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Uniform Plane Wave

1.1 Introduction to Uniform Plane Wave

With the emergence of new wireless technologies, electromagnetic wave propagation in nonconventional media has forced to revisit the classical electromagnetic theories in dynamic conditions. In this chapter, a few very important classical electromagnetic theories are examined in the light of wave propagations in time varying cases. The emerging applications are RF/microwave/millimeter-wave printed electronics on nonconventional materials, and study of propagation of electromagnetic waves in very complex composite media. The chapter first defines the uniform plane wave in the light of far field radiation from a dipole antenna followed by the uniform plane wave concept. Derivation of electromagnetic wave equations with the aid of Maxwell's equations and one-dimensional wave equation are presented next. The wave is characterized with its phase velocity meaning the velocity of wave propagation and the wave impedance, which is the function of the constitutive parameters of the medium. The above derivations are extended to the time harmonic field wave equation and the refractive index of medium and dispersion of medium. The time harmonic wave solution and polarization of the uniform plane wave are presented next. Finally, the energy balance equation which is called Poynting theorem is presented in detail for a few special cases followed by the conclusion of the chapter.

Maxwell unified all classical electromagnetic principles and laws into four potent equations. These equations are so universal that they can be used to characterize electromagnetic field wave equations for any frequency, for the static as well as the dynamic field conditions. In this regard, an electromagnetic field can be defined as: (i) every function that satisfies Maxwell's equations, which is finite, continuous,¹ and single valued, and (ii) propagates through homogeneous, linear, and isotropic medium. In this chapter, we shall develop the uniform plane wave concept of the electromagnetic fields using Maxwell's equations for any frequency in a medium as stated above. In this chapter, we examine this aspect in depth.

In solving the field wave equations, we need to deal with the complex phasor vector field quantities which are the functions of both space and time in double derivatives. In these situations, these vector functions are not easily solvable because of the following reasons: (i) no field solution is possible to get directly from scalar functions. Here, the functions are dependent on time and spatial coordinates, and (ii) the complexity of analyses of the vector functions increases as we move from one coordinate system to another coordinate system. As for example, the complexity of vector analysis of the electromagnetic fields increases as we move from the rectangular

1 Recall the boundary conditions that we have derived in the previous chapters for the dynamic fields.

coordinate system to the cylindrical coordinate system to the spherical coordinate system. There are also other seven coordinate systems in the vector analysis with which the electromagnetic fields are resolvable, but we only confine ourselves in the three main coordinate systems. Also note that electromagnetic field propagation in bounded and unbounded media behave differently. In unbounded medium, the antenna radiates spherically, whereas in bounded structure such as rectangular metallic waveguide we solve electromagnetic field problems with the rectangular wave equations. In optical fibers, we analyze the electromagnetic fields using cylindrical coordinate systems. With the simplest one, the rectangular coordinate system, we shall develop the electromagnetic field wave theory in this coordinate system in the chapter. Then gradually we define field solutions for more complex coordinate systems in cylindrical and spherical coordinate systems. (iii) It is also not possible to solve 3D scalar wave equations in arbitrary coordinate systems by the use of separation of variable method.² That is why, we keep our wave field analysis into simple configurations such as rectangular, circular, and spherical coordinate systems.³

Figure 1.2 illustrates how wireless communications networks established by the time varying electromagnetic field waves harness each and every device in the system for efficient wireless communications. As we have seen in the big picture of wireless communications shown in Figure 1.1, the wireless channel is established with the wave propagation. In this chapter, we shall study more

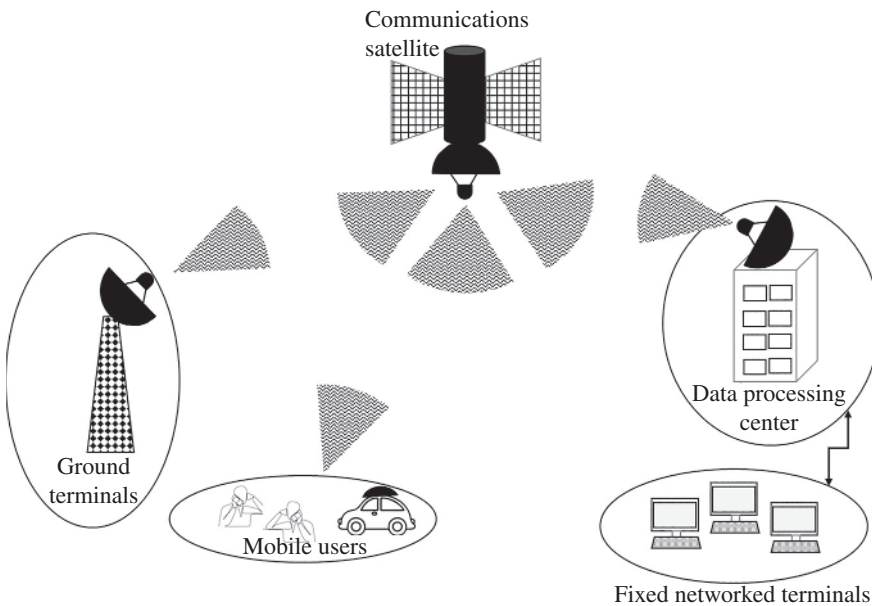


Figure 1.1 Wireless communications networks showing how the electromagnetic waves harness each and every device in the system for efficient wireless communications.

² It is a method to solve a vector field which is a function of multiple independent variables. In this method we derive the original function as the product of two functions of individual independent variables and then we solve the function one by one. Finally, we multiply the individual solutions to get the final solution.

³ Islam, M.A. (1969). *Electromagnetic Theory*. Dacca, East Pakistan: EPUET.

details of the wireless channel augmented with the theory of uniform plane wave propagation. A brief description of the topics covered in the chapter are as follows:

i) Definition of Uniform Plane Waves

In the definition of uniform plane waves in this chapter, the very beginning of the book, we already have given a very good understanding of the uniform plane wave. We have learned that in a uniform plane wave, the electric field \mathbf{E} , the magnetic field \mathbf{H} and, the Poynting vector $\mathbf{P} = \mathbf{E} \times \mathbf{H}$, which is also called the power flow density vector in (W/m^2), are orthogonal to each other, and the wave propagates orthogonally with respect to the directions of the two fields.

ii) Derivation of Field Wave Equations

After the proper definition of the uniform plane wave, we shall derive the electromagnetic plane wave equations. As we know, in a uniform plane wave the phase front is like a flat surface on which the phase of the signal is the same. From the derivation of the field wave equations, we shall now learn how to derive electromagnetic wave equations using simple differential equations. Uniform plane wave equation is a double derivative equation of both spatial and time coordinates. We shall see that the vector wave equation has two solutions, forward and reverse traveling wave fields.

iii) Wave Impedance

Every plane wave has its intrinsic wave impedance which is defined as the ratio of electric field and the magnetic field intensities at any point in space. The wave impedance is a function of the constitutive parameters such as medium permittivity ϵ (F/m), permeability μ (H/m), finite conductivity σ (S/m), and angular frequency ω (rad/m). We shall define the intrinsic wave impedance of the lossless and lossy media in the chapter. You may recall the wave impedance of the plane wave in free space is 377Ω .

iv) Polarizations of Uniform Plane Waves

The polarizations of uniform plane waves is a significant parameter that regulates the efficacy of electromagnetic communications. The polarization of the field wave is the spatial orientation of the electric field vector with respect to time. We shall learn more in details of various polarizations of the field waves and their significance in the wireless communications.

v) Poynting Theorem

And finally, Poynting Theorem, that tells us the power flow density vector $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ (w/m^2). Poynting Theorem is an energy balance equation that tells us that in a homogeneous, linear, and isotropic medium, the power input is equal to the electric and magnetic energy stored plus the power dissipated as the Ohmic loss as it propagates real power as the plane wave. The time average power is defined as

$$\mathbf{P}_{ave} = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*) \quad (1.1)$$

Finally, in this chapter, we shall solve many interesting problems as we go along with the theories on the uniform plane wave.

This chapter provides the very important concept of medium properties derived from the wave equation. There, we shall define the attenuation and phase constants, and the loss tangent for lossy medium. In defining those parameters, we shall go in deeper in defining the propagation of wave in good conductors and good dielectrics. These concepts are critically important in designing any practical RF and microwave circuits.

The aim of this chapter is to develop the concept of electromagnetic waves, and how they propagate in free space and in different complex media. It helps readers to understand the meaning of the polarization of plane waves and how introduction of a metallic or dielectric slab in their

propagation paths affects their behaviors.⁴ The reader will also obtain the concept of 3D and 2D vector plots of electromagnetic field distributions in various conditions via some illustrations generated with Computer Simulation Technologies (CST) Microwave Studio™. A few practical examples and worked out problems of the plane wave propagation is also provided to enhance understanding in a real-world scenario.

In this section, we investigate the different aspects of plane waves: how they look like, how they behave when encountered with different obstacles such as buildings, walls inside a room, on the roads, and in the free space when wireless propagations happen. Recall Maxwell's two curl equations, Faraday's and Ampere's laws that tell us that for a plane wave to be generated, a source of time varying electromagnetic fields must exist. According to Ampere's circuital law:

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \frac{\partial \mathbf{D}}{\partial t} \quad (1.2)$$

In other words, if the electric field, $\mathbf{E}(t)$, varies at a point in space resulting in a time-varying electric flux density, $\mathbf{D}(t)$, it produces a time-varying magnetic field in the region around that point instantaneously. On the other hand, Faraday's law states that:

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (1.3)$$

The law tells us about the induction of electric field $\mathbf{E}(t)$ due to the time varying magnetic field $\mathbf{B}(t)$. As for example if a time varying magnetic field cuts a conductor it generates an induced current that produces the induced electromotive force V_{emf} or electric potential or curling electric field. A good example is an antenna that cuts a time varying magnetic field and as a result induces V_{emf} . We amplify the received signal and process it for meaningful information in our radio receiver. This means we are always encountering Faraday's law in telecommunications engineering. Receiving a call via a mobile phone is a good example of Faraday's law in action.

1.2 Fundamental Concept of Wave Propagation

We consider at some point in space there is a radiating point source. Good example is an infinitesimal dipole antenna, also called a Hertzian dipole. This antenna isotropically radiates EM fields radially outward in all directions. We usually study dipole antennas as the first lesson of the antenna analysis. Assume the antenna carries a time varying uniform current density $\mathbf{J}(t)$ in (A/m). It is a legitimate guess because we assume a very tiny antenna on which the time varying current exists.

Review Questions 1.1: Introduction to Uniform Plane Wave

- Q1.** Explain how does the EM field waves propagate from one point to another point in space? What is the mechanism? What fundamental laws they follow?

⁴ This extra exercise helps develop concepts of wave incidence, transmission and reflection at the obstacles of the wave. We shall develop the concept of reflection and refraction of the wave at the discontinuities of media and derive the reflection and transmission coefficients when we shall cover the normal and oblique incidence cases. Also, the concept paves the way to understand the multipath fading which is a common problem of wireless mobile communications when the signal reflects from and penetrates buildings, walls, trees, and roads.

Let us observe the dipole antenna in Figures 1.2a and 1.3a. It is a source of EM wave propagation. By Ampere's circuital law, due to the current density $\mathbf{J}(t)$, which is present on the dipole, a time varying magnetic field, $\mathbf{H}(t)$, is induced in the surrounding region. By Faraday's law of the electromagnetic induction, the time varying magnetic field induces a time varying electric field $\mathbf{E}(t)$.

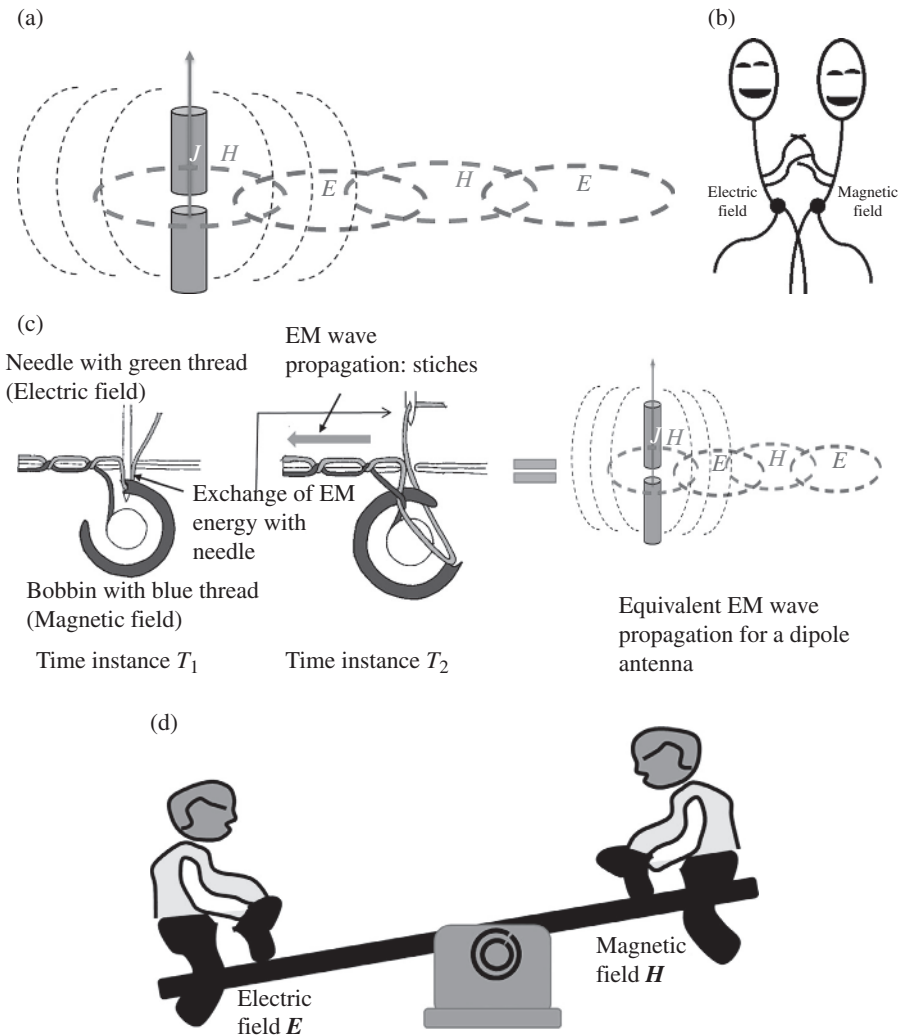


Figure 1.2 Depiction of electromagnetic wave propagation (a) wave propagation due to the time varying current $\mathbf{J}(t)$ that creates the time varying magnetic field $\mathbf{H}(t)$ by Ampere's law and then the time varying magnetic field creates the time varying electric field $\mathbf{E}(t)$ by Faraday's law. Then the oscillating fields alternate between the time varying electric and magnetic fields as they propagate away from the source $\mathbf{J}(t)$, the dipole, (b) a depiction of exchange between the electric and magnetic field with an analogy of tap dancers who exchange their positions of their legs in a rhythmic manner which resembles the oscillating frequency and exchange of energy between the electric and magnetic fields. *Source:* Ulaby, F.T. (2005). *EM for Engineers*. Upper Saddle River, NJ, USA: Pearson Prentice Hall, (c) a sewing machine's sewing mechanism has analogy of the electromagnetic field wave propagation, and (d) exchange of electric and magnetic energy in instantaneous electromagnetic field wave is equivalent to seesaw.

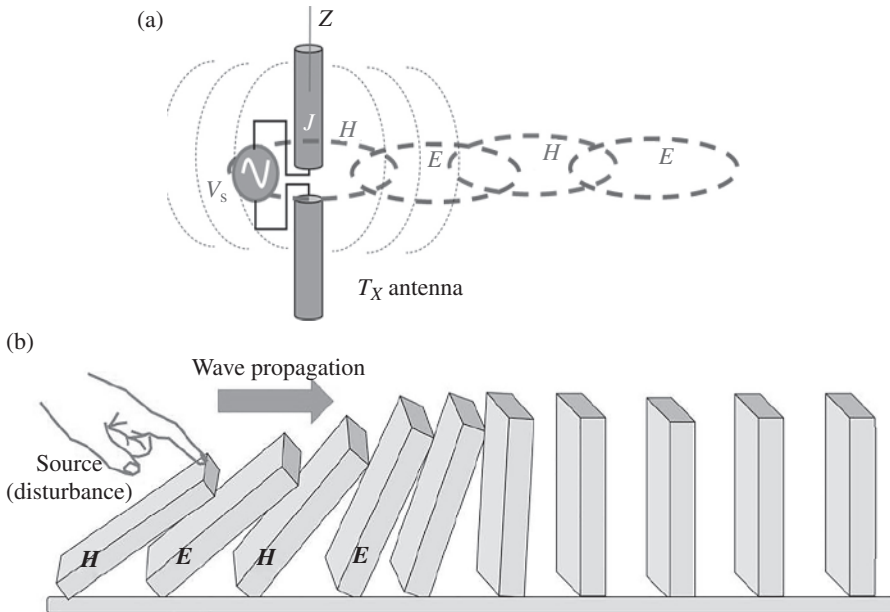


Figure 1.3 (a) Electromagnetic wave propagation via a dipole antenna connected with a generator, and (b) analogy of electromagnetic wave propagation as the domino effect of a series of vertical bricks made of alternate electric and magnetic fields. Here the figure push is the generator with alternating source voltage $V_s(t)$ that creates the induced current $J(t)$ on the dipole.

Energy is passing back and forth between electric and magnetic fields as they travel away from the source point at the speed of light. Note that away from the dipole there is no original current source $J(t)$ present, but still the electromagnetic wave fields carry real power in the direction of propagation due to these two laws in action.

It is an interesting analogy of the series of bricks vertically placed at a uniform distance like a domino. A slight push to the first brick propagates the mechanical energy to the next brick, and then after a moment, the energy is transferred to the third brick and the energy is propagating on and on. Now if we put the analogy of the domino effect in the electromagnetic system, the first push is equivalent to the source of the oscillation; here the current source that is generated by an oscillator that oscillates at an angular frequency $\omega = 2\pi f$ (rad/s) where, f is the operating frequency in Hz.⁵ Note that the source (hand) is sitting in its original position and does not need any further push but the signal is propagating indefinitely without any source present along with the propagation. This is the foundation for wireless propagation in which no source needs to be present in free space.

These source free oscillating fields away from the source in free space constitute the propagating waves of electromagnetic energy. As shown in Figure 1.3a, the electromagnetic energy exchanges hands between the electric and magnetic energies, and travels away from the source point creating electromagnetic wave propagation. And this alternating or periodic oscillation constitutes *the time*

⁵ See also the video of domino brick and the wave propagation due to push of a series of brick. It also tells that the wave energy propagates with finite velocity. The velocity is called the phase velocity of the wave propagation.

harmonic fields. We shall discuss the time harmonic fields after derivation of the generic wave equations. The sinusoidally varying electromagnetic fields are called the time harmonic fields. A signal waveform can be transformed into the sinusoidally varying field using Fourier series expansion. This assumption simplifies the electromagnetic field analysis.

Review Questions 1.2: Mechanism of Uniform Plane Wave

- Q1.** Discuss different analogies of the different mechanisms of creations of the uniform plane wave.
- Q2.** What is the analogy of seesaw in electromagnetic wave propagation?

1.3 Plane Wave Concept

As we discussed in this chapter, the wave radiates spherically, but at a remote distance away from the source the spherical wave resembles a uniform plane wave,⁶ UPW in short. The distance is called the far field distance in the antenna nomenclature. Usually the far field distance is defined as $R \geq (2D^2/\lambda)$, where D is the largest dimension of the radiating source, here is the active part of the antenna⁷ and λ is the wavelength in free space. As you know the wavelength λ is inversely proportional to the operating frequency f , therefore, the far field is also a function of the operating frequency. As the frequency goes up the wavelength gets shorter and the far field distance is also getting larger. However, due to the resonant nature of the antenna, the effective dimension D of the antenna is also proportional to the wavelength λ , hence the antenna is getting shorter almost linearly with the frequency. And the far field distance varies with the squared terms of D . Therefore, in general, the far field distance is almost inversely proportional to the operating frequency.

The far field distance R represents the distance at which the phase error between a perfect plane wave and the antenna's wavefront is equal to 22.5° or less. This phenomenon of the far field is shown in Figure 1.4. That means we can assume at this distance; the field's phase front resides on a plane as shown by the dotted line in Figure 1.4a. As we move further away from the source, the wave front becomes planar and then we can assume the wave is a uniform plane wave. We also defined that for a uniform plane wave, the electric and magnetic fields are oriented orthogonally to each other as well as the direction of wave propagation.

Figure 1.4b illustrates a uniform plane wave generated with a horn antenna. The illustration is generated from a 3D animation of CST Microwave Studio. The detailed field orientation and the direction of the wave propagation are shown in Figure 1.5. Both \mathbf{E} and \mathbf{H} fields are sinusoidal/time harmonic in nature. The electric (along the x -axis) and magnetic fields (along the y -axis) are orthogonal to each other and transverse to the direction of propagation (propagating along the z -axis) as

⁶ It follows the surface area of a sphere $S = 4\pi R^2$, where R is the radius of the sphere. As R increases, the spherical surface looks like a flat surface. Similarly, since the Earth surface is too large compared to a normal human, it looks like a flat surface to us.

⁷ The active part of the antenna is called effect length for a dipole antenna and effective area for a patch, a horn and a reflector antennas. Using the diagonal distance D of a rectangular horn aperture is considered in the calculation of the far field.

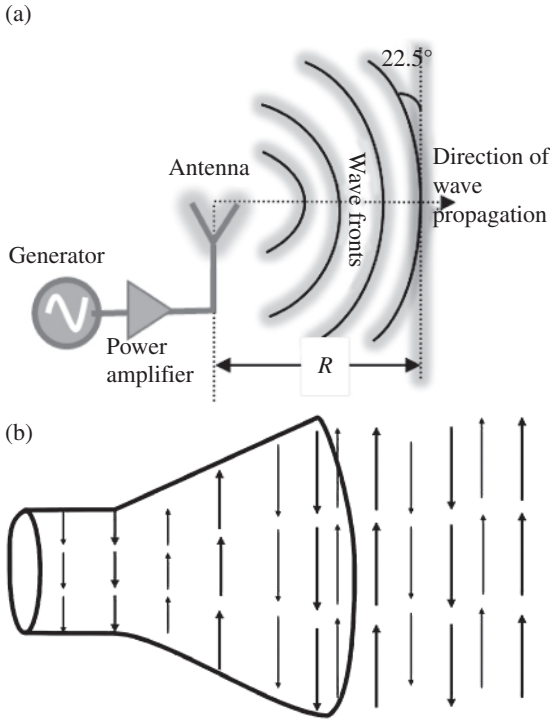


Figure 1.4 (a) Definitions of near field (spherical) wave and the far field (flat) waves at a distance from a radiating antenna, and (b) CST Microwave Studio rendering of plane wave creation from a horn antenna. Observe the wave front becomes planar as it moves away from the horn antenna.

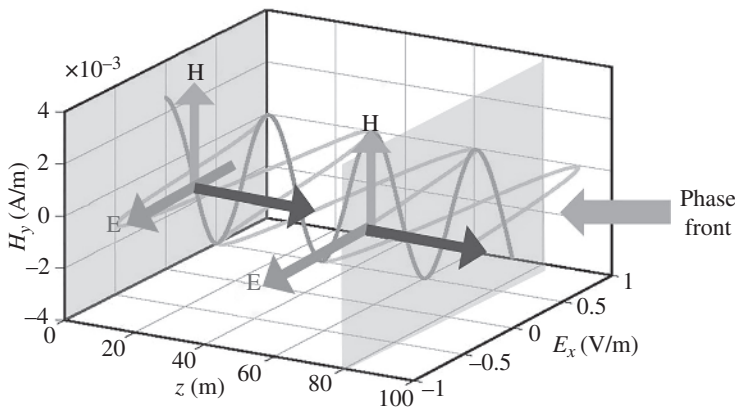


Figure 1.5 Definitions of near field (spherical) wave and the far field (flat) waves at a distance from a radiating antenna.

depicted in Figure 1.5. The power is propagating along the z -axis and is defined with the power density vector $\mathbf{P} = \mathbf{E} \times \mathbf{H}$. The phase on which the electric and magnetic fields reside is called the phase front or the wave front. The direction of the wave propagation follows the right-hand

rule: $\mathbf{P}_z = \mathbf{E}_x \times \mathbf{H}_y$. Therefore, the uniform plane wave is also called the transverse electromagnetic (TEM) wave. The magnitude and phase of the field vectors in a UPW are equal at every point on the wavefront. For simplicity we analyze the field wave problem in rectangular coordinate. We usually work on the spherical coordinate for the antenna analysis.

You can watch some YouTube video in the following link.⁸ You may find many videos and lecture notes on this topic on YouTube to enhance your understanding of UPW. You may also use free trial version of CST Microwave Studio™ animations software downloadable from website upon request from the vendor to obtain the visualization of the abstract nature of EM field in 3D animation via www.cst.com. In the next section, we shall derive the field wave equation using Maxwell's equations.

Review Questions 1.3: Introduction to Uniform Plane Wave

- Q1.** What is the far field of an electromagnetic wave?
- Q2.** Why do we need to consider the effective dimensions of the sources in calculating the far field distance?
- Q3.** Explain the salient features of a uniform plane wave.
- Q4.** What is the definition of the phase front? What is wavefront?
- Q5.** What is the far field distance of a radiating source? How do you determine the far field distance for a physical source?
- Q6.** The antenna operates at 1 GHz frequency and has the largest dimension of 210 mm. Calculate the far field distance.
- Q7.** Why a uniform plane is also called a TEM wave?

Example 1.1: Introduction to Uniform Plane Wave

The electric field $\mathbf{E}(z, t) = 10 \cos(10^8 \pi t - 4\pi z) \mathbf{a}_x$ (V/m) and the magnetic field is $\mathbf{H}(z, t) = 26.53 \cos(10^8 \pi t - 4\pi z) \mathbf{a}_y$ (mA/m). Calculate the power density or Poynting vector \mathbf{P} in (W/m²).

Answer: $\mathbf{P} = \mathbf{E} \times \mathbf{H} = 10 \cos(10^8 \pi t - 4\pi z) \mathbf{a}_x \times 26.53 \times 10^{-3} \cos(10^8 \pi t - 4\pi z) \mathbf{a}_y = 377 \cos^2(10^8 \pi t - 4\pi z) \mathbf{a}_z$ (W/m²).

Example 1.2: Uniform Plane Wave

Calculate the wave impedance of the wave in Example 1.1

Answer: The wave impedance is the ratio of the x -directed electric field and y -directed magnetic fields.

$$\eta = \frac{E_{x0}}{H_{y0}} = \frac{10 \left(\frac{\text{V}}{\text{m}}\right)}{26.53 \left(\frac{\text{mA}}{\text{m}}\right)} = 377 \text{ } (\Omega).$$

⁸ http://www.walter-fendt.de/html5/phen/electromagneticwave_en.htm (accessed 05 December 2017)

<http://www.edumedia-sciences.com/en/media/222> (accessed 05 December 2017)

https://www.youtube.com/watch?v=c_IUic_oatY (accessed 05 December 2017)

Example 1.3: Derivation of Electromagnetic Wave Equation

Now we are going to derive the wave equation from Maxwell's equations. Recall Maxwell's equations in point form.

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon} \quad (1.4)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (1.5)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (1.6)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (1.7)$$

In far away from the source, we legitimately consider the source free region where the field wave propagates with real energy. Therefore, in free space we assume $\rho_v = 0$; therefore, Gauss law for electric field case reduces to $\nabla \cdot \mathbf{E} = 0$.

Now, consider a homogeneous, linear and isotropic medium, which has time invariant constitutive parameters permittivity (ϵ), and permeability (μ) and the electromagnetic wave propagates in this medium. Then Gauss law for electric and magnetic cases are:

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{H} = 0.$$

The Gauss law for magnetic case is zero due to the fact that there is no magnetic source in reality, irrespective of the distance. Also, at far away from the source, the conduction current $\mathbf{J}_c = \sigma \mathbf{E}$ is zero. Using the above two source free assumptions in the free space, we can write the two curl equations of Maxwell:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \quad (1.8)$$

The symbols have their usual physical meanings. We shall be using these two source-free curl equations in deriving EM wave equations.

Review Questions 1.4: Introduction to Uniform Plane Wave
<p>Q1. Explain why do we consider J and ρ_v both zero in the derivation of the electromagnetic wave equation?</p> <p>Q2. Show that in a source free region $\nabla \times \mathbf{E} = -\mu(\partial \mathbf{H}/\partial t)$ and $\nabla \times \mathbf{H} = \epsilon(\partial \mathbf{E}/\partial t)$.</p>

Next, take curl on both sides of Faraday's law, we obtain,

$$\nabla \times \nabla \times \mathbf{E} = \nabla \times \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) \quad (1.9)$$

Observe on the right side of the above equation, the curl is a position derivative in $x, y,$ and z that is acting on a time derivative $\partial/\partial t$. A wave is a two independent dimensional entity with positions (x, y, z) and time (t). Now using the identity

$$\nabla \times \nabla \times \mathbf{E} = \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} \quad (1.10)$$

on the left-hand side and taking μ out of $\nabla \times$ in the right-hand side as it is not varying with space,⁹ yields

$$\nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} = -\mu \frac{\partial(\nabla \times \mathbf{H})}{\partial t} \quad (1.11)$$

Now if you substitute Ampere's law in point form $\nabla \times \mathbf{H} = \sigma \mathbf{E} + \varepsilon(\partial \mathbf{E}/\partial t)$ in the right-hand side and do the time derivative on it then we get the complete expression for the field wave equation as follow:

$$\nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} = -\mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (1.12)$$

Since in the source free region, no volume charge density ρ_v is present¹⁰ hence $\nabla \cdot \mathbf{E} = 0$. Taking the source free assumption in free space, we obtain the generic wave equation for the electric field:

$$\nabla^2 \mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (1.13)$$

where the Laplacian operator ∇^2 is a double derivation operator in spatial coordinates and is defined in the rectangular coordinate as:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (1.14)$$

Likewise, we can derive the generic wave equation for the magnetic field:

$$\nabla^2 \mathbf{H} - \mu\sigma \frac{\partial \mathbf{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad (1.15)$$

In the above two wave equations of \mathbf{E} and \mathbf{H} , we observe that \mathbf{E} and \mathbf{H} are the functions of both space (x, y, z) and time t . This wave equation is called the *Helmholtz wave equation*.¹¹ Similar equation is derived for \mathbf{H} . This derivation of \mathbf{H} is left as an exercise.

Review Questions 1.5: Introduction to Uniform Plane Wave

- Q1.** Derive the field wave Equation (1.15) for the time varying magnetic field from Maxwell's equations in free space.

Laplacian operator ∇^2 can be decomposed into independent three spatial dimensions x , y , and z . Therefore, each equation can be broken up into their individual spatial components: E_x , E_y , E_z , H_x , H_y , and H_z . This yields the corresponding six one dimensional field wave equations in x , y , and z directions. Each of the six equations is a second-order differential equation. Remember we shall no more use partial differential equation as we are working with one dimensional independent spatial

⁹ We assume the medium is linear, isotropic, and homogeneous.

¹⁰ As discussed above in free space away from the source there is not active source like volume charge density ρ_v , as well as current density J present.

¹¹ Helmholtz was the teacher of Heinrich Hertz and both worked in the same laboratory in Germany.

Table 1.1 One-dimensional electromagnetic field wave equations.

1-D electric field wave equations	1-D magnetic field wave equations
$\frac{d^2 \mathbf{E}}{dz^2} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$ (1.16a)	$\frac{d^2 \mathbf{H}}{dx^2} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$ (1.17a)
$\frac{d^2 \mathbf{E}}{dy^2} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$ (1.16b)	$\frac{d^2 \mathbf{H}}{dx^2} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$ (1.17b)
$\frac{d^2 \mathbf{E}}{dy^2} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$ (1.16c)	$\frac{d^2 \mathbf{H}}{dx^2} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$ (1.17c)

variable only. These equations can be solved in terms of one-dimensional position and time. The solution is an equation defining forward and reverse travelling field waves in these three respective coordinates. One example for the physical meaning of the field wave propagation of \mathbf{E}_x and \mathbf{H}_y fields along z -direction is shown in Figure 1.5. Table 1.1 summarizes the one-dimensional field wave equations for both electric and magnetic fields.

The above equations for the \mathbf{E} and \mathbf{H} fields are for the case of variations in one spatial coordinate only. As we have seen from the depiction of the uniform plane waves of the \mathbf{E} and \mathbf{H} fields components in Figure 1.5, that the uniform plane wave is transverse to the direction of the wave propagation, z . According to the illustration of the wave propagation along z -direction, the z -directed electromagnetic field components must be zero. Therefore, for the uniform plane wave we shall solve for the two transverse field components along x - and y -directions assuming that the direction of wave propagation is along z -axis. Let us derive the field solutions for \mathbf{E}_x field component first. The scalar field wave equation for \mathbf{E}_x is:

$$\frac{\partial^2 \mathbf{E}_x}{\partial z^2} = \mu\sigma \frac{\partial \mathbf{E}_x}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{E}_x}{\partial t^2} \quad (1.18)$$

We observe that \mathbf{E}_x is a function of z and t only. To solve for it, we resort the method of separation of variables. In this method we can decompose a function into the product of the two functions of individual independent variables separately as follow¹²:

$$E_x(z, t) = f(z)g(t) \quad (1.19)$$

where $f(z)$ is a function of z only and $g(t)$ is a function of t only.

Substitution for the Equation (1.19) in the 1D field wave Equation (1.18) above, we obtain

$$\frac{\partial^2 f(z)g(t)}{\partial z^2} = \mu\sigma \frac{\partial f(z)g(t)}{\partial t} + \mu\epsilon \frac{\partial^2 f(z)g(t)}{\partial t^2} \quad (1.20a)$$

$$g(t) \frac{\partial^2 f(z)}{\partial z^2} = \mu\sigma f(z) \frac{\partial g(t)}{\partial t} + \mu\epsilon f(z) \frac{\partial^2 g(t)}{\partial t^2} \quad (1.20b)$$

Now dividing both sides by $f(z)g(t)$, we obtain

$$\frac{1}{f(z)} \frac{d^2 f(z)}{dz^2} = \frac{\mu\sigma}{g(t)} \frac{dg(t)}{dt} + \frac{\mu\epsilon}{g(t)} \frac{d^2 g(t)}{dt^2} = -k^2 \text{ a constant} \quad (1.20c)$$

12 Similar derivation is also available in Islam, M.A. (1969). *Electromagnetic Theory*. Dacca, East Pakistan: EPUET.

k is a constant due to the fact that both sides are the singular functions of the respective independent variables z on the left-hand side and t on the right-hand side, respectively. The individual equations of the singular function are:

$$\frac{d^2 f(z)}{dz^2} + k^2 f(z) = 0 \quad (1.20d)$$

$$\frac{d^2 g(t)}{dt^2} + \frac{\sigma}{\epsilon} \frac{dg(t)}{dt} + \frac{k^2}{\mu\sigma} g(t) = 0 \quad (1.20e)$$

The general solution for $f(z)$ is:

$$f(z) = Ae^{-jkz} + Be^{jkz} \quad (1.21a)$$

where A and B are coefficients and usually represent the amplitudes of the forward and reverse functions of $f(z)$. For equation containing $g(t)$, we assume similar solution

$$g(t) = Ce^{-jpt} \quad (1.21b)$$

and substitute Equation (1.21b) in the above Equation (1.20e) then we obtain,

$$Ce^{-pt} \left(p^2 - \frac{\sigma}{\epsilon} p + \frac{k^2}{\mu\sigma} \right) = 0 \quad (1.22)$$

Since Ce^{-pt} cannot be zero, therefore

$$p^2 - \frac{\sigma}{\epsilon} p + \frac{k^2}{\mu\sigma} = 0 \quad (1.23)$$

In the above equation we observe that p and k are not independent of each other. Therefore, if one of them, for example p is specified then k can also be calculated. Now we combine solutions for $f(z)$ and $g(t)$ in the above two equations, we obtain the complete solution for E_x ,

$$E_x(z, t) = f(z)g(t) = (Ae^{-jkz} + Be^{jkz})(Ce^{-jpt}) \quad (1.24)$$

or,

$$E_x(z, t) = E_{1x}e^{(-jkz-pt)} + E_{2x}e^{(jkz-pt)} \quad (1.25)$$

where $E_{1x} = AC$ and $E_{2x} = BC$ are used.

In a similar mathematical manipulation, we can obtain the solution for the other transverse field components E_y ,

$$E_y(z, t) = E_{1y}e^{(-jkz-pt)} + E_{2y}e^{(jkz-pt)} \quad (1.26)$$

Now combining the individual complete we obtain the complete solution for the transverse electric field:

$$E(z, t) = E_1e^{(-jkz-pt)} + E_2e^{(jkz-pt)} \quad (1.27)$$

Likewise, we can derive the solution for the magnetic field,

$$H(z, t) = H_1e^{(-jkz-pt)} + H_2e^{(jkz-pt)} \quad (1.28)$$

We can easily develop the relationship between E_1 , E_2 , H_1 and H_2 and show that the field quantities are orthogonal to each and the direction of wave propagation.

Review Questions 1.6: Introduction to Uniform Plane Wave

- Q1.** Explain why we need to decouple 3D electromagnetic field wave equations into discrete 1D spatial field wave equation.
- Q2.** Sketch the procedure to calculate the 1D spatial field wave equation.

Exercise 1.1: Derivation of Uniform Plane Wave

- Q1.** Derive Equation (1.28) from Equation (1.17a).

1.4 One Dimensional Wave Equation Concept

In the beginning of the chapter we mentioned that the vector field analysis is a complex and challenging process, and it is not possible to get vector field solution using the conventional mathematical process. The complexity of the analysis increases with the coordinate systems. In this regard, the field components are decomposed into their individual one dimensional field components and the assumptions of variations are made to simplify the process. This process we have just witnessed in the above field wave analysis. Therefore, the simplest approach to this field analysis is to find the field solution in one dimensional wave equations. In this section, we shall develop the concept of one dimensional field wave equation and the relevant field solutions. This theory is also supported by the practical system. In guided structures such as transmission lines, waveguides and optical fibers; and radiation of waves via antennas, we always look for solutions for one dimensional fields, and develop the appropriate excitation mechanism to support the theory.¹³

Let us have a look at the relation of the field components and the direction of propagation as we have derived in the above section. Assume that the wave propagates along z-axis only and no variations along x and y axes. Let us assume the medium is lossless, and the region is charge free as we have assumed before. To solve the problem we take Maxwell's equations again as follows:

From Maxwell's equation,

$$\begin{bmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{bmatrix} = -\mu \frac{\partial}{\partial t} (\mathbf{a}_x H_x + \mathbf{a}_y H_y + \mathbf{a}_z H_z) \quad (1.29)$$

Simplification leads to the following differential equations:

$$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} \quad (1.30)$$

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \quad (1.31)$$

¹³ We have witnessed such excitation mechanism during the development of the field wave propagation via a dipole antenna. As we move along applications of the wave equation in specific cases such as waveguides, optical fibers and antennas, we shall encounter various specific excitation mechanisms.

$$0 = \mu \frac{\partial H_z}{\partial t} \quad (1.32)$$

From (1.2) with the conduction current $J_c = 0$ and $\mathbf{D} = \epsilon \mathbf{E}$ in 3D coordinates we obtain,

$$\begin{bmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ H_x & H_y & 0 \end{bmatrix} = \epsilon \frac{\partial}{\partial t} (\mathbf{a}_x E_x + \mathbf{a}_y E_y + \mathbf{a}_z E_z) \quad (1.33)$$

Simplification leads to the following differential equations:

$$\frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t} \quad (1.34)$$

$$\frac{\partial H_x}{\partial z} = \epsilon \frac{\partial E_y}{\partial t} \quad (1.35)$$

$$0 = \epsilon \frac{\partial E_z}{\partial t} \quad (1.36)$$

From the above derivations we observe that E_z and H_z must either be zero or time independent to satisfy the last bottom two equations. Since we are investigating the time varying field quantities, they must be zero. This confirms that if we assume the field quantities vary only in one direction then both \mathbf{E} and \mathbf{H} must be transverse to the direction of the field variation. In the present case the variation is assumed to be in z -direction only, and hence both electric and magnetic fields must have only x and y components. This is the fundamental definition of *the transverse electromagnetic field waves*.

Now we shall work on the field wave equations of the transverse wave that propagates only in the z direction. We start with

$$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t} \quad (1.37)$$

and differentiate with respect to z to obtain the field wave equation as follow:

$$\frac{\partial^2 E_y}{\partial z^2} = \mu \frac{\partial H_x}{\partial z \partial t} \quad (1.38)$$

Similarly for

$$\frac{\partial H_x}{\partial z} = \epsilon \frac{\partial E_y}{\partial t} \quad (1.39)$$

with partial derivate with respect to t we obtain,

$$\frac{\partial^2 H_x}{\partial t \partial z} = \epsilon \frac{\partial^2 E_y}{\partial t^2} \quad (1.40)$$

Now combining Equations (1.38) and (1.40), we obtain

$$\frac{\partial^2 E_y}{\partial z^2} = \mu \epsilon \frac{\partial^2 E_y}{\partial t^2} \quad (1.41)$$

Equation (1.41) is the field wave equation of E_y . In the above derivation we assume that the time varying magnetic field is a continuous function of both t and z . Therefore, the differentiation with respect to t and z can be done in any order that yields the same results.

Likewise we can derive the field wave equations for of E_x , H_x , and H_y , as follows:

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} \tag{1.42}$$

$$\frac{\partial^2 H_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 H_x}{\partial t^2} \tag{1.43}$$

$$\frac{\partial^2 H_y}{\partial z^2} = \mu\epsilon \frac{\partial^2 H_y}{\partial t^2} \tag{1.44}$$

Exercise 1.2: Derivations of One-Dimensional Wave Equation

Q1. Derive the one-dimensional field wave equations from the relevant Maxwell’s Equations (1.42)–(1.44).

We can generalize all the above one dimensional scalar field wave equations in the following form:

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \tag{1.45}$$

where $v = \frac{1}{\sqrt{\mu\epsilon}}$ is the velocity of wave propagation.

The wave of this type is called *the wave equation of vibrating string*. Figure 1.6 illustrates the waves for the stretched vibrating string: (a) stretched string, (b) fundamental mode ($n = 1$), (c) second harmonic ($n = 2$), and (d) third harmonic ($n = 3$).

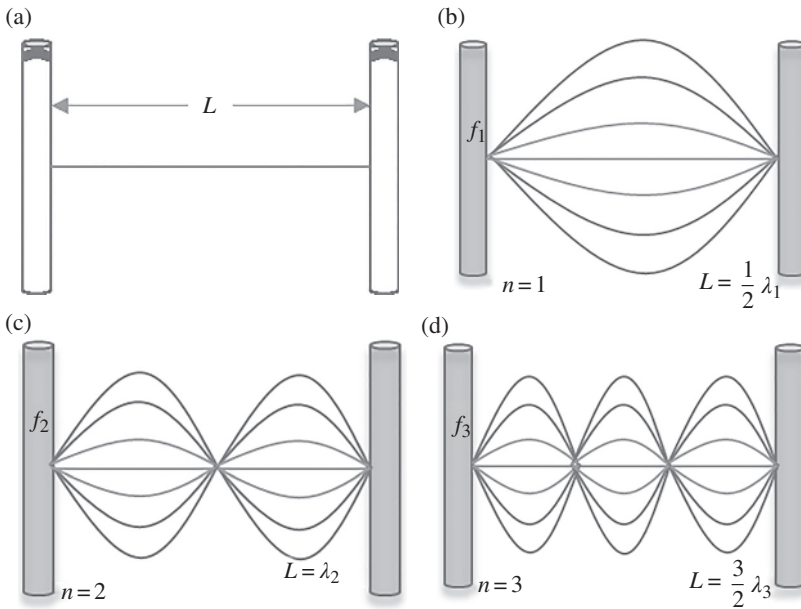


Figure 1.6 Illustration of string waves: (a) stretched string at equilibrium, (b) disturbance of fundamental mode frequency with wavelength λ , (c) 2nd harmonic, and (d) 3rd harmonic.

We solve for the one dimensional wave equation using separation of variable method and understand the salient feature of the one dimensional wave such as its two solutions of forward and reverse travelling natures, the velocity of wave propagation and definition of the uniform plane wave.

As per the method of separation of variables, we assume $f = Z(z)T(t)$ where, $Z(z)$ and $T(t)$ are the functions of z only and t only, respectively. Substitution for $f = Z(z)T(t)$ in the above Equation (1.45) and dividing both sides by v^2 we obtain,

$$\frac{d^2Z(z)}{dz^2} + \frac{c^2}{v^2}Z(z) = 0 \quad (1.46)$$

$$\frac{d^2T(t)}{dt^2} + c^2T(t) = 0 \quad (1.47)$$

Here c^2 is the separation variable. The solutions for the above two double derivative equations are:

$$Z(z) = A_1e^{-jcz/v} + A_2e^{jcz/v} \quad (1.48)$$

$$T(t) = Ce^{jct} \quad (1.49)$$

and the complete solution for $f(z, t)$ is obtained by the product of the two above Equations (1.48) and (1.49):

$$f(z, t) = (A_1e^{-jcz/v} + A_2e^{jcz/v})Ce^{jct} = Ae^{jct + (z/v)} + Be^{jct - (z/v)} \quad (1.50)$$

Review Questions 1.7: Derivation of General Wave Equation

- Q1. Derive the wave equation of vibrating strings.
- Q2. What is the velocity of the wave?

We find that the solution has two parts: (i) one with the function of $(t - (z/v))$ and can be defined as $F_1(t - (z/v))$, and (ii) second one with the function of $(t + (z/v))$ and that can be defined as $F_2(t + (z/v))$. These functions must satisfy the generic wave Equation (1.45).

The function $F_1(t - (z/v))$ represents the forward travelling wave along +ve z -direction and $F_2(t + (z/v))$ represents the reverse travelling wave in the $-ve$ z -direction, both are with the same velocity (m/s). Figure 1.7 illustrates the forward and reverse travelling wave function $F_f(t - (z/v))$ and $F_r(t + (z/v))$ in the sinusoidal form.

1.5 Wave Motion and Wave Front

We have defined the wave velocity $v = (1/\sqrt{\mu\epsilon})$ in a linear, isotropic, and homogeneous medium with the constitutive parameters ϵ and μ . Now we understand the physical significance of the wave motion and the wave front. These two concepts are vital for the uniform plane wave theory.

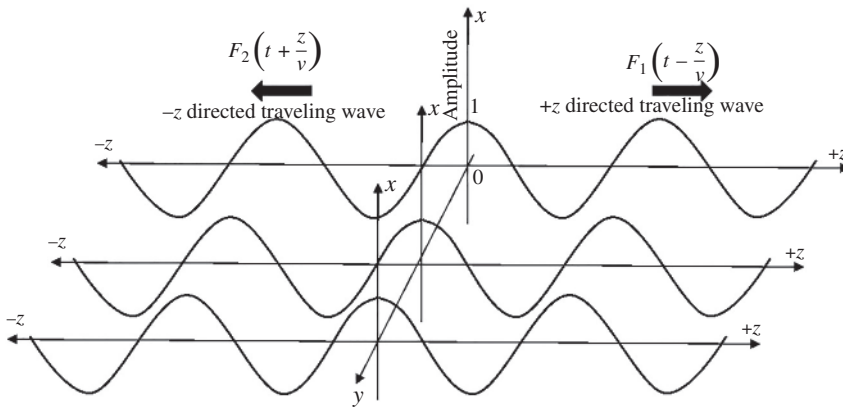


Figure 1.7 Concept of the field solution of the plane wave propagation with forward and reverse travelling waves on xz -plane with the propagation direction along z -axis.

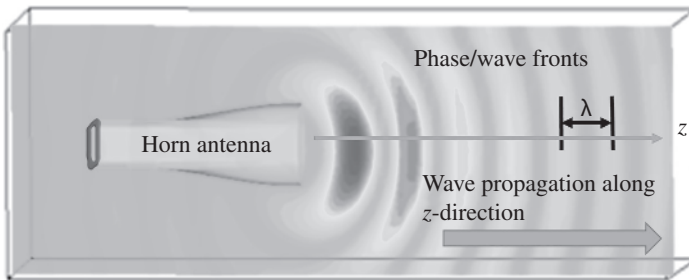


Figure 1.8 Concept of the wave front of the EM wave as it propagates away from a horn antenna (CST Microwave Studio rendition).

The wave motion is defined as the uniform propagation speed at which the constant *phase front*, also called *wave front*, of the wave energy travels. Each surface of the constant phase is called *the wave front*. Figure 1.8 illustrates such wave front propagates from a horn antenna as generated using CST Microwave Studio. As you may observe in the CST rendition of the wave propagation of an electromagnetic wave, for each wave there is a *characteristic distance* between two successive wave fronts of the same family of the wave front of which all members are simultaneously in the same phase. This characteristic distance is called *the wavelength*. The wavelength is related with the frequency of operation as $\lambda = v/f$ where f is the frequency of wave propagation.

Example 1.4: Derivation of Electromagnetic Wave Equation

If the horn antenna is propagating an electromagnetic wave at 9 GHz in air with $\epsilon = 8.854 \times 10^{-12}$ (F/m) and $\mu = 4.475 \times 10^{-7}$ (H/m). Calculate the wavelength of the wave.

Answer:

$$\lambda = \frac{v}{f} = \frac{c}{f} = \frac{3 \times 10^8 \text{ (m/s)}}{9 \times 10^9 \text{ (Hz)}} = 3.3 \text{ (cm)}$$

The above wave front is also called *the wave train*. This is a continuous sequence of wave fronts of which those intervals of the constant wavelength λ , and also called constant time period $T = 1/f$, and they are always in the same phase. As you can also see that the wave fronts near the antenna aperture is spherical and as the wave propagates away from the antenna aperture it become planar. In all cases the wave fronts are in the same phase. Depending on the shapes of the wave front they have different names. As for example, the wave fronts near the antenna consist of surfaces of concentric spherical surfaces. In this case, they are called *spherical waves*. This wave is also called *the near field wave* in antenna nomenclature.¹⁴ In a case, if we can make a wave front in cylindrical shape then the wave is called *cylindrical wave*. Therefore, depending on the shapes of the wave fronts we define the electromagnetic waves.

As the wave propagates away from the antenna, the field which is planar in shape is called *the far field wave* in the antenna nomenclature. The far field wave is also called *plane wave* meaning the phase front of the wave consisting of a plane surface as shown in Figure 1.8. If the amplitude of the field vectors remains constant over each plane which is perpendicular to the direction of propagation, then we call it as a *uniform plane wave* or in short UPW.

Review Questions 1.8: Introduction to Uniform Plane Wave

- Q1.** Define wavelength of a time varying electromagnetic propagating signal
- Q2.** What is wave motion?
- Q3.** Show that a uniform plane wave has two solutions, one is forward travelling wave and one is reverse travelling wave.
- Q4.** Define uniform plane wave.

1.6 Phase Velocity of UPW

In the preceding section, we have examined that the function $F_1(t - (z/v))$ is a solution of an electromagnetic wave equation that travels at a velocity v . Now we shall develop the physical concept of the wave velocity in this section. We all have examined that the wave equation has two solutions $F_1(t - (z/v))$ and $F_2(t + (z/v))$, the forward and reverse travelling wave functions, respectively. The wave function has two independent variables, the distance z along z -axis and time t . As the wave travels, the time increases with the distance irrespective of the direction of the wave propagation. To understand the physical concept appropriately, let us take an example of a car traveling with a certain uniform speed v along z -axis as shown in Figure 1.9. Assume that the location of the car is $z = z_1$ at an instant of time $t = t_1$. As the car travels with time, the location of the car is $z = z_2$ at the instant

¹⁴ The near field waves are of two types: (i) the spherical waves which are very close to the antenna aperture are called non-radiating near field wave, and (ii) the spherical waves which are in between the non-radiating near field wave and the far field waves are called radiating near field waves. In the first case of the waves, the non-radiating reactive energy is stored in the waves and does not contribute in the real power radiation. The radiating near field starts detaching the radiative field's energies in real power. However, the phase front of the wave remains curvilinear.

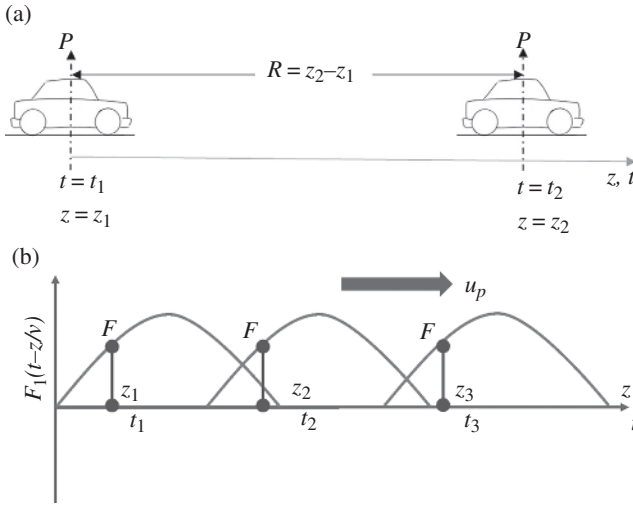


Figure 1.9 Concept of the phase velocity (a) analogy of car travelling a distance R at a velocity $v = ((z_2 - z_1)/(t_2 - t_1)) = (R/(t_2 - t_1))$, and (b) the constant phase point F of a wave function $F_1(t - z/v)$ travelling along z -axis with a phase velocity u_p .

of time $t = t_2$. By this time, the car travels a distance R with a uniform velocity v . Therefore, the magnitude of velocity of the car along z -axis is:

$$v = \frac{z_2 - z_1}{t_2 - t_1} = \frac{R}{t_2 - t_1} \text{ (m/s)} \tag{1.51}$$

If we want to measure the velocity of the car precisely, we need to measure the distance and time precisely as well. To measure the distance in precision details, we need to fix a point on the car as shown in Figure 1.9a. At the time instant t_1 , the location point P on the car is always fixed as for the time instants t_2 , and t_3 etc. The fixed-point P on the car is equivalent to the fixed or constant phase or the initial state of a uniform plane wave.¹⁵

Now look at the plots of the wave function $F_1(t - (z/v))$ at different locations along the z -axis as the time progresses from t_1 to t_2 to t_3 . At these time instances, the wave travels from z_1 to z_2 to z_3 along the z -axis, respectively. In the three progressive plots of the wave functions $F_1(t - (z/v))$ we set a fixed reference amplitude position F on the plots and set the three distances at the three distance–time instances (z_1, t_1) , (z_2, t_2) , and (z_3, t_3) , respectively, as shown in Figure 1.9b. This means as time increases, the same plot moves to a distance along the positive z -axis. The two independent variables are z and t that are denoted along the horizontal axis. In all cases, we assume the velocity v to be constant at all time. The above examination reveals that as time t increases the distance z must increase in order that $(t - (z/v))$ remains constant at all time.

¹⁵ The phase of the signal is such a concept that needs to be measured precisely. We may ignore small deviation of amplitude but not the phase. We hinted this issue in the development of the concept of the retarded potential. In the antenna radiation theory, we shall consider the minute phase variation very seriously, and include minute phase variation in the calculation. The direction of the radiated wave propagation and beamforming of the antenna are highly dependent on the phase excitation of individual antenna elements.

This means the solution of the wave function $F_1(t - (z/v))$ is constant as long as $(t - (z/v))$ is constant. And in this case, we designate $F_1(t - (z/v)) = F$ for all of the three-time instances and travelled distances.

Now to determine the velocity of the wave propagation, we focus our attention to the reference point F on the plot of $F_1(t - (z/v))$. We have observed that the fixed amplitude point F is moving with the plot of $F_1(t - (z/v))$ along the z -direction. Its height is equivalent to the phase state of the wave function and must remain constant all the time. Hence the argument $(t - (z/v))$ must remain constant all the time.

Now we are in a position to define the velocity of the wave in the physical sense as we can measure precisely the distance and time based on the above explanation. We set a reference point F on the plots of the wave function that is propagating along the z -axis at different time instances with a uniform velocity v . The velocity of the wave function is obtained simply taking the derivative of distance z with respect to time t . This means the velocity of the constant phase point F is:

$$v = \frac{dz}{dt} \quad (1.52)$$

This means the velocity with which a point of the constant phase of the wave is propagating is simply $v = (dz/dt)$. Thus, we conclude that a wave function $F_1(t - (z/v))$ that moves at a fixed velocity v along the positive z -direction. Likewise we can also show that $F_2(t + (z/v))$ is moving along the negative z -direction at a velocity $v = (dz/dt)$ to keep the argument $(t + (z/v))$ fixed. Here, as t increases z must increase negatively with a proportionality constant $1/v$ to keep the phase of the wave function fixed.

We have already defined $v = (1/\sqrt{\mu\epsilon})$ (m/s). The velocity is known as *the phase velocity* of the uniform plane wave, and in many instances, it is denoted as u_p , as shown in Figure 1.9b.

Example 1.5: Derivation of Phase Velocity

Show that the phase velocity has unit (m/s).

Answer:

Permittivity has unit $F/m = (\text{coulomb})^2/\{\text{N} \cdot (\text{m})\}^2$, permeability has unit $H/m = \text{V} \cdot \text{s}/(\text{A} \cdot \text{m})$, energy = $\text{N} \cdot \text{m} = \text{V} \cdot \text{A} \cdot \text{s}$, coulomb = $\text{A} \cdot \text{s}$

Answer:

$$\mu\epsilon = \frac{\text{V} \cdot \text{s}}{\text{A} \cdot \text{m}} \times \frac{(\text{coulomb})^2}{\text{N} \cdot (\text{m})^2} = \frac{\text{V} \cdot \text{s} \cdot (\text{A} \cdot \text{s})^2}{\text{A} \cdot (\text{V} \cdot \text{s} \cdot \text{A}) \cdot (\text{m})^2} = \frac{(\text{s})^2}{(\text{m})^2}$$

$$\therefore v = \frac{1}{\sqrt{\mu\epsilon}} \left(\frac{\text{m}}{\text{s}} \right)$$

Thus, v has the dimension of velocity (m/s), and is independent of frequency, but dependent on the constitutive parameters μ and ϵ . It represents the velocity of the phase or the original state of the signal. It is important to point out that it is not the velocity of energy of the wave. We shall discuss the energy velocity, which is called the *group velocity*, during developing the concept of uniform plane wave for time harmonic cases.

Review Questions 1.9: Phase Velocity to Uniform Plane Wave

- Q1.** Define phase velocity. What is the physical meaning of the phase velocity of the electromagnetic wave?
- Q2.** Who described that the light wave was an electromagnetic wave?
- Q3.** Explain the similarities of the electromagnetic and light waves in terms of the phase velocity
- Q4.** What is group velocity?

Exercise 1.3: Phase Velocity to Uniform Plane Wave

Q1. Show that in free space the phase velocity of the electromagnetic wave is $v = 3 \times 10^8$ (m/s).

Hints: $\epsilon \cong (1/36\pi) \times 10^{-9}$ (F/m) and $\mu = 4\pi \times 10^{-7}$ (H/m).

Q2. A coaxial cable has a Teflon filling which has $\epsilon_r = 2.3$ and $\mu_r = 1$. Calculate the electromagnetic signal's phase velocity in the coaxial cable. Compare the velocity with respect to the free space phase velocity.

Answer:

1.978×10^8 (m/s); 66% of free space velocity.

In free space and in vacuum, the electromagnetic wave travels with the velocity equal to the speed of light 3×10^8 (m/s).¹⁶ We therefore, both light and electromagnetic plane waves travel with the same velocity in vacuum. This tells us that light wave and electromagnetic waves are similar in nature. They are similar in many ways such as: (i) both waves are transverse to the direction of wave propagation, (ii) in free space both types of waves have exactly similar behavior, and (iii) the properties of reflection, refraction, and transmission of both wave types have similarities not only in free space or vacuum but also in any medium. These similarities laid the foundations for the electromagnetic wave theory of light wave by Maxwell. This textbook covers the theory of both electromagnetic and light waves for the advanced level applications of microwave passive design such as metallic waveguides, antennas, optical waveguides, and optical fibers, respectively.

So far, we have examined the wave motion of the plane wave. From the examination we have understood a few salient features of the plane wave: (i) a plane wave can travel in both -ve and +ve z -directions, (ii) the wave travels with a uniform velocity v , (iii) the reference phase point on the wave is always constant, and (iv) the velocity of the wave is independent of frequency of operation, but depends on the constitutive parameters of the medium of propagation.

Now let us go back to the wave equation of a particular wave function say the z -directed travelling electric field component E_x :

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon \frac{\partial^2 E_x}{\partial t^2} \quad (1.53)$$

¹⁶ Maxwell has discovered the phenomena of the similarity between the electromagnetic and light waves. Therefore, covering the light wave theory in terms of the electromagnetic phenomena and electromagnetic field solutions is a salient feature of the book compared to other traditional textbooks on electromagnetism.

Using the definition of the phase velocity $v = (1/\sqrt{\mu\epsilon})$, we can rewire the equation as

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2} \quad (1.54)$$

Based on our previous examination of the solution for E_x we can write the general solution of the one dimensional wave equations as

$$E_x = f_1\left(t - \frac{z}{v}\right) + f_2\left(t + \frac{z}{v}\right) \quad (1.55)$$

where, f_1 and f_2 are two arbitrary functions of $(t - (z/v))$ and $(t + (z/v))$, that travel along +ve and -ve z -directions, respectively. In electromagnetic nomenclature we denote the forward travelling wave as

$$E_x^+(z, t) = f_1\left(t - \frac{z}{v}\right) \quad (1.56a)$$

and the reverse travelling wave as

$$E_x^-(z, t) = f_2\left(t + \frac{z}{v}\right) \quad (1.56b)$$

Therefore, we can write the complete solution as:

$$E_x(z, t) = E_x^+(z, t) + E_x^-(z, t) \quad (1.56c)$$

Likewise we can develop the magnetic wave field solutions from the wave equation

$$\frac{\partial^2 H_y}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 H_y}{\partial t^2} \quad (1.57)$$

for H_y field component as:

$$H_y(z, t) = H_y^+(z, t) + H_y^-(z, t) \quad (1.58)$$

In the next section we shall use the above equation to develop the relationship between the corresponding magnetic field component to develop the concept of wave impedance. Again we shall resort to Maxwell's equations in the development.

Exercise 1.4: One-Dimensional Wave Equation

Q1. Show that the general solution of the one-dimensional wave equation for the magnetic field is:

$$H_y(z, t) = H_y^+(z, t) + H_y^-(z, t)$$

1.7 Wave Impedance

The concept of wave impedance is somewhat different than the impedance that we are used to in the theory of alternating current (AC) circuit. In the AC circuit, the impedance is fixed for a particular frequency, and is defined as the ratio of the voltage and current waveforms. The impedance

of an AC circuit is a complex quantity and is comprised of the real and reactive circuit elements. In free space electromagnetic wave propagation, we define the intrinsic wave impedance that relates the time varying electric and magnetic fields at any point in space. The impedance is a function of the constitutive parameters of the medium such as σ , ϵ , and μ . In lossless, homogeneous, and linear medium, the intrinsic impedance is a real quantity and does not change with frequency.

In this section, we shall develop the physical understanding of *the wave impedance*, also called *the intrinsic wave impedance*. As coined by Maxwell in his legendary theory, the electromagnetic wave carries real power as it propagates along the direction of propagation. Take the example of a radar as shown in Figure 1.10. The wave from the radar’s transmitter travels with the speed of light and hits a target at a certain distance in free space. As the electromagnetic wave hits the target, it experiences a different impedance than that for the free space. Therefore, the reflection of the electromagnetic wave is due to the fact that the intrinsic impedance of the object is different from that of the free space. The intrinsic impedance is defined by the ratio of the electric and magnetic field intensities at a point in space. Therefore, we understand that the reflection of the electromagnetic wave is due to the change in the intrinsic impedance of the medium in which it propagates. The reflected signal from the target starts propagating in the reverse direction and the radar’s antenna receives the returned echo of the target.¹⁷ The sensitive radar receiver processes the weak received signal and provides useful information of the target. Usually, a radar provides the target’s distance from the radar, the speed and direction of travel of the target, and the bearing of the target.

Now let us derive the intrinsic wave impedance using Maxwell’s equation that we derived in Section 1.6:

$$\frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \tag{1.31}$$

Substitution for

$$E_x(z, t) = E_x^+(z, t) + E_x^-(z, t) = f_1\left(t - \left(\frac{z}{v}\right)\right) + f_2\left(t + \left(\frac{z}{v}\right)\right) \tag{1.59}$$

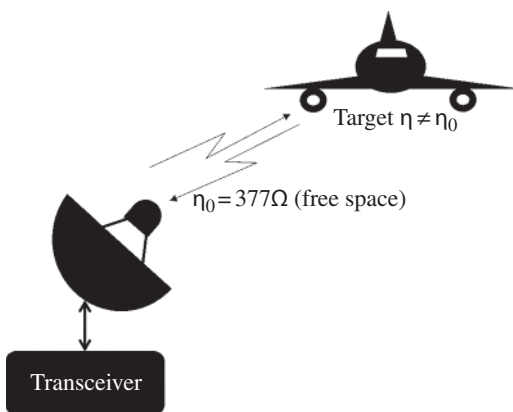


Figure 1.10 A radar hits a target in free space and the returned echo from the target is received by the radar receiver. The reflection is due to the fact that the intrinsic impedance of the object is different than that for the free space.

¹⁷ In previous section we solved wave equation and found the general solution has two components: the forward travelling wave function $F_1(t - (z/v))$ and the reverse travelling wave function $F_2(t + (z/v))$. The reflected signal is the reverse travelling wave in the radar example.

in the left-hand side of the above equation yields,

$$\frac{\partial E_x(z, t)}{\partial z} = -\frac{1}{v}f'_1\left(t - \left(\frac{z}{v}\right)\right) + \frac{1}{v}f'_2\left(t + \left(\frac{z}{v}\right)\right) = -\mu \frac{\partial H_y}{\partial t} \quad (1.60)$$

Now integrating Equation (1.60) with respect to z ,¹⁸ we obtain:

$$H_y(z, t) = H_y^+(z, t) + H_y^-(z, t) = \frac{1}{\mu v}E_x^+(z, t) - \frac{1}{\mu v}E_x^-(z, t) \quad (1.61)$$

As can be seen, H_y has two parts: (i) a forward travelling wave component $H_y^+(z, t)$, and (ii) a reverse travelling wave component: $H_y^-(z, t)$. Linking the respective forward and reverse travelling waves of both electric and magnetic field components, we can define the intrinsic wave impedance η as follow:

$$\eta = \frac{E_x^+(z, t)}{H_y^+(z, t)} = -\frac{E_x^-(z, t)}{H_y^-(z, t)} = \frac{1}{\mu v} = \sqrt{\frac{\mu}{\epsilon}} \quad (\Omega) \quad (1.62)$$

The intrinsic wave impedance is always positive and has dimension of ohms. It is very useful in wave analysis. In free space, the intrinsic wave impedance is:

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7}}{\frac{1}{36\pi} \times 10^{-12}}} \cong 377 \quad (\Omega) \quad (1.63)$$

Review Questions 1.10: Wave Impedance

- Q1.** What is intrinsic wave impedance?
- Q2.** What is the value of intrinsic wave impedance in vacuum?
- Q3.** Show that the unit of intrinsic wave impedance is Ohm.¹⁹

Exercise 1.5: One-dimensional Wave Equation

Q1. Derive the wave impedance using the following Maxwell's equation: $(\partial E_y / \partial z) = \mu(\partial H_x / \partial t)$

1.8 Time Harmonic Field Wave Equations

In the preceding chapter we have already introduced the time harmonic fields and its significance. In engineering applications, the sinusoidally varying or the time harmonic fields are of special significance. In laboratory and in the field, we measure such time varying field waveforms using oscilloscopes. Even if the field variation is not purely sinusoidal, any field function can be decomposed

¹⁸ Here we neglect any constant of integration which can give rise to at best a static field.

¹⁹ Use the proper dimensions of permittivity and permeability.

with Fourier series expansion as the combinations of purely sinusoidally varying field functions. In addition to these, the time harmonic field expression in the phasor vector formed with the aid of Euler's theorem simplifies the field analyses of complex problem so greatly that it becomes an integral tool for practical field solutions in the electromagnetic theory.

Now we go back to the Helmholtz generic wave equations for the electric and magnetic fields that we derived in the beginning of the chapter:

$$\nabla^2 \mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{1.13}$$

$$\nabla^2 \mathbf{H} - \mu\sigma \frac{\partial \mathbf{H}}{\partial t} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \tag{1.15}$$

In time harmonic field analysis, Helmholtz simplified the time harmonic case by simply substituting the time derivative $\partial/\partial t = j\omega$ in the above generic field wave equations. How we are getting this? Do you have a clue?

Euler's theorem defines:

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \tag{1.64}$$

Therefore, if we take time derivative of $e^{j\omega t}$, we obtain

$$\frac{de^{j\omega t}}{dt} = j\omega e^{j\omega t}; \frac{d^2 e^{j\omega t}}{dt^2} = (j\omega)(j\omega)e^{j\omega t} = -\omega^2 e^{j\omega t} \tag{1.65}$$

This gives us the substitution of generic time varying field quantities to time harmonic case ($\partial/\partial t = j\omega$). Thus, Maxwell's equations in source free region can be transformed into their time harmonic expressions as follows:

Time-varying field expression	Time-harmonic field expression
$\nabla \cdot \mathbf{E} = 0$ (1.66a)	$\nabla \cdot \mathbf{E}_s = 0$ (1.67a)
$\nabla \cdot \mathbf{H} = 0$ (1.66b)	$\nabla \cdot \mathbf{H}_s = 0$ (1.67b)
$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$ (1.66c)	$\nabla \times \mathbf{E}_s = -j\omega\mu \mathbf{H}_s$ (1.67c)
$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$ (1.66d)	$\nabla \times \mathbf{H}_s = j\omega\epsilon \mathbf{E}_s$ (1.67d)

We define the time harmonic field expressions with a subscript s where $s = j\omega$ is implied as shown above. Now let us go back to the two generic field wave equations and substitute $\partial/\partial t = j\omega$ then we obtain,

$$\nabla^2 \mathbf{E} - j\omega\mu\sigma \mathbf{E} + \omega^2 \mu\epsilon \mathbf{E} = \nabla^2 \mathbf{E} - j\omega\mu(\sigma + j\omega\epsilon) \mathbf{E} = 0 \tag{1.68}$$

$$\nabla^2 \mathbf{H} - j\omega\mu\sigma \mathbf{H} + \omega^2 \mu\epsilon \mathbf{H} = \nabla^2 \mathbf{H} - j\omega\mu(\sigma + j\omega\epsilon) \mathbf{H} = 0 \tag{1.69}$$

With the substitution, the previous generic wave equations with time derivatives become such a simplified form without any derivative or fraction, they just become a straight equation in one line. Therefore, substitution for $\partial/\partial t$ with $j\omega$ simplifies the wave equations significantly. In short, we can write the above equations as

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0 \tag{1.70}$$

$$\nabla^2 \mathbf{H} - \gamma^2 \mathbf{H} = 0 \tag{1.71}$$

The general solutions of the electric and magnetic field wave equations can be easily derived using the exponential form as follows:

$$\mathbf{E}_{\text{xs}} = E_0^+ e^{-\gamma z} + E_0^- e^{\gamma z} \quad (1.72)$$

and,

$$\mathbf{H}_{\text{ys}} = H_0^+ e^{-\gamma z} + H_0^- e^{\gamma z} = \frac{E_0^+}{\eta} e^{-\gamma z} - \frac{E_0^-}{\eta} e^{\gamma z} \quad (1.73)$$

where

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha + j\beta \quad (1.74)$$

is called the propagation constant with its unit (1/m). And the two components of γ are: α is called *the attenuation constant* (Np/m) and β is called *the phase constant* (rad/m).²⁰ Both are real positive numbers. For a lossy medium with finite conductivity σ , the propagation constant γ is a complex number. If the medium is lossless, then α is zero and $\gamma = j\beta$ is fully imaginary that represents the oscillatory nature and represents uninterrupted propagation of the wave. Now we derive the expressions for α and β in terms of the constitutive parameters.

Taking square of the propagation constant in (1.75), we obtain:

$$\gamma^2 = \alpha^2 - \beta^2 + j2\alpha\beta = j\omega\mu(\sigma + j\omega\epsilon) = -\omega^2\mu\epsilon + j\omega\mu\sigma \quad (1.75)$$

Separating the real and imaginary parts of the above equation, respectively, we obtain

$$\alpha^2 - \beta^2 = -\omega^2\mu\epsilon \quad (1.76a)$$

and

$$2\alpha\beta = \omega\mu\sigma \quad (1.76b)$$

Again, taking the square of the absolute value of γ in (1.75), we obtain,

$$|\gamma|^2 = \alpha^2 + \beta^2 = \omega^2\mu\epsilon\sqrt{1 + (\sigma/\omega\epsilon)^2} \quad (1.77)$$

Adding Equations (1.76a) and (1.77) we obtain:

$$2\alpha^2 = \omega^2\mu\epsilon \left\{ \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right\} \quad (1.78)$$

$$\therefore \alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right]} \quad (1.79)$$

Subtracting Equation (1.76a) from Equation (1.77) and with simple manipulation as above we obtain:

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]} \quad (1.80)$$

²⁰ The phase constant β (rad/m) is also called the wave number. The physical meaning of the wave number of a wave is that how many radians per meter the wave rotates in the phase plane where one wavelength λ represents 2π angular rotation. Therefore, β increases with frequency as well as with the constitute parameter ϵ and μ .

Example 1.6: Wave Propagation in Lossy Medium

Given, the operating frequency $f = 1$ GHz, the conductivity of the medium $\sigma = 0.1$ (S/m), relative permittivity $\epsilon_r = 2.2$, and the relative permeability $\mu_r = 1$, calculate the attenuation constant α in (Np/m) and the propagation constant β in (rad/m).

Answer:

We shall work out the problem step by step. First angular frequency $\omega = 2\pi f$. Then $\mu\epsilon/2$, $(\sigma/\omega\epsilon)$, $\sqrt{1 + (\sigma/\omega\epsilon)^2}$ and finally, we substitute all the calculated numerical values of the given parameters into expression for $\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + (\sigma/\omega\epsilon)^2} - 1 \right]}$ and

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right]}.$$

This yields $\alpha = 22.6$ (Np/m), the unit of attenuation constant. (Np/m) is in natural log scale because we are dealing with exponential form of the wave equation e^α . Np means at what distance the amplitude diminishes by $1/e$ of its original value. $1 \text{ Np} = 8.686 \text{ dB}$.

The angular frequency is derived as $\omega = 2\pi f = 2\pi \times 10^9$ (rad/s)

Now we calculate the parameters in the equations of α and β one by one

$$\frac{\mu\epsilon}{2} = \frac{4\pi \times 10^{-7} \times \left(\frac{10^{-9}}{36\pi}\right) \times 2.2}{2} = 1.22 \times 10^{-17}$$

$$\frac{\sigma}{\omega\epsilon} = \frac{0.1}{2\pi \times 10^9 \times \left(\frac{10^{-9}}{36\pi}\right)} = 1.8; \sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} = 2.06$$

$$\therefore \alpha = 2\pi \times 10^9 \sqrt{1.22 \times 10^{-17} \left[\sqrt{2.06} - 1 \right]} = 22.6 \text{ (Np/m)}$$

$$1 \text{ Np} = 8.686 \text{ dB}$$

$$\therefore \alpha = 22.6 \times 8.686 = 196.3 \text{ (dB/m)}$$

$$\text{The phase constant is: } \beta = 2\pi \times 10^9 \sqrt{1.22 \times 10^{-17} \left[\sqrt{2.06} + 1 \right]} = 38.4 \text{ (rad/m)}$$

As stated before, the phase constant is also called *the wave number*.

 α and β in Lossless Medium

For loss medium $\sigma = 0$, this simplifies the calculation significantly as follows:

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{0}{\omega\epsilon}\right)^2} - 1 \right]} = 0 \quad (1.81)$$

$$\beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left[\sqrt{1 + \left(\frac{0}{\omega\epsilon}\right)^2} + 1 \right]} = \omega \sqrt{\mu\epsilon} = \frac{2\pi f}{u_p} = \frac{2\pi}{\lambda} \quad (1.82)$$

$$\therefore \beta = \frac{\omega}{u_p} = \frac{2\pi}{\lambda} \quad (1.83)$$

The phase velocity is:

$$u_p = \frac{\omega}{\beta} \quad (1.84)$$

Substitutions for the value of angular frequency $\omega = 2\pi \times 10^9$ (rad/s) and the free space wave velocity

$$u_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ (m/s) yield:}$$

$$\beta = \frac{2\pi \times 10^9}{\frac{3 \times 10^8}{\sqrt{2.2}}} = 31.06 \text{ (rad/m)}$$

Review Questions 1.11: Wave Propagation in Lossy Medium

- Q1.** What is α ? What is β ? Can you explain them in terms of exponential form of these quantities?
- Q2.** We do not like a lossy medium as a wave loses its power due to the finite conductivity of the media. Explain from the above examinations of propagation and attenuation constants, and wave impedance, how we can know if a medium is lossy and how much power we are losing.
- Q3.** Explain the significance of lossy and low loss media in RF/microwave/mm-wave active and passive circuit design.
- Q4.** Explain if you prefer to use a low cost but highly lossy FR4 laminate for microwave design.
- Q5.** As a microwave design engineer, explain how do you modify the laminate for efficient design with low cost FR4 laminate.

1.8.1 Summary of Propagation Constant

- 1) If we put α , a real number as the exponent with its negative value along with the distance z , $e^{-\alpha z}$, this represents exponentially decaying factor as the wave propagates along the positive z -direction. Thus, α is called *the attenuation constant*.
- 2) If we put $j\beta$, an imaginary number as an exponent with its negative value along with the distance z , $e^{-j\beta z}$, this represents uninterrupted oscillating factor as the wave propagates infinitely along the z -direction. This is an oscillating quantity with phase velocity u_p , time period T , and wavelength λ . β is called *the phase constant* or *the wave number*.
- 3) The phase constant is also called *the wave number*, because it tells how many radians a wave cycles in one meter. This is a very important concept for practical design. Therefore, development of this concept of propagation constant, attenuation and phase constant is very significant in practical design exercises.
- 4) The propagation constant β is very important concept in wave propagation as it conveys many important pieces of information about the characteristics of the wave in the specific media and guided structures. We shall cover these elements in the subsequent chapters.

- 5) The concepts of the attenuation constant α and the phase constant β are integral parts of any wave type.
- 6) For simplicity, we, in most cases, neglect the attenuation constant α and only work on the phase constant β . For loss less media, the conductivity σ is zero, then the propagation constant γ reduces to $j\beta$, where $\beta = \omega\sqrt{\mu\epsilon} = 2\pi/\lambda$ (rad/m). This means the wave does not attenuate as it travels. We use β in every calculation of plane wave, transmission lines, waveguides, optical fibers and antennas.
- 7) Finally, the phase velocity, u_p , which is also called the velocity of wave propagation, or you can say the velocity of the wave front is $u_p = \omega/\beta$. This is another important expression that we shall be using in the above occasions.

1.9 Refractive Index of Medium and Dispersion

The refractive index of an optical medium is a very important design parameter in optical fiber design. It is considered to be one of the most important characteristic parameters of wave propagation in optical spectra. Maxwell hypothesized the concept of the light waves as electromagnetic waves from the definition of the refractive index n . The refractive index n of a medium is defined as the ratio of the phase velocity of light in vacuum to the phase velocity of light in that medium. Thus, the refractive index is defined mathematically as:

$$n = \frac{u_0}{u_p} = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}} = \sqrt{\frac{\mu_0\mu_r\epsilon_0\epsilon_r}{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r} \quad (1.85)$$

Most optical media are dielectric media, and the relative permeability $\mu_r \cong 1$, and the refractive index is defined as:

$$n = \sqrt{\epsilon_r} \quad (1.86)$$

For the above example with the relative permittivity $\epsilon_r = 2.2$, and the relative permeability $\mu_r = 1$, the refractive index is $n = \sqrt{\epsilon_r} = 1.48$. Note for most optical materials $\mu = \mu_0$, therefore, (1.86) is a valid expression. On the basis of the expression of n , the refractive index, Maxwell hypothesized that the light wave and the electromagnetic wave have the same wave phenomena.

As stated above, the refractive index of an optical medium is the key parameter in designing optical fibers in which the refractive index is gradually changed for avoidance of modal dispersion. See the two illustrations of two differently refractive index graded optical fibers in Figure 1.11a and observe how by changing the refractive index profile inside an optical fiber the signal propagation changes its profile. Also observe how the incoming pure impulse is distorted at the end of the optical fibers. The profiling of the refractive index inside the optical fiber helps reduce distortion of the light wave inside the optical fiber.

Likewise, our eyes are conceived as the optical devices that communicate optically with the outer world. Figure 1.11b shows the refractive indices of various parts of a human eye. With various eye diseases such as myopia, hyperopia, and astigmatism, the refractive indices of various layers of eyes change and cause focusing problem of optical signals. The rays of lights with a sound eye with 20/20 vision and unfocused ray with eye diseases are shown in Figure 1.11b. Using appropriate lenses in reading glasses we can correct the eyesight errors.

As you may observe in Figure 1.11a that with the change of the refractive index profile, the pulse distortion at a certain distance in the optical fiber can be controlled. This distortion of pulses due to the refractive index change is called *dispersion* and the medium in which the refractive index

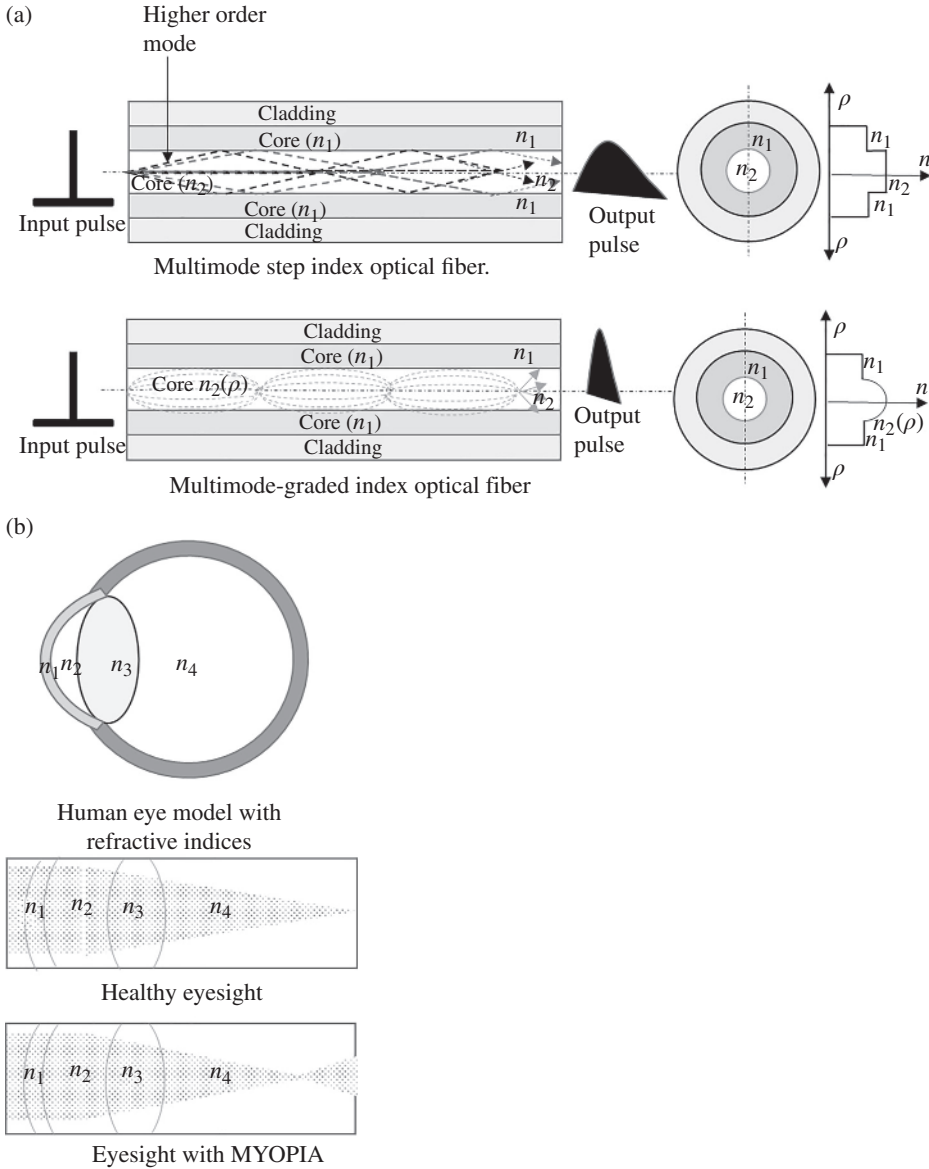


Figure 1.11 (a) Index graded optical fibers, and (b) refractive indices of various parts of a human eye and eyesight with health and diseased eyes.

changes with the frequency is called *the dispersive medium*.²¹ This pulse broadening happens with the change of the relative permittivity ϵ_r of the medium and hence the phase velocity u_p of the wave inside the medium with frequency f . For example, the refractive index of distilled water at microwave frequency is $n = 9$. At optical frequency, such as 500 THz, the frequency of sodium light,²²

21 The practical design of a low dispersion optical fiber consider various dispersion such as material and waveguide dispersion. Author encourages readers to investigate these topics via open sources.

22 Islam, M.A. (1969). *Electromagnetic Theory*, 348–349. Dacca, East Pakistan: EPUET.

the refractive index $n = 1.33$. This reveals that at the frequency in the optical range, the value of refractive index n does not remain the same as for the microwave frequency. In this example, there is a significant change in the relative permittivity $\epsilon_r = n^2$ from 81 in the microwave frequency to only 1.77 in the optical frequency. Since the phase velocity u_p is inversely proportional to $\sqrt{\epsilon_r}$ or $1/n$, the wave travels in different speeds in the medium. The wave has many harmonic components and suffers from relative displacement of the harmonic components in phase as the wave propagates along the medium. The resultant wave is a distorted version of the incoming wave at certain distance in the medium. We define this distorted version of the signal with harmonic components as *the dispersed signal*. This means the signal is scattered or spread out in space and time. This phenomenon of signal broadening is called *dispersion*. The medium as discussed above in which the phase velocity is a function of frequency $u_p = u_p(f)$ is called a *dispersive medium*.²³

1.9.1 Summary of Wave Propagation in Lossless Medium

Now we can summarize the findings about the EM wave in lossless media as follow.

- 1) Phase constant beta is $\beta = \omega/u_p = 2\pi/\lambda$ (rad/m). For the last example substitution for the angular frequency ω , free space permittivity ϵ_0 , the medium relative permittivity ϵ_r and the free space permeability μ_0 , we get $\beta = 31.06$ (rad/m).
- 2) For free space phase velocity, $u_p = c = 3 \times 10^8$ (m/s).
- 3) The refractive index of a medium is defined as the ratio of phase velocity of a signal in free space versus the velocity in the medium: $n = u_0/u_p$ is simplified by simple mathematical manipulation to $n = \sqrt{\mu_r \epsilon_r}$.
- 4) For most optical fibres which are made of dielectric media such as silica or plastic, the permeability $\mu = \mu_0$, gives rise to $n = \sqrt{\epsilon_r}$. For the above example $n = 1.48$.
- 5) We use Taconic TLX0 with dielectric constant or ϵ_r of 2.3. As homework, you find ϵ_r for most used commercial substrate such as Rogers Duroid, Taconic TLX series and RF4. Example includes fire resistant FR4 $\epsilon_r = 4.2$ is the low-cost and most-used laminate in low frequency RF while low loss Rogers Duroid, Taconic TLX laminates are used in microwave frequencies.

Review Questions 1.12: Wave Propagation in Lossy Medium

- Q1.** Define refractive index.
- Q2.** For an optical design what would be the refractive index?
- Q3.** Explain how the refractive index is graded in optical fiber design and why?
- Q4.** What is dispersive medium? How does the refractive index plays a significant role in mitigating dispersion in optical spectra?
- Q5.** Explain the contrasting difference in the refractive index of a medium in microwave and optical spectra.

²³ Chapter 7- Wave Propagation in Dispersive Medium deals with the analyses of waves in various dispersive medium.

1.10 Time Harmonic Wave Solution

As we have defined before that the time harmonic wave equation is a function of sine and cosine functions, and therefore, is oscillatory in nature with an angular frequency $\omega = 2\pi f$ and synchronised with a time period $T = 2\pi/\omega$. The wave also varies sinusoidally as it propagates along the direction of propagation. We have defined the solution of the plane wave with the wave function $F_1(t - (z/v))$, which is a two-dimensional function with space z and time t , respectively as shown in Figure 1.12. Therefore, the second periodicity comes with respect to space with a factor kz . Where $k = 2\pi/\lambda$, where k is called *the wave number* or *phase constant* and λ is *the wavelength*. Note there is a subtle difference between the phase constant β that we derive in the above theory of plane wave and the wave number k , that usually refers to the phase constant in free space. Sometimes in textbooks, k is also replaced with β_u , the free space wave number or propagation constant and β refers to the guided wave number or phase constant.

Let us consider the one-dimensional electric field wave E_x that is propagating along $\pm z$ -directions as time t progresses. Revisiting the generic form of the wave equation:

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 E_x}{\partial t^2} \quad (1.54)$$

with the propagation velocity $v = (1/\sqrt{\mu\epsilon})$

We already have derived the solution for the wave equation with the forward and backward travelling wave functions as follow²⁴:

$$E_x = f_1\left(t - \left(\frac{z}{v}\right)\right) + f_2\left(t + \left(\frac{z}{v}\right)\right) \quad (1.55)$$

We can express the time harmonic case in the sinusoidal function of a specific angular frequency ω and phase velocity u_p in the following form:

$$\begin{aligned} E_x(z, t) &= E_x^+(z, t) + E_x^-(z, t) \\ &= E_{x0}^+ \cos\{\omega(t - z/u_p) + \phi_1\} + E_{x0}^- \cos\{\omega(t + z/u_p) + \phi_2\} \\ &= E_{x0}^+ \cos\{\omega t - k_0 z + \phi_1\} + E_{x0}^- \cos\{\omega t + k_0 z + \phi_2\} \end{aligned} \quad (1.87)$$

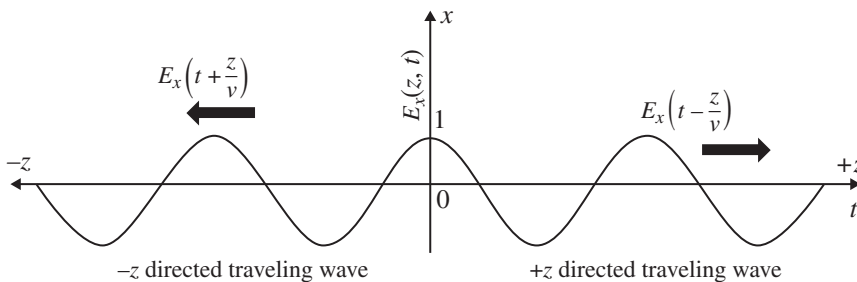


Figure 1.12 General wave solution of an x-directed field propagating in $\pm z$ -direction.

²⁴ Similar derivation is also available in Hayt, W.H. and Buck, J.A. (2001). *Engineering Electromagnetics*, 6e. Boston, Ma, USA: McGraw Hill.

Here,

$$k_o = \omega/u_p \text{ (rad/m)} \quad (1.88)$$

is called the free space *wave number* or *phase constant* of the wave.

As discussed before, the wave number k has a very practical significance in time harmonic electromagnetic analyses. The wave number defines the number of full wave cycles in radian²⁵ per meter as a wave cycle periodically with 2π radians per cycle. The solution of the time harmonic electric field in the Equation (1.87) is called *the instantaneous* or *time domain* form of the field. It has two components: the first part of the right-hand side is called *the forward travelling wave* and the second part is called *the reverse travelling wave*. Now we can analyze the functions in the parenthesis of cosine functions:

$$\cos \{ \omega t \pm k_o z + \phi_i \}$$

for $i = 1, 2$.

The simplest term is the phase terms ϕ_i for $i = 1$ and 2 , that represent the initial state of the signal in the forward and reverse directions, respectively. If properly referenced, the phase quantities can be made zero. For the first and second terms in the parenthesis both have units of angle and expressed in radians. The angular frequency $\omega = 2\pi f$ (rad/s) measures the phase shift of the signal in radians of a periodic signal per second. In a similar annotation, we can define k_o as *the spatial frequency* that measures the phase shift per unit length in radians per meter along the direction of propagation, in this case along z -axis. Assume the time is fixed, the wave travels a full wavelength is the distance over which the spatial phase shift is 2π (rad). At that time the spatial phase shift²⁶ in free space is defined as:

$$k_o z = k_o \lambda = 2\pi \text{ (rad)} \quad (1.89)$$

This understanding leads to the derivation of the wavelength in terms of the phase constant as follows:

$$\lambda = \frac{2\pi}{k_o} \text{ (m)} \quad (1.90)$$

As the distance progresses with time, the instantaneous field in Equation (1.87) above travels with the sinusoidal function in both space and time. Therefore, there are crests and troughs in the waveform. Neglecting the phase shifts $\phi_{1,2}$, which represent the initial conditions of the waves, the wave travels a full cycle of 2π from a crest to its successive one and this will continue to repeat. Therefore, we can define the spatial angular distance in radians as the integer multiple of 2π as

$$k_o z = 2\pi m \text{ (rad)} \quad (1.91)$$

Now we allow time to grow, then we can define the argument of the cosine functions as:

$$\omega t - k_o z = \omega \left(t - \frac{z}{c} \right) = 2 m \pi \text{ (rad)} \quad (1.92a)$$

²⁵ Note that 2π (rad) is equivalent to one cycle and a full wave length λ in meters.

²⁶ The spatial phase shift is also called the electric length in radians of the signal.

This means that as time progresses the distance must be positively increased to make the term in the parenthesis constant. This is the definition of *the forward travelling wave*. Using similar argument, we can say that

$$\omega t + k_0 z = \omega \left(t + \frac{z}{c} \right) = 2 m \pi \text{ (rad)} \quad (1.92b)$$

represents *the reverse travelling wave*.

Review Questions 1.13: Characteristics of electromagnetic waves

- Q1. Explain the difference in wave number k and the phase constant β in the electromagnetic wave theory.
- Q2. Define the instantaneous or time domain form of the field.
- Q3. Define spatial frequency and its unit.

1.11 Poynting Theorem²⁷

The above discussion on the fundamentals of electromagnetic laws and the electromagnetic wave theory can be derived from two aspects: (i) initially the action at a distance (force) such as the laws of Coulomb, Gauss, Biot-Savart, Ampere in electrostatic and magnetostatic cases, and Faraday in dynamic cases, and (ii) Maxwell's field theory due to the electromagnetic interaction. The legendary *continuity equation* of Maxwell had opened up the gate of the modern time varying electromagnetic wave theory. As you may recall that *the continuity equation* suggests that the time rate of diminishing charges in an enclosed surface generates the diverging current. This means the rate of the decrement of the stored charge is transformed into the diverging current. *The displacement current* concept, another legendary discovery of time varying electromagnetic field theory, is also derived from *the continuity equation* of Maxwell. Maxwell's original contribution leads to the wireless energy propagation through any medium either guided or wireless. In this section we shall discuss the energy balance equation in an enclosed surface that is derived by an English Physicist J. D. Poynting in 1883.²⁸ At the same time, Heaviside also reported similar theory of the energy balance equation.²⁹ To understand this energy balance equation, we shall first understand the energy stored and dissipated in the circuit elements as described in the circuit theory and also shown in Figure 1.13a. We shall draw an analogy of the energy stored in the circuit theory with those for the electrostatic and magnetostatic cases in a closed area which is defined by a differential volume ΔV enclosed by a closed surface area ΔS as shown in Figure 1.13b. Finally, we shall consider the dynamic electromagnetic cases and shall derive the energy balance equations for the time varying electromagnetic fields at any arbitrary time rate of changes.

²⁷ Similar derivation is also available in Islam, M.A. (1969). *Electromagnetic Theory*. Dacca, East Pakistan: EPUET.

²⁸ Poynting, J.D. (1883). On the transfer of energy in the electromagnetic field. *Phil. Trans.*, 174: 343 (quoted in Sadiku, M. N.O. (2001). *Elements of Electromagnetics*, 3e, 363. Oxford University Press.)

²⁹ Islam, M.A. (1969). *Electromagnetic Theory*. Dacca, East Pakistan: EPUET.

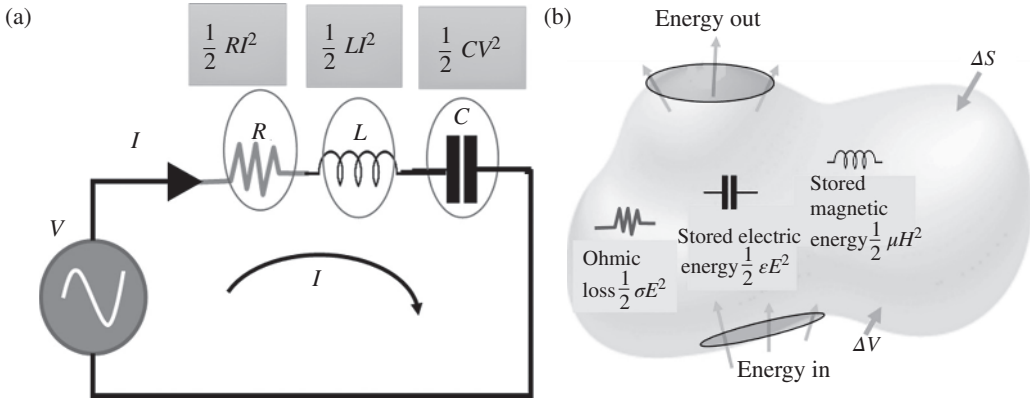


Figure 1.13 (a) An RLC AC circuit with impressed voltage V , and (b) equivalent Poynting theorem is a power balance equation for EM fields.

In guided electromagnetics, electromagnetic energy is transported from one point to another point via a transmission line.³⁰ We are more used to with the voltage V (equivalent to the electric field E) across the terminal and the current I (equivalent to the magnetic field H) through the circuit elements. Contrary to the guided energy case, based on Maxwell’s field wave theory, the electromagnetic energy is also propagated in free space as the unguided wave. However, in static and low frequency cases, we can assume that the energy is stored in separate cases of electric and magnetic energies in the respective fields in a medium with their respective finite constitutive parameters, permittivity ϵ and permeability μ . For a lossless medium the conductivity and Ohmic loss can be neglected.

First, we start with the AC circuit theory of the reactive elements with inductance L and capacitance C . In the circuit theory, the magnetic energy stored in the circuit element is defined as $(1/2) LI^2$ where, I is the current. Likewise, the electric energy stored in a capacitor is defined as $(1/2) CV^2$ where V is the voltage across the capacitor. If the transmission line and the circuit elements are lossy then the real power loss is represented by a resistor R . The time-average power dissipated as heat is defined as $(1/2) RI^2$.

Now if we draw the analogy of the static electric and magnetic cases³¹ with the above circuit theory, the electric and magnetic energies stored in a medium with permittivity ϵ and permeability μ can be defined as $(1/2)\epsilon E^2$ and $(1/2)\mu H^2$, respectively. If the medium is lossy then we can also derive the real power loss as heat dissipation as $(1/2)\sigma E^2$, where σ is the conductivity of the medium. The above definitions of the energy equations are based on the static and low frequency dynamic cases and is perceived that the energy accumulation is happening in a slowly building up form like a summation of the consecutive static states.³²

30 The transmission line is perceived as the distributive circuit elements with combinations of R , L , and C . We shall derive the wave equations based on the circuit elements using Telegraphist’s equations in Transmission Line theory chapter later on.

31 We reserve the term electromagnetic for the time varying case and here we define the static cases in separate definitions “Static electric and magnetic cases” instead of defining “static electromagnetic case.”

32 Revisit 3.9 Electric Energy and Joule’s Law; Section 3.11 Electrostatic Potential Energy and Section 3.19 Magnetic Energy and Inductance.

The objective of the above analogy is to get an understanding whether such energy accumulation as the stored electromagnetic energy is valid in any arbitrary rate in the time varying cases. To validate the theory, first we take a small volume ΔV within which we would like to evaluate the energy state as shown in Figure 1.13b. Assume that the total enclosed surface area of the volume is ΔS . Here we assume that the enclosed surface ΔS with a volume ΔV can accept energy as incident electromagnetic fields denoted as “Energy in,” and also energy flowing out as “Energy out.”³³ In between, there are some electromagnetic interaction based on the field theory of Maxwell, in the forms of stored electric energy (symbolized as capacitor) and magnetic energy (symbolized as inductor), and the real power dissipation (symbolized as resistor). This means in between the “energy in” and “energy out” there is some energy that can be stored in the form of electric and magnetic energies as shown with the symbols of capacitance and inductance, respectively, plus some real power loss as heat in the resistance.

Based on *the principle of the conservation of energy*, any increase of energy within the volume ΔV must be accompanied by the equal amount of energy flowing into the volume via the enclosed surface ΔS . Likewise, any decrease of energy within the volume ΔV must be accompanied by the equal amount of energy flowing out of the volume via the enclosed surface ΔS . First consider the energy flowing out of the surface per unit volume “per unit time” as a vector \mathbf{P} (W/m²). Also, for simplicity assume that the medium is loss less $\sigma = 0$. Then there must be diminishing of energy in the volume and we can define the energy balance with the following equation:

$$\int \mathbf{P} \cdot d\mathbf{S} = - \int_v \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) dV \quad (\text{W}) \quad (1.93)$$

Here the unit of \mathbf{P} is in (W/m²) and $(1/2)\epsilon E^2$ and $(1/2)\mu H^2$ are in (J/m²). The above equation is the energy balance equation in the integral form for lossless case. Now considering the differential form of the equation that can be written using the divergence theorem on the enclosed surface as follows:

$$\int_v (\nabla \cdot \mathbf{P}) dV = - \int_v \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) dV \quad (\text{W}) \quad (1.94a)$$

or

$$\nabla \cdot \mathbf{P} = - \frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) \quad (\text{W/m}^2) \quad (1.94b)$$

Here, \mathbf{P} is power density per unit area and is called *the Poynting vector*, after the name of J. D. Poynting.

Now consider the rapidly time varying field case in which the electric and magnetic fields are present simultaneously and the fields are governed by Maxwell’s equations. Recalling from Equations (1.4)–(1.7). The time varying electromagnetic field equations for any linear, isotropic, homogeneous medium are:

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon} \quad (1.4)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (1.5)$$

³³ Note that the figure shows the representations of area of in and out of energy. Energy can be in from any point/area and out from any point/area depending on the configuration of an electromagnetic system. As for example, for an antenna, the input energy is provided via its input point and the output energy is the radiation from its radiation aperture or length.

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (1.6)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \quad (1.7)$$

We observe that for a lossy medium with finite conductivity σ , we define Ohm's law $\mathbf{J} = \sigma \mathbf{E}$ that represents the Ohmic or real power loss. We drop subscript c for conduction current for simplicity. Therefore, $\mathbf{J} \cdot \mathbf{E} = \sigma E^2$ represents the real power loss as heat dissipation. To develop the generic energy balance equation with all energy stored and losses we start with the dot product of \mathbf{E} and Ampere's law $\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon(\partial \mathbf{E}/\partial t)$:

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{E} \cdot \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = \mathbf{E} \cdot \mathbf{J} + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \quad (1.95)$$

Likewise, if we take a dot product between Equation (1.6) and \mathbf{H} , we obtain

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} \quad (1.96)$$

Now subtracting Equation (1.95) from (1.96) we obtain:

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) = -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \mathbf{E} \cdot \mathbf{J} - \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \quad (1.97)$$

Using the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \quad (1.98)$$

we obtain

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = - \left(\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{E} \cdot \mathbf{J} \right) \quad (1.99)$$

Assuming the medium is linear, homogeneous, isotropic, and does not change with time, we can simplify the energy equations into the following forms:

$$\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2} \frac{\partial (\mu \mathbf{H} \cdot \mathbf{H})}{\partial t} = \frac{1}{2} \frac{\partial (\mu H^2)}{\partial t} \quad (1.100)$$

$$\varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2} \frac{\partial (\varepsilon \mathbf{E} \cdot \mathbf{E})}{\partial t} = \frac{1}{2} \frac{\partial (\varepsilon E^2)}{\partial t} \quad (1.101)$$

Substitution for Equations (1.100) and (1.101) into Equation (1.99) we obtain:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = - \frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) - \mathbf{E} \cdot \mathbf{J} \quad (1.102)$$

This is the differential form of the energy balance equation. The integral form of the energy balance equation can be obtained by integrating the above equation over a volume enclosed by the closed surface area S and using the divergence theorem as follows:

$$\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = - \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) - \mathbf{E} \cdot \mathbf{J} \right] dV \quad (1.103)$$

$$\int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = - \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) - \mathbf{E} \cdot \mathbf{J} \right] dV$$

or,

$$\oint_S \mathbf{P} \cdot d\mathbf{S} = - \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) - \mathbf{E} \cdot \mathbf{J} \right] dV \quad (1.104)$$

Here

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} \quad (\text{W/m}^2) \quad (1.105)$$

is used. Equation (1.104) is referred to as *the Poynting theorem*. As stated above, the equation was derived by J. D. Poynting in 1883 and also by Heaviside at about the same time. From the examination of the left side of the above equation we can easily state that the diverging energy density term $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ (W/m²) on an enclosed surface S equates with the decrement of the time rate of change of the electric and magnetic energies subtracted by the real power dissipation in the volume. By *the law of conservation of energy*, we can say that the decrement of energy must be equal to the power (time rate of energy) flowing out of the volume through the enclosed surface. The power flow density vector is designated as Equation (1.105). The vector \mathbf{P} is called *the Poynting vector*.

If we compare Maxwell's *current continuity equation* with the above *Poynting theorem* Equation (1.104), we can draw nice similarities between them. For *the continuity equation*, the negative time rate of change of the stored charge ($-(\partial Q/\partial t)$) in a volume must be equal to the diverging current density $\nabla \cdot \mathbf{J}$. For *the Poynting theorem*, the negative rate of change of the stored energy minus the Ohmic loss must equate with the diverging power flow density vector $\nabla \cdot \mathbf{P} = \nabla \cdot (\mathbf{E} \times \mathbf{H})$ where, the power flow density vector or *the Poynting vector* is defined as: $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ in (W/m²). In the following sections, we shall examine a few special cases of Poynting theorem.

Review Questions 1.14: Poynting Theorem

- Q1.** Draw a mud map to reflect how the energy balance equation evolved from Maxwell's current continuity equation.
- Q2.** Explain the difference between static and low frequency and time varying energy balance cases.
- Q3.** Derive the analogy between the circuit theory and the electromagnetic energy balance theory.
- Q4.** What is the equivalence of the circuit theory with the electromagnetic energy balance theory in the static and low frequency cases?
- Q5.** Derive the Poynting vector equations from the electric and magnetic potential energy theorems for the static case.
- Q6.** What is the relevance between the transmission line and an enclosed volume with electromagnetic phenomenon?
- Q7.** What is the difference between the concepts of the energy balance in the circuit theory and the theory developed by Poynting using Maxwell's equations?
- Q8.** Derive Poynting theorem from Maxwell's equations for the time varying case.
- Q9.** Relate Poynting theorem with Maxwell's current continuity equation and interpret each elements of the Poynting theorem.

1.12 Static Poynting Theorem

In static electromagnetic case such as DC or low frequency AC, there is no rate of change of the electromagnetic energy within the volume of investigation. Therefore, the Poynting theorem reduces to:

$$\int_S \mathbf{P} \cdot d\mathbf{S} = - \int_V \mathbf{E} \cdot \mathbf{J} dV \tag{1.106}$$

Assume the volume contains no source. Thus, in static case the net influx of power within the system is fully dissipated as Julian heat or power loss. In the following, an example is given for a conductor.

1.12.1 Poynting Theorem for a Wire

Figure 1.14 shows a wire with radius a and carrying a DC current I . The conductivity of the wire is σ and length is L . We investigate the Poynting vector and the total power loss due to the finite conductivity. We shall also define the real power loss in terms of the circuit parameter R .

Since the wire is a circularly symmetric configuration, we employ cylindrical coordinate system (ρ, ϕ, z) . From our understanding of the DC current I and resistance R , we can assume the current I is uniformly distributed over the cross-sectional area as shown in Figure 1.14. The current density is:

$$\mathbf{J} = \frac{I}{\pi a^2} \mathbf{a}_z \text{ (A/m}^2\text{)} \tag{1.107}$$

Applying Ohm's law the electric field is:

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{I}{\sigma \pi a^2} \mathbf{a}_z \text{ (A/m)} \tag{1.108}$$

Applying Ampere's law, the magnetic field on the surface of the wire is:

$$\mathbf{H} = \frac{I}{2\pi a} \mathbf{a}_\phi \text{ (A/m)} \tag{1.109}$$

The Poynting vector is the product of the electric and magnetic fields:

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = \frac{I}{\sigma \pi a^2} \mathbf{a}_z \times \frac{I}{2\pi a} \mathbf{a}_\phi = - \frac{I^2 \sigma}{2\pi^2 a^3} \mathbf{a}_\rho \text{ (W/m}^2\text{)} \tag{1.110}$$

Which is directed inside the wire along the radial direction, as shown in Figure 1.14.

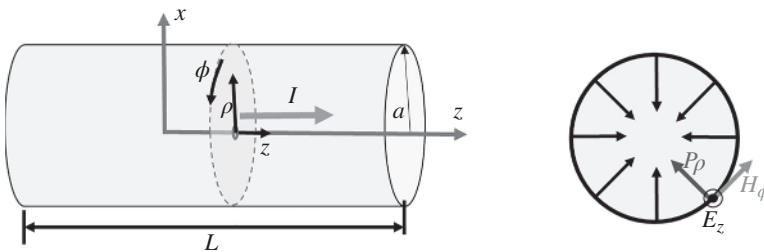


Figure 1.14 A wire with radius a which is carrying a DC current I . The conductivity of the wire is σ and length is L . We calculate the Poynting vector and the total power loss due to the finite conductivity σ . We shall also define the real power loss in term of the circuit parameter R .

Now to verify the Poynting theorem we integrate the Poynting vector \mathbf{P} over the entire volume or the entire enclosed surface which is $2\pi aL$. Let us do the surface integration. In the integration we can ignore the two vertical side surfaces at two ends of the wire of length L due to the fact that the normal of the surfaces is along $\pm z$ whereas the Poynting vector \mathbf{P} is along the radial direction, and therefore, the contributions of the integration of the two surfaces will be zero. Therefore, the integration of ρ over the cylindrical surface arc is:

$$\oint_S \mathbf{P} \cdot d\mathbf{S} = \oint_S -\frac{I^2}{2\sigma\pi^2 a^3} (\mathbf{a}_\rho \cdot \mathbf{a}_\rho) \rho d\phi d\rho = -\frac{I^2}{2\sigma\pi^2 a^3} \times 2\pi aL = -I^2 \left(\frac{1}{\sigma} \times \frac{L}{\pi a^2} \right) = -I^2 R \quad (1.111)$$

where, the resistance is defined as:

$$R = \frac{1}{\sigma} \times \frac{L}{\pi a^2} (\Omega). \quad (1.112)$$

If we draw the analogy of the *current continuity equation*, we can tell that the $-ve$ sign of the Poynting vector represents the power flowing out of the surface.

1.13 Energy Balance Equation in the Presence of a Generator: In-Flux and Out-Flow of Power

So far, we did not consider any source or generator that impressed energy inside the volume. Let us start with the term $\mathbf{E} \cdot \mathbf{J}$ that represents the Ohmic power loss per unit volume. As shown in the above figure now assume that the in-flux of energy generates an impressed electric field \mathbf{E}' .³⁴ Also assume that the conductivity of the medium is σ . Then the total current density \mathbf{J} generated by both the impressed electric field \mathbf{E}' and the electric field \mathbf{E} that is generated by the current and charges within the volume is:

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}') \quad (1.113)$$

This leads to the definition of the original electric field \mathbf{E} in terms of the current density and the impressed electric field as

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} - \mathbf{E}' \quad (1.114)$$

Therefore, the volume integration of $\mathbf{E} \cdot \mathbf{J}$ leads to the following expression:

$$\int_V (\mathbf{E} \cdot \mathbf{J}) dV = \int_V \frac{J^2}{\sigma} dV - \int_V (\mathbf{E}' \cdot \mathbf{J}) dV \quad (1.115)$$

Now examining the above Equation (1.115) we can easily deduce that the first term in the right-hand side which represents the energy dissipation as heat which is an irreversible process. The second term represents the impressed energy flowing inside the system. The $-ve$ sign represents some works need to be done by the external source to impress the electric field \mathbf{E}' .³⁵ This supplied power

³⁴ The prime sign is usually reserved to represented quantities and coordinate systems related to source.

³⁵ Recall the definition of electric potential $V = -\int \mathbf{E} \cdot d\mathbf{L}$ with the negative sign, it indicates work is needed to bring a unit charge in the influence of the electric field.

is balanced by the increase in the stored electromagnetic energy and heat loss in the system. The rest is flowing out of the surface as $\oint_S \mathbf{P} \cdot d\mathbf{S}$. Let us prove the above statement with the following mathematical manipulation.

$$\oint_S \mathbf{P} \cdot d\mathbf{S} = - \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) \right] dV - \int_V \frac{J^2}{\sigma} dV + \int_V (\mathbf{E}' \cdot \mathbf{J}) dV \tag{1.116}$$

or,

$$\int_V (\mathbf{E}' \cdot \mathbf{J}) dV = \oint_S \mathbf{P} \cdot d\mathbf{S} + \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) \right] dV + \int_V \frac{J^2}{\sigma} dV \tag{1.117}$$

Examination of the above Equation (1.117) reveals that the left-hand side term $\int_V (\mathbf{E}' \cdot \mathbf{J}) dV$ represents the in-flux energy in the system from a generator as shown in Figure 1.15. The first term on the right-hand side $\oint_S \mathbf{P} \cdot d\mathbf{S}$ represents the energy flowing out of the volume through the enclosed surface S . The Poynting vector \mathbf{P} represents energy flow at a point in the system. The second term on the right-hand side $\int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) \right] dV$ represents the rate of change of stored electromagnetic energy in any arbitrary time rate of change in fast varying and/or quasi-static state. The last term $\int_V (J^2/\sigma) dV$ is the irreversible Ohmic loss due to finite conductivity of the system.

A good analogy of the energy balance equation as developed via the Poynting theorem is an antenna as shown in Figure 1.4 in the beginning of the chapter. A source is added to the antenna. The antenna has stored reactive electromagnetic energy which changes with the operating frequency, some power is dissipated as heat due to the finite conductivity of the antenna material, and the rest of the power is radiated as the Poynting vector as $\mathbf{P} = \mathbf{E} \times \mathbf{H}$ (W/m²).

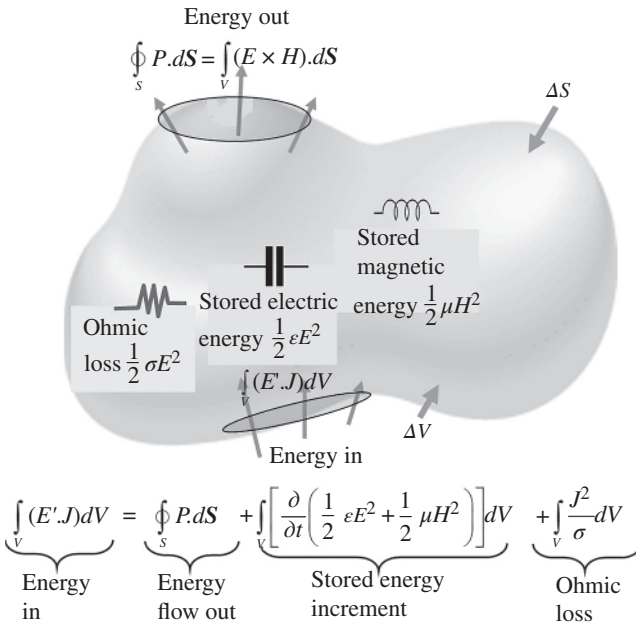


Figure 1.15 Depiction of energy balance Poynting theorem.

Review Questions 1.15: Poynting Theorem

- Q1.** Explain the difference of the energy balance equation without and with a generator.
- Q2.** Explain the difference of the Poynting theorem in static and dynamic cases in the light of Maxwell's equations.

1.14 Time Harmonic Poynting Vector

So far, we have discussed the generic time varying field quantities and developed the Poynting Theorem. However, when describing plane wave propagation, it is customary to express the field quantities in their complex phasor vector or time harmonic forms. As we know that the time domain or the instantaneous expressions of these complex field quantities are the real part of the product of the phasor vector quantity and $e^{j\omega t}$. The instantaneous expression of the field quantities that we can measure in the oscilloscope in the sine and cosine waveforms. In a lossless, linear, isotropic, and homogeneous medium, a positive z-directed electric and magnetic field can be expressed as follows:

$$E_x = \mathbf{a}_x E_0^+ \cos(\omega t - \beta z) \quad (1.118)$$

$$H_y = \mathbf{a}_y H_0^+ \cos(\omega t - \beta z) = \mathbf{a}_y \frac{E_0^+}{\eta} \cos(\omega t - \beta z) \quad (1.119)$$

where η is the intrinsic impedance of the medium.

The instantaneous Poynting vector is expressed as:

$$\mathbf{P}(z, t) = \mathbf{E} \times \mathbf{H} = [\mathbf{a}_x E_0^+ \cos(\omega t - \beta z)] \times \left[\mathbf{a}_y \frac{E_0^+}{\eta} \cos(\omega t - \beta z) \right] = \mathbf{a}_z \frac{[E_0^+]^2}{\eta} \cos^2(\omega t - \beta z) \quad (1.120)$$

Using the trigonometric identity of $\cos^2 \theta = (1/2)(1 + \cos 2\theta)$ we obtain the instantaneous expression for the Poynting vector as:

$$\mathbf{P}(z, t) = \mathbf{a}_z \frac{[E_0^+]^2}{2\eta} [1 + \cos 2(\omega t - \beta z)] \quad (1.121)$$

From the examination of the above expression (1.121), we can see that the time domain expression for the Poynting vector has two parts: (i) the first DC component with amplitude $P_{DC}(z, t) = \mathbf{a}_z \left([E_0^+]^2 / 2\eta \right)$, and (ii) the second AC component with double the frequency $P_{AC}(z, t) = \mathbf{a}_z \frac{[E_0^+]^2}{2\eta} \cos 2(\omega t - \beta z)$.

If we integrate the above AC component over the cycle (2π) then we obtain zero contribution from the AC component.

We are more interested in the time average quantity as this represents the real power loss as heat. The time average power is the real power loss in all resistive elements in the network over a time period or a cycle. The time average power is defined over a time period T or a cycle (2π) as

$$\mathbf{P}_{ave} = \frac{1}{T} \int_0^T \mathbf{P}(x, y, z, t) dt \quad (1.122)$$

or,

$$\mathbf{P}_{ave} = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{P}(x, y, z, t) d(\omega t) \quad (1.123)$$

Now substitution for the value of $\mathbf{P}(z, t) = \mathbf{a}_z \frac{[E_0^+]^2}{2\eta} [1 + \cos 2(\omega t - \beta z)]$ in the above Equation (1.123) we obtain:

$$\begin{aligned} \mathbf{P}_{ave} &= \mathbf{a}_z \frac{1}{2\pi} \int_0^{2\pi} P(x, y, z, t) d(\omega t) = \mathbf{a}_z \frac{1}{2\pi} \int_0^{2\pi} \frac{[E_0^+]^2}{2\eta} [1 + \cos 2(\omega t - \beta z)] d(\omega t) \\ &= \mathbf{a}_z \frac{1}{2\pi} \times \frac{[E_0^+]^2}{2\eta} \left\{ \int_0^{2\pi} d(\omega t) + \int_0^{2\pi} \cos 2(\omega t - \beta z) d(\omega t) \right\} = \mathbf{a}_z \frac{[E_0^+]^2}{2\eta} \end{aligned} \quad (1.124)$$

$$\therefore \mathbf{P}_{ave} = \mathbf{a}_z \frac{[E_0^+]^2}{2\eta} \quad (1.125)$$

The above expression can be deduced from their phasor vector expression using the following expression:

$$\mathbf{P}_{ave} = \frac{1}{2} \text{Re} [\mathbf{E}_s \times \mathbf{H}_s^*] = \frac{1}{2} \text{Re} [\mathbf{E}_s^* \times \mathbf{H}_s] \quad (1.126)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

The Phasor expressions of the electric and magnetic fields are:

$$\mathbf{E}_{xs} = \mathbf{a}_x E_0^+ e^{-j\beta z} \quad (1.127)$$

$$\mathbf{H}_y = \mathbf{a}_y H_0^+ e^{j\beta z} = \mathbf{a}_y \frac{E_0^+}{\eta} e^{-j\beta z} \quad (1.128)$$

Then the time average Poynting vector is defined³⁶ as:

$$\mathbf{P}_{ave} = \frac{1}{2} \text{Re} [\mathbf{E}_s \times \mathbf{H}_s^*] = \frac{1}{2} \text{Re} [\mathbf{E}_s^* \times \mathbf{H}_s] = \frac{1}{2} \text{Re} \left[\mathbf{a}_x E_0^+ e^{-j\beta z} \times \mathbf{a}_y \frac{E_0^+}{\eta} e^{j\beta z} \right] \quad (1.129)$$

$$\therefore \mathbf{P}_{ave} = \mathbf{a}_z \frac{[E_0^+]^2}{2\eta} \quad (1.130)$$

Observation from the above procedure of taking the real part of the electric and the conjugate of the magnetic fields or vice versa of the phasor expressions of the electric and magnetic fields we obtain the same results of the procedure followed in the integration over the full cycle of 2π . Therefore, we can experience the benefit of using the phasor vector calculation in time harmonic field analysis.³⁷

If the medium is lossy then we need to consider the loss term $e^{-\alpha z}$ in the above calculation. The phase difference between the field quantities due to the loss is also needed to be considered.³⁸ The instantaneous expressions for the field quantities are:

$$E_x = \mathbf{a}_x E_0^+ e^{-\alpha z} \cos(\omega t - \beta z) \quad (1.131)$$

³⁶ Usually we take complex conjugate of the magnetic field.

³⁷ Helmholtz devised the expressions of phasor vector method of calculation of time harmonic fields using Euler theorem.

³⁸ Recall the intrinsic impedance becomes complex for a lossy medium. Therefore, the phase difference between the electric and magnetic fields must be considered in the calculation.

$$\mathbf{H}_y = \mathbf{a}_y H_0^+ e^{-\alpha z} \cos(\omega t - \beta z - \theta) = \mathbf{a}_y e^{-\alpha z} \frac{E_0^+}{\eta} \cos(\omega t - \beta z - \theta) \quad (1.132)$$

where the phase angle θ represents the initial phase state of the magnetic field compared with the electric field due to the loss factor. The phasor expression of the above field quantities are:

$$\mathbf{E}_{xs} = \mathbf{a}_x E_0^+ e^{-\alpha z} e^{-j\beta z} \quad (1.133)$$

$$\mathbf{H}_{ys} = \mathbf{a}_y e^{-\alpha z} \frac{E_0^+}{\eta} e^{-j\beta z} e^{-j\theta} \quad (1.134)$$

Using the above procedure, the time average Poynting vector for the lossy medium is:

$$\mathbf{P}_{ave} = \frac{1}{2} \operatorname{Re} \left[\mathbf{a}_x E_0^+ e^{-\alpha z} e^{-j\beta z} \times \mathbf{a}_y e^{-\alpha z} \frac{E_0^+}{\eta} e^{j\beta z} e^{j\theta} \right] \quad (1.135)$$

$$\therefore \mathbf{P}_{ave} = \mathbf{a}_z \frac{[E_0^+]^2}{2\eta} e^{-2\alpha z} \cos \theta \quad (1.136)$$

The above expression uses Euler theorem: $\operatorname{Re}[e^{-j\theta}] = \operatorname{Re}[\cos \theta - j \sin \theta] = \cos \theta$

Review Questions 1.16: Poynting Vector and Time Average Power

- Q1.** Derive the Poynting vector for a lossy medium using the integration method and compared with the phasor vector method of calculation.
- Q2.** Explain the benefit of using the phasor vector notations of the field quantities in time harmonic electromagnetic analysis
- Q3.** Explain why there is a phase difference in the field quantities for a lossy medium.
- Q4.** Define the time average Poynting vector and its SI unit.
- Q5.** Explain why we are more interested in the time average values instead of the instantaneous values of the field quantities and Poynting vector.

Example 1.7: Poynting Theorem³⁹

A y -polarized electromagnetic field wave is propagating along $+z$ direction. The operating frequency is 6 GHz. The time average power density of the wave is 20 (mW/m²). Calculate the electric and magnetic fields in their instantaneous form using the above theorem. Assume that the medium is lossless.

³⁹ Adapted from Johnk, C.T.A. (1988). *Engineering Electromagnetic Fields and Waves*, 2e. NY, USA: Wiley.

Answer:

Since the electric field is propagating along +z direction with y-polarization, the generic expressions for the electric and magnetic fields are:

$$\mathbf{E}_s = \mathbf{a}_y E_0^+ e^{-j\beta z} \quad (1.137)$$

$$\mathbf{H}_s = -\mathbf{a}_x e^{-\alpha z} \frac{E_0^+}{\eta} e^{-j\beta z} \quad (1.138)$$

where the wave impedance $\eta = 120\pi$ (Ω)

The amplitude of the fields can be calculated using the time average power density P_{ave} as follow:

$$P_{ave} = \frac{[E_0^+]^2}{2\eta} = 20 \times 10^{-3} \text{ (W/m}^2\text{)}$$

Therefore, the peak amplitude of the electric field is:

$$E_0^+ = \sqrt{(20 \times 10^{-3} \times 2 \times 120\pi)} = 3.88 \text{ (V/m)}$$

The amplitude of the magnetic field is:

$$H_0^+ = \frac{E_0^+}{\eta} = \frac{3.88}{120\pi} = 0.01 \text{ (A/m)}$$

The angular frequency is: $\omega = 2\pi f = 2\pi \times 6 \times 10^9 = 37.7 \times 10^9$ (rad/s)

The Phase constant of the field wave is:

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{2\pi \times 6 \times 10^9}{3 \times 10^8} = 125.66 \text{ (rad/m)}$$

The expressions for the instantaneous electric and magnetic fields are:

$$\mathbf{E}(t) = 3.88 \cos(37.7 \times 10^9 t - 125.66z) \mathbf{a}_y \text{ (V/m)}$$

$$\mathbf{H}(t) = -0.01 \cos(37.7 \times 10^9 t - 125.66z) \mathbf{a}_x \text{ (A/m)}$$

Example 1.8: Poynting Theorem

A time varying incident wave is defined by

$$E_x(t) = \mathbf{a}_x E_0^+ \cos(\omega t - \beta z) \quad H_y = \mathbf{a}_y H_0^+ \cos(\omega t - \beta z) = \mathbf{a}_y \frac{E_0^+}{\eta} \cos(\omega t - \beta z)$$

Assume that the wave is emanating out of the aperture of a rectangular horn antenna with dimensions $a = 10$ cm, $b = 5$ cm as shown in Figure 1.16. Calculate the total power flux entering a rectangular region of the same dimensions of the aperture of the horn antenna. Also verify the Poynting theorem that $\oint_S \mathbf{P} \cdot d\mathbf{S} = -\int_V [(\partial/\partial t)((1/2)\epsilon E^2 + (1/2)\mu H^2)] dV$ assume the system is lossless $\sigma = 0$. The increase in the reactive power is a phenomenon of the near field of the antenna.

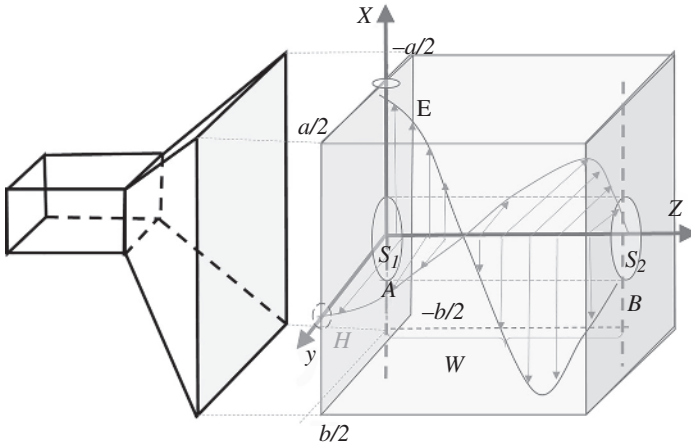


Figure 1.16 The Poynting theorem of near field of a horn antenna with dimensions ab .

Answer:

The contribution for surface S_1 is:

$$P_1(t) = - \int_{S_1} \mathbf{P} \cdot d\mathbf{S} \mathbf{a}_z = - \int_{y=-\frac{b}{2}}^{\frac{b}{2}} \int_{x=-\frac{a}{2}}^{\frac{a}{2}} \left[\mathbf{a}_z \frac{[E_0^+]^2}{\eta} \{ \cos^2(\omega t - \beta z) \} \right]_{z=0} \cdot (-\mathbf{a}_z dx dy) = \frac{[E_0^+]^2}{\eta} ab \cos^2(\omega t) \quad (1.139)$$

The contribution from surface S_2 is:

$$P_2(t) = - \int_{S_2} \mathbf{P} \cdot d\mathbf{S} \mathbf{a}_z = - \int_{y=-\frac{b}{2}}^{\frac{b}{2}} \int_{x=-\frac{a}{2}}^{\frac{a}{2}} \left[\mathbf{a}_z \frac{[E_0^+]^2}{\eta} \{ \cos^2(\omega t - \beta z) \} \right]_{z=W} \cdot (\mathbf{a}_z dx dy) = \frac{[E_0^+]^2}{\eta} ab \cos^2(\omega t - \beta W) \quad (1.140)$$

Total net power flux entering the closed surface S is:

$$P(t) = P_1(t) + P_2(t) = \frac{[E_0^+]^2}{2\eta} ab \{ \cos(2\omega t) - \cos 2(\omega t - \beta W) \} \quad (1.141)$$

where we use: $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$.

Now we shall prove $-\oint_{S_2} \mathbf{P} \cdot d\mathbf{S} = \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) \right] dV$

The right-hand side of the Equation (1.141) is:

$$\begin{aligned} \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) \right] dV &= \frac{\partial}{\partial t} \left[\frac{\epsilon [E_0^+]^2}{4} \int_{z=0}^W \int_{y=-b/2}^{b/2} \int_{x=-a/2}^{a/2} \{ 1 + \cos 2(\omega t - \beta z) \} dx dy dz \right] \\ &\quad + \frac{\partial}{\partial t} \left[\frac{\mu [E_0^+]^2}{4\eta^2} \int_{z=0}^W \int_{y=-b/2}^{b/2} \int_{x=-a/2}^{a/2} \{ 1 + \cos 2(\omega t - \beta z) \} dx dy dz \right] \end{aligned} \quad (1.142)$$

To simplify the calculation first we take the time derivative of the function⁴⁰ inside. Thus, the unity term becomes zero and $\cos\{2(\omega t - \beta z)\}$ becomes $-2\omega \sin 2\{(\omega t - \beta z)\}$. Also, we use the definition $\eta = \sqrt{\mu/\epsilon}$ so that $\mu/\eta^2 = \epsilon$ and $\omega\epsilon/\beta = \eta^{-1}$. Substitutions of these quantities into the above equation yields:

$$\begin{aligned} P(t) &= \int_V \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) \right] dV = \frac{\epsilon [E_0^+]^2}{2} \int_{z=0}^W -2\omega \sin 2(\omega t - \beta z) dz \\ &= \frac{\omega\epsilon [E_0^+]^2 ab}{2\beta} [-\cos 2(\omega t - \beta z)]_{z=0}^W \\ \therefore P(t) &= \frac{[E_0^+]^2 ab}{2\eta} \{ \cos 2\omega t - \cos 2(\omega t - \beta W) \} \end{aligned} \quad (1.143)$$

Substitution for $E_0^+ = 10$ (V/m), $\eta = 377$ (Ω). Operating frequency $f = 6$ GHz, and $W = 1$ m, we get the following closed form expression for the net power flux entering region is:

$$\begin{aligned} P(t) &= \frac{[E_0^+]^2}{2\eta} ab \{ \cos(2\omega t) - \cos 2(\omega t - \beta W) \} \\ P(t) &= \frac{10^2 \text{ (V/m)}^2}{2 \times 377} (0.10 \text{ (m)} \times 0.05 \text{ (m)}) \\ &\quad \times \left\{ \cos(2\pi \times 6 \times 10^9 \times t) - \cos 2 \left[2\pi \times 6 \times 10^9 \times t - \frac{2\pi \times 6 \times 10^9}{3 \times 10^8} \times 1 \text{ (m)} \right] \right\} \\ &= 6.63 \times 10^{-4} \times \left\{ \cos(3.77 \times 10^{10} \times t) - \cos[7.54 \times 10^{10} \times t - 251.3] \right\} \text{ (W)} \end{aligned}$$

Exercise 1.6: One Dimensional Wave Equation

Q1. Repeat the problem for finite conductivity of the medium σ .

1.15 Problems

Uniform Plane Wave Theory

P1.1 A half wavelength dipole is operating at 2.45 GHz Wi-Fi frequency band. Calculate the far field distance at which we consider the emanating out wave from the dipole is a plane wave.

P1.2 Is $v(t) = V_0 \sin \omega t$ a wave equation? If so what is the amplitude, frequency of operation, direction of propagation of the voltage wave? To complete the answer, write the complete equation of the voltage wave.

P1.3 In connection to P1.1, if the radiated far electric field has an amplitude $E_0 = 20$ mV/m that oscillates along x -axis, propagates along z -direction and the initial phase state is 0° , write the expression of the electric field with appropriate unit. Is the field a time harmonic field?

⁴⁰ Here we assume that the volume is not changing with neither time t nor the medium's constitutive parameters ϵ and μ .

P1.4 Using Maxwell's equation calculate the far magnetic field from P1.3 for the dipole antenna.

P1.5 For the complete vector time-domain expression for the electric field:

$$\mathbf{E}(z, t) = E_0^+ e^{-\alpha z} \cos(\omega t - \beta z) \mathbf{a}_x + E_0^- e^{+\alpha z} \cos(\omega t + \beta z) \mathbf{a}_x \text{ (V/m)}$$

what are the amplitude, phase, frequency of operation, direction of propagation of the electric field wave? Which ones are the forward travelling and reverse travelling waves' amplitudes?

P1.6 An electric field is defined as $\mathbf{E}(z, t) = 15e^{-10z} \cos(\omega t - \beta z) \mathbf{a}_x - 20e^{-\alpha z} \cos(\omega t - \beta z) \mathbf{a}_y$, (V/m). Determine the amplitude and direction of the field at $t = 0$ and $z = \lambda/2$.

P1.7 We know that curling electric and magnetic fields generate time varying magnetic and electric fields respectively. The fields support propagation of electromagnetic wave. Explain the justification against your belief.

P1.8 Draw a conceptual map (flow chart) to derive the general wave equations for the electric and magnetic fields, and finally, show that the time-domain field expressions for both fields are similar in form.

P1.9 A uniform plane wave which is defined with an electric field as $\mathbf{E}(z, t) = 250 \sin(2\pi \times 10^3 t - \beta z) \mathbf{a}_y$ (V/m). Calculate the magnetic field if the medium of propagation is free space.

P1.10 Write the complete time domain expression of the magnetic field for the following parameters: the peak amplitude, $H_0 = 20$ (A/m), the phase constant, $\beta = 37.7$ (rad/m), the direction of propagation is along $-z$ direction and the field vibrates along $-\mathbf{a}_y$ at its initial half cycle. Derive the time domain expression for the electric field from the magnetic field. Identify the frequency operation that has commercial significance.

Propagation in Dielectric and Lossy Media

P1.11 Explain how food is cooked in a microwave oven in the perspective of wave propagation in lossy dielectric as shown in Figure P1.11.

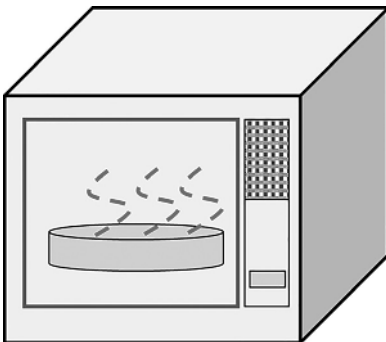


Figure P1.11 A microwave oven launches signal at 2.45 GHz from a high power microwave source called magnetron.

P1.12 What are the real power losses in the lossy dielectric materials such as foods as shown in the figure above?

- 1) How does the finite conductivity contribute to the loss in the dielectric when an EM wave propagates?
- 2) What is the other factor that contributes to the loss in the dielectric media?
- 3) Write the expression of loss tangent of a loss dielectric.
- 4) Find the data sheets for a few microwave laminates and study the dielectric constant in terms of ϵ' , ϵ'' and $\tan\delta$ (Some manufactures of microwave laminates are Rogers; Taconic; etc.)

P1.13 Using general expressions for the propagation constant: $\gamma = \alpha + j\beta$; where

$$\alpha = \omega \sqrt{\frac{\mu\epsilon}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} - 1 \right)}; \beta = \omega \sqrt{\frac{\mu\epsilon}{2} \left(\sqrt{1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2} + 1 \right)}$$

- 1) Find the attenuation constant α and the propagation constant β in:
 - i) low-loss medium with $(\sigma/\omega\epsilon) < < 1$;
 - ii) a good dielectric?
 - iii) a good conductor?

P1.14 A high intensity Electromagnetic bomb (electric bomb) may cripple a nation within a moment and could be as dangerous as atomic bomb, but without any human casualty. Design an electric bomb shield to protect the computers of a strategic information management center of Melbourne with attenuation 100 (dB/m). Figure P1.14 illustrates the attenuation of wave propagation in the conducting shield. Hints: $1 \text{ (Np/m)} = 8.686 \text{ (dB/m)}$.

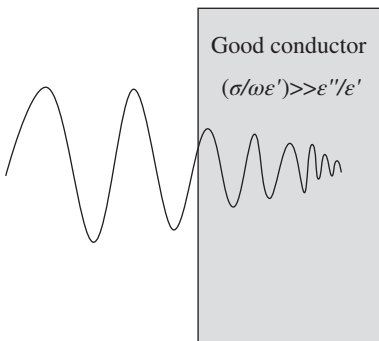


Figure P1.14 Propagation in good conductor as a shield.

P1.15 A plane wave incident from free space upon a dielectric medium with the constitutive parameters: μ_r , ϵ_r , σ . Calculate the propagation constant of a wave in a medium with $\mu_r = 1$, $\epsilon_r = 10.2$, $\sigma = 24 \times 10^{-12} \text{ (S/m)}$ at an operating frequency of 1 GHz.

P1.16 A plane wave is propagating from free space to a dielectric medium. Due to the medium change, the wavelength reduces from 5 to 2.5 mm. Determine the frequency, dielectric constant, and phase velocity of the medium.

P1.17 From a mobile tower an x-polarised uniform plane wave is launched to a partially conducting wall with $\epsilon_r = 10.2$, $\mu_r = 1000$, and $\sigma = 0.4 \text{ (S/m)}$. The wave is propagating along z-direction with an operating frequency of 1800 MHz. Calculate attenuation constant α ,

phase constant, β , wave impedance η , phase velocity u_p of the wave inside the medium. If the electric field has a peak amplitude of 230 (V/m), derive the expression for both the electric and magnetic fields.

- P1.18** A dual band GSM smart phone operates in 900 and 1800 MHz. The phone is protected with an aluminium cage so that the electromagnetic compatibility is maintained for underwater leakage of signal to and from the phone circuitry. Find the skin depth δ , propagation constant γ , phase velocity u_p , and the thickness of the aluminum cage so that less than 1% signal can penetrate the cage.
- P1.19** If the fields from a GSM 1800 MHz mobile base station in P1.9 propagate through a thick wall of a nearby building with the following constitutive parameters, $\mu_r = 1$, $\epsilon_r = 9$, $\sigma = 0.2$ (nS/m). Determine the propagation constant of the wave in the wall. Also determine if the wall is conductor or a dielectric in nature. If the thickness of the wall is 1 m, calculate the attenuation in the wall in dB.
- P1.20** An x -polarised free space plane wave with amplitude 125 (mV/m) travelling along $+z$ -direction from a GSM dual band mobile tower is normally incident on the aluminum cage of the smart phone in P1.18, calculate the wave impedance, the electric field, and the magnetic field propagating inside the aluminum cage for both bands. Assume cosine reference of the electric field and the cage is at reference position $z = 0$.
- P1.21** Using the definition of attenuation constant in (Np/m) for a plane wave in lossy medium, show that 1 Np is 8.686 dB.
- P1.22** If the wall shows high conductivity $\sigma = 20 \times 10^{-4}$ (S/m) due to steel frames inside the wall, but other constitutive parameters remain the same for P1.19, calculate the frequencies at which the wall is assumed to be a perfect dielectric. For a perfect dielectric use the threshold $(\sigma/\omega\epsilon) \leq 1\%$. What is the attenuation constant at these frequencies? Does the attenuation depend on the frequency?
- P1.23** The property of a material can be characterized with the attenuation profile with frequency. This can be used to detect adulteration in substances such as milk and powder for uniformity. If the amplitude of the incident electric field just at the edge of sample of substance is 20 (V/m) and the constitutive parameters of the substance is $\mu_r = 1$, $\epsilon_r = 22$ and $\sigma = 0.58$ (S/m). Calculate the amplitude of the electric field for a 25 cm thick sample for frequencies (a) 800, (b) 1200, and (c) 1600 MHz. Draw a plot for amplitude versus frequency.
- P1.24** A uniform plane wave with an electric field that vibrates along x -axis and propagates along z -direction has a maximum amplitude of 10 (mV/m) at $t = 0$ and $z = 0$. The medium of propagation is a lossless dielectric medium of dielectric constant of 10.2. The operating frequency is 850 MHz. Calculate the followings:
- the instantaneous expression of the electric field for any t and z .
 - the instantaneous expression of the magnetic field for any t and z .
 - the time at which the electric field is the maximum for $t > 0$.

- P1.25** Repeat P1.24 for which the maximum of the electric field is at $t = 0$, $z = 0.25$. (Hints: first calculate the initial phase ϕ).
- P1.26** A radar hits a target as shown Figure P1.26. Explain in what condition the radar receive the signal. How do the stealth bombers avoid the radar?

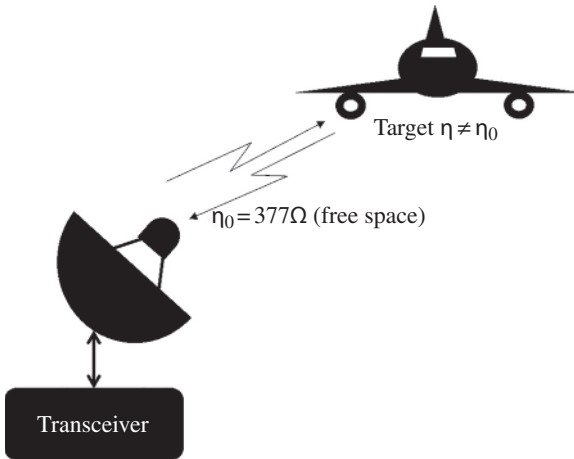


Figure P1.26 A radar targets a plane with different wave impedance than that for the free space.

- P1.27** What is the intrinsic impedance of the EM wave in a lossy dielectric with a finite conductivity σ , permittivity ϵ , and permeability μ ?
- P1.28** For a propagation medium that has conductivity $\sigma = 0.02$ (S/m), dielectric constant $\epsilon_r = 4$, and relative permeability $\mu_r = 1$. The operating frequency of a signal is 100 MHz. Calculate the propagation constant γ , attenuation constant α , phase constant β , and the wave impedance η .
- P1.29** What is the phase velocity of an EM wave in a medium with the relative permittivity $\epsilon_r = 2.2$, and the relative permeability $\mu_r = 1$?
- P1.30** Show that the attenuation constant α and the propagation constant β of a good conductor are the same. Calculate the skin depth of a good conductor for which the signal attenuates to e^{-1} of its original value. And find the thickness at which the signal attenuates less than 1%.
- P1.31** Show that the wave impedance of a good conductor is $\eta = \sqrt{\omega\mu/\sigma} \angle 45^\circ$.
- P1.32** A mobile phone cage is made of aluminum sheet to stop leakage of electromagnetic wave for electromagnetic compatibility. The incident electric field is $\mathbf{E}_i(z, t) = 20e^{-\alpha z} \cos(\omega t - \beta z)\mathbf{a}_x$ (V/m), the operating frequency is 540 MHz. The aluminum conductivity is $\sigma = 3.69 \times 10^7$ (S/m) and $\mu_r = 1$. Calculate the skin depth of aluminum at 450 MHz and find thickness of the aluminum shield so that less than 1% of the incident wave can leak through the shield.

- P1.33** The phasor vector expression of a composite uniform plane wave is given by the electric field: $\mathbf{E}_s(x, z) = 123e^{j(23x + 15z)}\mathbf{a}_y$ (V/m). Find the frequency f and wavelength λ and write the instantaneous expression for the electric and the associated magnetic fields.

Poynting Theorem

- P1.34** Draw a conceptual map (flow chart) of the Poynting Theorem. (Hints: take divergence of Poynting vector: $\mathbf{E} \times \mathbf{H}$). Describe the physical meaning of the Poynting Theorem.
- P1.35** Energy can be transmitted wirelessly from one point in space to another point. Most recent example is the microwave power transmission from the outer orbit to the earth. About $3-9 \times 10^9$ W are expected to be collected via the rectifying antennas as the renewable energy. Read more in Wikipedia: Microwave power transmission.
- P1.36** If an incident electric field is 2 (V/m) with x -polarization and a magnetic field is 2 (A/m) with y -polarization hit an antenna; what is the average power intensity at the aperture of the antenna in (W/m^2)? What is the direction of the power intensity? [Hints: $\mathbf{P}_{ave} = (1/2) \text{Re}(\mathbf{E} \times \mathbf{H}^*)$]
- P1.37** A circular patch antenna of radius 150 mm is receiving a plane wave with the instantaneous electric field $\mathbf{E}(z, t) = 250 \cos(\omega t - \beta z)\mathbf{a}_x$ (V/m). Find the time average power incident upon the aperture of the patch antenna. Note the circular patch antenna has the diameter for the fundamental mode is very close to half of the wavelength.
- P1.38** A time harmonic electromagnetic wave with electric field amplitude of 125 (V/m) from a mobile tower is launched in a highly foggy weather with relative permittivity of 9 and loss tangent of 0.2. Calculate the time average power dissipation in the atmosphere in (W/m^3). The operating frequencies are 900 and 1800 MHz for GSM mobile bands in Australia.
- P1.39** An x -polarized electromagnetic field wave is propagating along $+z$ direction from a mobile tower with an operating frequency of 900 MHz. The time average power density of the wave is 30 (mW/cm^2). Calculate the electric and magnetic fields in their instantaneous form using the Poynting theorem.
- P1.40** A square patch antenna is operating at 2.45 GHz wi-fi frequency band. The patch is resonant at half the wavelength of the operating frequency and designed on a microwave laminate of dielectric constant $\epsilon_r = 4.2$. (a) Determine the patch dimension in mm. (b) If the launched average power density from the patch antenna is 50 (W/m^2), calculate the total power flux entering a square region of the same dimensions and length of 1 m of the radiating aperture of the patch antenna.
- P1.41** A circular patch antenna radiates spherically symmetric electromagnetic fields. If the patch antenna radiated field is defined in the spherical coordinates as follows: $\mathbf{E}(r, \theta, t) = \frac{E_0}{r} \cos^{1.2}\theta \cos(\omega t - \beta r)\mathbf{a}_\theta$ (V/m), where $E_0 = 250$ (mV/m), calculate the radiated

magnetic field and the time average power crossing the hemispherical shell of the radius $r = 1 \text{ km}$, $0 \leq \theta \leq (\pi/2)$. Also calculate the power density if no leakage in the bottom side of the antenna as shown in the Figure P1.41.

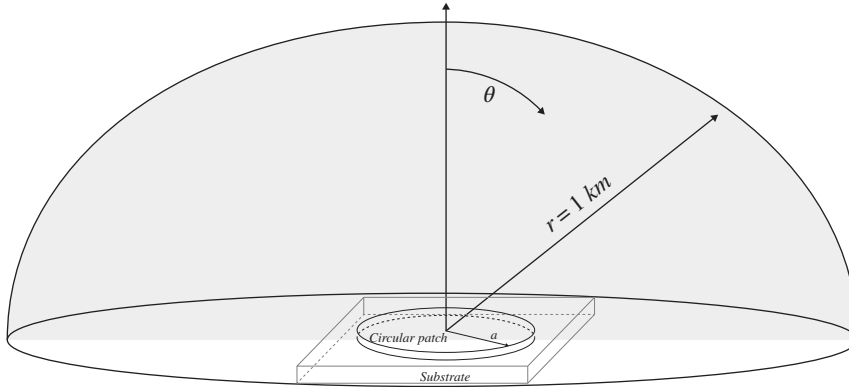


Figure P1.41 Circular patch antenna radiation mechanism.

- P1.42** A uniform plane wave which is defined with $\mathbf{E}(z, t) = 125 \sin(\omega t - \beta z) \mathbf{a}_x$ (V/m) is propagating in free space. (a) Calculate the time average power density in (W/m^2). (b) Calculate the power crossing a rectangular area of sides 25 mm by 15 mm normal to the direction of power propagation.