table above, and thus each one must appear exactly once in the second column. Thus there are only two functions whose domain is the set $\{2, 9\}$ and whose range is the set $\{4, 6\}$; these functions are given by the following two tables:

x	$f(x)$
2	4
9	6

x	$f(x)$
2	6
9	4

The first function above is the function f defined by $f(2) = 4$ and $f(9) = 6$; the second function above is the function f defined by $f(2) = 6$ and $f(9) = 4$.

57 Find all functions (displayed as tables) whose domain is $\{1, 2, 4\}$ and whose range is $\{-2, 1, \sqrt{3}\}$.

solution Because we seek functions f whose domain is $\{1, 2, 4\}$, the first column of the table for any such function must have 1 appear once, must have 2 appear once, and must have 4 appear once. The order of the rows in a table that defines a function does not matter. For convenience, we put the first column in numerical order 1, 2, 4.

Because the range must be $\{-2, 1, \sqrt{3}\}$, the second column must contain -2, 1, and $\sqrt{3}$. There are only three slots in which to put these three numbers, and thus each one must appear exactly once in the second column. There are six ways in which these three numbers can be ordered. Thus the six functions whose domain is $\{1, 2, 4\}$ and whose range is $\{-2, 1, \sqrt{3}\}$ are given by the following tables:

x	$f(x)$
1	-2
2	1
4	$\sqrt{3}$

x	$f(x)$
1	-2
2	$\sqrt{3}$
4	1

x	$f(x)$
1	1
2	-2
4	$\sqrt{3}$

x	$f(x)$
1	1
2	$\sqrt{3}$
4	-2

x	$f(x)$
1	$\sqrt{3}$
2	-2
4	1

x	$f(x)$
1	$\sqrt{3}$
2	1
4	-2

59 Find all functions (displayed as tables) whose domain is $\{3, 5, 9\}$ and whose range is $\{2, 4\}$.

solution Because we seek functions f whose domain is $\{3, 5, 9\}$, the first column of the table for any such function must have 3 appear once, must have 5 appear once, and must have 9 appear once. The order of the rows in a table that defines a function does not matter. For convenience, we put the first column in numerical order 3, 5, 9.

Because the range must be $\{2, 4\}$, the second column must contain 2 and 4. There are three slots in which to put these three numbers, and thus one of them must be repeated. There are six ways to do this. Thus the six functions whose domain is $\{3, 5, 9\}$ and whose range is $\{2, 4\}$ are given by the following tables:

x	$f(x)$
3	2
5	2
9	4

x	$f(x)$
3	2
5	4
9	2

x	$f(x)$
3	4
5	2
9	2

x	$f(x)$
3	4
5	4
9	2

x	$f(x)$
3	4
5	2
9	4

x	$f(x)$
3	2
5	4
9	4

1.2 The Coordinate Plane and Graphs

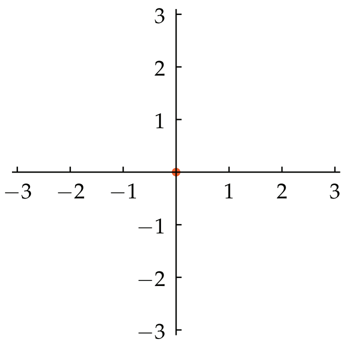
Learning Objectives

By the end of this section you should be able to

- locate points in the coordinate plane;
- sketch the graph of a function, possibly using technology;
- estimate values of a function from its graph;
- determine the domain of a function from its graph;
- determine the range of a function from its graph;
- use the vertical line test to determine whether or not a curve is the graph of some function.

The Coordinate Plane

The coordinate plane is constructed in a fashion similar to our construction of the real line (see Section 0.1). However, in constructing the coordinate plane we use a horizontal line and a vertical line rather than just a horizontal line.



The coordinate plane, with a dot at the origin.

The coordinate plane

- The **coordinate plane** is constructed by starting with a horizontal line and a vertical line in a plane. These lines are called the **coordinate axes**.
- The intersection point of the coordinate axes is called the **origin**; it receives a label of 0 on both axes.
- On the horizontal axis, pick a point to the right of the origin and label it 1. Then label other points on the horizontal axis using the scale determined by the origin and 1.
- Similarly, on the vertical axis, pick a point above the origin and label it 1. Then label other points on the vertical axis using the scale determined by the origin and 1.

The same scale has been used on both axes in the figure here. However, sometimes it is more convenient to have different scales on the two axes.

A point in the plane is identified with its coordinates. The coordinates are written as an ordered pair of numbers surrounded by parentheses, as described below.

The plane with this system of labeling is often called the **Cartesian plane** *in honor of the French mathematician René Descartes (1596–1650), who described this technique in his 1637 book* DISCOURSE ON METHOD.

Coordinates

- The first coordinate indicates the horizontal distance from the origin, with positive numbers corresponding to points right of the origin and negative numbers corresponding to points left of the origin.
- The second coordinate indicates the vertical distance from the origin, with positive numbers corresponding to points above the origin and negative numbers corresponding to points below the origin.

The coordinates of the origin are $(0,0)$.

The coordinate axes divide the plane into four pieces, which are called **quadrants**. The next example shows one point in each of the four quadrants.

Example 1

Locate on a coordinate plane the following points:
(a) $(2, 1)$; (b) $(-1, 2.5)$; (c) $(-2.5, -2.5)$; (d) $(3, -2)$.

solution

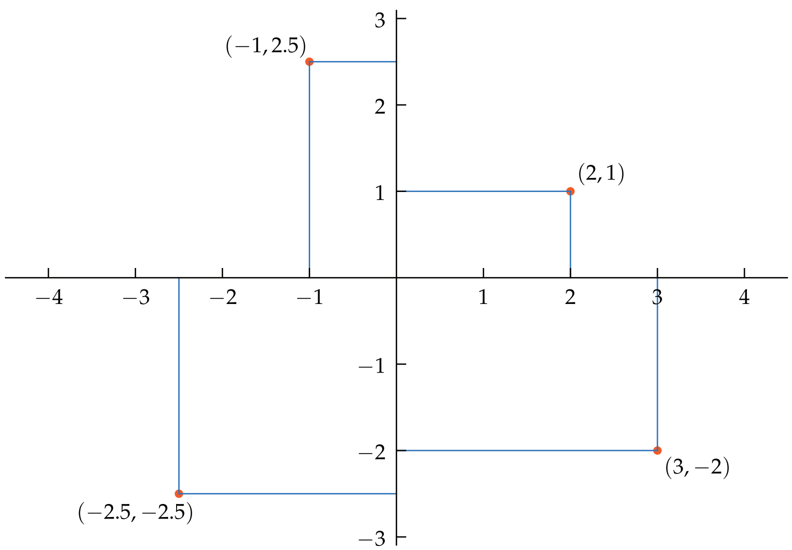
(a) The point $(2, 1)$ can be located by starting at the origin, moving 2 units to the right along the horizontal axis, and then moving up 1 unit; see the figure below.

(b) The point $(-1, 2.5)$ can be located by starting at the origin, moving 1 unit to the left along the horizontal axis, and then moving up 2.5 units; see the figure below.

The notation $(-1, 2.5)$ could denote either the point with coordinates $(-1, 2.5)$ or the open interval $(-1, 2.5)$. You should be able to tell from the context which meaning is intended.

(c) The point $(-2.5, -2.5)$ can be located by starting at the origin, moving 2.5 units to the left along the horizontal axis, and then moving down 2.5 units; see the figure below.

(d) The point $(3, -2)$ can be located by starting at the origin, moving 3 units to the right along the horizontal axis, and then moving down 2 units; see the figure below.



These coordinates are sometimes called **rectangular coordinates** *because each point's coordinates are determined by a rectangle, as shown in this figure.*

The horizontal axis is often called the ***x*-axis** and the vertical axis is often called the ***y*-axis**. In this case, the coordinate plane can be called the ***xy*-plane**. However, other variables can also be used, depending on the problem at hand.

The terms **horizontal axis** *and* **vertical axis** *are often better than the terms* ***x*-axis** *and* ***y*-axis**.

If the horizontal axis has been labeled the x-axis, then the first coordinate of a point is often called the ***x*-coordinate**. Similarly, if the vertical axis has been labeled the y-axis, then the second coordinate is often called the ***y*-coordinate**.

The potential confusion of this terminology becomes apparent when we want to consider a point whose coordinates are (y, x); here y is the x-coordinate and x is the y-coordinate. Furthermore, always calling the first coordinate the x-coordinate will lead to confusion when the horizontal axis is labeled with another variable such as t or θ. Regardless of the names of the axes, keep in mind the following:

Similarly, the terms **first coordinate** *and* **second coordinate** *are often better than the terms* ***x*-coordinate** *and* ***y*-coordinate**.

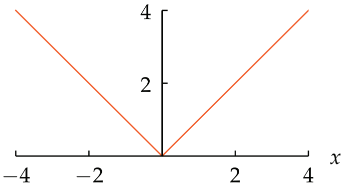
- The first coordinate corresponds to horizontal distance from the origin. A positive first coordinate indicates a point to the right of the origin; a negative first coordinate indicates a point to the left of the origin.

- The second coordinate corresponds to vertical distance from the origin. A positive second coordinate indicates a point above the origin; a negative second coordinate indicates a point below the origin.

The Graph of a Function

A function can be visualized by its graph, which we now define.

> ***The graph of a function***
>
> The **graph** of a function f is the set of points of the form $(x, f(x))$ as x varies over the domain of f.



The graph of $|x|$ *on the interval* $[-4, 4]$.

Thus in the xy-plane, the graph of a function f is the set of points (x, y) satisfying the equation $y = f(x)$ with x in the domain of f.

The figure here shows the graph of the function f whose domain is $[-4, 4]$, with f defined by

$$f(x) = |x|.$$

The point $(-2, 2)$ is on this graph because -2 is in the domain of f and $f(-2) = 2$. Similarly, the origin $(0, 0)$ is on this graph because 0 is in the domain of f and $f(0) = 0$. Note that this graph has a corner at the origin.

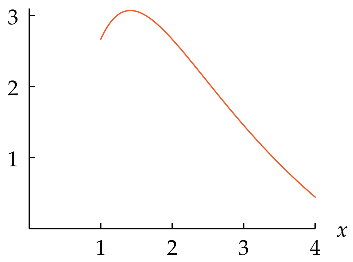
In the next chapter we will learn how to graph linear and quadratic functions, so we will not take up those topics here.

Sketching the graph of a complicated function usually requires the aid of a computer or calculator. The next example uses *WolframAlpha* in the solution, but you could use a graphing calculator or any other technology instead.

Example 2

Let f be the function defined by

$$f(x) = \frac{4(5x - x^2 - 2)}{x^2 + 2}.$$



The graph of $\frac{4(5x - x^2 - 2)}{x^2 + 2}$ *on the interval* $[1, 4]$.

(a) Sketch the graph of f on the interval $[1, 4]$.

(b) Sketch the graph of f on the interval $[-4, 4]$.

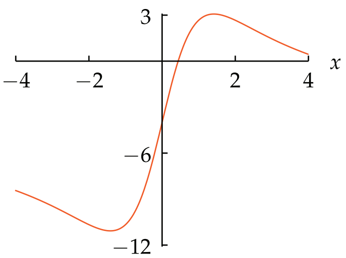
solution

(a) Point a web browser to www.wolframalpha.com. In the one-line entry box, type

`graph 4(5x – x^2 – 2)/(x^2 + 2) from x=1 to 4`

and then press the `enter` key on your keyboard or click the `=` box on the right side of the *WolframAlpha* entry box. This procedure produces a graph on your computer screen that should allow you to sketch a figure like the one shown here.

(b) In the *WolframAlpha* entry box from part (a), change `1` to `–4`, producing a graph like the one shown here.



The graph of the same function $\frac{4(5x - x^2 - 2)}{x^2 + 2}$ *on the interval* $[-4, 4]$.

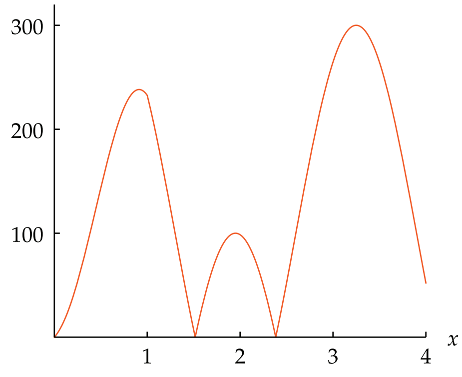
The horizontal and vertical axes have different scales in this graph. The graph would become too large in the vertical direction if the same scale was used on both axes.

Using different scales on the two axes makes the size of the graph more appropriate, but be aware that it changes the apparent shape of the curve. Specifically, the part of the graph on the interval $[1, 4]$ appears flatter in part (b) than the graph in part (a).

Sometimes the only information we have about a function is a sketch of its graph. The next example illustrates the procedure for finding approximate values of a function from a sketch of its graph.

Example 3

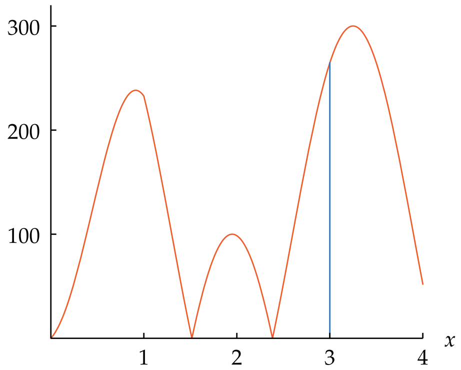
The web site for the 4-mile HillyView race shows the graph below of the function f, where $f(x)$ is the altitude in feet when the race path is x miles from the starting line. Estimate the altitude when the race path is three miles from the starting line.



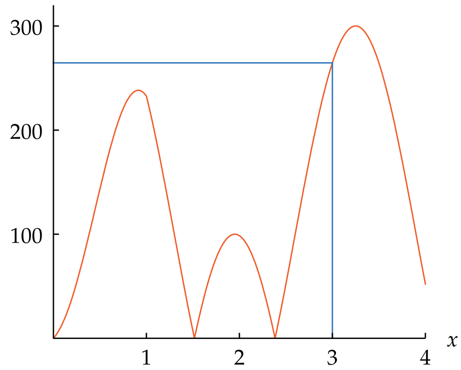
The graph of the function that gives altitude.

Here the scale on the horizontal axis is in miles, and the scale on the vertical axis is in feet.

solution We need to estimate the value of $f(3)$. To do this, draw a vertical line segment from the point 3 on the x-axis until it intersects the graph. The length of that line segment will equal $f(3)$, as shown below on the left.



The vertical line segment has length $f(3)$.



$f(3)$ is approximately 265.

This procedure gives only an estimate of the value of $f(3)$. You cannot find the intersection points exactly if the only information you have is a sketch of the graph.

Usually the easiest way to estimate the value of $f(3)$ is to draw the horizontal line shown above on the right. The point where this horizontal line intersects the vertical axis then gives the value of $f(3)$.

From the figure on the right, we see that $f(3)$ is a bit more than halfway between 200 and 300. Thus 265 is a good estimate of $f(3)$. In other words, the altitude is about 265 feet when the race path is three miles from the starting line.

The standard procedure that was used in the example above can be summarized as follows.

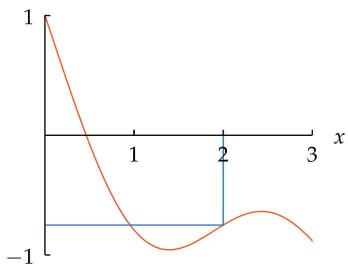
Finding values of a function from its graph

To evaluate $f(b)$ given only the graph of f in the xy-plane,

(a) find the point where the vertical line $x = b$ intersects the graph of f;

(b) draw a horizontal line from that point to the y-axis;

(c) the intersection of that horizontal line with the y-axis gives the value of $f(b)$.



In this graph, we see that $f(2) \approx -0.8$. The vertical line $x = 2$ intersects the graph below the horizontal axis, and thus in this case $f(2)$ is the negative of the length of the vertical line segment.

Determining the Domain and Range from a Graph

The next example shows how the domain of a function can be determined from its graph.

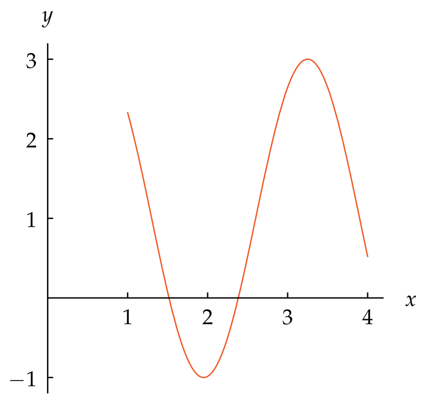
Example 4

Suppose all we know about a function f is the sketch of its graph shown here.

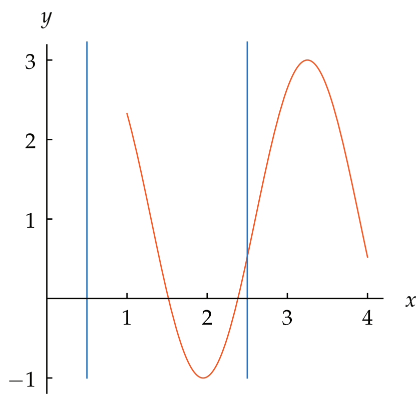
(a) Is 0.5 in the domain of f?

(b) Is 2.5 in the domain of f?

(c) Make a reasonable guess of the domain of f.



solution Recall that the graph of f consists of all points of the form $(b, f(b))$ as b varies over the domain of f. Thus the line $x = b$ in the xy-plane intersects the graph of f if and only if b is in the domain of f. The next figure, in addition to the graph of f, contains the lines $x = 0.5$ and $x = 2.5$.



The vertical lines that intersect the graph correspond to numbers in the domain.

Unlikely but possible: Perhaps a tiny hole in this graph, too small for us to see, implies that 2.5 is not in the domain of this function. Thus caution must be used when working with graphs.

(a) The figure above shows that the line $x = 0.5$ does not intersect the graph of f. Thus 0.5 is not in the domain of f.

(b) The line $x = 2.5$ does intersect the graph of f. Thus 2.5 is in the domain of f.

(c) A reasonable guess for the domain of f is the interval $[1, 4]$. The open interval $(1, 4)$ would also be a reasonable guess for the domain of f. A graph can give only a good approximation of the domain. The actual domain of f might be $[1, 4.001)$ or an even more unusual set such as all numbers in the interval $[1, 4]$ except $\sqrt{2}$ and 2.5; our eyes could not detect such subtle differences from a sketch of the graph.

Do not be afraid to draw reasonable conclusions that would be valid unless something weird is happening.

The technique used above can be summarized as follows.

Determining the domain from the graph

A number b is in the domain of a function f if and only if the line $x = b$ in the xy-plane intersects the graph of f.

Recall that the range of a function is the set of all values taken on by the function. Thus the range of a function can be determined by the horizontal lines that intersect the graph of the function, as shown by the next example.

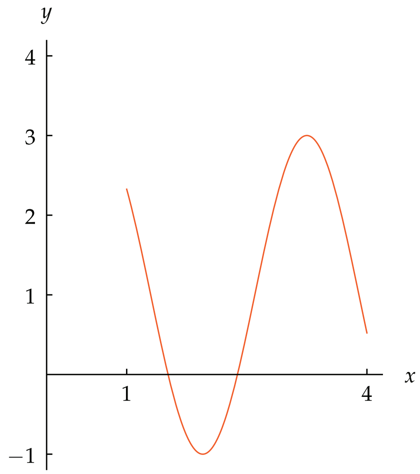
Example 5

Suppose f is the function with domain $[1,4]$ whose graph is shown here.

(a) Is 1.5 in the range of f?

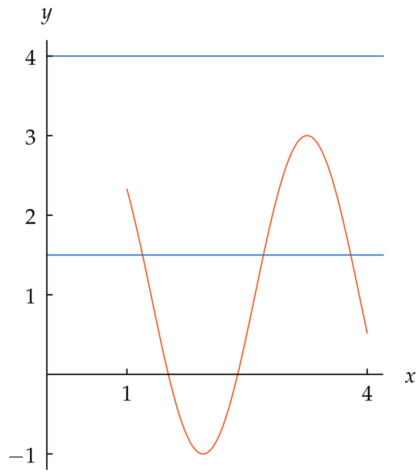
(b) Is 4 in the range of f?

(c) Make a reasonable guess of the range of f.



solution

(a) The figure below shows that the line $y = 1.5$ intersects the graph of f in three points. Thus 1.5 is in the range of f.



Horizontal lines that intersect the graph of f correspond to numbers in the range of f. The lower blue line $y = 1.5$ intersects the graph; thus 1.5 is in the range of f. The upper blue line $y = 4$ does not intersect the graph; thus 1.5 is not in the range of f.

The lower blue line shows that the equation

$$f(x) = 1.5$$

has three solutions for x in the domain of f. We need one or more such solutions for 1.5 to be in the range of f.

(b) The figure above shows that the line $y = 4$ does not intersect the graph of f. In other words, the equation $f(x) = 4$ has no solutions for x in the domain of f. Thus 4 is not in the range of f.

(c) By drawing horizontal lines, we can see that the range of this function appears to be the interval $[-1,3]$.

The actual range of this function might be slightly different from the interval $[-1,3]$ given in the solution to part (c)—we would not be able to notice the difference from the sketch of the graph if the range actually equaled the interval $[-1.02, 3.001]$.

The technique used above can be summarized as follows.

Determining the range from the graph

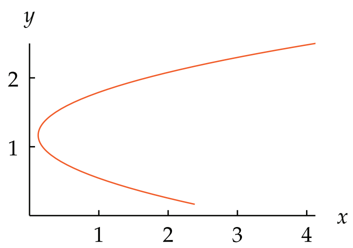
A number c is in the range of a function f if and only if the horizontal line $y = c$ in the xy-plane intersects the graph of f.

Which Sets Are Graphs of Functions?

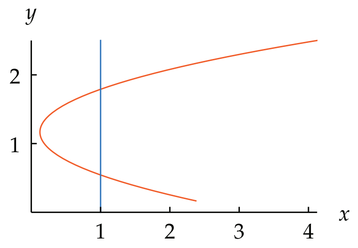
Not every curve in the plane is the graph of some function, as illustrated by the following example.

Example 6

Is this curve shown here the graph of some function?



solution If this curve were the graph of some function f, then we could find the value of $f(1)$ by seeing where the line $x = 1$ intersects the curve. However, the figure below shows that the line $x = 1$ intersects the curve at two points. The definition of a function requires that $f(1)$ be a single number, not a pair of numbers. Thus this curve is not the graph of any function.



The line $x = 1$ intersects the curve at two points. Thus this curve is not the graph of a function.

More generally, any set in the coordinate plane that intersects some vertical line in more than one point cannot be the graph of a function. Conversely, a set in the plane that intersects each vertical line in at most one point is the graph of some function f, with the values of f determined as in Example 3 and the domain of f determined as in Example 4.

The condition for a set in the coordinate plane to be the graph of some function can be summarized as follows.

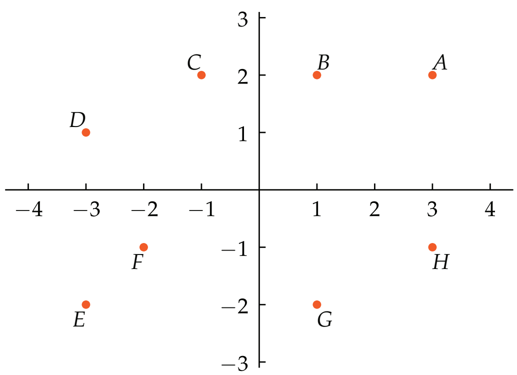
The vertical line test shows, for example, that no function has a graph that is a circle.

Vertical line test

A set of points in the coordinate plane is the graph of some function if and only if every vertical line intersects the set in at most one point.

Exercises

For Exercises 1–8, give the coordinates of the specified point using the figure below:



1 A **3** C **5** E **7** G

2 B **4** D **6** F **8** H

9 Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(1, 2)$, $(-2, 2)$, $(-3, -1)$, $(2, -3)$.

10 Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(2.5, 1)$, $(-1, 3)$, $(-1.5, -1.5)$, $(1, -3)$.

11 Suppose F is the function defined by $F(x) = x^2 + 5$. Find a number t such that $(3, t)$ is on the graph of F.

12 Suppose G is the function defined by $G(x) = 2x^2 - 3$. Find a number r such that $(5, r)$ is on the graph of G.

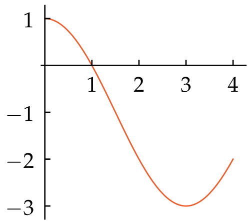
13 Suppose G is the function defined by $G(x) = 3x + 2$. Find a number r such that $(r, 17)$ is on the graph of G.

14 Suppose F is the function defined by $F(x) = 5x - 3$. Find a number t such that $(t, -23)$ is on the graph of F.

15 Shown below is the graph of a function f.

(a) What is the domain of f?

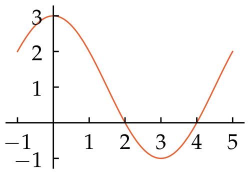
(b) What is the range of f?



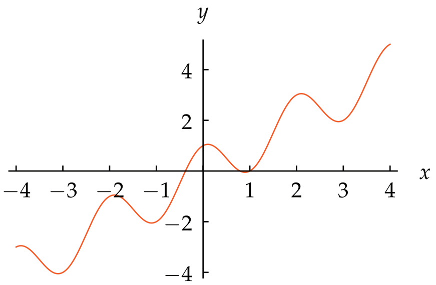
16 Shown below is the graph of a function f.

(a) What is the domain of f?

(b) What is the range of f?



For Exercises 17–28, assume f is the function with domain $[-4, 4]$ whose graph is shown below:



The graph of f.

17 Estimate the value of $f(-4)$.

18 Estimate the value of $f(-3)$.

19 Estimate the value of $f(-2)$.

20 Estimate the value of $f(-1)$.

21 Estimate the value of $f(2)$.

22 Estimate the value of $f(0)$.

23 Estimate the value of $f(4)$.

24 Estimate the value of $f(3)$.

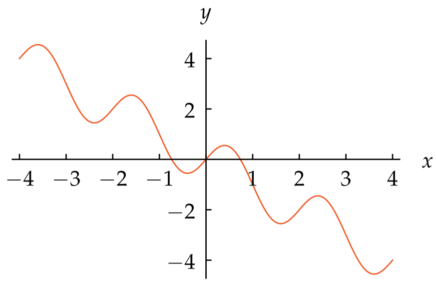
25 Estimate a number b such that $f(b) = 4$.

26 Estimate a negative number b such that $f(b) = 0.5$.

27 How many values of x satisfy the equation $f(x) = \frac{1}{2}$?

28 How many values of x satisfy the equation $f(x) = -3.5$?

For Exercises 29–40, assume g is the function with domain $[-4, 4]$ whose graph is shown below:



The graph of g.

29 Estimate the value of $g(-4)$.

30 Estimate the value of $g(-3)$.

31 Estimate the value of $g(-2)$.

32 Estimate the value of $g(-1)$.

33 Estimate the value of $g(2)$.

34 Estimate the value of $g(1)$.

35 Estimate the value of $g(2.5)$.

36 Estimate the value of $g(1.5)$.

37 Estimate a number b such that $g(b) = 3.5$.

38 Estimate a number b such that $g(b) = -3.5$.

39 How many values of x satisfy the equation $g(x) = -2$?

40 How many values of x satisfy the equation $g(x) = 0$?

For Exercises 41–44, use appropriate technology to sketch the graph of the function f defined by the given formula on the given interval.

41 $f(x) = 2x^3 - 9x^2 + 12x - 3$ on the interval $[\frac{1}{2}, \frac{5}{2}]$

42 $f(x) = 0.6x^5 - 7.5x^4 + 35x^3 - 75x^2 + 72x - 20$ on the interval $[\frac{1}{2}, \frac{9}{2}]$

43 $f(t) = \dfrac{t^2 + 1}{t^5 + 2}$

on the interval $[-\frac{1}{2}, 2]$

44 $f(t) = \dfrac{8t^3 - 5}{t^4 + 2}$

on the interval $[-1, 3]$

For Exercises 45–50, assume g and h are the functions completely defined by the tables below:

x	$g(x)$
−3	−1
−1	1
1	2.5
3	−2

x	$h(x)$
−4	2
−2	−3
2	−1.5
3	1

45 What is the domain of g?

46 What is the domain of h?

47 What is the range of g?

48 What is the range of h?

49 Draw the graph of g.

50 Draw the graph of h.

Problems

51 Sketch the graph of a function whose domain is the interval $[1, 3]$ and whose range is the interval $[-2, 4]$.

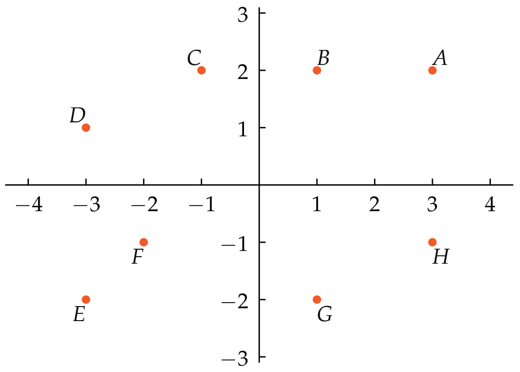
52 Sketch the graph of a function whose domain is the interval $[0, 4]$ and whose range is the set of two numbers $\{2, 3\}$.

53 Give an example of a line in the coordinate plane that is not the graph of any function.

54 Give an example of a set consisting of two points in the coordinate plane that is not the graph of any function.

Worked-Out Solutions to Odd-Numbered Exercises

For Exercises 1–8, give the coordinates of the specified point using the figure below:



1 A

solution To get to the point A starting at the origin, we must move 3 units right and 2 units up. Thus A has coordinates $(3, 2)$.

Numbers obtained from a figure should be considered approximations. Thus the actual coordinates of A might be $(3.01, 1.98)$.

3 C

solution To get to the point C starting at the origin, we must move 1 unit left and 2 units up. Thus C has coordinates $(-1, 2)$.

5 E

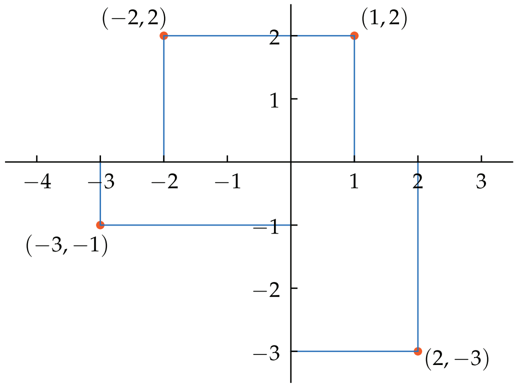
solution To get to the point E starting at the origin, we must move 3 units left and 2 units down. Thus E has coordinates $(-3, -2)$.

7 G

solution To get to the point G starting at the origin, we must move 1 unit right and 2 units down. Thus G has coordinates $(1, -2)$.

9 Sketch a coordinate plane showing the following four points, their coordinates, and the rectangles determined by each point (as in Example 1): $(1, 2)$, $(-2, 2)$, $(-3, -1)$, $(2, -3)$.

solution



11 Suppose F is the function defined by $F(x) = x^2 + 5$. Find a number t such that $(3, t)$ is on the graph of F.

solution Because $(3, t)$ is on the graph of F, we see that $t = F(3) = 3^2 + 5$. Thus $t = 14$.

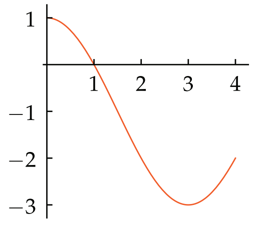
13 Suppose G is the function defined by $G(x) = 3x + 2$. Find a number r such that $(r, 17)$ is on the graph of G.

solution Because $(r, 17)$ is on the graph of G, we see that $17 = G(r) = 3r + 2$. Solving this equation for r, we have $r = 5$.

15 Shown below is the graph of a function f.

(a) What is the domain of f?

(b) What is the range of f?

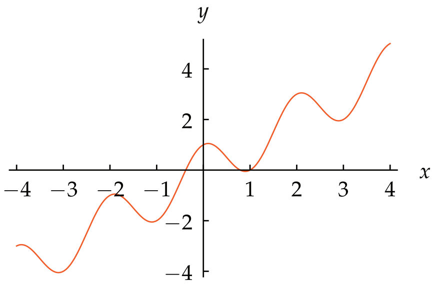


solution

(a) From the figure, it appears that the domain of f is $[0, 4]$. The word "appears" is used here because a figure cannot provide precision. The actual domain of f might be $[0, 4.001]$ or $[0, 3.99]$ or $(0, 4)$.

(b) From the figure, it appears that the range of f is $[-3, 1]$.

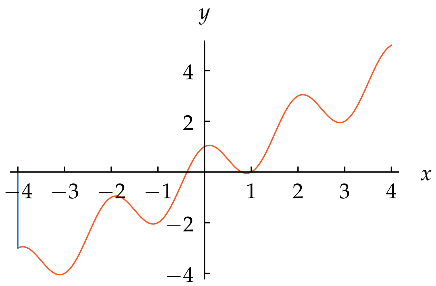
For Exercises 17–28, assume f is the function with domain $[-4, 4]$ whose graph is shown below:



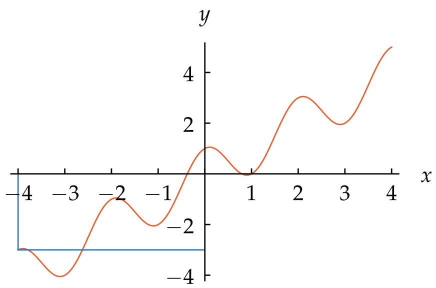
The graph of f.

17 Estimate the value of $f(-4)$.

solution To estimate the value of $f(-4)$, draw a vertical line from the point -4 on the x-axis to the graph, as shown below:



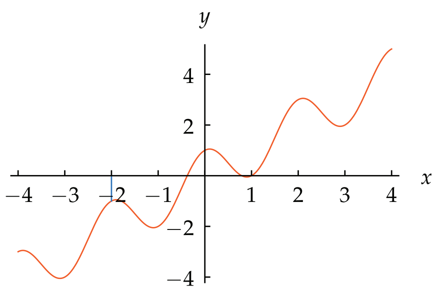
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



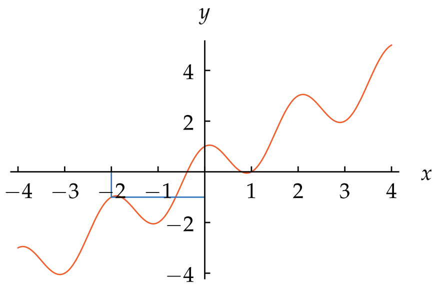
The intersection of the horizontal line with the y-axis gives the value of $f(-4)$. Thus we see that $f(-4) \approx -3$ (the symbol $\approx$ means "is approximately", which is the best that can be done when using a graph).

19 Estimate the value of $f(-2)$.

solution To estimate the value of $f(-2)$, draw a vertical line from the point -2 on the x-axis to the graph, as shown below:



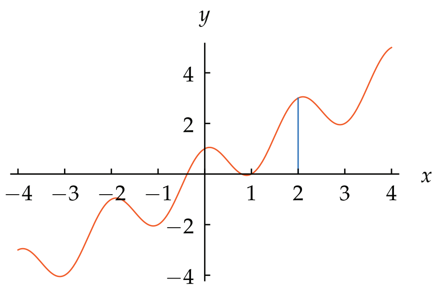
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



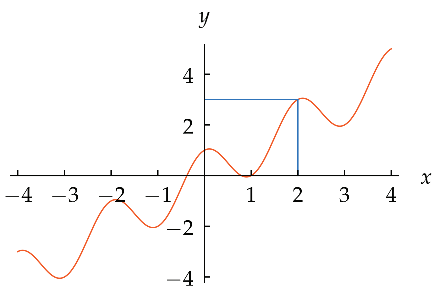
The intersection of the horizontal line with the y-axis gives the value of $f(-2)$. Thus we see that $f(-2) \approx -1$.

21 Estimate the value of $f(2)$.

solution To estimate the value of $f(2)$, draw a vertical line from the point 2 on the x-axis to the graph, as shown below:



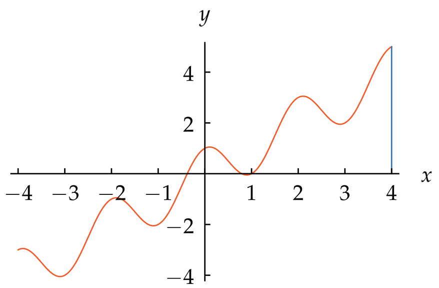
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



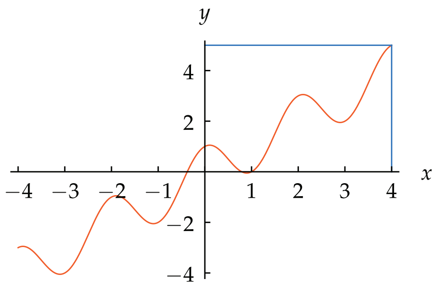
The intersection of the horizontal line with the y-axis gives the value of $f(2)$. Thus we see that $f(2) \approx 3$.

23 Estimate the value of $f(4)$.

solution To estimate the value of $f(4)$, draw a vertical line from the point 4 on the x-axis to the graph, as shown below:



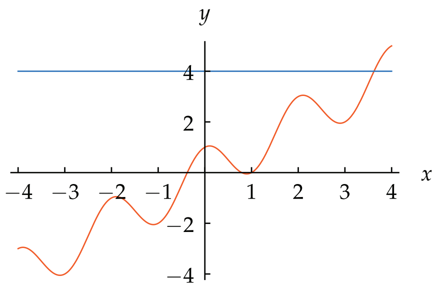
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



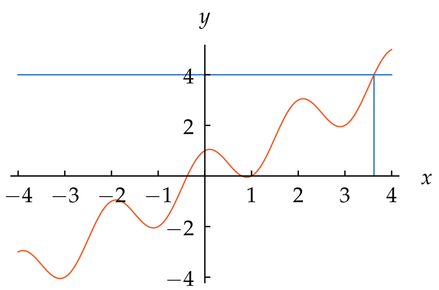
The intersection of the horizontal line with the y-axis gives the value of $f(4)$. Thus we see that $f(4) \approx 5$.

25 Estimate a number b such that $f(b) = 4$.

solution Draw the horizontal line $y = 4$, as shown below:



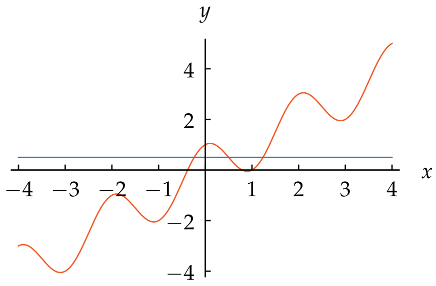
Then draw a vertical line from where this horizontal line intersects the graph to the x-axis:



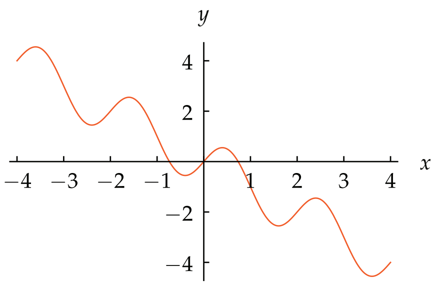
The intersection of the vertical line with the x-axis gives the value of b such that $f(b) = 4$. Thus we see that $b \approx 3.6$.

27 How many values of x satisfy the equation $f(x) = \frac{1}{2}$?

solution Draw the horizontal line $y = \frac{1}{2}$, as shown below. This horizontal line intersects the graph at three points. Thus there exist three values of x such that $f(x) = \frac{1}{2}$.



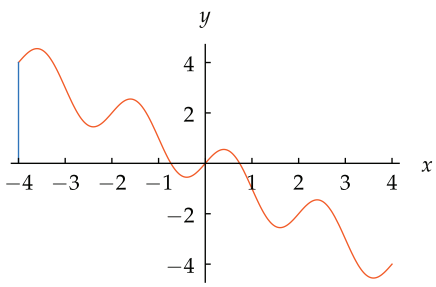
For Exercises 29–40, assume g is the function with domain [−4, 4] whose graph is shown below:



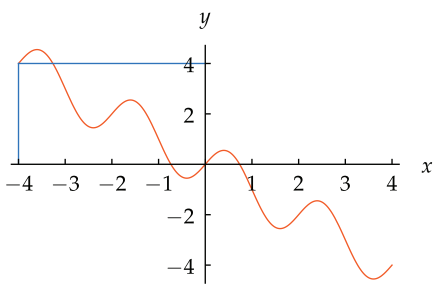
The graph of g.

29 Estimate the value of $g(-4)$.

solution To estimate the value of $g(-4)$, draw a vertical line from the point -4 on the x-axis to the graph, as shown below:



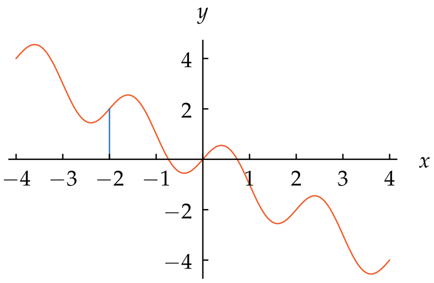
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



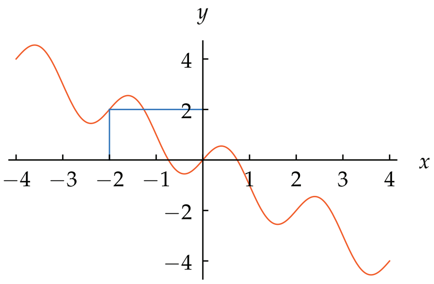
The intersection of the horizontal line with the y-axis gives the value of $g(-4)$. Thus we see that $g(-4) \approx 4$.

31 Estimate the value of $g(-2)$.

solution To estimate the value of $g(-2)$, draw a vertical line from the point -2 on the x-axis to the graph, as shown below:



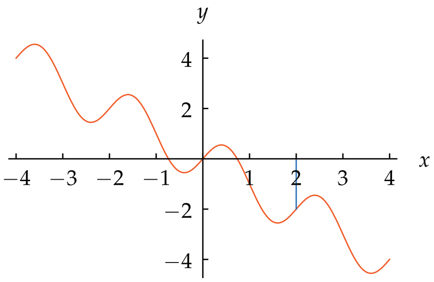
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



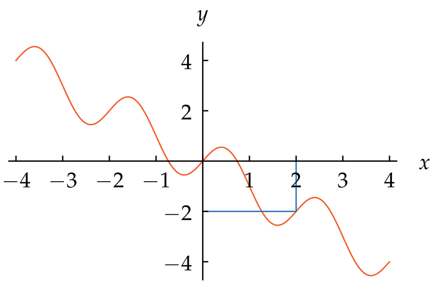
The intersection of the horizontal line with the y-axis gives the value of $g(-2)$. Thus we see that $g(-2) \approx 2$.

33 Estimate the value of $g(2)$.

solution To estimate the value of $g(2)$, draw a vertical line from the point 2 on the x-axis to the graph, as shown below:



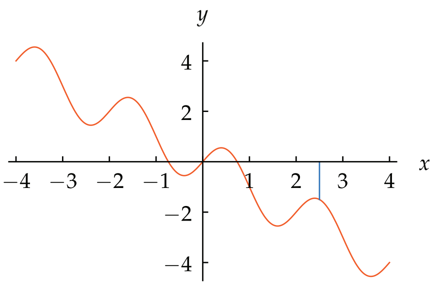
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



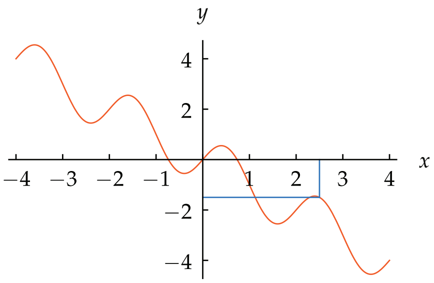
The intersection of the horizontal line with the y-axis gives the value of $g(2)$. Thus $g(2) \approx -2$.

35 Estimate the value of $g(2.5)$.

solution To estimate the value of $g(2.5)$, draw a vertical line from the point 2.5 on the x-axis to the graph, as shown below:



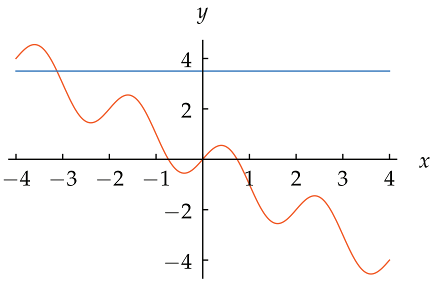
Then draw a horizontal line from where the vertical line intersects the graph to the y-axis:



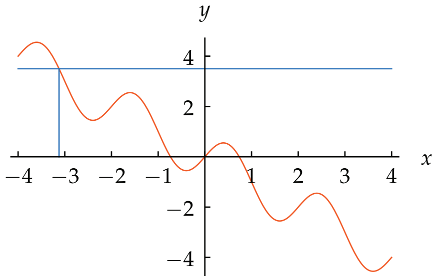
The intersection of the horizontal line with the y-axis gives the value of $g(2.5)$. Thus we see that $g(2.5) \approx -1.5$.

37 Estimate a number b such that $g(b) = 3.5$.

solution Draw the horizontal line $y = 3.5$, as shown below:



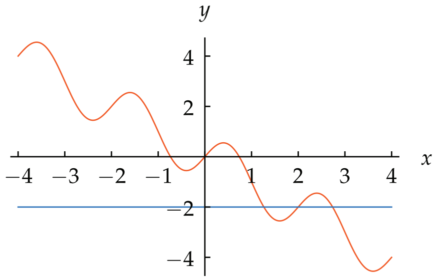
Then draw a vertical line from where this horizontal line intersects the graph to the x-axis:



The intersection of the vertical line with the x-axis gives the value of b such that $g(b) = 3.5$. Thus we see that $b \approx -3.1$.

39 How many values of x satisfy the equation $g(x) = -2$?

solution Draw the horizontal line $y = -2$, as shown here. This horizontal line intersects the graph at three points. Thus there exist three values of x such that $g(x) = -2$.



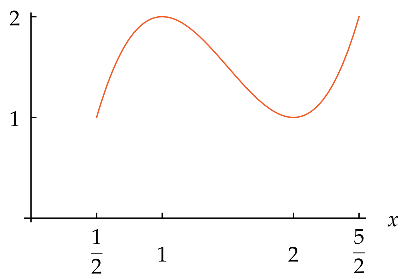
For Exercises 41–44, use appropriate technology to sketch the graph of the function f defined by the given formula on the given interval.

41 $f(x) = 2x^3 - 9x^2 + 12x - 3$ on the interval $[\frac{1}{2}, \frac{5}{2}]$

solution If using *WolframAlpha*, type

graph 2x^3 – 9x^2 + 12x – 3 from x=1/2 to 5/2,

or use your preferred software or graphing calculator to produce a graph like the one below.

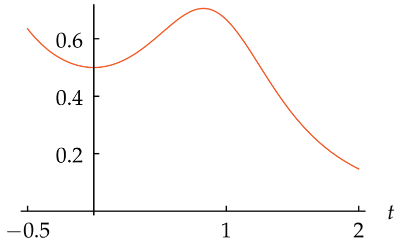


43 $f(t) = \dfrac{t^2 + 1}{t^5 + 2}$ on the interval $[-\frac{1}{2}, 2]$

solution If using *WolframAlpha*, type

graph (t^2 + 1)/(t^5 + 2) from t=–1/2 to 2,

or use your preferred software or graphing calculator to produce a graph like the one here.



For Exercises 45–50, assume g and h are the functions completely defined by the tables below:

x	$g(x)$
−3	−1
−1	1
1	2.5
3	−2

x	$h(x)$
−4	2
−2	−3
2	−1.5
3	1

45 What is the domain of g?

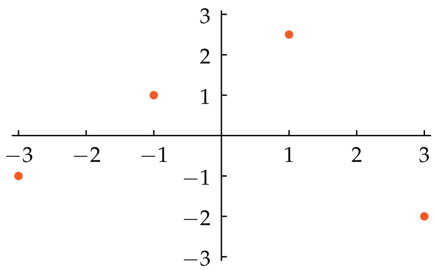
solution The domain of g is the set of numbers in the first column of the table defining g. Thus the domain of g is the set $\{-3, -1, 1, 3\}$.

47 What is the range of g?

solution The range of g is the set of numbers that appear in the second column of the table defining g. Thus the range of g is the set $\{-1, 1, 2.5, -2\}$.

49 Draw the graph of g.

solution The graph of g consists of the four points with coordinates $(-3, -1)$, $(-1, 1)$, $(1, 2.5)$, $(3, -2)$, as shown below:



1.3 Function Transformations and Graphs

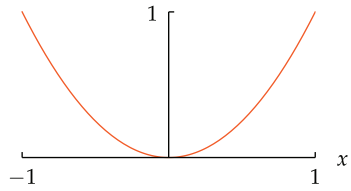
Learning Objectives

By the end of this section you should be able to

- use the vertical function transformations, which shift a graph up or down, stretch a graph vertically, or flip a graph across the horizontal axis;
- use the horizontal function transformations, which shift a graph left or right, stretch a graph horizontally, or flip a graph across the vertical axis;
- combine multiple function transformations;
- determine the domain, range, and graph of a transformed function;
- recognize even functions and odd functions.

In this section we investigate various transformations of functions and learn the effect of such transformations on the domain, range, and graph of a function. To illustrate these ideas, throughout this section we will use the function f defined by $f(x) = x^2$, with the domain of f being the interval $[-1, 1]$. Thus the graph of f is part of a familiar parabola.

In Section 5.7 we will revisit many of the ideas in this section, using trigonometric functions as the examples.



The graph of $f(x) = x^2$, with domain $[-1, 1]$. The range of f is $[0, 1]$.

Vertical Transformations: Shifting, Stretching, and Flipping

This subsection focuses on vertical function transformations, which change the vertical shape or location of the graph of a function. Because vertical function transformations affect the graph only vertically, the vertical function transformations do not change the domain of the function.

We begin with an example showing the procedure for shifting the graph of a function up.

Example 1

Define a function g by

$$g(x) = f(x) + 1,$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) Find the domain of g.

(b) Find the range of g.

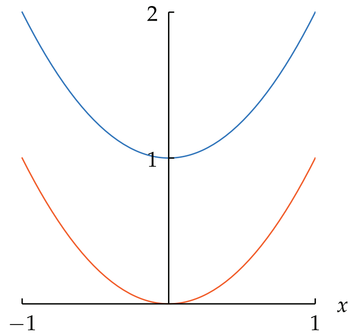
(c) Sketch the graph of g.

solution

(a) The formula defining g shows that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the domain of g equals the domain of f. Thus the domain of g is the interval $[-1, 1]$.

(b) Recall that the range of g is the set of values taken on by $g(x)$ as x varies over the domain of g. Because $g(x)$ equals $f(x) + 1$, the range of g is obtained by adding 1 to each number in the range of f. Thus the range of g is the interval $[1, 2]$.

(c) A typical point on the graph of f has the form (x, x^2), where x is in the interval $[-1, 1]$. Because $g(x) = x^2 + 1$, a typical point on the graph of g has the form $(x, x^2 + 1)$, where x is in the interval $[-1, 1]$. In other words, each point on the graph of g is obtained by adding 1 to the second coordinate of a point on the graph of f. Thus the graph of g is obtained by shifting the graph of f up 1 unit, as shown here.



The graphs of $f(x) = x^2$ (orange) and $g(x) = x^2 + 1$ (blue), each with domain $[-1, 1]$.

Shifting the graph of a function down follows a similar pattern, with a minus sign replacing the plus sign, as shown in the following example.

Example 2

Define a function h by

$$h(x) = f(x) - 1,$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) Find the domain of h.

(b) Find the range of h.

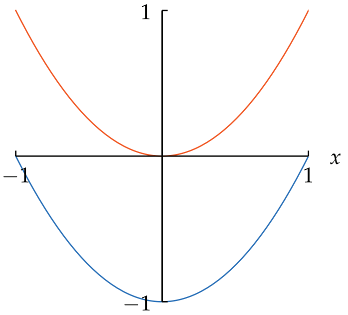
(c) Sketch the graph of h.

solution

(a) The formula above shows that $h(x)$ is defined precisely when $f(x)$ is defined. In other words, the domain of h equals the domain of f. Thus the domain of h is the interval $[-1, 1]$.

(b) Because $h(x)$ equals $f(x) - 1$, the range of h is obtained by subtracting 1 from each number in the range of f. Thus the range of h is the interval $[-1, 0]$.

(c) Because $h(x) = x^2 - 1$, a typical point on the graph of h has the form $(x, x^2 - 1)$, where x is in the interval $[-1, 1]$. Thus the graph of h is obtained by shifting the graph of f down 1 unit, as shown here.



The graphs of $f(x) = x^2$ (orange) and $h(x) = x^2 - 1$ (blue), each with domain $[-1, 1]$.

We could have used any positive number a instead of 1 in these examples when defining $g(x)$ as $f(x) + 1$ and defining $h(x)$ as $f(x) - 1$. Similarly, there is nothing special about the particular function f that we used. Thus the following results hold in general.

Instead of memorizing the conclusions in all the result boxes in this section, try to understand how these conclusions arise. Then you can figure out what you need depending on the problem at hand.

Shifting a graph up or down

Suppose f is a function and $a > 0$. Define functions g and h by

$$g(x) = f(x) + a \quad \text{and} \quad h(x) = f(x) - a.$$

Then

- the graph of g is obtained by shifting the graph of f up a units;
- the graph of h is obtained by shifting the graph of f down a units.

The next example shows how to stretch (or shrink) the graph of a function in the vertical direction.

Example 3

Define functions g and h by

$$g(x) = 2f(x) \quad \text{and} \quad h(x) = \tfrac{1}{2}f(x),$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) Find the domain of g and the domain of h.

(b) Find the range of g.

(c) Find the range of h.

(d) Sketch the graphs of g and h.

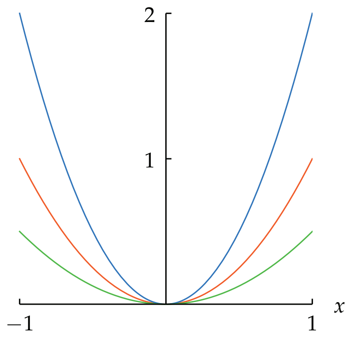
solution

(a) The formulas defining g and h show that $g(x)$ and $h(x)$ are defined precisely when $f(x)$ is defined. In other words, the domain of g and the domain of h both equal the domain of f. Thus the domain of g and the domain of h both equal the interval $[-1, 1]$.

(b) Because $g(x)$ equals $2f(x)$, the range of g is obtained by multiplying each number in the range of f by 2. Thus the range of g is the interval $[0, 2]$.

(c) Because $h(x)$ equals $\frac{1}{2}f(x)$, the range of h is obtained by multiplying each number in the range of f by $\frac{1}{2}$. Thus the range of h is the interval $[0, \frac{1}{2}]$.

(d) For each x in the interval $[-1, 1]$, the point $(x, 2x^2)$ is on the graph of g and the point $(x, \frac{1}{2}x^2)$ is on the graph of h. Thus the graph of g is obtained by vertically stretching the graph of f by a factor of 2, and the graph of h is obtained by vertically stretching the graph of f by a factor of $\frac{1}{2}$, as shown here.



The graphs of $f(x) = x^2$ (orange) and $g(x) = 2x^2$ (blue) and $h(x) = \frac{1}{2}x^2$ (green), each with domain $[-1, 1]$.

In the last part of the example above, we noted that the graph of h is obtained by vertically stretching the graph of f by a factor of $\frac{1}{2}$. This terminology may seem a bit strange because the word "stretch" often has the connotation of something getting larger. However, we will find it convenient to use the word "stretch" in the wider sense of multiplying by some positive number, which might be less than 1.

Perhaps the word "shrink" would be more appropriate here.

We could have used any positive number c instead of 2 or $\frac{1}{2}$ in the example above. Similarly, there is nothing special about the particular function f that we used. Thus the following result holds in general.

Stretching a graph vertically

Suppose f is a function and $c > 0$. Define a function g by

$$g(x) = cf(x).$$

Then the graph of g is obtained by vertically stretching the graph of f by a factor of c.

The procedure for flipping the graph of a function across the horizontal axis is illustrated by the following example.

Example 4

Define a function g by

$$g(x) = -f(x),$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) Find the domain of g.

(b) Find the range of g.

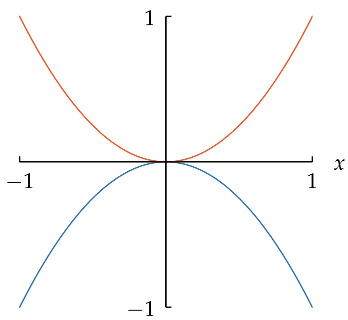
(c) Sketch the graph of g.

solution

(a) The formula defining g shows that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the domain of g equals the domain of f. Thus the domain of g is the interval $[-1, 1]$.

(b) Because $g(x)$ equals $-f(x)$, the values taken on by g are the negatives of the values taken on by f. Thus the range of g is the interval $[-1, 0]$.

(c) Note that $g(x) = -x^2$ for each x in the interval $[-1, 1]$. For each point (x, x^2) on the graph of f, the point $(x, -x^2)$ is on the graph of g. Thus the graph of g is obtained by flipping the graph of f across the horizontal axis, as shown here.



The graphs of $f(x) = x^2$ (orange) and $g(x) = -x^2$ (blue), each with domain $[-1, 1]$.

The following result holds for every function f.

Many books use the word **reflect** *instead of* **flip**. *However, flipping seems to be a more accurate description of how the blue graph above is obtained from the orange graph. Reflecting, as happens with a mirror, would not produce the desired graph.*

Flipping a graph across the horizontal axis

Suppose f is a function. Define a function g by

$$g(x) = -f(x).$$

Then the graph of g is obtained by flipping the graph of f across the horizontal axis.

Horizontal Transformations: Shifting, Stretching, Flipping

Vertical transformations work pretty much as you would expect. As you will soon see, the actions of horizontal transformation are less intuitive.

Now we focus on horizontal function transformations, which change the horizontal shape or location of the graph of a function. Because horizontal function transformations affect the graph only horizontally, the horizontal function transformations do not change the range of the function.

We begin with an example showing the procedure for shifting the graph of a function to the left.

Example 5

Define a function g by

$$g(x) = f(x+1),$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) Find the domain of g.

(b) Find the range of g.

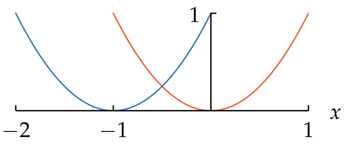
(c) Sketch the graph of g.

solution

(a) The formula defining g shows that $g(x)$ is defined precisely when $f(x+1)$ is defined, which means that $x+1$ must be in the interval $[-1, 1]$, which means that x must be in the interval $[-2, 0]$. Thus the domain of g is the interval $[-2, 0]$.

(b) Because $g(x)$ equals $f(x+1)$, the values taken on by g are the same as the values taken on by f. Thus the range of g equals the range of f, which is the interval $[0, 1]$.

(c) Note that $g(x) = (x+1)^2$ for each x in the interval $[-2, 0]$. For each point (x, x^2) on the graph of f, the point $(x-1, x^2)$ is on the graph of g (because $g(x-1) = x^2$). Thus the graph of g is obtained by shifting the graph of f left 1 unit, as shown here.



The graphs of $f(x) = x^2$ (orange, with domain $[-1, 1]$) and $g(x) = (x+1)^2$ (blue, with domain $[-2, 0]$). The graph of g is obtained by shifting the graph of f left 1 unit.

Suppose we define a function h by

$$h(x) = f(x-1),$$

where f is again the function defined by $f(x) = x^2$, with the domain of f the interval $[-1, 1]$. Then everything works as in the example above, except that the domain and graph of h are obtained by shifting the domain and graph of f right 1 unit (instead of left 1 unit as in the example above).

More generally, we could have used any positive number b instead of 1 in these examples when defining $g(x)$ as $f(x+1)$ and defining $h(x)$ as $f(x-1)$. Similarly, there is nothing special about the particular function f that we used. Thus the following results hold in general.

Shifting a graph left or right

Suppose f is a function and $b > 0$. Define functions g and h by

$$g(x) = f(x+b) \quad \text{and} \quad h(x) = f(x-b).$$

Then

- the graph of g is obtained by shifting the graph of f left b units;
- the graph of h is obtained by shifting the graph of f right b units.

The next example shows the procedure for horizontally stretching a graph.

Example 6

Define functions g and h by

$$g(x) = f(2x) \quad \text{and} \quad h(x) = f(\tfrac{1}{2}x),$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) Find the domain of g.

(b) Find the domain of h.

(c) Find the range of g and the range of h.

(d) Sketch the graphs of g and h.

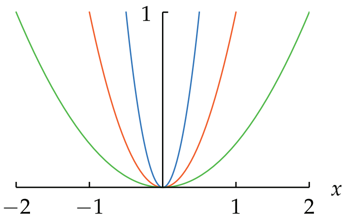
solution

(a) The formula defining g shows that $g(x)$ is defined precisely when $f(2x)$ is defined, which means that $2x$ must be in the interval $[-1, 1]$, which means that x must be in the interval $[-\frac{1}{2}, \frac{1}{2}]$. Thus the domain of g is the interval $[-\frac{1}{2}, \frac{1}{2}]$.

(b) The formula defining h shows that $h(x)$ is defined precisely when $f(\frac{1}{2}x)$ is defined, which means that $\frac{1}{2}x$ must be in the interval $[-1, 1]$, which means that x must be in the interval $[-2, 2]$. Thus the domain of h is the interval $[-2, 2]$.

(c) The formulas defining g and h show that the values taken on by g and h are the same as the values taken on by f. Thus the range of g and the range of h both equal the range of f, which is the interval $[0, 1]$.

(d) For each point (x, x^2) on the graph of f, the point $(\frac{x}{2}, x^2)$ is on the graph of g (because $g(\frac{x}{2}) = x^2$) and the point $(2x, x^2)$ is on the graph of h (because $h(2x) = x^2$) . Thus the graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$, and the graph of h is obtained by horizontally stretching the graph of f by a factor of 2, as shown here.



The graphs of $f(x) = x^2$ (orange, with domain $[-1, 1]$) and $g(x) = (2x)^2$ (blue, with domain $[-\frac{1}{2}, \frac{1}{2}]$) and $h(x) = (\frac{1}{2}x)^2$ (green, with domain $[-2, 2]$).

We could have used any positive number c instead of 2 or $\frac{1}{2}$ when defining $g(x)$ as $f(2x)$ and defining $h(x)$ as $f(\frac{1}{2}x)$ in the example above. Similarly, nothing special about the particular function f was used. Thus the following result holds.

Stretching a graph horizontally

Suppose f is a function and $c > 0$. Define a function g by

$$g(x) = f(cx).$$

Then the graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{c}$.

Note that if $c > 1$, then $\frac{1}{c} < 1$, which means that stretching the graph by a factor of $\frac{1}{c}$ actually shrinks the graph.

The procedure for flipping the graph of a function across the vertical axis is illustrated by the following example. To show the ideas more clearly, the domain of f has been changed to the interval $[\frac{1}{2}, 1]$.

Example 7

Define a function g by

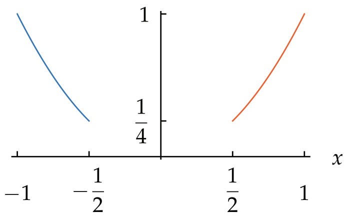
$$g(x) = f(-x),$$

where $f(x) = x^2$ and the domain of f is the interval $[\frac{1}{2}, 1]$.

(a) Find the domain of g.

(b) Find the range of g.

(c) Sketch the graph of g.



The graphs of $f(x) = x^2$ (orange, with domain $[\frac{1}{2}, 1]$) and $g(x) = (-x)^2 = x^2$ (blue, with domain $[-1, -\frac{1}{2}]$). The graph of g is obtained by flipping the graph of f across the vertical axis.

solution

(a) The formula defining g shows that $g(x)$ is defined precisely when $f(-x)$ is defined, which means that $-x$ must be in the interval $[\frac{1}{2}, 1]$, which means that x must be in the interval $[-1, -\frac{1}{2}]$. Thus the domain of g is the interval $[-1, -\frac{1}{2}]$.

(b) Because $g(x)$ equals $f(-x)$, the values taken on by g are the same as the values taken on by f. Thus the range of g equals the range of f, which is the interval $[\frac{1}{4}, 1]$.

(c) Note that $g(x) = (-x)^2 = x^2$ for each x in the interval $[-1, -\frac{1}{2}]$. For each point (x, x^2) on the graph of f, the point $(-x, x^2)$ is on the graph of g (because $g(-x) = x^2$). Thus the graph of g is obtained by flipping the graph of f across the vertical axis.

The result in the example holds for any function f, not just the specific function in the example above. In other words, we have the following result.

The values taken on by g are the same as the values taken on by f. In other words, the range of g equals the range of f.

Flipping a graph across the vertical axis

Suppose f is a function. Define a function g by

$$g(x) = f(-x).$$

Then the graph of g is obtained by flipping the graph of f across the vertical axis. Also, the domain of g is obtained by multiplying each number in the domain of f by -1.

Combinations of Vertical Function Transformations

When dealing with combinations of vertical function transformations, the order in which those transformations are applied can be crucial. The following simple procedure can be used to find the graph.

Combinations of vertical function transformations

To obtain the graph of a function defined by combinations of vertical function transformations, apply the transformations in the same order as the corresponding operations when evaluating the function.

Example 8

Define a function g by

$$g(x) = -2f(x) + 1,$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) List the order of operations used to evaluate $g(x)$ after $f(x)$ has been evaluated.

(b) Find the domain of g.

(c) Find the range of g.

(d) Sketch the graph of g.

solution

(a) Because $g(x) = -2f(x) + 1$, operations should be done in the following order to evaluate $g(x)$:

1. Multiply $f(x)$ by 2.
2. Multiply the number obtained in the previous step by -1.
3. Add 1 to the number obtained in the previous step.

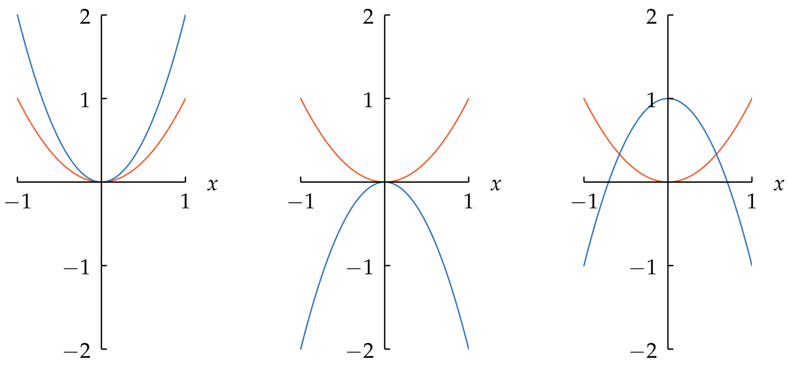
The order of steps 1 and 2 could be interchanged. However, the operation of adding 1 must be the last step.

(b) The formula defining g shows that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the domain of g equals the domain of f. Thus the domain of g is the interval $[-1, 1]$.

(c) The range of g is obtained by applying the operations from the answer to part (a), in the same order, to the range of f, which is the interval $[0, 1]$:

1. Multiplying each number in $[0, 1]$ by 2 gives the interval $[0, 2]$.
2. Multiplying each number in $[0, 2]$ by -1 gives the interval $[-2, 0]$.
3. Adding 1 to each number in $[-2, 0]$ gives the interval $[-1, 1]$, which is the range of g.

(d) Applying function transformations in the same order as in the answer to part (a), we see that the graph of g is obtained from the graph of f by vertically stretching the graph of f by a factor of 2, then flipping the resulting graph across the horizontal axis, then shifting the resulting graph up 1 unit, as follows.



The operations listed in the solutions to part (a) of this example and the next example are the same, differing only in the order. However, the different order produces different graphs.

The graphs of $f(x) = x^2$ (orange), $2f(x)$ (blue, left), $-2f(x)$ (blue, center), and $-2f(x) + 1$ (blue, right), each with domain $[-1, 1]$.

Comparing the example above to the next example shows the importance of applying the operations in the appropriate order.

Example 9

Define a function h by

$$h(x) = -2\big(f(x) + 1\big),$$

where $f(x) = x^2$ and the domain of f is the interval $[-1, 1]$.

(a) List the order of operations used to evaluate $h(x)$ after $f(x)$ has been evaluated.

(b) Find the domain of h.

(c) Find the range of h.

(d) Sketch the graph of h.

solution

(a) Because $h(x) = -2\big(f(x) + 1\big)$, operations should be done in the following order to evaluate $h(x)$:

1. Add 1 to $f(x)$.
2. Multiply the number obtained in the previous step by 2.
3. Multiply the number obtained in the previous step by -1.

The order of steps 2 and 3 could be interchanged. However, the operation of adding 1 must be the first step.

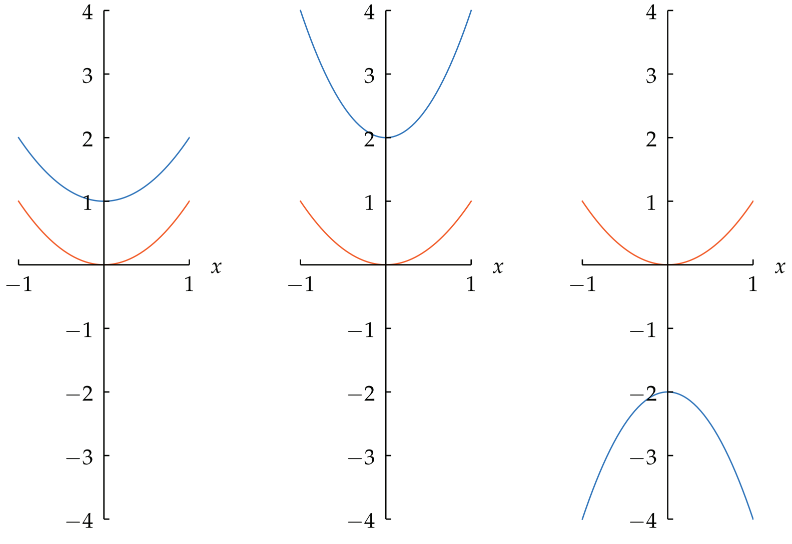
(b) The formula defining h shows that $h(x)$ is defined precisely when $f(x)$ is defined. In other words, the domain of h equals the domain of f. Thus the domain of h is the interval $[-1, 1]$.

(c) The range of h is obtained by applying the operations from the answer to part (a), in the same order, to the range of f, which is the interval $[0, 1]$:

1. Adding 1 to each number in $[0, 1]$ gives the interval $[1, 2]$.
2. Multiplying each number in $[1, 2]$ by 2 gives the interval $[2, 4]$.
3. Multiplying each number in $[2, 4]$ by -1 gives the interval $[-4, -2]$, which is the range of h.

(d) Applying function transformations in the same order as in the answer to part (a), we see that the graph of g is obtained by shifting the graph of f up 1 unit, then vertically stretching the resulting graph by a factor of 2, then flipping the resulting graph across the horizontal axis, producing the graph shown here.

Notice how the graph of $-2\big(f(x) + 1\big)$ in this example differs from the graph of $-2f(x) + 1$ in the previous example.



The graphs of $f(x) = x^2$ (orange), $f(x) + 1$ (blue, left), $2\big(f(x) + 1\big)$ (blue, center), and $-2\big(f(x) + 1\big)$ (blue, right), each with domain $[-1, 1]$.

When dealing with a combination of a vertical function transformation and a horizontal function transformation, the transformations can be applied in either order. For a combination of multiple vertical function transformations and a single horizontal function transformation, be sure to apply the vertical function transformations in the proper order; the horizontal function transformation can be applied either after or before the vertical function transformations.

Combinations of multiple horizontal function transformations, possibly combined with vertical function transformations, are more complicated. If you are interested in these kinds of multiple function transformations, then see the worked-out solutions to Exercises 45, 47, 49, 51, 53, and 55 for examples of the proper technique.

Even Functions

Suppose $f(x) = x^2$ for every real number x. Notice that

$$f(-x) = (-x)^2 = x^2 = f(x).$$

This property is sufficiently important that we give it a name.

Even function

A function f is called **even** if

$$f(-x) = f(x)$$

for every x in the domain of f.

If n is an integer, then the function f defined by $f(x) = x^n$ is an even function if and only if n is an even integer.

For the equation $f(-x) = f(x)$ to hold for every x in the domain of f, the expression $f(-x)$ must make sense. Thus $-x$ must be in the domain of f for every x in the domain of f. For example, a function whose domain is the interval $[-3, 5]$ cannot possibly be an even function, but a function whose domain is the interval $(-4, 4)$ may or may not be an even function.

As we have already observed, x^2 is an even function. Here is another simple example.

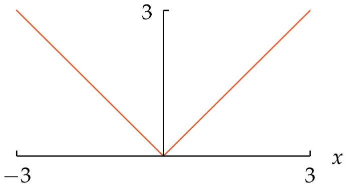
Example 10

Show that the function f defined by $f(x) = |x|$ for every real number x is an even function.

solution This function is even because

$$f(-x) = |-x| = |x| = f(x)$$

for every real number x.



The graph of $|x|$ on the interval $[-3, 3]$.

Suppose f is an even function. As we know, flipping the graph of f across the vertical axis gives the graph of the function h defined by $h(x) = f(-x)$. Because f is even, we actually have $h(x) = f(-x) = f(x)$, which implies that $h = f$.

In other words, flipping the graph of an even function f across the vertical axis gives us back the graph of f. Thus the graph of an even function is symmetric about the vertical axis. This symmetry can be seen, for example, in the graph shown above of $|x|$ on the interval $[-3, 3]$.

The graph of an even function

A function is even if and only if its graph is unchanged when flipped across the vertical axis.

Odd Functions

Consider now the function defined by $f(x) = x^3$ for every real number x. Notice that

$$f(-x) = (-x)^3$$
$$= -(x^3)$$
$$= -f(x).$$

This property is sufficiently important that we also give it an appropriate descriptive name.

If n is an integer, then the function f defined by $f(x) = x^n$ is an odd function if and only if n is an odd integer.

Odd function

A function f is called **odd** if

$$f(-x) = -f(x)$$

for every x in the domain of f.

As was the case for even functions, for a function to be odd $-x$ must be in the domain of f for every x in the domain of f, because otherwise there is no possibility that the equation $f(-x) = -f(x)$ can hold for every x in the domain of f. Thus, for example, the interval $[-2, 2]$ can be the domain of an odd function, but the interval $[-2, 3]$ cannot be the domain of an odd function.

As we have already observed, x^3 is an odd function. Here is another simple example.

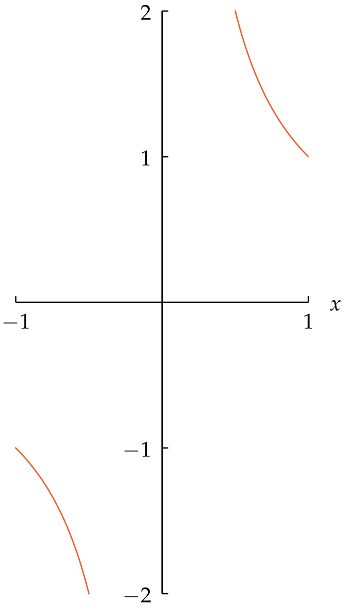
Example 11

Show that the function f defined by $f(x) = \frac{1}{x}$ for every real number $x \neq 0$ is an odd function.

solution This function is odd because

$$f(-x) = \frac{1}{-x}$$
$$= -\frac{1}{x}$$
$$= -f(x)$$

for every real number $x \neq 0$.



The graph of $\frac{1}{x}$ on $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$.

Suppose f is an odd function. If x is a number in the domain of f, then $\big(x, f(x)\big)$ is a point on the graph of f. Because $f(-x) = -f(x)$, the point $\big(-x, -f(x)\big)$ also is on the graph of f.

In other words, flipping a point $\big(x, f(x)\big)$ on the graph of an odd function f across the origin gives a point $\big(-x, -f(x)\big)$ that is also on the graph of f. Thus the graph of an odd function is symmetric about the origin. This symmetry can be seen, for example, in the graph shown here of $\frac{1}{x}$ on $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$.

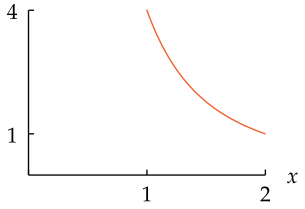
Flipping a point (c, d) across the origin gives the point $(-c, -d)$.

The graph of an odd function

A function is odd if and only if its graph is unchanged when flipped across the origin.

Exercises

For Exercises 1–14, assume f is the function defined on the interval $[1,2]$ by the formula $f(x) = \frac{4}{x^2}$. Thus the domain of f is the interval $[1,2]$ and the range of f is the interval $[1,4]$. The graph of f is shown here.



The graph of the function f defined on the interval $[1,2]$ by $f(x) = \frac{4}{x^2}$.

For each function g described below:

- **(a)** *Sketch the graph of g.*
- **(b)** *Find the domain of g (the endpoints of this interval should be shown on the horizontal axis of your sketch of the graph of g).*
- **(c)** *Give a formula for g.*
- **(d)** *Find the range of g (the endpoints of this interval should be shown on the vertical axis of your sketch of the graph of g).*

1 The graph of g is obtained by shifting the graph of f up 1 unit.

2 The graph of g is obtained by shifting the graph of f up 3 units.

3 The graph of g is obtained by shifting the graph of f down 3 units.

4 The graph of g is obtained by shifting the graph of f down 2 units.

5 The graph of g is obtained by vertically stretching the graph of f by a factor of 2.

6 The graph of g is obtained by vertically stretching the graph of f by a factor of 3.

7 The graph of g is obtained by shifting the graph of f left 3 units.

8 The graph of g is obtained by shifting the graph of f left 4 units.

9 The graph of g is obtained by shifting the graph of f right 1 unit.

10 The graph of g is obtained by shifting the graph of f right 3 units.

11 The graph of g is obtained by horizontally stretching the graph of f by a factor of 2.

12 The graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$.

13 The graph of g is obtained by flipping the graph of f across the horizontal axis.

14 The graph of g is obtained by flipping the graph of f across the vertical axis.

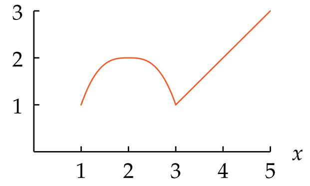
15 Suppose g is an even function whose domain is $[-2,-1] \cup [1,2]$ and whose graph on the interval $[1,2]$ is the graph used in the instructions for Exercises 1–14. Sketch the graph of g on $[-2,-1] \cup [1,2]$.

16 Suppose g is an even function whose domain is $[-5,-1] \cup [1,5]$ and whose graph on the interval $[1,5]$ is the graph used in the instructions for Exercises 19–56. Sketch the graph of g on $[-5,-1] \cup [1,5]$.

17 Suppose h is an odd function whose domain is $[-2,-1] \cup [1,2]$ and whose graph on the interval $[1,2]$ is the graph used in the instructions for Exercises 1–14. Sketch the graph of h on $[-2,-1] \cup [1,2]$.

18 Suppose h is an odd function whose domain is $[-5,-1] \cup [1,5]$ and whose graph on the interval $[1,5]$ is the graph used in the instructions for Exercises 19–56. Sketch the graph of h on $[-5,-1] \cup [1,5]$.

For Exercises 19–56, assume f is a function whose domain is the interval $[1,5]$, whose range is the interval $[1,3]$, and whose graph is the figure below.



The graph of f.

For each given function g:

- **(a)** *Find the domain of g.*
- **(b)** *Find the range of g.*
- **(c)** *Sketch the graph of g.*

19 $g(x) = f(x) + 1$

20 $g(x) = f(x) + 3$

21 $g(x) = f(x) - 3$

22 $g(x) = f(x) - 5$

23 $g(x) = 2f(x)$

24 $g(x) = \frac{1}{2}f(x)$

25 $g(x) = f(x+2)$

26 $g(x) = f(x+3)$

27 $g(x) = f(x-1)$

28 $g(x) = f(x-2)$

29 $g(x) = f(2x)$

30 $g(x) = f(3x)$

31 $g(x) = f\left(\frac{x}{2}\right)$

32 $g(x) = f\left(\frac{5x}{8}\right)$

33 $g(x) = 2f(x) + 1$

34 $g(x) = 3f(x) + 2$

35 $g(x) = \frac{1}{2}f(x) - 1$

36 $g(x) = \frac{2}{3}f(x) - 2$

37 $g(x) = 3 - f(x)$

38 $g(x) = 2 - f(x)$

39 $g(x) = -f(x-1)$

40 $g(x) = -f(x-3)$

41 $g(x) = f(x+1) + 2$

42 $g(x) = f(x+2) + 1$

43 $g(x) = f(2x) + 1$

44 $g(x) = f(3x) + 2$

45 $g(x) = f(2x+1)$

46 $g(x) = f(3x+2)$

47 $g(x) = 3f(2x+1)$

48 $g(x) = 2f(3x+2)$

49 $g(x) = 2f\left(\frac{x}{2}+1\right)$

50 $g(x) = 3f\left(\frac{2x}{5}+2\right)$

51 $g(x) = 2f\left(\frac{x}{2}+1\right) - 3$

52 $g(x) = 3f\left(\frac{2x}{5}+2\right) + 1$

53 $g(x) = 2f\left(\frac{x}{2}+3\right)$

54 $g(x) = 3f\left(\frac{2x}{5}-2\right)$

55 $g(x) = 6 - 2f\left(\frac{x}{2}+3\right)$

56 $g(x) = 1 - 3f\left(\frac{2x}{5}-2\right)$

For Exercises 57–60, suppose f is a function whose domain is the interval $[-5,5]$ and

$$f(x) = \frac{x}{x+3}$$

for every x in the interval $[0,5]$.

57 Suppose f is an even function. Evaluate $f(-2)$.

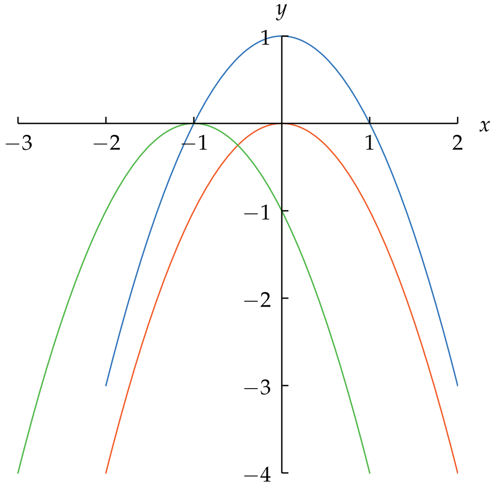
58 Suppose f is an even function. Evaluate $f(-3)$.

59 Suppose f is an odd function. Evaluate $f(-2)$.

60 Suppose f is an odd function. Evaluate $f(-3)$.

Problems

61 The figure below shows the graphs of $-x^2$, $-x^2+1$, and $-(x+1)^2$.



The graphs of $-x^2$, $-x^2+1$, and $-(x+1)^2$.

(a) What color is the graph of $-x^2$?

(b) What color is the graph of $-x^2+1$?

(c) What color is the graph of $-(x+1)^2$?

For Problems 62–65, suppose that to provide additional funds for higher education, the federal government adopts a new income tax plan that consists of the 2016 income tax plus an additional $100 per taxpayer. Let g be the function such that $g(x)$ is the 2016 federal income tax for a single person with taxable income x dollars, and let h be the corresponding function for the new income tax plan.

62 Is h obtained from g by a vertical function transformation or by a horizontal function transformation?

63 Write a formula for $h(x)$ in terms of $g(x)$.

64 Using the explicit formula for $g(x)$ given in Example 2 in Section 1.1, give an explicit formula for $h(x)$.

65 Under the new income tax plan, what will be the income tax for a single person whose annual taxable income is $75,000?

For Problems 66–69, suppose that to pump more money into the economy during a recession, the federal government adopts a new income tax plan that makes income taxes 90% of the 2016 income tax. Let g be the function such that $g(x)$ is the 2016 federal income tax for a single person with taxable income x dollars, and let h be the corresponding function for the new income tax plan.

66 Is h obtained from g by a vertical function transformation or by a horizontal function transformation?

67 Write a formula for $h(x)$ in terms of $g(x)$.

68 Using the explicit formula for $g(x)$ given in Example 2 in Section 1.1, give an explicit formula for $h(x)$.

69 Under the new income tax plan, what will be the income tax for a single person whose annual taxable income is $40,000?

70 Find the only function whose domain is the set of real numbers and is both even and odd.

71 Show that if f is an odd function such that 0 is in the domain of f, then $f(0)=0$.

72 The result box following Example 2 could have been made more complete by including explicit information about the domain and range of the functions g and h. For example, the more complete result box might have looked like the one shown here:

> ***Shifting a graph up or down***
>
> Suppose f is a function and $a>0$. Define functions g and h by
>
> $$g(x)=f(x)+a \quad \text{and} \quad h(x)=f(x)-a.$$
>
> Then
>
> - g and h have the same domain as f;
> - the range of g is obtained by adding a to every number in the range of f;
> - the range of h is obtained by subtracting a from every number in the range of f;
> - the graph of g is obtained by shifting the graph of f up a units;
> - the graph of h is obtained by shifting the graph of f down a units.

Construct similar complete result boxes, including explicit information about the domain and range of the functions g and h, for each of the other five result boxes in this section that deal with function transformations.

73 True or false: If f is an odd function whose domain is the set of real numbers and a function g is defined by

$$g(x)=\begin{cases} f(x) & \text{if } x\geq 0 \\ -f(x) & \text{if } x<0, \end{cases}$$

then g is an even function. Explain your answer.

74 True or false: If f is an even function whose domain is the set of real numbers and a function g is defined by

$$g(x)=\begin{cases} f(x) & \text{if } x\geq 0 \\ -f(x) & \text{if } x<0, \end{cases}$$

then g is an odd function. Explain your answer.

75 (a) True or false: Just as every integer is either even or odd, every function whose domain is the set of integers is either an even function or an odd function.

(b) Explain your answer to part (a). This means that if the answer is "true", then you should explain why every function whose domain is the set of integers is either an even function or an odd function; if the answer is "false", then you should give an example of a function whose domain is the set of integers but is neither even nor odd.

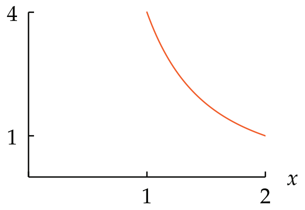
76 Show that the function f defined by $f(x)=mx+b$ is an odd function if and only if $b=0$.

77 Show that the function f defined by $f(x)=mx+b$ is an even function if and only if $m=0$.

78 Show that the function f defined by $f(x)=ax^2+bx+c$ is an even function if and only if $b=0$.

Worked-Out Solutions to Odd-Numbered Exercises

For Exercises 1–14, assume f is the function defined on the interval $[1,2]$ by the formula $f(x) = \frac{4}{x^2}$. Thus the domain of f is the interval $[1,2]$ and the range of f is the interval $[1,4]$. The graph of f is shown here.



The graph of the function f defined on the interval $[1,2]$ by $f(x) = \frac{4}{x^2}$.

For each function g described below:

(a) ***Sketch the graph of g.***

(b) ***Find the domain of g (the endpoints of this interval should be shown on the horizontal axis of your sketch of the graph of g).***

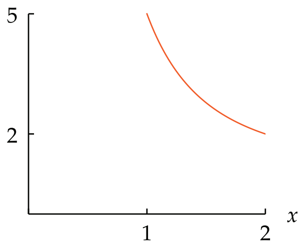
(c) ***Give a formula for g.***

(d) ***Find the range of g (the endpoints of this interval should be shown on the vertical axis of your sketch of the graph of g).***

1 The graph of g is obtained by shifting the graph of f up 1 unit.

solution

(a)



Shifting the graph of f up 1 unit gives this graph.

(b) The domain of g is the same as the domain of f. Thus the domain of g is the interval $[1,2]$.

(c) Because the graph of g is obtained by shifting the graph of f up 1 unit, we have $g(x) = f(x) + 1$. Thus

$$g(x) = \frac{4}{x^2} + 1$$

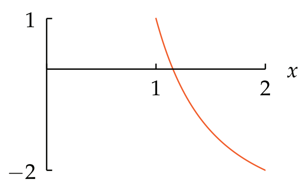
for each number x in the interval $[1,2]$.

(d) The range of g is obtained by adding 1 to each number in the range of f. Thus the range of g is the interval $[2,5]$.

3 The graph of g is obtained by shifting the graph of f down 3 units.

solution

(a)



Shifting the graph of f down 3 units gives this graph.

(b) The domain of g is the same as the domain of f. Thus the domain of g is the interval $[1,2]$.

(c) Because the graph of g is obtained by shifting the graph of f down 3 units, we have $g(x) = f(x) - 3$. Thus

$$g(x) = \frac{4}{x^2} - 3$$

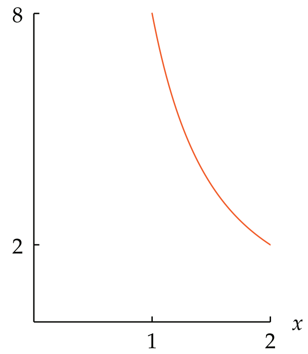
for each number x in the interval $[1,2]$.

(d) The range of g is obtained by subtracting 3 from each number in the range of f. Thus the range of g is the interval $[-2,1]$.

5 The graph of g is obtained by vertically stretching the graph of f by a factor of 2.

solution

(a)



Vertically stretching the graph of f by a factor of 2 gives this graph.

(b) The domain of g is the same as the domain of f. Thus the domain of g is the interval $[1,2]$.

(c) Because the graph of g is obtained by vertically stretching the graph of f by a factor of 2, we have $g(x) = 2f(x)$. Thus

$$g(x) = \frac{8}{x^2}$$

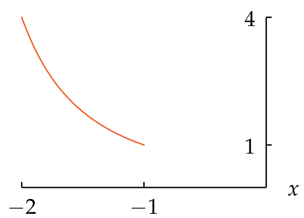
for each number x in the interval $[1,2]$.

(d) The range of g is obtained by multiplying every number in the range of f by 2. Thus the range of g is the interval $[2,8]$.

7 The graph of g is obtained by shifting the graph of f left 3 units.

solution

(a)



Shifting the graph of f left 3 units gives this graph.

(b) The domain of g is obtained by subtracting 3 from every number in domain of f. Thus the domain of g is the interval $[-2,-1]$.

(c) Because the graph of g is obtained by shifting the graph of f left 3 units, we have $g(x) = f(x+3)$. Thus

$$g(x) = \frac{4}{(x+3)^2}$$

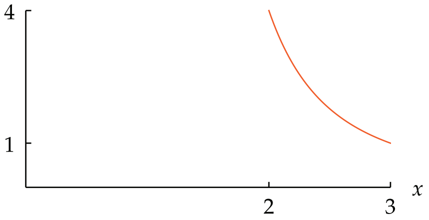
for each number x in the interval $[-2,-1]$.

(d) The range of g is the same as the range of f. Thus the range of g is the interval $[1, 4]$.

9 The graph of g is obtained by shifting the graph of f right 1 unit.

solution

(a)



Shifting the graph of f right 1 unit gives this graph.

(b) The domain of g is obtained by adding 1 to every number in domain of f. Thus the domain of g is the interval $[2, 3]$.

(c) Because the graph of g is obtained by shifting the graph of f right 1 unit, we have $g(x) = f(x - 1)$. Thus

$$g(x) = \frac{4}{(x-1)^2}$$

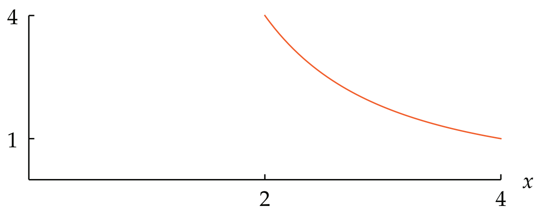
for each number x in the interval $[2, 3]$.

(d) The range of g is the same as the range of f. Thus the range of g is the interval $[1, 4]$.

11 The graph of g is obtained by horizontally stretching the graph of f by a factor of 2.

solution

(a)



Horizontally stretching the graph of f by a factor of 2 gives this graph.

(b) The domain of g is obtained by multiplying every number in the domain of f by 2. Thus the domain of g is the interval $[2, 4]$.

(c) Because the graph of g is obtained by horizontally stretching the graph of f by a factor of 2, we have $g(x) = f(x/2)$. Thus

$$g(x) = \frac{4}{(x/2)^2} = \frac{16}{x^2}$$

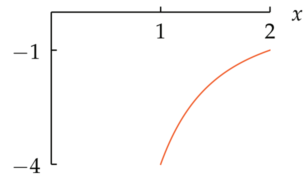
for each number x in the interval $[2, 4]$.

(d) The range of g is the same as the range of f. Thus the range of g is the interval $[1, 4]$.

13 The graph of g is obtained by flipping the graph of f across the horizontal axis.

solution

(a)



Flipping the graph of f across the horizontal axis gives this graph.

(b) The domain of g is the same as the domain of f. Thus the domain of g is the interval $[1, 2]$.

(c) Because the graph of g is obtained by flipping the graph of f across the horizontal axis, we have $g(x) = -f(x)$. Thus

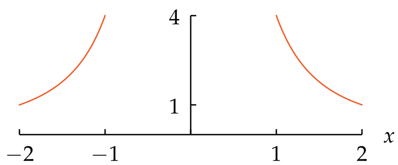
$$g(x) = -\frac{4}{x^2}$$

for each number x in the interval $[1, 2]$.

(d) The range of g is obtained by multiplying every number in the range of f by -1. Thus the range of g is the interval $[-4, -1]$.

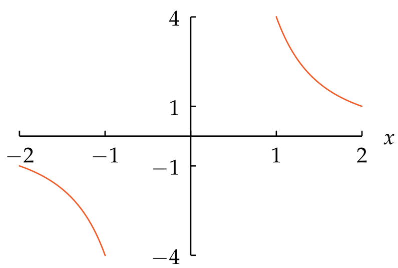
15 Suppose g is an even function whose domain is $[-2, -1] \cup [1, 2]$ and whose graph on the interval $[1, 2]$ is the graph used in the instructions for Exercises 1–14. Sketch the graph of g on $[-2, -1] \cup [1, 2]$.

solution Because g is an even function, its graph is unchanged when flipped across the vertical axis. Thus we can find the graph of g on the interval $[-2, -1]$ by flipping the graph on the interval $[1, 2]$ across the vertical axis, producing the following graph of g:

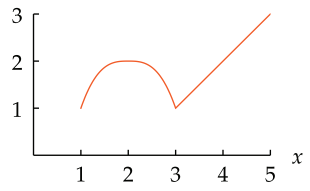


17 Suppose h is an odd function whose domain is $[-2, -1] \cup [1, 2]$ and whose graph on the interval $[1, 2]$ is the graph used in the instructions for Exercises 1–14. Sketch the graph of h on $[-2, -1] \cup [1, 2]$.

solution Because h is an odd function, its graph is unchanged when flipped across the origin. Thus we can find the graph of h on the interval $[-2, -1]$ by flipping the graph on the interval $[1, 2]$ across the origin, producing the following graph of h:



For Exercises 19–56, assume f is a function whose domain is the interval $[1, 5]$, whose range is the interval $[1, 3]$, and whose graph is the figure below.



The graph of f.

For each given function g:

(a) *Find the domain of g.*

(b) *Find the range of g.*

(c) *Sketch the graph of g.*

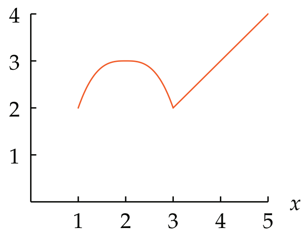
19 $g(x) = f(x) + 1$

solution

(a) Note that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the function g has the same domain as f. Thus the domain of g is the interval $[1, 5]$.

(b) The range of g is obtained by adding 1 to every number in the range of f. Thus the range of g is the interval $[2, 4]$.

(c) The graph of g, shown here, is obtained by shifting the graph of f up 1 unit.



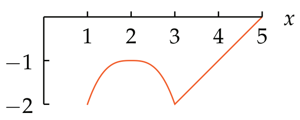
21 $g(x) = f(x) - 3$

solution

(a) Note that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the function g has the same domain as f. Thus the domain of g is the interval $[1, 5]$.

(b) The range of g is obtained by subtracting 3 from each number in the range of f. Thus the range of g is the interval $[-2, 0]$.

(c) The graph of g, shown here, is obtained by shifting the graph of f down 3 units.



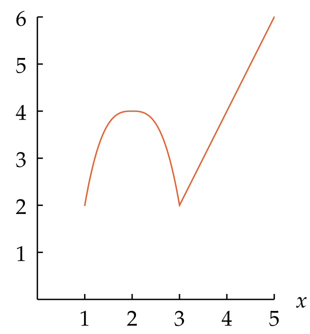
23 $g(x) = 2f(x)$

solution

(a) Note that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the function g has the same domain as f. Thus the domain of g is the interval $[1, 5]$.

(b) The range of g is obtained by multiplying each number in the range of f by 2. Thus the range of g is the interval $[2, 6]$.

(c) The graph of g, shown here, is obtained by vertically stretching the graph of f by a factor of 2.



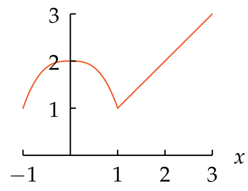
25 $g(x) = f(x + 2)$

solution

(a) Note that $g(x)$ is defined when $x + 2$ is in the interval $[1, 5]$, which means that x must be in the interval $[-1, 3]$. Thus the domain of g is the interval $[-1, 3]$.

(b) The range of g is the same as the range of f. Thus the range of g is the interval $[1, 3]$.

(c) The graph of g, shown here, is obtained by shifting the graph of f left 2 units.



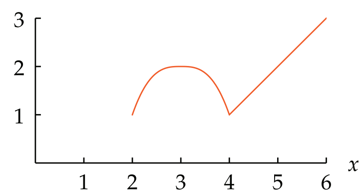
27 $g(x) = f(x - 1)$

solution

(a) Note that $g(x)$ is defined when $x - 1$ is in the interval $[1, 5]$, which means that x must be in the interval $[2, 6]$. Thus the domain of g is the interval $[2, 6]$.

(b) The range of g is the same as the range of f. Thus the range of g is the interval $[1, 3]$.

(c) The graph of g, shown here, is obtained by shifting the graph of f right 1 unit.



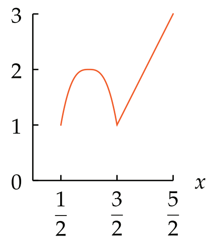
29 $g(x) = f(2x)$

solution

(a) Note that $g(x)$ is defined when $2x$ is in the interval $[1, 5]$, which means that x must be in the interval $[\frac{1}{2}, \frac{5}{2}]$. Thus the domain of g is the interval $[\frac{1}{2}, \frac{5}{2}]$.

(b) The range of g is the same as the range of f. Thus the range of g is the interval $[1, 3]$.

(c) The graph of g, shown here, is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$.



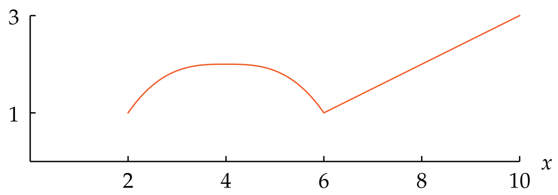
31 $g(x) = f(\frac{x}{2})$

solution

(a) Note that $g(x)$ is defined when $\frac{x}{2}$ is in the interval $[1, 5]$, which means that x must be in the interval $[2, 10]$. Thus the domain of g is the interval $[2, 10]$.

(b) The range of g is the same as the range of f. Thus the range of g is the interval $[1, 3]$.

(c) The graph of g, shown below, is obtained by horizontally stretching the graph of f by a factor of 2.



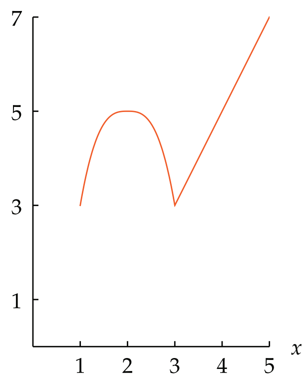
33 $g(x) = 2f(x) + 1$

solution

(a) Note that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the function g has the same domain as f. Thus the domain of g is the interval $[1, 5]$.

(b) The range of g is obtained by multiplying each number in the range of f by 2, which gives the interval $[2, 6]$, and then adding 1 to each number in this interval, which gives the interval $[3, 7]$. Thus the range of g is the interval $[3, 7]$.

(c) Note that $g(x)$ is evaluated by evaluating $f(x)$, then multiplying by 2, and then adding 1. Applying function transformations in the same order, we see that the graph of g, shown here, is obtained by vertically stretching the graph of f by a factor of 2, then shifting up 1 unit.



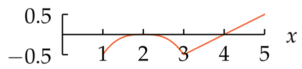
35 $g(x) = \frac{1}{2}f(x) - 1$

solution

(a) Note that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the function g has the same domain as f. Thus the domain of g is the interval $[1, 5]$.

(b) The range of g is obtained by multiplying each number in the range of f by $\frac{1}{2}$, which gives the interval $[\frac{1}{2}, \frac{3}{2}]$, and then subtracting 1 from each number in this interval, which gives the interval $[-\frac{1}{2}, \frac{1}{2}]$. Thus the range of g is the interval $[-\frac{1}{2}, \frac{1}{2}]$.

(c) Note that $g(x)$ is evaluated by evaluating $f(x)$, then multiplying by $\frac{1}{2}$, and then subtracting 1. Applying function transformations in the same order, we see that the graph of g, shown here, is obtained by vertically stretching the graph of f by a factor of $\frac{1}{2}$, then shifting down 1 unit.



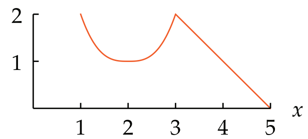
37 $g(x) = 3 - f(x)$

solution

(a) Note that $g(x)$ is defined precisely when $f(x)$ is defined. In other words, the function g has the same domain as f. Thus the domain of g is the interval $[1, 5]$.

(b) The range of g is obtained by multiplying each number in the range of f by -1, which gives the interval $[-3, -1]$, and then adding 3 to each number in this interval, which gives the interval $[0, 2]$. Thus the range of g is the interval $[0, 2]$.

(c) Note that $g(x)$ is evaluated by evaluating $f(x)$, then multiplying by -1, and then adding 3. Applying function transformations in the same order, we see that the graph of g, shown here, is obtained by flipping the graph of f across the horizontal axis, then shifting up 3 units.



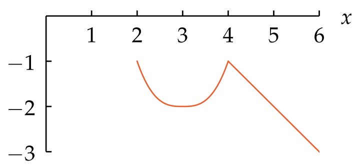
39 $g(x) = -f(x - 1)$

solution

(a) Note that $g(x)$ is defined when $x - 1$ is in the interval $[1, 5]$, which means that x must be in the interval $[2, 6]$. Thus the domain of g is the interval $[2, 6]$.

(b) The range of g is obtained by multiplying each number in the range of f by -1. Thus the range of g is the interval $[-3, -1]$.

(c) The graph of g, shown here, is obtained by shifting the graph of f right 1 unit, then flipping across the horizontal axis.



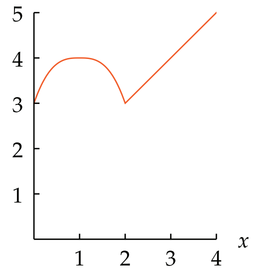
41 $g(x) = f(x + 1) + 2$

solution

(a) Note that $g(x)$ is defined when $x + 1$ is in the interval $[1, 5]$, which means that x must be in the interval $[0, 4]$. Thus the domain of g is the interval $[0, 4]$.

(b) The range of g is obtained by adding 2 to each number in the range of f. Thus the range of g is the interval $[3,5]$.

(c) The graph of g, shown here, is obtained by shifting the graph of f left 1 unit, then shifting up 2 units.



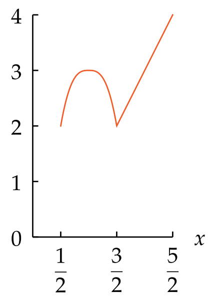
43 $g(x) = f(2x) + 1$

solution

(a) Note that $g(x)$ is defined when $2x$ is in the interval $[1,5]$, which means that x must be in the interval $[\frac{1}{2}, \frac{5}{2}]$. Thus the domain of g is the interval $[\frac{1}{2}, \frac{5}{2}]$.

(b) The range of g is obtained by adding 1 to each number in the range of f. Thus the range of g is the interval $[2,4]$.

(c) The graph of g, shown here, is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$, then shifting up 1 unit.



45 $g(x) = f(2x + 1)$

solution

(a) Note that $g(x)$ is defined when $2x + 1$ is in the interval $[1,5]$, which means that

$$1 \le 2x + 1 \le 5.$$

Adding -1 to each part of the inequality above gives $0 \le 2x \le 4$, and then dividing each part by 2 produces $0 \le x \le 2$. Thus the domain of g is the interval $[0,2]$.

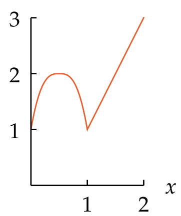
(b) The range of g equals the range of f. Thus the range of g is the interval $[1,3]$.

(c) Define a function h by $h(x) = f(2x)$. The graph of h is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$. Note that

$$g(x) = f(2x+1) = f\big(2(x+\tfrac{1}{2})\big) = h\big(x+\tfrac{1}{2}\big).$$

Thus the graph of g is obtained by shifting the graph of h left $\frac{1}{2}$ unit.

Putting this together, we see that the graph of g, shown here, is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$, then shifting left $\frac{1}{2}$ unit.



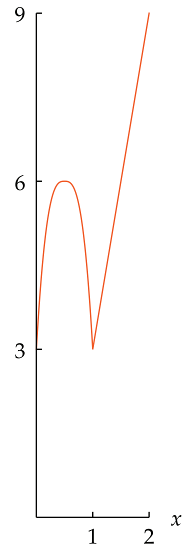
47 $g(x) = 3f(2x + 1)$

solution

(a) The function in this exercise is 3 times the function in Exercise 45. Thus this function has the same domain as the function in Exercise 45. In other words, the domain of this function g is the interval $[0,2]$.

(b) The range of g is obtained by multiplying each number in the range of f by 3. Thus the range of g is the interval $[3,9]$.

(c) The graph of this function g is obtained by horizontally stretching the graph of the function in Exercise 45 by a factor of 3, obtaining the graph shown here.



49 $g(x) = 2f(\frac{x}{2} + 1)$

solution

(a) Note that $g(x)$ is defined when $\frac{x}{2} + 1$ is in the interval $[1,5]$, which means that

$$1 \le \frac{x}{2} + 1 \le 5.$$

Adding -1 to each part of the inequality above gives $0 \le \frac{x}{2} \le 4$, and then multiplying each part by 2 produces $0 \le x \le 8$. Thus the domain of g is the interval $[0,8]$.

(b) The range of g is obtained by multiplying each number in the range of f by 2. Thus the range of g is the interval $[2,6]$.

(c) Define a function h by

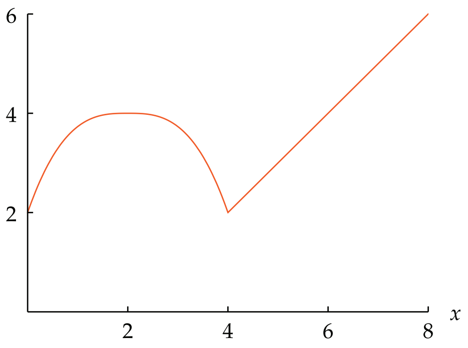
$$h(x) = f\big(\tfrac{x}{2}\big).$$

The graph of h is obtained by horizontally stretching the graph of f by a factor of 2. Note that

$$g(x) = 2f\big(\tfrac{x}{2}+1\big) = 2f\big(\tfrac{x+2}{2}\big) = 2h(x+2).$$

Thus the graph of g is obtained from the graph of h by shifting left 2 units and stretching vertically by a factor of 2.

Putting this together, we see that the graph of g, shown below, is obtained by horizontally stretching the graph of f by a factor of 2, shifting left 2 units, and then vertically stretching by a factor of 2.



51 $g(x) = 2f\left(\frac{x}{2} + 1\right) - 3$

solution

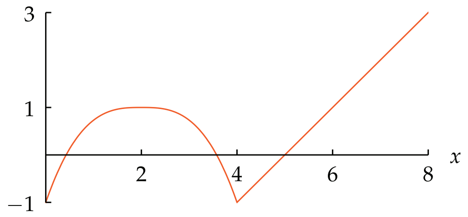
(a) Note that $g(x)$ is defined when $\frac{x}{2} + 1$ is in $[1, 5]$, which means that

$$1 \le \frac{x}{2} + 1 \le 5.$$

Adding -1 to each part of this inequality gives the inequality $0 \le \frac{x}{2} \le 4$, and then multiplying by 2 gives $0 \le x \le 8$. Thus the domain of g is the interval $[0, 8]$.

(b) The range of g is obtained by multiplying each number in the range of f by 2 and then subtracting 3. Thus the range of g is the interval $[-1, 3]$.

(c) The graph of g, shown below, is obtained by shifting the graph in the solution to Exercise 49 down 3 units.



53 $g(x) = 2f\left(\frac{x}{2} + 3\right)$

solution

(a) Note that $g(x)$ is defined when $\frac{x}{2} + 3$ is in $[1, 5]$, which means that

$$1 \le \frac{x}{2} + 3 \le 5.$$

Adding -3 to each part of this inequality gives the inequality $-2 \le \frac{x}{2} \le 2$, and then multiplying by 2 gives $-4 \le x \le 4$. Thus the domain of g is the interval $[-4, 4]$.

(b) The range of g is obtained by multiplying each number in the range of f by 2. Thus the range of g is $[2, 6]$.

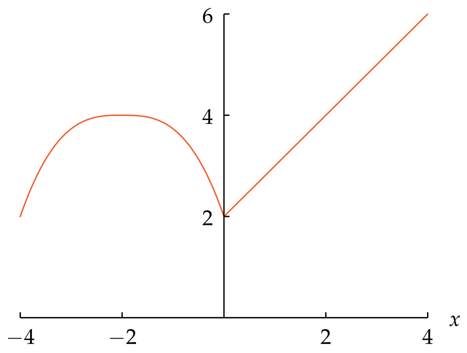
(c) Define a function h by

$$h(x) = f\left(\tfrac{x}{2}\right).$$

The graph of h is obtained by horizontally stretching the graph of f by a factor of 2. Note that

$$g(x) = 2f\left(\tfrac{x}{2} + 3\right) = 2f\left(\tfrac{x+6}{2}\right) = 2h(x + 6).$$

Thus the graph of g is obtained from the graph of h by shifting left 6 units and stretching vertically by a factor of 2. Putting this together, we see that the graph of g, shown below, is obtained by horizontally stretching the graph of f by a factor of 2, shifting left 6 units, and then vertically stretching by a factor of 2.



55 $g(x) = 6 - 2f\left(\frac{x}{2} + 3\right)$

solution

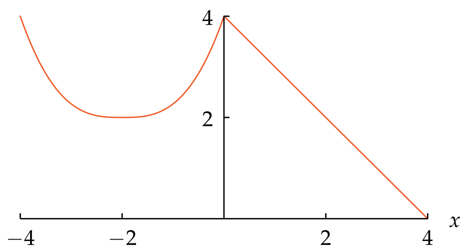
(a) Note that $g(x)$ is defined when $\frac{x}{2} + 3$ is in the interval $[1, 5]$, which means that

$$1 \le \frac{x}{2} + 3 \le 5.$$

Adding -3 to each part of this inequality gives the inequality $-2 \le \frac{x}{2} \le 2$, and then multiplying by 2 gives $-4 \le x \le 4$. Thus the domain of g is the interval $[-4, 4]$.

(b) The range of g is obtained by multiplying each number in the range of f by -2, giving the interval $[-6, -2]$, and then adding 6 to each number in this interval, which gives the interval $[0, 4]$.

(c) The graph of g, shown below, is obtained by flipping across the horizontal axis the graph obtained in the solution to Exercise 53, then shifting up 6 units.



For Exercises 57–60, suppose f is a function whose domain is the interval $[-5, 5]$ and

$$f(x) = \frac{x}{x + 3}$$

for every x in the interval $[0, 5]$.

57 Suppose f is an even function. Evaluate $f(-2)$.

solution Because 2 is in the interval $[0, 5]$, we can use the formula above to evaluate $f(2)$. We have

$$f(2) = \frac{2}{2+3} = \frac{2}{5}.$$

Because f is an even function, we have

$$f(-2) = f(2) = \frac{2}{5}.$$

59 Suppose f is an odd function. Evaluate $f(-2)$.

solution Because f is an odd function, we have

$$f(-2) = -f(2) = -\frac{2}{5}.$$

1.4 Composition of Functions

Learning Objectives

By the end of this section you should be able to

- combine two functions using the usual algebraic operations;
- compute the composition of two functions;
- write a complicated function as the composition of simpler functions;
- express a function transformation as a composition of functions.

Combining Two Functions

Suppose f and g are functions. We define a new function, called the **sum** of f and g and denoted by $f+g$, by letting $f+g$ be the function whose value at a number x is given by the equation

$$(f+g)(x) = f(x) + g(x).$$

For $f(x)+g(x)$ to make sense, both $f(x)$ and $g(x)$ must make sense. Thus the domain of $f+g$ is the intersection of the domain of f and the domain of g.

Similarly, we can define the difference, product, and quotient of two functions in the expected manner. Here are the formal definitions.

Algebra of functions

Suppose f and g are functions. Then the **sum**, **difference**, **product**, and **quotient** of f and g are the functions denoted $f+g$, $f-g$, fg, and $\frac{f}{g}$ defined by

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

Addition and multiplication of functions are commutative and associative operations; subtraction and division of functions have neither of these properties.

The domain of the first three functions above is the intersection of the domain of f and the domain of g. To avoid division by 0, the domain of $\frac{f}{g}$ is the set of numbers x such that x is in the domain of f and x is in the domain of g and $g(x) \neq 0$.

Example 1

Suppose f and g are the functions defined by

$$f(x) = \sqrt{x-3} \quad \text{and} \quad g(x) = \sqrt{8-x}.$$

(a) Evaluate $(f+g)(5)$.

(b) Find a formula for $(f+g)(x)$.

(c) Evaluate $(fg)(5)$.

(d) Find a formula for $(fg)(x)$.

Negative numbers do not have square roots in the real number system. Thus the domain of f is the interval $[3,\infty)$ and the domain of g is the interval $(-\infty, 8]$.

solution

(a) Using the definition of $f+g$, we have

$$\begin{aligned}(f+g)(5) &= f(5) + g(5)\\ &= \sqrt{5-3} + \sqrt{8-5}\\ &= \sqrt{2} + \sqrt{3}.\end{aligned}$$

The domain of $f+g$ is the intersection of the domain of f and the domain of g. Thus the domain of $f+g$ is the interval $[3, 8]$.

(b) Using the definition of $f+g$, we have

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ &= \sqrt{x-3}+\sqrt{8-x}.\end{aligned}$$

(c) Using the definition of fg, we have

$$\begin{aligned}(fg)(5) &= f(5)g(5)\\ &= \sqrt{5-3}\,\sqrt{8-5}\\ &= \sqrt{2}\,\sqrt{3}.\end{aligned}$$

(d) Using the definition of fg, we have

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= \sqrt{x-3}\,\sqrt{8-x}.\end{aligned}$$

Definition of Composition

Now we turn to a new way of combining two functions.

Example 2

The domain of h is the interval $[-3,\infty)$.

Consider the function h defined by

$$h(x)=\sqrt{x+3}.$$

The value of $h(x)$ is computed by carrying out two steps: first add 3 to x, and then take the square root of that sum. Write h in terms of two simpler functions that correspond to these two steps.

solution Define

$$f(x)=\sqrt{x} \quad \text{and} \quad g(x)=x+3.$$

Then

$$h(x)=\sqrt{x+3}=\sqrt{g(x)}=f\big(g(x)\big).$$

In the last term above, $f\big(g(x)\big)$, we evaluate f at $g(x)$. This kind of construction occurs so often that it has been given a name and notation.

When evaluating $(f\circ g)(x)$, first we evaluate $g(x)$, then we evaluate $f\big(g(x)\big)$.

Composition

If f and g are functions, then the **composition** of f and g, denoted $f\circ g$, is the function defined by

$$(f\circ g)(x)=f\big(g(x)\big).$$

The domain of $f\circ g$ is the set of numbers x such that $f\big(g(x)\big)$ makes sense. Thus the domain of $f\circ g$ is the set of numbers x in the domain of g such that $g(x)$ is in the domain of f.

Not everyone follows these conventions. Thus elsewhere you may sometimes see $f(g(x))$, which means the same as $f\big(g(x)\big)$ but with slightly different typography.

In this book, outer parentheses are slightly larger than inner parentheses, to provide a visual clue of the grouping. Also, in the expression $f\big(g(x)\big)$, and in similar functional evaluations with parentheses inside parentheses, this book adds a bit of space after the first open parenthesis and before the last close parenthesis to emphasize that we are evaluating f at some number.

Example 3

Suppose

$$f(x) = \frac{1}{x-4} \quad \text{and} \quad g(x) = x^2.$$

(a) Evaluate $(f \circ g)(3)$.

(b) Evaluate $(g \circ f)(3)$.

(c) Find a formula for the composition $f \circ g$.

(d) Find a formula for the composition $g \circ f$.

(e) What is the domain of $f \circ g$?

(f) What is the domain of $g \circ f$?

solution

(a) Using the definition of composition, we have

$$(f \circ g)(3) = f\big(g(3)\big) = f(3^2) = f(9) = \tfrac{1}{9-4} = \tfrac{1}{5}.$$

(b) Using the definition of composition, we have

$$(g \circ f)(3) = g\big(f(3)\big) = g\big(\tfrac{1}{3-4}\big) = g(-1) = (-1)^2 = 1.$$

Comparing the results of parts (a) and (b), we see that composition is not commutative—order matters.

(c) Using the definition of composition, we have

$$(f \circ g)(x) = f\big(g(x)\big) = f(x^2) = \frac{1}{x^2-4}.$$

(d) Using the definition of composition, we have

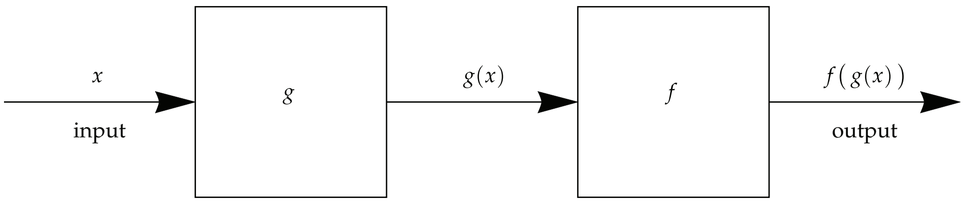
$$(g \circ f)(x) = g\big(f(x)\big) = g\Big(\frac{1}{x-4}\Big) = \Big(\frac{1}{x-4}\Big)^2.$$

Comparing the results of parts (c) and (d), we see again that composition is not commutative—order matters.

(e) The domain of f equals the set of real numbers except 4, and the domain of g equals the set of real numbers. From part (c), we see that $f\big(g(x)\big)$ makes sense provided $x^2 \neq 4$. Thus the domain of $f \circ g$ equals the set of all real numbers except -2 and 2.

(f) From part (d), we see that $g\big(f(x)\big)$ makes sense provided $x \neq 4$. Thus the domain of $g \circ f$ equals the set of all real numbers except 4.

In Section 1.1 we saw that a function g can be thought of as a machine that takes an input x and produces an output $g(x)$. The composition $f \circ g$ can then be thought of as the machine that feeds the output of the g machine into the f machine:



The composition $f \circ g$ as the combination of two machines.

Here $g(x)$ is the output of the g machine and $g(x)$ is also the input of the f machine.

Example 4

*In this example, the functions are called p and t to help you remember that p is the function giving the amount charged by your **phone** company and t is the function giving the **total** cost, including tax.*

(a) Suppose your cell phone company charges \$0.05 per minute plus \$0.47 for each call to China. Find a function p that gives the amount charged by your cell phone company for a call to China as a function of the number of minutes.

(b) Suppose the tax on cell phone bills is 6% plus \$0.01 for each call. Find a function t that gives your total cost, including tax, for a call to China as a function of the amount charged by your cell phone company.

(c) Explain why the composition $t \circ p$ gives your total cost, including tax, of making a cell phone call to China as a function of the number of minutes.

(d) Compute a formula for $t \circ p$.

(e) What is your total cost for a ten-minute call to China?

solution

Most cell phone companies round up any extra seconds to the next highest minute. Thus only positive integer values for m should be used in this formula.

(a) For a call of m minutes to China, the amount $p(m)$ in dollars charged by your cell phone company will be

$$p(m) = 0.05m + 0.47,$$

consisting of \$0.05 times the number of minutes plus \$0.47.

(b) For an amount of y dollars charged by your phone company for a phone call, your total cost in dollars, including tax, will be

$$t(y) = 1.06y + 0.01.$$

(c) Your total cost for a call of m minutes to China is computed by first calculating the amount $p(m)$ charged by the phone company, and then calculating the total cost (including tax) of $t\big(p(m)\big)$, which equals $(t \circ p)(m)$.

(d) The composition $t \circ p$ is given by the formula

$$\begin{aligned}(t \circ p)(m) &= t\big(p(m)\big)\\ &= t(0.05m + 0.47)\\ &= 1.06(0.05m + 0.47) + 0.01\\ &= 0.053m + 0.5082.\end{aligned}$$

(e) Using the formula above from part (d), we have

$$(t \circ p)(10) = 0.053 \times 10 + 0.5082 = 1.0382.$$

Thus your total cost for a ten-minute call to China is \$1.04.

The simple function defined below plays an important role with composition.

Identity function

The **identity function** is the function I defined by

$$I(x) = x$$

for every number x.

If f is any function and x is any number in the domain of f, then

$$(f \circ I)(x) = f\big(I(x)\big) = f(x) \quad \text{and} \quad (I \circ f)(x) = I\big(f(x)\big) = f(x).$$

Thus we have the following result.

The function I is the identity for composition

If f is any function, then $f \circ I = I \circ f = f$.

This result explains why I is called the identity function.

Decomposing Functions

The following example illustrates the process of starting with a function and writing it as the composition of two simpler functions.

Example 5

Suppose

$$T(y) = \left|\frac{y^2 - 3}{y^2 - 7}\right|.$$

Write T as the composition of two simpler functions. In other words, find two functions f and g, each of them simpler than T, such that $T = f \circ g$.

solution There is no rigorous definition of "simpler". Certainly it is easy to write T as the composition of two functions, because $T = T \circ I$, where I is the identity function, but that decomposition is unlikely to be useful.

Typically a function can be decomposed into the composition of other functions in many different ways.

Because evaluating an absolute value is the last operation done in computing $T(y)$, one reasonable possibility is to define

$$f(y) = |y| \quad \text{and} \quad g(y) = \frac{y^2 - 3}{y^2 - 7}.$$

You should verify that with these definitions of f and g, we indeed have $T = f \circ g$. Furthermore, both f and g seem to be simpler functions than T.

Because y appears in the formula defining T only in the expression y^2, another reasonable possibility is to define

Both potential solutions discussed here are correct. Choosing one or the other may depend on the context or on one's taste.

$$f(y) = \left|\frac{y - 3}{y - 7}\right| \quad \text{and} \quad g(y) = y^2.$$

Again you should verify that with these definitions of f and g, we have $T = f \circ g$; both f and g seem to be simpler functions than T.

See Example 6, where T is decomposed into three simpler functions.

Composing More than Two Functions

Although composition is not commutative, it is associative.

Composition is associative

If f, g, and h are functions, then

$$(f \circ g) \circ h = f \circ (g \circ h).$$

To prove the associativity of composition, note that

$$((f \circ g) \circ h)(x) = (f \circ g)\big(h(x)\big) = f\big(g(h(x))\big)$$

and

$$\big(f \circ (g \circ h)\big)(x) = f\big((g \circ h)(x)\big) = f\big(g(h(x))\big).$$

The domain of $f \circ g \circ h$ is the set of numbers x in the domain of h such that $h(x)$ is in the domain of g and $g\big(h(x)\big)$ is in the domain of f.

The equations above show that the functions $(f \circ g) \circ h$ and $f \circ (g \circ h)$ have the same value at every number x in their domain. Thus $(f \circ g) \circ h = f \circ (g \circ h)$.

Because composition is associative, we can dispense with the parentheses and simply write $f \circ g \circ h$, which is the function whose value at a number x is $f\big(g(h(x))\big)$.

Example 6

Suppose

$$T(x) = \left|\frac{x^2 - 3}{x^2 - 7}\right|.$$

Write T as the composition of three simpler functions.

solution We want to choose reasonably simple functions f, g, and h such that $T = f \circ g \circ h$. Probably the best choice here is to take

$$f(x) = |x|, \quad g(x) = \frac{x-3}{x-7}, \quad h(x) = x^2.$$

With these choices, we have

$$\begin{aligned} f\big(g(h(x))\big) &= f\big(g(x^2)\big) \\ &= f\Big(\frac{x^2-3}{x^2-7}\Big) \\ &= \left|\frac{x^2-3}{x^2-7}\right|, \end{aligned}$$

as desired.

Here is how to come up with the choices made above for f, g, and h: Because x appears in the formula defining T only in the expression x^2, we start by taking $h(x) = x^2$. To make $(g \circ h)(x)$ equal $\frac{x^2-3}{x^2-7}$, we then take $g(x) = \frac{x-3}{x-7}$. Finally, because evaluating an absolute value is the last operation done in computing $T(x)$, we take $f(x) = |x|$.

Function Transformations as Compositions

All the function transformations discussed in Section 1.3 can be considered to be compositions with linear functions, which we now define.

Linear function

A **linear function** is a function h of the form

$$h(x) = mx + b,$$

where m and b are numbers.

*The term **linear function** is used because the graph of any such function is a line, as we will see in Chapter 2.*

Vertical function transformations can be expressed as compositions with a linear function on the left, as shown in the next example.

Example 7

Suppose f is a function. Define a function g by

$$g(x) = -2f(x) + 1.$$

(a) Write g as the composition of a linear function with f.

(b) Describe how the graph of g is obtained from the graph of f.

solution

(a) Define a linear function h by

$$h(x) = -2x + 1.$$

For x in the domain of f, we have

$$\begin{aligned} g(x) &= -2f(x) + 1 \\ &= h\big(f(x)\big) \\ &= (h \circ f)(x). \end{aligned}$$

Thus $g = h \circ f$.

(b) As discussed in the solution to Example 8 in Section 1.3, the graph of g is obtained by vertically stretching the graph of f by a factor of 2, then flipping the resulting graph across the horizontal axis, then shifting the resulting graph up 1 unit.

Example 8 in Section 1.3 shows a figure of this function transformation.

Horizontal function transformations can be expressed as compositions with a linear function on the right, as shown in the next example.

Example 8

Suppose f is a function. Define a function g by

$$g(x) = f(2x).$$

(a) Write g as the composition of f with a linear function.

(b) Describe how the graph of g is obtained from the graph of f.

solution

(a) Define a linear function h by

$$h(x) = 2x.$$

For x in the domain of f, we have

$$\begin{aligned} g(x) &= f(2x) \\ &= f\big(h(x)\big) \\ &= (f \circ h)(x). \end{aligned}$$

Thus $g = f \circ h$.

(b) As discussed in the solution to Example 6 in Section 1.3, the graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$.

Example 6 in Section 1.3 shows a figure of this function transformation.

Combinations of vertical function transformations and horizontal function transformations can be expressed as compositions with a linear function on the left and a linear function on the right, as shown in the next example.

Example 9

Suppose f is a function. Define a function g by

$$g(x) = f(2x) + 1.$$

(a) Write g as the composition of a linear function, f, and another linear function.

(b) Describe how the graph of g is obtained from the graph of f.

solution

(a) Define linear functions h and p by

$$h(x) = x + 1 \quad \text{and} \quad p(x) = 2x.$$

For x in the domain of g, we have

$$g(x) = f(2x) + 1 = h\big(f(2x)\big) = h\big(f(p(x))\big) = (h \circ f \circ p)(x).$$

Thus $g = h \circ f \circ p$.

The solution to Exercise 43 in Section 1.3 shows a figure of this function transformation.

(b) As discussed in the solution to Exercise 43 in Section 1.3, the graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{2}$, then shifting up 1 unit.

Exercises

For Exercises 1–20, evaluate the indicated expression assuming that

$$f(x) = \sqrt{x}, \quad g(x) = \frac{x+1}{x+2}, \quad h(x) = |x-1|.$$

1 $(f+g)(3)$

2 $(g+h)(6)$

3 $(gh)(7)$

4 $(fh)(9)$

5 $\left(\frac{f}{h}\right)(10)$

6 $\left(\frac{h}{g}\right)(11)$

7 $(f \circ g)(4)$

8 $(f \circ g)(5)$

9 $(g \circ f)(4)$

10 $(g \circ f)(5)$

11 $(f \circ h)(-3)$

12 $(f \circ h)(-15)$

13 $(f \circ g \circ h)(0)$

14 $(h \circ g \circ f)(0)$

15 $(f \circ g)(0.23)$

16 $(f \circ g)(3.85)$

17 $(g \circ f)(0.23)$

18 $(g \circ f)(3.85)$

19 $(h \circ f)(0.3)$

20 $(h \circ f)(0.7)$

In Exercises 21–26, for the given functions f and g find formulas for (a) $f \circ g$ and (b) $g \circ f$. Simplify your results as much as possible.

21 $f(x) = x^2 + 1, \quad g(x) = \frac{1}{x}$

22 $f(x) = (x+1)^2, \quad g(x) = \frac{3}{x}$

23 $f(x) = \frac{x-1}{x+1}, \quad g(x) = x^2 + 2$

24 $f(x) = \frac{x+2}{x-3}, \quad g(x) = \frac{1}{x+1}$

25 $f(t) = \frac{t-1}{t^2+1}, \quad g(t) = \frac{t+3}{t+4}$

26 $f(t) = \frac{t-2}{t+3}, \quad g(t) = \frac{1}{(t+2)^2}$

For Exercises 27–36, evaluate the indicated expression assuming that f, g, and h are the functions completely defined by these tables:

x	$f(x)$
1	4
2	1
3	2
4	2

x	$g(x)$
1	2
2	4
3	1
4	3

x	$h(x)$
1	3
2	3
3	4
4	1

27 $(f \circ g)(1)$

28 $(f \circ g)(3)$

29 $(g \circ f)(1)$

30 $(g \circ f)(3)$

31 $(f \circ f)(2)$

32 $(f \circ f)(4)$

33 $(g \circ g)(4)$

34 $(g \circ g)(2)$

35 $(f \circ g \circ h)(2)$

36 $(h \circ g \circ f)(2)$

37 Find a number b such that $f \circ g = g \circ f$, where $f(x) = 2x + b$ and $g(x) = 3x + 4$.

38 Find a number c such that $f \circ g = g \circ f$, where $f(x) = 5x - 2$ and $g(x) = cx - 3$.

39 Suppose

$$h(x) = \Big(\frac{x^2+1}{x-1} - 1\Big)^3.$$

(a) If $f(x) = x^3$, then find a function g such that $h = f \circ g$.

(b) If $f(x) = (x-1)^3$, then find a function g such that $h = f \circ g$.

40 Suppose

$$h(x) = \sqrt{\frac{1}{x^2+1} + 2}.$$

(a) If $f(x) = \sqrt{x}$, then find a function g such that $h = f \circ g$.

(b) If $f(x) = \sqrt{x+2}$, then find a function g such that $h = f \circ g$.

41 Suppose

$$h(t) = 2 + \sqrt{\frac{1}{t^2+1}}.$$

(a) If $g(t) = \frac{1}{t^2+1}$, then find a function f such that $h = f \circ g$.

(b) If $g(t) = t^2$, then find a function f such that $h = f \circ g$.

42 Suppose

$$h(t) = \Big(\frac{t^2+1}{t-1} - 1\Big)^3.$$

(a) If $g(t) = \frac{t^2+1}{t-1} - 1$, then find a function f such that $h = f \circ g$.

(b) If $g(t) = \frac{t^2+1}{t-1}$, then find a function f such that $h = f \circ g$.

In Exercises 43–46, find functions f and g, each simpler than the given function h, such that $h = f \circ g$.

43 $h(x) = (x^2 - 1)^2$

44 $h(x) = \sqrt{x^2 - 1}$

45 $h(x) = \dfrac{3}{2 + x^2}$

46 $h(x) = \dfrac{2}{3 + \sqrt{1+x}}$

In Exercises 47–50, find functions f, g, and h, each simpler than the given function T, such that $T = f \circ g \circ h$.

47 $T(x) = \dfrac{4}{5 + x^2}$

48 $T(x) = \sqrt{4 + x^2}$

49 $T(x) = \sqrt{6 + \sqrt{x}}$

50 $T(x) = \dfrac{3}{\sqrt{7 + \sqrt{x}}}$

For Exercises 51–56, suppose f is a function and a function g is defined by the given expression.

(a) ***Write g as the composition of f and one or two linear functions.***

(b) ***Describe how the graph of g is obtained from the graph of f.***

51 $g(x) = 3f(x) - 2$

52 $g(x) = -4f(x) - 7$

53 $g(x) = f(5x)$

54 $g(x) = f\big(-\tfrac{2}{3}x\big)$

55 $g(x) = 2f(3x) + 4$

56 $g(x) = -5f\big(-\tfrac{4}{3}x\big) - 8$

Problems

For Problems 57–61, suppose you are exchanging currency in the London airport. The currency exchange service there only makes transactions in which one of the two currencies is British pounds, but you want to exchange dollars for Euros. Thus you first need to exchange dollars for British pounds, then exchange British pounds for Euros. At the time you want to make the exchange, the function f for exchanging dollars for British pounds is given by the formula

$$f(d) = 0.69d - 1$$

and the function g for exchanging British pounds for Euros is given by the formula

$$g(p) = 1.28p - 2.$$

The subtraction of 1 or 2 in the number of British pounds or Euros that you receive is the fee charged by the currency exchange service for each transaction.

57 Is the function describing the exchange of dollars for Euros $f \circ g$ or $g \circ f$? Explain your answer in terms of which function is evaluated first when computing a value for a composition.

58 Find a formula for the function given by your answer to Problem 57.

59 How many Euros would you receive for exchanging \$100 after going through this two-step exchange process?

60 How many Euros would you receive for exchanging \$200 after going through this two-step exchange process?

61 Which process gives you more Euros: exchanging \$100 for Euros twice or exchanging \$200 for Euros once?

62 Suppose $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are numbers. Show that $f \circ g = g \circ f$ if and only if $d(a-1) = b(c-1)$.

63 Suppose f and g are functions. Show that the composition $f \circ g$ has the same domain as g if and only if the range of g is contained in the domain of f.

64 Show that the sum of two even functions (with the same domain) is an even function.

65 Show that the product of two even functions (with the same domain) is an even function.

66 True or false: The product of an even function and an odd function (with the same domain) is an odd function. Explain your answer.

67 True or false: The sum of an even function and an odd function (with the same domain) is an odd function. Explain your answer.

68 Suppose g is an even function and f is any function. Show that $f \circ g$ is an even function.

69 Suppose f is an even function and g is an odd function. Show that $f \circ g$ is an even function.

70 Suppose f and g are both odd functions. Is the composition $f \circ g$ even, odd, or neither? Explain.

71 Show that if f, g, and h are functions, then

$$(f+g) \circ h = f \circ h + g \circ h.$$

72 Find functions f, g, and h such that

$$f \circ (g+h) \neq f \circ g + f \circ h.$$

Worked-Out Solutions to Odd-Numbered Exercises

For Exercises 1–20, evaluate the indicated expression assuming that

$$f(x) = \sqrt{x}, \quad g(x) = \frac{x+1}{x+2}, \quad h(x) = |x-1|.$$

1 $(f+g)(3)$

solution $(f+g)(3) = f(3) + g(3) = \sqrt{3} + \frac{4}{5}$

3 $(gh)(7)$

solution $(gh)(7) = g(7)h(7) = \frac{8}{9} \cdot 6 = \frac{16}{3}$

5 $\left(\frac{f}{h}\right)(10)$

solution $\left(\frac{f}{h}\right)(10) = \frac{f(10)}{h(10)} = \frac{\sqrt{10}}{9}$

7 $(f \circ g)(4)$

solution $(f \circ g)(4) = f\big(g(4)\big) = f\left(\frac{5}{6}\right) = \sqrt{\frac{5}{6}}$

9 $(g \circ f)(4)$

solution

$$(g \circ f)(4) = g\big(f(4)\big)$$
$$= g(\sqrt{4}) = g(2) = \frac{2+1}{2+2} = \frac{3}{4}$$

11 $(f \circ h)(-3)$

solution

$$(f \circ h)(-3) = f\big(h(-3)\big) = f(|-3-1|)$$
$$= f(|-4|) = f(4) = \sqrt{4} = 2$$

13 $(f \circ g \circ h)(0)$

solution

$$(f \circ g \circ h)(0) = f\big(g(h(0))\big)$$
$$= f\big(g(1)\big) = f\left(\frac{2}{3}\right) = \sqrt{\frac{2}{3}}$$

15 $(f \circ g)(0.23)$

solution

$$(f \circ g)(0.23) = f\big(g(0.23)\big) = f\left(\frac{0.23+1}{0.23+2}\right)$$
$$\approx f(0.55157) = \sqrt{0.55157} \approx 0.74268$$

17 $(g \circ f)(0.23)$

solution

$$(g \circ f)(0.23) = g\big(f(0.23)\big) = g(\sqrt{0.23})$$
$$\approx g(0.47958) = \frac{0.47958+1}{0.47958+2}$$
$$\approx 0.59671$$

19 $(h \circ f)(0.3)$

solution

$$(h \circ f)(0.3) = h\big(f(0.3)\big) = h(\sqrt{0.3})$$
$$\approx h(0.547723) = |0.547723 - 1|$$
$$= |-0.452277| = 0.452277$$

In Exercises 21–26, for the given functions f and g find formulas for (a) $f \circ g$ and (b) $g \circ f$. Simplify your results as much as possible.

21 $f(x) = x^2 + 1, \quad g(x) = \frac{1}{x}$

solution

(a)
$$(f \circ g)(x) = f\big(g(x)\big)$$
$$= f\left(\frac{1}{x}\right)$$
$$= \left(\frac{1}{x}\right)^2 + 1$$
$$= \frac{1}{x^2} + 1$$

(b)
$$(g \circ f)(x) = g\big(f(x)\big)$$
$$= g(x^2+1)$$
$$= \frac{1}{x^2+1}$$

23 $f(x) = \dfrac{x-1}{x+1}$, $g(x) = x^2 + 2$

solution

(a)
$$\begin{aligned}(f \circ g)(x) &= f\big(g(x)\big)\\ &= f(x^2+2)\\ &= \frac{(x^2+2)-1}{(x^2+2)+1}\\ &= \frac{x^2+1}{x^2+3}\end{aligned}$$

(b)
$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big)\\ &= g\Big(\frac{x-1}{x+1}\Big)\\ &= \Big(\frac{x-1}{x+1}\Big)^2 + 2\end{aligned}$$

25 $f(t) = \dfrac{t-1}{t^2+1}$, $g(t) = \dfrac{t+3}{t+4}$

solution

(a) We have

$$\begin{aligned}(f \circ g)(t) &= f\big(g(t)\big)\\ &= f\big(\tfrac{t+3}{t+4}\big)\\ &= \frac{\frac{t+3}{t+4} - 1}{\big(\frac{t+3}{t+4}\big)^2 + 1}\\ &= \frac{(t+3)(t+4) - (t+4)^2}{(t+3)^2 + (t+4)^2}\\ &= \frac{t^2 + 7t + 12 - t^2 - 8t - 16}{t^2 + 6t + 9 + t^2 + 8t + 16}\\ &= \frac{-t-4}{2t^2 + 14t + 25}.\end{aligned}$$

In going from the third line above to the fourth line, both numerator and denominator were multiplied by $(t+4)^2$.

(b) We have

$$\begin{aligned}(g \circ f)(t) &= g\big(f(t)\big)\\ &= g\big(\tfrac{t-1}{t^2+1}\big)\\ &= \frac{\frac{t-1}{t^2+1} + 3}{\frac{t-1}{t^2+1} + 4}\\ &= \frac{t - 1 + 3(t^2+1)}{t - 1 + 4(t^2+1)}\\ &= \frac{3t^2 + t + 2}{4t^2 + t + 3}.\end{aligned}$$

In going from the third line above to the fourth line, both numerator and denominator were multiplied by $t^2 + 1$.

For Exercises 27–36, evaluate the indicated expression assuming that f, g, and h are the functions completely defined by these tables:

x	$f(x)$
1	4
2	1
3	2
4	2

x	$g(x)$
1	2
2	4
3	1
4	3

x	$h(x)$
1	3
2	3
3	4
4	1

27 $(f \circ g)(1)$

solution $(f \circ g)(1) = f\big(g(1)\big) = f(2) = 1$

29 $(g \circ f)(1)$

solution $(g \circ f)(1) = g\big(f(1)\big) = g(4) = 3$

31 $(f \circ f)(2)$

solution $(f \circ f)(2) = f\big(f(2)\big) = f(1) = 4$

33 $(g \circ g)(4)$

solution $(g \circ g)(4) = g\big(g(4)\big) = g(3) = 1$

35 $(f \circ g \circ h)(2)$

solution

$$\begin{aligned}(f \circ g \circ h)(2) &= f\big(g(h(2))\big)\\ &= f\big(g(3)\big) = f(1) = 4\end{aligned}$$

37 Find a number b such that $f \circ g = g \circ f$, where $f(x) = 2x + b$ and $g(x) = 3x + 4$.

solution We will compute $(f \circ g)(x)$ and $(g \circ f)(x)$, then set those two expressions equal to each other and solve for b. We begin with $(f \circ g)(x)$:

$$\begin{aligned}(f \circ g)(x) &= f\big(g(x)\big) = f(3x+4)\\ &= 2(3x+4) + b = 6x + 8 + b.\end{aligned}$$

Next we compute $(g \circ f)(x)$:

$$\begin{aligned}(g \circ f)(x) &= g\big(f(x)\big) = g(2x+b)\\ &= 3(2x+b) + 4 = 6x + 3b + 4.\end{aligned}$$

Looking at the expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$, we see that they equal each other if

$$8 + b = 3b + 4.$$

Solving this equation for b, we get $b = 2$.

39 Suppose

$$h(x) = \Big(\frac{x^2+1}{x-1} - 1\Big)^3.$$

(a) If $f(x) = x^3$, then find a function g such that $h = f \circ g$.

(b) If $f(x) = (x-1)^3$, then find a function g such that $h = f \circ g$.

solution

(a) We want the following equation to hold: $h(x) = f\big(g(x)\big)$. Replacing h and f with the formulas for them, we have

$$\Big(\frac{x^2+1}{x-1} - 1\Big)^3 = \big(g(x)\big)^3.$$

Looking at the equation above, we see that we want

$$g(x) = \frac{x^2+1}{x-1} - 1.$$

(b) We want the following equation to hold: $h(x) = f\big(g(x)\big)$. Replacing h and f with the formulas for them, we have

$$\Big(\frac{x^2+1}{x-1} - 1\Big)^3 = \big(g(x) - 1\big)^3.$$

Looking at the equation above, we see that we want

$$g(x) = \frac{x^2+1}{x-1}.$$

41 Suppose

$$h(t) = 2 + \sqrt{\frac{1}{t^2+1}}.$$

(a) If $g(t) = \frac{1}{t^2+1}$, then find a function f such that $h = f \circ g$.

(b) If $g(t) = t^2$, then find a function f such that $h = f \circ g$.

solution

(a) We want the following equation to hold: $h(t) = f\big(g(t)\big)$. Replacing h and g with the formulas for them, we have

$$2 + \sqrt{\frac{1}{t^2+1}} = f\Big(\frac{1}{t^2+1}\Big).$$

Looking at the equation above, we see that we want to choose $f(t) = 2 + \sqrt{t}$.

(b) We want the following equation to hold: $h(t) = f\big(g(t)\big)$. Replacing h and g with the formulas for them, we have

$$2 + \sqrt{\frac{1}{t^2+1}} = f(t^2).$$

Looking at the equation above, we see that we want to choose

$$f(t) = 2 + \sqrt{\frac{1}{t+1}}.$$

In Exercises 43–46, find functions f and g, each simpler than the given function h, such that $h = f \circ g$.

43 $h(x) = (x^2 - 1)^2$

solution The last operation performed in the computation of $h(x)$ is squaring. Thus the most natural way to write h as a composition of two functions f and g is to choose $f(x) = x^2$, which then suggests that we choose $g(x) = x^2 - 1$.

45 $h(x) = \dfrac{3}{2+x^2}$

solution The last operation performed in the computation of $h(x)$ is dividing 3 by a certain expression. Thus the most natural way to write h as a composition of two functions f and g is to choose $f(x) = \frac{3}{x}$, which then requires that we choose $g(x) = 2 + x^2$.

In Exercises 47–50, find functions f, g, and h, each simpler than the given function T, such that $T = f \circ g \circ h$.

47 $T(x) = \dfrac{4}{5+x^2}$

solution A good solution is to take

$$f(x) = \frac{4}{x}, \quad g(x) = 5 + x, \quad h(x) = x^2.$$

49 $T(x) = \sqrt{6 + \sqrt{x}}$

solution A good solution is to take

$$f(x) = \sqrt{x}, \quad g(x) = 6 + x, \quad h(x) = \sqrt{x}.$$

For Exercises 51–56, suppose f is a function and a function g is defined by the given expression.

(a) *Write g as the composition of f and one or two linear functions.*

(b) *Describe how the graph of g is obtained from the graph of f.*

51 $g(x) = 3f(x) - 2$

solution

(a) Define a linear function h by

$$h(x) = 3x - 2.$$

Then $g = h \circ f$.

(b) The graph of g is obtained by vertically stretching the graph of f by a factor of 3, then shifting down 2 units.

53 $g(x) = f(5x)$

solution

(a) Define a linear function h by

$$h(x) = 5x.$$

Then $g = f \circ h$.

(b) The graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{5}$.

55 $g(x) = 2f(3x) + 4$

solution

(a) Define linear functions h and p by

$$h(x) = 2x + 4 \quad \text{and} \quad p(x) = 3x$$

Then $g = h \circ f \circ p$.

(b) The graph of g is obtained by horizontally stretching the graph of f by a factor of $\frac{1}{3}$, then vertically stretching by a factor of 2, then shifting up 4 units.

1.5 Inverse Functions

Learning Objectives

By the end of this section you should be able to

- determine which functions have inverses;
- find a formula for an inverse function (when possible);
- use the composition of a function and its inverse to check that an inverse function has been found correctly;
- find the domain and range of an inverse function.

The Inverse Problem

The concept of an inverse function will play a key role in this book in defining roots, logarithms, and inverse trigonometric functions. To motivate this concept, we begin with some simple examples.

Suppose f is the function defined by $f(x) = 3x$. Given a value of x, we can find the value of $f(x)$ by using the formula defining f. For example, taking $x = 5$, we see that $f(5)$ equals 15.

In the inverse problem, we are given the value of $f(x)$ and asked to find the value of x. The following example illustrates the idea of an inverse problem.

Example 1

Suppose f is the function defined by

$$f(x) = 3x.$$

(a) Find x such that $f(x) = 6$.

(b) Find x such that $f(x) = 300$.

(c) For each number y, find a number x such that $f(x) = y$.

solution

(a) Solving the equation $3x = 6$ for x, we get $x = 2$.

(b) Solving the equation $3x = 300$ for x, we get $x = 100$.

(c) Solving the equation $3x = y$ for x, we get $x = \frac{y}{3}$.

For each number y, part (c) of the example above asks for the number x such that $f(x) = y$. That number x is called $f^{-1}(y)$ (pronounced "f inverse of y"). The example above shows that if $f(x) = 3x$, then $f^{-1}(6) = 2$ and $f^{-1}(300) = 100$ and, more generally, $f^{-1}(y) = \frac{y}{3}$ for every number y.

Inverse functions will be defined more precisely after we work through some examples.

To see how inverse functions can arise in real-world problems, suppose you know that a temperature of x degrees Celsius corresponds to $\frac{9}{5}x + 32$ degrees Fahrenheit (we will derive this formula in Example 5 in Section 2.1). In other words, you know that the function f that converts the Celsius temperature scale to the Fahrenheit temperature scale is given by the formula

For example, because $f(20) = 68$, this formula shows that 20 degrees Celsius corresponds to 68 degrees Fahrenheit.

$$f(x) = \frac{9}{5}x + 32.$$

If you are given a temperature on the Fahrenheit scale and asked to convert it to Celsius, then you are facing the problem of finding the inverse of the function above, as shown in the following example.

Example 2

The Fahrenheit temperature scale was invented around 1712 by the German physicist and engineer Daniel Fahrenheit.

(a) Convert 95 degrees Fahrenheit to the Celsius scale.

(b) For each temperature y on the Fahrenheit scale, what is the corresponding temperature on the Celsius scale?

solution Let

$$f(x) = \frac{9}{5}x + 32.$$

Thus x degrees Celsius corresponds to $f(x)$ degrees Fahrenheit.

(a) We need to find x such that $f(x) = 95$. Solving the equation $\frac{9}{5}x + 32 = 95$ for x, we get $x = 35$. Thus 35 degrees Celsius corresponds to 95 degrees Fahrenheit.

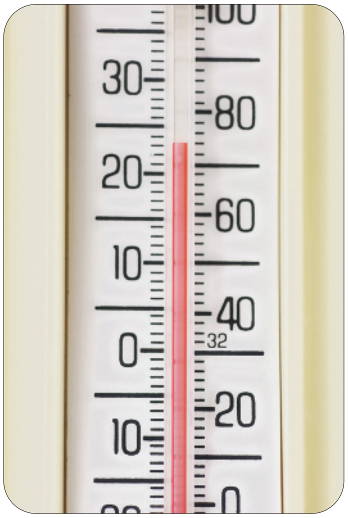
(b) For each number y, we need to find x such that $f(x) = y$. Solving the equation

$$\frac{9}{5}x + 32 = y$$

for x, we get

$$x = \frac{5}{9}(y - 32).$$

Thus $\frac{5}{9}(y - 32)$ degrees Celsius corresponds to y degrees Fahrenheit.



This thermometer shows Celsius degrees on the left, Fahrenheit degrees on the right.

In the example above we have $f(x) = \frac{9}{5}x + 32$. For each number y, part (b) of the example above asks for the number x such that $f(x) = y$. We call that number $f^{-1}(y)$. Part (a) of the example above shows that $f^{-1}(95) = 35$; part (b) shows more generally that

$$f^{-1}(y) = \frac{5}{9}(y - 32).$$

In this example, the function f converts from Celsius to Fahrenheit, and the function f^{-1} goes in the other direction, converting from Fahrenheit to Celsius.

One-to-one Functions

To see the difficulties that can arise with inverse problems, consider the function g, with domain the set of real numbers, defined by the formula

$$g(x) = x^2.$$

Suppose we are told that x is a number such that $g(x) = 16$, and we are asked to find the value of x. Of course $g(4) = 16$, but also $g(-4) = 16$. Thus with the information given we have no way to determine a unique value of x such that $g(x) = 16$. Hence in this case an inverse function does not exist.

The difficulty with the lack of a unique solution to an inverse problem can often be fixed by changing the domain. For example, consider the function f, with domain the set of nonnegative numbers (a number is called **nonnegative** if it is either positive or 0), defined by the formula


The house in what is now the Polish city of Gdańsk in which Daniel Fahrenheit lived as a child.

$$f(x) = x^2.$$

Note that f is defined by the same formula as g in the previous paragraph, but these two functions are not the same because they have different domains. Now if we are told that x is a number in the domain of f such that $f(x) = 16$ and we are asked to find x, we can assert that $x = 4$. More generally, given any nonnegative number y, we can ask for the number x in the domain of f such that $f(x) = y$. This number x, which depends on y, is denoted $f^{-1}(y)$, and is given by the formula

$$f^{-1}(y) = \sqrt{y}.$$

We saw earlier that the function g defined by $g(x) = x^2$ (and with domain equal to the set of real numbers) does not have an inverse because, in particular, the equation $g(x) = 16$ has more than one solution. A function is called **one-to-one** if this situation does not arise.

One-to-one

A function f is called **one-to-one** if for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.

Functions that are one-to-one are precisely the functions that have inverses.

For example, the function g, with domain the set of real numbers, defined by $g(x) = x^2$ is not one-to-one because there are two distinct numbers x in the domain of g such that $f(x) = 16$ (we could have used any positive number instead of 16 to show that g is not one-to-one). In contrast, the function f, with domain the set of nonnegative numbers, defined by $f(x) = x^2$ is one-to-one.

The set of nonnegative numbers is the interval $[0, \infty)$.

The Definition of an Inverse Function

We are now ready to give the formal definition of an inverse function.

Inverse function

Suppose f is a one-to-one function.

- If y is in the range of f, then $f^{-1}(y)$ is defined to be the number x such that $f(x) = y$.
- The function f^{-1} is called the **inverse function** of f.

Short version:

- $f^{-1}(y) = x$ means $f(x) = y$.

Example of the short version:

$\sqrt{y} = x$ *means* $x^2 = y$.

Here f is the function whose domain is the set of nonnegative numbers, $f(x) = x^2$, and $f^{-1}(y) = \sqrt{y}$.

Example 3

Suppose $f(x) = 2x + 3$.

(a) Evaluate $f^{-1}(11)$.

(b) Find a formula for $f^{-1}(y)$.

solution

(a) To evaluate $f^{-1}(11)$, we must find the number x such that $f(x) = 11$. In other words, we must solve the equation $2x + 3 = 11$. The solution to this equation is $x = 4$. Thus $f(4) = 11$, and hence $f^{-1}(11) = 4$.

(b) Fix a number y. To find a formula for $f^{-1}(y)$, we must find the number x such that $f(x) = y$. In other words, we must solve the equation

$$2x + 3 = y$$

for x. The solution to this equation is $x = \frac{y-3}{2}$. Thus $f\left(\frac{y-3}{2}\right) = y$, and hence

$$f^{-1}(y) = \frac{y-3}{2}.$$

The inverse function is not defined for a function that is not one-to-one.

If f is a one-to-one function, then for each y in the range of f we have a uniquely defined number $f^{-1}(y)$. Thus f^{-1} is itself a function.

Think of f^{-1} as undoing whatever f does. This list gives some examples of a function f and its inverse f^{-1}.

f	f^{-1}
$f(x) = x + 2$	$f^{-1}(y) = y - 2$
$f(x) = 3x$	$f^{-1}(y) = \frac{y}{3}$
$f(x) = x^2$	$f^{-1}(y) = \sqrt{y}$
$f(x) = \sqrt{x}$	$f^{-1}(y) = y^2$

The first entry in the list above shows that if f is the function that adds 2 to a number, then f^{-1} is the function that subtracts 2 from a number.

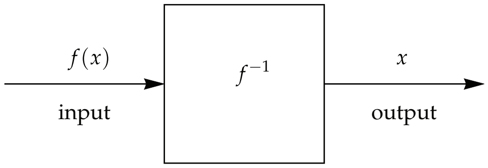
The second entry in the list above shows that if f is the function that multiplies a number by 3, then f^{-1} is the function that divides a number by 3.

Similarly, the third entry in the list above shows that if f is the function that squares a number, then f^{-1} is the function that takes the square root of a number (here the domain of f is assumed to be the nonnegative numbers, so that we have a one-to-one function).

Finally, the fourth entry in the list above shows that if f is the function that takes the square root of a number, then f^{-1} is the function that squares a number (here the domain of f is assumed to be the nonnegative numbers, because the square root of a negative number is not defined as a real number).

In Section 1.1 we saw that a function f can be thought of as a machine that takes an input x and produces an output $f(x)$. Similarly, we can think of f^{-1} as a machine that takes an input $f(x)$ and produces an output x.

Thought of as a machine, f^{-1} reverses the action of f.



The procedure for finding a formula for an inverse function can be described as follows.

Finding a formula for an inverse function

Suppose f is a one-to-one function. To find a formula for $f^{-1}(y)$, solve the equation $f(x) = y$ for x in terms of y.

Example 4

Suppose

$$f(x) = \frac{4x + 5}{2x + 3}$$

for every $x \neq -\frac{3}{2}$. Find a formula for f^{-1}.

solution To find a formula for $f^{-1}(y)$, we need to solve the equation

$$\frac{4x + 5}{2x + 3} = y$$

for x in terms of y. This can be done be multiplying both sides of the equation above by $2x + 3$, getting

$$4x + 5 = 2xy + 3y,$$

which can then be rewritten as

$$(4-2y)x = 3y - 5,$$

which can then be solved for x, getting

$$x = \frac{3y-5}{4-2y}.$$

Thus

$$f^{-1}(y) = \frac{3y-5}{4-2y}$$

for every $y \neq 2$.

The equation $f(x) = 2$ has no solution (try to solve it to see why), and thus $f^{-1}(2)$ is not defined.

The Domain and Range of an Inverse Function

The domain and range of a one-to-one function are nicely related to the domain and range of its inverse. To understand this relationship, consider a one-to-one function f. Note that $f^{-1}(y)$ is defined precisely when y is in the range of f. Thus the domain of f^{-1} equals the range of f.

Similarly, because f^{-1} reverses the action of f, a moment's thought shows that the range of f^{-1} equals the domain of f. We can summarize the relationship between the domains and ranges of functions and their inverses as follows.

Domain and range of an inverse function

If f is a one-to-one function, then

- the domain of f^{-1} equals the range of f;
- the range of f^{-1} equals the domain of f.

Example 5

Suppose the domain of f is the interval $[0, 2]$, with f defined on this domain by the equation $f(x) = x^2$.

(a) What is the range of f?

(b) Find a formula for the inverse function f^{-1}.

(c) What is the domain of the inverse function f^{-1}?

(d) What is the range of the inverse function f^{-1}?

solution

(a) The range of f is the interval $[0, 4]$ because that interval is equal to the set of squares of numbers in the interval $[0, 2]$.

(b) Suppose y is in the range of f, which is the interval $[0, 4]$. To find a formula for $f^{-1}(y)$, we have to solve for x in the equation $f(x) = y$. In other words, we have to solve the equation $x^2 = y$ for x. The solution x must be in the domain of f, which is $[0, 2]$, and in particular x must be nonnegative. Thus we have $x = \sqrt{y}$. In other words, $f^{-1}(y) = \sqrt{y}$.

(c) The domain of the inverse function f^{-1} is the interval $[0, 4]$, which is the range of f.

(d) The range of the inverse function f^{-1} is the interval $[0, 2]$, which is the domain of f.

This example illustrates how the inverse function interchanges the domain and range of the original function.

The Composition of a Function and Its Inverse

The following example will help motivate our next result.

Example 6

Suppose f is the function whose domain is the set of real numbers, with f defined by $f(x) = 2x + 3$.
(a) Find a formula for $f \circ f^{-1}$. (b) Find a formula for $f^{-1} \circ f$.

solution As we saw in Example 3, $f^{-1}(y) = \frac{y-3}{2}$. Thus we have the following:

(a) $(f \circ f^{-1})(y) = f\big(f^{-1}(y)\big) = f\big(\frac{y-3}{2}\big) = 2\big(\frac{y-3}{2}\big) + 3 = y$

(b) $(f^{-1} \circ f)(x) = f^{-1}\big(f(x)\big) = f^{-1}(2x+3) = \frac{(2x+3)-3}{2} = x$

Similar equations hold for the composition of any one-to-one function and its inverse.

The composition of a function and its inverse

Suppose f is a one-to-one function. Then

- $f\big(f^{-1}(y)\big) = y$ for every y in the range of f;
- $f^{-1}\big(f(x)\big) = x$ for every x in the domain of f.

Example:

$$(\sqrt{y})^2 = y$$
$$\sqrt{x^2} = x$$

for all nonnegative numbers x and y. Here f is the function whose domain is the set of nonnegative numbers, $f(x) = x^2$, and $f^{-1}(y) = \sqrt{y}$.

To see why these results hold, first suppose y is a number in the range of f. Let $x = f^{-1}(y)$. Then $f(x) = y$. Thus

$$f\big(f^{-1}(y)\big) = f(x) = y,$$

as claimed above.

To verify the second conclusion in the box above, suppose x is a number in the domain of f. Let $y = f(x)$. Then $f^{-1}(y) = x$. Thus

$$f^{-1}\big(f(x)\big) = f^{-1}(y) = x,$$

as claimed.

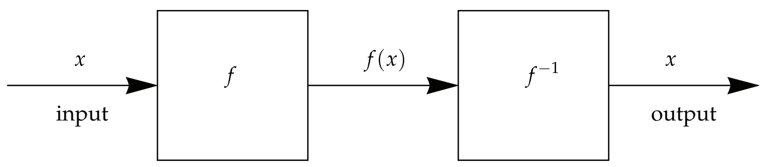
Recall that I is the identity function defined by $I(x) = x$ (where we have left the domain vague), or we could equally well define I by the equation $I(y) = y$. The results in the box above could be expressed by the equations

$$f \circ f^{-1} = I \quad \text{and} \quad f^{-1} \circ f = I.$$

Here the I in the first equation above has domain equal to the range of f (which equals the domain of f^{-1}), and the I in the second equation above has the same domain as f. The equations above explain why the terminology "inverse" is used for the inverse function: f^{-1} is the inverse of f under composition in the sense that the composition of f and f^{-1} in either order gives the identity function.

The figure below illustrates the equation $f^{-1} \circ f = I$, thinking of f and f^{-1} as machines.

Here we start with x as input and end with x as output. Thus this figure illustrates the equation $f^{-1} \circ f = I$.



We start with x as the input. The first machine produces output $f(x)$, which then becomes the input for the second machine. When $f(x)$ is input into the second machine, the output is x because the second machine, which is based on f^{-1}, reverses the action of f.

Suppose you need to compute the inverse of a function f. As discussed earlier, to find a formula for f^{-1} you need to solve the equation $f(x) = y$ for x in terms of y. Once you have obtained a formula for f^{-1}, a good way to check your result is to verify one or both of the equations in the box above.

Example 7

Suppose

$$f(x) = \tfrac{9}{5}x + 32,$$

which is the formula for converting the Celsius temperature scale to the Fahrenheit scale. We computed earlier that the inverse to this function is given by the formula

$$f^{-1}(y) = \tfrac{5}{9}(y - 32).$$

Check that this formula is correct by verifying that $f\big(f^{-1}(y)\big) = y$ for every real number y.

solution To check that we have the right formula for f^{-1}, we compute as follows:

$$\begin{aligned} f\big(f^{-1}(y)\big) &= f\big(\tfrac{5}{9}(y-32)\big) \\ &= \tfrac{9}{5}\big(\tfrac{5}{9}(y-32)\big) + 32 \\ &= (y-32) + 32 \\ &= y. \end{aligned}$$

To be doubly sure that we are not making an algebraic manipulation error, we could also verify that

$$f^{-1}\big(f(x)\big) = x$$

for every real number x. However, one check is usually good enough.

Thus $f\big(f^{-1}(y)\big) = y$, which means that our formula for f^{-1} is correct. If our computation of $f\big(f^{-1}(y)\big)$ had simplified to anything other than y, we would know that we had made a mistake in computing f^{-1}.

Comments About Notation

The notation $y = f(x)$ leads naturally to the notation $f^{-1}(y)$. Recall, however, that in defining a function the variable is simply a placeholder. Thus we could use other letters, including x, as the variable for the inverse function. For example, consider the function f, with domain equal to the set of positive numbers, defined by the equation

$$f(x) = x^2.$$

As we know, the inverse function is given by the formula

$$f^{-1}(y) = \sqrt{y}.$$

However, the inverse function could also be characterized by the formula

$$f^{-1}(x) = \sqrt{x}.$$

Other letters could also be used as the placeholder. For example, we might also characterize the inverse function by the formula $f^{-1}(t) = \sqrt{t}$.

The notation f^{-1} for the inverse of a function (which means the inverse under composition) should not be confused with the multiplicative inverse $\frac{1}{f}$. In other words, $f^{-1} \neq \frac{1}{f}$. However, if the exponent -1 is placed anywhere other than immediately after a function symbol, then it should probably be interpreted as a multiplicative inverse.

Do not confuse $f^{-1}(y)$ with $\big(f(y)\big)^{-1}$.

Example 8

Suppose $f(x) = x^2 - 1$, with the domain of f being the set of positive numbers.

(a) Evaluate $f^{-1}(8)$.

(b) Evaluate $\bigl(f(8)\bigr)^{-1}$.

(c) Evaluate $f(8^{-1})$.

solution

Notice that the answers to (a), (b), and (c) differ from one another.

(a) To evaluate $f^{-1}(8)$, we must find a positive number x such that $f(x) = 8$. In other words, we must solve the equation $x^2 - 1 = 8$. The solution to this equation is $x = 3$. Thus $f(3) = 8$, and hence $f^{-1}(8) = 3$.

(b)

$$\bigl(f(8)\bigr)^{-1} = \frac{1}{f(8)} = \frac{1}{8^2-1} = \frac{1}{63}$$

(c)

$$f(8^{-1}) = f\bigl(\tfrac{1}{8}\bigr) = \bigl(\tfrac{1}{8}\bigr)^2 - 1 = \frac{1}{64} - 1 = -\frac{63}{64}$$

When dealing with real-world problems, you may want to choose the notation to reflect the context. The next example illustrates this idea, with the use of the variable d to denote distance and t to denote time.

Example 9

Suppose you ran a marathon (26.2 miles) in exactly 4 hours. Let f be the function with domain $[0, 26.2]$ such that $f(d)$ is the number of minutes since the start of the race at which you reached distance d miles from the starting line.

(a) What is the range of f?

(b) What is the domain of the inverse function f^{-1}?

(c) What is the meaning of $f^{-1}(t)$ for a number t in the domain of f^{-1}?

solution

(a) Because 4 hours equals 240 minutes, the range of f is the interval $[0, 240]$.

(b) As usual, the domain of f^{-1} is the range of f. Thus the domain of f^{-1} is the interval $[0, 240]$.

(c) The function f^{-1} reverses the roles of the input and the output as compared to the function f. Thus $f^{-1}(t)$ is the distance in miles you had run from the starting line at time t minutes after the start of the race.

Philippides giving news to the people of Athens in 490 BC of the Greek victory over Persia at the Battle of Marathon. According to legend, Philippides ran the 26 miles from Marathon to Athens, leading to our modern term for a race of that length.

Exercises

For Exercises 1–8, check your answer by evaluating the appropriate function at your answer.

1 Suppose $f(x) = 4x + 6$. Evaluate $f^{-1}(5)$.

2 Suppose $f(x) = 7x - 5$. Evaluate $f^{-1}(-3)$.

3 Suppose $g(x) = \frac{x+2}{x+1}$. Evaluate $g^{-1}(3)$.

4 Suppose $g(x) = \frac{x-3}{x-4}$. Evaluate $g^{-1}(2)$.

5 Suppose $f(x) = 3x + 2$. Find a formula for f^{-1}.

6 Suppose $f(x) = 8x - 9$. Find a formula for f^{-1}.

7 Suppose $h(t) = \frac{1+t}{2-t}$. Find a formula for h^{-1}.

8 Suppose $h(t) = \frac{2-3t}{4+5t}$. Find a formula for h^{-1}.

9 Suppose $f(x) = 2 + \frac{x-5}{x+6}$.

(a) Evaluate $f^{-1}(4)$.

(b) Evaluate $\big(f(4)\big)^{-1}$.

(c) Evaluate $f(4^{-1})$.

10 Suppose $h(x) = 3 - \frac{x+4}{x-7}$.

(a) Evaluate $h^{-1}(9)$.

(b) Evaluate $\big(h(9)\big)^{-1}$.

(c) Evaluate $h(9^{-1})$.

11 Suppose $g(x) = x^2 + 4$, with the domain of g being the set of positive numbers. Evaluate $g^{-1}(7)$.

12 Suppose $g(x) = 3x^2 - 5$, with the domain of g being the set of positive numbers. Evaluate $g^{-1}(8)$.

13 Suppose $h(x) = 5x^2 + 7$, where the domain of h is the set of positive numbers. Find a formula for h^{-1}.

14 Suppose $h(x) = 3x^2 - 4$, where the domain of h is the set of positive numbers. Find a formula for h^{-1}.

For each of the functions f given in Exercises 15–24:

(a) ***Find the domain of f.***

(b) ***Find the range of f.***

(c) ***Find a formula for f^{-1}.***

(d) ***Find the domain of f^{-1}.***

(e) ***Find the range of f^{-1}.***

You can check your solutions to part (c) by verifying that $f^{-1} \circ f = I$ and $f \circ f^{-1} = I$ (recall that I is the function defined by $I(x) = x$).

15 $f(x) = 3x + 5$

16 $f(x) = 2x - 7$

17 $f(x) = \dfrac{1}{3x+2}$

18 $f(x) = \dfrac{4}{5x-3}$

19 $f(x) = \dfrac{2x}{x+3}$

20 $f(x) = \dfrac{3x-2}{4x+5}$

21 $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ 4x & \text{if } x \geq 0 \end{cases}$

22 $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$

23 $f(x) = x^2 + 8$, where the domain of f equals $(0, \infty)$.

24 $f(x) = 2x^2 + 5$, where the domain of f equals $(0, \infty)$.

25 Suppose $f(x) = x^5 + 2x^3$. Which of the numbers listed below equals $f^{-1}(8.10693)$?

1.1, 1.2, 1.3, 1.4

[For this particular function, it is not possible to find a formula for $f^{-1}(y)$.]

26 Suppose $f(x) = 3x^5 + 4x^3$. Which of the numbers listed below equals $f^{-1}(0.28672)$?

0.2, 0.3, 0.4, 0.5

[For this particular function, it is not possible to find a formula for $f^{-1}(y)$.]

For Exercises 27–28, use the U.S. 2016 federal income tax function for a single person as defined in Example 2 of Section 1.1.

27 What is the taxable income of a single person who paid \$10,000 in federal taxes for 2016?

28 What is the taxable income of a single person who paid \$20,000 in federal taxes for 2016?

29 Suppose $F(x) = x^8 + 5x^2$. Evaluate

$$\big(F^{-1}(3)\big)^8 + 5\big(F^{-1}(3)\big)^2.$$

30 Suppose $G(x) = 9x^5 - 4x$. Evaluate

$$9\big(G^{-1}(6)\big)^5 - 4G^{-1}(6).$$

31 Suppose $g(x) = x^7 + x^3$. Evaluate

$$\big(g^{-1}(4)\big)^7 + \big(g^{-1}(4)\big)^3 + 1.$$

32 Suppose $g(x) = 8x^9 + 7x^3$. Evaluate

$$8\big(g^{-1}(5)\big)^9 + 7\big(g^{-1}(5)\big)^3 - 3.$$

33 Suppose $F^{-1}(y) = 2y + 3$. Evaluate $F(4)$.

34 Suppose $F^{-1}(y) = 3y - 5$. Evaluate $F(2)$.

Problems

35 The exact number of meters in y yards is $f(y)$, where f is the function defined by

$$f(y) = 0.9144y.$$

(a) Find a formula for $f^{-1}(m)$.

(b) What is the meaning of $f^{-1}(m)$?

36 The exact number of kilometers in M miles is $f(M)$, where f is the function defined by

$$f(M) = 1.609344M.$$

(a) Find a formula for $f^{-1}(k)$.

(b) What is the meaning of $f^{-1}(k)$?

37 A temperature F degrees Fahrenheit corresponds to $g(F)$ degrees on the Kelvin temperature scale, where

$$g(F) = \frac{5}{9}F + 255.37.$$

(a) Find a formula for $g^{-1}(K)$.

(b) What is the meaning of $g^{-1}(K)$?

(c) Evaluate $g^{-1}(0)$. (This is absolute zero, the lowest possible temperature, because all molecular activity stops at 0 degrees Kelvin.)

38 Suppose g is the federal income tax function given by Example 2 of Section 1.1. What is the meaning of the function g^{-1}?

39 Suppose f is the function whose domain is the set of real numbers, with f defined on this domain by the formula

$$f(x) = |x + 6|.$$

Explain why f is not a one-to-one function.

40 Suppose g is the function whose domain is the interval $[-2, 2]$, with g defined on this domain by the formula

$$g(x) = (5x^2 + 3)^{7777}.$$

Explain why g is not a one-to-one function.

41 Show that if f is the function defined by $f(x) = mx + b$, where $m \neq 0$, then f is a one-to-one function.

42 Show that if f is the function defined by $f(x) = mx + b$, where $m \neq 0$, then the inverse function f^{-1} is defined by the formula $f^{-1}(y) = \frac{1}{m}y - \frac{b}{m}$.

43 Consider the function h whose domain is the interval $[-4, 4]$, with h defined on this domain by the formula

$$h(x) = (2 + x)^2.$$

Does h have an inverse? If so, find it, along with its domain and range. If not, explain why not.

44 Consider the function h whose domain is the interval $[-3, 3]$, with h defined on this domain by the formula

$$h(x) = (3 + x)^2.$$

Does h have an inverse? If so, find it, along with its domain and range. If not, explain why not.

45 Suppose f is a one-to-one function. Explain why the inverse of the inverse of f equals f. In other words, explain why

$$(f^{-1})^{-1} = f.$$

46 The function f defined by

$$f(x) = x^5 + x^3$$

is one-to-one (here the domain of f is the set of real numbers). Compute $f^{-1}(y)$ for four different values of y of your choice.
[For this particular function, it is not possible to find a formula for $f^{-1}(y)$.]

47 Suppose f is a function whose domain equals $\{2, 4, 7, 8, 9\}$ and whose range equals $\{-3, 0, 2, 6, 7\}$. Explain why f is a one-to-one function.

48 Suppose f is a function whose domain equals $\{2, 4, 7, 8, 9\}$ and whose range equals $\{-3, 0, 2, 6\}$. Explain why f is not a one-to-one function.

49 Show that the composition of two one-to-one functions is a one-to-one function.

50 Give an example to show that the sum of two one-to-one functions is not necessarily a one-to-one function.

51 Give an example to show that the product of two one-to-one functions is not necessarily a one-to-one function.

52 Give an example of a function f such that the domain of f and the range of f both equal the set of integers, but f is not a one-to-one function.

53 Give an example of a one-to-one function whose domain equals the set of integers and whose range equals the set of positive integers.

Worked-Out Solutions to Odd-Numbered Exercises

For Exercises 1–8, check your answer by evaluating the appropriate function at your answer.

1 Suppose $f(x) = 4x + 6$. Evaluate $f^{-1}(5)$.

solution We need to find a number x such that $f(x) = 5$. In other words, we need to solve the equation

$$4x + 6 = 5.$$

This equation has solution $x = -\frac{1}{4}$. Thus we have $f^{-1}(5) = -\frac{1}{4}$.

check To check that $f^{-1}(5) = -\frac{1}{4}$, we need to verify that $f(-\frac{1}{4}) = 5$. We have

$$f(-\tfrac{1}{4}) = 4(-\tfrac{1}{4}) + 6 = 5,$$

as desired.

3 Suppose $g(x) = \frac{x+2}{x+1}$. Evaluate $g^{-1}(3)$.

solution We need to find a number x such that $g(x) = 3$. In other words, we need to solve the equation

$$\frac{x+2}{x+1} = 3.$$

Multiplying both sides of this equation by $x+1$ gives the equation

$$x + 2 = 3x + 3,$$

which has solution $x = -\frac{1}{2}$. Thus $g^{-1}(3) = -\frac{1}{2}$.

check To check that $g^{-1}(3) = -\frac{1}{2}$, we need to verify that $g(-\frac{1}{2}) = 3$. We have

$$g(-\tfrac{1}{2}) = \frac{-\frac{1}{2}+2}{-\frac{1}{2}+1} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3,$$

as desired.

5 Suppose $f(x) = 3x + 2$. Find a formula for f^{-1}.

solution For each number y, we need to find a number x such that $f(x) = y$. In other words, we need to solve the equation

$$3x + 2 = y$$

for x in terms of y. Subtracting 2 from both sides of the equation above and then dividing both sides by 3 gives

$$x = \frac{y-2}{3}.$$

Thus

$$f^{-1}(y) = \frac{y-2}{3}$$

for every number y.

check To check that $f^{-1}(y) = \frac{y-2}{3}$, we need to verify that $f(\frac{y-2}{3}) = y$. We have

$$\begin{aligned} f\left(\frac{y-2}{3}\right) &= 3\left(\frac{y-2}{3}\right) + 2 \\ &= y, \end{aligned}$$

as desired.

7 Suppose $h(t) = \frac{1+t}{2-t}$. Find a formula for h^{-1}.

solution For each number y, we need to find a number t such that $h(t) = y$. In other words, we need to solve the equation

$$\frac{1+t}{2-t} = y$$

for t in terms of y. Multiplying both sides of this equation by $2-t$ and then collecting all the terms with t on one side gives

$$t + yt = 2y - 1.$$

Rewriting the left side as $(1+y)t$ and then dividing both sides by $1+y$ gives $t = \frac{2y-1}{1+y}$. Thus

$$h^{-1}(y) = \frac{2y-1}{1+y}$$

for every number $y \neq -1$.

check To check that $h^{-1}(y) = \frac{2y-1}{1+y}$, we need to verify that $h\left(\frac{2y-1}{1+y}\right) = y$. We have

$$h\left(\frac{2y-1}{1+y}\right) = \frac{1+\frac{2y-1}{1+y}}{2-\frac{2y-1}{1+y}}.$$

Multiplying numerator and denominator of the expression on the right by $1+y$ gives

$$h\left(\frac{2y-1}{1+y}\right) = \frac{1+y+2y-1}{2+2y-2y+1} = \frac{3y}{3} = y,$$

as desired.

9 Suppose $f(x) = 2 + \frac{x-5}{x+6}$.

(a) Evaluate $f^{-1}(4)$.

(b) Evaluate $\big(f(4)\big)^{-1}$.

(c) Evaluate $f(4^{-1})$.

solution

(a) We need to find a number x such that $f(x) = 4$. In other words, we need to solve the equation

$$2 + \frac{x-5}{x+6} = 4.$$

Subtracting 2 from both sides and then multiplying both sides by $x+6$ gives the equation

$$x - 5 = 2x + 12,$$

which has solution $x = -17$. Thus $f^{-1}(4) = -17$.

(b) Note that

$$f(4) = 2 + \frac{4-5}{4+6} = \frac{20}{10} - \frac{1}{10} = \frac{19}{10}.$$

Thus $\big(f(4)\big)^{-1} = \frac{10}{19}$.

(c)

$$f(4^{-1}) = f(\tfrac{1}{4}) = 2 + \frac{\frac{1}{4}-5}{\frac{1}{4}+6} = \frac{31}{25}$$

11 Suppose $g(x) = x^2 + 4$, with the domain of g being the set of positive numbers. Evaluate $g^{-1}(7)$.

solution We need to find a positive number x such that $g(x) = 7$. In other words, we need to find a positive solution to the equation

$$x^2 + 4 = 7,$$

which is equivalent to the equation

$$x^2 = 3.$$

The equation above has solutions

$$x = \sqrt{3} \quad \text{and} \quad x = -\sqrt{3}.$$

Because the domain of g is the set of positive numbers, the value of x that we seek must be positive. The second solution above is negative; thus it can be discarded, giving $g^{-1}(7) = \sqrt{3}$.

13 Suppose $h(x) = 5x^2 + 7$, where the domain of h is the set of positive numbers. Find a formula for h^{-1}.

solution For each number y, we need to find a number x such that $h(x) = y$. In other words, we need to solve the equation

$$5x^2 + 7 = y$$

for x in terms of y. Subtracting 7 from both sides of the equation above and then dividing both sides by 5 and taking square roots gives

$$x = \sqrt{\frac{y-7}{5}},$$

where we chose the positive square root because x is required to be a positive number. Thus

$$h^{-1}(y) = \sqrt{\frac{y-7}{5}}$$

for every number $y > 7$ (the restriction that $y > 7$ is required to ensure that we get a positive number when evaluating the formula above).

For each of the functions f given in Exercises 15–24:

(a) ***Find the domain of f.***
(b) ***Find the range of f.***
(c) ***Find a formula for f^{-1}.***
(d) ***Find the domain of f^{-1}.***
(e) ***Find the range of f^{-1}.***

You can check your solutions to part (c) by verifying that $f^{-1} \circ f = I$ and $f \circ f^{-1} = I$ (recall that I is the function defined by $I(x) = x$).

15 $f(x) = 3x + 5$

solution

(a) The expression $3x + 5$ makes sense for all real numbers x. Thus the domain of f is the set of real numbers.

(b) To find the range of f, we need to find the numbers y such that

$$y = 3x + 5$$

for some x in the domain of f. In other words, we need to find the values of y such that the equation above can be solved for a real number x. Solving the equation above for x, we get

$$x = \frac{y-5}{3}.$$

The expression above on the right makes sense for every real number y. Thus the range of f is the set of real numbers.

(c) The expression above shows that f^{-1} is given by the formula

$$f^{-1}(y) = \frac{y-5}{3}.$$

(d) The domain of f^{-1} equals the range of f. Thus the domain of f^{-1} is the set of real numbers.

(e) The range of f^{-1} equals the domain of f. Thus the range of f^{-1} is the set of real numbers.

17 $f(x) = \dfrac{1}{3x+2}$

solution

(a) The expression $\frac{1}{3x+2}$ makes sense except when $3x + 2 = 0$. Solving this equation for x gives $x = -\frac{2}{3}$. Thus the domain of f is the set $\{x : x \neq -\frac{2}{3}\}$.

(b) To find the range of f, we need to find the numbers y such that

$$y = \frac{1}{3x+2}$$

for some x in the domain of f. In other words, we need to find the values of y such that the equation above can be solved for a real number $x \neq -\frac{2}{3}$. To solve this equation for x, multiply both sides by $3x + 2$, getting

$$3xy + 2y = 1.$$

Now subtract $2y$ from both sides, then divide by $3y$, getting

$$x = \frac{1-2y}{3y}.$$

The expression above on the right makes sense for every real number $y \neq 0$ and produces a number $x \neq -\frac{2}{3}$ (because the equation $-\frac{2}{3} = \frac{1-2y}{3y}$ leads to nonsense, as you can verify if you try to solve it for y). Thus the range of f is the set $\{y : y \neq 0\}$.

(c) The expression above shows that f^{-1} is given by the formula

$$f^{-1}(y) = \frac{1-2y}{3y}.$$

(d) The domain of f^{-1} equals the range of f. Thus the domain of f^{-1} is the set $\{y : y \neq 0\}$.

(e) The range of f^{-1} equals the domain of f. Thus the range of f^{-1} is the set $\{x : x \neq -\frac{2}{3}\}$.

19 $f(x) = \dfrac{2x}{x+3}$

solution

(a) The expression $\frac{2x}{x+3}$ makes sense except when $x = -3$. Thus the domain of f is the set $\{x : x \neq -3\}$.

(b) To find the range of f, we need to find the numbers y such that

$$y = \frac{2x}{x+3}$$

for some x in the domain of f. In other words, we need to find the values of y such that the equation above can be solved for a real number $x \neq -3$. To solve this equation for x, multiply both sides by $x + 3$, getting

$$xy + 3y = 2x.$$

Now subtract xy from both sides, getting

$$3y = 2x - xy = x(2 - y).$$

Dividing by $2 - y$ gives

$$x = \frac{3y}{2-y}.$$

The expression above on the right makes sense for every real number $y \neq 2$ and produces a number $x \neq -3$ (because the equation $-3 = \frac{3y}{2-y}$ leads to nonsense, as you can verify if you try to solve it for y). Thus the range of f is the set $\{y : y \neq 2\}$.

(c) The expression above shows that f^{-1} is given by the formula
$$f^{-1}(y) = \frac{3y}{2-y}.$$

(d) The domain of f^{-1} equals the range of f. Thus the domain of f^{-1} is the set $\{y : y \neq 2\}$.

(e) The range of f^{-1} equals the domain of f. Thus the range of f^{-1} is the set $\{x : x \neq -3\}$.

21 $f(x) = \begin{cases} 3x & \text{if } x < 0 \\ 4x & \text{if } x \geq 0 \end{cases}$

solution

(a) The expression defining $f(x)$ makes sense for all real numbers x. Thus the domain of f is the set of real numbers.

(b) To find the range of f, we need to find the numbers y such that $y = f(x)$ for some real number x. From the definition of f, we see that if $y < 0$, then $y = f(\frac{y}{3})$, and if $y \geq 0$, then $y = f(\frac{y}{4})$. Thus every real number y is in the range of f. In other words, the range of f is the set of real numbers.

(c) From the paragraph above, we see that f^{-1} is given by the formula
$$f^{-1}(y) = \begin{cases} \frac{y}{3} & \text{if } y < 0 \\ \frac{y}{4} & \text{if } y \geq 0. \end{cases}.$$

(d) The domain of f^{-1} equals the range of f. Thus the domain of f^{-1} is the set of real numbers.

(e) The range of f^{-1} equals the domain of f. Thus the range of f^{-1} is the set of real numbers.

23 $f(x) = x^2 + 8$, where the domain of f equals $(0, \infty)$.

solution

(a) As part of the definition of the function f, the domain has been specified to be the interval $(0, \infty)$, which is the set of positive numbers.

(b) To find the range of f, we need to find the numbers y such that
$$y = x^2 + 8$$
for some x in the domain of f. In other words, we need to find the values of y such that the equation above can be solved for a positive number x. To solve this equation for x, subtract 8 from both sides and then take square roots of both sides, getting
$$x = \sqrt{y-8},$$
where we chose the positive square root of $y - 8$ because x is required to be a positive number.
The expression above on the right makes sense and produces a positive number x for every number $y > 8$. Thus the range of f is the interval $(8, \infty)$.

(c) The expression above shows that $f^{-1}(y) = \sqrt{y-8}$.

(d) The domain of f^{-1} equals the range of f. Thus the domain of f^{-1} is the interval $(8, \infty)$.

(e) The range of f^{-1} equals the domain of f. Thus the range of f^{-1} is the interval $(0, \infty)$, which is the set of positive numbers.

25 Suppose $f(x) = x^5 + 2x^3$. Which of the numbers listed below equals $f^{-1}(8.10693)$?

1.1, 1.2, 1.3, 1.4

solution First we test whether or not $f^{-1}(8.10693)$ equals 1.1 by checking whether or not $f(1.1)$ equals 8.10693. Using a calculator, we find that $f(1.1) = 4.27251$, which means that $f^{-1}(8.10693) \neq 1.1$.

Next we test whether or not $f^{-1}(8.10693)$ equals 1.2 by checking whether or not $f(1.2)$ equals 8.10693. Using a calculator, we find that $f(1.2) = 5.94432$, which means that $f^{-1}(8.10693) \neq 1.2$.

Next we test whether or not $f^{-1}(8.10693)$ equals 1.3 by checking whether or not $f(1.3)$ equals 8.10693. Using a calculator, we find that
$$f(1.3) = 8.10693,$$
which means that $f^{-1}(8.10693) = 1.3$.

For Exercises 27–28, use the U.S. 2016 federal income tax function for a single person as defined in Example 2 of Section 1.1.

27 What is the taxable income of a single person who paid \$10,000 in federal taxes for 2016?

solution Let g be the income tax function as defined in Example 2 of Section 1.1. We need to evaluate $g^{-1}(10000)$. Letting $t = g^{-1}(10000)$, this means that we need to solve the equation $g(t) = 10000$ for t.

Determining which formula to apply requires a bit of experimentation. Using the definition of g, we can calculate that $g(9275) = 927.50$, $g(37650) = 5183.75$, and $g(91150) = 18558.75$. Because 10000 is between 5183.75 and 18558.75, this means that t is between 37650 and 91150. Thus $g(t) = 0.25t - 4228.75$. Solving the equation
$$0.25t - 4228.75 = 10000$$
for t, we get $t = 56915$. Thus a single person whose federal tax bill was \$10,000 had a taxable income of \$56,915.

29 Suppose $F(x) = x^8 + 5x^2$. Evaluate
$$\big(F^{-1}(3)\big)^8 + 5\big(F^{-1}(3)\big)^2.$$

solution The expression above equals $F\big(F^{-1}(3)\big)$, which equals 3.

31 Suppose $g(x) = x^7 + x^3$. Evaluate
$$\big(g^{-1}(4)\big)^7 + \big(g^{-1}(4)\big)^3 + 1.$$

solution We are asked to evaluate $g\big(g^{-1}(4)\big) + 1$. Because $g\big(g^{-1}(4)\big) = 4$, the quantity above equals 5.

33 Suppose $F^{-1}(y) = 2y + 3$. Evaluate $F(4)$.

solution If $F(4) = y$, then
$$4 = F^{-1}(y) = 2y + 3.$$
Solving the equation above for y, we have $y = \frac{1}{2}$. Thus $F(4) = \frac{1}{2}$.

1.6 A Graphical Approach to Inverse Functions

Learning Objectives

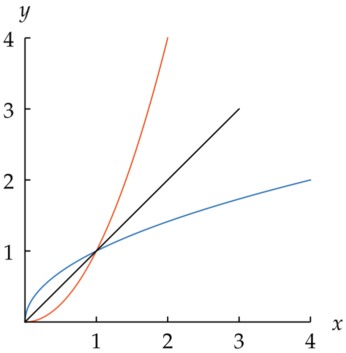
By the end of this section you should be able to

- sketch the graph of f^{-1} from the graph of f;
- use the horizontal line test to determine whether a function has an inverse;
- recognize increasing functions and decreasing functions;
- compute an inverse function for a function defined by a table.

The Graph of an Inverse Function

We begin with an example that illustrates how the graph of an inverse function is related to the graph of the original function.

Example 1



The graphs of x^2 (orange) and its inverse $\sqrt{x}$ (blue) are symmetric about the line $y = x$ (black).

Suppose f is the function with domain $[0, 2]$ defined by

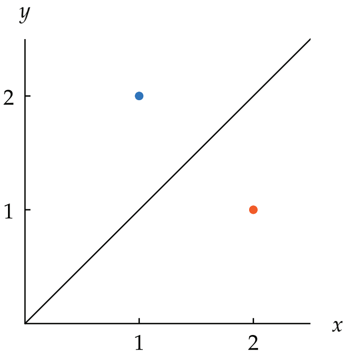
$$f(x) = x^2.$$

What is the relationship between the graph of f and the graph of f^{-1}?

solution The graph of f is part of the familiar parabola defined by $y = x^2$. The range of f is the interval $[0, 4]$. The inverse function f^{-1} has domain $[0, 4]$, with

$$f^{-1}(x) = \sqrt{x}.$$

The graphs of f and f^{-1}, shown here, are symmetric about the line $y = x$, meaning that we could obtain either graph by flipping the other graph across this line.



Flipping $(2, 1)$ (orange) across the line $y = x$ gives $(1, 2)$ (blue).

Here we are assuming that we are working in the xy-plane and that the same scale is used on both axes.

The relationship above between the graphs of x^2 and its inverse $\sqrt{x}$ holds for the graphs of any one-to-one function and its inverse. For example, suppose the point $(2, 1)$ is on the graph of some one-to-one function f. This means that $f(2) = 1$, which is equivalent to the equation $f^{-1}(1) = 2$, which means that $(1, 2)$ is on the graph of f^{-1}. The point $(1, 2)$ can be obtained by flipping the point $(2, 1)$ across the line $y = x$.

More generally, a point (a, b) is on the graph of a one-to-one function f if and only if (b, a) is on the graph of its inverse function f^{-1}. In other words, the graph of f^{-1} can be obtained by interchanging the first and second coordinates of each point on the graph of f. If we are working in the xy-plane, then interchanging first and second coordinates amounts to flipping across the line $y = x$.

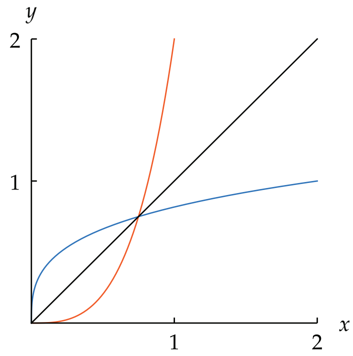
The graph of a one-to-one function and its inverse

- A point (a, b) is on the graph of a one-to-one function if and only if (b, a) is on the graph of its inverse function.
- The graph of a one-to-one function and the graph of its inverse are symmetric about the line $y = x$.
- Each graph can be obtained from the other by flipping across the line $y = x$.

Sometimes an explicit formula cannot be found for f^{-1} because the equation $f(x) = y$ cannot be solved for x even though f is a one-to-one function. However, even in such cases we can obtain the graph of f^{-1}.

Example 2

Suppose f is the function with domain $[0, 1]$ defined by $f(x) = \frac{1}{2}x^5 + \frac{3}{2}x^3$. Sketch the graphs of f and f^{-1}.



The graph of f^{-1} (blue) can be obtained by flipping the graph of f (orange) across the line $y = x$.

solution The graph of f shown here (in orange) was produced by a computer program that can graph a function if given a formula for the function.

Even though f is a one-to-one function, neither humans nor computers can solve the equation

$$\tfrac{1}{2}x^5 + \tfrac{3}{2}x^3 = y$$

for x in terms of y. Thus in this case there is no formula for f^{-1} that a computer can use to produce the graph of f^{-1}.

However, we can find the graph of f^{-1} by flipping the graph of f across the line $y = x$ (shown in black), producing the blue curve as the graph of f^{-1}. Specifically, a typical point on the graph of f has the form $(x, \frac{1}{2}x^5 + \frac{3}{2}x^3)$ for x in the interval $[0, 1]$. Interchanging the coordinates, we thus see that a typical point on the graph of f^{-1} has the form $(\frac{1}{2}x^5 + \frac{3}{2}x^3, x)$ for x in the interval $[0, 1]$. Hence the following command in *WolframAlpha* produces the graph of f^{-1} (without a formula for f^{-1}!).

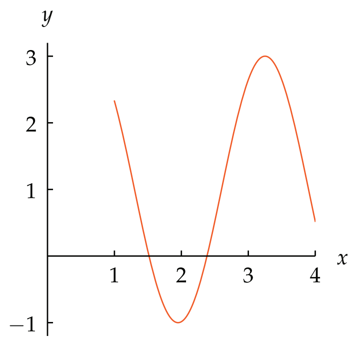
```
ParametricPlot ( (1/2)x^5 + (3/2)x^3, x ) from x = 0 to 1
```

Graphical Interpretation of One-to-One

The graph of a function can be used to determine whether or not the function is one-to-one (and thus whether or not the function has an inverse).

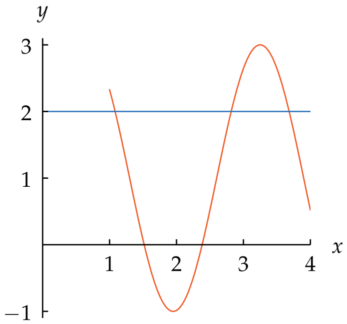
Example 3

Suppose f is the function with domain $[1, 4]$ whose graph is shown here. Is f a one-to-one function?



The graph of f.

solution For f to be one-to-one, for each number y there must be at most one number x such that $f(x) = y$. Draw the line $y = 2$ on the same coordinate plane as the graph, as shown below.



The line $y = 2$ intersects the graph of f in three points, as shown here. Thus there are three numbers x in the domain of f such that $f(x) = 2$. Hence f is not a one-to-one function.

The method used above can be used with the graph of any function.

Horizontal line test

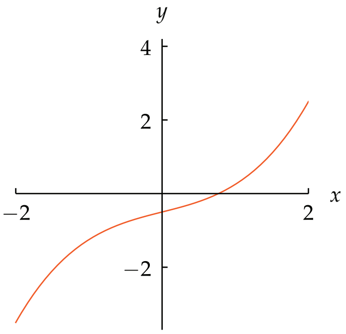
A function is one-to-one if and only if every horizontal line intersects the graph of the function in at most one point.

The functions that have inverses are precisely the one-to-one functions. Thus the horizontal line test can be used to determine whether or not a function has an inverse.

If you find even one horizontal line that intersects the graph at more than one point, then the function is not one-to-one. However, finding one horizontal line that intersects the graph at most once does not imply anything concerning whether or not the function is one-to-one. For the function to be one-to-one, *every* horizontal line must intersect the graph at most once.

Example 4

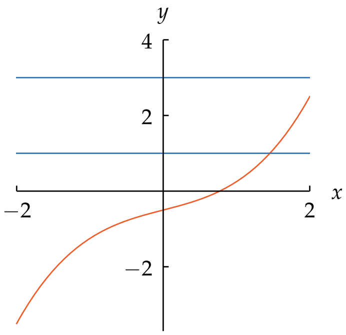
Suppose f is the function with domain $[-2, 2]$ whose graph is shown here. Is f a one-to-one function?



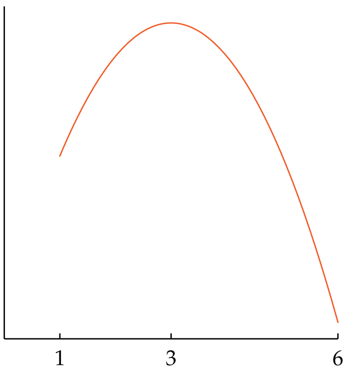
The graph of f.

solution For f to be one-to-one, each horizontal line must intersect the graph of f in at most one point. The figure below shows the graph of f along with the horizontal lines $y = 1$ and $y = 3$.

The line $y = 1$ intersects the graph of f at one point, and the line $y = 3$ does not intersect the graph, as shown here. Furthermore, as can be seen from the figure, each horizontal line intersects the graph at most once. Hence we conclude that f is a one-to-one function.



Increasing and Decreasing Functions



The graph of a function that is increasing on the interval $[1, 3]$, decreasing on the interval $[3, 6]$.

The domain of the function shown here is the interval $[1, 6]$. On the interval $[1, 3]$, the graph of this function gets higher from left to right; thus we say that this function is **increasing** on the interval $[1, 3]$. On the interval $[3, 6]$, the graph of this function gets lower from left to right; thus we say that this function is **decreasing** on the interval $[3, 6]$. Here are the formal definitions.

Increasing on an interval

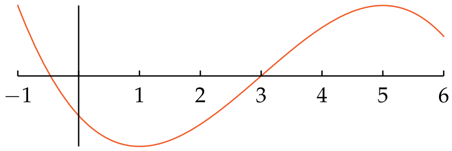
A function f is called **increasing** on an interval if $f(a) < f(b)$ whenever $a < b$ and a, b are in the interval.

Decreasing on an interval

A function f is called **decreasing** on an interval if $f(a) > f(b)$ whenever $a < b$ and a, b are in the interval.

Example 5

The function f whose graph is shown here has domain $[-1, 6]$.



(a) Find the largest interval on which f is increasing.

(b) Find the largest interval on which f is decreasing.

(c) Find the largest interval containing 6 on which f is decreasing.

solution

(a) We see that $[1, 5]$ is the largest interval on which f is increasing.

(b) We see that $[-1, 1]$ is the largest interval on which f is decreasing.

(c) We see that $[5, 6]$ is the largest interval containing 6 on which f is decreasing.

A function is called **increasing** if its graph gets higher from left to right on its entire domain. Here is the formal definition.

Increasing function

A function f is called **increasing** if $f(a) < f(b)$ whenever $a < b$ and a, b are in the domain of f.

Sometimes the terms "increasing" and "decreasing" are used without referring to an interval, as explained here.

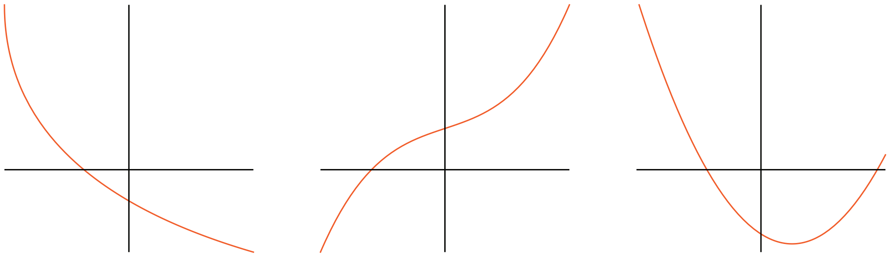
Similarly, a function is called **decreasing** if its graph gets lower from left to right on its entire domain, as defined below.

Decreasing function

A function f is called **decreasing** if $f(a) > f(b)$ whenever $a < b$ and a, b are in the domain of f.

Example 6

Shown below are the graphs of three functions; each function is graphed on its entire domain.



The graph of f. *The graph of g.* *The graph of h.*

(a) Is f increasing, decreasing, or neither?

(b) Is g increasing, decreasing, or neither?

(c) Is h increasing, decreasing, or neither?

solution

(a) The graph of f gets lower from left to right on its entire domain. Thus f is decreasing.

(b) The graph of g gets higher from left to right on its entire domain. Thus g is increasing.

(c) The graph of h gets lower from left to right on part of its domain and gets higher from left to right on another part of its domain. Thus h is neither increasing nor decreasing.

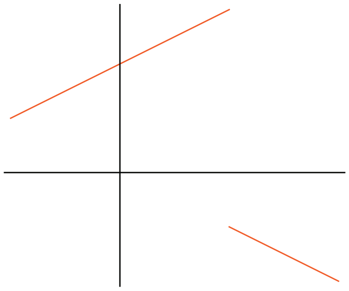
Some functions are neither increasing nor decreasing.

Every horizontal line intersects the graph of an increasing function in at most one point, and similarly for the graph of a decreasing function. The horizontal line test thus implies the following result.

Increasing and decreasing functions are one-to-one

- Every increasing function is one-to-one.
- Every decreasing function is one-to-one.

This result implies that a function that is either increasing or decreasing has an inverse.



The graph of a one-to-one function that is neither increasing nor decreasing.

The result above raises the question of whether every one-to-one function must be increasing or decreasing. The graph shown here answers this question. Specifically, this function is one-to-one because each horizontal line intersects the graph in at most one point. However, this function is neither increasing nor decreasing.

The graph in the example shown here is not one connected piece—you cannot sketch it without lifting your pencil from the paper. A one-to-one function whose graph consists of just one connected piece must be either increasing or decreasing. However, a rigorous explanation of why this result holds requires tools from calculus.

Suppose f is an increasing function and a and b are numbers in the domain of f with $a < b$. Thus $f(a) < f(b)$. Recall that $f(a)$ and $f(b)$ are numbers in the domain of f^{-1}. We have

$$f^{-1}\big(f(a)\big) < f^{-1}\big(f(b)\big)$$

because $f^{-1}\big(f(a)\big) = a$ and $f^{-1}\big(f(b)\big) = b$. The inequality above shows that f^{-1} is an increasing function.

In other words, we have just shown that the inverse of an increasing function is increasing. The figure of a function and its inverse in Example 2 illustrates this result graphically. A similar result holds for decreasing functions.

Inverses of increasing and decreasing functions

- The inverse of an increasing function is increasing.
- The inverse of a decreasing function is decreasing.

Inverse Functions via Tables

For functions whose domain consists of only finitely many numbers, tables provide good insight into the notion of an inverse function.

Example 7

x	$f(x)$
$\sqrt{2}$	3
8	-5
17	6
18	1

Suppose f is the function whose domain is the set of four numbers $\{\sqrt{2}, 8, 17, 18\}$, with the values of f given in the table shown here.

(a) What is the range of f?

(b) Explain why f is a one-to-one function.

(c) What is the table for the function f^{-1}?

solution

(a) The range of f is the set of numbers appearing in the second column of the table defining f. Thus the range of f is the set $\{3, -5, 6, 1\}$.

(b) A function is one-to-one if and only if each number in its range corresponds to only one number in the domain. This means that a function defined by a table is one-to-one if and only if no number is repeated in the second column of the table defining the function. Because the second column of the table above contains no repetitions, we conclude that f is a one-to-one function.

(c) To evaluate $f^{-1}(3)$, we need to find a number x such that $f(x) = 3$. Looking in the table above at the column labeled $f(x)$, we see that $f(\sqrt{2}) = 3$. Thus $f^{-1}(3) = \sqrt{2}$, which means that in the table for f^{-1} the positions of $\sqrt{2}$ and 3 should be interchanged from their positions in the table for f.

More generally, the table for f^{-1} is obtained by interchanging the columns in the table for f, producing the table shown here.

y	$f^{-1}(y)$
3	$\sqrt{2}$
-5	8
6	17
1	18

The ideas used in the example above apply to any function defined by a table.

Inverse functions via tables

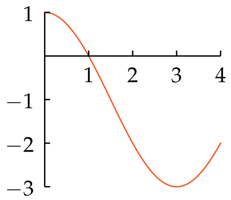
Suppose f is a function defined by a table. Then:

- f is one-to-one if and only if the table defining f has no repetitions in the second column.
- If f is one-to-one, then the table for f^{-1} is obtained by interchanging the columns of the table defining f.

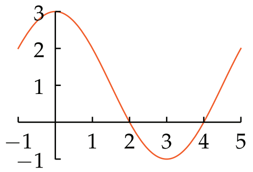
Here we are assuming, as usual throughout this book, that the numbers in the domain of f are listed in the first column of the table, with the corresponding values of f in the second column.

Exercises

For Exercises 1–12, use the following graphs:



The graph of f.



The graph of g.

Here f has domain $[0,4]$ and g has domain $[-1,5]$.

1 What is the largest interval contained in the domain of f on which f is increasing?

2 What is the largest interval contained in the domain of g on which g is increasing?

3 Let F denote the function obtained from f by restricting the domain to the interval in Exercise 1. What is the domain of F^{-1}?

4 Let G denote the function obtained from g by restricting the domain to the interval in Exercise 2. What is the domain of G^{-1}?

5 With F as in Exercise 3, what is the range of F^{-1}?

6 With G as in Exercise 4, what is the range of G^{-1}?

7 What is the largest interval contained in the domain of f on which f is decreasing?

8 What is the largest interval contained in the domain of g on which g is decreasing?

9 Let H denote the function obtained from f by restricting the domain to the interval in Exercise 7. What is the domain of H^{-1}?

10 Let J denote the function obtained from g by restricting the domain to the interval in Exercise 8. What is the domain of J^{-1}?

11 With H as in Exercise 9, what is the range of H^{-1}?

12 With J as in Exercise 10, what is the range of J^{-1}?

13 Suppose $f(x) = x^3 + x$. Which of the following points is on the graph of f^{-1}?

$$(4,1), \quad (2,10), \quad (10,2)$$

14 Suppose $f(x) = x^4 - 3x$. Which of the following points is on the graph of f^{-1}?

$$(2,1), \quad (-1,4), \quad (4,-1)$$

For Exercises 15–38 suppose f and g are functions, each with domain of four numbers, with f and g defined by the tables below:

x	$f(x)$
1	**4**
2	**5**
3	**2**
4	**3**

x	$g(x)$
2	**3**
3	**2**
4	**4**
5	**1**

15 What is the domain of f?

16 What is the domain of g?

17 What is the range of f?

18 What is the range of g?

19 Sketch the graph of f.

20 Sketch the graph of g.

21 Give the table of values for f^{-1}.

22 Give the table of values for g^{-1}.

23 What is the domain of f^{-1}?

24 What is the domain of g^{-1}?

25 What is the range of f^{-1}?

26 What is the range of g^{-1}?

27 Sketch the graph of f^{-1}.

28 Sketch the graph of g^{-1}.

29 Give the table of values for $f^{-1} \circ f$.

30 Give the table of values for $g^{-1} \circ g$.

31 Give the table of values for $f \circ f^{-1}$.

32 Give the table of values for $g \circ g^{-1}$.

33 Give the table of values for $f \circ g$.

34 Give the table of values for $g \circ f$.

35 Give the table of values for $(f \circ g)^{-1}$.

36 Give the table of values for $(g \circ f)^{-1}$.

37 Give the table of values for $g^{-1} \circ f^{-1}$.

38 Give the table of values for $f^{-1} \circ g^{-1}$.

Problems

39 Suppose f is the function whose domain is the interval $[-2, 2]$, with f defined by the following formula:

$$f(x) = \begin{cases} -\frac{x}{3} & \text{if } -2 \le x < 0 \\ 2x & \text{if } 0 \le x \le 2. \end{cases}$$

(a) Sketch the graph of f.

(b) Explain why the graph of f shows that f is not a one-to-one function.

(c) Give an explicit example of two distinct numbers a and b such that $f(a) = f(b)$.

40 Draw the graph of a function that is increasing on the interval $[-2, 0]$ and decreasing on the interval $[0, 2]$.

41 Draw the graph of a function that is decreasing on the interval $[-2, 1]$ and increasing on the interval $[1, 5]$.

42 Give an example of an increasing function whose domain is the interval $[0, 1]$ but whose range does not equal the interval $[f(0), f(1)]$.

43 Show that the sum of two increasing functions is increasing.

44 Give an example of two increasing functions whose product is not increasing.
[*Hint:* There are no such examples where both functions are positive everywhere.]

45 Give an example of two decreasing functions whose product is increasing.

46 Show that the composition of two increasing functions is increasing.

47 Explain why it is important as a matter of social policy that the income tax function g in Example 2 of Section 1.1 be an increasing function.

48 Suppose the income tax function in Example 2 of Section 1.1 is changed so that

$$g(x) = 0.15x - 500 \quad \text{if } 9275 < x \le 37650,$$

with the other parts of the definition of g left unchanged. Show that if this change is made, then the income tax function g would no longer be an increasing function.

49 Suppose the income tax function in Example 2 of Section 1.1 is changed so that

$$g(x) = 0.14x - 463.75 \quad \text{if } 9275 < x \le 37650,$$

with the other parts of the definition of g left unchanged. Show that if this change is made, then the income tax function g would no longer be an increasing function.

50 Explain why an even function whose domain contains a nonzero number cannot be a one-to-one function.

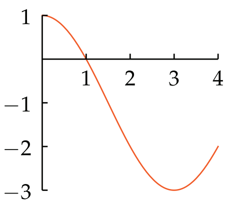
51 The solutions to Exercises 35 and 37 are the same, suggesting that

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1}.$$

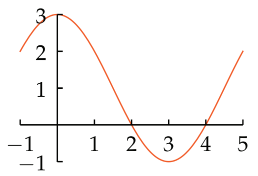
Explain why the equation above holds whenever f and g are one-to-one functions such that the range of g equals the domain of f.

Worked-Out Solutions to Odd-Numbered Exercises

For Exercises 1–12, use the following graphs:



The graph of f.



The graph of g.

Here f has domain $[0, 4]$ and g has domain $[-1, 5]$.

1 What is the largest interval contained in the domain of f on which f is increasing?

solution As can be seen from the graph, $[3, 4]$ is the largest interval on which f is increasing.

As usual when obtaining information solely from graphs, this answer (as well as the answers to the other parts of this exercise) should be considered an approximation. An expanded graph at a finer scale might show that $[2.99, 4]$ or $[3.01, 4]$ would be a more accurate answer than $[3, 4]$.

3 Let F denote the function obtained from f by restricting the domain to the interval in Exercise 1. What is the domain of F^{-1}?

solution The domain of F^{-1} equals the range of F. Because F is the function f with domain restricted to the interval $[3, 4]$, we see from the graph above that the range of F is the interval $[-3, -2]$. Thus the domain of F^{-1} is the interval $[-3, -2]$.

5 With F as in Exercise 3, what is the range of F^{-1}?

solution The range of F^{-1} equals the domain of F. Thus the range of F^{-1} is the interval $[3, 4]$.

7 What is the largest interval contained in the domain of f on which f is decreasing?

solution As can be seen from the graph, $[0, 3]$ is the largest interval on which f is decreasing.

9 Let H denote the function obtained from f by restricting the domain to the interval in Exercise 7. What is the domain of H^{-1}?

solution The domain of H^{-1} equals the range of H. Because H is the function f with domain restricted to the interval $[0, 3]$, we see from the graph above that the range of H is the interval $[-3, 1]$. Thus the domain of H^{-1} is the interval $[-3, 1]$.

11 With H as in Exercise 9, what is the range of H^{-1}?

solution The range of H^{-1} equals the domain of H. Thus the range of H^{-1} is the interval $[0,3]$.

13 Suppose $f(x) = x^3 + x$. Which of the following points is on the graph of f^{-1}?

$$(4,1), \quad (2,10), \quad (10,2)$$

solution A point (c,d) is on the graph of f^{-1} if and only if $f^{-1}(c) = d$, which happens if and only if $f(d) = c$.

Thus to check whether $(4,1)$ is on the graph of f^{-1}, we ask whether $f(1)$ equals 4. However, $f(1) = 2$; thus $(4,1)$ is not on the graph of f^{-1}. Similarly, $f(10) = 1010$; thus $(2,10)$ is not on the graph of f^{-1}. However, $f(2) = 10$; thus $(10,2)$ is on the graph of f^{-1}.

For Exercises 15–38 suppose f and g are functions, each with domain of four numbers, with f and g defined by the tables below:

x	$f(x)$
1	4
2	5
3	2
4	3

x	$g(x)$
2	3
3	2
4	4
5	1

15 What is the domain of f?

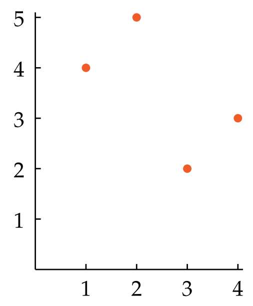
solution The domain of f equals the set of numbers in the left column of the table defining f. Thus the domain of f equals $\{1,2,3,4\}$.

17 What is the range of f?

solution The range of f equals the set of numbers in the right column of the table defining f. Thus the range of f equals $\{2,3,4,5\}$.

19 Sketch the graph of f.

solution The graph of f consists of all points of the form $\big(x, f(x)\big)$ as x varies over the domain of f. Thus the graph of f, shown below, consists of the four points $(1,4)$, $(2,5)$, $(3,2)$, and $(4,3)$.



21 Give the table of values for f^{-1}.

solution The table for the inverse of a function is obtained by interchanging the two columns of the table for the function (after which one can, if desired, reorder the rows, as has been done below):

y	$f^{-1}(y)$
2	3
3	4
4	1
5	2

23 What is the domain of f^{-1}?

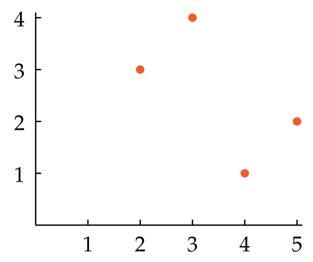
solution The domain of f^{-1} equals the range of f. Thus the domain of f^{-1} is the set $\{2,3,4,5\}$.

25 What is the range of f^{-1}?

solution The range of f^{-1} equals the domain of f. Thus the range of f^{-1} is the set $\{1,2,3,4\}$.

27 Sketch the graph of f^{-1}.

solution The graph of f^{-1} consists of all points of the form $\big(x, f^{-1}(x)\big)$ as x varies over the domain of f^{-1}. Thus the graph of f^{-1}, shown below, consists of the four points $(4,1)$, $(5,2)$, $(2,3)$, and $(3,4)$.



29 Give the table of values for $f^{-1} \circ f$.

solution We know that $f^{-1} \circ f$ is the identity function on the domain of f; thus no computations are necessary. However, because this function f has only four numbers in its domain, it may be instructive to compute $(f^{-1} \circ f)(x)$ for each value of x in the domain of f. Here is that computation:

$$(f^{-1} \circ f)(1) = f^{-1}\big(f(1)\big) = f^{-1}(4) = 1$$

$$(f^{-1} \circ f)(2) = f^{-1}\big(f(2)\big) = f^{-1}(5) = 2$$

$$(f^{-1} \circ f)(3) = f^{-1}\big(f(3)\big) = f^{-1}(2) = 3$$

$$(f^{-1} \circ f)(4) = f^{-1}\big(f(4)\big) = f^{-1}(3) = 4$$

Thus, as expected, the table of values for $f^{-1} \circ f$ is as shown here.

x	$(f^{-1} \circ f)(x)$
1	1
2	2
3	3
4	4

31 Give the table of values for $f \circ f^{-1}$.

solution We know that $f \circ f^{-1}$ is the identity function on the range of f (which equals the domain of f^{-1}); thus no computations are necessary. However, because this function f has only four numbers in its range, it may be instructive to compute $(f \circ f^{-1})(y)$ for each value of y in the range of f. Here is that computation:

$$(f \circ f^{-1})(2) = f\big(f^{-1}(2)\big) = f(3) = 2$$

$$(f \circ f^{-1})(3) = f\big(f^{-1}(3)\big) = f(4) = 3$$

$$(f \circ f^{-1})(4) = f\big(f^{-1}(4)\big) = f(1) = 4$$

$$(f \circ f^{-1})(5) = f\big(f^{-1}(5)\big) = f(2) = 5$$

Thus, as expected, the table of values for $f \circ f^{-1}$ is as shown here.

y	$(f \circ f^{-1})(y)$
2	2
3	3
4	4
5	5

33 Give the table of values for $f \circ g$.

solution We need to compute $(f \circ g)(x)$ for every x in the domain of g. Here is that computation:

$$(f \circ g)(2) = f\big(g(2)\big) = f(3) = 2$$

$$(f \circ g)(3) = f\big(g(3)\big) = f(2) = 5$$

$$(f \circ g)(4) = f\big(g(4)\big) = f(4) = 3$$

$$(f \circ g)(5) = f\big(g(5)\big) = f(1) = 4$$

Thus the table of values for $f \circ g$ is as shown here.

x	$(f \circ g)(x)$
2	2
3	5
4	3
5	4

35 Give the table of values for $(f \circ g)^{-1}$.

solution The table of values for $(f \circ g)^{-1}$ is obtained by interchanging the two columns of the table for $(f \circ g)$ (after which one can, if desired, reorder the rows, as has been done below).

Thus the table for $(f \circ g)^{-1}$ is as shown here.

y	$(f \circ g)^{-1}(y)$
2	2
3	4
4	5
5	3

37 Give the table of values for $g^{-1} \circ f^{-1}$.

solution We need to compute $(g^{-1} \circ f^{-1})(y)$ for every y in the domain of f^{-1}. Here is that computation:

$$(g^{-1} \circ f^{-1})(2) = g^{-1}\big(f^{-1}(2)\big) = g^{-1}(3) = 2$$

$$(g^{-1} \circ f^{-1})(3) = g^{-1}\big(f^{-1}(3)\big) = g^{-1}(4) = 4$$

$$(g^{-1} \circ f^{-1})(4) = g^{-1}\big(f^{-1}(4)\big) = g^{-1}(1) = 5$$

$$(g^{-1} \circ f^{-1})(5) = g^{-1}\big(f^{-1}(5)\big) = g^{-1}(2) = 3$$

Thus the table of values for $g^{-1} \circ f^{-1}$ is as shown here.

y	$(g^{-1} \circ f^{-1})(y)$
2	2
3	4
4	5
5	3

Chapter Summary

To check that you have mastered the most important concepts and skills covered in this chapter, make sure that you can do each item in the following list:

- Explain the concept of a function, including its domain.
- Define the range of a function.
- Locate points on the coordinate plane.
- Explain the relationship between a function and its graph.
- Determine the domain and range of a function from its graph.
- Use the vertical line test to determine if a curve is the graph of some function.
- Determine whether a function transformation shifts the graph up, down, left, or right.
- Determine whether a function transformation stretches the graph vertically or horizontally.
- Determine whether a function transformation flips the graph vertically or horizontally.
- Determine the domain, range, and graph of a transformed function.
- Determine whether a function is even or odd or neither.
- Compute the composition of two functions.
- Write a complicated function as the composition of simpler functions.
- Explain the concept of an inverse function.
- Explain which functions have inverses.
- Find a formula for an inverse function (when possible).
- Sketch the graph of f^{-1} from the graph of f.
- Use the horizontal line test to determine whether a function has an inverse.
- Construct a table of values of f^{-1} from a table of values of f.
- Recognize from a graph whether a function is increasing or decreasing or neither on an interval.

To review a chapter, go through the list above to find items that you do not know how to do, then reread the material in the chapter about those items. Then try to answer the chapter review questions below without looking back at the chapter.

Chapter Review Questions

1 Suppose f is a function. Explain what it means to say that $\frac{3}{2}$ is in the domain of f.

2 Suppose f is a function. Explain what it means to say that $\frac{3}{2}$ is in the range of f.

3 Write the domain of

$$\frac{x^7 + 4}{(x+2)(x-3)}$$

as a union of intervals.

4 Give an example of a function whose domain consists of seven numbers and whose range consists of three numbers.

5 Give an example of a function whose domain is the set of real numbers and whose range is not an interval.

6 Suppose f is defined by $f(x) = x^2$.

(a) What is the range of f if the domain of f is the interval $[1, 3]$?

(b) What is the range of f if the domain of f is the interval $[-2, 3]$?

(c) What is the range of f if the domain of f is the set of positive numbers?

(d) What is the range of f if the domain of f is the set of negative numbers?

(e) What is the range of f if the domain of f is the set of real numbers?

7 Explain how to find the domain of a function from its graph.

8 Explain how to find the range of a function from its graph.

9 Explain why no function has a graph that is a circle.

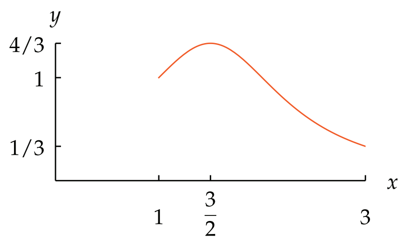
10 Explain how to use the vertical line test to determine whether or not a curve in the plane is the graph of some function.

11 Sketch a curve in the coordinate plane that is not the graph of any function.

For Questions 12–19, assume f is the function defined on the interval $[1, 3]$ by the formula

$$f(x) = \frac{1}{x^2 - 3x + 3}.$$

The domain of f is the interval $[1, 3]$, the range of f is the interval $[\frac{1}{3}, \frac{4}{3}]$, and the graph of f is shown below.



The graph of f.

For each function g described below:

- **(a)** *Sketch the graph of g.*
- **(b)** *Find the domain of g (the endpoints of this interval should be shown on the horizontal axis of your sketch of the graph of g).*
- **(c)** *Give a formula for g.*
- **(d)** *Find the range of g (the endpoints of this interval should be shown on the vertical axis of your sketch of the graph of g).*

12 The graph of g is obtained by shifting the graph of f up 2 units.

13 The graph of g is obtained by shifting the graph of f down 2 units.

14 The graph of g is obtained by shifting the graph of f left 2 units.

15 The graph of g is obtained by shifting the graph of f right 2 units.

16 The graph of g is obtained by vertically stretching the graph of f by a factor of 3.

17 The graph of g is obtained by horizontally stretching the graph of f by a factor of 2.

18 The graph of g is obtained by flipping the graph of f across the horizontal axis.

19 The graph of g is obtained by flipping the graph of f across the vertical axis.

20 Suppose f is a one-to-one function with domain $[1, 3]$ and range $[2, 5]$. Define functions g and h by

$$g(t) = 3f(t) \quad \text{and} \quad h(t) = f(4t).$$

(a) What is the domain of g?
(b) What is the range of g?
(c) What is the domain of h?
(d) What is the range of h?
(e) What is the domain of f^{-1}?
(f) What is the range of f^{-1}?

21 Suppose f is a one-to-one function such that the point $(3, 2)$ is on the graph of f. Which two of the following statements must be true?

(a) $f(2) = 3$
(b) $f(3) = 2$
(c) $f^{-1}(2) = 3$
(d) $f^{-1}(3) = 2$

22 Suppose the domain of f is the set of four numbers $\{1, 2, 3, 4\}$, with f defined by the table shown here. Draw the graph of f.

x	$f(x)$
1	2
2	3
3	−1
4	1

23 Show that the sum of two odd functions is also an odd function.

24 Explain why the product of two odd functions is an even function.

25 Define the composition of two functions.

26 Suppose f and g are functions. Explain why the domain of $f \circ g$ is contained in the domain of g.

27 Suppose f and g are functions. Explain why the range of $f \circ g$ is contained in the range of f.

28 Suppose f is a function and g is a linear function. Explain why the domain of $g \circ f$ is the same as the domain of f.

29 Suppose

$$h(x) = \sqrt{\frac{1 + \sqrt{x}}{2 + \sqrt{x}}} \quad \text{and} \quad g(x) = \sqrt{x}.$$

(a) Find a function f such that $h = f \circ g$.
(b) Find a function f such that $h = g \circ f$.

30 Suppose

$$f(x) = \frac{x^2 + 3}{5x^2 - 9}.$$

Find two functions g and h, each simpler than f, such that $f = g \circ h$.

31 Find a number c such that the function g defined by

$$g(t) = 2t + c$$

has the property that $(g \circ g)(3) = 17$.

32 Explain how to use the horizontal line test to determine whether or not a function is one-to-one.

33 Draw the graph of a function that is decreasing on the interval $[1, 2]$ and increasing on the interval $[2, 5]$.

For Questions 34–39, suppose

$$h(x) = |2x+3| + x^2 \quad \textit{and} \quad f(x) = 3x - 5.$$

34 Evaluate $\left(\frac{h}{f}\right)(-2)$.

35 Find a formula for $(f+h)(4+t)$.

36 Evaluate $(h \circ f)(3)$.

37 Evaluate $(f \circ h)(-4)$.

38 Find a formula for $h \circ f$.

39 Find a formula for $f \circ h$.

40 Suppose

$$f(x) = \frac{2x+1}{3x-4}.$$

Evaluate $f^{-1}(-5)$.

41 Suppose

$$g(x) = 3 + \frac{x}{2x-3}.$$

Find a formula for g^{-1}.

42 Suppose f is a one-to-one function. Explain the relationship between the graph of f and the graph of f^{-1}.

43 Suppose f is a one-to-one function. Explain the relationship between the domain and range of f and the domain and range of f^{-1}.

44 Explain the different meanings of the notations $f^{-1}(x)$, $\big(f(x)\big)^{-1}$, and $f(x^{-1})$.

45 The function f defined by

$$f(x) = x^5 + 2x^3 + 2$$

is an increasing function and thus is one-to-one (here the domain of f is the set of real numbers).

(a) Evaluate $f^{-1}\big(f(1)\big)$

(b) Evaluate $f\big(f^{-1}(4)\big)$.

(c) Compute $f^{-1}(y)$ for four different values of y of your choice.

46 Make up a table that defines a one-to-one function g whose domain consists of five numbers.

(a) Sketch the graph of g.

(b) Give the table for g^{-1}.

(c) Sketch the graph of g^{-1}.

47 The exact number of yards in c centimeters is $f(c)$, where f is the function defined by

$$f(c) = \frac{c}{91.44}.$$

(a) Find a formula for $f^{-1}(y)$.

(b) What is the meaning of $f^{-1}(y)$?

48 A sign at the currency exchange booth in the Rome airport states that if you exchange d dollars you will receive $f(d)$ Euros, where f is the function defined by

$$f(d) = 0.88d - 3.$$

Here 0.88 represents the exchange rate and the subtraction of 3 represents the commission of the currency exchange booth. Suppose you want to end up with 200 Euros.

(a) Explain why you should compute $f^{-1}(200)$.

(b) Evaluate $f^{-1}(200)$.