CHAPTER 1

MAGNETIC AND MAGNETICALLY COUPLED CIRCUITS

1.1 INTRODUCTION

Before diving into the analysis of electromechanical motion devices, it is helpful to review briefly some of our previous work in physics and in basic electric circuit analysis. In particular, the analysis of magnetic circuits, the basic properties of magnetic materials, and the derivation of equivalent circuits of stationary, magnetically coupled circuits are topics presented in this chapter. Much of this material will be a review for most, since it is covered either in a sophomore physics course for engineers or in introductory electrical engineering courses in circuit theory. Nevertheless, reviewing this material and establishing concepts and terms for later use sets the appropriate stage for our study of electromechanical motion devices.

Perhaps the most important new concept presented in this chapter is the fact that in all electromechanical devices, mechanical motion must occur, either translational or rotational, and this motion is reflected into the electric system either as a change of flux linkages in the case of an electromagnetic system or as a change of charge in the case of an electrostatic system. We will deal primarily with electromagnetic systems. If the magnetic system is linear, then the change in flux linkages results, owing to a change in the inductance. In other words, we will find that the inductances of the electric circuits associated with electromechanical motion devices are functions of the mechanical motion. In this chapter, we shall learn to express the self- and mutual inductances for simple translational and rotational electromechanical devices, and to handle these changing inductances in the voltage equations describing the electric circuits associated with the electromechanical system.

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Throughout this text, we will give short problems (SPs) with answers following most sections. If we have done our job, each SP should take less than ten minutes to solve. Also, it may be appropriate to skip or de-emphasize some material in this chapter depending upon the background or interest of the students. At the close of each chapter, we shall take a moment to look back over some of the important aspects of the material that we have just covered and mention what is coming next and how we plan to fit things together as we go along.

1.2 PHASOR ANALYSIS

Phasors are used to analyze steady-state performance of ac circuits and devices. This concept can be readily established by expressing a steady-state sinusoidal variable as

$$F_a = F_p \cos \theta_{ef} \tag{1.2-1}$$

where capital letters are used to denote steady-state quantities and F_p is the peak value of the sinusoidal variation, which is generally voltage or current but could be any electrical or mechanical sinusoidal variable. For steady-state conditions, θ_{ef} may be written as

$$\theta_{ef} = \omega_e t + \theta_{ef}(0) \tag{1.2-2}$$

where ω_e is the electrical angular velocity and $\theta_{ef}(0)$ is the time-zero position of the electrical variable. Substituting (1.2-2) into (1.2-1) yields

$$F_a = F_p \cos\left[\omega_e t + \theta_{ef}(0)\right] \tag{1.2-3}$$

Since

$$e^{j\alpha} = \cos\alpha + j\sin\alpha \tag{1.2-4}$$

equation (1.2-3) may also be written as

$$F_a = Re\left\{F_p e^{j\left[\omega_e t + \theta_{ef}(0)\right]}\right\} \tag{1.2-5}$$

where Re is shorthand for the "real part of." Equations (1.2-3) and (1.2-5) are equivalent. Let us rewrite (1.2-5) as

$$F_a = Re\left\{F_p e^{j\theta_{ef}(0)} e^{j\omega_e t}\right\} \tag{1.2-6}$$

Thus, we need to take a moment to define what is referred to as the root mean square (rms) of a sinusoidal variation. In particular, the rms value is defined as

$$F = \left(\frac{1}{T}\int_0^T F_a^2(t)dt\right)^{\frac{1}{2}} \tag{1.2-7}$$

where F is the rms value of $F_a(t)$ and T is the period of the sinusoidal variation. It is left to the reader to show that the rms value of (1.2-3) is $F_p/\sqrt{2}$. Therefore, we can express (1.2-6) as

$$F_a = Re\left[\sqrt{2}Fe^{j\theta_{ef}(0)}e^{j\omega_e t}\right] \tag{1.2-8}$$

By definition, the phasor representing F_a, which is denoted with a raised tilde, is

$$\tilde{F}_a = Fe^{j\theta_{ef}(0)} \tag{1.2-9}$$

which is a complex number. The reason for using the rms value as the magnitude of the phasor will be addressed later in this section. Equation (1.2-6) may now be written as

$$F_a = Re\left[\sqrt{2}\tilde{F}_a e^{j\omega_e t}\right] \tag{1.2-10}$$

A shorthand notation for (1.2-9) is

$$\tilde{F}_a = F\underline{/\theta_{ef}(0)} \tag{1.2-11}$$

Equation (1.2-11) is commonly referred to as the polar form of the phasor. The Cartesian form is

$$\tilde{F}_a = F\cos\theta_{ef}(0) + jF\sin\theta_{ef}(0) \tag{1.2-12}$$

When using phasors to calculate steady-state voltages and currents, we think of the phasors as being stationary at $t = 0$. On the other hand, a phasor is related to the instantaneous value of the sinusoidal quantity it represents. Let us take a moment to consider this aspect of the phasor and, thereby, give some physical meaning to it. From (1.2-4), we realize that $e^{j\omega_e t}$ is a constant-amplitude line of unity length rotating counterclockwise at an angular velocity of ω_e. Therefore,

$$\sqrt{2}\tilde{F}_a e^{j\omega_e t} = \sqrt{2}F\left\{\cos\left[\omega_e t + \theta_{ef}(0)\right] + j\sin\left[\omega_e t + \theta_{ef}(0)\right]\right\} \tag{1.2-13}$$

is a constant-amplitude line $\sqrt{2}F$ in length rotating counterclockwise at an angular velocity of ω_e with a time-zero displacement from the positive real axis of $\theta_{ef}(0)$. Since $\sqrt{2}F$ is the peak value of the sinusoidal variation, the instantaneous value of F_a is the real part of (1.2-13). In other words, the real projection of the phasor $\tilde{F}_a$ is the instantaneous value of $F_a/\sqrt{2}$ at time zero. As time progresses, $\tilde{F}_a e^{j\omega_e t}$ rotates at ω_e in the counterclockwise direction, and its real projection, in accordance with (1.2-10), is the instantaneous value of $F_a/\sqrt{2}$. Thus, for

$$F_a = \sqrt{2}F\cos\omega_e t \tag{1.2-14}$$

the phasor representing F_a is

$$\tilde{F}_a = Fe^{j0} = F\underline{/0°} = F + j0 \tag{1.2-15}$$

For

$$F_a = \sqrt{2}F \sin \omega_e t \tag{1.2-16}$$

the phasor is

$$\tilde{F}_a = Fe^{-j\pi/2} = F\underline{/-90^\circ} = 0 - jF \tag{1.2-17}$$

Although there are several ways to arrive at (1.2-17) from (1.2-16), it is helpful to ask yourself where must the rotating phasor be positioned at time zero so that, when it rotates counterclockwise at ω_e, its real projection is $\left(1/\sqrt{2}\right)F_p \sin \omega_e t$? Is it clear that a phasor of amplitude F positioned at $\frac{\pi}{2}$ represents $-\sqrt{2}F \sin \omega_e t$?

In order to show the facility of the phasor in the analysis of steady-state performance of ac circuits and devices, it is useful to consider a series circuit consisting of a resistance, an inductance, and a capacitance. Thus,

$$v_a = Ri_a + L\frac{di_a}{dt} + \frac{1}{C}\int i_a dt \tag{1.2-18}$$

For steady-state operation, let

$$V_a = \sqrt{2}V \cos\left[\omega_e t + \theta_{ev}(0)\right] \tag{1.2-19}$$

$$I_a = \sqrt{2}I \cos\left[\omega_e t + \theta_{ei}(0)\right] \tag{1.2-20}$$

where the subscript a is used to distinguish the instantaneous value from the rms value of the steady-state variable. The steady-state voltage equation may be obtained by substituting (1.2-19) and (1.2-20) into (1.2-18), whereupon we can write

$$\begin{aligned}\sqrt{2}V \cos\left[\omega_e t + \theta_{ev}(0)\right] = R\sqrt{2}I \cos\left[\omega_e t + \theta_{ei}(0)\right] + \omega_e L\sqrt{2}I \cos\left[\omega_e t + \frac{1}{2}\pi + \theta_{ei}(0)\right] \\ + \frac{1}{\omega_e C}\sqrt{2}I \cos\left[\omega_e t - \frac{1}{2}\pi + \theta_{ei}(0)\right]\end{aligned} \tag{1.2-21}$$

The second term on the right-hand side of (1.2-21), which is $L\frac{dI_a}{dt}$, can be written as

$$\omega_e L\sqrt{2}I \cos\left[\omega_e t + \frac{1}{2}\pi + \theta_{ei}(0)\right] = \omega_e L Re\left[\sqrt{2}Ie^{j\frac{1}{2}\pi}e^{j\theta_{ei}(0)}e^{j\omega_e t}\right] \tag{1.2-22}$$

Since $\tilde{I}_a = Ie^{j\theta_{ei}(0)}$, we can write

$$\widetilde{L\frac{dI_a}{dt}} = \omega_e L e^{j\frac{1}{2}\pi}\tilde{I}_a \tag{1.2-23}$$

Since $e^{j\frac{1}{2}\pi} = j$, (1.2-23) may be written as

$$L\widetilde{\frac{dI_a}{dt}} = j\omega_e L\widetilde{I}_a \tag{1.2-24}$$

If we follow a similar procedure, we can show that

$$\frac{1}{C}\widetilde{\int I_a dt} = -j\frac{1}{\omega_e C}\widetilde{I}_a \tag{1.2-25}$$

It is interesting that differentiation of a steady-state sinusoidal variable rotates the phasor counterclockwise by $\frac{1}{2}\pi$, whereas integration rotates the phasor clockwise by $\frac{1}{2}\pi$.

The steady-state voltage equation given by (1.2-21) can be written in phasor form as

$$\widetilde{V}_a = \left[R + j\left(\omega_e L - \frac{1}{\omega_e C}\right)\right]\ \widetilde{I}_a \tag{1.2-26}$$

We can express (1.2-26) compactly as

$$\tilde{V}_a = Z\tilde{I}_a \tag{1.2-27}$$

where Z, the impedance, is a complex number; it is not a phasor. It is often expressed as

$$Z = R + j(X_L - X_C) \tag{1.2-28}$$

where $X_L = \omega_e L$ is the inductive reactance and $X_C = \frac{1}{\omega_e C}$ is the capacitive reactance.

The instantaneous power is

$$\begin{aligned} P &= V_a I_a \\ &= \sqrt{2}V\cos\left[\omega_e t + \theta_{ev}(0)\right]\sqrt{2}I\cos\left[\omega_e t + \theta_{ei}(0)\right] \end{aligned} \tag{1.2-29}$$

After some manipulation, we can write (1.2-29) as

$$P = VI\cos\left[\theta_{ev}(0) - \theta_{ei}(0)\right] + VI\cos\left[2\omega_e t + \theta_{ev}(0) + \theta_{ei}(0)\right] \tag{1.2-30}$$

Therefore, the average power P_{ave} may be written as

$$P_{\text{ave}} = |\widetilde{V}_a||\widetilde{I}_a|\ \cos\left[\theta_{ev}(0) - \theta_{ei}(0)\right] \tag{1.2-31}$$

where $|\widetilde{V}_a|$ and $|\widetilde{I}_a|$ are the magnitude of the phasors (rms value), $\theta_{ev}(0) - \theta_{ei}(0)$ is the power factor angle φ_{pf}, and $\cos\left[\theta_{ev}(0) - \theta_{ei}(0)\right]$ is referred to as the power factor. If current is positive in the direction of voltage drop then (1.2-31) is positive if power is consumed and negative if power is generated. It is interesting to point out that in going from (1.2-29) to (1.2-30), the coefficient of the two right-hand terms is $\frac{1}{2}\left(\sqrt{2}V\sqrt{2}I\right)$ or one-half the product of the peak values of the sinusoidal variables. Therefore, it was considered more convenient to use the rms values for the phasors, whereupon average power could be calculated by the product of the magnitude of the voltage and current phasors as given by (1.2-31).

We see from (1.2-30) that the instantaneous power of a single-phase ac circuit oscillates at $2\omega_e t$ about an average value. Let us take a moment to calculate the steady-state power of a two-phase ac system. Balanced, steady-state, two-phase variables (a and b phase) may be expressed as

$$V_a = \sqrt{2}V \cos\left[\omega_e t + \theta_{ev}(0)\right] \tag{1.2-32}$$

$$I_a = \sqrt{2}I \cos\left[\omega_e t + \theta_{ei}(0)\right] \tag{1.2-33}$$

$$V_b = \sqrt{2}V \cos\left[\omega_e t - \frac{1}{2}\pi + \theta_{ev}(0)\right] \tag{1.2-34}$$

$$I_b = \sqrt{2}I \cos\left[\omega_e t - \frac{1}{2}\pi + \theta_{ei}(0)\right] \tag{1.2-35}$$

The total instantaneous power is

$$P = V_a I_a + V_b I_b \tag{1.2-36}$$

Substituting (1.2-32) through (1.2-35) into (1.2-36) and after some trigonometric manipulation, the total power for a balanced two-phase system becomes

$$P = 2\,|\,\tilde{V}_a\,|\,|\,\tilde{I}_a\,|\,\cos\varphi_{pf} \tag{1.2-37}$$

It is important to note that the $2\omega_e t$ oscillation is not present. In other words, the total instantaneous steady-state power is constant. In the case of a three-phase balanced system, the phasors of the three voltages or currents are displaced 120° and the instantaneous steady-state power is also constant and three times the average power of one phase. In other words, the 2 in (1.2-37) becomes 3 when considering a three-phase system.

Example 1A Phasor Diagram

It is often instructive to construct a phasor diagram. For example, let us consider a voltage equation of the form

$$\tilde{V} = Z\tilde{I} + \tilde{E} \tag{1A-1}$$

where Z is given by (1.2-28). Let us assume that $\tilde{V}$ and $\tilde{I}$ are known and that we are to calculate $\tilde{E}$. The phasor diagram may be used as a rough check on these calculations. Let us construct this phasor diagram by assuming that $|\,X_L\,| > |\,X_C\,|$ and $\tilde{V}$ and $\tilde{I}$ are known as shown in Fig. 1A-1. Solving (1A-1) for $\tilde{E}$ yields

$$\tilde{E} = \tilde{V} - [R + j(X_L - X_C)]\tilde{I} \tag{1A-2}$$

To perform this graphically, start at the origin in Fig. 1A-1 and walk to the terminus of $\tilde{V}$. Now, we want to subtract $R\tilde{I}$. To achieve the proper orientation to do this, stand at the terminus of $\tilde{V}$, turn, and look in the $\tilde{I}$ direction which is at the angle

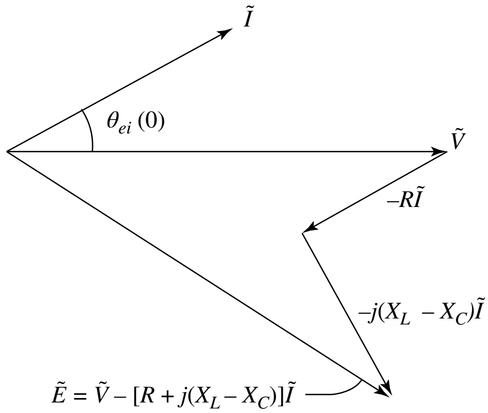

Figure 1A-1 Phasor diagram for (1A-2).

$\theta_{ei}(0)$. But we must subtract $R\widetilde{I}$; hence, $-\widetilde{I}$ is 180° from $\widetilde{I}$, so do an about-face and now we are headed in the $-\widetilde{I}$ direction, which is $\theta_{ei}(0) - 180°$. Start walking in the direction of $-\widetilde{I}$ for the distance $R\,|\,\widetilde{I}\,|$ and then stop. While still facing in the $-\widetilde{I}$ direction, let us consider the next term. Now, since we have assumed that $|\,X_L\,| > |\,X_C\,|$, we must subtract $j(X_L - X_C)\widetilde{I}$, so let us face in the direction of $-j\widetilde{I}$. We are still looking in the $-\widetilde{I}$ direction, so we need only to j ourselves. Thus, we must rotate 90° in the counterclockwise direction, whereupon we are standing at the end of $\widetilde{V} - R\widetilde{I}$ looking in the direction of $\theta_{ei}(0) - 180° + 90°$. Start walking in this direction for the distance of $(X_L - X_C)\,|\,\widetilde{I}\,|$, whereupon we are at the terminus of $\widetilde{V} - [R + j(X_L - X_C)]\widetilde{I}$. According to (1A-2), $\widetilde{E}$ is the phasor drawn from the origin of the phasor diagram to where we are.

The average steady-state power for a single-phase circuit may be calculated using (1.2-31). We will mention in passing that the reactive power is defined as

$$Q = |\,\widetilde{V}\,||\,\widetilde{I}\,|\,\sin[\theta_{ev}(0) - \theta_{ei}(0)] \tag{1A-3}$$

The units of Q are var (voltampere reactive). An inductance is said to absorb reactive power and thus, by definition, Q is positive for an inductor and negative for a capacitor. Actually, Q is a measure of the interchange of energy stored in the electric (capacitor) and magnetic (inductance) fields.

SP1.2-1 If $\widetilde{V} = 1\underline{/0°}$ and $\widetilde{I} = 1\underline{/180°}$ in the direction of the voltage drop, calculate Z and P_{ave}. Is power generated or consumed? [$(-1 + j0)$ ohms, 1 watt, generated]

SP1.2-2 For SP1.2-1, express instantaneous voltage, current, and power if the frequency is 60 Hz. [$V = \sqrt{2}\cos 377t$, $I = \sqrt{2}\cos(377t + \pi)$, $P = -1 + 1\cos(754t + \pi)$]

SP1.2-3 $A = \sqrt{2}\underline{/0°}$, $B = \sqrt{2}\underline{/90°}$. Calculate $A + B$ and $A \times B$. [$1\,\underline{/45°}$, $2\,\underline{/90°}$]

SP1.2-4 In Example 1A, $X_L > X_C$ and yet $\widetilde{I}$ was given as leading $\widetilde{V}$. How can this be? [$\widetilde{E}$]

1.3 MAGNETIC CIRCUITS

An elementary magnetic circuit is shown in Fig. 1.3-1. This system consists of an electric conductor wound N times about the magnetic member, which is generally some type of ferromagnetic material. In this example system, the magnetic member contains an air gap of uniform length between points a and b. We will assume that the magnetic system (circuit) consists only of the magnetic member and the air gap. Recall that Ampere's law states that the line integral of the field intensity **H** about a closed path is equal to the net current enclosed within this closed path of integration. That is,

$$\oint \mathbf{H} \cdot d\mathbf{L} = i_n \tag{1.3-1}$$

where i_n is the net current enclosed. Let us apply Ampere's law to the closed path depicted as a dashed line in Fig. 1.3-1. In particular,

$$\int_a^b H_i dL + \int_b^a H_g dL = Ni \tag{1.3-2}$$

where the path of integration is assumed to be in the clockwise direction. This equation requires some explanation. First, we are assuming that the field intensity exists only in the direction of the given path, hence we have dropped the vector notation. The subscript i denotes the field intensity (H_i) in the ferromagnetic material (iron or steel) and g denotes the field intensity (H_g) in the air gap. The path of integration is taken as the mean length about the magnetic member, for purposes we shall explain later. The right-hand side of (1.3-2) represents the net current enclosed. In particular, we have enclosed the current i, N times. This has the units of amperes but is commonly referred to as ampere-turns (A·t) or magnetomotive force (mmf). We will find that the mmf in magnetic circuits is analogous to the electromotive force (emf) in electric circuits. Note that the current enclosed is positive in (1.3-2) if the current i is positive. The sign of the right-hand side of (1.3-2)

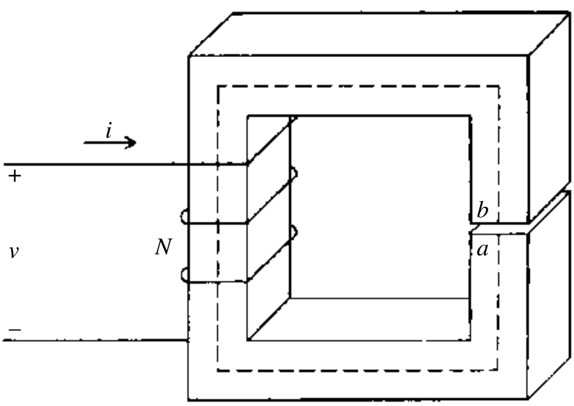

Figure 1.3-1 Elementary magnetic circuit.

may be determined by the so-called "corkscrew" rule. That is, the current enclosed is positive if its assumed positive direction is in the same direction as the advance of a right-hand screw if it were turned in the direction of the path of integration, which in Fig. 1.3-1 is clockwise. Before continuing, it should be mentioned that we refer to **H** as the field intensity; however, some authors prefer to call **H** the field strength.

If we carry out the line integration, (1.3-2) can be written as

$$H_i l_i + H_g l_g = Ni \tag{1.3-3}$$

where l_i is the mean length of the magnetic material and l_g is the length across the air gap. Now, we have some explaining to do. We have assumed that the magnetic circuit consists only of the ferromagnetic material and the air gap, and that the magnetic field intensity is always in the direction of the path of integration or, in other words, perpendicular to a cross section of the magnetic material taken in the same sense as the air gap is cut through the material. The assumed direction of the magnetic field intensity is valid except in the vicinity of the corners. The direction of the field intensity changes gradually rather than abruptly at the corners. Nevertheless, the "mean length approximation" is widely used as an adequate means of analyzing this type of magnetic circuit.

Let us now take a cross section of the magnetic material as shown in Fig. 1.3-2. From our study of physics, we know that for linear, isotropic magnetic materials the flux density **B** is related to the field intensity as

$$\mathbf{B} = \mu \mathbf{H} \tag{1.3-4}$$

where μ is the permeability of the medium. Hence, we can write (1.3-3) in terms of flux density as

$$\frac{B_i}{\mu_i} l_i + \frac{B_g}{\mu_g} l_g = Ni \tag{1.3-5}$$

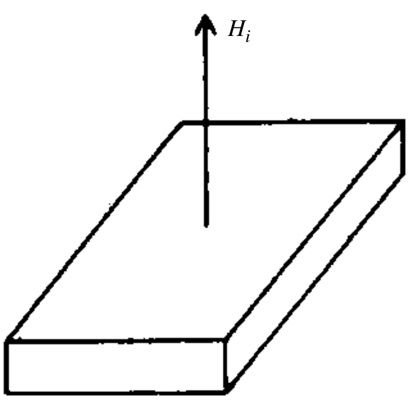

Figure 1.3-2 Cross section of magnetic material.

The surface integral of the flux density is equal to the flux Φ, thus

$$\Phi = \int_A \mathbf{B} \cdot d\mathbf{S} \tag{1.3-6}$$

If we assume that the flux density is uniform over the cross-sectional area, then

$$\Phi_i = B_i A_i \tag{1.3-7}$$

where Φ_i is the total flux in the magnetic material and A_i is the associated cross-sectional area. In the air gap,

$$\Phi_g = B_g A_g \tag{1.3-8}$$

where A_g is the cross-sectional area of the gap. From physics, it is known that the streamlines of flux density **B** are closed; hence, the flux in the air gap is equal to the flux in the core. That is, $\Phi_i = \Phi_g$, and, if the air gap is small, $A_i \cong A_g$, and, therefore, $B_i \cong B_g$. However, the effective area of the air gap is larger than that of the magnetic material, since the flux will tend to balloon or spread out (fringing effect), covering a maximum area midway across the air gap. Generally, this is taken into account by assuming that $A_g = kA_i$, where k, which is greater than unity, is determined primarily by the length of the air gap. Although we shall keep this in mind, it is sufficient for our purposes to assume $A_g = A_i$. If we let $\Phi_i = \Phi_g = \Phi$ and substitute (1.3-7) and (1.3-8) into (1.3-5), we obtain

$$\frac{l_i}{\mu_i A_i}\Phi + \frac{l_g}{\mu_g A_g}\Phi = Ni \tag{1.3-9}$$

The analogy to Ohm's law is at hand. Ni (mmf) is analogous to the voltage (emf), and the flux Φ is analogous to the current. We can complete this analogy if we recall that the resistance of a conductor is proportional to its length and inversely proportional to its conductivity and cross-sectional area. Similarly, $l_i/\mu_i A_i$ and $l_g/\mu_g A_g$ are the reluctances of the magnetic material and air gap, respectively. Generally, the permeability is expressed in terms of relative permeability as

$$\mu_i = \mu_{ri}\mu_0 \tag{1.3-10}$$

$$\mu_g = \mu_{rg}\mu_0 \tag{1.3-11}$$

where μ_0 is the permeability of free space ($4\pi \times 10^{-7}$ Wb/A · m or $4\pi \times 10^{-7}$ H/m, since Wb/A is a henry) and μ_{ri} and μ_{rg} are the relative permeability of the magnetic material and the air gap, respectively. For all practical purposes, $\mu_{rg} = 1$; however, μ_{ri} may be as large as 500–4000 depending upon the type of ferromagnetic material. We will use $\mathcal{R}$ to denote reluctance so as to distinguish reluctance from resistance, which will be denoted by r or R. We can now write (1.3-9) as

$$\left(\mathcal{R}_i + \mathcal{R}_g\right)\Phi = Ni \tag{1.3-12}$$

where $\mathcal{R}_i$ and $\mathcal{R}_g$ are the reluctance of the iron and air gap, respectively.

Example 1B Magnetic Circuit – dc Source
A magnetic system is shown in Fig. 1B-1. The total number of turns is 100, the relative permeability of the iron is 1000, and the current is 10 A. Calculate the total flux in the center leg.

Let us draw the electric circuit analog of this magnetic system for which we will need to calculate the reluctance of the various paths:

$$\mathcal{R}_{ab} = \frac{l_{ab}}{\mu_{ri}\mu_0 A_i} = \frac{0.22}{1000\left(4\pi \times 10^{-7}\right)(0.04)^2} = 109,419\ \mathrm{H}^{-1} \tag{1B-1}$$

Similarly,

$$\mathcal{R}_{bcda} = \frac{0.25 + 0.22 + 0.25}{(1000)\left(4\pi \times 10^{-7}\right)(0.04)^2} = 358,099\ \mathrm{H}^{-1} \tag{1B-2}$$

Neglecting the air gap length,

$$\mathcal{R}_{bef} = \mathcal{R}_{gha} = \frac{1}{2}\mathcal{R}_{bcda} = 179,049\ \mathrm{H}^{-1} \tag{1B-3}$$

The reluctance of the air gap is

$$\mathcal{R}_{fg} = \frac{0.002}{\left(4\pi \times 10^{-7}\right)(0.04)^2} = 994,718\ \mathrm{H}^{-1} \tag{1B-4}$$

The electric circuit analog is given in Fig. 1B-2. The polarity of the mmf is determined by the right-hand rule. That is, if we grasp one of the turns of the

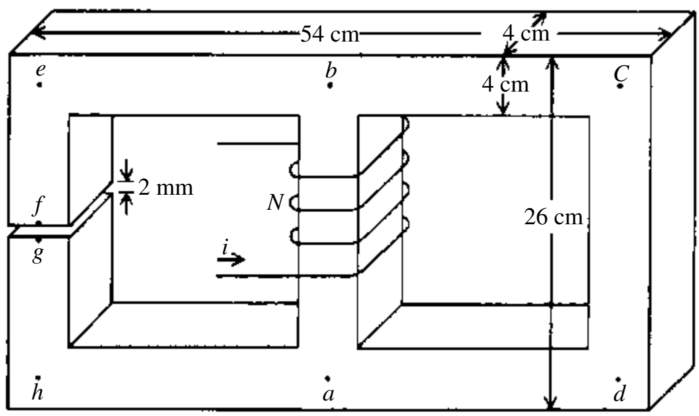

Figure 1B-1 Single-winding magnetic system.

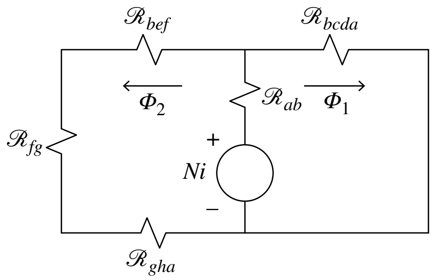

Figure 1B-2 Electric-circuit analog of Fig. 1B-1.

winding with our right hand with the thumb pointed in the direction of positive current, then our fingers will point in the direction of positive flux which flows in the direction of an mmf rise. Or, if we grasp the winding (center leg) with the fingers of our right hand in the direction of positive current, then our thumb will be in the direction of positive flux and in the direction of a rise in mmf.

We can now apply dc circuit theory to solve for the total flux, $\Phi_1 + \Phi_2$, flowing in the center leg. For example, we can use loop equations or, as we will do here, reduce the series–parallel circuit to an equivalent reluctance. The equivalent reluctance of the parallel combination is

$$\begin{aligned}\mathcal{R}_{eq} &= \frac{(\mathcal{R}_{bcda})(\mathcal{R}_{bef} + \mathcal{R}_{fg} + \mathcal{R}_{gha})}{\mathcal{R}_{bcda} + \mathcal{R}_{bef} + \mathcal{R}_{fg} + \mathcal{R}_{gha}} \\ &= \frac{(358,099)(179,049 + 994,718 + 179,049)}{358,099 + 179,049 + 994,718 + 179,049} \\ &= \frac{(358,099)(1,352,816)}{1,710,915} = 283,148 \text{ H}^{-1}\end{aligned} \tag{1B-5}$$

$$\begin{aligned}\Phi_1 + \Phi_2 &= \frac{Ni}{\mathcal{R}_{ab} + \mathcal{R}_{eq}} \\ &= \frac{(100)(10)}{109,419 + 283,148} = 2.547 \times 10^{-3} \text{ Wb}\end{aligned} \tag{1B-6}$$

Example 1C Magnetic Circuit – ac Source

Consider the magnetic system shown in Fig. 1C-1. The windings are supplied from ac sources and, in the steady state, $I_1 = \sqrt{2}\cos\omega_e t$ and $I_2 = \sqrt{2}\,0.3\cos(\omega_e t + 45^\circ)$, where capital letters are used to denote steady-state conditions. $N_1 = 150$ turns, $N_2 = 90$ turns, and $\mu_r = 3000$. Calculate the flux in the center leg.

The electric circuit analog is given in Fig. 1C-2. The reluctance $\mathcal{R}_x$ is the reluctance of the center leg and $\mathcal{R}_y$ is the reluctance of one of the two parallel paths

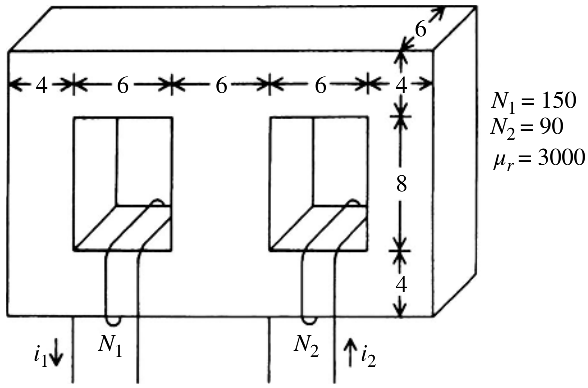

Figure 1C-1 A two-winding magnetic system with dimensions in centimeters.

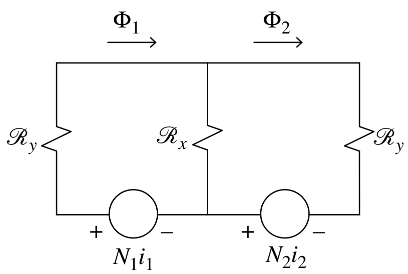

Figure 1C-2 Electric-circuit analog of Fig. 1C-1.

from the top of the center leg through an outside leg to the bottom of the center leg. In particular,

$$\mathcal{R}_y = \frac{2(0.03 + 0.06 + 0.02) + 0.12}{3000\left(4\pi \times 10^{-7}\right)(0.06)(0.04)} = 37,578\ \mathrm{H}^{-1} \tag{1C-1}$$

$$\mathcal{R}_x = \frac{0.12}{3000\left(4\pi \times 10^{-7}\right)0.06^2} = 8842\ \mathrm{H}^{-1} \tag{1C-2}$$

Since the currents are sinusoidal, the mmfs will be sinusoidal. Thus, it is convenient to use phasors to solve for Φ_1 and Φ_2. The loop equations are

$$\widetilde{\mathrm{mmf}}_1 = \mathcal{R}_y\tilde{\Phi}_1 + \mathcal{R}_x\left(\tilde{\Phi}_1 - \tilde{\Phi}_2\right) \tag{1C-3}$$

$$\widetilde{\mathrm{mmf}}_2 = \mathcal{R}_x\left(\tilde{\Phi}_2 - \tilde{\Phi}_1\right) + \mathcal{R}_y\tilde{\Phi}_2 \tag{1C-4}$$

which may be written in matrix form as

$$\begin{bmatrix} \widetilde{\mathrm{mmf}}_1 \\ \widetilde{\mathrm{mmf}}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{R}_x + \mathcal{R}_y & -\mathcal{R}_x \\ -\mathcal{R}_x & \mathcal{R}_x + \mathcal{R}_y \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} \tag{1C-5}$$

Now,

$$\tilde{\text{mmf}}_1 = N_1\tilde{I}_1 = (150)(1\angle 0^\circ) = 150\angle 0^\circ \text{At} \tag{1C-6}$$

$$\tilde{\text{mmf}}_2 = N_2\tilde{I}_2 = (90)(0.3\angle 45^\circ) = 27\angle 45^\circ \text{At} \tag{1C-7}$$

Solving (1C-5) yields

$$\tilde{\Phi}_1 = (3.434 + j0.081) \times 10^{-3} \text{ Wb} \tag{1C-8}$$

$$\tilde{\Phi}_2 = (1.065 + j0.427) \times 10^{-3} \text{ Wb} \tag{1C-9}$$

The flux flowing down through the center leg is

$$\begin{aligned} \tilde{\Phi}_1 - \tilde{\Phi}_2 &= (2.369 - j0.346) \times 10^{-3} \\ &= 2.39 \times 10^{-3}\angle -8.3^\circ \text{ Wb} \end{aligned} \tag{1C-10}$$

SP1.3-1 Calculate Φ_1 in Example 1B. $[\Phi_1 = 2.014 \times 10^{-3} \text{ Wb}]$

SP1.3-2 Calculate $\tilde{\Phi}_1 + \tilde{\Phi}_2$ in Example 1B when $I = \sqrt{2}\,10\cos(\omega_e t - 30^\circ)$. $[\tilde{\Phi}_1 + \tilde{\Phi}_2 = 2.547 \times 10^{-3}\angle -30^\circ \text{ Wb, rms}]$

SP1.3-3 Remove the center leg of the magnetic system shown in Fig. 1C-1. Calculate the total flux when $I_1 = 9$ A and $I_2 = -15$ A. [Zero]

SP1.3-4 Express the sinusoidal variation represented by $\tilde{\Phi}_2$ given by (1C-9). $[\sqrt{2}\,1.147 \times 10^{-3}\cos(\omega_e t + 21.8^\circ)]$

1.4 PROPERTIES OF MAGNETIC MATERIALS

We may be aware from our study of physics that, when ferromagnetic materials such as iron, nickel, cobalt, or alloys of these elements, such as various types of steels, are placed in a magnetic field, the flux produced is markedly larger (500–4000 times, for example) than that which would be produced when a non-magnetic material is subjected to the same magnetic field. We must take some time to review briefly the basic properties of ferromagnetic materials and to establish terminology for later use.

Let us begin by considering the relationship between B and H shown in Fig. 1.4-1 which is typical of silicon steel used in transformers. We will assume that the ferromagnetic core is initially completely demagnetized (both B and H are zero). As we apply an external H field by increasing the current in a winding wound around the core, the flux density B also increases, but nonlinearly, as shown in Fig. 1.4-1. After H reaches a value of approximately 150 A·t/m, the flux density rises more slowly and the material begins to saturate.

In ferromagnetic materials, the combination of the magnetic moments produced by the electrons orbiting the nucleus of an atom and the electron itself spinning on its axis produce a net magnetic moment of the atom that is not canceled

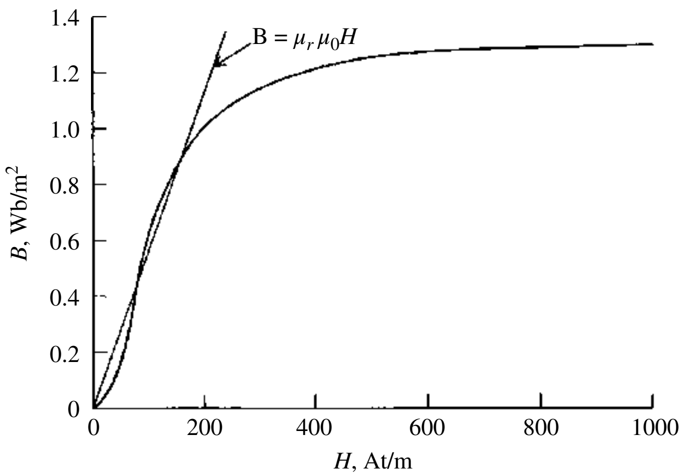

Figure 1.4-1 *B–H* curve for typical silicon steel used in transformers.

by an opposing magnetic moment of a neighboring atom. Ferromagnetic materials have been found to be divided into magnetic domains wherein all magnetic moments (dipoles) are aligned. Although the magnetic moments are all aligned within a magnetic domain, the direction of this alignment will differ from one domain to another.

When a ferromagnetic material is subjected to an external magnetic field, those domains, which originally tend to be aligned with the applied magnetic field, grow at the expense of those domains with magnetic moments that are less aligned. Thereby, the flux is increased from that which would occur with a nonmagnetic material. This is known as *domain-wall motion* [1]. As the strength of the magnetic field increases, the aligned domains continue to grow in nearly a linear fashion. Thus, a nearly linear *B–H* curve results $(B \cong \mu_r \mu_0 H)$ until the ability of the aligned domains to take from the unaligned domains starts to slow. This gives rise to the knee of the *B–H* curve and the beginning of saturation. At this point, the displacements of the domain walls are complete. That is, there are no longer unaligned domains from which to take. However, the remaining domains may still not be in perfect alignment with the external *H* field. A further increase in *H* will cause a rotation of the atomic dipole moments within the remaining domains toward a more perfect alignment. However, the marginal increase in *B* due to rotation is less than the original increase in *B* due to domain-wall motion, resulting in a decrease in slope of the *B–H* curve. The magnetic material is said to be completely saturated when the remaining domains are perfectly aligned. In this case, the slope of the *B–H* curve becomes μ_0 [1]. If it is assumed that the magnetic flux is uniform through most of the magnetic material, then *B* is proportional to Φ and *H* is proportional to mmf. Hence, a plot of flux versus current is of the same shape as the *B–H* curve.

A transformer is generally designed so that some saturation occurs during normal operation. Electric machines are also designed similarly in that a machine generally operates slightly in the saturated region during normal, rated operating conditions. Since saturation causes the coefficients of the differential equations describing the behavior of an electromagnetic device to be functions of the winding currents, a transient analysis is difficult without the aid of a computer. However, it is not our purpose to set forth methods of analyzing nonlinear magnetic systems.

In the previous discussion, we have assumed that the ferromagnetic material is initially demagnetized and that the applied field intensity is gradually increased from zero. However, if a ferromagnetic material is subjected to an alternating field intensity, the resulting *B–H* curve exhibits hysteresis. For example, let us assume that a ferromagnetic material is subjected to an alternating field intensity (alternating current flowing in the winding) and initially the flux density and field intensity are both zero. As *H* increases from zero, *B* increases along the initial *B–H* curve, as shown in Fig. 1.4-2. However, the field intensity varies sinusoidally and, when *H* decreases from a maximum, *B* does not follow back down the original *B–H* curve. After several cycles, the magnetic system will reach a steady-state condition and the plot of *B* versus *H* will form a hysteresis loop or a double-valued function, as shown in Fig. 1.4-2. What is happening is very complex. In simple terms, the growth of aligned domains for an incremental change in *H* in one direction is not equal to the growth of oppositely aligned domains if this change in *H* were suddenly reversed. We could become quite involved by discussing minor hysteresis loops which would occur if, during the sinusoidal variation of *H*, it were suddenly stopped at some nonzero value then reversed, stopped, and reversed again [1]. We shall only mention this phenomenon in passing.

A family of hysteresis loops is shown in Fig. 1.4-3. In each case, the applied *H* is sinusoidal; however, the amplitude of the *H* field is varied to give the family of

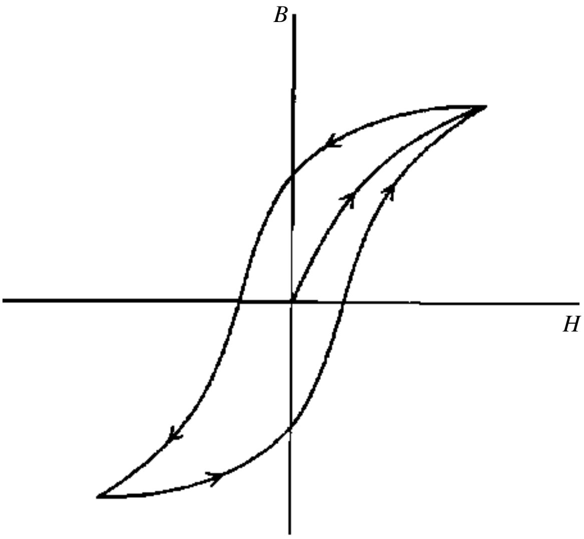

Figure 1.4-2 Hysteresis loop.

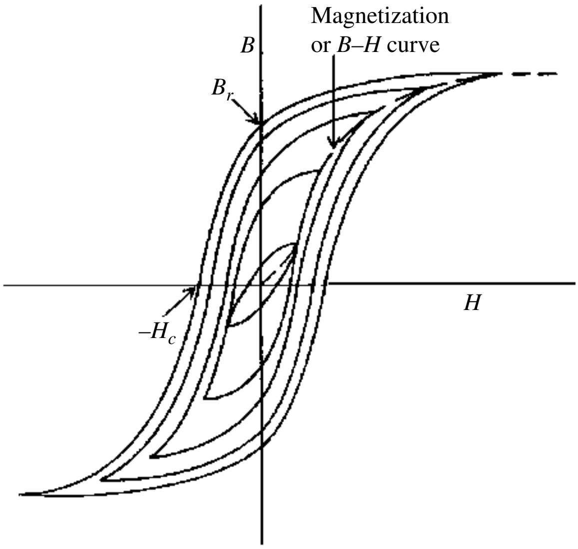

Figure 1.4-3 Family of steady-state hysteresis loops.

loops shown in Fig. 1.4-3. A magnetization or *B–H* curve for a given material is obtained by connecting the tips of the hysteresis loops, as shown by the dashed line in Fig. 1.4-3. The locus of the tips of the hysteresis curves is about the same as the original *B–H* curve in Fig. 1.4-1, which corresponds to a gradual increase of *H* in an initially demagnetized material. If *H* were suddenly stopped at zero, the flux density remaining in the ferromagnetic material is called the residual flux density (B_r). The negative field intensity necessary to bring this residual flux density to zero is called the coercive force (H_c). These two quantities are indicated in Fig. 1.4-3 for the largest hysteresis loop shown.

Energy is required to increase the size of the magnetic domains of the ferromagnetic material. It can be shown that the energy necessary to align alternately the magnetic domains is equal to the area enclosed by the hysteresis loop. This energy causes a rise in the temperature of the magnetic material, and the power associated with this energy loss is called the hysteresis loss.

When a solid block of magnetic material such as that shown in Fig. 1.3-1 is subjected to an alternating field intensity, the resulting alternating flux induces current in the solid magnetic material, which will circulate in a loop perpendicular to the flux density (**B**) inducing it. These so-called eddy currents have two undesirable effects. First, the mmf established by these circulating currents opposes the mmf produced by the winding, and this opposition is greatest at the center of the material because that tends to be also the center of the current loops. Thus, the flux would tend not to flow through the center of the solid magnetic member, thereby not utilizing the full benefits of the ferromagnetic material. Second, there is an i^2r loss associated with these currents, called *eddy current loss*, which is dissipated as heat. These two adverse effects can be minimized in several ways, but the most common is to build the ferromagnetic core of laminations (thin strips) insulated from each other and oriented in the direction of the magnetic field (**B** or **H**). These thin strips offer a much smaller area in which the eddy currents can flow; hence, smaller currents and smaller losses result.

The core losses associated with ferromagnetic materials are the combination of the hysteresis and eddy current losses. Electromagnetic devices are designed to minimize these losses; however, they are always present and are often taken into account in a linear system analysis by assuming that their effects on the electric system can be represented by a resistance.

SP1.4-1 The magnetic circuit of Fig. 1.3-1 is constructed by using silicon sheet steel. Its magnetization curve is given by Fig. 1.4-1. The gap length l_g is 1 mm, the mean core length l_i is 100 cm, $N = 500$, and $A_i = A_g = 25\ \text{cm}^2$. Determine the current needed to produce a flux Φ of 2.5×10^{-3} Wb. [*Hint*: First establish H_i, H_g, and use (1.3-3).] [$I = 1.99$ A]

1.5 STATIONARY MAGNETICALLY COUPLED CIRCUITS

Magnetically coupled electric circuits are central to the operation of transformers and electromechanical motion devices. In the case of transformers, stationary circuits are magnetically coupled for the purpose of changing the ac voltage and current levels. In the case of electromechanical devices, circuits in relative motion are magnetically coupled for the purpose of transferring energy between the mechanical and electric systems. Since magnetically coupled circuits play such an important role in energy conversion, it is important to establish the equations that describe their behavior and to express these equations in a form convenient for analysis. Many of these goals may be achieved by considering two stationary electric circuits that are magnetically coupled, as shown in Fig. 1.5-1. The two windings consist of turns N_1 and N_2, and they are wound on a common core, which is a ferromagnetic material with a permeability large relative to that of air. The magnetic core is not illustrated in three dimensions.

Before proceeding, a comment or two is in order. Generally, the concept of an ideal transformer is introduced in a basic circuit course. In the ideal case, v_2 in Fig. 1.5-1 is $(N_2/N_1)v_1$ and i_2 is $-(N_1/N_2)i_1$. Only the turns ratio of the transformer is considered. However, this treatment is often not sufficient for a detailed analysis of transformers, and it is seldom appropriate in the analysis of electromechanical motion devices, since an air gap is necessary for motion to occur; hence, the windings are not as tightly coupled as in the case of transformers and the leakage flux must be taken into account.

In general, the flux produced by each winding can be separated into two components: a leakage component denoted with the subscript l and a magnetizing component denoted by the subscript m. Each of these components is depicted by a single streamline with the positive direction determined by applying the right-hand

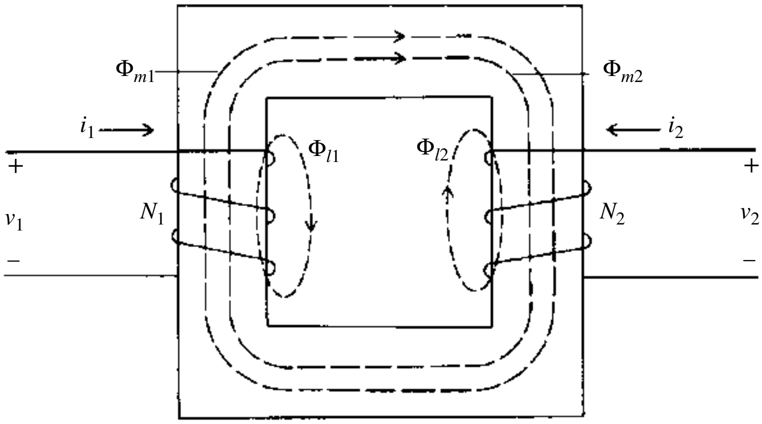

Figure 1.5-1 Magnetically coupled circuits.

rule to the directions of current flow in the winding. (The right-hand rule was reviewed in Example 1B.) The leakage flux associated with a given winding links only that winding, whereas the magnetizing flux, whether it is due to current in winding 1 or winding 2, links both windings. In some cases, i_2 is selected positive out of the top of winding 2 and a dot is placed at that terminal. Although the "dot notation" is convenient for transformers, it is seldom used in the case of electromechanical devices.

The flux linking each winding may be expressed as

$$\Phi_1 = \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \tag{1.5-1}$$

$$\Phi_2 = \Phi_{l2} + \Phi_{m2} + \Phi_{m1} \tag{1.5-2}$$

The leakage flux Φ_{l1} is produced by current flowing in winding 1 and it links only the turns of winding 1. Likewise, the leakage flux Φ_{l2} is produced by current flowing in winding 2 and it links only the turns of winding 2. The flux Φ_{m1} is produced by current flowing in winding 1 and it links all turns of windings 1 and 2. Similarly, the magnetizing flux Φ_{m2} is produced by current flowing in winding 2 and it also links all turns of windings 1 and 2. Both Φ_{m1} and Φ_{m2} are called *magnetizing fluxes*. With the selected positive directions of current flow and the manner in which the windings are wound, magnetizing flux produced by positive current flowing in one winding adds to the magnetizing flux produced by positive current flowing in the other winding. For this case, we will find that the mutual inductance is positive.

It is appropriate to point out that this is an idealization of the actual magnetic system. It seems logical that all of the leakage flux will not link all the turns of the winding producing it; hence, Φ_{l1} and Φ_{l2} are "equivalent" leakage fluxes. Similarly, all of the magnetizing flux of one winding may not link all of the turns of the other winding. To acknowledge this practical aspect of the magnetic system,

N_1 and N_2 are often considered to be the equivalent number of turns rather than the actual number.

The voltage equations may be expressed as

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt} \tag{1.5-3}$$

$$v_2 = r_2 i_2 + \frac{d\lambda_2}{dt} \tag{1.5-4}$$

In matrix form,

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{1.5-5}$$

The resistances r_1 and r_2 and the flux linkages λ_1 and λ_2 are related to windings 1 and 2, respectively. Since it is assumed that Φ_1 links the equivalent turns of winding 1 (N_1) and Φ_2 links the equivalent turns of winding 2 (N_2), the flux linkages may be written as

$$\lambda_1 = N_1 \Phi_1 \tag{1.5-6}$$

$$\lambda_2 = N_2 \Phi_2 \tag{1.5-7}$$

where Φ_1 and Φ_2 are given by (1.5-1) and (1.5-2), respectively.

If we assume that the magnetic system is linear, we may apply Ohm's law for magnetic circuits to express the fluxes. Thus, the fluxes may be written as

$$\Phi_{l1} = \frac{N_1 i_1}{\mathcal{R}_{li}} \tag{1.5-8}$$

$$\Phi_{m1} = \frac{N_1 i_1}{\mathcal{R}_m} \tag{1.5-9}$$

$$\Phi_{l2} = \frac{N_2 i_2}{\mathcal{R}_{l2}} \tag{1.5-10}$$

$$\Phi_{m2} = \frac{N_2 i_2}{\mathcal{R}_m} \tag{1.5-11}$$

where $\mathcal{R}_{l1}$ and $\mathcal{R}_{l2}$ are the reluctances of the leakage paths, and $\mathcal{R}_m$ is the reluctance of the path of magnetizing fluxes. Typically, the reluctances associated with leakage paths are much larger than the reluctance of the magnetizing path. The reluctance associated with an individual leakage path is difficult to determine exactly, and it is usually approximated from test data or by using the computer to solve the field equations numerically. On the other hand, the reluctance of the magnetizing path of the core shown in Fig. 1.5-1 may be computed with sufficient accuracy as in Example 1B.

Substituting (1.5-8) through (1.5-11) into (1.5-1) and (1.5-2) yields

$$\Phi_1 = \frac{N_1 i_1}{\mathcal{R}_{l1}} + \frac{N_1 i_1}{\mathcal{R}_m} + \frac{N_2 i_2}{\mathcal{R}_m} \tag{1.5-12}$$

$$\Phi_2 = \frac{N_2 i_2}{\mathcal{R}_{l2}} + \frac{N_2 i_2}{\mathcal{R}_m} + \frac{N_1 i_1}{\mathcal{R}_m} \tag{1.5-13}$$

Substituting (1.5-12) and (1.5-13) into (1.5-6) and (1.5-7) yields

$$\lambda_1 = \frac{N_1^2}{\mathcal{R}_{l1}} i_1 + \frac{N_1^2}{\mathcal{R}_m} i_1 + \frac{N_1 N_2}{\mathcal{R}_m} i_2 \tag{1.5-14}$$

$$\lambda_2 = \frac{N_2^2}{\mathcal{R}_{l2}} i_2 + \frac{N_2^2}{\mathcal{R}_m} i_2 + \frac{N_2 N_1}{\mathcal{R}_m} i_1 \tag{1.5-15}$$

When the magnetic system is linear, the flux linkages are generally expressed in terms of inductances and the currents. We see that the coefficients of the first two terms on the right-hand side of (1.5-14) depend upon N_1 and the reluctance of the magnetic system, independent of the existence of winding 2. An analogous statement may be made regarding (1.5-15) with the roles of winding 1 and winding 2 reversed. Hence, the self-inductances are defined as

$$L_{11} = \frac{N_1^2}{\mathcal{R}_{l1}} + \frac{N_1^2}{\mathcal{R}_m} = L_{l1} + L_{m1} \tag{1.5-16}$$

$$L_{22} = \frac{N_2^2}{\mathcal{R}_{l2}} + \frac{N_2^2}{\mathcal{R}_m} = L_{l2} + L_{m2} \tag{1.5-17}$$

where L_{l1} and L_{l2} are the leakage inductances and L_{m1} and L_{m2} are the magnetizing inductances of windings 1 and 2, respectively. From (1.5-16) and (1.5-17), it follows that the magnetizing inductances may be related as

$$\frac{L_{m2}}{N_2^2} = \frac{L_{m1}}{N_1^2} \tag{1.5-18}$$

The mutual inductances are defined as the coefficient of the third term on the right-hand side of (1.5-14) and (1.5-15). In particular,

$$L_{12} = \frac{N_1 N_2}{\mathcal{R}_m} \tag{1.5-19}$$

$$L_{21} = \frac{N_2 N_1}{\mathcal{R}_m} \tag{1.5-20}$$

We see that $L_{12} = L_{21}$ and, with the assumed positive direction of current flow and the manner in which the windings are wound, the mutual inductances are positive. If, however, the assumed positive directions of current were such that Φ_{m1} opposed Φ_{m2}, then the mutual inductances would be negative.

The mutual inductances may be related to the magnetizing inductances. Comparing (1.5-16) and (1.5-17) with (1.5-19) and (1.5-20), we see that

$$L_{12} = \frac{N_2}{N_1} L_{m1} = \frac{N_1}{N_2} L_{m2} \tag{1.5-21}$$

The flux linkages may now be written as

$$\lambda_1 = L_{11} i_1 + L_{12} i_2 \tag{1.5-22}$$

$$\lambda_2 = L_{21} i_1 + L_{22} i_2 \tag{1.5-23}$$

where L_{11} and L_{22} are defined by (1.5-16) and (1.5-17), respectively, and L_{12} and L_{21}, by (1.5-21). The self-inductances L_{11} and L_{22} are always positive; however, the mutual inductances $L_{12}\,(L_{21})$ may be positive or negative, as previously mentioned.

Although the voltage equations given by (1.5-3) and (1.5-4) may be used for purposes of analysis, it is customary to perform a change of variables which yields the well-known equivalent *T* circuit of two windings coupled by a linear magnetic circuit. To set the stage for this derivation, let us express the flux linkages from (1.5-22) and (1.5-23) as

$$\lambda_1 = L_{l1} i_1 + L_{m1}\left(i_1 + \frac{N_2}{N_1} i_2\right) \tag{1.5-24}$$

$$\lambda_2 = L_{l2} i_2 + L_{m2}\left(\frac{N_1}{N_2} i_1 + i_2\right) \tag{1.5-25}$$

With λ_1 in terms of L_{m1} and λ_2 in terms of L_{m2}, we see two logical candidates for substitute variables, in particular, $(N_2/N_1)i_2$ or $(N_1/N_2)i_1$. If we let

$$i_2' = \frac{N_2}{N_1} i_2 \tag{1.5-26}$$

then we are using the substitute variable i_2', which, when flowing through winding 1, produces the same mmf as the actual i_2 flowing through winding 2; $N_1 i_2' = N_2 i_2$. This is said to be referring the current in winding 2 to winding 1 or to a winding with N_1 turns, whereupon winding 1 becomes the reference winding. On the other hand, if we let

$$i_1' = \frac{N_1}{N_2} i_1 \tag{1.5-27}$$

then i_1' is the substitute variable that produces the same mmf when flowing through winding 2 as i_1 does when flowing in winding 1; $N_2 i_1' = N_1 i_1$. This change of variables is said to refer the current of winding 1 to winding 2 or to a winding with N_2 turns, whereupon winding 2 becomes the reference winding.

We will demonstrate the derivation of the equivalent *T* circuit by referring the current of winding 2 to a winding with N_1 turns; thus, i_2' is expressed by (1.5-26). We want the instantaneous power to be unchanged by this substitution of variables. Therefore,

$$v_2' i_2' = v_2 i_2 \tag{1.5-28}$$

Hence,

$$v_2' = \frac{N_1}{N_2} v_2 \tag{1.5-29}$$

Flux linkages, which have the units of V · s, are related to the substitute flux linkages in the same way as voltages. In particular,

$$\lambda_2' = \frac{N_1}{N_2}\lambda_2 \tag{1.5-30}$$

Now, replace $(N_2/N_1)i_2$ with i_2' in the expression for λ_1, given by (1.5-24). Next, solve (1.5-26) for i_2 and substitute it into λ_2 given by (1.5-25). Now, multiply this result by N_1/N_2 to obtain λ_2' and then substitute $(N_2/N_1)^2 L_{m1}$ for L_{m2} in λ_2'. If we do all this, we will obtain

$$\lambda_1 = L_{l1} i_1 + L_{m1}\left(i_1 + i_2'\right) \tag{1.5-31}$$

$$\lambda_2' = L_{l2}' i_2' + L_{m1}\left(i_1 + i_2'\right) \tag{1.5-32}$$

where

$$L_{l2}' = \left(\frac{N_1}{N_2}\right)^2 L_{l2} \tag{1.5-33}$$

The flux linkage equations given by (1.5-31) and (1.5-32) may also be written as

$$\lambda_1 = L_{11} i_1 + L_{m1} i_2' \tag{1.5-34}$$

$$\lambda_2' = L_{m1} i_1 + L_{22}' i_2' \tag{1.5-35}$$

where

$$L_{22}' = \left(\frac{N_1}{N_2}\right)^2 L_{22} = L_{l2}' + L_{m1} \tag{1.5-36}$$

and L_{22} is defined by (1.5-17).

If we multiply (1.5-4) by N_1/N_2 to obtain v_2', the voltage equations become

$$\begin{bmatrix} v_1 \\ v_2' \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2' \end{bmatrix} \begin{bmatrix} i_1 \\ i_2' \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \lambda_1 \\ \lambda_2' \end{bmatrix} \tag{1.5-37}$$

where

$$r_2' = \left(\frac{N_1}{N_2}\right)^2 r_2 \tag{1.5-38}$$

The previous voltage equations, (1.5-37), together with the flux linkage equations, (1.5-34) through (1.5-35), suggest the equivalent *T* circuit shown in Fig. 1.5-2. This method may be extended to include any number of windings wound on the same core.

Earlier in this section, we mentioned that in the case of an ideal transformer only the turns ratio is considered, that is, $v_2 = (N_2/N_1)v_1$ and $i_2 = -(N_1/N_2)i_1$. We can now more fully appreciate the assumptions that are made in this type of

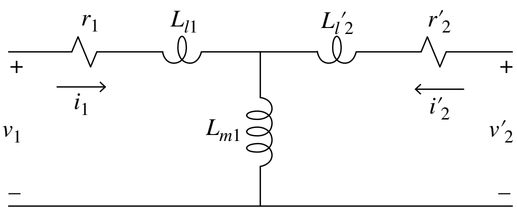

Figure 1.5-2 Equivalent *T* circuit with winding 1 selected as reference winding.

analysis. In particular, the resistances r_1 and r_2 and the leakage inductances L_{l1} and L_{l2} are neglected, and it is assumed that the magnetizing inductance is large so that the magnetizing current $i_1 + i'_2$ is negligibly small.

The information presented in this section forms the basis of the equivalent circuits for many types of electric machines. Using a turns ratio to refer the voltages and currents of rotor circuits of electric machines to a winding with the same number of turns as the stator windings is common practice.

Example 1D The Equivalent *T* Circuit

It is instructive to illustrate the method of deriving an equivalent *T* circuit from open- and short-circuit measurements. When winding 2 of the two-winding transformer shown in Fig. 1.5-2 is open circuited and a voltage of 110 V (rms) at 60 Hz is applied to winding 1, the average power supplied to winding 1 is 6.66 W. The measured current in winding 1 is 1.05 A (rms). Next, with winding 2 short-circuited, the current flowing in winding 1 is 2 A (rms) when the applied voltage is 30 V at 60 Hz. The average input power is 44 W. If we assume $L_{l1} = L'_{l2}$, an approximate equivalent *T* circuit can be determined from these measurements with winding 1 selected as the reference winding.

The average power supplied to winding 1 may be expressed as

$$P_1 = |\widetilde{V}_1||\widetilde{I}_1| \cos\varphi_{pf} \tag{1D-1}$$

where

$$\varphi_{pf} = \theta_{ev}(0) - \theta_{ei}(0) \tag{1D-2}$$

Here, $\widetilde{V}_1$ and $\widetilde{I}_1$ are phasors with the positive direction of $\widetilde{I}_1$ taken in the direction of the voltage drop, and $\theta_{ev}(0)$ and $\theta_{ei}(0)$ are the phase angles of $\widetilde{V}_1$ and $\widetilde{I}_1$, respectively. Solving for φ_{pf} during the open-circuit test, we have

$$\varphi_{pf} = \cos^{-1}\frac{P_1}{|\widetilde{V}_1||\widetilde{I}_1|} = \cos^{-1}\frac{6.66}{(110)(1.05)} = 86.7^\circ \tag{1D-3}$$

Although $\varphi_{pf} = -86.7^\circ$ is also a legitimate solution of (1D-3), the positive solution is taken since $\widetilde{V}_1$ leads $\widetilde{I}_1$ in an inductive circuit. With winding 2 open-circuited, the input impedance of winding 1 is

$$Z = \frac{\widetilde{V}_1}{\widetilde{I}_1} = r_1 + j(X_{l1} + X_{m1}) \tag{1D-4}$$

With $\widetilde{V}_1$ as the reference phasor, $\widetilde{V}_1 = 110\angle 0°$, $\widetilde{I}_1 = 1.05\angle -86.7°$. Thus,

$$r_1 + j(X_{l1} + X_{m1}) = \frac{110\angle 0°}{1.05\angle -86.7°} = 6 + j104.6 \;\; \Omega \tag{1D-5}$$

If we neglect core losses, then, from (1D-5), $r_1 = 6\,\Omega$. We also see from (1D-5) that $X_{l1} + X_{m1} = 104.6\,\Omega$.

For the short-circuit test, we will assume that $\widetilde{I}_1 = -I'_2$ since transformers are designed so that at rated frequency $X_{m1} \gg | r'_2 + jX'_{l2} |$. Hence, using (1D-1) again,

$$\varphi_{pf} = \cos^{-1}\frac{44}{(30)(2)} = 42.8° \tag{1D-6}$$

In this case, the input impedance is $Z = (r_1 + r'_2) + j(X_{l1} + X'_{l2})$. This may be determined as

$$Z = \frac{30\angle 0°}{2\angle -42.8°} = 11 + j10.2 \;\; \Omega \tag{1D-7}$$

Hence, $r'_2 = 11 - r_1 = 5\ \Omega$ and, since it is assumed that $X_{l1} = X'_{l2}$, both are $10.2/2 = 5.1\ \ \Omega$. Therefore, $X_{m1} = 104.6 - 5.1 = 99.5\,\Omega$. In summary, $r_1 = 6\ \Omega$, $L_{l1} = 13.5\,\text{mH}$, $L_{m1} = 263.9\,\text{mH}$, $r'_2 = 5\,\Omega$, and $L'_{l2} = 13.5\ \text{mH}$. Make sure we converted from X's to L's correctly.

SP1.5-1 Remove the center leg of the magnetic system shown in Fig. 1C-1. Calculate L_{11}, L_{22}, and L_{12}. Neglect the leakage inductances. $[L_{11} = 299.4\,\text{mH}$, $L_{22} = 107.8\,\text{mH}$, $L_{12} = 179.5\,\text{mH}]$

SP1.5-2 Consider the transformer and parameters calculated in Example 1D. Winding 2 is short-circuited and 12 V (dc) is applied to winding 1. Calculate the steady-state values of i_1 and i_2. Repeat with winding 2 open-circuited. [$I_1 = 2$ A and $I_2 = 0$ in both cases]

SP1.5-3 Calculate the no-load (winding 2 open-circuited) current for the transformer given in Example 1D if $V_1 = \sqrt{2}\ 10\cos 100t$. $[\widetilde{I}_1 = 0.352\ \angle -77.8°\ \text{A}]$

1.6 OPEN- AND SHORT-CIRCUIT CHARACTERISTICS OF STATIONARY MAGNETICALLY COUPLED CIRCUITS

It is instructive to observe the open- and short-circuit characteristics of a transformer with two windings. For this purpose, a transformer with the parameters given in Example 1D was simulated on a computer. The open-circuit characteristics are shown in Figs. 1.6-1 and 1.6-2. The variables plotted are λ, v_1, i_1, v'_2, and i'_2. The variable λ is equal to $L_{m1}(i_1 + i'_2)$, which is the last term on the right-hand side of (1.5-31) and (1.5-32). This is the flux linkage of winding 1 due to the flux in the transformer iron. It is often referred to as the *magnetizing flux linkage(s)* and denoted λ_m, λ_{mag}, or λ_φ, whereas $i_1 + i'_2$ is called the *magnetizing current*.

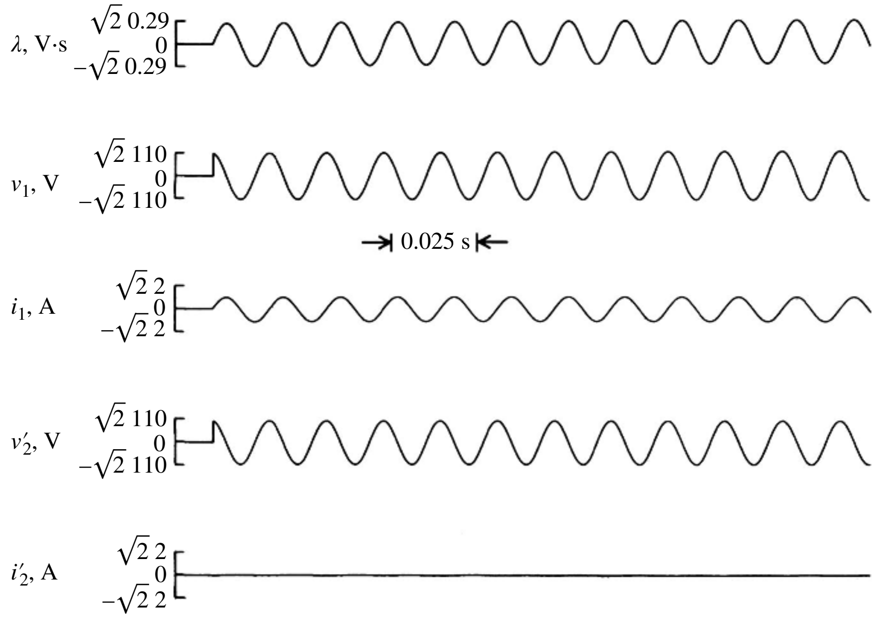

Figure 1.6-1 Open-circuit conditions of a two-winding transformer with $v_1 = \sqrt{2}\ 110 \cos 377t$.

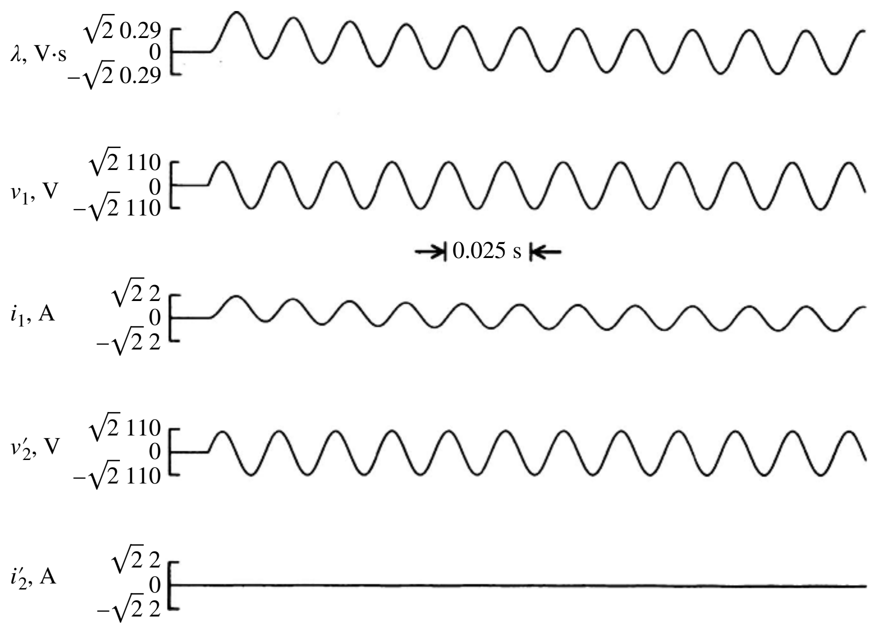

Figure 1.6-2 Open-circuit conditions of a two-winding transformer with $v_1 = \sqrt{2}\ 110 \sin 377t$.

Initially, the windings are unexcited. At time zero $(t = 0)$, the voltage applied to winding 1 with winding 2 open-circuited is $v_1 = \sqrt{2}\ 110\cos 377t$ in Fig. 1.6-1 and $v_1 = \sqrt{2}\ 110\sin 377t$ in Fig. 1.6-2. The waveforms of the steady-state current i_1 are identical in Figs. 1.6-1 and 1.6-2; however, since the inductive reactance is large, applying a sine wave voltage for v_1 at time zero results in a much larger transient offset in i_1 than when $v_1 = \sqrt{2}\ 110\cos 377t$. Since $v_1 = \sqrt{2}\ 110\sin 377t$ causes a larger transient offset, it makes it easier for us to identify the transient period. Therefore, we shall continue with v_1 as a sine wave. Although it is difficult to determine the time constant for the offset of the current i_1 (or λ) to decay to one-third of its original value, it is on the order of 50 ms. The calculated value of the no-load time constant is $\tau_{nl} = (L_{l1} + L_{m1})/r_1 = 46.2$ ms. Before leaving Figs. 1.6-1 and 1.6-2, note that, during steady-state conditions, I_1 lags V_1 by something close to 90° (86.7°, from Example 1D).

Let us now go to the short-circuit characteristics. The transient and steady-state response with $v_1 = \sqrt{2}\ 110\sin 377t$ and with $v_2' = 0$ are shown in Fig. 1.6-3. There are several things to note. From Fig. 1.6-3, it appears that the time constant associated with the decay of i_1 is small, less than 5 ms. Now let us look at the magnetizing flux linkage λ. We see that it is smaller in amplitude than in the no-load case. We would expect this since during short-circuit conditions $i_1 \cong -i_2'$, whereby the mmfs of the two windings oppose, and the resulting flux

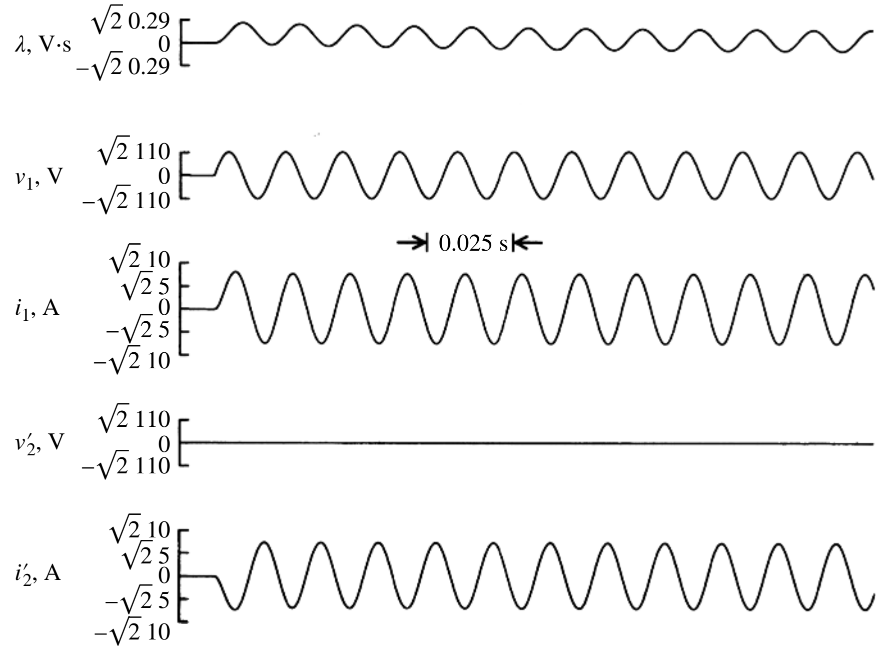

Figure 1.6-3 Short-circuit conditions of a two-winding transformer with $v_1 = \sqrt{2}\ 110\sin 377t$.

in the transformer iron is less than for the no-load condition where $i_2' = 0$. Looking at this in another way, we realize that i_1 and $-i_2'$ will be essentially equal during short-circuit conditions whenever the impedance of the magnetizing branch $(j\omega_e L_{m1})$ is much larger (say 8–10 times larger) than $r_2' + j\omega_e L_{l2}'$. Here, $\omega_e = 377$ rad/s.

It is interesting to note that the decay of the magnetizing flux linkage λ is much slower than the apparent decay of the currents. As we mentioned, the time constant associated with i_1 is small; however, there is indeed a small difference between i_1 and $-i_2'$, and this small current (magnetizing current), which is actually a small part of i_1, must flow in the large inductance L_{m1}. Hence, the magnetizing current is associated with a longer time constant than the much larger component of the current i_1, which circulates through the series r_2' and L_{l2}'.

SP1.6-1 Use the plot of λ in Fig. 1.6-3 to approximate $|\tilde{I}_1 + \tilde{I}_2'|$. $\left[|\tilde{I}_1 + \tilde{I}'_2| \cong \frac{1}{2}\text{ A}\right]$

SP1.6-2 Calculate, using reasonable approximations, the phase angle between the steady-state current $\tilde{I}_1$ and voltage $\tilde{V}_1$ for the conditions of Fig. 1.6-3. Check your answer from the plots. [$\tilde{V}_1$ leads $\tilde{I}_1$ by 42.8°]

SP1.6-3 Consider the transformer given in Example 1D. Assume $V_1 = \sqrt{2}\ 110\cos 1000t$, and a load is connected across winding 2. The impedance of this load referred to winding 1 is $21 + j5\ \Omega$. Calculate $\tilde{I}_2'$. Make valid approximations to reduce your work. $\left[\tilde{I}_2' \cong -2.4\underline{/-45^\circ}\right]$

1.7 MAGNETIC SYSTEMS WITH MECHANICAL MOTION

In Chapter 2, relationships are derived for determining the electromagnetic force or torque established in electromechanical systems. Once this development is completed, three examples of elementary electromechanical systems are considered. It is convenient to introduce these three systems here for the purpose of establishing the voltage equations and expressions for the self- and mutual inductances, thereby setting the stage for the analysis to follow in Chapter 2. The first of these electromechanical systems is an elementary version of an electromagnet. It consists of a magnetic core, part of which is movable. The electric system exerts an electromagnetic force upon this movable member, thereby moving it relative to the stationary member. We shall analyze this device, and in Chapter 2 we shall observe its operating characteristics by computer traces. The second system is a rotational device commonly referred to as a reluctance machine. A large number of stepper motors operate on the reluctance-torque principle. The third device is also a rotational device that has two windings: one on the stationary member and one on the rotational member. This device, although somewhat impracticable, illustrates the concept of windings or magnetic systems in relative motion.

Elementary Electromagnet

An elementary electromagnet that we will consider is shown in Fig. 1.7-1. This system consists of a stationary core with a winding of N turns and a block of magnetic material that is free to slide relative to the stationary member. It is shown in more detail in Chapter 2, wherein a spring, a damper, and an external force are associated with the movable member. We do not need to consider that level of detail here; instead, we will assume that the movable member is at a distance x from the stationary member, which may be a function of time, that is $x = x(t)$.

The voltage equation that describes the electric system is

$$v = ri + \frac{d\lambda}{dt} \tag{1.7-1}$$

where the flux linkages are expressed as

$$\lambda = N\Phi \tag{1.7-2}$$

The flux may be written as

$$\Phi = \Phi_l + \Phi_m \tag{1.7-3}$$

where Φ_l is the leakage flux and Φ_m is the magnetizing flux that is common to both the stationary and movable members. If the magnetic system is considered to be linear (saturation neglected), then, as in the case of the stationary coupled circuits, we can express the fluxes in terms of reluctances. That is,

$$\Phi_l = \frac{Ni}{\mathcal{R}_l} \tag{1.7-4}$$

$$\Phi_m = \frac{Ni}{\mathcal{R}_m} \tag{1.7-5}$$

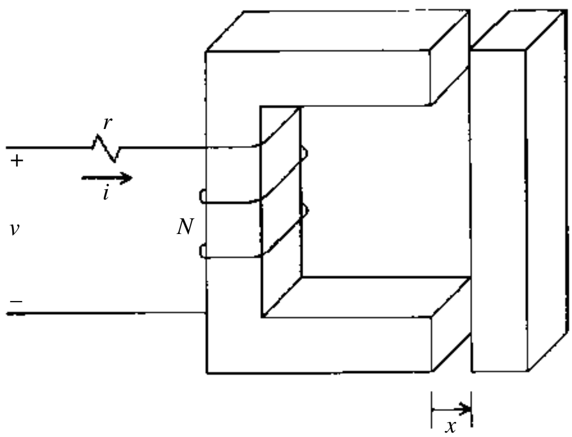

Figure 1.7-1 Elementary electromagnet.

where $\mathcal{R}_l$ and $\mathcal{R}_m$ are the reluctances of the leakage and magnetizing paths, respectively.

The flux linkages may now be written as

$$\lambda = \left(\frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_m}\right)i \tag{1.7-6}$$

where the leakage inductance is

$$L_l = \frac{N^2}{\mathcal{R}_l} \tag{1.7-7}$$

and the magnetizing inductance is

$$L_m = \frac{N^2}{\mathcal{R}_m} \tag{1.7-8}$$

The reluctance of the magnetizing path is

$$\mathcal{R}_m = \mathcal{R}_i + 2\mathcal{R}_g \tag{1.7-9}$$

where $\mathcal{R}_i$ is the total reluctance of the magnetic material of the stationary and movable members and $\mathcal{R}_g$ is the reluctance of one of the air gaps. If the cross-sectional area of the stationary and movable members is assumed to be equal and of the same material, the reluctances may be expressed as

$$\mathcal{R}_i = \frac{l_i}{\mu_{ri}\mu_0 A_i} \tag{1.7-10}$$

$$\mathcal{R}_g = \frac{x}{\mu_0 A_g} \tag{1.7-11}$$

We will assume that $A_g = A_i$. Even though, as we have mentioned previously, this may be somewhat of an oversimplification, it is sufficient for our purposes. Hence, $\mathcal{R}_m$ may be written as

$$\mathcal{R}_m = \frac{1}{\mu_0 A_i}\left(\frac{l_i}{\mu_{ri}} + 2x\right) \tag{1.7-12}$$

The magnetizing inductance now becomes

$$L_m = \frac{N^2}{(1/\mu_0 A_i)(l_i/\mu_{ri} + 2x)} \tag{1.7-13}$$

In this analysis, the leakage inductance is assumed to be constant. The magnetizing inductance is clearly a function of displacement. That is, $x = x(t)$ and $L_m = L_m(x)$. Heretofore, when dealing with linear magnetic circuits wherein mechanical motion is not present as in the case of a transformer, the change of flux linkages with respect to time was simply $L(di/dt)$. This is not the case here. When the inductance is a function of $x(t)$,

$$\lambda(i, x) = L(x)i = [L_l + L_m(x)]i \tag{1.7-14}$$

and

$$\frac{d\lambda(i,x)}{dt} = \frac{\partial\lambda}{\partial i}\frac{di}{dt} + \frac{\partial\lambda}{\partial x}\frac{dx}{dt} \tag{1.7-15}$$

With (1.7-15) in mind, we see that the voltage equation (1.7-1) becomes

$$v = ri + [L_l + L_m(x)]\frac{di}{dt} + i\frac{dL_m(x)}{dx}\frac{dx}{dt} \tag{1.7-16}$$

Equation (1.7-16) is a nonlinear differential equation owing to the last two terms on the right-hand side.

In preparation for our work in Chapter 2, let us write (1.7-13) as

$$L_m(x) = \frac{k}{k_0 + x} \tag{1.7-17}$$

where

$$k = \frac{N^2\mu_0 A_i}{2} \tag{1.7-18}$$

$$k_0 = \frac{l_i}{2\mu_{ri}} \tag{1.7-19}$$

When $x = 0$, $L_m(x)$ is determined by the reluctance of the magnetic material. That is, for $x = 0$,

$$L_m(0) = \frac{k}{k_0} = \frac{N^2\mu_0\mu_{ri}A_i}{l_i} \tag{1.7-20}$$

Depending upon the parameters of the magnetic material, $L_m(x)$ may be adequately predicted by

$$L_m(x) = \frac{k}{x} \qquad \text{for } x > 0 \tag{1.7-21}$$

We will use this approximation in Chapter 2.

Elementary Reluctance Machine

An elementary reluctance machine is shown in Fig. 1.7-2. It consists of a stationary core with a winding of N turns and a movable member that rotates at an angular displacement and angular velocity of θ_r and ω_r, respectively. The displacement is defined as

$$\theta_r = \omega_r t + \theta_r(0) \tag{1.7-22}$$

The voltage equation is of the form given by (1.7-1). Similarly, the flux may be divided into a leakage and magnetizing flux, as given by (1.7-3). It is convenient to express the flux linkages as

$$\lambda = (L_l + L_m)i \tag{1.7-23}$$

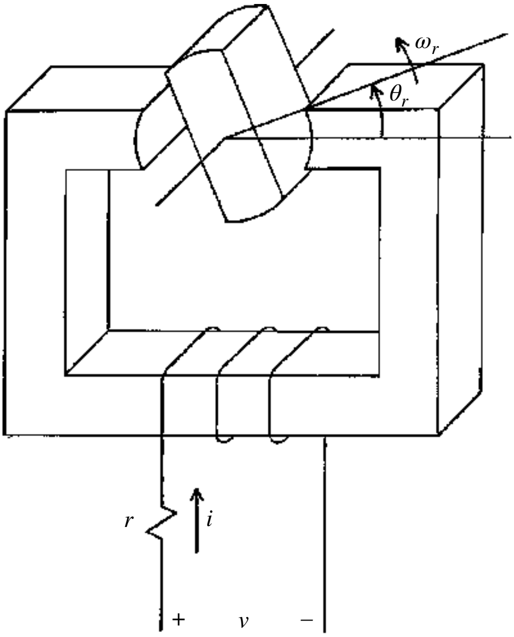

Figure 1.7-2 Elementary reluctance machine.

where L_l is the leakage inductance and L_m is the magnetizing inductance. The leakage inductance is essentially constant, independent of θ_r; however, the magnetizing inductance is a periodic function of θ_r. That is, $L_m = L_m(\theta_r)$. In particular, with $\theta_r = 0$,

$$L_m(0) = \frac{N^2}{\mathcal{R}_m(0)} \tag{1.7-24}$$

Here, the reluctance of the magnetizing path $\mathcal{R}_m$ is maximum due to the large air gap when the rotor is in the vertical (unaligned) position. Hence, L_m is a minimum in this position. Note that this same situation occurs not only at $\theta_r = 0$ but also when $\theta_r = \pi, 2\pi$, and so on.

Now, $\theta_r = \frac{1}{2}\pi$

$$L_m\left(\frac{1}{2}\pi\right) = \frac{N^2}{\mathcal{R}_m\left(\frac{1}{2}\pi\right)} \tag{1.7-25}$$

Here, $\mathcal{R}_m$ is a minimum and, thus, L_m is a maximum. This same situation occurs at $\theta_r = \frac{3}{2}\pi, \frac{5}{2}\pi$, and so on. Hence, the magnetizing inductance varies between maximum and minimum positive values twice per revolution of the rotating member (rotor). Let us make it easy for ourselves and assume that this variation may be adequately approximated by a sinusoidal function. In particular, let $L_m(\theta_r)$ be expressed as

$$L_m(\theta_r) = L_A - L_B \cos 2\theta_r \tag{1.7-26}$$

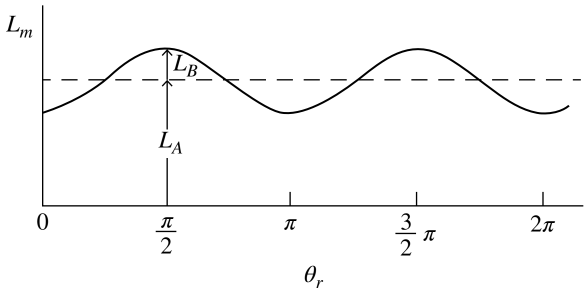

Figure 1.7-3 Approximation of magnetizing inductance of an elementary reluctance machine.

whereupon

$$L_m(0) = L_A - L_B \tag{1.7-27}$$

$$L_m\left(\frac{1}{2}\pi\right) = L_A + L_B \tag{1.7-28}$$

and $L_A > L_B$. The average value is L_A, illustrated in Fig. 1.7-3. The self-inductance may now be expressed as

$$\begin{aligned} L(\theta_r) &= L_l + L_m(\theta_r) \\ &= L_l + L_A - L_B \cos 2\theta_r \end{aligned} \tag{1.7-29}$$

The voltage equation is of the form given by (1.7-16) with x replaced by θ_r.

Windings in Relative Motion

The rotational device shown in Fig. 1.7-4 will be used to illustrate windings in relative motion. This device consists of two windings each containing several turns of a conductor. Winding 1 has N_1 turns and it is on the stationary member (stator); winding 2 has N_2 turns and it is on the rotating member (rotor). The $\otimes$ indicates that the assumed direction of positive current flow in the conductors is into the paper, whereas $\odot$ indicates positive current flow in the conductors is out of the paper. In a practical device, the turns of a winding are distributed over an arc (often 30–60°) of the stator and rotor. However, in this introductory consideration, it is sufficient to assume that the turns are concentrated in one position, as shown in Fig. 1.7-4. Also, the length of the air gap between the stator and rotor is shown exaggerated relative to the inside diameter of the stator.

The voltage equations may be written as

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt} \tag{1.7-30}$$

$$v_2 = r_2 i_2 + \frac{d\lambda_2}{dt} \tag{1.7-31}$$

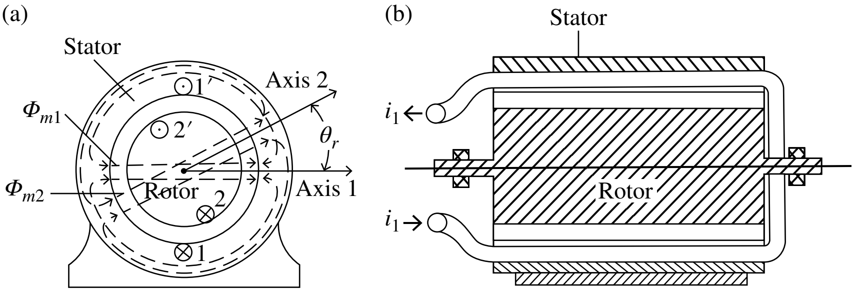

Figure 1.7-4 Elementary rotational electromechanical device. (a) End view; (b) cross-sectional view.

where r_1 and r_2 are the resistances of winding 1 and winding 2, respectively. The magnetic system is assumed linear; therefore, the flux linkages may be expressed as

$$\lambda_1 = L_{11}i_1 + L_{12}i_2 \tag{1.7-32}$$

$$\lambda_2 = L_{21}i_1 + L_{22}i_2 \tag{1.7-33}$$

The self-inductances L_{11} and L_{22} are constants and may be expressed in terms of leakage and magnetizing inductances as

$$\begin{aligned} L_{11} &= L_{l1} + L_{m1} \\ &= \frac{N_1^2}{\mathcal{R}_{l1}} + \frac{N_1^2}{\mathcal{R}_m} \end{aligned} \tag{1.7-34}$$

$$\begin{aligned} L_{22} &= L_{l2} + L_{m2} \\ &= \frac{N_2^2}{\mathcal{R}_{l2}} + \frac{N_2^2}{\mathcal{R}_m} \end{aligned} \tag{1.7-35}$$

where $\mathcal{R}_m$ is the reluctance of the complete magnetic path of Φ_{m1} and Φ_{m2}, which is through the rotor and stator iron and twice across the air gap. Clearly, it is the same for the magnetic system established by either winding 1 or winding 2.

Take a moment to note the designation of axis 1 and axis 2 in Fig. 1.7-4. These axes denote the positive direction of the respective magnetic systems with the assumed positive direction of current flow in the windings (right-hand rule). Now let us consider L_{12}. (Is it clear that $L_{12} = L_{21}$?) When θ_r, which is defined by (1.7-22), is zero, then the coupling between windings 1 and 2 is maximum. In particular, with $\theta_r = 0$ the magnetic system of winding 1 aids that of winding 2 with positive currents assumed. Hence, the mutual inductance is positive and

$$L_{12}(0) = \frac{N_1N_2}{\mathcal{R}_m} \tag{1.7-36}$$

When $\theta_r = \frac{1}{2}\pi$, the windings are orthogonal. The mutual coupling is zero. Hence,

$$L_{12}\left(\frac{1}{2}\pi\right) = 0 \tag{1.7-37}$$

Again let us make it as simple as possible by assuming that the mutual inductance may be adequately predicted by

$$L_{12}(\theta_r) = L_{sr}\cos\theta_r \tag{1.7-38}$$

where L_{sr} is the amplitude of the sinusoidal mutual inductance between the stator and rotor windings as given by (1.7-36).

In writing the voltage equations from (1.7-30) and (1.7-31), the total derivative of the flux linkages is required. This is accomplished by taking the partial derivative of both λ_1 and λ_2 with respect to i_1, i_2, and θ_r.

SP1.7-1 Let $L_m(x) = k/x$, $i = t$, and $x = t$. Express $d[L_m(x)i]/dt$. [Zero]

SP1.7-2 Express $L(\theta_r)$ of the elementary reluctance machine if minimum reluctance occurs at $\theta_r = 0$. $[L(\theta_r) = L_l + L_A + L_b\cos 2\theta_r]$

SP1.7-3 Express L_{11}, L_{22}, and L_{12} if positive i_2 is reversed from that shown in Fig. 1.7-4. $[L_{11}$ and L_{22} are unchanged; $L_{12} = -L_{sr}\cos\theta_r]$

SP1.7-4 Consider Fig. 1.7-4. $I_1 = 1$ A, $L_{sr} = 0.1$ H, $\omega_r = 100$ rad/s, $\theta_r(0) = 0$, and winding 2 is open-circuited. Express V_2. $[V_2 = -10\sin 100t]$

1.8 RECAPPING

We will analyze electromechanical motion devices from the coupled-circuit viewpoint. Although the coupled windings of many electromechanical devices are in relative motion, the equivalent circuit of stationary coupled windings (the transformer) is the beginning of the equivalent circuits that we will develop for these devices in later chapters. We will find the concept of referring variables from one winding to the other very useful as we proceed.

The first step in the analysis of electromechanical motion devices of the electromagnetic type is to express the voltage and flux linkage equations in terms of self- and mutual inductances. We will not consider saturation in our analysis; instead we will restrict our work to linear magnetic systems and leave the analytical treatment of saturation to a more advanced study of these devices. In this chapter, we learned that electromagnetic, electromechanical motion devices are characterized by self- or mutual inductances that vary with displacement of the movable member.

In the next chapter, we will first develop an analytical means of determining the electromagnetic force or torque in electromechanical motion devices. Once we have accomplished this, we will be able to express the electromagnetic force in the elementary electromagnet and the electromagnetic torque in the elementary rotational devices that we have just considered.

1.9 PROBLEMS

1. Consider the magnetic system shown in Fig. 1.3-1. Let $\mu_r = 1500$, $N = 100$ turns, and $I = 2$ A. The cross section of the iron is square, each side 4 cm in length. The air gap is 4 mm in length. The mean length of the iron is 200 times the air gap length. Neglect leakage flux and assume $A_i = A_g$. Calculate the flux.

2. Repeat Example 1B with a second air gap of 2 mm in length cut midway between c and d. Neglect leakage flux and assume $A_i = A_g$.

3. An iron-core transformer that has two windings is shown in Fig. 1.9-1. $N_1 = 50$ turns, $N_2 = 100$ turns, and $\mu_r = 4000$. Calculate L_{12}, L_{m1}, and L_{m2}.

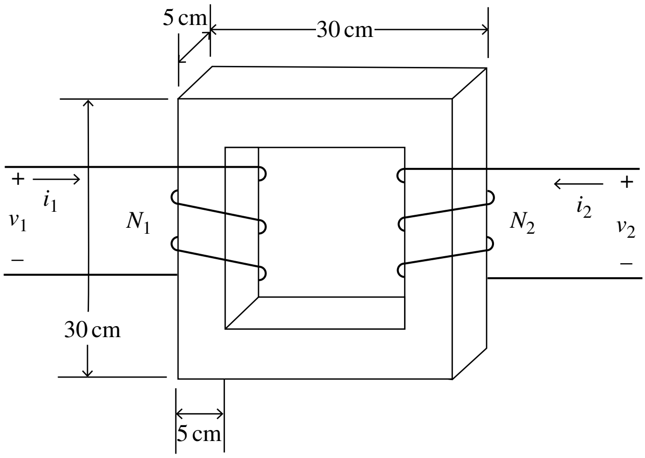

Figure 1.9-1 A two-winding iron-core transformer.

4. An iron "doughnut" (toroid) with two coils is shown in Fig. 1.9-2. $N_1 = 100$ turns and $N_2 = 200$ turns, $\mu_r = 10^4/4\pi$. Calculate L_{12}.

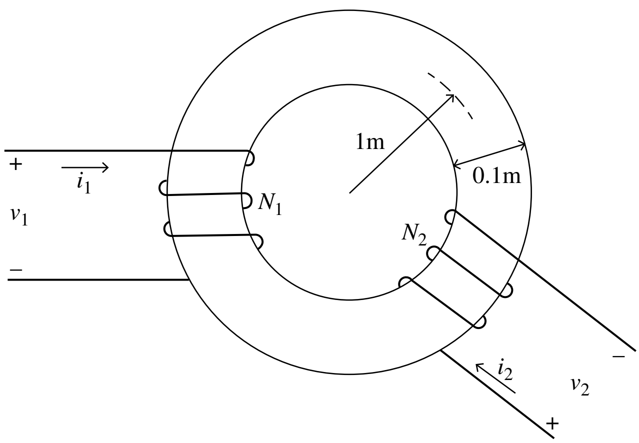

Figure 1.9-2 A two-winding iron-core toroid.

5. An air gap is cut through the left leg of the magnetic system shown in Fig. 1C-1 so that the associated reluctance is $10\mathcal{R}_y$ rather than $\mathcal{R}_y$. Express L_{12} and L_{21} in terms of N_1, N_2, $\mathcal{R}_x$, and $\mathcal{R}_y$.

6. Two coupled coils have the following parameters: $r_1 = 10\,\Omega$, $L_{l1} = 0.1L_{11}$, $r_2 = 2.5\,\Omega$, $L_{l2} = 0.1L_{22}$, $L_{11} = 100$ mH, $N_1 = 100$ turns, $L_{22} = 25$ mH, and $N_2 = 50$ turns. Develop an equivalent T circuit with (a) winding 1 as the reference winding and (b) winding 2 as reference winding.

7. Assume that the direction of positive current is reversed in winding 2 of Fig. 1.5-1. Express (a) L_{12} in terms of N_1, N_2, and $\mathcal{R}_m$; (b) λ_1 and λ_2 in the form of (1.5-22) and (1.5-23); (c) λ_1 and λ_2' in the form of (1.5-31) and (1.5-32); and (d) v_1 and v_2' in the form of (1.5-37).

8. The parameters of a transformer are: $r_1 = r_2' = 10\ \Omega$, $L_{m1} = 300$ mH, and $L_{l1} = L_{l2}' = 30$ mH. A 10-V peak-to-peak 30-Hz sinusoidal voltage is applied to winding 1. Winding 2 is short-circuited. Assume $i_1 = -i_2'$. Calculate the phasor $\widetilde{I}_1$ with $\widetilde{V}_1$ at zero degrees.

9. A transformer with two windings has the following parameters: $r_1 = r_2 = 1\,\Omega$, $L_{m1} = 1$ H, $L_{l1} = L_{l2} = 0.01$ H, and $N_1 = N_2$. A 2-Ω load resistance R_L is connected across winding 2. $V_1 = 2\cos 400t$. (a) Calculate $\widetilde{I}_1$. (b) Express I_1.

10. A transformer with two windings has the following parameters: $r_1 = 1\,\Omega$, $L_{l1} = 0.01$ H, $L_{m1} = 0.2$ H, $N_2 = 2N_1$, $r_2 = 2\,\Omega$, $L_{l2} = 0.04$ H, and $L_{m2} = 0.08$ H. A 4 - Ω resistance R_L is connected across the terminals of winding 2 and a voltage $V_1 = \sqrt{2}\,2\cos 400t$ is applied to winding 1. Calculate and draw the phasor diagram showing $\widetilde{V}_1$, $\widetilde{I}_1$, $\widetilde{V}_2'$, and $\widetilde{I}_2'$. Neglect the magnetizing current.

11. For the elementary electromagnet shown in Fig. 1.7-1, assume that the cross-sectional area of the stationary and movable member is the same and $A_i = A_g = 4\ \text{cm}^2$. Assume $l_i = 20$ cm, $N = 500$ turns, and $\mu_{ri} = 1000$. Express $L_m(x)$ given by (1.7-17) and the approximation for $x > 0$ given by (1.7-21). Determine the minimum value of x when this approximate expression for $L_m(x)$ is less than 1.1 the value given by (1.7-17).

12. Express the voltage v of the elementary electromagnet given by (1.7-16) for $L_m(x)$ given by (1.7-17), $i = \sqrt{2}I_s\cos\omega_e t$, and $x = t$. Approximate v when t is large.

13. Express the voltage equation for the elementary reluctance machine shown in Fig. 1.7-2. Use (1.7-29) for $L(\theta_r)$.

14. Write the voltage equations for the coils in relative motion shown in Fig. 1.7-4. Use L_{11}, L_{22}, and L_{12} as expressed by (1.7-38). ■

1.10 REFERENCES

1. G. R. Slemon and A. Straughen, *Electric Machines*, Addison-Wesley Publishing Company, Reading, Mass., 1980.
2. P. C. Krause, *Analysis of Electric Machinery*, McGraw-Hill Book Company, New York, 1986.