1

Basic Concepts and Quick Review

A standard scientific practice is to formulate an explanation for an observed phenomenon and then test this formulation by projecting the outcomes of various experiments under pertinent conditions. Projections are generally compared with experimental data. If there is agreement, the explanation can be accepted as a valid theory, whereas discrepancies point to a need for reformulation of the explanation. A model that describes the main features of the phenomenon, often represented mathematically, can be iteratively improved in the process of reformulation to resolve its discrepancies with observations or experimental data. This iterative process is known as the *modelling cycle* (Figure 1.1).

In simple terms, mathematical modelling is a process by which we derive a model to describe a phenomenon that may or may not be observable. For example, the movement of a pendulum is an observable phenomenon, but the transmission of a disease in the population may not be observable. In the latter case, the outcomes of infection and illness indicate that the epidemic phenomenon may be taking place and the disease is being transmitted among individuals. The process of modelling consists of several important steps. In general, the model represents a framework that includes simplification, assumptions, and approximation to describe the phenomenon under investigation. This framework can be expressed by mathematical equations and analyzed using the theory of dynamical systems and computational tools for model validation and comparison with available data (Figure 1.1).

Before proceeding further, let us present an example of developing a simple mathematical model. In this example, we wish to calculate the volume of sand that falls from the top half to the bottom half of a conical hourglass within a period of time (Figure 1.2). Suppose that the sand flows at the rate of 4 cm^3 per second from the top half to bottom half of the hourglass. We remember from calculus that the volume of a cone with height h and radius R is given by $V = \pi R^2 h/3$. Here, we will first find the volume of sand in the bottom half of

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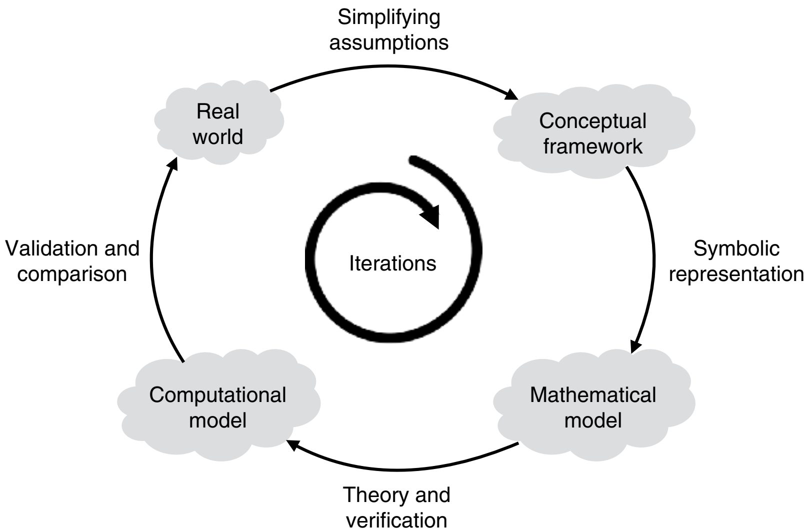

Figure 1.1 The process of model development, analysis, and validation.

the conical hourglass. From the dimensions given in Figure 1.2, this volume is given by:

$$V_h(t) = \frac{\pi 6^2 \times 12}{3} - \frac{\pi r^2(12-h)}{3} = 144\pi - \frac{\pi r^2(12-h)}{3}.$$

Using the property of similar triangles, we can write r in terms of h as $r = 6 - h/2$. Substituting this into V_h and taking the derivative of V_h with respect to t, we get:

$$\frac{dV_h}{dt} = \pi\left(6 - \frac{h}{2}\right)^2 \frac{dh}{dt},$$

where we consider h to be a function of time t. Given the flow rate of sand (i.e., $V'_h = 4$), the change in the height of sand in the bottom-half of the conical hourglass with respect to time is:

$$\frac{dh}{dt} = \frac{4}{\pi\left(6 - \frac{h}{2}\right)^2}.$$

Using separation of variables and integrating the height equation gives:

$$\int_{h(0)}^{h(t)} \left(6 - \frac{h}{2}\right)^2 dh = \int_0^t \frac{4}{\pi}\, dt.$$

Thus,

$$-(12-h)^3 \Big|_{h(0)}^{h(t)} = \frac{48t}{\pi}.$$

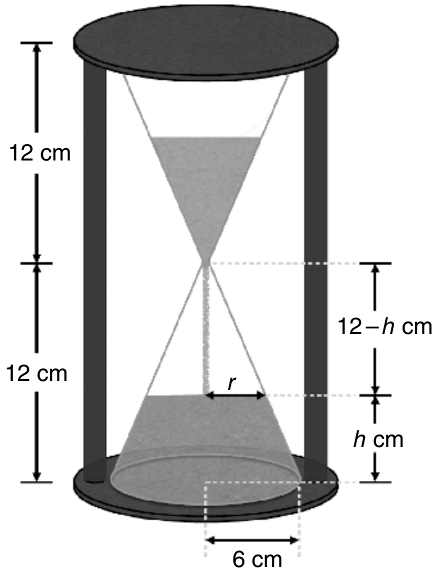

Figure 1.2 Representation of a conical hourglass.

Since $h(0) = 0$, we have $h(t) = 12 - \sqrt[3]{12^3 - 48t/\pi}$. Substituting $h(t)$ into the equation for $V_h(t)$, we can calculate the amount of sand that falls from the top half to the bottom half of the conical hourglass within a certain time period. For example, between $t = 0$ and $t = 36\pi$, the bottom half of the hourglass will be filled, that is, $V(36\pi) = 144\pi$. This simple example shows how a model can be used to describe the outcomes of a process that changes with time.

Mathematical models are often used to explore the dynamics of a system over time. Let us present another example from classical mechanics. A simple mechanical oscillating system can be illustrated by a weight attached to a linear spring subject to only weight and tension, representing a harmonic oscillator. In mechanics and physics, simple harmonic motion is a type of oscillation where the restoring force is proportional to the displacement and acts in the direction opposite to that of displacement. Ignoring the damping behavior, the restoring force (given by the product of mass and acceleration according to Newton's second law of motion for a constant mass) in a linear spring can be modelled by:

$$F = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mx'' = -kx, \tag{1.1}$$

where m is the mass attached to the spring, k is the spring constant, and x represents the displacement of the mass from its equilibrium state (Figure 1.3). A solution of the equation $mx'' = -kx$ (with the initial condition $x(0) = 0$) is given by:

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right),$$

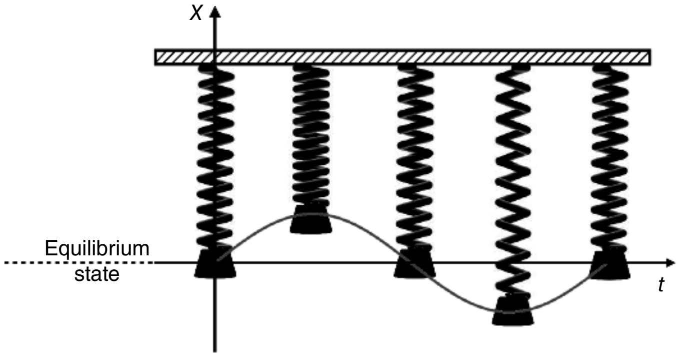

Figure 1.3 Representation of spring motion in a sinusoidal form.

where A is a constant representing the amplitude of sinusoidal motion of the spring. The displacement x is represented as a function of t in Figure 1.3.

1.1 Modelling Types

Broadly speaking, mathematical models can be classified based on three major characteristics that may depend on the nature of the phenomenon: (i) deterministic or stochastic; (ii) dynamic or static; and (iii) discrete or continuous (Figure 1.4). To better understand this classification, we provide some specific examples as follows.

- Deterministic of dynamic-continuous type: steam engine.
- Deterministic of static-continuous type: snapshot of pendulum.
- Deterministic of dynamic-discrete type: the percentage of computer processing unit in use upon startup.
- Deterministic of static-discrete type: clock cycles for a computer program to run on a given input.
- Stochastic of dynamic-continuous type: weather.
- Stochastic of static-continuous type: noise in an electronic circuit.
- Stochastic of dynamic-discrete type: random arrivals.
- Stochastic of static-discrete type: flipping a coin.

In this textbook, we present various examples of mathematical models within this classification and analyze their behavior. In our analyses of such models, we use essential analytical tools from the theory of dynamical systems. Here, we briefly review some techniques from the theory of differential equations and linear algebra that are useful in understanding the analytical tools and their applications to mathematical modelling in subsequent chapters.

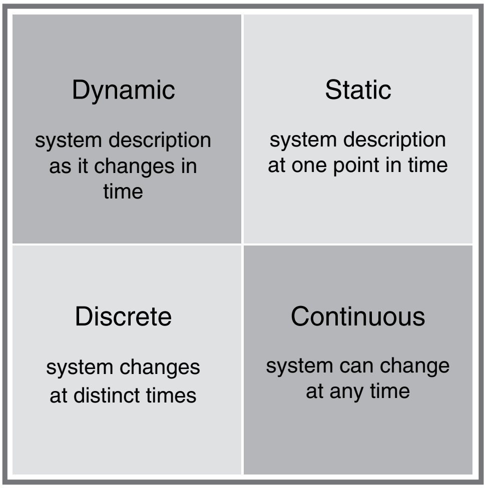

Figure 1.4 Types of deterministic and stochastic models.

1.2 Quick Review

We begin by reviewing methods for solving first- and second-order linear differential equations [7].

1.2.1 First-order Differential Equations

The general form of a first-order linear differential equation is given by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x), \tag{1.2}$$

where $P(x)$ and $Q(x)$ are continuous real-valued functions. To solve this equation for a solution in the form of $y(x)$, we use the integration factor:

$$R(x) = \exp\left(\int_a^x P(t)\mathrm{d}t\right). \tag{1.3}$$

Multiplying (1.2) by $R(x)$ gives:

$$\frac{\mathrm{d}(R(x)y(x))}{\mathrm{d}x} = R(x)Q(x).$$

Integrating both sides of this equation with respect to x gives:

$$R(x)y(x) = \int R(t)Q(t)\mathrm{d}t + C, \tag{1.4}$$

where C is a constant. Since $R(x) \neq 0$ for all $x \in \mathbb{R}$, we can divide each side of (1.4) by $R(x)$ to obtain the solution of $y(x)$. To illustrate this method, we provide the following example.

Example 1.1 Consider the following differential equation:

$$y' = 2x(y+1),$$

where we represent $\frac{dy}{dx}$ by y'. Rewriting this equation in general form of (1.2) gives:

$$y' - 2xy = 2x.$$

From (1.3), we obtain the integration factor:

$$R(x) = Ae^{-x^2},$$

where A is a constant. Multiplying the equation by $R(x)$ gives:

$$(e^{-x^2}y)' = 2xe^{-x^2}.$$

Integrating this equation with respect to x leads to the solution:

$$y = -1 + Ce^{x^2},$$

where C is a constant.

☞ **Bernoulli equation.** The general form of a first-order differential equation of Bernoulli type is given by [7]:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \tag{1.5}$$

where $P(x)$ and $Q(x)$ are continuous real-valued functions, and $n = 0, 1, 2, \ldots$. If $n > 1$, then we can use the change of variable $u = y^{1-n}$. Thus, equation (1.5) reduces to:

$$\begin{aligned} \frac{du}{dx} &= (1-n)y^{-n}\frac{dy}{dx} \\ &= (1-n)y^{-n}(-P(x)y + Q(x)y^n) \\ &= (1-n)(-P(x)y^{1-n} + Q(x)) \\ &= (1-n)(-P(x)u + Q(x)). \end{aligned} \tag{1.6}$$

Equation (1.6) can now be solved using an integration factor.

Example 1.2 Consider the following differential equation:

$$y' - \frac{y}{x} = y^2 \ln x. \tag{1.7}$$

Letting $u = y^{-1}$, we get:

$$u' + \frac{u}{x} = -\ln x.$$

Using the integration factor $R(x) = x$, we get $(xu)' = -x \ln x$, and therefore:

$$u = -\frac{x}{2}(\ln x - \frac{1}{2}) + \frac{C}{x},$$

where C is a constant. Thus, the solution of (1.7) is:

$$y = \frac{4x}{-x^2(2\ln x + 1) + C}.$$

1.2.2 Second-order Differential Equations

The general form of a second-order linear differential equation is given by:

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = F(x), \tag{1.8}$$

where $F(x)$, $a(x)$ and $b(x)$ are continuous real-valued functions. Here, we assume that $a(x) = a$ and $b(x) = b$ are constants. We can consider homogeneous and inhomogeneous cases.

☞ **Homogeneous case.** In this case, $F(x) \equiv 0$, and (1.8) reduces to:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0. \tag{1.9}$$

We look for a solution of the form $y(x) = e^{\lambda x}$, in which λ is yet to be determined. Substituting this solution into (1.9) gives the characteristic equation:

$$(\lambda^2 + a\lambda + b)e^{\lambda t} = 0.$$

Solving this equation for λ will provide different types of solutions for (1.9), depending on whether the characteristic equation has distinct real roots, repeated roots, or complex roots. The general form of the solution of (1.9) for distinct roots is then given by:

$$y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x},$$

and for repeated roots by:

$$y(x) = C_1e^{\lambda x} + C_2xe^{\lambda x}.$$

Example 1.3 Consider the following second-order differential equation:

$$y'' - y' - 2y = 0.$$

Solving the characteristic equation $\lambda^2 - \lambda - 2 = 0$ gives the solutions $\lambda_1 = -1$, and $\lambda_2 = 2$. Therefore, the general solution of (1.9) can be expressed by:

$$y(x) = C_1e^{-x} + C_2e^{2x},$$

where C_1 and C_2 are constants.

Example 1.4 Consider the following second-order differential equation:

$$y'' - 2y' + y = 0.$$

Solving the characteristic equation $\lambda^2 - 2\lambda + 1 = 0$ gives the solutions $\lambda_1 = \lambda_2 = 1$. Therefore, the general solution of (1.9) can be expressed by:

$$y(x) = C_1 e^x + C_2 x e^x,$$

where C_1 and C_2 are constants.

☞ **Inhomogeneous case.** In this case, $F(x) \neq 0$. We first solve the homogeneous equation (setting $F(x) = 0$), and then extend this solution to the inhomogeneous case. Suppose $y_h(x)$ represents the solution for the homogeneous equation. We consider a particular solution $y_p(x)$ in a similar functional form to $F(x)$ with unknown constants. The general form of the solution for inhomogeneous case is then given by:

$$y(x) = y_h(x) + y_p(x).$$

In the last step, we find the coefficients of the particular solution by substituting $y_p(x)$ into equation (1.8).

Example 1.5 Consider the following second-order differential equation:

$$y'' - y' - 2y = x^2. \tag{1.10}$$

Solving the homogeneous case $y'' - y' - 2y = 0$ as described above gives the solution:

$$y_h(x) = C_1 e^{-x} + C_2 e^{2x}.$$

We now assume that the particular solution has the form of a polynomial of degree 2 similar to the functional form of $F(x) = x^2$:

$$y_p(x) = Ax^2 + Bx + C.$$

Substituting this particular solution into equation (1.10), we get:

$$2A - (2Ax + B) - 2Ax^2 - 2Bx - 2C = x^2.$$

Rearranging this equation, we find that the constant term and the coefficient of x must be zero, and the coefficient of x^2 must be 1, so that the equation holds for all $x \in \mathbb{R}$. This implies that:

$$\begin{aligned} -2A &= 1, \\ -2B - 2A &= 0, \\ 2A - B - 2C &= 0. \end{aligned}$$

Thus, $A = -\frac{1}{2}$, $B = \frac{1}{2}$, and $C = -\frac{3}{4}$. Hence, we obtain the general solution of (1.10):

$$y(x) = C_1 e^{-x} + C_2 e^{2x} - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}.$$

Example 1.6 Consider the following differential equation:

$$y'' - y = 2e^{5x}. \tag{1.11}$$

Solving the characteristic equation for the homogeneous case gives the solution $y_h(x) = C_1 e^{-x} + C_2 e^{x}$. Assuming $y_p(x) = Ae^{5x} + B$, and substituting back into (1.11), we find $A = \frac{1}{12}$ and $B = 0$. Thus, the general solution of (1.11) is:

$$y(x) = C_1 e^{-x} + C_2 e^{x} + \frac{1}{12}e^{5x}.$$

Example 1.7 Consider the following inhomogeneous differential equation:

$$y'' + y' + y = x + \cos x. \tag{1.12}$$

The characteristic equation for the homogeneous case is $\lambda^2 + \lambda + 1 = 0$, which has the solutions $\lambda_{\pm} = (-1 \pm \mathrm{i}\sqrt{3})/2$. This gives the solutions in the complex domain as $y(x) = e^{-\frac{x}{2}}\left(C_1 e^{\mathrm{i}\frac{\sqrt{3}x}{2}} + C_2 e^{-\mathrm{i}\frac{\sqrt{3}x}{2}}\right)$. Therefore, we find the solution:

$$y_h(x) = C_1 e^{-\frac{x}{2}} \sin\frac{\sqrt{3}}{2}x + C_2 e^{-\frac{x}{2}} \cos\frac{\sqrt{3}}{2}x.$$

We now consider a particular solution of the form $y_p(x) = Ax + B + D\sin x + E\cos x$. Substituting $y_p(x)$ into (1.12) gives:

$$-E\sin x + D\cos x + Ax + A + B = x + \cos x,$$

which implies:

$$A = 1, \quad B = -1, \quad E = 0, \quad D = 1.$$

Thus, the general solution of the equation is:

$$y(x) = C_1 e^{-\frac{x}{2}} \sin\frac{\sqrt{3}}{2}x + C_2 e^{-\frac{x}{2}} \cos\frac{\sqrt{3}}{2}x + x - 1 + \sin x.$$

1.2.3 Linear Algebra

A number of analytical tools that we introduce in the subsequent chapters apply fundamental concepts from matrix theory and linear algebra [49]. Here, we provide an overview of these concepts.

A square matrix $A_{n\times n}$ (with equal number of rows and columns) is invertible, if there exists a square matrix $B_{n\times n}$ such that $AB = BA = I_{n\times n}$, where

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{n\times n}.$$

The matrix B is often denoted by A^{-1} (referred to as the inverse of A), and it is unique. It is important to note that the concept of inverse applies only to square matrices, while it is possible to find matrices A and B that satisfy the condition of $AB = I$ but are not square. For example, let:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Then multiplication of A and B gives:

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}.$$

However, neither A nor B is invertible because:

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq AB.$$

For a square matrix $A_{n\times n}$, there is a number called the *determinant*, denoted by $\det(A)$. According to the matrix theory, a square matrix is invertible (i.e., A^{-1} exists) if and only if $\det(A) \neq 0$. Using the inverse of a matrix, when it exists, it is possible to solve systems of linear equations. To illustrate this, let us consider the general form of a system of n linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n. \end{aligned} \tag{1.13}$$

We can rewrite (1.13) in matrix form as $AX = B$, where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

If $\det(A) \neq 0$, then A^{-1} exists, and we can multiply each side of the equation by A^{-1} to get:

$$A^{-1}(AX) = A^{-1}B.$$

Using the associative property of matrix multiplication, and $AA^{-1} = I$, we obtain the solution of the system as $X = A^{-1}B$.

For a square matrix $A_{n\times n}$, a scalar λ is called an eigenvalue if there is a nonzero solution X of $AX = \lambda X$. Such an X is called an *eigenvector* corresponding to the *eigenvalue* λ. The eigenvalues and the corresponding eigenvectors are obtained by solving the characteristic equation:

$$\det(A - \lambda I) = 0,$$

where λ is an eigenvalue of matrix A. For each λ, a vector $V \neq 0$ is an eigenvector if:

$$(A - \lambda I)V = 0.$$

Example 1.8 Find the eigenvalues and the corresponding eigenvectors of the matrix:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$$

For this purpose, we solve the characteristic equation:

$$\det\left(\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} 1-\lambda & 4 \\ 2 & -1-\lambda \end{bmatrix} = 0,$$

which is $-(1+\lambda)(1-\lambda) - 8 = 0$. Thus, there are two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -3$. To find an eigenvector V_1 corresponding to λ_1, we consider the equation $(A - 3I)V_1 = 0$, which gives the linear system:

$$\begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This system provides only one equation with two unknowns given by $-2x_1 + 4x_2 = 0$. Since we are looking for a nonzero vector, assuming $x_1 = 1$, we find $x_2 = 1/2$, and therefore:

$$V_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

In a similar way, we can find an eigenvector corresponding to the eigenvalue $\lambda_2 = -3$. In this case, we need to solve the linear system $(A + 3I)V_2 = 0$. This system provides only one equation, given by $2x_1 + 2x_2 = 0$. Assuming $x_1 = 1$, we obtain $x_2 = -1$, and therefore a nonzero eigenvector is obtained as:

$$V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

1.2.4 Scaling

Scaling is a useful technique in mathematical modelling to simplify the model for analysis, without changing the theoretical structure of the system or its behavior. Scaling is often used to make independent and dependent variables dimensionless, normalize the size associated with the system variables, and reduce the number of independent parameters in the model [25]. Here, we explain this technique using the following examples.

Example 1.9 Consider the following differential equation:

$$\frac{dN}{dt} = r\,N\left(1 - \frac{N}{K}\right),$$

where r and K are positive numbers, and $N \geq 0$. This equation is known as the *logistic model*, and we will detail its properties in the next chapter. The variable N represents the size of a population which changes with time, and depends on the growth rate r and the carrying capacity K. We may simplify this equation by scaling through a new variable and a new parameter, defined by:

$$n = \frac{N}{K}, \qquad \tau = rt.$$

Taking the derivative of n with respect to τ and using the chain rule, we get:

$$\begin{aligned}
\frac{dn}{d\tau} &= \frac{dn}{dt} \cdot \frac{dt}{d\tau} \\
&= \frac{d}{dt}\left(\frac{N}{K}\right) \cdot \left(\frac{1}{r}\right) = \frac{1}{rK}\frac{dN}{dt} \\
&= \frac{1}{rK}(rN)\left(1 - \frac{N}{K}\right) \\
&= \frac{N}{K}\left(1 - \frac{N}{K}\right) = n(1 - n).
\end{aligned}$$

Thus, the logistic equation can be simplified to the equation $n' = n(1 - n)$, where $n \geq 0$.

Example 1.10 Consider the following system of nonlinear differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= -\beta S(t)I(t), \\
\frac{dI}{dt} &= \beta S(t)I - \gamma I(t).
\end{aligned}$$

This system may represent the spread of a disease in a population of susceptible individuals (S), with a transmission rate of β through contacts with infected individuals (I). Infected individuals recover at a rate of γ. We will delineate epidemic models in the following chapters. For this *SI epidemic model*, we set $S_0 = S(0)$, and define:

$$u = \frac{S(t)}{S_0}, \quad v = \frac{I(t)}{S_0}, \quad \tau = \gamma t.$$

Differentiating u with respect to τ gives:

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= \frac{\mathrm{d}u}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{S}{S_0}\right).\frac{1}{\gamma} = \frac{1}{S_0\gamma}(-\beta SI) \\
&= \frac{-\beta}{S_0\gamma}(uS_0)(vS_0) \\
&= \frac{-\beta S_0}{\gamma}uv.
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}\tau} &= \frac{\mathrm{d}v}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{I}{S_0}\right).\frac{1}{\gamma} \\
&= \frac{1}{S_0\gamma}[\beta SI - \gamma I] \\
&= \frac{1}{S_0\gamma}[\beta(uS_0)(vS_0) - \gamma vS_0] \\
&= \frac{\beta S_0}{\gamma}uv - v = \left(\frac{\beta S_0 u}{\gamma} - 1\right)v.
\end{aligned}$$

If we define $R_0 = \frac{\beta S_0}{\gamma}$, then the SI epidemic model reduces to:

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}\tau} &= -R_0 uv, \\
\frac{\mathrm{d}v}{\mathrm{d}\tau} &= (R_0 u - 1)v.
\end{aligned}$$

This simplified model, which depends on a single parameter R_0, could help us understand the behavior of the epidemic dynamics at the early stages of disease onset (i.e., for $t > 0$ and sufficiently small). We note that at the early stage of an epidemic, the number of infected individuals is small compared to the size of the susceptible population. Thus, it is reasonable to assume that for small t, $S(t) \approx S_0$ or $u \approx 1$. This assumption can be used to solve the equation $v' = (R_0 - 1)v$, representing the dynamics of the infected population at the early stages of the epidemic. Solving this equation, with an initial value of $v_0 = I(0)/S(0)$ gives:

$$\ln v(\tau) - \ln v_0 = (R_0 - 1)\tau.$$

Thus,

$$v(\tau) = v_0 e^{(R_0-1)\tau}.$$

This solution suggests that $v(\tau)$ decreases if $R_0 < 1$, and increases exponentially if $R_0 > 1$.

Exercises

1 Solve the following differential equations:
a) $y' = \mu y + x^5$, where $\mu \in \mathbb{R}$ is a constant.
b) $y'' - y = 2x + e^x$.
c) $y'' - y = x + \sin x$.

2 Solve the following differential equation for distinct roots if $\alpha < 1$, repeated roots if $\alpha = 1$, and complex roots if $\alpha > 1$:

$$2y'' - 2\sqrt{2}y' + \alpha y = -\frac{1}{\alpha x^2} - \frac{\sqrt{2}}{\alpha x} + \frac{1}{2}\ln x + 5,$$

for $x > 0$. *Hint*: define $y_p(x) = A\ln x + B$.

3 Find the values of a for which the following matrix is invertible:

$$A = \begin{bmatrix} 0 & 1 & a \\ 1 & 3 & 0 \\ 0 & 2 & 2 \end{bmatrix}.$$

4 Find the eigenvalues and the corresponding eigenvectors of the matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

5 Solve the linear system:

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 3 \end{bmatrix}.$$

6 Consider the following differential equation:

$$mx''(t) + kx(t) = 0, \quad x(0) = a, \quad x'(0) = b,$$

where m, k, a, and b are constant numbers. Use an appropriate change of variables to scale this equation into an equation with dimensionless independent and dependent variables.