1

Basic Prerequisites

We shall investigate EM radiation, scattering, and reciprocity properties of antenna systems, an example of which is shown in Figure 1.1. To analyze such space–time problems, it is necessary to localize the point in the problem configuration and register the instant at which a given wave field quantity occurs in its evolution following the source excitation. To this end, we will employ the Cartesian reference frame that is defined via its basis vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ and the origin denoted by $\mathcal{O}$. Consequently, the (standard) basis allows specifying the position of an observer via the linear combination:

$$x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3 \tag{1.1}$$

where $\{x_1, x_2, x_3\}$ are the scalar and real-valued components of the position vector (i.e., spatial coordinates, in short). Such linear combinations of the Cartesian basis vectors can be represented as 1D arrays and will be further represented by boldface symbols. In particular, the position vector will be denoted by $\boldsymbol{x}$ and its components are denoted by x_k for $k \in \{1, 2, 3\}$. A natural extension in this respect is a Cartesian tensor of rank 2 that can be represented by a 2D array. Such quantities will be denoted by underlined boldface symbols. An example from the category is a quantity denoted by $\underline{\boldsymbol{\eta}}$, for instance, whose components are $\eta_{q,r}$ for $q, r \in \{1, 2, 3\}$. Consequently, we shall use the following notation for the products between arrays:

$$\boldsymbol{u} \cdot \boldsymbol{v} \quad \Leftrightarrow \quad \sum_{k=1}^{3} u_k v_k \tag{1.2}$$

$$(\underline{\boldsymbol{\zeta}} \cdot \boldsymbol{v}) \cdot \boldsymbol{i}_j \quad \Leftrightarrow \quad \sum_{k=1}^{3} \zeta_{j,k} v_k \text{ for } j \in \{1, 2, 3\} \tag{1.3}$$

Electromagnetic Reciprocity in Antenna Theory, First Edition. Martin Stumpf.

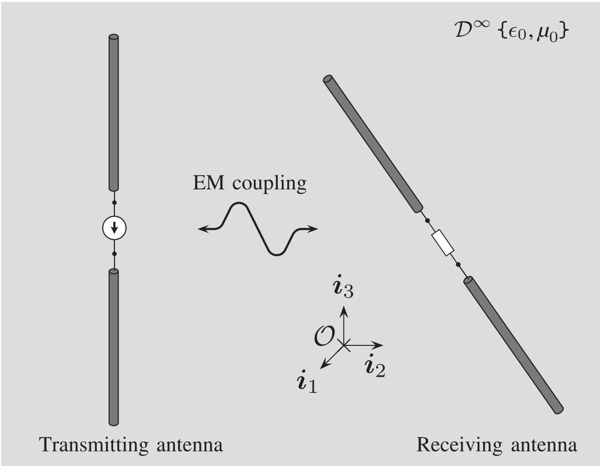

Figure 1.1 Two wire antennas.

$$\boldsymbol{u} \cdot \underline{\boldsymbol{\zeta}} \cdot \boldsymbol{v} \quad \Leftrightarrow \quad \sum_{j=1}^{3}\sum_{k=1}^{3} \zeta_{j,k} u_j v_k \tag{1.4}$$

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{i}_j \quad \Leftrightarrow \quad \sum_{k=1}^{3}\sum_{l=1}^{3} \varepsilon_{j,k,l} u_k v_l \ \text{ for } \ j \in \{1,2,3\} \tag{1.5}$$

$$\underline{\boldsymbol{\alpha}} \cdot \underline{\boldsymbol{\zeta}} \cdot \underline{\boldsymbol{\beta}} \quad \Leftrightarrow \quad \sum_{p=1}^{3}\sum_{q=1}^{3} \alpha_{k,p} \zeta_{p,q} \beta_{q,n} \ \text{ for } \ k,n \in \{1,2,3\} \tag{1.6}$$

$$\boldsymbol{u} \times \underline{\boldsymbol{\zeta}} \times \boldsymbol{v} \quad \Leftrightarrow \quad \sum_{r=1}^{N}\sum_{s=1}^{N}\sum_{q=1}^{N}\sum_{j=1}^{N} \varepsilon_{k,r,s} \varepsilon_{l,q,j} \zeta_{s,q} u_r v_j$$

$$\text{for } \ k,l \in \{1,2,3\} \tag{1.7}$$

where $\varepsilon_{j,k,l}$ is the Levi-Civita tensor (= the completely antisymmetrical unit tensor of rank 3) defined as $\varepsilon_{j,k,l} = 1$ for $\{j,k,l\}$ = even permutation of $\{1,2,3\}$, $\varepsilon_{j,k,l} = -1$ for $\{j,k,l\}$ = odd permutation of $\{1,2,3\}$, and $\varepsilon_{j,k,l} = 0$ in all other cases ([16], Sec. A.7). Finally, for a tensor of rank 2, say $\underline{\boldsymbol{\zeta}}$, the tensor transpose is denoted by $\underline{\boldsymbol{\zeta}}^{\mathcal{T}}$ and its components are found according to

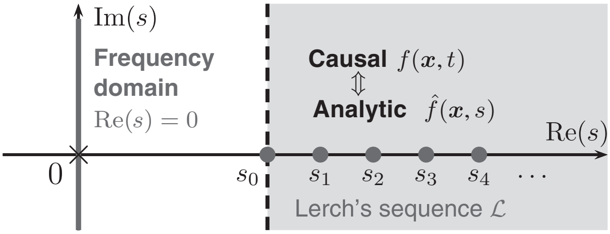

Figure 1.2 Complex frequency plane.

$$\zeta^{\mathcal{T}}_{k,l} = \zeta_{l,k} \tag{1.8}$$

for all $k, l \in \{1, 2, 3\}$.

The time coordinate is real-valued and will be denoted by t. Since we assume that the sources generating the EM wave fields are switched on at the origin $t = 0$, we will analyze the excited EM field quantities in $\{t \in \mathbb{R}; t > 0\}$ only, which is possible in virtue of the universal property of causality.

1.1 Laplace Transformation

The one-sided Laplace transformation of a wave quantity $f(\boldsymbol{x}, t)$ is defined by the following integral:

$$\hat{f}(\boldsymbol{x}, s) = \mathsf{L}\{f(\boldsymbol{x}, t)\} = \int_{t=0}^{\infty} \exp(-st)f(\boldsymbol{x}, t)\mathrm{d}t \tag{1.9}$$

In the definition of Eq. (1.9), we shall limit ourselves to physical wave quantities that are bounded such that $f(\boldsymbol{x}, t) = O[\exp(s_0 t)]$[1] with $s_0 \in \mathbb{R}$ being its exponential order . For such functions, the Laplace integral converges if s is either real-valued and positive with $s > s_0$ or complex-valued with $\mathrm{Re}(s) > s_0$. Accordingly, the right-half plane $\mathrm{Re}(s) > s_0$ is the domain of regularity of the causal wave quantity (see Figure 1.2). If Eq. (1.9) is viewed as an integral equation to be solved for the unknown function $f(\boldsymbol{x}, t)$, a natural question as to its uniqueness arises. The question has been convincingly settled by Lerch who proved that the image function known along the Lerch sequence $\mathcal{L} = \{s \in \mathbb{R}; s = s_0 + nh, h > 0, n = 1, 2, ...\}$ results in one and the same causal original function. Since single-point discontinuities are of no practical importance, the ambiguity brought about by null functions may be ignored for our

1 By $\phi(x) = O[h(x)]$, we mean that $|\phi(x)| < A|h(x)|$ for $\{A \in \mathbb{R}; A > 0\}$. In particular, $\phi(x) = O(1)$ represents a bounded function.

purposes (see Appendix A). The solution of Eq. (1.9) can be expressed via the Bromwich inversion integral:

$$f(\boldsymbol{x},t) = \frac{1}{2\pi \mathrm{i}} \int_{s\in\mathcal{B}_r} \exp(st)\hat{f}(\boldsymbol{x},s)\mathrm{d}s \tag{1.10}$$

where $\mathcal{B}_r = \{s \in \mathbb{C}; \mathrm{Re}(s) = s_0, -\infty < \mathrm{Im}(s) < \infty\}$ and we have tacitly assumed that Eq. (1.9) converges absolutely at $s = s_0$ (and hence for all $\mathrm{Re}(s) > s_0$). A special, yet frequently used, case arises for (bounded) functions of exponential order $s_0 = 0$ for which the Bromwich integral (1.10) can be rewritten using the limit $s = \delta + \mathrm{i}\omega$ as $\delta \downarrow 0$, where ω is the (real-valued) angular frequency. Under this limit, Eqs. (1.9) and (1.10) express the Fourier transform of a causal wave quantity. For more details about the Laplace transformation, we refer the reader to Refs. [16,20,47,52].

1.2 Time Convolution

The time convolution between the *causal* wave quantities $f(\boldsymbol{x},t)$ and $g(\boldsymbol{x},t)$ is defined as

$$[f * g](\boldsymbol{x},t) = \int_{\tau\in\mathbb{R}} f(\boldsymbol{x},\tau)g(\boldsymbol{x},t-\tau)\mathrm{d}\tau \tag{1.11}$$

An important property of the time-convolution operator is its commutativity allowing to rewrite the latter as

$$\begin{aligned}[g * f](\boldsymbol{x},t) &= \int_{\tau\in\mathbb{R}} f(\boldsymbol{x},t-\tau)g(\boldsymbol{x},\tau)\mathrm{d}\tau \\ &= \int_{\tau\in\mathbb{R}} f(\boldsymbol{x},\tau)g(\boldsymbol{x},t-\tau)\mathrm{d}\tau = [f * g](\boldsymbol{x},t)\end{aligned} \tag{1.12}$$

Applying the Laplace transformation to Eq. (1.11) yields

$$\mathsf{L}\{[f * g](\boldsymbol{x},t)\} = \hat{f}(\boldsymbol{x},s)\hat{g}(\boldsymbol{x},s) \tag{1.13}$$

For Eq. (1.13) to make any sense, there must be at least one value of s for which the Laplace integrals for $\hat{f}(\boldsymbol{x},s)$ and $\hat{g}(\boldsymbol{x},s)$ do converge simultaneously. If $f(\boldsymbol{x},t) = O[\exp(s_0 t)]$ and $g(\boldsymbol{x},t) = O[\exp(\sigma_0 t)]$, the region of convergence for Eq. (1.13) is found in the domain extending to the right of $\max\{s_0,\sigma_0\}$ (see Figure 1.3 for $\sigma_0 > s_0$). Consequently, the right half-plane of the complex frequency plane $\mathrm{Re}(s) > \max\{s_0,\sigma_0\}$ is the domain of regularity of the Laplace-transformed time convolution given in Eq. (1.13).

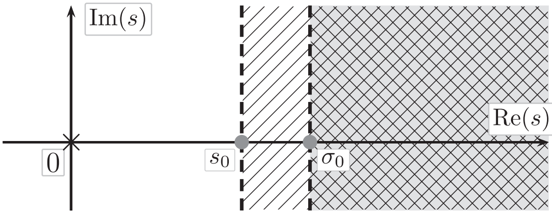

Figure 1.3 Region of convergence of $\hat{f}(\boldsymbol{x}, s)\hat{g}(\boldsymbol{x}, s)$ that is found as the intersection of $\text{Re}(s) > s_0$ and $\text{Re}(s) > \sigma_0$.

1.3 Time Correlation

The time correlation between the *causal* wave quantities $f(\boldsymbol{x}, t)$ and $g(\boldsymbol{x}, t)$ is defined as

$$[f \star g](\boldsymbol{x}, t) = \int_{\tau\in\mathbb{R}} f(\boldsymbol{x}, \tau)g(\boldsymbol{x}, \tau - t)\mathrm{d}\tau \tag{1.14}$$

In contrast to the time-convolution operator, the time-correlation one is not commutative. Indeed, Eq. (1.14) can be rewritten as

$$[f \star g](\boldsymbol{x}, t) = \int_{\tau\in\mathbb{R}} f(\boldsymbol{x}, \tau + t)g(\boldsymbol{x}, \tau)\mathrm{d}\tau \tag{1.15}$$

which implies that

$$[f \star g](\boldsymbol{x}, t) = [g \star f](\boldsymbol{x}, -t) \tag{1.16}$$

Also, by inspection of Eqs. (1.11) and (1.14), we may write

$$[f \star g](\boldsymbol{x}, t) = [f * g^{\circledast}](\boldsymbol{x}, t)\mathrm{d}\tau \tag{1.17}$$

where superscript $\circledast$ applied to a *space–time* field quantity represents the time-reversal operator, that is,

$$g^{\circledast}(\boldsymbol{x}, t) = g(\boldsymbol{x}, -t) \tag{1.18}$$

Obviously, the time-reversed causal wave quantity has its support in $\{t \in \mathbb{R}; t < 0\}$, which in accordance with Eq. (1.9) implies that the domain of convergence of its image function extends over a left half of the complex s-plane. Combination of Eqs. (1.17) and (1.18) with (1.13) and (1.9) then implies that the Laplace transformation of Eq. (1.14) can be, yet formally only, written as

$$\mathsf{L}\{[f \star g](\boldsymbol{x}, t)\} = \hat{f}(\boldsymbol{x}, s)\hat{g}^{\circledast}(\boldsymbol{x}, s) \tag{1.19}$$

where superscript $\circledast$ applied to a *complex frequency domain* field quantity represents the following operation:

$$\hat{g}^{\circledast}(\boldsymbol{x}, s) = \hat{g}(\boldsymbol{x}, -s) \tag{1.20}$$

For Eq. (1.19) to make any sense, there must be at least one value of s for which the Laplace integrals for $\hat{f}(\boldsymbol{x}, s)$ and $\hat{g}(\boldsymbol{x}, -s)$ do converge simultaneously. Again, assuming $f(\boldsymbol{x}, t) = O[\exp(s_0 t)]$ and $g(\boldsymbol{x}, t) = O[\exp(\sigma_0 t)]$, the region of convergence for Eq. (1.19) is found as the intersection of the domains of convergence corresponding to $\hat{f}(\boldsymbol{x}, s)$ and $\hat{g}(\boldsymbol{x}, -s)$. Hence, this region of convergence is at most a strip of the complex frequency plane bounded by two verticals $\{s_0 < \mathrm{Re}(s) < \sigma_0\}$ (see Figure 1.4). In this respect it should be emphasized that the vast majority of wave field quantities we will deal with are bounded functions of exponential order $s_0 = 0$. For such a class of functions, Eq. (1.19) makes a sense along the imaginary axis in the complex frequency plane only, that is, in the limiting real-frequency domain for $s = \delta + \mathrm{i}\omega$ with $\delta \downarrow 0$ and $\omega \in \mathbb{R}$. Note that in such a case, superscript $\circledast$ in Eq. (1.20) has the meaning of complex conjugate, that is, $\hat{g}^{\circledast}(\boldsymbol{x}, \mathrm{i}\omega) = \hat{g}(\boldsymbol{x}, -\mathrm{i}\omega)$ since $g(\boldsymbol{x}, t)$ is real-valued.

1.4 EM Reciprocity Theorems

In this section, a brief review concerning EM reciprocity theorems is given. In accordance with Sec. 28 of Ref. [16], we shall distinguish between the reciprocity theorems of the time-convolution and time-correlation types. The reciprocity relations will be given in the complex-frequency domain.

To find the reciprocity theorems in their generic form, we shall interrelate two states of EM fields, say A and B, that are governed by the EM field (Maxwell) equations ([16], Sec. 24.4):

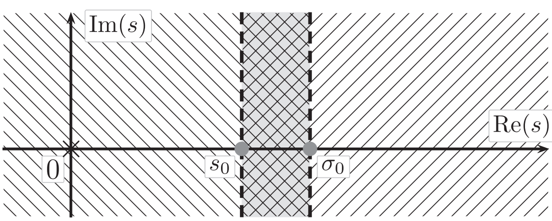

Figure 1.4 Region of convergence of $\hat{f}(\boldsymbol{x}, s)\hat{g}(\boldsymbol{x}, -s)$ that is found as the intersection of $\mathrm{Re}(s) > s_0$ and $\mathrm{Re}(s) < \sigma_0$.

$$\nabla \times \hat{\boldsymbol{H}}^{\mathrm{A,B}} - \underline{\hat{\boldsymbol{\eta}}}^{\mathrm{A,B}} \cdot \hat{\boldsymbol{E}}^{\mathrm{A,B}} = \hat{\boldsymbol{J}}^{\mathrm{A,B}} \tag{1.21}$$

$$\nabla \times \hat{\boldsymbol{E}}^{\mathrm{A,B}} + \underline{\hat{\boldsymbol{\zeta}}}^{\mathrm{A,B}} \cdot \hat{\boldsymbol{H}}^{\mathrm{A,B}} = -\hat{\boldsymbol{K}}^{\mathrm{A,B}} \tag{1.22}$$

for $\boldsymbol{x} \in \mathcal{D}$, where

- $\hat{\boldsymbol{E}}^{\mathrm{A,B}}$ = electric field strength (V/m),
- $\hat{\boldsymbol{H}}^{\mathrm{A,B}}$ = magnetic field strength (A/m),
- $\hat{\boldsymbol{J}}^{\mathrm{A,B}}$ = electric current volume density (A/m^2),
- $\hat{\boldsymbol{K}}^{\mathrm{A,B}}$ = magnetic current volume density (V/m^2),
- $\underline{\hat{\boldsymbol{\eta}}}^{\mathrm{A,B}}$ = transverse admittance per length of the medium (S/m),
- $\underline{\hat{\boldsymbol{\zeta}}}^{\mathrm{A,B}}$ = longitudinal impedance per length of the medium (Ω/m).

Equations (1.21) and (1.22) are in $\mathcal{D}$ further supplemented with the constitutive relations:

$$\underline{\hat{\boldsymbol{\eta}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) = \underline{\hat{\boldsymbol{\sigma}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) + s\underline{\hat{\boldsymbol{\epsilon}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) \tag{1.23}$$

$$\underline{\hat{\boldsymbol{\zeta}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) = s\underline{\hat{\boldsymbol{\mu}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) \tag{1.24}$$

where

- $\underline{\hat{\boldsymbol{\sigma}}}^{\mathrm{A,B}}$ = electric conductivity (S/m),
- $\underline{\hat{\boldsymbol{\epsilon}}}^{\mathrm{A,B}}$ = electric permittivity (F/m), and
- $\underline{\hat{\boldsymbol{\mu}}}^{\mathrm{A,B}}$ = magnetic permeability (H/m).

The medium described by such constitutive relations can be inhomogeneous, anisotropic, and dispersive in its EM behavior. A special yet useful case of Eqs. (1.23) and (1.24) describes an instantaneously reacting (dispersion-free) medium:

$$\underline{\hat{\boldsymbol{\eta}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) = s\underline{\boldsymbol{\epsilon}}^{\mathrm{A,B}}(\boldsymbol{x}) \tag{1.25}$$

$$\underline{\hat{\boldsymbol{\zeta}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) = s\underline{\boldsymbol{\mu}}^{\mathrm{A,B}}(\boldsymbol{x}) \tag{1.26}$$

For isotropic materials the latter relations further simplify to

$$\underline{\hat{\boldsymbol{\eta}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) = s\epsilon^{\mathrm{A,B}}(\boldsymbol{x})\underline{\boldsymbol{I}} \tag{1.27}$$

$$\underline{\hat{\boldsymbol{\zeta}}}^{\mathrm{A,B}}(\boldsymbol{x}, s) = s\mu^{\mathrm{A,B}}(\boldsymbol{x})\underline{\boldsymbol{I}} \tag{1.28}$$

where $\underline{\boldsymbol{I}}$ denotes the 3×3 identity matrix.

1.4.1 Reciprocity Theorem of the Time-Convolution Type

The reciprocity theorem of the time-convolution type is constructed from the following local interaction quantity:

$$\nabla \cdot \left[\hat{\boldsymbol{E}}^{\mathrm{A}}(\boldsymbol{x},s) \times \hat{\boldsymbol{H}}^{\mathrm{B}}(\boldsymbol{x},s) - \hat{\boldsymbol{E}}^{\mathrm{B}}(\boldsymbol{x},s) \times \hat{\boldsymbol{H}}^{\mathrm{A}}(\boldsymbol{x},s)\right] \tag{1.29}$$

for $\boldsymbol{x} \in \mathcal{D}$, that is with the help of Eqs. (1.21) and (1.22) and Gauss' theorem rearranged to its global form ([16], Eq. (28.4-7)):

$$\boxed{\begin{aligned}
&\int_{\boldsymbol{x}\in\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}} - \hat{\boldsymbol{E}}^{\mathrm{B}} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&\quad = \int_{\boldsymbol{x}\in\mathcal{D}} \Big\{ \hat{\boldsymbol{H}}^{\mathrm{A}} \cdot \left[\hat{\underline{\boldsymbol{\zeta}}}^{\mathrm{B}} - (\hat{\underline{\boldsymbol{\zeta}}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{H}}^{\mathrm{B}} \\
&\qquad\qquad - \hat{\boldsymbol{E}}^{\mathrm{A}} \cdot \left[\hat{\underline{\boldsymbol{\eta}}}^{\mathrm{B}} - (\hat{\underline{\boldsymbol{\eta}}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{E}}^{\mathrm{B}} \Big\}\mathrm{d}V \\
&\qquad\qquad + \int_{\boldsymbol{x}\in\mathcal{D}} \Big(\hat{\boldsymbol{J}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}} - \hat{\boldsymbol{K}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}} \\
&\qquad\qquad\qquad - \hat{\boldsymbol{J}}^{\mathrm{B}} \cdot \hat{\boldsymbol{E}}^{\mathrm{A}} + \hat{\boldsymbol{K}}^{\mathrm{B}} \cdot \hat{\boldsymbol{H}}^{\mathrm{A}} \Big)\mathrm{d}V \qquad (1.30)
\end{aligned}}$$

The first integral on the right-hand side represents the interaction of the field and material states. As this interaction is proportional to the contrast in the EM constitutive properties of the two states, this integral vanishes whenever

$$\hat{\underline{\boldsymbol{\eta}}}^{\mathrm{B}}(\boldsymbol{x},s) = (\hat{\underline{\boldsymbol{\eta}}}^{\mathrm{A}})^{\mathcal{T}}(\boldsymbol{x},s) \tag{1.31}$$

$$\hat{\underline{\boldsymbol{\zeta}}}^{\mathrm{B}}(\boldsymbol{x},s) = (\hat{\underline{\boldsymbol{\zeta}}}^{\mathrm{A}})^{\mathcal{T}}(\boldsymbol{x},s) \tag{1.32}$$

for all $\boldsymbol{x} \in \mathcal{D}$. In such a case, the media are denoted as each other's adjoint. Note in this respect that the latter conditions boil down to $\underline{\boldsymbol{\epsilon}}^{\mathrm{B}}(\boldsymbol{x}) = (\underline{\boldsymbol{\epsilon}}^{\mathrm{A}})^{\mathcal{T}}(\boldsymbol{x})$ and $\underline{\boldsymbol{\mu}}^{\mathrm{B}}(\boldsymbol{x}) = (\underline{\boldsymbol{\mu}}^{\mathrm{A}})^{\mathcal{T}}(\boldsymbol{x})$ throughout $\mathcal{D}$ for instantaneously reacting media described by Eqs. (1.25)–(1.26) and to $\epsilon^{\mathrm{B}}(\boldsymbol{x}) = \epsilon^{\mathrm{A}}(\boldsymbol{x})$ and $\mu^{\mathrm{B}}(\boldsymbol{x}) = \mu^{\mathrm{A}}(\boldsymbol{x})$ throughout $\mathcal{D}$ for instantaneously reacting, isotropic media described via Eqs. (1.27) and (1.28). Finally, the key ingredients constituting the time-convolution field–field, material–field, and source–field interactions in domain $\mathcal{D}$ are summarized in Table 1.1. Constitutive relations (1.25) and (1.26) describing the EM behavior of an instantaneously reacting, anisotropic, and inhomogeneous medium lead to

Table 1.1 Application of the reciprocity theorem of the time-convolution type.

Domain $\mathcal{D}$		
Time-convolution	State (A)	State (B)
Source	$\{\hat{\boldsymbol{J}}^{\mathrm{A}}, \hat{\boldsymbol{K}}^{\mathrm{A}}\}$	$\{\hat{\boldsymbol{J}}^{\mathrm{B}}, \hat{\boldsymbol{K}}^{\mathrm{B}}\}$
Field	$\{\hat{\boldsymbol{E}}^{\mathrm{A}}, \hat{\boldsymbol{H}}^{\mathrm{A}}\}$	$\{\hat{\boldsymbol{E}}^{\mathrm{B}}, \hat{\boldsymbol{H}}^{\mathrm{B}}\}$
Material	$\{\underline{\hat{\eta}}^{\mathrm{A}}, \underline{\hat{\zeta}}^{\mathrm{A}}\}$	$\{\underline{\hat{\eta}}^{\mathrm{B}}, \underline{\hat{\zeta}}^{\mathrm{B}}\}$

$$
\begin{aligned}
&\int_{\boldsymbol{x}\in\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}} - \hat{\boldsymbol{E}}^{\mathrm{B}} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&= s\int_{\boldsymbol{x}\in\mathcal{D}} \Big\{ \hat{\boldsymbol{H}}^{\mathrm{A}} \cdot \left[\underline{\boldsymbol{\mu}}^{\mathrm{B}} - (\underline{\boldsymbol{\mu}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{H}}^{\mathrm{B}} \\
&\qquad - \hat{\boldsymbol{E}}^{\mathrm{A}} \cdot \left[\underline{\boldsymbol{\epsilon}}^{\mathrm{B}} - (\underline{\boldsymbol{\epsilon}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{E}}^{\mathrm{B}} \Big\}\mathrm{d}V \\
&\qquad + \int_{\boldsymbol{x}\in\mathcal{D}} \Big(\hat{\boldsymbol{J}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}} - \hat{\boldsymbol{K}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}} \\
&\qquad\qquad - \hat{\boldsymbol{J}}^{\mathrm{B}} \cdot \hat{\boldsymbol{E}}^{\mathrm{A}} + \hat{\boldsymbol{K}}^{\mathrm{B}} \cdot \hat{\boldsymbol{H}}^{\mathrm{A}}\Big)\mathrm{d}V
\end{aligned} \tag{1.33}
$$

which can be further simplified for an isotropic medium to

$$
\begin{aligned}
&\int_{\boldsymbol{x}\in\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}} - \hat{\boldsymbol{E}}^{\mathrm{B}} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&= s\int_{\boldsymbol{x}\in\mathcal{D}} \left[\left(\mu^{\mathrm{B}} - \mu^{\mathrm{A}}\right)\hat{\boldsymbol{H}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}} - \left(\epsilon^{\mathrm{B}} - \epsilon^{\mathrm{A}}\right)\hat{\boldsymbol{E}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}}\right]\mathrm{d}V \\
&+ \int_{\boldsymbol{x}\in\mathcal{D}} \left(\hat{\boldsymbol{J}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}} - \hat{\boldsymbol{K}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}} - \hat{\boldsymbol{J}}^{\mathrm{B}} \cdot \hat{\boldsymbol{E}}^{\mathrm{A}} + \hat{\boldsymbol{K}}^{\mathrm{B}} \cdot \hat{\boldsymbol{H}}^{\mathrm{A}}\right)\mathrm{d}V
\end{aligned} \tag{1.34}
$$

in accordance with Eqs. (1.27) and (1.28). The literature on the subject is frequently limited to the simplest form of the reciprocity theorem of the time-convolution type (e.g., Refs. [27], Sec. 3.8; [29], Sec. 2.11; and [30], Sec. 5.5).

1.4.2 Reciprocity Theorem of the Time-Correlation Type

The reciprocity theorem of the time-correlation type is constructed from the following local interaction quantity:

$$
\begin{aligned}
&\boldsymbol{\nabla} \cdot \left[\hat{\boldsymbol{E}}^{\mathrm{A}}(\boldsymbol{x}, s) \times \hat{\boldsymbol{H}}^{\mathrm{B}\circledast}(\boldsymbol{x}, s) + \hat{\boldsymbol{E}}^{\mathrm{B}\circledast}(\boldsymbol{x}, s) \times \hat{\boldsymbol{H}}^{\mathrm{A}}(\boldsymbol{x}, s)\right] \\
&= \boldsymbol{\nabla} \cdot \left[\hat{\boldsymbol{E}}^{\mathrm{A}}(\boldsymbol{x}, s) \times \hat{\boldsymbol{H}}^{\mathrm{B}}(\boldsymbol{x}, -s) + \hat{\boldsymbol{E}}^{\mathrm{B}}(\boldsymbol{x}, -s) \times \hat{\boldsymbol{H}}^{\mathrm{A}}(\boldsymbol{x}, s)\right]
\end{aligned} \tag{1.35}
$$

for $\boldsymbol{x} \in \mathcal{D}$, that is with the help of Eqs. (1.21) and (1.22) and Gauss' theorem rearranged to its global form ([16], Eq. (28.5-7)):

$$
\boxed{
\begin{aligned}
&\int_{\boldsymbol{x}\in\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&\quad = -\int_{\boldsymbol{x}\in\mathcal{D}} \Big\{ \hat{\boldsymbol{H}}^{\mathrm{A}} \cdot \left[\underline{\hat{\zeta}}^{\mathrm{B}\circledast} + (\underline{\hat{\zeta}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} \\
&\qquad\qquad + \hat{\boldsymbol{E}}^{\mathrm{A}} \cdot \left[\underline{\hat{\eta}}^{\mathrm{B}\circledast} + (\underline{\hat{\eta}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \Big\}\mathrm{d}V \\
&\qquad\qquad - \int_{\boldsymbol{x}\in\mathcal{D}} \Big(\hat{\boldsymbol{J}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{K}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} \\
&\qquad\qquad\qquad + \hat{\boldsymbol{J}}^{\mathrm{B}\circledast} \cdot \hat{\boldsymbol{E}}^{\mathrm{A}} + \hat{\boldsymbol{K}}^{\mathrm{B}\circledast} \cdot \hat{\boldsymbol{H}}^{\mathrm{A}}\Big)\mathrm{d}V \qquad (1.36)
\end{aligned}
}
$$

The first integral on the right hand-side represents the interaction of the field and material states. As this interaction is proportional to the contrast in the EM constitutive properties of the two states, this integral vanishes whenever

$$\underline{\hat{\eta}}^{\mathrm{B}}(\boldsymbol{x}, -s) = -(\underline{\hat{\eta}}^{\mathrm{A}})^{\mathcal{T}}(\boldsymbol{x}, s) \tag{1.37}$$

$$\underline{\hat{\zeta}}^{\mathrm{B}}(\boldsymbol{x}, -s) = -(\underline{\hat{\zeta}}^{\mathrm{A}})^{\mathcal{T}}(\boldsymbol{x}, s) \tag{1.38}$$

for all $\boldsymbol{x} \in \mathcal{D}$. In such a case, the media are denoted as each other's time-reverse adjoint. Note in this respect that for instantaneously-reacting media described by Eqs. (1.25)–(1.28), the latter conditions have the same form as the one applying to adjoint media (see Eqs. (1.39) and (1.40)). Finally, the key ingredients constituting the time-correlation field-field, material–field, and source–field interactions in domain $\mathcal{D}$ are summarized in Table 1.2.

Table 1.2 Application of the reciprocity theorem of the time-correlation type.

Domain $\mathcal{D}$		
Time-correlation	State (A)	State (B)
Source	$\{\hat{\boldsymbol{J}}^{\mathrm{A}}, \hat{\boldsymbol{K}}^{\mathrm{A}}\}$	$\{\hat{\boldsymbol{J}}^{\mathrm{B}}, \hat{\boldsymbol{K}}^{\mathrm{B}}\}$
Field	$\{\hat{\boldsymbol{E}}^{\mathrm{A}}, \hat{\boldsymbol{H}}^{\mathrm{A}}\}$	$\{\hat{\boldsymbol{E}}^{\mathrm{B}}, \hat{\boldsymbol{H}}^{\mathrm{B}}\}$
Material	$\{\underline{\hat{\eta}}^{\mathrm{A}}, \underline{\hat{\zeta}}^{\mathrm{A}}\}$	$\{\underline{\hat{\eta}}^{\mathrm{B}}, \underline{\hat{\zeta}}^{\mathrm{B}}\}$

Constitutive relations (1.25) and (1.26) describing the EM behavior of an instantaneously reacting, anisotropic, and inhomogeneous medium leads to

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&\quad = s\int_{\boldsymbol{x}\in\mathcal{D}} \Big\{ \hat{\boldsymbol{H}}^{\mathrm{A}} \cdot \left[\underline{\boldsymbol{\mu}}^{\mathrm{B}} - (\underline{\boldsymbol{\mu}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} \\
&\qquad\qquad + \hat{\boldsymbol{E}}^{\mathrm{A}} \cdot \left[\underline{\boldsymbol{\epsilon}}^{\mathrm{B}} - (\underline{\boldsymbol{\epsilon}}^{\mathrm{A}})^{\mathcal{T}}\right] \cdot \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \Big\}\mathrm{d}V \\
&\qquad\qquad - \int_{\boldsymbol{x}\in\mathcal{D}} \Big(\hat{\boldsymbol{J}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{K}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} \\
&\qquad\qquad\qquad + \hat{\boldsymbol{J}}^{\mathrm{B}\circledast} \cdot \hat{\boldsymbol{E}}^{\mathrm{A}} + \hat{\boldsymbol{K}}^{\mathrm{B}\circledast} \cdot \hat{\boldsymbol{H}}^{\mathrm{A}} \Big)\mathrm{d}V
\end{aligned} \tag{1.39}$$

which can be further simplified for an isotropic medium to

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\partial\mathcal{D}} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&= s\int_{\boldsymbol{x}\in\mathcal{D}} \left\{ \left(\mu^{\mathrm{B}} - \mu^{\mathrm{A}}\right)\hat{\boldsymbol{H}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} + \left(\epsilon^{\mathrm{B}} - \epsilon^{\mathrm{A}}\right)\hat{\boldsymbol{E}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \right\}\mathrm{d}V \\
&- \int_{\boldsymbol{x}\in\mathcal{D}} \left(\hat{\boldsymbol{J}}^{\mathrm{A}} \cdot \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{K}}^{\mathrm{A}} \cdot \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{J}}^{\mathrm{B}\circledast} \cdot \hat{\boldsymbol{E}}^{\mathrm{A}} + \hat{\boldsymbol{K}}^{\mathrm{B}\circledast} \cdot \hat{\boldsymbol{H}}^{\mathrm{A}} \right)\mathrm{d}V
\end{aligned} \tag{1.40}$$

in accordance with Eqs. (1.27) and (1.28). The importance of the nature of the temporal behavior of the interacting wave field quantities has been stressed by Bojarski [3] who has introduced the time-convolution-and time-correlation-type reciprocity theorems applying to homogeneous, isotropic, and lossless media. This has been later clearly unified by de Hoop [14] who has further generalized the reciprocity theorems by including general inhomogeneous, anisotropic, and dispersive media.

1.4.3 Application of the Reciprocity Theorems to an Unbounded Domain

Whenever a reciprocity theorem is applied to an unbounded domain exterior to an antenna system, the surface integrals that appear in Eqs. (1.30) and (1.36) will be carried out over the outer bounding surface $\partial\mathcal{D}^{\Delta}$ under the limit $\Delta \to \infty$ (see Figure 1.5). To evaluate this contribution for *causal* EM field states, we observe that the EM wave field radiated into the homogeneous, isotropic embedding $\mathcal{D}^{\infty}$, whose EM properties are described by (real-valued and positive) electric permittivity ϵ_0 and magnetic permeability μ_0, has the form of a spherical wave

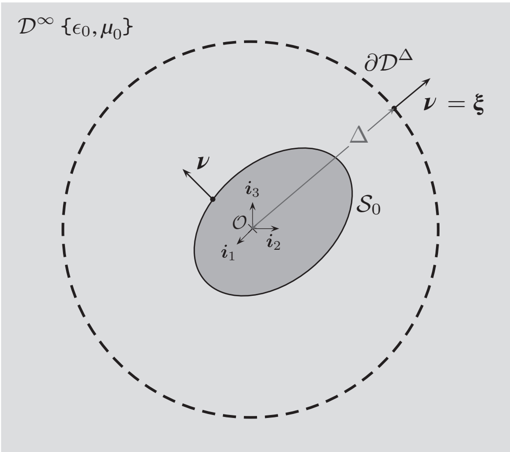

Figure 1.5 Unbounded domain to which the reciprocity theorems are applied.

expanding away from the origin ([16], Sec. 26.11):

$$\{\hat{\boldsymbol{E}}, \hat{\boldsymbol{H}}\}(\boldsymbol{x}, s) = \{\hat{\boldsymbol{E}}^\infty, \hat{\boldsymbol{H}}^\infty\}(\boldsymbol{\xi}, s) \exp(-s|\boldsymbol{x}|/c_0)(4\pi|\boldsymbol{x}|)^{-1} \times \left[1 + O(|\boldsymbol{x}|^{-1})\right] \tag{1.41}$$

as $|\boldsymbol{x}| \to \infty$, where $\{\hat{\boldsymbol{E}}^\infty, \hat{\boldsymbol{H}}^\infty\}$ are the electric- and magnetic-type amplitude radiation characteristics, $\boldsymbol{\xi} = \boldsymbol{x}/|\boldsymbol{x}|$ is the unit vector in the direction of observation, and $c_0 = (\epsilon_0\mu_0)^{-1/2} > 0$ is the EM wave speed. Now, making use of the far-field behavior (1.41) in the surface integral of the time-convolution type, we get

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}^\Delta} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}} - \hat{\boldsymbol{E}}^{\mathrm{B}} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A = O(\Delta^{-1}) \tag{1.42}$$

as $\Delta \to \infty$, since the leading terms in the integrand that are of order Δ^{-2} cancel each other. Hence, owing to causality of the interrelated EM wave fields, the time-convolution-type surface integral over $\partial\mathcal{D}^\Delta$ vanishes as $\Delta \to \infty$. On the other hand, the time-correlation type of the surface interaction integral leads to a non-vanishing contribution:

$$\int_{\boldsymbol{x}\in\partial\mathcal{D}^\Delta} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A = \left(\eta_0/8\pi^2\right) \int_{\boldsymbol{\xi}\in\Omega} \hat{\boldsymbol{E}}^{\mathrm{A};\infty}(\boldsymbol{\xi}, s) \cdot \hat{\boldsymbol{E}}^{\mathrm{B};\infty}(\boldsymbol{\xi}, -s)\mathrm{d}\Omega \times \left[1 + O(\Delta^{-1})\right] \tag{1.43}$$

as $\Delta \to \infty$, where the integration on the right-hand side is carried out over $\Omega = \{\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 1\}$ defining a unit sphere. Consequently, if the medium exterior to $\mathcal{S}_0$ is source-free, we may, in view of its self-adjointness, write

$$\int_{\boldsymbol{x}\in\mathcal{S}_0} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}} - \hat{\boldsymbol{E}}^{\mathrm{B}} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A = 0 \tag{1.44}$$

$$\begin{aligned} &\int_{\boldsymbol{x}\in\mathcal{S}_0} \left(\hat{\boldsymbol{E}}^{\mathrm{A}} \times \hat{\boldsymbol{H}}^{\mathrm{B}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{B}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{A}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\ &= \left(\eta_0/8\pi^2\right) \int_{\boldsymbol{\xi}\in\Omega} \hat{\boldsymbol{E}}^{\mathrm{A};\infty}(\boldsymbol{\xi}, s) \cdot \hat{\boldsymbol{E}}^{\mathrm{B};\infty}(\boldsymbol{\xi}, -s)\mathrm{d}\Omega \\ &= \left(8\pi^2\eta_0\right)^{-1} \int_{\boldsymbol{\xi}\in\Omega} \hat{\boldsymbol{H}}^{\mathrm{A};\infty}(\boldsymbol{\xi}, s) \cdot \hat{\boldsymbol{H}}^{\mathrm{B};\infty}(\boldsymbol{\xi}, -s)\mathrm{d}\Omega \end{aligned} \tag{1.45}$$

where we have tacitly taken the limit $\Delta \to \infty$ (see Figure 1.5). In conclusion, the surface integral contribution from the outer bounding surface $\partial\mathcal{D}^{\Delta}$ is vanishing only for the *time-convolution* interaction of two *causal* wave fields. The time-correlation interaction results in the nonzero contribution (1.43) that for the lossless embedding described by two positive scalar constants $\{\epsilon_0, \mu_0\}$ should be approached via the real-frequency domain (see Section 1.3).

1.5 Description of the Antenna Configuration

We will not limit our further analysis to a particular antenna geometry; instead, we will take the advantage of the generic antenna model introduced in Ref. [12], which encompasses all EM antennas used in practice (see Figure 1.6). The antenna system occupies a bounded domain $\mathcal{A} \subset \mathbb{R}^3$ that is terminated by surfaces $\mathcal{S}_0$ and $\mathcal{S}_1$. Surface $\mathcal{S}_0$ separates the antenna system from the exterior domain denoted by $\mathcal{D}^{\infty}$, while surface $\mathcal{S}_1$ represents the terminal surface where the antenna system is accessible via its N-ports. The maximum diameter of the domain enclosed by $\mathcal{S}_1$ is supposed to be small with respect to the pulse time width of the excited EM wave fields. The bounding surfaces $\mathcal{S}_0$ and $\mathcal{S}_1$ may partially overlap.

The antenna system consists of a linear and passive media, whose EM properties can be described by the (tensorial) transverse admittance and the longitudinal impedance, $\underline{\hat{\boldsymbol{\eta}}} = \underline{\hat{\boldsymbol{\eta}}}(\boldsymbol{x}, s)$ and $\underline{\hat{\boldsymbol{\zeta}}} = \underline{\hat{\boldsymbol{\zeta}}}(\boldsymbol{x}, s)$, respectively (see Eqs. (1.23) and (1.24)). These constitutive functions are piecewise continuous functions with respect to the position vector $\boldsymbol{x}$, that is, they may show finite-jump discontinuities across bounded interfaces, and, in view of the uniqueness theorem given in Chapter 2, they are positive definite tensors of rank 2 for all real and

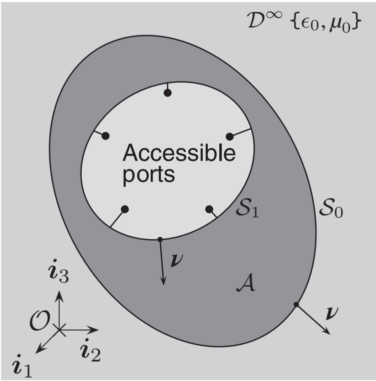

Figure 1.6 Generic antenna configuration.

positive values of s. The antenna system may also contain perfectly conducting surfaces. The antenna structure itself is placed in the linear, homogeneous, and isotropic embedding $\mathcal{D}^\infty$, whose EM properties are defined via its (real-valued and positive) electric permittivity ϵ_0 and magnetic permeability μ_0.

1.5.1 Antenna Power Conservation

The power-reciprocity theorem in antenna theory [18] calls for the application of the reciprocity theorem of the time-correlation type to the domain occupied by the antenna system according to Table 1.3. Taking into account the orientation of the outer normal vector along the bounding surfaces, one may arrive at

Table 1.3 Application of the reciprocity theorem of the time-correlation type

Domain $\mathcal{A}$		
Time-correlation	State (A)	State (A)
Source	0	0
Field	$\{\hat{\boldsymbol{E}}^{\mathrm{A}}, \hat{\boldsymbol{H}}^{\mathrm{A}}\}$	$\{\hat{\boldsymbol{E}}^{\mathrm{A}}, \hat{\boldsymbol{H}}^{\mathrm{A}}\}$
Material	$\{\underline{\hat{\eta}}, \underline{\hat{\zeta}}\}$	$\{\underline{\hat{\eta}}, \underline{\hat{\zeta}}\}$

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{S}_0}\left(\hat{\boldsymbol{E}}^{\mathrm{A}}\times\hat{\boldsymbol{H}}^{\mathrm{A}\circledast}+\hat{\boldsymbol{E}}^{\mathrm{A}\circledast}\times\hat{\boldsymbol{H}}^{\mathrm{A}}\right)\cdot\boldsymbol{\nu}\,\mathrm{d}A\\
&=\int_{\boldsymbol{x}\in\mathcal{S}_1}\left(\hat{\boldsymbol{E}}^{\mathrm{A}}\times\hat{\boldsymbol{H}}^{\mathrm{A}\circledast}+\hat{\boldsymbol{E}}^{\mathrm{A}\circledast}\times\hat{\boldsymbol{H}}^{\mathrm{A}}\right)\cdot\boldsymbol{\nu}\,\mathrm{d}A\\
&\qquad-\int_{\boldsymbol{x}\in\mathcal{A}}\left\{\hat{\boldsymbol{H}}^{\mathrm{A}}\cdot\left[\underline{\hat{\boldsymbol{\zeta}}}^{\circledast}+\underline{\hat{\boldsymbol{\zeta}}}^{\mathcal{T}}\right]\cdot\hat{\boldsymbol{H}}^{\mathrm{A}\circledast}\right.\\
&\qquad\qquad\left.+\hat{\boldsymbol{E}}^{\mathrm{A}}\cdot\left[\underline{\hat{\boldsymbol{\eta}}}^{\circledast}+\underline{\hat{\boldsymbol{\eta}}}^{\mathcal{T}}\right]\cdot\hat{\boldsymbol{E}}^{\mathrm{A}\circledast}\right\}\mathrm{d}V
\end{aligned}\tag{1.46}$$

In case that antenna's losses can be entirely included in the electric-conduction relaxation function, while its (s-independent) electric permittivity and magnetic permeability relaxation functions are symmetrical tensors of rank 2, Eq. (1.46) has the following form:

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{S}_0}\left(\hat{\boldsymbol{E}}^{\mathrm{A}}\times\hat{\boldsymbol{H}}^{\mathrm{A}\circledast}+\hat{\boldsymbol{E}}^{\mathrm{A}\circledast}\times\hat{\boldsymbol{H}}^{\mathrm{A}}\right)\cdot\boldsymbol{\nu}\,\mathrm{d}A\\
&=\int_{\boldsymbol{x}\in\mathcal{S}_1}\left(\hat{\boldsymbol{E}}^{\mathrm{A}}\times\hat{\boldsymbol{H}}^{\mathrm{A}\circledast}+\hat{\boldsymbol{E}}^{\mathrm{A}\circledast}\times\hat{\boldsymbol{H}}^{\mathrm{A}}\right)\cdot\boldsymbol{\nu}\,\mathrm{d}A\\
&\qquad-\int_{\boldsymbol{x}\in\mathcal{A}}\left\{\hat{\boldsymbol{E}}^{\mathrm{A}}\cdot\left[\underline{\hat{\boldsymbol{\sigma}}}^{\circledast}+\underline{\hat{\boldsymbol{\sigma}}}^{\mathcal{T}}\right]\cdot\hat{\boldsymbol{E}}^{\mathrm{A}\circledast}\right\}\mathrm{d}V
\end{aligned}\tag{1.47}$$

No matter whether the antenna system operates in the transmitting or receiving state, the volume integrals in Eqs. (1.46) and (1.47) are for $\{s=\delta+\mathrm{i}\omega,\delta\downarrow 0,\omega\in\mathbb{R}\}$ proportional to the (time-averaged) power dissipated in between the exterior bounding surface $\mathcal{S}_0$ and the interface surface $\mathcal{S}_1$. In line with Eqs. (1.37) and (1.38), this contribution vanishes whenever the medium in $\mathcal{A}$ is time-reverse self-adjoint in its EM behavior. This includes, in particular, the instantaneously reacting media whose electrical permittivity and magnetic permeability are symmetrical tensors of rank 2, specifically

$$\underline{\boldsymbol{\epsilon}}(\boldsymbol{x})=\underline{\boldsymbol{\epsilon}}^{\mathcal{T}}(\boldsymbol{x})\tag{1.48}$$

$$\underline{\boldsymbol{\mu}}(\boldsymbol{x})=\underline{\boldsymbol{\mu}}^{\mathcal{T}}(\boldsymbol{x})\tag{1.49}$$

for all $\boldsymbol{x}\in\mathcal{A}$, as well as (idealized) perfectly electrically conducting (PEC) antenna models. An important class of antennas in this category is represented by a wire antenna. For this antenna system, the external surface $\mathcal{S}_0$ is formed by a closed cylindrical surface closely surrounding the PEC arms of the wire antenna including the (vanishing) volume of the excitation gap. The latter is enclosed by the cylindrical surface $\mathcal{S}_1$ that is at its ends crossed by the antenna ports. Obviously, in this antenna configuration, the terminal surface $\mathcal{S}_1$ partially overlaps with $\mathcal{S}_0$ (see Figure 1.7).

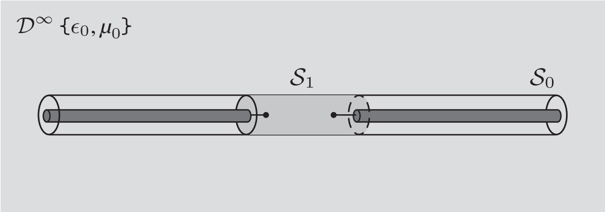

Figure 1.7 Wire antenna and its bounding surfaces.

1.5.2 Antenna Interface Relations

The EM wave quantities on the terminal surface $\mathcal{S}_1$ of the antenna system can be expressed in terms of its Kirchhoff-type quantities (see Ref. [16], Sec. 30.1). To illustrate the procedure that leads to such a relation, we shall associate state (A) from the previous section with the receiving situation (R) in which the antenna system is externally irradiated by the incident EM wave fields. Since the maximum diameter of the domain enclosed by $\mathcal{S}_1$ is small with respect to the pulse time width of the excited fields, the electric field strength $\hat{\boldsymbol{E}}^{\mathrm{R}}$ can be expressed as (the opposite of) the gradient of the scalar potential $\hat{\phi}^{\mathrm{R}}$. Consequently, the surface integral over $\mathcal{S}_1$ from Eq. (1.46) can be written as

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{S}_1} \left(\hat{\boldsymbol{E}}^{\mathrm{R}} \times \hat{\boldsymbol{H}}^{\mathrm{R}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{R}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{R}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&\simeq -\int_{\boldsymbol{x}\in\mathcal{S}_1} \left(\nabla\hat{\phi}^{\mathrm{R}} \times \hat{\boldsymbol{H}}^{\mathrm{R}\circledast} + \nabla\hat{\phi}^{\mathrm{R}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{R}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&= -\int_{\boldsymbol{x}\in\mathcal{S}_1} \left[\nabla \times \left(\hat{\phi}^{\mathrm{R}}\hat{\boldsymbol{H}}^{\mathrm{R}\circledast}\right) + \nabla \times \left(\hat{\phi}^{\mathrm{R}\circledast}\hat{\boldsymbol{H}}^{\mathrm{R}}\right)\right] \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&\quad + \int_{\boldsymbol{x}\in\mathcal{S}_1} \left[\hat{\phi}^{\mathrm{R}}\left(\nabla \times \hat{\boldsymbol{H}}^{\mathrm{R}\circledast}\right) + \hat{\phi}^{\mathrm{R}\circledast}\left(\nabla \times \hat{\boldsymbol{H}}^{\mathrm{R}}\right)\right] \cdot \boldsymbol{\nu}\,\mathrm{d}A
\end{aligned} \tag{1.50}$$

where we have used integration by parts and $\simeq$ indicates here the low-frequency approximation. Since the first integral on the right-hand side is in view of Stokes' theorem zero, we may, upon using the first Maxwell's equation (see Eq. (1.21)) in the second one, write

$$\begin{aligned}
&\int_{\boldsymbol{x}\in\mathcal{S}_1} \left(\hat{\boldsymbol{E}}^{\mathrm{R}} \times \hat{\boldsymbol{H}}^{\mathrm{R}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{R}\circledast} \times \hat{\boldsymbol{H}}^{\mathrm{R}}\right) \cdot \boldsymbol{\nu}\,\mathrm{d}A \\
&\qquad \simeq \int_{\boldsymbol{x}\in\mathcal{S}_1} \left(\hat{\phi}^{\mathrm{R}}\,\boldsymbol{\nu} \cdot \hat{\boldsymbol{J}}^{\mathrm{R}\circledast} + \hat{\phi}^{\mathrm{R}\circledast}\,\boldsymbol{\nu} \cdot \hat{\boldsymbol{J}}^{\mathrm{R}}\right)\mathrm{d}A \\
&\qquad\qquad \simeq -\sum_{n=1}^{N} \left[\hat{V}_n^{\mathrm{R}}(s)\hat{I}_n^{\mathrm{R}}(-s) + \hat{V}_n^{\mathrm{R}}(-s)\hat{I}_n^{\mathrm{R}}(s)\right]
\end{aligned} \tag{1.51}$$

where we have used the fact that the electric current volume density on $\mathcal{S}_1$ is dominated by the conduction current flowing in the perfect conductors of the N-port termination and that these electric currents are in the receiving state oriented into the load. Furthermore, in the interior of the terminal surface $\mathcal{S}_1$, we have chosen a reference point where the scalar potential $\hat{\phi}^{\mathrm{R}}$ has the value zero. Consequently, the scalar potential at the nth PEC port, that is related to this reference point, is denoted as $\hat{V}_n^{\mathrm{R}}$. Employing these results, the time-averaged power absorbed by the antenna load can be written as

$$\begin{aligned}\hat{P}^{\mathrm{L}}(s) &= \tfrac{1}{4}\sum_{n=1}^{N}\left[\hat{V}_n^{\mathrm{R}}(s)\hat{I}_n^{\mathrm{R}}(-s) + \hat{V}_n^{\mathrm{R}}(-s)\hat{I}_n^{\mathrm{R}}(s)\right] \\ &\simeq -\tfrac{1}{4}\int_{\boldsymbol{x}\in\mathcal{S}_1}\left(\hat{\boldsymbol{E}}^{\mathrm{R}}\times\hat{\boldsymbol{H}}^{\mathrm{R}\circledast} + \hat{\boldsymbol{E}}^{\mathrm{R}\circledast}\times\hat{\boldsymbol{H}}^{\mathrm{R}}\right)\cdot\boldsymbol{\nu}\,\mathrm{d}A\end{aligned} \tag{1.52}$$

for $\{s = \delta + \mathrm{i}\omega, \delta \downarrow 0, \omega \in \mathbb{R}\}$. Equation (1.52) thus makes possible to clearly interpret the integral over $\mathcal{S}_1$ in the antenna power conservation relations (1.46) and (1.47) in the receiving (R) state.