1

Kinematics of Particles

Kinematics is the description of the motion of material bodies without referring to their inertia or the forces that caused their motion. In particular, in kinematics we are interested in defining the velocity, acceleration, angular velocity, and angular acceleration of a given body. To achieve this objective, this chapter introduces the important concept of inertial and non-inertial frames of reference and uses them to illustrate how to fully describe the kinematics of particles.

1.1 Inertial Frames

We embark on this journey of learning dynamics by learning about inertial frames. In describing the position, velocity, and acceleration of a moving point in space, it is essential to define a frame of reference. A common example used in undergraduate textbooks of dynamics to explain the concept of the reference frame is that of two cars moving in the same direction at the same speed. An observer standing at the side of the road sees both cars moving at a certain speed relative to him, while an observer riding in one of the cars will see the other car as stationary. The difference in the observed speed stems from the observer's point of view, referred to in dynamics as the observer's "reference frame".

In his early work, Newton realized the importance of the frame of reference in deriving his first and second laws of dynamics. Therefore, he used the fixed stars as his reference frame. However, it was later shown using the theory of relativity that this choice of reference can yield discrepancies, especially for systems moving at a very high speed close to the speed of light.

In an attempt to overcome this problem, an inertial frame of reference, also known as a Galilean frame, was introduced to define the acceleration of points in space. An inertial frame was defined as a frame that can only undergo pure translation at a constant velocity without any rotation with respect to an *absolute* space. However, as our understanding of the universe evolved, it became apparent that the notion of absolute space does not really exist, because everything we know is moving and rotating with respect to something else. As such, referring to an absolute space to define an inertial frame is fundamentally incorrect. Today, most scientists

---

Dynamics of Particles and Rigid Bodies: A Self-Learning Approach, First Edition. Mohammed F. Daqaq.

Companion website: www.wiley.com/go/daqaq/dynamics

working in the field of classical mechanics define an inertial frame of reference as one in which the motion of a particle not subject to forces is in a straight line at constant speed. In other words, an inertial frame is a frame in which the motion of particles follows Newton's first law of dynamics.

Inertial Frame

An inertial frame of reference is one in which the motion of a particle not subject to forces is in a straight line at constant speed. In other words, an inertial frame is a frame in which the motion of particles follows Newton's first law of dynamics.

The existence of an inertial frame is extremely important because Newton's first and second laws hold true *only* if such a frame exists. In particular, Newton's first law of dynamics, which states that any free motion of a particle has a constant magnitude and direction, is true *only* when the observer of this particle is not rotating or accelerating. Similarly, his second and widely celebrated law of dynamics concerning particles, given by

$$\mathbf{F} = m\mathbf{a}, \tag{1.1}$$

which means that the net force, $\mathbf{F}$, acting on a particle is equal to its mass, m, times its acceleration, $\mathbf{a}$, is true in this simple widely-used form *only* when the acceleration is measured with respect to an observer standing in an inertial frame. Otherwise, one has to account for other fictitious forces resulting from other types of acceleration, such as the Coriolis, centrifugal, and tangential forces.

1.2 Rotating Frames

In addition to the inertial frame of reference, rotating frames, which do not obey Newton's laws in their simple form, are also used quite frequently in kinematics to describe positions, velocities, and accelerations. This is usually done to reduce complexities that could arise when describing the kinematics of bodies involved in complex rotational motions. To show the importance of rotating frames, we consider the kinematics of a ball attached to a rigid cable of fixed length, l, forming a simple pendulum, as shown in Figure 1.1. The position, velocity, and acceleration of the ball, P, with respect to an observer standing in an inertial Cartesian frame, N-frame, with unit vectors $(\hat{n}_1, \hat{n}_2, \hat{n}_3)$ located at O can be easily obtained by defining the position vector **OP** then differentiating it as follows:

$$\begin{aligned}
\text{Position: } \mathbf{OP} &= l\sin\theta\hat{n}_1 - l\cos\theta\hat{n}_2,\\
\text{Velocity: } \mathbf{v} &= \frac{\mathrm{d}\mathbf{OP}}{\mathrm{d}t} = l\cos\theta\dot{\theta}\hat{n}_1 + l\sin\theta\dot{\theta}\hat{n}_2,\\
\text{Acceleration: } \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = (l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta)\hat{n}_1 + (l\ddot{\theta}\sin\theta + l\dot{\theta}^2\cos\theta)\hat{n}_2.
\end{aligned}$$

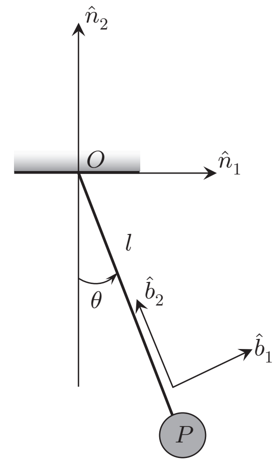


Figure 1.1 Schematic of a simple pendulum.

It is evident that the complexity of the kinematic description of motion increases substantially as more and more derivatives are taken, despite the very simple nature of the pendulum motion. However, it turns out that it is possible to simplify these expressions considerably, if we describe the motion of the pendulum in a rotating frame. For instance, let us form a new Cartesian frame and call it the B-frame, with unit vectors ($\hat{b}_1, \hat{b}_2, \hat{b}_3$). This rotates with the pendulum such that the $\hat{b}_1$ unit vector is always perpendicular to the cable and the $\hat{b}_2$ unit vector is always parallel to it, as shown in Figure 1.1. As we will learn later in this chapter, we can find the velocity and acceleration of point P with respect to point O and express it in the rotating frame as

$$\text{Position: } \mathbf{OP} = -l\hat{b}_2,$$

$$\text{Velocity: } \mathbf{v} = l\dot{\theta}\hat{b}_1,$$

$$\text{Acceleration: } \mathbf{a} = l\ddot{\theta}\hat{b}_l + l\dot{\theta}^2\hat{b}_2.$$

The reader should not be concerned yet about how we obtained the velocity and acceleration expressions; we use them here only to see how the acceleration expression in the rotating B-frame is much simpler than the one obtained using the inertial, N-frame. As such, for more complex motions involving multiple rotations, describing kinematics in an inertial frame is often the wrong approach to take.

Another reason for describing motion in a rotating frame stems from the simplicity of measuring angular velocities, angular accelerations, and the mass moment of inertia of rigid bodies in such frames. The importance of rotating frames in dynamics will become clearer as the reader delves into the subsequent chapters of this book.

Rotating Frame

Rotating frames are frames that can rotate with respect to the inertial frame of reference. Such frames are often used to simplify the kinematic description of particles and rigid bodies.

1.3 Rotation Matrices

Since both of the inertial and rotating frames are critical in kinematics, it is quiet important to learn how to switch back and forth between two different frames in a simple way. To this end, we consider Figure 1.2a, which depicts two different frames: the N-frame and the B-frame. The B-frame is formed by rotating the N-frame around an unknown axis. Our goal in this section is to find a set of equations that allows us to easily go back and forth between these two frames. As will be shown next, this set of equations can be used to construct a matrix known as the *rotation* or *transformation* matrix.

To relate the B-frame to the N-frame, we consider the general rotation shown in Figure 1.2a and decompose it into three successive rotations around the unit vectors of the Cartesian coordinate system. The first rotation, shown in Figure 1.2b, is a rotation of angle θ about the $\hat{n}_1$ unit vector. This rotation creates a new intermediate Cartesian coordinate system, which is denoted as the E-frame, such that $\hat{e}_1 \equiv \hat{n}_1$.

Next, we take a second rotation, as shown in Figure 1.2c, this time about the $\hat{e}_2$ unit vector through an angle ϕ forming a second intermediate frame, denoted as the C-frame, such that $\hat{c}_2 \equiv \hat{e}_2$. Finally, as shown in Figure 1.2d, the C-frame is rotated by an angle ψ about the $\hat{c}_3$ unit vector to form the B-frame such that $\hat{b}_3 \equiv \hat{c}_3$. As such, it is clear now that going from the original N-frame to the B-frame can be done by performing three successive rotations:[1]

- a rotation θ around $\hat{n}_1$ to form the E frame;
- a rotation ϕ around $\hat{e}_2$ to form the C frame;
- a rotation ψ around $\hat{c}_3$ to form the B frame.

It is now possible to relate the orientation of the B-frame to the original N-frame by using these three successive rotations. First, we refer back to Figure 1.2b to relate the E-frame to the N-frame. Using simple vector projections, we can write

$$\begin{aligned}
\hat{e}_1 &= \hat{n}_1, \\
\hat{e}_2 &= \cos\theta \hat{n}_2 + \sin\theta \hat{n}_3, \\
\hat{e}_3 &= -\sin\theta \hat{n}_2 + \cos\theta \hat{n}_3.
\end{aligned}$$

Note that $\hat{e}_1 = \hat{n}_1$ because the first rotation occurs about the $\hat{n}_1$ unit vector, and hence the $\hat{e}_1$ unit vector remains in the same direction as the $\hat{n}_1$ unit vector. The relationship between $\hat{e}_1$, $\hat{e}_2$ and $\hat{n}_1$, $\hat{n}_2$ can be clarified by referring to Figure 1.3, which shows a planar projection

[1] Note that this is not a unique transformation. One could go from the N-frame to the B-frame by performing many other different rotations using different angles.

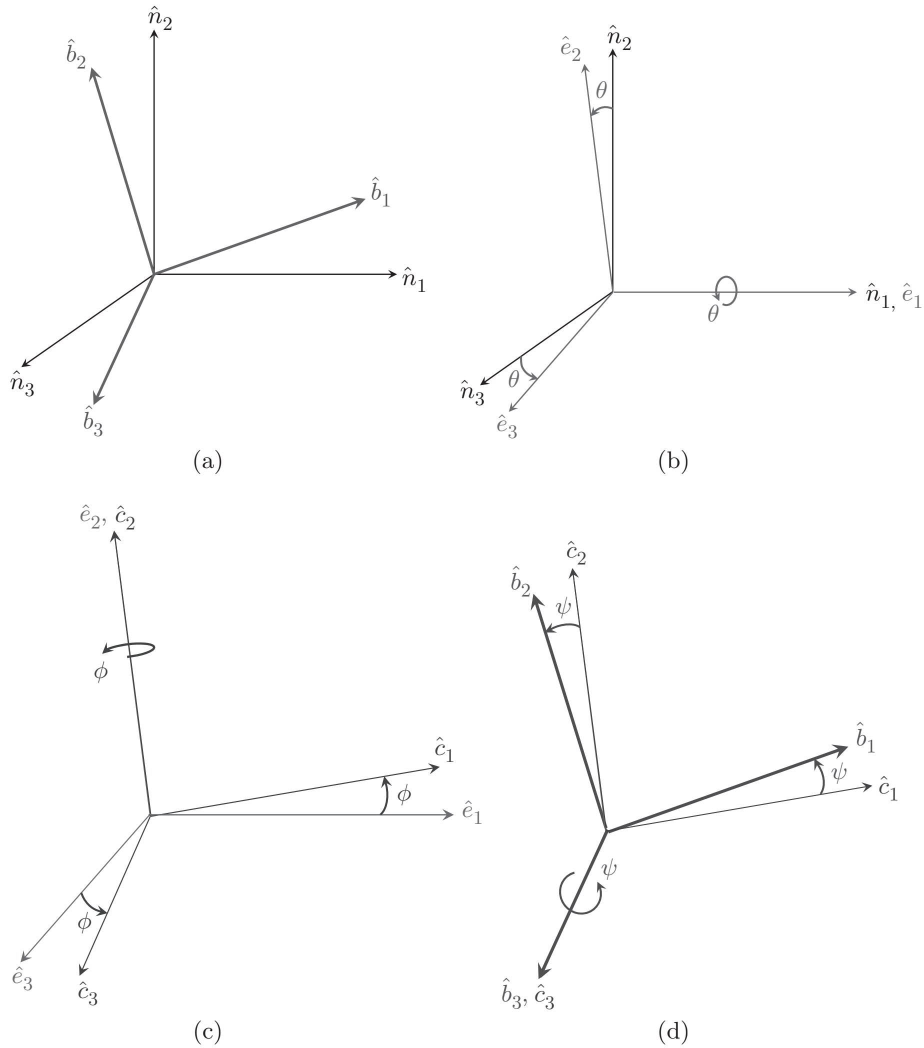


Figure 1.2 A 1-2-3 rotation performed to orient the N-frame in the unit coordinates of the B-frame. (*See color plate section for the color representation of this figure*.)

of the $(\hat{n}_2 - \hat{n}_3)$ plane shown in Figure 1.2b. Using Figure 1.3 it can be shown that $\hat{e}_2 = \cos\theta\hat{n}_2 + \sin\theta\hat{n}_3$. Similarly, using Figure 1.3 one can project the unit vector $\hat{e}_3$ in the N-frame by letting $\hat{e}_3 = -\sin\theta\hat{n}_2 + \cos\theta\hat{n}_3$.

The previous projections can be described in matrix form as

$$\begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}.$$

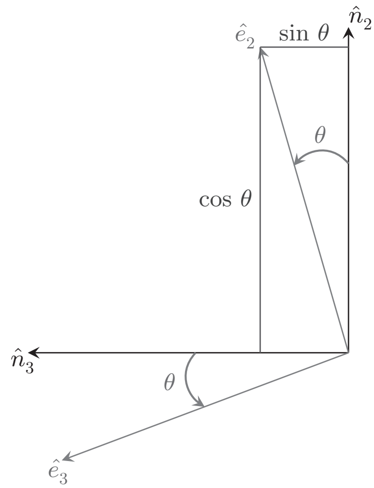


Figure 1.3 Relationship between the unit vectors of the E-frame and the unit vectors of the N-frame. (*See color plate section for the color representation of this figure*.)

where the matrix relating the $E-$frame to the N-frame is known as a rotation matrix. Furthermore, since this rotation matrix is formed by a single rotation about the $\hat{n}_1$ axis, it is commonly referred to as a 1-rotation.

Following similar reasoning, we can relate the C-frame to the E-frame via a 2-rotation of angle ϕ; such that

$$\begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix} = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix},$$

and the B-frame to the C-frame via a 3-rotation of angle ψ

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \end{bmatrix}.$$

To describe the rotation from the original N-frame to the final B-frame we combine the previous rotations, letting

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix}.$$

Multiplying the previous matrices yields the following general transformation matrix that can be used to go from the N-frame to the B-frame via a 1-2-3 rotation.

$$L = \begin{pmatrix} \cos\phi\cos\psi & \cos\theta\sin\psi + \sin\theta\sin\phi\cos\psi & \sin\theta\sin\psi - \cos\theta\cos\phi\sin\psi \\ -\cos\phi\sin\psi & \cos\theta\cos\psi - \sin\theta\sin\phi\sin\psi & \sin\theta\cos\psi + \cos\theta\sin\phi\sin\psi \\ \sin\phi & -\sin\theta\cos\phi & \cos\theta\cos\phi \end{pmatrix}.$$

Example 1.1 Properties of a Rotation Matrix

Prove that any rotation matrix has the following two properties:

- $L^{-1} = L^T$
- $|L| = \pm 1$.

1. A rotation matrix is a matrix that preserves the length of the vectors involved in the rotation and the angle between them. In other words, a Cartesian coordinate system formed by three normal unit vectors undergoing a rotation remains a Cartesian coordinate system with unit vectors normal to each other. A matrix that preserves length and direction between vectors is known as an orthogonal matrix, and satisfies the property $LL^T = \mathcal{I}$, where $\mathcal{I}$ is the identity matrix. It follows that $L^{-1} = L^T$ for a rotation matrix (see Property 5 in Section I.2.1.1).
2. Using $LL^T = \mathcal{I}$, and taking the determinant of both sides, we obtain:

$$|LL^T| = |\mathcal{I}|, \qquad |LL^T| = 1.$$

Using $|LL^T| = |L|\ |L^T| = |L|^2$ because $|L^T| = |L|$, we obtain:

$$|L|^2 = 1, \qquad |L| = \pm 1.$$

Properties of a Rotation Matrix

A rotation matrix L is an orthogonal matrix that satisfies the following conditions:

1. $L^{-1} = L^T$
2. $|L| = \pm 1$.

When using the right-hand rule to define the successive rotations, it can be shown that $|L| = +1$.

Flipped Classroom Exercise 1.1

Find the rotation matrix necessary to take you from a certain frame, N, to another frame B by performing a successive 2-1-3 rotation using angles (θ, ϕ, ψ).

To answer this exercise, follow the following steps:

1. Which rotation takes place first? What is the rotation matrix associated with it?
2. Which rotation takes place second? What is the rotation matrix associated with it?
3. Which rotation takes place third? What is the rotation matrix associated with it?
4. Multiply the rotation matrices obtained in steps 1, 2, and 3. Since the 2-rotation occurs first, the matrix obtained in step 1 must be on the far left. Show that the transformation matrix from N to B can be written as

$$L = \begin{pmatrix} \cos\phi\cos\psi + \sin\theta\sin\phi\sin\psi & \cos\theta\sin\psi & -\cos\psi\sin\phi + \cos\phi\sin\theta\sin\psi \\ \cos\psi\sin\theta\sin\phi - \cos\phi\sin\psi & \cos\theta\cos\psi & \cos\phi\cos\psi\sin\theta + \sin\phi\sin\psi \\ \cos\theta\sin\phi & -\sin\theta & \cos\theta\cos\phi \end{pmatrix}$$

1.4 Velocity of a Particle in a Three-dimensional Space

The kinematics of a point moving in space is fully described using three vectorial quantities:

- its position
- its velocity (being the time rate of change of the position)
- its acceleration (being the time rate of change of the velocity).

When the position vector is defined in an inertial frame, the velocity and acceleration can be easily obtained by differentiating the position vector with respect to time. On the other hand, the process is not as simple when the position vector is defined in a rotating frame. *This is because the unit vectors that are used to describe directions in the rotating frame are continuously changing their orientation with respect to the inertial frame*.

In what follows, we explain in detail how to find the velocity and acceleration of a point whose position is described in a rotating frame when the observer is standing in an inertial frame of reference. To this end, we consider a hypothetical situation in which a student named Joe is trying to understand kinematics. Joe is standing at point O and observing point P, as shown in Figure 1.4. The position vector **OP** is given by $\mathbf{r} = r_1\hat{n}_1 + r_2\hat{n}_2 + r_3\hat{n}_3$. Here, r_i are the lengths of the vector **OP** projected in the directions of $\hat{n}_i$, where $\hat{n}_i$ are the unit vectors of the inertial Cartesian frame denoted as the N-frame.

When point P is moving such that the vector **OP** does not change orientation with respect to the N-frame, then, from Joe's perspective, the vector **OP** only changes length. As such, the velocity of point P with respect to point O, as measured in the inertial N-frame, is the derivative of the vector **r** with respect to time, t; that is,

$$^{N}\mathbf{v}^{P/O} = \dot{r}_1\hat{n}_1 + \dot{r}_2\hat{n}_2 + \dot{r}_3\hat{n}_3, \tag{1.2}$$

where the dot is a derivative with respect to time. At this point, it is worth decoding the notation on the left-hand side of Equation (1.2), as similar notation will be used throughout this book. The right-hand superscript, P/O, on the vector **v**, means P with respect to O. The left-hand superscript, N on the vector **v** refers to the frame of the observer. As such, $^{N}\mathbf{v}^{P/O}$ reads as "the velocity of point P with respect to point O as observed in the N frame".

Notation

Throughout this book, the following notation will be used to describe kinematic quantities:

- Bold face **v** and **a** represent, respectively, the velocity and acceleration vectors. Such vectors will always appear with superscripts on their right- and left-hand sides. For

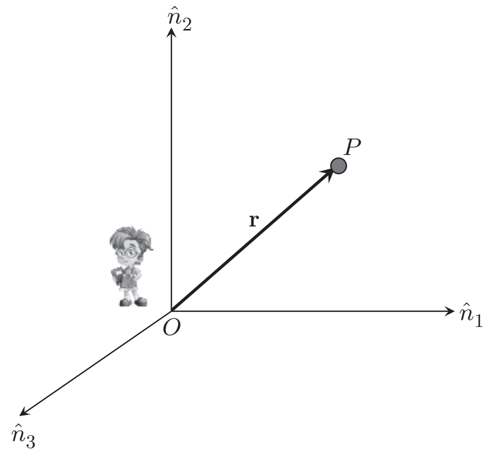


Figure 1.4 Kinematics of a particle in an inertial frame.

instance, you will often see velocities described in the form ${}^{N}\mathbf{v}^{P/O}$. The right-hand superscript, P/O, on the vector $\mathbf{v}$, means P with respect to O. The left-hand superscript, N on the vector $\mathbf{v}$ refers to the frame of the observer. As such, ${}^{N}\mathbf{v}^{P/O}$ reads as "the velocity of point P with respect to point O as observed in the N frame". Along similar lines, ${}^{B}\mathbf{a}^{Q/P}$ reads as "the acceleration of point Q with respect to point P as observed in the B-frame".

- Bold face $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ represent, respectively, the angular velocity and angular acceleration vectors. Such vectors will always appear with superscripts on their right- and left-hand sides. For instance, you will often see angular velocities described in the form ${}^{N}\boldsymbol{\omega}^{B}$, which reads as "the angular velocity of the B-frame with respect to the N-frame". Similarly, ${}^{C}\boldsymbol{\alpha}^{A}$, reads as "the angular velocity of the A-frame with respect to the C-frame".

Now, Joe allows the vector $\mathbf{r}$ to change direction, but he also decides to sit in a three-degrees-of-freedom chair, which changes its orientation (pitch, roll, yaw), such that at any instant his orientation is always in the direction of $\mathbf{r}$. In other words, again Joe can *only* observe changes in length. To reflect the fact that Joe's frame of reference is now rotating in space, we define a rotating frame called the B-frame at point O, as shown in Figure 1.5. In this rotating frame, the position and velocity of point P with respect to O can be written as

$$\begin{aligned} \mathbf{OP} &= r_1\hat{b}_1 + r_2\hat{b}_2 + r_3\hat{b}_3, \\ {}^{B}\mathbf{v}^{P/O} &= \dot{r}_1\hat{b}_1 + \dot{r}_2\hat{b}_2 + \dot{r}_3\hat{b}_3. \end{aligned} \tag{1.3}$$

Next, Joe decides to step out of the three degrees-of-freedom chair and observe the velocity of particle, P, from the inertial frame. In this case, Joe observes that the vector r is simultaneously changing its length and direction. To describe the velocity of point P from Joe's perspective,

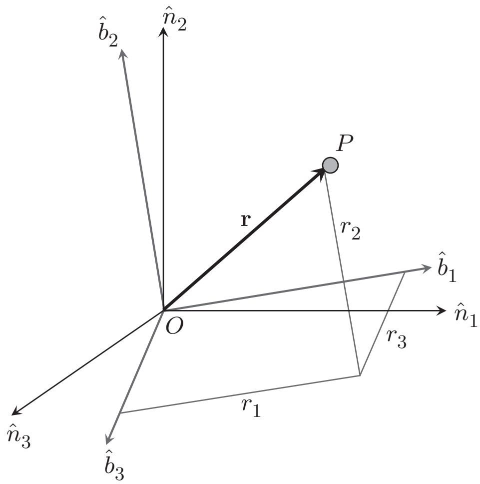


Figure 1.5 Kinematics of a particle in a rotating frame.

we differentiate the position vector **OP** once with respect to time, taking into account that the unit vectors $\hat{b}_i$ are also changing orientation with respect to the inertial frame. This yields

$$^N\mathbf{v}^{P/O} = \underbrace{\dot{r}_1\hat{b}_1 + \dot{r}_2\hat{b}_2 + \dot{r}_3\hat{b}_3}_{\text{Translation}} + \underbrace{r_1\dot{\hat{b}}_1 + r_2\dot{\hat{b}}_2 + r_3\dot{\hat{b}}_3}_{\text{Rotation}}, \tag{1.4}$$

where the first three terms on the right-hand side represent the translational component of the velocity, while the last three terms represent the rotational component.

In Equation (1.4), the derivatives of the unit vectors $\hat{b}_i$ emerge to reflect the fact that the B-frame is rotating with respect to the inertial N-frame. The derivative of the unit vectors depends on the rate at which the rotation occurs; in other words, the angular velocity between the two frames. To this end, in order to capture the change of orientation of the B-frame with respect to the N-frame, we define the angular velocity vector $^N\boldsymbol{\omega}^B$, which describes the angular rate at which the B-frame rotates with respect to the N-frame.

Next, we turn our attention to describing the derivative of the unit vectors in terms of the angular velocity $^N\boldsymbol{\omega}^B$. For simplicity, consider Figure 1.6, where we have assumed that the B-frame is rotating around one axis, chosen here to be the third axis $\hat{b}_3$, at an angular velocity $^N\boldsymbol{\omega}^B = \omega_3\hat{b}_3$. The rotation of the B-frame at this angular velocity causes the tip of the unit vectors $\hat{b}_1$ and $\hat{b}_2$ to rotate with a velocity equal to their length – unity in this case – times the angular velocity ω_3. The direction of their rotation will be perpendicular to the unit vectors themselves; that is, in the direction of $\hat{b}_2$ for unit vector $\hat{b}_1$, and in the direction of $-\hat{b}_1$ for unit vector $\hat{b}_2$. Mathematically, we can write

$$\begin{aligned} \frac{\mathrm{d}\hat{b}_1}{\mathrm{d}t} &= \omega_3\hat{b}_2, \\ \frac{\mathrm{d}\hat{b}_2}{\mathrm{d}t} &= -\omega_3\hat{b}_1, \\ \frac{\mathrm{d}\hat{b}_3}{\mathrm{d}t} &= 0. \end{aligned} \tag{1.5}$$

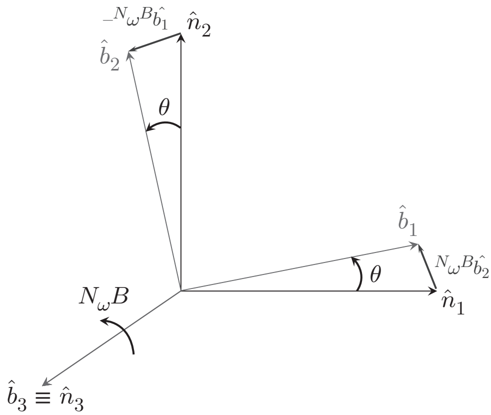


Figure 1.6 Time rate of change of the unit vectors as a result of their rotation.

Substituting Equation (1.15) into Equation (1.4), we obtain

$$^N\mathbf{v}^{P/O} = (\dot{r}_1\hat{b}_1 + \dot{r}_2\hat{b}_2 + \dot{r}_3\hat{b}_3) + \omega_3(r_1\hat{b}_2 - r_2\hat{b}_1). \quad (1.6)$$

Note that the term $\omega_3(r_1\hat{b}_2 - r_2\hat{b}_1)$ is nothing but $\omega_3\hat{b}_3 \times \mathbf{r}$, where $\times$ refers to the cross product. Thus, we can write

$$^N\mathbf{v}^{P/O} = {}^B\mathbf{v}^{P/O} + \omega_3\hat{b}_3 \times \mathbf{r}, \quad (1.7)$$

where the first term on the right-hand side represents the change in the velocity of point P with respect to O as observed in the rotating frame.

By using the concept of successive rotations described earlier, the same conclusion can be achieved, even when the rotation is not restricted to one axis; that is, when ${}^N\boldsymbol{\omega}^B = \omega_1\hat{b}_1 + \omega_2\hat{b}_1 + \omega_3\hat{b}_3$. In other words, one can generalize Equation (1.7), such that

$$\boxed{^N\mathbf{v}^{P/O} = {}^B\mathbf{v}^{P/O} + {}^N\boldsymbol{\omega}^B \times \mathbf{r}} \quad (1.8)$$

In words, Equation (1.8) states that "the velocity of point P with respect to point O as observed in the N-frame is equal to the velocity of point P with respect to point O as observed in the rotating B-frame (translation) plus a rotational component resulting from the angular velocity of the B-frame with respect to the N-frame".

Equation (1.8) can also be generalized to any kinematic quantity defined in two different frames (velocity, acceleration, angular velocity, or angular acceleration). In general, the temporal derivative of the vector described in one frame can be expressed as its derivative in another frame plus the cross product of the angular velocity vector between the two frames and the vector itself.

Differentiating in a Rotating Frame

When taking the time derivative of a vector, $\mathbf{r}$, described in a rotating frame – say the B-frame – with respect to an observer standing in another frame – say the C-frame – do not forget to add the term ${}^C\boldsymbol{\omega}^B \times \mathbf{r}$:

$$ {}^C\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}) = {}^B\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}) + {}^C\boldsymbol{\omega}^B \times \mathbf{r} $$

Example 1.2 Velocity of a Particle 1

Consider a particle P whose position with respect to a fixed point O is given by $\mathbf{r} = 2t\hat{n}_1 + 3t^3\hat{n}_2$ m, where $\hat{n}_i$ are the unit vectors of a Cartesian inertial frame. Find the velocity of the particle after 2 s, as measured by an observer standing in the N-frame.

Since the components of the vector $\mathbf{r}$ are all described in the N-frame and the observer is also in the $N-$frame, the velocity is just a direct derivative of $\mathbf{r}$:

$$ {}^N\mathbf{v}^{P/O} = 2\hat{n}_1 + 9t^2\hat{n}_2 = 2\hat{n}_1 + 36\hat{n}_2 \text{ m/s}. $$

Example 1.3 Velocity of a Particle 2

Consider a particle P whose position with respect to a fixed point O is given by $\mathbf{r} = 2t\hat{b}_1 + 3t^3\hat{b}_2$ m, where the $\hat{b}_i$ are the unit vectors of a Cartesian rotating frame, B, which rotates with respect to a stationary frame N at a constant angular velocity ${}^N\boldsymbol{\omega}^B = 2\hat{b}_1 + 3\hat{b}_2$ rad/s. Find the velocity of the particle after 2 s as measured by:

(a) an observer standing in the N-frame;
(b) an observer standing in the B-frame.

(a) Observer standing in the N-frame Note that the position vector is defined in the rotating frame. Hence, when taking a derivative with respect to the inertial frame, we need to take into account the rotation of the unit vectors. As such, for each component of the vector $\mathbf{r}$, we have to take a direct derivative as well as account for the rotation resulting from the angular velocity; that is,

$$ \begin{aligned} {}^N\mathbf{v}^{P/O} &= 2\hat{b}_1 + 9t^2\hat{b}_2 + {}^N\boldsymbol{\omega}^{\mathbf{B}} \times (2t\hat{b}_1 + 3t^3\hat{b}_2), \\ &= 2\hat{b}_1 + 9t^2\hat{b}_2 + (6t^3 - 6t)\hat{b}_3, \\ &= 2\hat{b}_1 + 36\hat{b}_2 + 36\hat{b}_3 \text{ m/s}. \end{aligned} $$

(b) Observer standing in the B-frame Note that the position vector and the observer are in the same frame. As such, we can just directly differentiate the position vector to obtain

$$ \begin{aligned} {}^N\mathbf{v}^{P/O} &= 2\hat{b}_1 + 9t^2\hat{b}_2, \\ &= 2\hat{b}_1 + 36\hat{b}_2 \text{ m/s}. \end{aligned} $$

Example 1.4 Velocity of a Particle 3

Consider a particle P whose position with respect to a fixed point O is given by $\mathbf{r} = 2t\hat{n}_1 + 3t^3\hat{b}_2$ m, where the $\hat{n}_i$ are the unit vectors of a Cartesian inertial frame, N, and the $\hat{b}_i$ are the unit vectors of a Cartesian rotating reference, B, which rotates with respect to a stationary frame N at a constant angular velocity ${}^N\boldsymbol{\omega}^B = 2\hat{b}_1 + 3\hat{b}_2$ rad/s. Find the velocity of the particle after 2 s as measured by an observer standing in the N-frame.

Notice that, in this case, only a part of the position vector is described in the B-frame, hence when finding the velocity of the particle as measured by a stationary observer, we need to account *only* for the rotation associated with that component, namely, $3t^3\hat{b}_2$. In other words, we can write

$$\begin{aligned}
{}^N\mathbf{v}^{P/O} &= 2\hat{n}_1 + 9t^2\hat{b}_2 + (2\hat{b}_1 + 3\hat{b}_2) \times 3t^3\hat{b}_2 \\
&= 2\hat{n}_1 + 9t^2\hat{b}_2 + 6t^3\hat{b}_3 = 2\hat{n}_1 + 36\hat{b}_2 + 48\hat{b}_3 \text{ m/s}.
\end{aligned}$$

Flipped Classroom Exercise 1.2

In this exercise, we will walk through the process of finding the derivative of the unit vectors $\frac{\mathrm{d}\hat{b}_i}{\mathrm{d}t}$ for a general three-dimensional rotation. The rotation is carried out through three successive angles (θ, ϕ, ψ). To this end, following Figure 1.2, we will assume that the rotation is a 1-2-3 (θ, ϕ, ψ) rotation. The first rotation goes from the N- to the E-frame. The second rotation goes from the E- to the C-frame while the third rotation goes from the C- to the B-frame.

Using the assumed definition of the successive rotations, you can write

$$\begin{aligned}
{}^N\boldsymbol{\omega}^E &= \dot{\theta}\hat{e}_1 = \omega_1\hat{e}_1, \\
{}^E\boldsymbol{\omega}^C &= \dot{\phi}\hat{c}_1 = \omega_2\hat{c}_2, \\
{}^C\boldsymbol{\omega}^B &= \dot{\psi}\hat{b}_3 = \omega_3\hat{b}_2,
\end{aligned}$$

Thus, ${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^E + {}^E\boldsymbol{\omega}^C + {}^C\boldsymbol{\omega}^B = \dot{\theta}\hat{e}_1 + \dot{\phi}\hat{c}_2 + \dot{\psi}\hat{b}_3 = \omega_1\hat{e}_1 + \omega_2\hat{c}_2 + \omega_3\hat{b}_3$.

Next, you need to do the following:

1. Express the unit vector **n** in terms of the unit vector **b** using the 1-2-3 rotation derived previously.
2. Assume small angles $(\mathrm{d}\theta, \mathrm{d}\phi, \mathrm{d}\psi)$ and linearize the rotation matrix as well as the angular velocity vector. Note that this assumption is not restrictive, since any general rotation can be described in terms of series of tiny successive rotations. Show that upon linearizing you obtain

$$\begin{bmatrix} \hat{n}_1 \\ \hat{n}_2 \\ \hat{n}_3 \end{bmatrix} = \mathcal{I} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} + \begin{pmatrix} 0 & -\mathrm{d}\psi & \mathrm{d}\phi \\ \mathrm{d}\psi & 0 & -\mathrm{d}\theta \\ -\mathrm{d}\phi & \mathrm{d}\theta & 0 \end{pmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}$$

and that for small angles ${}^N\boldsymbol{\omega}^B = \dot{\theta}\hat{b}_1 + \dot{\phi}\hat{b}_2 + \dot{\psi}\hat{b}_3$.

3. Using the previous equations, show that we can write

$$\begin{bmatrix} \mathrm{d}\hat{b}_1 \\ \mathrm{d}\hat{b}_2 \\ \mathrm{d}\hat{b}_3 \end{bmatrix} = \begin{pmatrix} 0 & -\mathrm{d}\psi & \mathrm{d}\phi \\ \mathrm{d}\psi & 0 & -\mathrm{d}\theta \\ -\mathrm{d}\phi & \mathrm{d}\theta & 0 \end{pmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}.$$

4. Divide the previous equation by $\mathrm{d}t$ and show that

$$\begin{bmatrix} \dot{\hat{b}}_1 \\ \dot{\hat{b}}_2 \\ \dot{\hat{b}}_3 \end{bmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_2 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}.$$

5. Show that the previous equation can be written as

$$\dot{\mathbf{b}} = {}^N\boldsymbol{\omega}^B \times \mathbf{b}.$$

1.5 Acceleration of a Particle in a Three-dimensional Space

To find a general expression for the acceleration of a particle in a three-dimensional space with respect to an inertial frame, we differentiate Equation (1.8) once with respect to time, noting that the term on the left-hand side is defined in the stationary N-frame while all terms on the right-hand side are described in the rotating B frame. In other words, we need to find ${}^N\frac{\mathrm{d}}{\mathrm{d}t}({}^N\mathbf{v}^{P/O})$, where the superscript on the left-hand side of the derivative is used to denote that the change of the quantity included within the derivative is observed from the inertial frame. Differentiating each term in Equation (1.8) yields

$$\begin{aligned} {}^N\frac{\mathrm{d}}{\mathrm{d}t}({}^N\mathbf{v}^{P/O}) &= {}^N\mathbf{a}^{P/O}, \\ {}^N\frac{\mathrm{d}}{\mathrm{d}t}({}^B\mathbf{v}^{P/O}) &= {}^B\frac{\mathrm{d}}{\mathrm{d}t}({}^B\mathbf{v}^{P/O}) + {}^N\boldsymbol{\omega}^{\mathbf{B}} \times {}^B\mathbf{v}^{P/O} = {}^B\mathbf{a}^{P/O} + {}^N\boldsymbol{\omega}^{\mathbf{B}} \times {}^B\mathbf{v}^{P/O}, \\ {}^N\frac{\mathrm{d}}{\mathrm{d}t}({}^N\boldsymbol{\omega}^B \times \mathbf{r}) &= {}^B\frac{\mathrm{d}}{\mathrm{d}t}({}^N\boldsymbol{\omega}^B \times \mathbf{r}) + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}), \\ &= {}^N\boldsymbol{\alpha}^B \times \mathbf{r} + {}^N\boldsymbol{\omega}^{\mathbf{B}} \times {}^B\mathbf{v}^{P/O} + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r}), \end{aligned}$$

where ${}^N\boldsymbol{\alpha}^B = {}^B\frac{\mathrm{d}}{\mathrm{d}t}({}^N\boldsymbol{\omega}^B)$ is the angular acceleration vector. Using the previous equation, we can write

$$\boxed{{}^N\mathbf{a}^{P/O} = {}^B\mathbf{a}^{P/O} + 2{}^N\boldsymbol{\omega}^B \times {}^B\mathbf{v}^{P/O} + {}^N\boldsymbol{\alpha}^B \times \mathbf{r} + {}^N\boldsymbol{\omega}^B \times ({}^N\boldsymbol{\omega}^B \times \mathbf{r})} \tag{1.9}$$

Equation (1.9) represents the general formula for the acceleration of a particle in space. The first term on the right-hand side represents the acceleration of the particle in the rotating frame. The second term represents the *Coriolis* acceleration, which, as evident, is perpendicular to the velocity of the particle as measured in the rotating frame and the direction of the frame's rotation. The third term is the tangential acceleration, and finally the fourth term is the normal acceleration.

While it is possible to use the acceleration formula as described in the form shown in Equation (1.9), it is absolutely unnecessary to memorize it since one can use the basic understanding utilized throughout its derivation to calculate the acceleration of any particle in space. To demonstrate this, consider the following series of examples:

Example 1.5 Acceleration of a Particle

Consider a particle P whose position with respect to a fixed point O is given by $\mathbf{r} = 2t\hat{b}_1 + 3t^3\hat{b}_2$ m, where the $\hat{n}_i$ are the unit vectors of a Cartesian inertial frame, N, and the $\hat{b}_i$ are the unit vectors of a Cartesian rotating reference, B, which rotates with respect to the stationary frame N at a constant angular velocity ${}^N\boldsymbol{\omega}^B = 2\hat{b}_1 + 3\hat{b}_2$ rad/s. Find the acceleration of the particle after 2 s as measured by an observer standing in the N-frame.

In Example 1.3, the velocity of the particle with respect to the inertial observer was found to be

$$ {}^N\mathbf{v}^{P/O} = 2\hat{b}_1 + 9t^2\hat{b}_2 + (6t^3 - 6t)\hat{b}_3 \text{ m/s}. $$

To calculate the acceleration in the reference frame, we differentiate the previous expression, taking into account the rotational component of any term described in a rotating frame. Note also that the angular acceleration ${}^N\boldsymbol{\alpha}^B$ vanishes because the angular velocity is constant. As such, we can write

$$ \begin{aligned} {}^N\mathbf{a}^{P/O} &= 18t\hat{b}_2 + (18t^2 - 6)\hat{b}_3 + (2\hat{b}_1 + 3\hat{b}_2) \times (2\hat{b}_1 + 9t^2\hat{b}_2 + (6t^3 - 6t)\hat{b}_3), \\ &= 18(t^3 - t)\hat{b}_1 + (30t - 12t^3)\hat{b}_2 + (36t^2 - 12)\hat{b}_3, \\ &= 108\hat{b}_1 - 36\hat{b}_2 + 132\hat{b}_3 \text{ m/s}^2. \end{aligned} $$

Example 1.6 Simple Pendulum Kinematics

For the simple pendulum of rigid cable length l shown in Figure 1.7, find the velocity and acceleration of particle P with respect to point O. Choose your rotating B-frame such that ${}^N\boldsymbol{\omega}^B = \dot{\theta}\hat{b}_3$ and express all your answers in the B-frame.

We start by defining the position vector from O to P as

$$ \mathbf{OP} = -l\hat{b}_2. $$

It follows that the velocity of point P as observed by an inertial observer at point O is

$$ {}^N\mathbf{v}^{P/O} = \frac{\mathrm{d}\mathbf{OP}}{\mathrm{d}t} = -\dot{l}\hat{b}_2 + {}^N\boldsymbol{\omega}^B \times \mathbf{OP}. $$

Since the pendulum does not change length, $\dot{l} = 0$. Therefore we can write

$$ {}^N\mathbf{v}^{P/O} = \dot{\theta}\hat{b}_3 \times \mathbf{OP} = l\dot{\theta}\hat{b}_1. $$

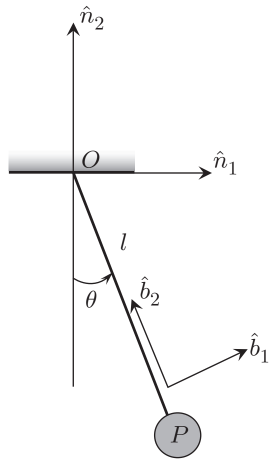


Figure 1.7 Kinematics of a simple pendulum.

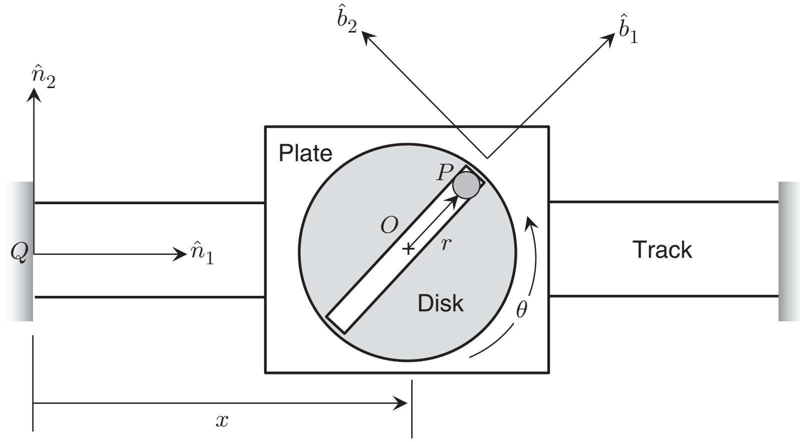


Figure 1.8 Two-dimensional motion of a particle.

The acceleration of point P with respect to point O can be written as

$$^N\mathbf{a}^{P/O} = \frac{\mathrm{d}^N\mathbf{v}^{P/O}}{\mathrm{d}t} = l\ddot{\theta}\hat{b}_1 + {}^N\boldsymbol{\omega}^B \times {}^N\mathbf{v}^{P/O} = l\ddot{\theta}\hat{b}_1 + l\dot{\theta}^2\hat{b}_2.$$

Example 1.7 Single-rotation Kinematics

Consider the system shown in Figure 1.8, which consists of a plate sliding on a track. A disk is mounted on top of the moving plate and is free to rotate through an angle θ about point O. A particle, P, is free to move in a groove drilled across the disk as shown. Define two frames: an inertial frame located at point Q and a rotating frame such that $\hat{b}_3 = \hat{n}_3$ and $^N\boldsymbol{\omega}^B = \dot{\theta}\hat{b}_3$ to obtain

- the velocity of the particle P with respect to O as measured by an observer standing in the B-frame, ${}^B\mathbf{v}^{P/O}$;
- the velocity of the particle P with respect to O as measured by an observer standing in the N-frame, ${}^N\mathbf{v}^{P/O}$;
- the acceleration of the particle P with respect to Q as measured by an observer standing in the N-frame, ${}^N\mathbf{a}^{P/Q}$.

Describe your answers in the B-coordinate system.

To find ${}^B\mathbf{v}^{P/O}$, we define the position vector **OP** and differentiate it once with respect to time; that is,

$$\mathbf{OP} = r\hat{b}_1, \qquad {}^B\mathbf{v}^{P/O} = \dot{r}\hat{b}_1.$$

To find ${}^N\mathbf{v}^{P/O}$, we carry out the same procedure but account for the relative rotation between the B- and N-frame; that is,

$$\begin{aligned}
\mathbf{OP} &= r\hat{b}_1, \\
{}^N\mathbf{v}^{P/O} &= \dot{r}\hat{b}_1 + \dot{\theta}\hat{b}_3 \times r\hat{b}_1, \\
{}^N\mathbf{v}^{P/O} &= \dot{r}\hat{b}_1 + r\dot{\theta}\hat{b}_2.
\end{aligned}$$

To find ${}^N\mathbf{a}^{P/Q}$, we define the position vector **QP** and differentiate it twice with respect to time, accounting for the relative rotation between the B- and N-frame. This yields

$$\begin{aligned}
\mathbf{QP} &= x\hat{n}_1 + r\hat{b}_1, \\
{}^N\mathbf{v}^{P/Q} &= \dot{x}\hat{n}_1 + \dot{r}\hat{b}_1 + \dot{\theta}\hat{b}_3 \times r\hat{b}_1, \\
{}^N\mathbf{v}^{P/Q} &= \dot{x}\hat{n}_1 + \dot{r}\hat{b}_1 + r\dot{\theta}\hat{b}_2, \\
{}^N\mathbf{a}^{P/Q} &= \ddot{x}\hat{n}_1 + (\ddot{r} - r\dot{\theta}^2)\hat{b}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{b}_2.
\end{aligned}$$

To express the previous answer in the B-frame, we still need to express $\hat{n}_1$ in terms of the B-coordinates. Note that the B-frame was formed by carrying out a counter-clockwise rotation around the $\hat{n}_3$ axis. Hence, using the definition of the 3-rotation, we can write $\hat{n}_1 = \cos\theta\hat{b}_1 - \sin\theta\hat{b}_2$. This yields

$${}^N\mathbf{a}^{P/Q} = (\ddot{x}\cos\theta + \ddot{r} - r\dot{\theta}^2)\hat{b}_1 + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - \ddot{x}\sin\theta)\hat{b}_2.$$

Example 1.8 Radar-tracking Kinematics

A radar station is tracking a rocket that has just lifted off vertically with a velocity v_r and acceleration a_r. The station is located such that the distance between the radar station and the rocket is R (Figure 1.9). Calculate $\dot{R}$, $\ddot{R}$, $\dot{\theta}$, $\ddot{\theta}$ in terms of v_r, a_r, R, and θ only.
We begin by calculating ${}^N\mathbf{v}^{P/O}$. To this end, we let

$$\mathbf{OP} = R\hat{b}_1, \qquad {}^N\mathbf{v}^{P/O} = \dot{R}\hat{b}_1 + R\dot{\theta}\hat{b}_2.$$

Using $v_r\hat{n}_1 = v_r\sin\theta\hat{b}_1 + v_r\cos\theta\hat{b}_2$, we conclude that $\dot{R} = v_r\sin\theta$ and $\dot{\theta} = v_r\cos\theta/R$.

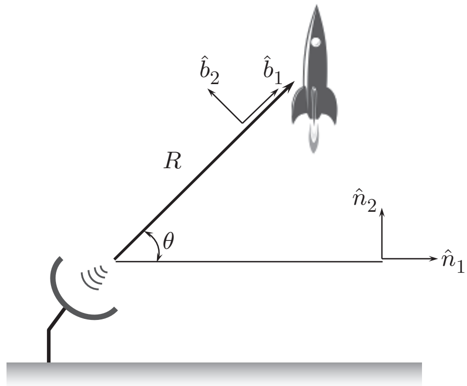


Figure 1.9 A rocket detected by a radar station.

Next, we calculate the acceleration vector

$$^N\mathbf{a}^{P/O} = (\ddot{R} - R\dot{\theta}^2)\hat{b}_1 + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{b}_2.$$

Using $a_r\hat{n}_1 = a_r \sin\theta\hat{b}_1 + a_r \cos\theta\hat{b}_2$, we obtain

$$\ddot{R} = a_r \sin\theta + \frac{v_r^2}{R}\cos^2\theta,$$
$$\ddot{\theta} = \frac{a_r}{R}\cos\theta - \frac{v_r^2}{R^2}\sin 2\theta.$$

Example 1.9 Cylindrical Coordinates

Obtain the general velocity and acceleration expressions for the motion of a particle as described in a cylindrical coordinate system.

We start by defining a stationary frame, the N-frame, as shown in Figure 1.10. Subsequently, we define a rotating frame, the B-frame, which is formed through a rotation angle ϕ, such that $\hat{b}_2 \equiv \hat{n}_2$ is always aligned with the direction of z, and $\hat{b}_3$ is always aligned with the direction of r. Using this understanding, we can write $^N\boldsymbol{\omega}^B = \dot{\phi}\hat{n}_2 = \dot{\phi}\hat{b}_2$, $^N\boldsymbol{\alpha}^B = \ddot{\phi}\hat{n}_2 = \ddot{\phi}\hat{b}_2$. In the B-frame, the position vector from O to P can be written as

$$\mathbf{OP} = z\hat{b}_2 + r\hat{b}_3.$$

The velocity of the particle as measured by a stationary observer at O can be written as

$$^N\mathbf{v}^{P/O} = \dot{z}\hat{b}_2 + \dot{r}\hat{b}_3 + {}^N\boldsymbol{\omega}^B \times (z\hat{b}_2 + r\hat{b}_3),$$
$$^N\mathbf{v}^{P/O} = r\dot{\phi}\hat{b}_1 + \dot{z}\hat{b}_2 + \dot{r}\hat{b}_3.$$

The acceleration of the particle as measured by a stationary observer at O can be obtained by differentiating the previous equation once with respect to time, taking into account the relative

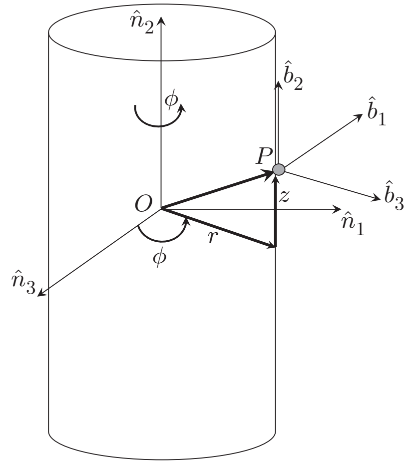


Figure 1.10 Description of the motion of a particle in a cylindrical coordinate system.

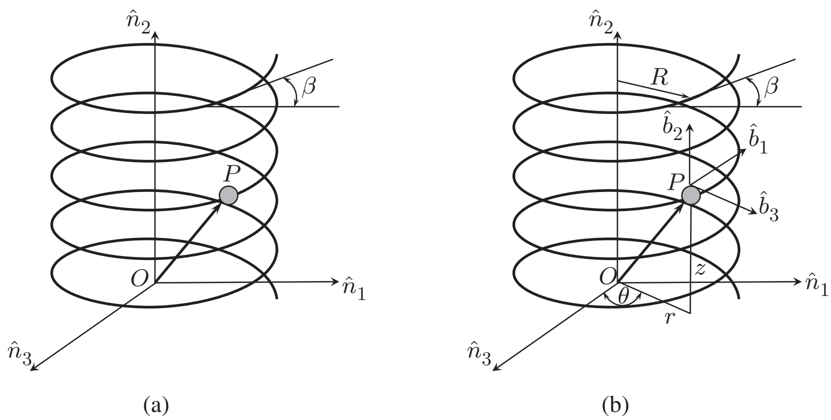


Figure 1.11 Motion on a helix.

rotations between the two frames. This yields

$$
\begin{aligned}
{}^N\mathbf{a}^{P/O} &= (r\ddot{\phi} + \dot{r}\dot{\phi})\hat{b}_1 + \ddot{z}\hat{b}_2 + \ddot{r}\hat{b}_3 + {}^N\boldsymbol{\omega}^B \times (r\dot{\phi}\hat{b}_1 + \dot{z}\hat{b}_2 + \dot{r}\hat{b}_3) \\
{}^N\mathbf{a}^{P/O} &= (r\ddot{\phi} + 2\dot{r}\dot{\phi})\hat{b}_1 + \ddot{z}\hat{b}_2 + (\ddot{r} - r\dot{\phi}^2)\hat{b}_3.
\end{aligned}
$$

Example 1.10 Helix

Consider the motion of a particle along the helical path shown in Figure 1.11a, where R and β are respectively its radius and angle. Find the velocity and acceleration of the particle with respect to point O.

To describe the kinematics of the problem, we use cylindrical coordinates and define the rotating frame as shown in Figure 1.11b, where ${}^N\boldsymbol{\omega}^B = \dot{\theta}\hat{b}_2$. We define the position vector **OP** as

$$\mathbf{OP} = r\hat{b}_3 + z\hat{b}_2 = R\hat{b}_3 + R\theta\tan\beta\hat{b}_2.$$

The velocity can then be written as

$${}^N\mathbf{v}^{P/O} = (R\dot{\theta}\tan\beta)\hat{b}_2 + \dot{\theta}\hat{b}_2 \times (R\hat{b}_3 + R\theta\tan\beta\hat{b}_2) = R\dot{\theta}\hat{b}_1 + R\dot{\theta}\tan\beta\hat{b}_2,$$

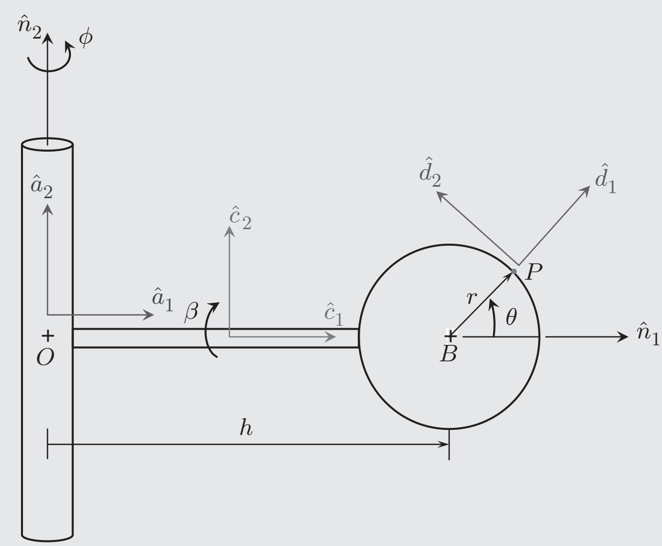
and the acceleration is

$${}^N\mathbf{a}^{P/O} = R\ddot{\theta}\hat{b}_1 + R\ddot{\theta}\tan\beta\hat{b}_2 - R\dot{\theta}^2\hat{b}_3.$$

Flipped Classroom Exercise 1.3

Point P on a thin disk of radius r rotates about its own axis at point B through an angle θ. The yoke where it is mounted also rotates about the line OB through an angle β. The entire assembly rotates about the $\hat{n}_2$-axis shown in the figure through angle ϕ. To fully describe the kinematics of the problem, define three rotating frames such that ${}^N\boldsymbol{\omega}^A = \dot{\phi}\hat{n}_2$, ${}^A\boldsymbol{\omega}^C = -\dot{\beta}\hat{a}_1$, and ${}^C\boldsymbol{\omega}^D = \dot{\theta}\hat{c}_3$, then find the following:

- ${}^C\mathbf{a}^{P/B}$
- ${}^A\mathbf{a}^{P/B}$



To find ${}^C\mathbf{a}^{P/B}$, you need to do the following:

1. Define the position vector from B to P in the easiest possible way. Note that regardless of how this complex system rotates, r will always be in the direction of $\hat{d}_1$.
2. Find the velocity as seen by an observer in the C frame. Note that $\dot{r} = 0$ because the radius of the circle does not change length
3. Show that the acceleration is given by

$$\begin{aligned} {}^C\mathbf{a}^{P/B} &= r\ddot{\theta}\hat{d}_2 + {}^C\boldsymbol{\omega}^D \times r\dot{\theta}\hat{d}_2, \\ &= r\ddot{\theta}\hat{d}_2 - r\dot{\theta}\hat{d}_1. \end{aligned}$$

To find ${}^A\mathbf{a}^{P/B}$, you need to do the following:

1. Define the position vector from B to P.
2. Show that the velocity of point P with respect to point B as observed in the A-frame is given by

$$ {}^A\mathbf{v}^{P/B} = {}^A\boldsymbol{\omega}^D \times r\hat{d}_1 = (-\dot{\beta}\hat{c}_1 + \dot{\theta}\hat{d}_3) \times r\hat{d}_1. $$

   Note that, to carry the cross product between vectors described in two different frames, it is much more convenient to rotate one of them such that they are both described in the same frame.
3. Show that the acceleration of point P with respect to point B as observed in the A frame is given by

$$\begin{aligned} {}^A\mathbf{a}^{P/B} &= r\ddot{\theta}\hat{d}_2 - (r\ddot{\beta}\sin\theta + r\dot{\beta}\dot{\theta}\cos\theta)\hat{d}_3 + {}^C\boldsymbol{\omega}^D \times (r\dot{\theta}\hat{d}_2 - r\dot{\beta}\sin\theta\hat{d}_3), \\ &= -(r\dot{\theta}^2 + r\dot{\beta}^2\sin^2\theta)\hat{d}_1 + (r\ddot{\theta} - r\dot{\beta}^2\cos\theta\sin\theta)\hat{d}_2, \\ &\quad - (r\ddot{\beta}\sin\theta + 2r\dot{\theta}\dot{\beta}\cos\theta)\hat{d}_3. \end{aligned}$$

Exercises

1.1 Find the rotation matrix necessary to transform a frame N to another frame B by performing a 1-2-1 rotation using angles (θ, ϕ, ψ).

1.2 A position vector from point O to point P is defined as $\mathbf{OP} = 2t\hat{n}_1 + 3\sin t\hat{c}_2 + t^3\hat{b}_1$ m, where ${}^N\boldsymbol{\omega}^C = 2\hat{c}_2 + 3t\hat{c}_1$ rad/s and ${}^C\boldsymbol{\omega}^B = 2\sin t\hat{b}_2 + 3\hat{b}_3$ rad/s. Here, N is an inertial frame, while B and C are rotating frames. Find the velocity and acceleration of point P with respect to point O as observed in the inertial frame at $t = \pi$ s.

1.3 A position vector from point O to point P is defined as $\mathbf{OP} = 2t\hat{a}_1 + 3\sin t\hat{a}_2 + t^3\hat{b}_1$ m, where ${}^N\boldsymbol{\omega}^A = 2\hat{a}_2 + 3t\hat{a}_1$ rad/s and ${}^A\boldsymbol{\omega}^B = 3\hat{b}_3$ rad/s. Here, N is an inertial frame, while B and A are rotating frames. Find the velocity and acceleration of point P with respect to point O as observed in the inertial frame at $t = \frac{\pi}{2}$ s.

1.4 Derive the velocity and acceleration of a particle in a spherical coordinate system. Use Figure 1.12 for the definition of the rotation angles.

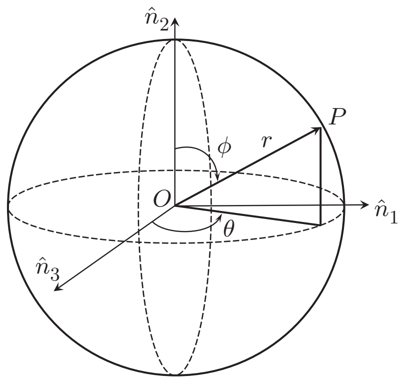


Figure 1.12 Exercise 1.4.

1.5 In kinematics, the derivative of the acceleration is defined as the Jerk, **J**. Imagine we live in a world in which the Jerk replaces the acceleration in Newton's second law. Using the general acceleration formula we derived in the chapter, derive a general formula for the Jerk, ${}^N\mathbf{J}^{P/O}$ of a particle P with respect to an inertial point, O.

1.6 In the two-dimensional problem shown in Figure 1.13, a particle P of mass m slides in a pipe which rotates with an angle ϕ relative to a rod of length l as shown in the figure. The rod, in turn, rotates with an angle θ relative to the vertical. Define r as the distance from point Q to point P and your rotating coordinates systems such that ${}^N\boldsymbol{\omega}^A = \dot{\theta}\hat{n}_3$ and ${}^A\boldsymbol{\omega}^B = \dot{\phi}\hat{a}_3$, then find:

(a) ${}^A\mathbf{v}^{P/O}$
(b) ${}^N\mathbf{v}^{Q/O}$
(c) ${}^A\mathbf{a}^{P/Q}$
(d) ${}^B\mathbf{a}^{P/Q}$

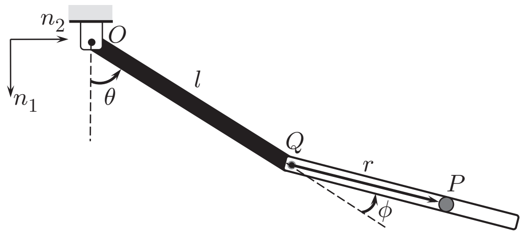


Figure 1.13 Exercise 1.6.

1.7 For the system shown in Figure 1.14, find ${}^N\mathbf{v}^{P/Q}$, ${}^N\mathbf{a}^{P/Q}$, ${}^N\boldsymbol{\alpha}^B$, and ${}^D\mathbf{a}^{P/O}$. Express all your answers in the B-frame and define your frames such that ${}^N\boldsymbol{\omega}^D = \dot{\phi}\hat{n}_2$, ${}^D\boldsymbol{\omega}^B = \dot{\theta}\hat{d}_3$.

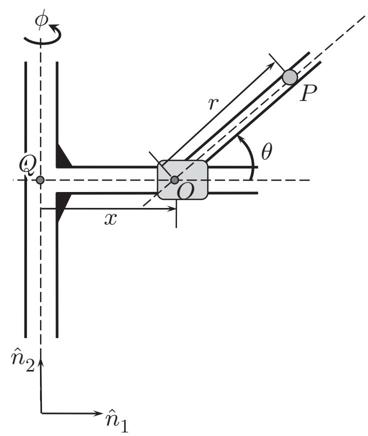


Figure 1.14 Exercise 1.7.

1.8 For the system shown in Figure 1.15, the column is forced to rotate with constant angular velocity $\dot{\phi}$. A rod of length d is welded onto one end to the column (point O) and is pinned at the other end to a rod of length l (point G). The rod, of length l, is welded to a frictionless pipe at point S. A particle P of mass m is free to slide inside the pipe. Define your coordinate system such that ${}^N\boldsymbol{\omega}^A = \dot{\phi}\,\hat{a}_2$ and ${}^A\boldsymbol{\omega}^B = \dot{\theta}\,\hat{b}_3$, then find the following:

(a) ${}^A\mathbf{v}^{P/S}$

(b) ${}^A\mathbf{a}^{P/S}$

(c) ${}^A\mathbf{a}^{P/O}$

Express all your answers in the B coordinate system.

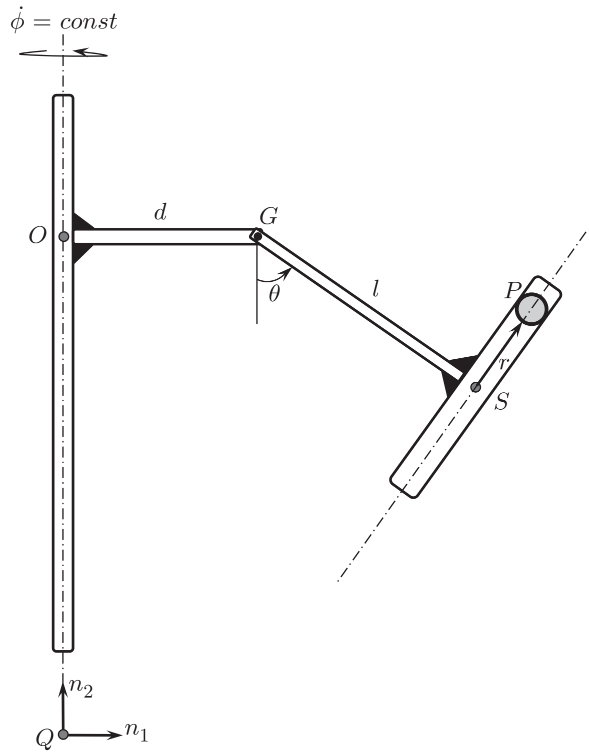


Figure 1.15 Exercise 1.8.

1.9 The bead shown in Figure 1.16 is constrained to move along the path shown, the radius of which changes with θ according to $r = \frac{a}{1+\theta}$. Obtain the velocity and acceleration of the particle with respect to point O as observed in the inertial frame. Assuming $\dot{\theta} = 2$ rad/s, what are the velocity and acceleration at $\theta = \pi/2$?

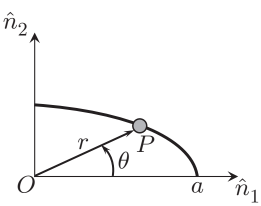


Figure 1.16 Exercise 1.9.

1.10 Two particles P_1 and P_2 are rotating with constant angular velocities on the circular paths shown. The angular velocities of P_1 and P_2 with respect to an inertial frame are $\dot{\theta}$ and $\dot{\phi}$, respectively. Find the velocity and acceleration of P_1 with respect to an observer moving with P_2.

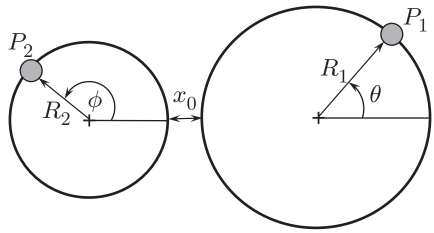


Figure 1.17 Exercise 1.10.

1.11 A particle moves on the inside surface of a cone of half angle α. The axis of the cone is vertical with the vertex pointing downwards. Find the velocity and acceleration of the particle with respect to an inertial point.

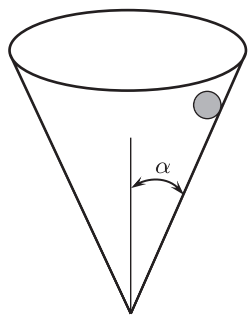


Figure 1.18 Exercise 1.11.