Chapter 1

LINEAR FUNCTIONS AND CHANGE

CONTENTS

1.1 FUNCTIONS AND FUNCTION NOTATION

In everyday language, the word *function* expresses the idea of dependence. For example, we might say that election results are a function of the economy, meaning that the winner is determined by how the economy is doing. Or we might claim that car sales are a function of the weather, meaning that the number of cars sold is affected by the weather.

In mathematics, the meaning of the word *function* is similar, but more precise. A function is a relationship between two quantities. If the value of the first quantity determines exactly one value of the second quantity, we say the second quantity is a function of the first:

> A **function** is a rule that takes numbers as inputs and assigns to each input number exactly one output number. The output is a function of the input.

The inputs and outputs are also called *variables*.

Representing Functions: Words, Tables, Graphs, and Formulas

A function can be described in several ways.

Example 1 It is a surprising biological fact that the rate at which the snowy tree cricket (*Oecanthus fultoni*) chirps increases with temperature. The relationship between temperature and chirp rate is so reliable that this cricket is called the thermometer cricket. We find the temperature (in degrees Fahrenheit) by counting the number of times the cricket chirps in a minute, dividing by four, and adding 40.[1]

The rule used to find the temperature T (in °F) from the chirp rate R (in chirps per minute) is an example of a function. The input is chirp rate and the output is temperature. Describe this function using words, a table, a graph, and a formula.

Solution

- **Words**: To find the temperature, we count the number of chirps in a minute, divide by four, and add forty. For instance, if we count 80 chirps in a minute, then we estimate the temperature to be $80/4 + 40 = 20 + 40 = 60$°F.
- **Table**: Table 1.1 gives the estimated temperature, T, as a function of R, the number of chirps per minute. Notice the pattern in Table 1.1: each time the chirp rate, R, goes up by 20 chirps per minute, the temperature, T, goes up by 5°F.
- **Graph**: The data from Table 1.1 are plotted on the *Cartesian plane* in Figure 1.1. For instance, the pair of values $R = 80$, $T = 60$ is plotted as the point with coordinates $P = (80, 60)$, which is 80 units along the horizontal axis and 60 units up the vertical axis.

Table 1.1 *Chirp rate and temperature*

R, chirp rate (chirps/minute)	T, predicted temperature (°F)
20	45
40	50
60	55
80	60
100	65
120	70
140	75
160	80

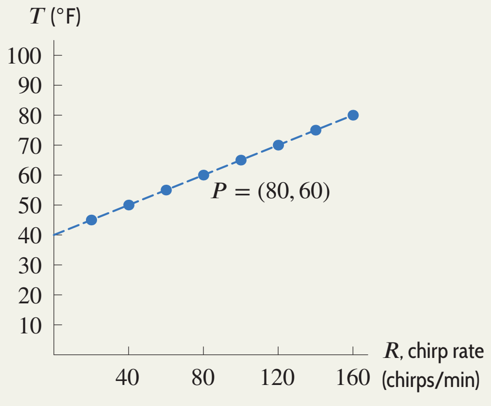


Figure 1.1: Chirp rate and temperature

[1] This relationship is often called Dolbear's Law, as it was first proposed in Amos Dolbear, "The Cricket as a Thermometer," in *The American Naturalist,* 31(1897), pp. 970–971.

- **Formula**: The verbal description leads to the following formula giving T in terms of R:

$$\underbrace{\text{Estimated temperature (in °F)}}_{T} = \frac{1}{4} \cdot \underbrace{\text{Chirp rate (in chirps/min)}}_{R} + 40.$$

Rewriting this using the variables T and R gives the formula:

$$T = \frac{1}{4}R + 40.$$

Let's check the formula. Substituting $R = 80$, we have, as expected,

$$T = \frac{1}{4} \cdot 80 + 40 = 60.$$

The formula $T = \frac{1}{4}R + 40$ also tells us that if $R = 0$, then $T = 40$. Thus, the dashed line in Figure 1.1 crosses (or intersects) the T-axis at $T = 40$; we say the T*-intercept* is 40.

All the descriptions in Example 1 provide the same information, but with a different emphasis.

Mathematical Models

When we use a function to describe a situation, the function is called a **mathematical model**. The formula $T = \frac{1}{4}R + 40$ is a mathematical model of the relationship between the temperature and the cricket's chirp rate. Such models can be powerful tools for understanding phenomena and making predictions.

In everyday language, saying that T is a function of R suggests that making the cricket chirp faster somehow makes the temperature change. Clearly, this is not the case. In mathematics, saying that the temperature "depends" on the chirp rate means only that knowing the chirp rate is sufficient to tell us the temperature.

Function Notation

To indicate that a quantity Q is a function of a quantity t, we abbreviate using function notation and write

$$Q = f(t).$$

Thus, applying the rule f to the input value, t, gives the output value, $f(t)$, which is a value of Q. Here Q is called the *dependent variable* and t is called the *independent variable*. In other words,

$$\text{Output} = f(\text{Input}) \qquad \text{or} \qquad \text{Dependent} = f(\text{Independent}).$$

We could have used any letter, not just f, to represent the rule. Notice that the parentheses do *not* mean multiplication.

Example 2 Example 1 gives the formula for estimating air temperature, T, from the chirp rate, R, of the snowy tree cricket. Since T depends on R, we write $T = f(R)$ to indicate that the relationship is a function:

$$T = f(R) = \frac{1}{4}R + 40.$$

Example 3 The number of gallons of paint needed to paint a house depends on the size of the house. A gallon of paint typically covers 250 square feet. Thus, the number of gallons of paint, n, is a function of the area to be painted, A ft^2. We write $n = f(A)$.

(a) Find a formula for f.
(b) Explain in words what the statement $f(10{,}000) = 40$ tells us about painting houses.

Solution (a) If $A = 250$, the house requires one gallon of paint. If $A = 500$, it requires $500/250 = 2$ gallons of paint, if $A = 750$ it requires $750/250 = 3$ gallons of paint, and so on. We see that a house of area A requires $A/250$ gallons of paint, so n and A are related by the formula

$$n = f(A) = \frac{A}{250}.$$

(b) The input of the function $n = f(A)$ is an area and the output is an amount of paint. The statement $f(10{,}000) = 40$ tells us that an area of $A = 10{,}000$ ft^2 requires $n = 40$ gallons of paint.

Functions Don't Have to Be Defined by Formulas

Not all functions can be represented by formulas. Some are given only by tables or graphs.

Example 4 The average monthly rainfall, R, at Chicago's O'Hare airport is given in Table 1.2, where time, t, is in months and $t = 1$ is January, $t = 2$ is February, and so on. The rainfall is a function of the month, so we write $R = f(t)$. However, there is no formula that gives R when t is known. Evaluate $f(1)$ and $f(11)$. Explain what your answers mean.

Table 1.2 *Average monthly rainfall at Chicago's O'Hare airport*[2]

Month, t	1	2	3	4	5	6	7	8	9	10	11	12
Rainfall, R (inches)	1.7	1.8	2.5	3.4	3.7	3.5	3.7	4.9	3.2	3.2	3.2	2.2

Solution The value of $f(1)$ is the average rainfall in inches at Chicago's O'Hare airport in a typical January. From the table, $f(1) = 1.7$ inches. Similarly, $f(11) = 3.2$ means that in a typical November, there are 3.2 inches of rain at O'Hare.

When Is a Relationship Not a Function?

It is possible for two quantities to be related and yet for neither quantity to be a function of the other.

Example 5 A national park contains foxes that prey on rabbits. Table 1.3 gives the two populations, F and R, over a 12-month period, where $t = 0$ means January 1, $t = 1$ means February 1, and so on.

Table 1.3 *Number of foxes and rabbits in a national park, by month*

t, month	0	1	2	3	4	5	6	7	8	9	10	11
R, rabbits	1000	750	567	500	567	750	1000	1250	1433	1500	1433	1250
F, foxes	150	143	125	100	75	57	50	57	75	100	125	143

(a) Is F a function of t? Is R a function of t?
(b) Is F a function of R? Is R a function of F?

Solution (a) Remember that for a relationship to be a function, an input can give only a single output. Both F and R are functions of t. For each value of t, there is exactly one value of F and exactly one value of R. For example, Table 1.3 shows that if $t = 5$, then $R = 750$ and $F = 57$. This means that on June 1 there are 750 rabbits and 57 foxes in the park. If we write $R = f(t)$ and $F = g(t)$, then $f(5) = 750$ and $g(5) = 57$.

[2]http://www.usclimatedata.com, accessed March 13, 2017.

(b) No, F is not a function of R. For example, suppose $R = 750$, meaning there are 750 rabbits. This happens both at $t = 1$ (February 1) and at $t = 5$ (June 1). In the first instance, there are 143 foxes; in the second instance, there are 57 foxes. Since there are R-values which correspond to more than one F-value, F is not a function of R.

Similarly, R is not a function of F. At time $t = 5$, we have $R = 750$ when $F = 57$, while at time $t = 7$, we have $R = 1250$ when $F = 57$ again. Thus, the value of F does not uniquely determine the value of R.

How to Tell if a Graph Represents a Function: Vertical Line Test

What does it mean graphically for y to be a function of x? Look at a graph of y against x, with y on the vertical axis and x on the horizontal axis. For a function, each x-value corresponds to exactly one y-value. This means that the graph intersects any vertical line at most once (either once or not at all). If a vertical line cuts the graph twice, the graph contains two points with different y-values but the same x-value; this violates the definition of a function. Thus, we have the following criterion:

Vertical Line Test: If there is a vertical line that intersects a graph in more than one point, then the graph does not represent a function.

Example 6 In which of the graphs in Figures 1.2 and 1.3 could y be a function of x?

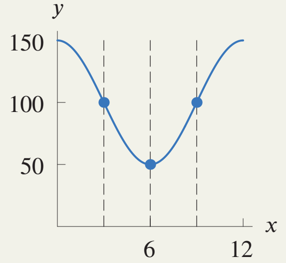


Figure 1.2: Since no vertical line intersects this curve at more than one point, y could be a function of x

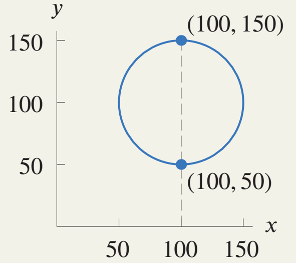


Figure 1.3: Since one vertical line intersects this curve at more than one point, y is not a function of x

Solution The graph in Figure 1.2 could represent y as a function of x because no vertical line intersects this curve in more than one point. The graph in Figure 1.3 does not represent a function because the vertical line shown intersects the curve at two points.

A graph fails the vertical line test if at least one vertical line cuts the graph more than once, as in Figure 1.3. However, if a graph represents a function, then *every* vertical line must intersect the graph at no more than one point.

Summary for Section 1.1

- **Definition of function**: a rule which takes numbers as inputs and assigns to each input exactly one output number.
- **Function notation:** $y = f(x)$, where x is the **independent variable** and y is the **dependent variable**. We say y depends on x and we graph y against x, with y on the vertical axis.
- **Vertical line test**: If a vertical line cuts the graph of y against x more than once, then y is not a function of x.

Exercises and Problems for Section 1.1

Skill Refresher

In Exercises S1–S6, simplify each expression.

S1. $4x + 8x$

S2. $8w - 5w$

S3. $c + \frac{1}{2}c$

S4. $P + 0.07P + 0.02P$

S5. $2\pi r^2 + 2\pi r \cdot 2r$

S6. $\dfrac{12\pi - 2\pi}{6\pi}$

In Exercises S7–S12, find the value of the expression for the given value of x and y.

S7. $5x - 2$ for $x = 3$

S8. $3x^2 + 5$ for $x = 2$

S9. $x - 5y$ for $x = \frac{1}{2}$, $y = -5$.

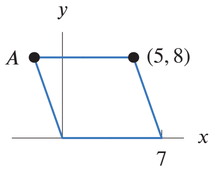
S10. $1 - 12x + x^2$ for $x = 3$.

S11. $\dfrac{3}{2 - x^3}$ for $x = -1$.

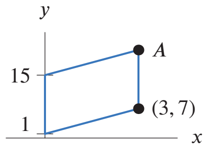
S12. $\dfrac{4}{1 + 1/x}$ for $x = -\frac{3}{4}$.

The figures in Exercises S13–S14 are parallelograms. Find the coordinates of the point A.

S13.



S14.



In Exercises S15–S16, for a graph of the given function, which variable goes on the horizontal axis and which variable on the vertical axis?

S15. $w = f(r)$

S16. $T = f(A)$

In Exercises S17–S18, use the given information about the function f to determine the coordinates of a point on the graph of f.

S17. $f(2) = 6$

S18. $5 = f(-8)$

EXERCISES

In Exercises 1–2, write the relationship using function notation (that is, y is a function of x is written $y = f(x)$).

1. Number of molecules, m, in a gas, is a function of the volume of the gas, v.

2. Weight, w, is a function of caloric intake, c.

In Exercises 3–6, label the axes for a sketch to illustrate the given statement.

3. "Over the past century we have seen changes in the population, P (in millions), of the city."

4. "Sketch a graph of the cost of manufacturing q items."

5. "Graph the pressure, p, of a gas as a function of its volume, v, where p is in pounds per square inch and v is in cubic inches."

6. "Graph D in terms of y."

7. Figure 1.4 gives the depth of the water at Montauk Point, New York, for a day in November.

(a) How many high tides took place on this day?
(b) How many low tides took place on this day?
(c) How much time elapsed in between high tides?

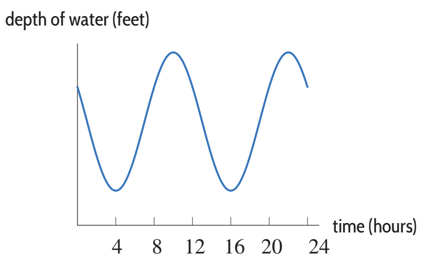


Figure 1.4

8. Using Table 1.4, graph $n = f(A)$, the number of gallons of paint needed to cover walls of area A. Identify the independent and dependent variables.

Table 1.4

A	0	250	500	750	1000	1250	1500
n	0	1	2	3	4	5	6

9. Use Figure 1.5 to fill in the missing values:

(a) $f(0) = ?$ **(b)** $f(?) = 0$

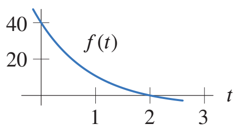


Figure 1.5

10. Use Table 1.5 to fill in the missing values. (There may be more than one answer.)

(a) $f(0) = ?$ **(b)** $f(?) = 0$
(c) $f(1) = ?$ **(d)** $f(?) = 1$

Table 1.5

x	0	1	2	3	4
$f(x)$	4	2	1	0	1

Exercises 11–14 use Figure 1.6.

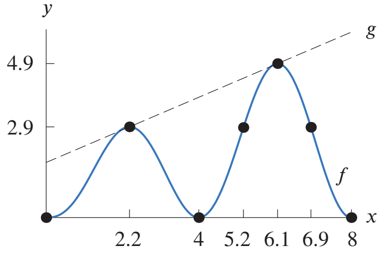


Figure 1.6

11. Find $f(6.9)$.

12. Give the coordinates of two points on the graph of g.

13. Solve $f(x) = 0$ for x.

14. Solve $f(x) = g(x)$ for x.

15. **(a)** You are going to graph $p = f(w)$. Which variable goes on the horizontal axis?
(b) If $10 = f(-4)$, give the coordinates of a point on the graph of f.
(c) If 6 is a solution of the equation $f(w) = 1$, give a point on the graph of f.

16. If $f(x) = 3x + 8$, find:

(a) $f(0)$ **(b)** $f(2)$ **(c)** $f(-1)$

17. If $f(x) = px + q$, find:

(a) $f(0)$ **(b)** $f(1)$
(c) $f(2)$ **(d)** $f(-1)$

18. If $f(x) = 4x - 5$, find the following when $x = 2$:

(a) $f(x)$ **(b)** $3f(x)$ **(c)** $f(3x)$

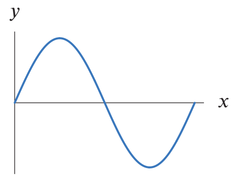
In Exercises 19–22 a relationship is given between two quantities. Are both quantities functions of the other one, or is one or neither a function of the other? Explain.

19. $7w^2 + 5 = z^2$ 20. $y = x^4 - 1$ 21. $m = \sqrt{t}$

22. The number of gallons of gas, g, at \$3 per gallon and the number of pounds of coffee, c, at \$10 per pound that can be bought for a total of \$100.

In Exercises 23–28, could the graph represent y as a function of x?

23.



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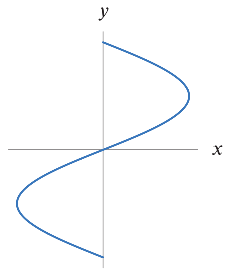

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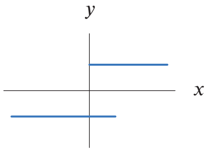

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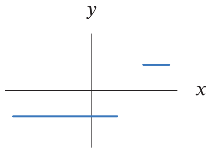

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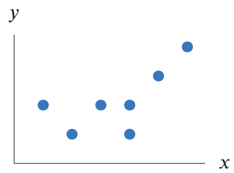

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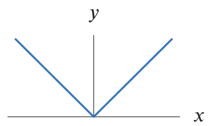


PROBLEMS

29. At the end of a semester, students' math grades are listed in a table which gives each student's ID number in the left column and the student's grade in the right column. Let N represent the ID number and G represent the grade. Which quantity, N or G, must necessarily be a function of the other?

30. A person's blood sugar level at a particular time of the day is partially determined by the time of the most recent meal. After a meal, blood sugar level increases rapidly, then slowly comes back down to a normal level. Sketch a person's blood sugar level as a function of time over the course of a day. Label the axes to indicate normal blood sugar level and the time of each meal.

31. When a parachutist jumps out of a plane, the speed of her fall increases until she opens her parachute, at which time her falling speed suddenly decreases and stays constant until she reaches the ground. Sketch a possible graph of the height H of the parachutist as a function of time t, from the time when she jumps from the plane to the time when she reaches the ground.

32. A buzzard is circling high overhead when it spies some road kill. It swoops down, lands, and eats. Later it takes off sluggishly, and resumes circling overhead, but at a lower altitude. Sketch a possible graph of the height of the buzzard as a function of time.

33. Table 1.6 gives the ranking r_o for Olivia, r_c for Charlotte, and r_m for Madison of names for girls born in each of the years 2010 ($t = 0$) to 2015 ($t = 5$). For example in the left column, $r_o = 4$, tells us that when $t = 0$, Olivia was the 4^{th}-most-common girl's name.[3] Of the three names, which were most and least common in

(a) 2010? **(b)** 2015?

Table 1.6

t	0	1	2	3	4	5
r_o	4	4	4	3	2	2
r_c	46	27	19	11	10	9
r_m	8	8	9	9	9	11

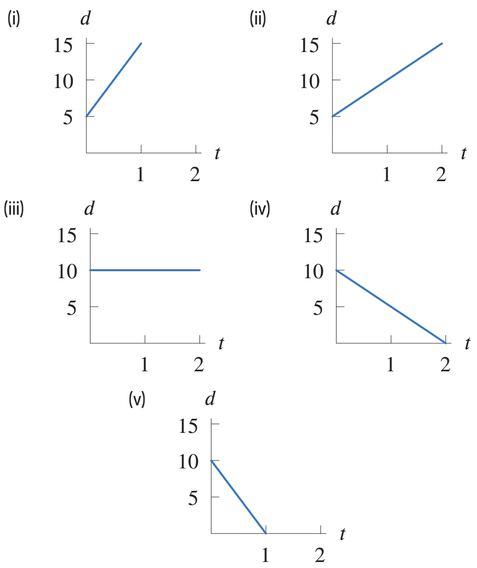
[3] Data from the SSA website at www.ssa.gov, accessed March 9, 2017.

34. Table 1.6 gives information about the popularity of the names Olivia, Charlotte, and Madison.

(a) Find $r_c(0)-r_o(0)$. What does this value tell us about names?
(b) Find $r_m(5) - r_c(5)$. What does this value tell us about names?
(c) Solve $r_m(t) > r_c(t)$. What does the solution tell us about names?

35. Match each story about a bike ride to one of the graphs (i)–(v), where d represents distance from home and t is time in hours since the start of the ride. (A graph may be used more than once.)

(a) Starts 5 miles from home and rides 5 miles per hour away from home.
(b) Starts 5 miles from home and rides 10 miles per hour away from home.
(c) Starts 10 miles from home and arrives home one hour later.
(d) Starts 10 miles from home and is halfway home after one hour.
(e) Starts 5 miles from home and is 10 miles from home after one hour.



36. Figure 1.7 shows the fuel consumption (in miles per gallon, mpg) of a car traveling at various speeds (in mph).

(a) How much gas is used on a 300-mile trip at 40 mph?
(b) How much gas is saved by traveling 60 mph instead of 70 mph on a 200-mile trip?
(c) According to this graph, what is the most fuel-efficient speed to travel? Explain.

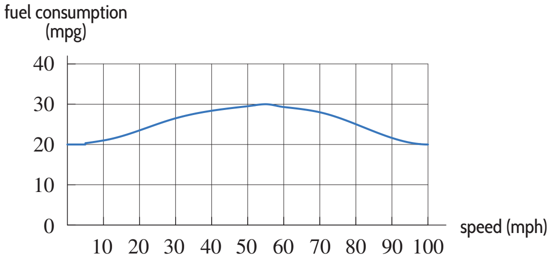


Figure 1.7

37. Figure 1.8 shows the mass of water in air, in grams of water per kilogram of air, as a function of air temperature in °C, for two different levels of relative humidity.

(a) Find the mass of water in 1 kg of air at 30°C if the relative humidity is
(a) 100% **(b)** 50% **(c)** 75%
(b) How much water is in a room containing 300 kg of air if the relative humidity is 50% and the temperature is 20°C?
(c) The density of air is approximately 1.2 kg/m^3. If the relative humidity in your classroom is 50% and the temperature is 20°C, estimate the amount of water in the air.

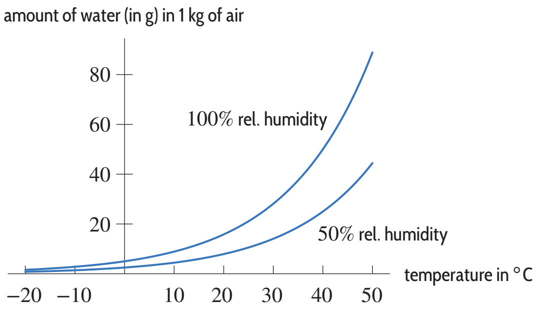


Figure 1.8

38. Let $f(t)$ be the number of people, in millions, who own cell phones t years after 1990. Explain the meaning of the following statements.

(a) $f(10) = 100.3$ **(b)** $f(a) = 20$
(c) $f(20) = b$ **(d)** $n = f(t)$

39. **(a)** Ten inches of snow is equivalent to about one inch of rain.[4] Write an equation for the amount of precipitation, measured in inches of rain, $r = f(s)$, as a function of the number of inches of snow, s.
(b) Evaluate and interpret $f(5)$.
(c) Find s such that $f(s) = 5$ and interpret your result.

[4]http://mo.water.usgs.gov/outreach/rain, accessed March 13, 2017.

40. An 8-foot-tall cylindrical water tank has a base of diameter 6 feet.

(a) How much water can the tank hold?
(b) How much water is in the tank if the water is 5 feet deep?
(c) Write a formula for the volume of water as a function of its depth in the tank.

41. Data for the observed rainfall, $R = f(t)$, in Tucson, Arizona is given in Table 1.7, where time, t, is in months with $t = 1$ being January.

(a) Find and interpret $f(5)$.
(b) Solve $f(t) = 0$ and interpret the meaning of your answer.
(c) Solve $f(t) = 0.1$, and interpret the meaning of your answer.

Table 1.7

t (months)	1	2	3	4	5
R (inches)	0	0.1	0.54	0.1	0.35

42. Table 1.8 gives $A = f(d)$, the amount of money in bills of denomination d circulating in US currency in 2016.[5] For example, there were \$83.5 billion worth of \$50 bills in circulation.

(a) Find $f(100)$. What does this tell you about money?
(b) Are there more \$1 bills or \$5 bills in circulation?

Table 1.8

Denomination (\$)	1	2	5	10	20	50	100
Circulation (\$bn)	11.7	2.3	14.2	19.2	177.2	83.5	1154.8

43. Table 1.9 shows the daily low temperature for a one-week period in New York City during July.

(a) What was the low temperature on July 19?
(b) When was the low temperature 73°F?
(c) Is the daily low temperature a function of the date?
(d) Is the date a function of the daily low temperature?

Table 1.9

Date	17	18	19	20	21	22	23
Low temp (°F)	73	77	69	73	75	75	70

44. Use the data from Table 1.3 on page 4.

(a) Plot R on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that R is a function of t.
(b) Plot F on the vertical axis and t on the horizontal axis. Use this graph to explain why you believe that F is a function of t.
(c) Plot F on the vertical axis and R on the horizontal axis. From this graph show that F is not a function of R.
(d) Plot R on the vertical axis and F on the horizontal axis. From this graph show that R is not a function of F.

45. Since Roger Bannister broke the 4-minute mile on May 6, 1954, the record has been lowered by over sixteen seconds. Table 1.10 shows the year and times (as min:sec) of new world records for the one-mile run.[6] (Official records for the mile ended in 1999.)

(a) Is the time a function of the year? Explain.
(b) Is the year a function of the time? Explain.
(c) Let $y(r)$ be the year in which the world record, r, was set. Explain what is meant by the statement $y(3{:}47.33) = 1981$.
(d) Evaluate and interpret $y(3{:}51.1)$.

Table 1.10

Year	Time	Year	Time	Year	Time
1954	3:59.4	1966	3:51.3	1981	3:48.53
1954	3:58.0	1967	3:51.1	1981	3:48.40
1957	3:57.2	1975	3:51.0	1981	3:47.33
1958	3:54.5	1975	3:49.4	1985	3:46.31
1962	3:54.4	1979	3:49.0	1993	3:44.39
1964	3:54.1	1980	3:48.8	1999	3:43.13
1965	3:53.6				

46. The sales tax on an item is 6%. Express the total cost, C, in terms of the price of the item, P.

47. A price increases 5% due to inflation and is then reduced 10% for a sale. Express the final price as a function of the original price, P.

48. Write a formula for the area of a circle as a function of its radius and determine the percent increase in the area if the radius is increased by 10%.

49. There are x male job-applicants at a certain company and y female applicants. Suppose that 15% of the men are accepted and 18% of the women are accepted. Write an expression in terms of x and y representing each of the following quantities:

(a) The total number of applicants to the company.
(b) The total number of applicants accepted.
(c) The percentage of all applicants accepted.

[5] http://www.federalreserve.gov/paymentsystems/coin_currcircvalue.htm, accessed March 13, 2017.
[6] www.infoplease.com/ipsa/A0112924.html, accessed March 13, 2017.

50. A chemical company spends \$2 million to buy machinery before it starts producing chemicals. Then it spends \$0.5 million on raw materials for each million liters of chemical produced.

(a) The number of liters produced ranges from 0 to 5 million. Make a table showing the relationship between the number of million liters produced, l, and the total cost, C, in millions of dollars, to produce that number of million liters.

(b) Find a formula that expresses C as a function of l.

51. A person leaves home and walks due west for a time and then walks due north.

(a) The person walks 10 miles in total. If w represents the (variable) distance west she walks, and D represents her (variable) distance from home at the end of her walk, is D a function of w? Why or why not?

(b) Suppose now that x is the distance that she walks in total. Is D a function of x? Why or why not?

1.2 RATE OF CHANGE

Worldwide sales of smartphones have increased each year since they were introduced. To measure how fast sales increase, we calculate a *rate of change* in number of units sold annually:

$$\frac{\text{Change in units sold}}{\text{Change in time}}.$$

Due to the rising popularity of smartphones, sales of regular cell phones, or *feature phones*, have declined. See Table 1.11.

We calculate the rate of change of smartphone and feature phone sales between 2012 and 2015. Writing ΔS for the change in the sales of smart phones and Δt for the change in time,[7] we have

$$\begin{array}{c}\text{Average rate of change of}\\ \text{smartphone sales from 2012 to 2015}\end{array} = \frac{\Delta S}{\Delta t} = \frac{1462 - 681}{2015 - 2012} \approx 260.3 \text{ mn/year.}$$

Thus, the number of smartphones sold has increased on average by 260.3 million units per year between 2012 and 2015. See Figure 1.9.

Similarly, writing ΔF for the change of sales of feature phones, we have

$$\begin{array}{c}\text{Average rate of change of feature phone}\\ \text{sales from 2012 to 2015}\end{array} = \frac{\Delta F}{\Delta t} = \frac{481 - 1110}{2015 - 2012} \approx -209.7 \text{ mn/year.}$$

Thus, the number of feature phones sold has decreased on average by 209.7 million units per year between 2012 and 2015. See Figure 1.10.

Table 1.11 *Worldwide annual sales of smartphones and feature phones*[8]

Year	2012	2013	2014	2015
Smartphone sales (millions of units per year)	681	970	1245	1462
Feature phone sales (millions of units per year)	1110	845	634	481

[7] The Greek letter Δ, delta, is often used in mathematics to represent change. In this book, we use rate of change to mean average rate of change across an interval. In calculus, rate of change means something called instantaneous rate of change.

[8] www.statista.com/statistics/263445/global-smartphone-sales-by-operating-system-since-2009, accessed March, 2017.

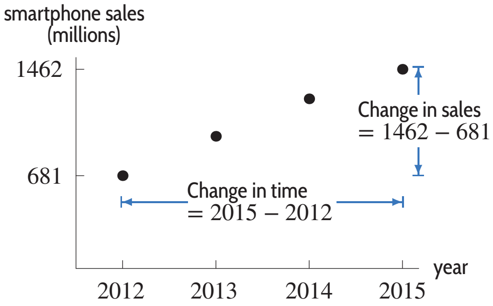


Figure 1.9: Smartphone sales

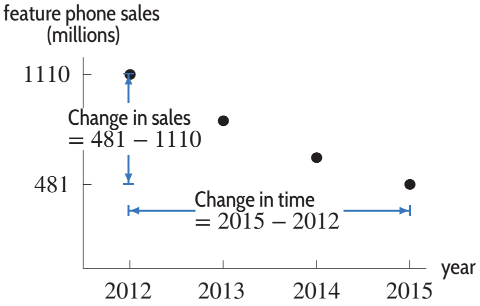


Figure 1.10: Feature phone sales

Rate of Change of a Function

The average rate of change of sales is an example of the rate of change of a function. In general, for a function $Q = f(t)$, the change in Q over the interval $a \le t \le b$ is

$$\Delta Q = f(b) - f(a)$$

and the change in t by

$$\Delta t = b - a.$$

Therefore, we have the *difference quotient*:

$$\begin{array}{c}\text{Average rate of change of } Q = f(t) \\ \text{over the interval } a \le t \le b\end{array} = \frac{\text{Change in } Q}{\text{Change in } t} = \frac{\Delta Q}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$

The average rate of change of the function $Q = f(t)$ over an interval tells us how much Q changes, on average, for each unit change in t within that interval. On some parts of the interval, Q may be changing rapidly, while on other parts Q may be changing slowly. The average rate of change evens out these variations.

Often, we drop the word "average" and instead talk about the rate of change over an interval.

Increasing and Decreasing Functions

In the previous example, the average rate of change of smartphone sales is positive on the interval from 2012 to 2015 since the number of smartphones sold increased then. Similarly, the average rate of change of feature phone sales is negative on the same interval since the number of feature phones sold decreased then. We make the following definition and observations:

If $Q = f(t)$ for t in the interval $a \le t \le b$,

- f is an **increasing function** if the values of f increase as t increases in this interval. For an increasing function $Q = f(t)$:
  - The graph of f rises when read from left to right.
  - The average rate of change of Q with respect to t is positive on every interval.
- f is a **decreasing function** if the values of f decrease as t increases in this interval. For a decreasing function $Q = f(t)$:
  - The graph of f falls when read from left to right.
  - The average rate of change of Q with respect to t is negative on every interval.

Example 1 The function $A = q(r) = \pi r^2$ gives the area, A, of a circle as a function of its radius, r. Graph q. Explain how you can see from the graph that q is an increasing function. What does this tell you about its average rate of change on an interval?

Solution The area increases as the radius increases, so $A = q(r)$ is an increasing function. We see this in Figure 1.11 because the graph climbs as we move from left to right. The average rate of change, $\Delta A/\Delta r$, is positive on every interval.

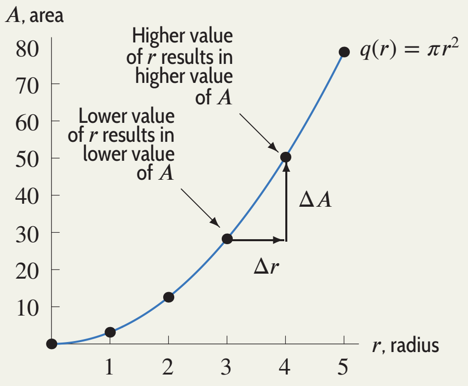


Figure 1.11: The graph of an increasing function, $A = q(r)$, rises when read from left to right

Example 2 Carbon-14 is a radioactive element that exists naturally in the atmosphere and is absorbed by living organisms. When an organism dies, the carbon-14 present at death begins to decay. Let $L = g(t)$ represent the quantity of carbon-14 (in micrograms, μg) in a tree t years after its death. See Table 1.12. Explain why we expect g to be a decreasing function of t and how the graph displays this.

Table 1.12 *Quantity of carbon-14 as a function of time*

t, time (years)	0	1000	2000	3000	4000	5000
L, quantity of carbon-14 (μg)	200	177	157	139	123	109

Solution Since the amount of carbon-14 is decaying over time, g is a decreasing function. In Figure 1.12, the graph falls as we move from left to right and the average rate of change in the level of carbon-14 with respect to time, $\Delta L/\Delta t$, is negative on every interval.

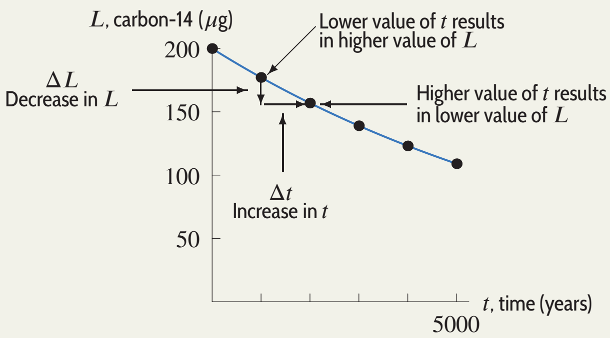


Figure 1.12: The graph of a decreasing function, $L = g(t)$, falls when read from left to right

Many functions have some intervals on which they are increasing and other intervals on which they are decreasing. These intervals can often be identified from the graph.

Example 3 On what intervals is the function graphed in Figure 1.13 increasing? Decreasing?

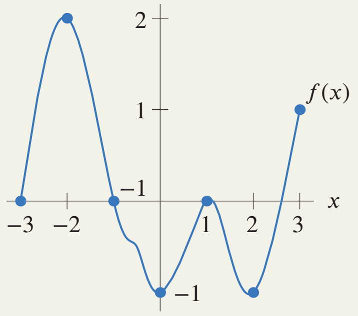


Figure 1.13: Graph of a function that is increasing on some intervals and decreasing on others

Solution The function appears to be increasing for values of x between -3 and -2, for x between 0 and 1, and for x between 2 and 3. The function appears to be decreasing for x between -2 and 0 and for x between 1 and 2. Using inequalities, we say that f is increasing for $-3 < x < -2$, for $0 < x < 1$, and for $2 < x < 3$. Similarly, f is decreasing for $-2 < x < 0$ and $1 < x < 2$.

Rate of Change and Slope

In Figure 1.14, notice that the average rate of change is the slope of the dashed line segment given by the ratio of the rise, $f(b) - f(a)$, to the run, $b - a$.

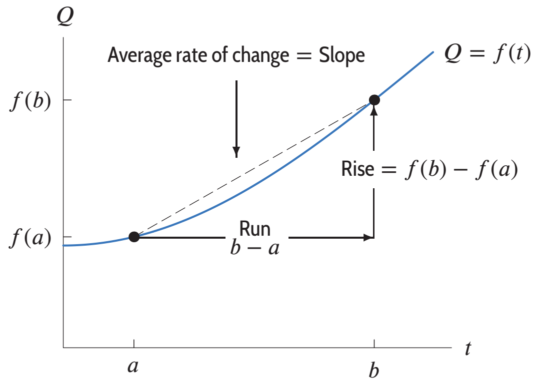


Figure 1.14: The average rate of change is the ratio rise/run

In previous examples we calculated the average rate of change from data. We now calculate average rates of change for functions given by formulas.

Example 4 Calculate the average rates of change of the function $f(x) = x^2$ between $x = 1$ and $x = 3$ and between $x = -2$ and $x = 1$. Show your results on a graph.

Solution Between $x = 1$ and $x = 3$, we have

$$\begin{aligned}\begin{array}{c}\text{Average rate of change of } f(x)\\ \text{over the interval } 1 \le x \le 3\end{array} &= \frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{f(3) - f(1)}{3 - 1}\\ &= \frac{3^2 - 1^2}{3 - 1} = \frac{9 - 1}{2} = 4.\end{aligned}$$

Between $x = -2$ and $x = 1$, we have

$$\begin{aligned}\text{Average rate of change of } f(x) \\ \text{over the interval } -2 \le x \le 1\end{aligned} = \frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{f(1) - f(-2)}{1-(-2)}$$
$$= \frac{1^2 - (-2)^2}{1-(-2)} = \frac{1-4}{3} = -1.$$

The average rate of change between $x = 1$ and $x = 3$ is positive because $f(x)$ is increasing on this interval. See Figure 1.15. However, on the interval from $x = -2$ and $x = 1$, the function is partly decreasing and partly increasing. The average rate of change on this interval is negative because the decrease on the interval is larger than the increase.

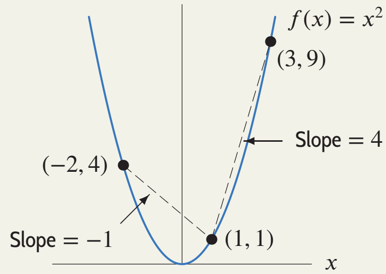


Figure 1.15: Average rate of change of $f(x)$ on an interval is the slope of the dashed line on that interval

Summary for Section 1.2

- **Average rate of change** of $Q = f(t)$ on $a \le t \le b$ is
$$\frac{\Delta Q}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$
Average rate of change can be visualized as the **slope** of the line between the points on the graph of f from $t = a$ to $t = b$.
- **Increasing functions** have positive average rates of change; **decreasing functions** have negative average rates of change.

Exercises and Problems for Section 1.2

Skill Refresher

In Exercises **S1–S10**, simplify each expression.

S1. $\dfrac{4-6}{3-2}$

S2. $\dfrac{1-3}{2^2-(-3)^2}$

S3. $\dfrac{-3-(-9)}{-1-2}$

S4. $\dfrac{(1-3^2)-(1-4^2)}{3-4}$

S5. $\dfrac{\frac{1}{2}-(-4)^2-\left(\frac{1}{2}-(5^2)\right)}{-4-5}$

S6. $2(x+a)-3(x-b)$

S7. $x^2-(2x+a)^2$

S8. $4x^2-(x-b)^2$

S9. $\dfrac{x^2-\frac{3}{4}-\left(y^2-\frac{3}{4}\right)}{x-y}$

S10. $\dfrac{2(x+h)^2-2x^2}{(x+h)-x}$

S11. Table 1.13 gives values of the function f.

(a) Which is larger: $f(1)$ or $f(3)$?
(b) Find $f(3) - f(1)$.

Table 1.13

x	0	1	2	3
$f(x)$	20	25	27	28

S12. Table 1.14 gives values of the function g.

(a) Which is larger: $g(0)$ or $g(10)$?
(b) Find $g(10) - g(0)$.

Table 1.14

x	0	5	10	15
$g(x)$	20	17	12	5

S13. Figure 1.16 shows a graph of the function g.

(a) Which is larger: $g(5)$ or $g(0)$?
(b) Find $g(5) - g(0)$.

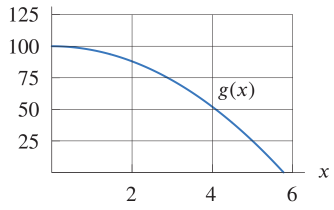


Figure 1.16

S14. Figure 1.17 shows a graph of the function f.

(a) Which is larger: $f(1)$ or $f(2)$?
(b) Find $f(2) - f(1)$.

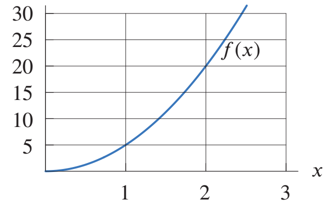
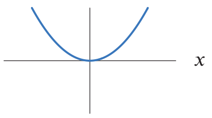


Figure 1.17

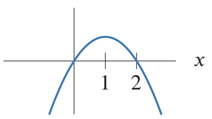
EXERCISES

In Exercises 1–4, on what intervals is the function increasing? Decreasing?

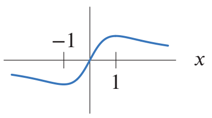
1.



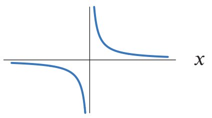
2.



3.



4.



5. Table 1.11 on page 10 gives the annual sales (in millions) of smartphones and feature phones. What was the average rate of change of annual sales of each of them between

(a) 2012 and 2014? **(b)** 2014 and 2015?
(c) Interpret these results in terms of sales.

6. Table 1.11 on page 10 shows that feature phone sales are a function of smartphone sales. Is this function increasing or decreasing?

7. In 2010, you have 120 songs in your favorite iTunes playlist. In 2013, you have 210 songs. In 2019, you have 120. What is the average rate of change per year in the number of songs in your favorite iTunes playlist between

(a) 2010 and 2013? **(b)** 2013 and 2019?
(c) 2010 and 2019?

8. Table 1.15 gives the populations of two cities (in thousands) over a 17-year period.

(a) Find the average rate of change of each population on the following intervals:
(i) 2002 to 2012 (ii) 2002 to 2019
(iii) 2007 to 2019

(b) What do you notice about the average rate of change of each population? Explain what the average rate of change tells you about each population.

Table 1.15

Year	2002	2004	2007	2012	2019
P_1	42	46	52	62	76
P_2	82	80	77	72	65

9. The amount of electrical charge remaining in a capacitor t seconds after it is discharged through a circuit is given by $q = f(t)$ microcoulombs (μC). Values of the function f are given in Table 1.16.

(a) Find the average rate of change of f between the given times. Include units.
(i) $t = 0$ and $t = 1$ (ii) $t = 1$ and $t = 2$
(iii) $t = 2$ and $t = 3$ (iv) $t = 3$ and $t = 4$

(b) What do the average rates of change you calculated in part (a) tell you about the behavior of the electrical charge dissipation in the capacitor?

Table 1.16

t (sec)	0	1	2	3	4
q (μC)	60	46.7	36.4	28.3	22.1

10. Table 1.17 gives the population of California, in millions, in various years.[9]

(a) Within which time period did the population of California increase by the greater amount: between 1900 and 1950 or 2010 and 2015?
(b) Within which time period did the population of California grow at a faster rate: between 1900 and 1950 or 2010 and 2015?

Table 1.17

Year	1900	1950	1990	2010	2015
P (millions)	1.49	10.6	29.8	37.3	39.0

11. Figure 1.18 shows distance traveled as a function of time.

(a) Find ΔD and Δt between:

(i) $t = 2$ and $t = 5$ (ii) $t = 0.5$ and $t = 2.5$
(iii) $t = 0.5$ and $t = 0.5$

(b) Compute the rate of change, $\Delta D/\Delta t$, over each of the intervals in part (a), and interpret its meaning.

(i) $t = 2$ and $t = 5$
(ii) $t = 0.5$ and $t = 2.5$
(iii) $t = 0.5$ and $t = 13$

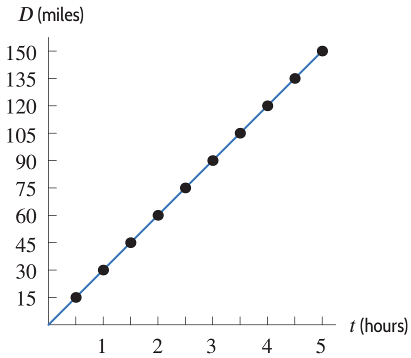


Figure 1.18

12. Figure 1.19 shows the power $P = f(I)$, measured in watts, transformed by an electrical appliance, where I is the electrical current flowing through the appliance, measured in amperes. Estimate and interpret the average rate of change of f between $I = 0$ and $I = 10$.

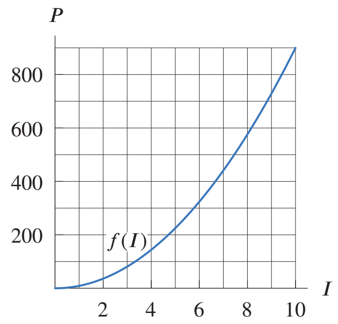


Figure 1.19

Exercises **13–18** use Figure 1.20.

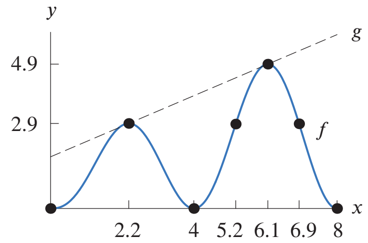


Figure 1.20

13. On what intervals is f increasing? Decreasing?

14. Find the average rate of change of f for $2.2 \le x \le 6.1$.

15. Give two different intervals on which $\Delta f(x)/\Delta x = 0$.

16. What is the average rate of change of g between $x = 2.2$ and $x = 6.1$?

17. What is the relation between the average rate of change of f and the average rate of change of g between $x = 2.2$ and $x = 6.1$?

18. Is the rate of change of f positive or negative on the following intervals?

(a) $2.2 \le x \le 4$ **(b)** $5 \le x \le 6$

19. If F is a decreasing function, what can you say about $F(-2)$ compared to $F(2)$?

20. If G is an increasing function, what can you say about $G(3) - G(-1)$?

PROBLEMS

21. Figure 1.21 shows the percent of the side of the moon toward the earth illuminated by the sun at different times during the year 2016.[10] Use the figure to answer the following questions.

[9] Data from worldpopulationreview.com/states/california-population/, accessed March 20, 2018.
[10] http://aa.usno.navy.mil/cgi-bin/aa_moonill2.pl, accessed March 15, 2017.

(a) During which time intervals is the function increasing?

(b) During which time intervals is the function decreasing?

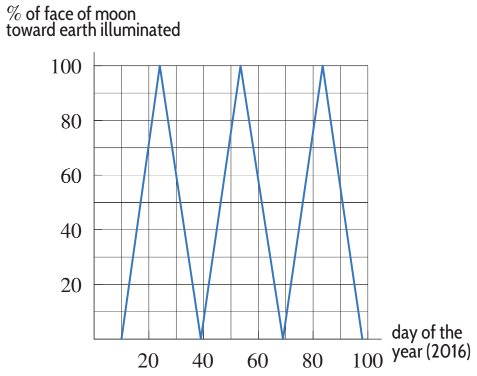


Figure 1.21: Moon phases

22. Figure 1.22 gives the population of two different towns over a 50-year period of time.

(a) Which town starts (in year $t = 0$) with the most people?

(b) Which town is growing faster over these 50 years?

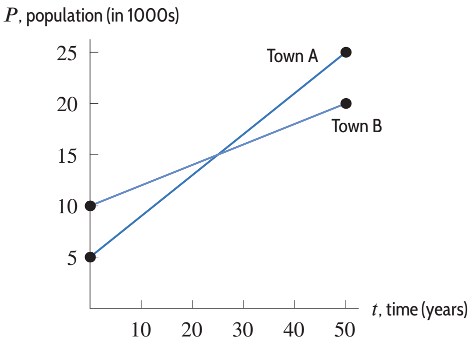


Figure 1.22

23. **(a)** What is the average rate of change of $g(x) = 2x - 3$ between the points $(-2, -7)$ and $(3, 3)$?

(b) The function g is either increasing or decreasing everywhere. Explain how your answer to part (a) tells you which.

(c) Graph the function.

24. **(a)** Let $f(x) = 16 - x^2$. Compute each of the following expressions, and interpret each as an average rate of change.

(i) $\dfrac{f(2) - f(0)}{2 - 0}$ (ii) $\dfrac{f(4) - f(2)}{4 - 2}$

(iii) $\dfrac{f(4) - f(0)}{4 - 0}$

(b) Graph $f(x)$. Illustrate each ratio in part (a) by sketching the line segment with the given slope. Over which interval is the average rate of decrease the greatest?

25. The kinetic energy of a truck having a mass of 36,000 kg is given by $E = f(v) = 18v^2$ kilojoules, where v is the speed of the truck, in meters per second. Find and interpret the average rate of change of f between $v = 30$ and $v = 35$.

26. The pressure of 5 moles of an ideal gas is given by $P = f(V) = 120/V$ atmospheres, where V is the volume of the gas, in liters. Find and interpret each of the following quantities.

(a) $f(10)$ **(b)** $f(20) - f(10)$

(c) $\dfrac{f(20) - f(10)}{20 - 10}$

27. Imagine you constructed a list of the world record times for a particular event—such as the mile footrace, or the 100-meter freestyle swimming race—in terms of when they were established. Is the world record time a function of the date when it was established? If so, is this function increasing or decreasing? Explain. Could a world record be established twice in the same year? Is the world record time a function of the year it was established?

28. The graph of $P = f(t)$ in Figure 1.23 gives the population of a town, in thousands, after t years.

(a) Find the average rate of change of the population of the town during the first 10 years.

(b) Does the population of the town grow more between $t = 5$ and $t = 10$ years, or between $t = 15$ and $t = 30$ years? Explain.

(c) Does the population of the town grow faster between $t = 5$ and $t = 10$ years, or between $t = 15$ and $t = 30$ years? Explain.

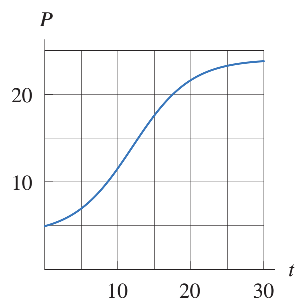


Figure 1.23

29. The most freakish change in temperature ever recorded was from −4°F to 45°F between 7:30 am and 7:32 am on January 22, 1943 at Spearfish, South Dakota.[11] What was the average rate of change of the temperature for this time period?

30. You have zero dollars now and the average rate of change in your net worth is \$5000 per year. How much money will you have in forty years?

[11] www.weather.gov/unr/1943-01-22, accessed March 13, 2017.

31. The surface of the sun has dark areas, known as sunspots, that are cooler than the rest of the sun's surface. The number of sunspots[12] fluctuates with time, as shown in Figure 1.24.

(a) Explain how you know the number of sunspots, s, in year t is a function of t.
(b) Approximate the time intervals on which s is an increasing function of t.

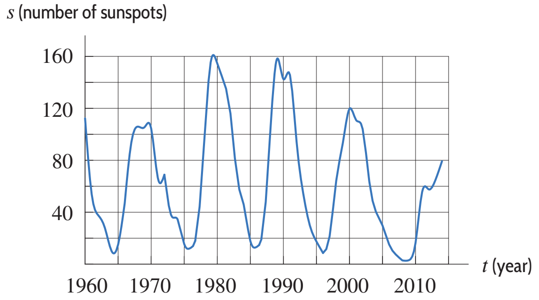


Figure 1.24

32. Table 1.18 shows the number of calories used per minute as a function of body weight for three sports.[13]

(a) Determine the number of calories that a 200-lb person uses in one half-hour of walking.
(b) Who uses more calories, a 120-lb person swimming for one hour or a 220-lb person bicycling for a half-hour?
(c) Does the number of calories used by a person walking increase or decrease as weight increases?

Table 1.18

Activity	100 lb	120 lb	150 lb	170 lb	200 lb	220 lb
Walking	2.7	3.2	4.0	4.6	5.4	5.9
Bicycling	5.4	6.5	8.1	9.2	10.8	11.9
Swimming	5.8	6.9	8.7	9.8	11.6	12.7

33. Because scientists know how much carbon-14 a living organism should have in its tissues, they can measure the amount of carbon-14 present in the tissue of a fossil and then calculate how long it took for the original amount to decay to the current level, thus determining the time of the organism's death. A tree fossil is found to contain 130 μg of carbon-14, and scientists determine from the size of the tree that it would have contained 200 μg of carbon-14 at the time of its death. Using Table 1.12 on page 12, approximately how long ago did the tree die?

34. Figure 1.25 shows the graph of the function $g(x)$.

(a) Estimate $\dfrac{g(4)-g(0)}{4-0}$.
(b) The ratio in part (a) is the slope of a line segment joining two points on the graph. Sketch this line segment on the graph.
(c) Estimate $\dfrac{g(b)-g(a)}{b-a}$ for $a=-9$ and $b=-1$.
(d) On the graph, sketch the line segment whose slope is given by the ratio in part (c).

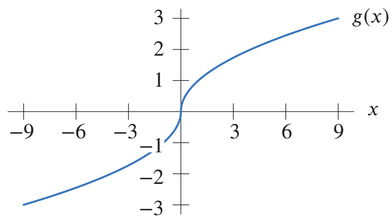


Figure 1.25

35. Find the average rate of change of $f(x)=3x^2+1$ between the points

(a) $(1,4)$ and $(2,13)$ **(b)** (j,k) and (m,n)
(c) $(x, f(x))$ and $(x+h, f(x+h))$

36. A water company employee measured the water level (in centimeters) in a reservoir during March and computed the average rates of change in Table 1.19.

(a) What are the units of the average rates of change in Table 1.19?
(b) Interpret the average rates of change in context.
(c) For each time interval, what was the total change in water level?
(d) Draw a possible graph of the water level as a function of time.

Table 1.19

Interval (days)	$1 \le t < 11$	$11 \le t < 16$	$16 \le t < 31$
Average rate of change	7	5	3

37. Table 1.20 gives the amount of garbage, G, in millions of tons, produced[14] in the US in year t.

(a) What is the value of Δt for consecutive entries in this table?

[12] http://www.sws.bom.gov.au/Educational/2/3/6, accessed March 15, 2017.
[13] From *1993 World Almanac.*
[14] www.epa.gov/osw/nonhaz/municipal/pubs/msw_2010_rev_factsheet.pdf, accessed March 13, 2017.

(b) Calculate the value of ΔG for each pair of consecutive entries in this table.

(c) Are all the values of ΔG you found in part (b) the same? What does this tell you?

(d) The function G changed from increasing to decreasing between 2005 and 2010. To what might this be attributed?

Table 1.20

t	1960	1970	1980	1990	2000	2005	2010
G	88.1	121.1	151.6	208.3	242.5	252.7	249.9

38. Table 1.21 shows the times, t, in sec, achieved every 10 meters by Carl Lewis in the 100-meter final of the World Championship in Rome in 1987.[15] Distance, d, is in meters.

(a) For each successive time interval, calculate the average rate of change of distance. What is a common name for the average rate of change of distance?

(b) Where did Carl Lewis attain his maximum speed during this race? Some runners are running their fastest as they cross the finish line. Does that seem to be true in this case?

Table 1.21

t	0.00	1.94	2.96	3.91	4.78	5.64
d	0	10	20	30	40	50
t	6.50	7.36	8.22	9.07	9.93	
d	60	70	80	90	100	

1.3 LINEAR FUNCTIONS

Constant Rate of Change

In the previous section, we introduced the average rate of change of a function on an interval. For many functions, the average rate of change is different on different intervals. For the remainder of this chapter, we consider functions that have the same average rate of change on every interval. Such a function is called *linear* and has a graph that is a line.

Population Growth

Mathematical models of population growth are used by city planners to project the growth of towns and states. Biologists model the growth of animal populations and physicians model the spread of an infection in the bloodstream. One possible model, a linear model, assumes that the population changes at the same average rate on every time interval.

Example 1 A town of 30,000 people grows by 2000 people every year. Since the population, P, is growing at the constant rate of 2000 people per year, P is a linear function of time, t, in years.

(a) What is the average rate of change of P over every time interval?

(b) Make a table that gives the town's population every five years over a 20-year period. Graph the population.

(c) Find a formula for P as a function of t.

Solution

(a) The average rate of change of population with respect to time is 2000 people per year.

(b) The initial population in year $t = 0$ is $P = 30{,}000$ people. Since the town grows by 2000 people every year, after five years we have

$$\text{Increase in population} = 2000 \text{ people/year} \cdot 5 \text{ years} = 10{,}000 \text{ people}.$$

Thus, in year $t = 5$ the population is given by

$$P = \text{Initial population} + \text{Increase in population} = 30{,}000 + 10{,}000 = 40{,}000.$$

[15] W. G. Pritchard, "Mathematical Models of Running", *SIAM Review* 35, 1993, pp. 359–379.

In year $t = 10$ the population is given by

$$P = 30{,}000 + \underbrace{2000 \text{ people/year} \cdot 10 \text{ years}}_{20{,}000 \text{ new people}} = 50{,}000.$$

Similar calculations for $t = 15$ and $t = 20$ give the values in Table 1.22. In Figure 1.26, we see that the data points lie on a line whose slope is the same over every interval, 2000 people/year.

Table 1.22 *Population over 20 years*

t, years	P, population
0	30,000
5	40,000
10	50,000
15	60,000
20	70,000

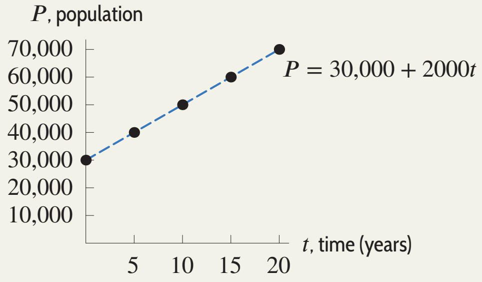


Figure 1.26: Town's population over 20 years

(c) From part (b), we see that the size of the population is given by

$$\begin{aligned} P &= \text{Initial population} + \text{Number of new people} \\ &= 30{,}000 + 2000 \text{ people/year} \cdot \text{Number of years,} \end{aligned}$$

so a formula for P in terms of t is

$$P = 30{,}000 + 2000t.$$

As in Example 1, any linear function has the same average rate of change over every interval. Thus, we talk about *the* rate of change of a linear function. In general:

- A **linear function** has a constant rate of change.
- The graph of any linear function is a straight line.

Financial Models

For tax purposes, the value of equipment is considered to decrease, or depreciate, over time. For example, computer equipment may be state-of-the-art today, but after several years it is outdated. Economists and accountants use linear functions for *straight-line depreciation*, which assumes that the rate of change of value with respect to time is constant.

Example 2 A small business spends \$20,000 on new computer equipment and, for tax purposes, chooses to depreciate it to \$0 at a constant rate over a five-year period.

(a) Make a table and a graph of the value, V, in dollars, of the equipment over the five-year period.
(b) Give a formula for value as a function of time, t.
(c) Find and interpret the vertical intercept where $t = 0$.
(d) Find and interpret the horizontal intercept where $V = 0$.

Solution (a) After five years, the equipment is valued at \$0. If V is the value in dollars and t is the number of years, we see that

$$\begin{array}{c}\text{Rate of change of value} \\ \text{from } t = 0 \text{ to } t = 5\end{array} = \frac{\text{Change in value}}{\text{Change in time}} = \frac{\Delta V}{\Delta t} = \frac{-\$20{,}000}{5 \text{ years}} = -\$4000 \text{ per year.}$$

Thus, the value drops at the constant rate of \$4000 per year. (Notice that ΔV is negative because the value of the equipment decreases.) See Table 1.23 and Figure 1.27. Since V changes at a constant rate, $V = f(t)$ is a linear function; its slope is $-\$4000$ per year.

Table 1.23 *Value of equipment depreciated over a 5-year period*

t, year	V, value (\$)
0	20,000
1	16,000
2	12,000
3	8000
4	4000
5	0

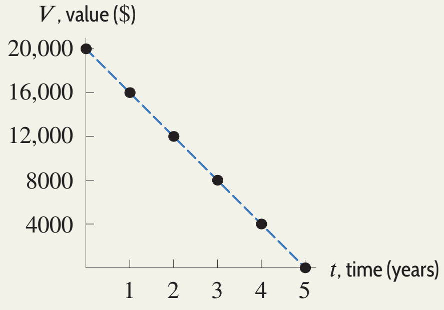


Figure 1.27: Value of equipment depreciated over a 5-year period

(b) After t years have elapsed,

$$\text{Decrease in value of equipment} = \$4000 \cdot \text{Number of years} = \$4000t.$$

The initial value of the equipment is \$20,000, so at time t,

$$V = 20{,}000 - 4000t.$$

(c) Substituting $t = 0$ gives the vertical intercept, $V = 20{,}000$, which is the initial value of the equipment.

(d) To find the horizontal intercept, we set $V = 0$, giving

$$20{,}000 - 4000t = 0.$$

Solving for t, we get $t = 5$, which is the number of years for the value of the equipment to depreciate to \$0.

A General Formula for the Family of Linear Functions

Example 1 involved a town whose population is growing at a constant rate with formula

$$\text{Current population} = \underbrace{\text{Initial population}}_{30{,}000 \text{ people}} + \underbrace{\text{Growth rate}}_{2000 \text{ people per year}} \times \underbrace{\text{Number of years}}_{t}$$

so

$$P = 30{,}000 + 2000t.$$

In Example 2, the value, V, as a function of t is given by

$$\text{Current value} = \underbrace{\text{Initial value}}_{\$20{,}000} + \underbrace{\text{Change per year}}_{-\$4000 \text{ per year}} \times \underbrace{\text{Number of years}}_{t}$$

so

$$V = 20{,}000 + (-4000)t.$$

Using x for time and y for the current population or the current value, and using b and m for constants, we see that the formulas for both linear functions follow the same pattern:

$$\underbrace{\text{Output}}_{y} = \underbrace{\text{Initial value}}_{b} + \underbrace{\text{Rate of change}}_{m} \times \underbrace{\text{Input}}_{x}.$$

Summarizing, we get the following results:

If $y = f(x)$ is a linear function, then for some constants b and m:

$$y = b + mx.$$

- b is called the **vertical intercept**, or **y-intercept**, and gives the value of y for $x = 0$. In mathematical models, b typically represents an initial, or starting, value of the output.
- m is called the **slope**, and gives the rate of change of y with respect to x. Thus,

$$m = \frac{\Delta y}{\Delta x}.$$

If (x_0, y_0) and (x_1, y_1) are any two distinct points on the graph of f, then

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}.$$

- The **horizontal intercept**, or **x-intercept**, is the value of x at which $y = 0$.

Every linear function can be written in the form $y = b + mx$. Different linear functions have different values for b and m. The constants b and m in the formula for a linear function are called *parameters*; the set of all linear functions is called a *family*.

Example 3 In Example 1, the population function, $P = 30{,}000 + 2000t$, has vertical intercept $b = 30{,}000$ and slope $m = 2000$. In Example 2, the value of the computer equipment, $V = 20{,}000 - 4000t$, has vertical intercept $b = 20{,}000$ and slope $m = -4000$.

Tables for Linear Functions

A table of values could represent a linear function if the rate of change is constant for all pairs of points in the table; that is,

$$\text{Rate of change of linear function} = \frac{\text{Change in output}}{\text{Change in input}} = \text{Constant.}$$

Thus, if the value of x goes up by equal steps in a table for a linear function, then the value of y goes up (or down) by equal steps as well.

Example 4 Table 1.24 gives values of two functions, p and q. Could either of these functions be linear?

Table 1.24 *Values of two functions p and q*

x	50	55	60	65	70
$p(x)$	0.10	0.11	0.12	0.13	0.14
$q(x)$	0.01	0.03	0.06	0.14	0.15

Solution The value of x goes up by equal steps of $\Delta x = 5$. The value of $p(x)$ also goes up by equal steps of $\Delta p = 0.01$, so $\Delta p/\Delta x$ is a constant. See Table 1.25. Thus, p could be a linear function.

Table 1.25 *Values of* $\Delta p/\Delta x$

x	$p(x)$	Δp	$\Delta p/\Delta x$
50	0.10		
		0.01	0.002
55	0.11		
		0.01	0.002
60	0.12		
		0.01	0.002
65	0.13		
		0.01	0.002
70	0.14		

Table 1.26 *Values of* $\Delta q/\Delta x$

x	$q(x)$	Δq	$\Delta q/\Delta x$
50	0.01		
		0.02	0.004
55	0.03		
		0.03	0.006
60	0.06		
		0.08	0.016
65	0.14		
		0.01	0.002
70	0.15		

In contrast, the value of $q(x)$ does not go up by equal steps. The value climbs by 0.02, then by 0.03, and so on. See Table 1.26. This means that $\Delta q/\Delta x$ is not constant. Thus, q is not a linear function.

It is possible to have data from a linear function where neither the x-values nor the y-values change by equal steps. However, if a function is linear the rate of change is constant, as in the following example.

Example 5 The former Republic of Yugoslavia exported cars called Yugos to the US between 1985 and 1989. The car is now a collector's item.[16] Table 1.27 gives the quantity of Yugos sold, Q, and the price, p, for each year from 1985 to 1988.

(a) Using Table 1.27, explain why Q could be a linear function of p.
(b) What does the rate of change of this function tell you about the sale of Yugos?

Table 1.27 *Price and sales of Yugos in the US*

Year	Price in \$, p	Number sold, Q
1985	3990	49,000
1986	4110	43,000
1987	4200	38,500
1988	4330	32,000

Solution (a) We are interested in Q as a function of p, so we plot Q on the vertical axis and p on the horizontal axis. The data points in Figure 1.28 appear to lie on a straight line, suggesting a linear function.

For further evidence that Q is a linear function, we check that the rate of change of Q with respect to p is constant for the points given. When the price of a Yugo rose from \$3990 to \$4110, sales fell from 49,000 to 43,000. Thus,

$$\Delta p = 4110 - 3990 = 120,$$

$$\Delta Q = 43{,}000 - 49{,}000 = -6000.$$

Since the number of Yugos sold decreased, ΔQ is negative. Thus,

$$\text{Rate of change of quantity as price increases} = \frac{\Delta Q}{\Delta p} = \frac{-6000}{120} = -50 \text{ cars per dollar.}$$

[16]www.inet.hr/˜pauric/epov.htm, accessed January 16, 2006.

Next, we calculate the rate of change as the price increased from \$4110 to \$4200:

$$\text{Rate of change} = \frac{\Delta Q}{\Delta p} = \frac{38{,}500 - 43{,}000}{4200 - 4110} = \frac{-4500}{90} = -50 \text{ cars per dollar,}$$

and as the price increased from \$4200 to \$4330:

$$\text{Rate of change} = \frac{\Delta Q}{\Delta p} = \frac{32{,}000 - 38{,}500}{4330 - 4200} = \frac{-6500}{130} = -50 \text{ cars per dollar.}$$

Since the rate of change, -50, is constant for the values in the table, Q could be a linear function of p.

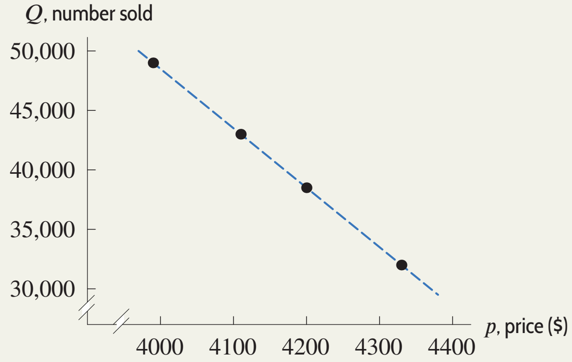


Figure 1.28: Since the data from Table 1.27 falls on a straight line, the table could represent a linear function

(b) Since ΔQ is the change in the number of cars sold and Δp is the change in price, the rate of change is -50 cars per dollar. Thus, the number of Yugos sold decreased by 50 each time the price increased by \$1.

Warning: Not All Graphs That Look Like Lines Represent Linear Functions

The graph of any linear function is a line. However, a graph can look like a line without the function actually being linear. Consider the following example.

Example 6 The function $P = 100(1.02)^t$ approximates the population of Mexico in the early 2000s. Here P is the population (in millions) and t is the number of years since 2000. Table 1.28 and Figure 1.29 show values of P over a 5-year period. Is P a linear function of t?

Table 1.28 *Population of Mexico t years after 2000*

t (years)	P (millions)
0	100
1	102
2	104.04
3	106.12
4	108.24
5	110.41

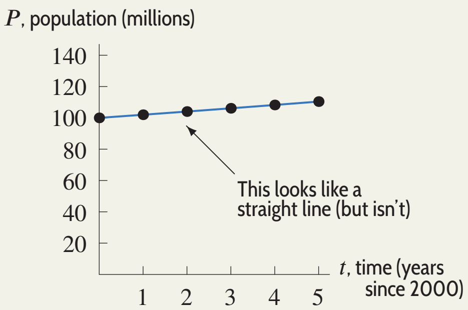


Figure 1.29: Graph of $P = 100(1.02)^t$ over 5-year period: Looks linear (but is not)

Solution The formula $P = 100(1.02)^t$ cannot be written in the form $P = b+mt$, so P is not a linear function of t. However, the graph of P in Figure 1.29 appears to be a straight line. We check P's rate of change in Table 1.28. When $t = 0$, $P = 100$ and when $t = 1$, $P = 102$. Thus, between 2000 and 2001,

$$\text{Rate of change of population} = \frac{\Delta P}{\Delta t} = \frac{102 - 100}{1 - 0} = 2.$$

For the interval from 2001 to 2002, we have

$$\text{Rate of change} = \frac{\Delta P}{\Delta t} = \frac{104.04 - 102}{2 - 1} = 2.04,$$

and for the interval from 2004 to 2005, we have

$$\text{Rate of change} = \frac{\Delta P}{\Delta t} = \frac{110.41 - 108.24}{5 - 4} = 2.17.$$

Thus, P's rate of change is not constant. In fact, P appears to be increasing at a faster and faster rate. Table 1.29 and Figure 1.30 show values of P over a longer (60-year) period. On this scale, these points do not appear to fall on a straight line. However, the graph of P curves upward so gradually at first that over the short interval shown in Figure 1.29, it barely curves at all. The graphs of many nonlinear functions, when viewed on a small scale, appear to be linear.

Table 1.29 *Population over 60 years*

t (years since 2000)	P (millions)
0	100
10	121.90
20	148.59
30	181.14
40	220.80
50	269.16
60	328.10

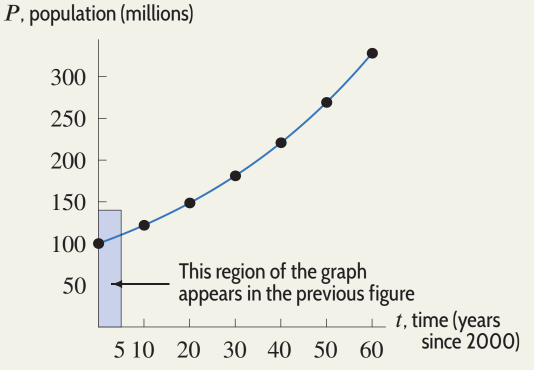


Figure 1.30: Graph of $P = 100(1.02)^t$ over 60 years: Not linear

Summary for Section 1.3

- A **linear function** has values of y that change at a constant rate with the values of x.
- **Formula** for linear functions:

$$y = \underbrace{b}_{\text{Initial value}} + \underbrace{m}_{\text{Slope}} \cdot x$$

- The graph of a linear function is a **line**.
  - b is the **vertical intercept**, or **y-intercept**, and gives the value of y for $x = 0$.
  - m is the **slope** of the line, and gives the rate of change of y with respect to x:

$$m = \frac{\Delta y}{\Delta x}.$$

  - The **horizontal intercept**, or **x-intercept**, is the value of x where $y = 0$.
- **Tables** for linear functions with equally-spaced x-values have equally-spaced y-values.

Exercises and Problems for Section 1.3

Skill Refresher

In Exercises S1–S2, find $f(0)$ and $f(3)$.

S1. $f(x) = \frac{2}{3}x + 5$

S2. $f(t) = 17 - 4t$

In Exercises S3–S4, find $f(2) - f(0)$.

S3.

x	0	1	2	3
$f(x)$	−2	0	3	4

S4.

t	−1	0	1	2
$f(t)$	0	2	7	−1

In Exercises S5–S6, find the coordinates of the x and y intercepts.

S5. $y = -4x + 3$

S6. $5x - 2y = 4$

For each of the linear expressions in x in Exercises S7–S16, give the constant term and the coefficient of x.

S7. $8 + 5x$

S8. $3x - 7$

S9. $1.35x - 4.18$

S10. $8.29 - 4.64x$

S11. $28x$

S12. 73

S13. $3 - 2x + \frac{1}{2}$

S14. $4 - 3(x + 2) + 6(2x - 1)$

S15. $ax - ab - 3x + a + 3$

S16. $5(x - 1) + 3$

EXERCISES

In Exercises 1–3, which line has the greater

(a) Slope? **(b)** y-intercept?

1. $y = -1 + 2x;\quad y = -2 + 3x$

2. $y = 3 + 4x;\quad y = 5 - 2x$

3. $y = \frac{1}{4}x;\quad y = 1 - 6x$

In Exercises 4–9, could the data represent a linear function? If so, give the rate of change.

4.

x	0	5	10	15
$f(x)$	10	20	30	40

5.

x	0	10	20	30
$h(x)$	20	40	50	55

6.

x	0	100	300	600
$g(x)$	50	100	150	200

7.

t	1	2	3	4	5
$g(t)$	5	4	5	4	5

8.

x	−3	−1	0	3
$j(x)$	5	1	−1	−7

9.

y	9	8	7	6	5
$p(y)$	42	52	62	72	82

In Exercises 10–13, identify the vertical intercept and the slope, and explain their meanings in practical terms.

10. The population of a town can be represented by the formula $P(t) = 54.25 - \frac{2}{7}t$, where $P(t)$ represents the population, in thousands, and t represents the time, in years, since 1970.

11. A stalactite grows according to the formula $L(t) = 17.75 + \frac{1}{250}t$, where $L(t)$ represents the length of the stalactite, in inches, and t represents the time, in years, since the stalactite was first measured.

12. The profit, in dollars, of selling n items is given by $P(n) = 0.98n - 3000$.

13. A phone company charges according to the formula $C(n) = 29.99 + 0.05n$, where n is the number of minutes, and $C(n)$ is the monthly phone charge, in dollars.

PROBLEMS

14. Table 1.30 shows the cost C, in dollars, of selling x cups of coffee per day from a cart.

(a) Using the table, show that the relationship appears to be linear.
(b) Plot the data in the table.
(c) Find the slope of the line. Explain what this means in the context of the given situation.
(d) Why should it cost \$50 to serve zero cups of coffee?

Table 1.30

x	0	5	10	50	100	200
C	50.00	51.25	52.50	62.50	75.00	100.00

15. Table 1.31 gives the proposed fine $r = f(v)$ to be imposed on a motorist for speeding, where v is the motorist's speed and 55 mph is the speed limit.

(a) Decide whether f appears to be linear.
(b) What would the rate of change represent in practical terms for the motorist?
(c) Plot the data points.

Table 1.31

v (mph)	60	65	70	75	80	85
r (dollars)	75	100	125	150	175	200

16. In 2003, the number, N, of cases of SARS (Severe Acute Respiratory Syndrome) reported in Hong Kong[17] was initially approximated by $N = 78.9 + 30.1t$, where t is the number of days since March 17. Interpret the constants 78.9 and 30.1.

17. A new Toyota RAV4 costs $26,500. The car's value depreciates linearly to $19,999 in three years' time.[18] Write a formula which expresses its value, V, in terms of its age, t, in years.

18. In 2016, the population of a town was 23,520 and growing by 58 people per year. Find a formula for P, the town's population, in terms of t, the number of years since 2016.

19. A flight costs $10,000 to operate, regardless of the number of passengers. Each ticket costs $127. Express profit, π, as a linear function of the number of passengers, n, on the flight.

20. Owners of an inactive quarry in Australia have decided to resume production. They estimate that it will cost them $10,000 per month to maintain and insure their equipment and that monthly salaries will be $30,000. It costs $800 to mine a ton of rocks. Write a formula that expresses the total cost each month, c, as a function of r, the number of tons of rock mined per month.

21. In each case, graph a linear function with the given rate of change. Label and put scales on the axes.

(a) Increasing at 2.1 inches/day
(b) Decreasing at 1.3 gallons/mile

22. A small café sells coffee for $3.50 per cup. On average, it costs the café $0.50 to make a cup of coffee (for grounds, hot water, filters). The café also has a fixed daily cost of $450 (for rent, wages, utilities).

(a) Let R, C, and P be the café's daily revenue, costs, and profit, respectively, for selling x cups of coffee in a day. Find formulas for R, C, and P as functions of x. [Hint: The revenue, R, is the total amount of money that the café brings in. The cost, C, includes the fixed daily cost as well as the cost for all x cups of coffee sold. P is the café's profit after costs have been accounted for.]
(b) Plot P against x. For what x-values is the graph of P below the x-axis? Above the x-axis? Interpret your results.
(c) Interpret the slope and both intercepts of your graph in practical terms.

23. Table 1.32 gives the area and perimeter of a square as a function of the length of its side.

(a) From the table, decide if either area or perimeter could be a linear function of side length.
(b) From the data make two graphs, one showing area as a function of side length, the other showing perimeter as a function of side length. Connect the points.
(c) If you find a linear relationship, give its corresponding rate of change and interpret its significance.

Table 1.32

Length of side	0	1	2	3	4	5	6
Area of square	0	1	4	9	16	25	36
Perimeter of square	0	4	8	12	16	20	24

24. Make two tables, one comparing the radius of a circle to its area, the other comparing the radius of a circle to its circumference. Repeat parts (a), (b), and (c) from Problem 23, this time comparing radius with circumference, and radius with area.

25. Australia experienced approximately linear population growth from 1960 to 2010. On the other hand, Afghanistan was torn by warfare in the 1900s and did not experience linear or near-linear growth.[19]

(a) Table 1.33 gives the population of these two countries, in millions. Which of these two countries is A and which is B? Explain.
(b) What is the approximate rate of change of the linear function? What does the rate of change represent in practical terms?
(c) Estimate the population of Australia in 2018.

Table 1.33

Year	1960	1970	1980	1990	2000	2010
Population of country A	9.00	11.12	13.21	12.07	19.70	27.96
Population of country B	10.29	12.90	14.71	17.10	19.11	22.16

26. Table 1.34 gives the average temperature, T, at a depth d, in a borehole in Belleterre, Quebec.[20] Evaluate $\Delta T/\Delta d$ on the following intervals, and explain what your answers tell you about borehole temperature.

(a) $25 \le d \le 150$ **(b)** $25 \le d \le 75$
(c) $100 \le d \le 200$

[17] World Health Organization, www.who.int/csr/sars/country/en, accessed September, 2005.
[18] www.motortrend.com/used_cars/11/toyota/rav4/pricing/, accessed March 17, 2017.
[19] http://www.worldometers.info/world-population/, accessed March 17, 2017.
[20] Hugo Beltrami of St. Francis Xavier University and David Chapman of the University of Utah posted this data at http://geophysics.stfx.ca/public/borehole/borehole.html, accessed April 22, 2017.

Table 1.34

d, depth (m)	25	50	75	100
T, temp (°C)	5.50	5.20	5.10	5.10
d, depth (m)	125	150	175	200
T, temp (°C)	5.30	5.50	5.75	6.00
d, depth (m)	225	250	275	300
T, temp (°C)	6.25	6.50	6.75	7.00

27. Table 1.34 gives the temperature-depth profile, $T = f(d)$, in a borehole in Belleterre, Quebec, where T is the average temperature at a depth d.

(a) Could f be linear?
(b) Graph f. What do you notice about the graph for $d \geq 150$?
(c) What can you say about the average rate of change of f for $d \geq 150$?

28. A company finds that there is a linear relationship between the amount of money that it spends on advertising and the number of units it sells. If it spends no money on advertising, it sells 300 units. For each additional \$5000 spent, an additional 20 units are sold.

(a) If x is the amount of money that the company spends on advertising, find a formula for y, the number of units sold as a function of x.
(b) How many units does the firm sell if it spends \$25,000 on advertising? \$50,000?
(c) How much advertising money must be spent to sell 700 units?
(d) What is the slope of the line you found in part (a)? Give an interpretation of the slope that relates units sold and advertising costs.

29. Tuition cost T (in dollars) for part-time students at Stonewall College is given by $T = 300 + 200C$, where C represents the number of credits taken.

(a) Find the tuition cost for eight credits.
(b) How many credits were taken if the tuition was \$1700?
(c) Make a table showing costs for taking from one to twelve credits. For each value of C, give both the tuition cost, T, and the cost per credit, T/C. Round to the nearest dollar.
(d) Which of these values of C has the smallest cost per credit?
(e) What does the 300 represent in the formula for T?
(f) What does the 200 represent in the formula for T?

30. The summit of Africa's largest peak, Mt. Kilimanjaro, consists of the northern and southern ice fields and the Furtwangler glacier. In 2000 the Furtwangler glacier covered an area of 60,000 m^2. By 2012, the area of the glacier had shrunk[21] to approximately 25,000 m^2.

(a) If this decline is modeled by a linear function, find $A = f(t)$, the formula giving the area of the glacier as a function of time, t, in years since 2000. Explain what the slope and A-intercept mean in terms of the area of the Furtwangler glacier.
(b) Evaluate $f(20)$.
(c) If this model is correct, in what year would you expect the Furtwangler glacier to disappear?

31. When each of the following equations are written in the form $y = b + mx$, the result is $y = 5 + 4x$. Find the constants r, s, k, j in these equations.

(a) $y = 2r + x\sqrt{s}$
(b) $y = \dfrac{1}{k} - (j-1)x$.

32. Graph the following function in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Is this graph a line? Explain.

$$y = -x\left(\frac{x-1000}{900}\right)$$

33. Graph $y = 2x + 400$ using the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Describe what happens, and how you can fix it by using a better window.

34. Graph $y = 200x + 4$ using the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Describe what happens and how you can fix it by using a better window.

35. Figure 1.31 shows the graph of $y = x^2/1000 + 5$ in the window $-10 \leq x \leq 10, -10 \leq y \leq 10$. Discuss whether this is a linear function.

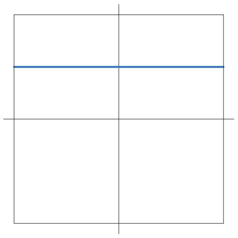


Figure 1.31

36. The cost of a cab ride is given by the function $C = 2.50 + 2d$, where d is the number of miles traveled and C is in dollars. Choose an appropriate window and graph the cost of a ride for a cab that travels no farther than a 10-mile radius from the center of the city.

1.4 FORMULAS FOR LINEAR FUNCTIONS

To find a formula for a linear function we find values for the slope, m, and the vertical intercept, b, in the formula $y = b + mx$.

[21] www.geo.umass.edu/climate/tanzania/furtwangler.html, accessed May, 2017.

Slope-Intercept Form for a Linear Function

If a table of data represents a linear function, we first calculate m and then determine b.

Example 1 A grapefruit is thrown into the air. Its velocity, v, is a linear function of t, the time since it was thrown. A positive velocity indicates the grapefruit is rising and a negative velocity indicates it is falling. Check that the data in Table 1.35 corresponds to a linear function. Find a formula for v in terms of t.

Table 1.35 *Velocity of a grapefruit t seconds after being thrown into the air*

t, time (sec)	1	2	3	4
v, velocity (ft/sec)	48	16	−16	−48

Solution Figure 1.32 shows the data in Table 1.35. The points appear to fall on a line. To check that the velocity function is linear, calculate the rates of change of v and see that they are constant. From time $t = 1$ to $t = 2$, we have

$$\text{Average rate of change of velocity with time} = \frac{\Delta v}{\Delta t} = \frac{16 - 48}{2 - 1} = -32.$$

For the next second, from $t = 2$ to $t = 3$, we have

$$\text{Average rate of change} = \frac{\Delta v}{\Delta t} = \frac{-16 - 16}{3 - 2} = -32.$$

You can check that the rate of change from $t = 3$ to $t = 4$ is also -32.

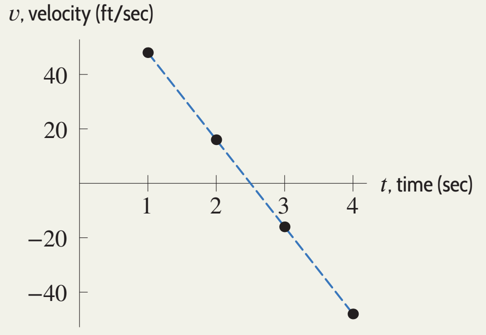


Figure 1.32: Velocity of a grapefruit is a linear function of time

A formula for v is of the form $v = b + mt$. Since m is the rate of change, we have $m = -32$, so

$$v = b - 32t.$$

The value of b is the initial velocity at $t = 0$. We are not given this value of v, but we can use any data point to calculate b. For example, $v = 48$ when $t = 1$, so

$$48 = b - 32 \cdot 1,$$

which gives

$$b = 80.$$

Thus, a formula for the velocity is $v = 80 - 32t$.

What does the rate of change, m, in Example 1 tell us about the grapefruit? Think about the units:

$$m = \frac{\Delta v}{\Delta t} = \frac{\text{Change in velocity}}{\text{Change in time}} = \frac{-32 \text{ ft/sec}}{1 \text{ sec}} = -32 \text{ ft/sec per second.}$$

The value of m, -32 ft/sec per second, tells us that the grapefruit's velocity is decreasing by 32 ft/sec for every second that passes. We say the grapefruit is accelerating at -32 ft/sec per second. (We often write ft/sec^2 for ft/sec per second.[22])

The formula $y = b + mx$ is called the **slope-intercept** form of a line since the slope, m, and vertical intercept, b, are given explicitly. We can calculate m using two points on the graph of a linear function, as in Example 2. Having found m, we can use either of the points to calculate b.

Example 2 Figure 1.33 shows oxygen consumption as a function of heart rate (that is, pulse) for two people.

(a) Assuming linearity, find formulas for these two functions.
(b) Interpret the slope of each graph in terms of oxygen consumption.

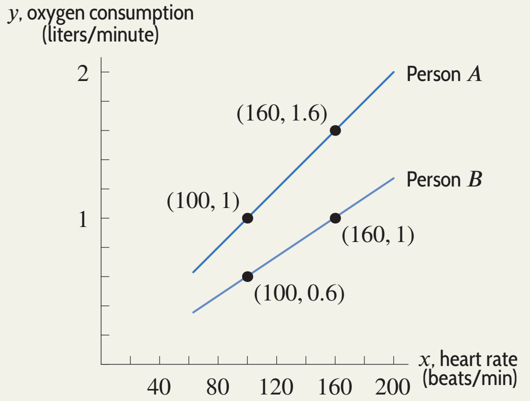


Figure 1.33: Oxygen consumption of two people running on treadmills

Solution (a) Let x be heart rate and let y be oxygen consumption. Since we are assuming linearity, $y = b+mx$. The two points on person A's line, (100, 1) and (160, 1.6), give

$$\text{Slope of A's line } = m = \frac{\Delta y}{\Delta x} = \frac{1.6 - 1}{160 - 100} = 0.01.$$

Thus $y = b + 0.01x$. To find b, use the fact that $y = 1$ when $x = 100$:

$$1 = b + 0.01(100)$$
$$1 = b + 1$$
$$b = 0,$$

so $y = 0.01x$. Alternatively, b can be found using the fact that $y = 1.6$ if $x = 160$.

For person B, we again begin with the formula $y = b + mx$. In Figure 1.33, two points on B's line are (100, 0.6) and (160, 1), so

$$\text{Slope of B's line } = m = \frac{\Delta y}{\Delta x} = \frac{1 - 0.6}{160 - 100} = \frac{0.4}{60} = 0.0067.$$

To find b, use the fact that $y = 1$ when $x = 160$:

$$1 = b + (0.4/60) \cdot 160$$

[22]The notation ft/sec^2 is shorthand for ft/sec per second; it does not mean a "square second" in the same way that areas are measured in square feet or square meters.

$$1 = b + 1.067$$
$$b = -0.067.$$

Thus, for person B, we have $y = -0.067 + 0.0067x$.

(b) The slope for person A is $m = 0.01$, so

$$m = \frac{\text{Change in oxygen consumption}}{\text{Change in heart rate}} = \frac{\text{Change in liters/min}}{\text{Change in beats/min}} = 0.01\frac{\text{liters}}{\text{heartbeat}}.$$

Every additional heartbeat (per minute) for person A translates to an additional 0.01 liters (per minute) of oxygen consumed.

The slope for person B is $m = 0.0067$. Thus, for every additional beat (per minute), person B consumes an additional 0.0067 liter of oxygen (per minute). Since the slope for person B is smaller than for person A, person B consumes less additional oxygen than person A for the same increase in pulse.

What do the y-intercepts of the functions in Example 2 say about oxygen consumption? The y-intercept would be the oxygen consumption of a person whose pulse is zero (i.e. $x = 0$). Since a person running on a treadmill must have a pulse, it does not make sense to interpret the y-intercept this way. The formula for oxygen consumption is useful only for realistic values of the pulse.

Point-Slope Form for a Linear Function

If we know its slope and the coordinates of a point, we can find a formula for a line without having to find its intercept.

Example 3 In Example 2, we found the slope of person A's line to be $m = 0.01$. Use this information and the fact that (100, 1) lies on the line to find an equation of the line without evaluating the y-intercept.

Solution We can compute the slope $m = 0.01$ using any two points (100, 1) and (x, y) on the line. Hence

$$\text{Slope} = \frac{y-1}{x-100} = 0.01.$$

Thus an equation for the line is

$$y - 1 = 0.01(x - 100).$$

To check that this gives the same equation we got in Example 2, we multiply out and simplify:

$$y - 1 = 0.01x - 1$$
$$y = 0.01x.$$

Alternatively, we could have used the point (160, 1.6) instead of (100, 1), giving

$$y - 1.6 = 0.01(x - 160).$$

Multiplying out again gives $y = 0.01x$.

The formula $y - y_0 = m(x - x_0)$ is called the **point-slope** form of a line since the slope, m, and the coordinates of one point, (x_0, y_0) are given explicitly. Different points on a line give different point-slope equations, but they are all valid equations for the line.

Standard Form for a Linear Function

Sometimes the verbal description of a linear function is less straightforward than those in Section 1.3. Consider the following example.

Example 4 We have \$48 to spend on soda and chips for a party. A six-pack of soda costs \$6 and a bag of chips costs \$4. The number of six-packs we can afford, y, is a function of the number of bags of chips we decide to buy, x.

(a) Find an equation relating x and y.
(b) Graph the equation. Interpret the intercepts and the slope in the context of the party.

Solution (a) If we spend all \$48 on soda and chips, then we have the following equation:

$$\text{Amount spent on chips} + \text{Amount spent on soda} = \$48.$$

If we buy x bags of chips at \$4 per bag, then the amount spent on chips is \$$4x$. Similarly, if we buy y six-packs of soda at \$6 per six-pack, then the amount spent on soda is \$$6y$. Thus,

$$4x + 6y = 48.$$

We can solve for y, giving

$$6y = 48 - 4x$$
$$y = 8 - \frac{2}{3}x.$$

This is a linear function with slope $m = -2/3$ and y-intercept $b = 8$.

(b) The graph of this function is a discrete set of points, since the number of bags of chips and the number of six-packs of soda must be (nonnegative) integers. See Figure 1.34.

To find the y-intercept, we set $x = 0$, giving

$$4 \cdot 0 + 6y = 48.$$

So $6y = 48$, giving $y = 8$.

Substituting $y = 0$ gives the x-intercept,

$$4x + 6 \cdot 0 = 48.$$

So $4x = 48$, giving $x = 12$. Thus the points $(0, 8)$ and $(12, 0)$ are on the graph in Figure 1.34.

The point $(0, 8)$ indicates that we can buy 8 six-packs of soda if we buy no chips. The point $(12, 0)$ indicates that we can buy 12 bags of chips if we buy no soda.

To interpret the slope, notice that

$$m = \frac{\Delta y}{\Delta x} = \frac{\text{Change in number of six-packs}}{\text{Change in number of bags of chips}},$$

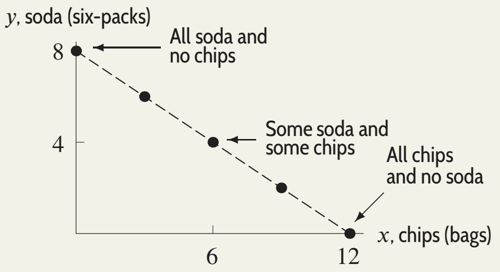


Figure 1.34: Relation between the number of six-packs, y, and the number of bags of chips, x

so the units of m are six-packs of soda per bag of chips. The fact that $m = -2/3$ means that for each additional 3 bags of chips purchased, we can purchase 2 fewer six-packs of soda. This occurs because 2 six-packs cost \$12, the same as 3 bags of chips. Thus, $m = -2/3$ is the rate at which the amount of soda we can buy decreases as we buy more chips.

In Example 4, the graph of $4x + 6y = 48$ is a line even though the equation is not in the form $y = b + mx$. When the equation of a line is written in the form $Ax + By = C$ where A, B, and C are constants, we say it is in **standard form**.

Formulas for the Equation of a Line

Summarizing, the following equations are all used to represent lines:

- The *slope-intercept form* is
  $y = b + mx$ where m is the slope and b is the y-intercept.
- The *point-slope form* is
  $y - y_0 = m(x - x_0)$ where m is the slope and (x_0, y_0) is a point on the line.
- The *standard form* is
  $Ax + By = C$ where A, B, and C are constants.

Equations of Horizontal and Vertical Lines

The slope m of a line $y = b + mx$ gives the rate of change of y with respect to x. What about a line with slope $m = 0$? If the rate of change of a quantity is zero, then the quantity does not change. Thus, if the slope of a line is zero, the value of y must be constant. Such a line is horizontal.

Example 5 Explain why the equation $y = 4$ represents a horizontal line and the equation $x = 4$ represents a vertical line.

Solution The equation $y = 4$ represents a linear function with slope $m = 0$. To see this, notice that this equation can be rewritten as $y = 4 + 0 \cdot x$. Thus, the value of y is 4 no matter what the value of x is. See Figure 1.35. Similarly, the equation $x = 4$ means that x is 4 no matter what the value of y is. Every point on the line in Figure 1.36 has x equal to 4, so this line is the graph of $x = 4$.

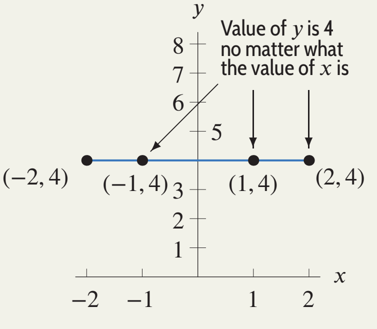


Figure 1.35: The horizontal line $y = 4$ has slope 0

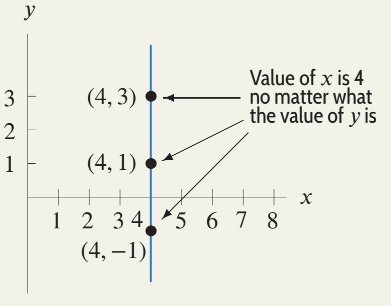


Figure 1.36: The vertical line $x = 4$ has an undefined slope

What is the slope of a vertical line? Figure 1.36 shows three points, $(4, -1)$, $(4, 1)$, and $(4, 3)$, on a vertical line. Calculating the slope gives

$$m = \frac{\Delta y}{\Delta x} = \frac{3 - 1}{4 - 4} = \frac{2}{0}.$$

The slope is undefined because the denominator, Δx, is 0. The slope of every vertical line is undefined for the same reason. A vertical line is not the graph of a function, since it fails the vertical line test. It does not have an equation of the form $y = b + mx$.

In summary,

> For any constant k:
> - The graph of the equation $y = k$ is a horizontal line and its slope is zero.
> - The graph of the equation $x = k$ is a vertical line and its slope is undefined.

Slopes of Parallel and Perpendicular Lines

Figure 1.37 shows two parallel lines. These lines are parallel because they have equal slopes.

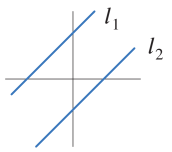


Figure 1.37: Parallel lines: l_1 and l_2 have equal slopes

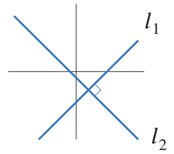


Figure 1.38: Perpendicular lines: Slopes of l_1 and l_2 are negative reciprocals of each other

What about perpendicular lines? Two perpendicular lines are graphed in Figure 1.38. We can see that if one line has a positive slope, then a perpendicular line must have a negative slope. In fact, if l_1 and l_2 are two perpendicular lines with slopes, m_1 and m_2, then m_1 is the negative reciprocal of m_2. If m_1 and m_2 are not zero, we have the following result:

> Let l_1 and l_2 be two lines having slopes m_1 and m_2.Then:
> - The lines are parallel if and only if $m_1 = m_2$.
> - The lines are perpendicular if and only if $m_2 = -\dfrac{1}{m_1}$.

Any two horizontal lines are parallel and $m_1 = m_2 = 0$. Any two vertical lines are parallel and m_1 and m_2 are undefined. A horizontal line is perpendicular to a vertical line. See Figures 1.39–1.41.

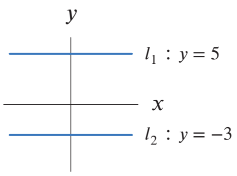


Figure 1.39: Any two horizontal lines are parallel

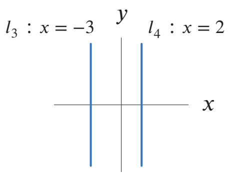


Figure 1.40: Any two vertical lines are parallel

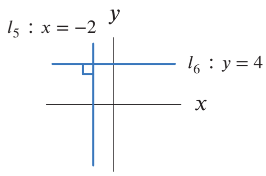


Figure 1.41: A horizontal line and a vertical line are perpendicular

Justification of Formula for Slopes of Perpendicular Lines

Figure 1.42 shows l_1 and l_2, two perpendicular lines with slopes m_1 and m_2. Neither line is horizontal or vertical, so m_1 and m_2 are both defined and nonzero.

Using right triangle ΔPQR with side lengths a and b, we see that

$$m_1 = \frac{b}{a}.$$

Rotating ΔPQR by 90° about the point P produces triangle ΔPST. Using ΔPST, we see that

$$m_2 = -\frac{a}{b} = -\frac{1}{b/a} = -\frac{1}{m_1}.$$

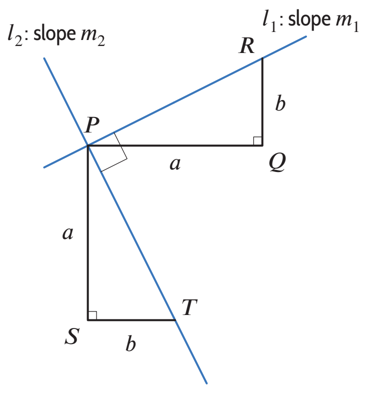


Figure 1.42: Perpendicular lines

Summary for Section 1.4

- **Formulas for linear functions:**
  - **Slope-intercept** form: With $b =$ y-intercept and $m =$ slope,
  $$y = b + mx$$
  - **Point-slope** form: With $m =$ slope and the point (x_0, y_0) on the line,
  $$y - y_0 = m(x - x_0)$$
  - **Standard** form: With A, B, C constants,
  $$Ax + By = C$$
- Equations of
  - **Horizontal** lines: $y = c$, where c is a constant
  - **Vertical** lines: $x = k$, where k is a constant
  - **Parallel** lines: Equal slopes $m_1 = m_2$
  - **Perpendicular** lines: Negative reciprocal slopes $m_1 = -\dfrac{1}{m_2}$

Exercises and Problems for Section 1.4

Skill Refresher

Solve the equations in Exercises S1–S5.

S1. $y - 5 = 21$

S2. $2x - 5 = 13$

S3. $2x - 5 = 4x - 9$

S4. $17 - 28y = 13y + 24$

S5. $\frac{5}{3}(y + 2) = \frac{1}{2} - y$

In Exercises S6–S10, solve for the indicated variable.

S6. $I = Prt$, for P.

S7. $C = \frac{5}{9}(F - 32)$, for F.

S8. $C = 2\pi r$, for r.

S9. $ab + ax = c - ax$, for x.

S10. $by - d = ay + c$, for y.

In Exercises S11–S14, solve the equation $f(x) = 0$ for x.

S11. $f(x) = 2x + 7$

S12. $f(x) = 3x - 12$

S13. $f(x) = 7x + 1$

S14. $f(x) = 5x - 3$

EXERCISES

Find formulas for the linear functions in Exercises 1–8.

1. Slope -4 and x-intercept 7

2. Slope 3 and y-intercept 8

3. Passes through the points $(-1, 5)$ and $(2, -1)$

4. Slope $2/3$ and passes through the point $(5, 7)$

5. Has x-intercept 3 and y-intercept -5

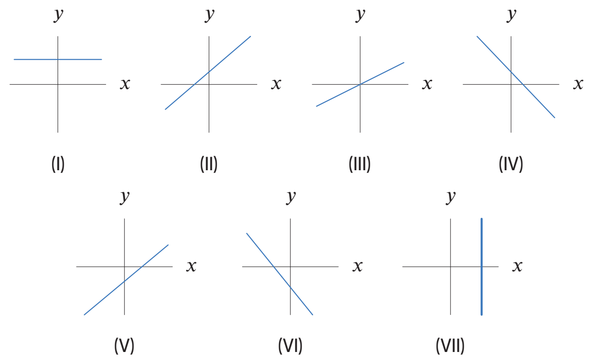
6. Slope 0.1, passes through $(-0.1, 0.02)$

7. Function f has $f(0.3) = 0.8$ and $f(0.8) = -0.4$

8. Function f has $f(-2) = 7$ and $f(3) = -3$

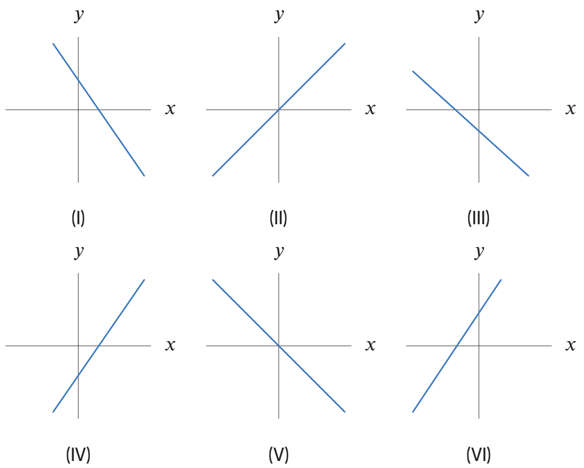
9. Without a calculator, match the equations (a)–(g) to the graphs (I)–(VII).

(a) $y = x - 5$ **(b)** $-3x + 4 = y$
(c) $5 = y$ **(d)** $y = -4x - 5$
(e) $y = x + 6$ **(f)** $y = x/2$
(g) $5 = x$



10. Without using a calculator, match the equations (a) – (f) to the graphs (I) – (VI).

(a) $y = -2.72x$ **(b)** $y = 0.01 + 0.001x$
(c) $y = 27.9 - 0.1x$ **(d)** $y = 0.1x - 27.9$
(e) $y = -5.7 - 200x$ **(f)** $y = x/3.14$



Exercises **11–17** give data from a linear function. Find a formula for the function.

11.

Year, t	0	1	2
Value of computer, \$$V = f(t)$	2000	1500	1000

12.

Price per bottle, p (\$)	0.50	0.75	1.00
Number of bottles sold, $q = f(p)$	1500	1000	500

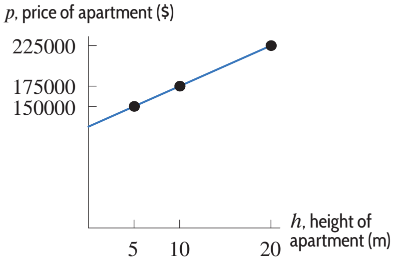
13.

Temperature, $y = f(x)$ (°C)	0	5	20
Temperature, x (°F)	32	41	68

14.

Temperature, $y = f(x)$, (°R)	459.7	469.7	489.7
Temperature, x (°F)	0	10	30

15.



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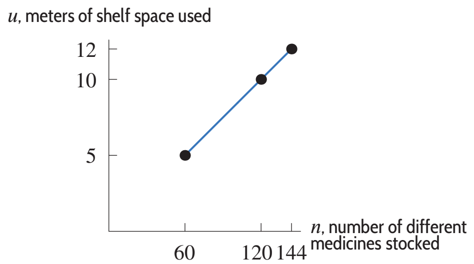

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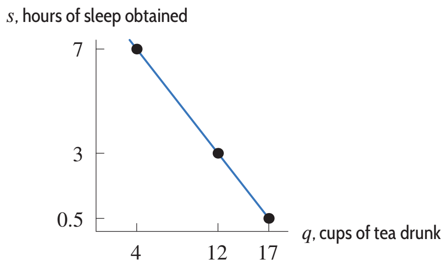


In Exercises **18–26**, if possible rewrite the equation in slope-intercept form, $y = b + mx$.

18. $5(x + y) = 4$
19. $3x + 5y = 20$
20. $0.1y + x = 18$
21. $5x - 3y + 2 = 0$
22. $y - 0.7 = 5(x - 0.2)$
23. $y = 5$
24. $3x + 2y + 40 = x - y$
25. $x = 4$
26. $\dfrac{x + y}{7} = 3$

In Exercises **27–32**, is the function linear? If so, rewrite it in slope-intercept form.

27. $g(w) = -\dfrac{1 - 12w}{3}$
28. $F(P) = 13 - \dfrac{2^{-1}}{4}P$
29. $j(s) = 3s^{-1} + 7$
30. $C(r) = 2\pi r$
31. $h(x) = 3^x + 12$
32. $f(x) = m^2x + n^2$

In Exercises 33–38, are the lines perpendicular? Parallel? Neither?

33. $y = 5x - 7;\quad y = 5x + 8$

34. $y = 4x + 3;\quad y = 13 - \frac{1}{4}x$

35. $y = 2x + 3;\quad y = 2x - 7$

36. $y = 4x + 7;\quad y = \frac{1}{4}x - 2$

37. $f(q) = 12q + 7;\quad g(q) = \frac{1}{12}q + 96$

38. $2y = 16 - x;\quad 4y = -8 - 2x$

PROBLEMS

In Problems 39–42, find a formula for the linear function.

39. The graph of f contains $(-3, -8)$ and $(5, -20)$.

40. $g(100) = 2000$ and $g(400) = 3800$

41. $P = h(t)$ gives the size of a population that begins with 12,000 members and grows by 225 members each year.

42. The graph of h intersects the graph of $y = x^2$ at $x = -2$ and $x = 3$.

43. **(a)** By hand, graph $y = 3$ and $x = 3$.
(b) Can the equations in part (a) be written in slope-intercept form?

44. Find the equation of the line parallel to $3x+5y = 6$ and passing through the point $(0, 6)$.

45. Find the equation of the line passing through the point $(2, 1)$ and perpendicular to the line $y = 5x - 3$.

46. Find the equations of the lines parallel to and perpendicular to the line $y + 4x = 7$, and through the point $(1, 5)$.

47. In Table 1.36, data for two functions f and g are given. One of the functions is linear, and the other is not.

(a) Which of the two functions is linear? Explain how you know.
(b) Find the equation of the linear function. Write your answer in slope-intercept form.

Table 1.36

x	1	1.5	2	2.5	3
$f(x)$	−1.22	−0.64	−0.06	0.52	1.10
$g(x)$	9.71	4.86	2.43	1.22	0.61

48. Find the equation of the line l_2 in Figure 1.43.

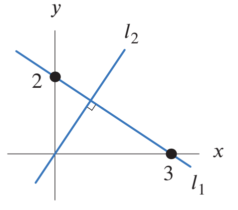


Figure 1.43

49. Line l in Figure 1.44 is parallel to the line $y = 2x + 1$. Find the coordinates of the point P.

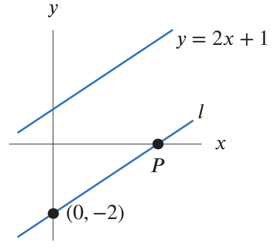


Figure 1.44

50. Line l is given by $y = 3 - \frac{2}{3}x$ and point P has coordinates $(6, 5)$.

(a) Find the equation of the line containing P and parallel to l.
(b) Find the equation of the line containing P and perpendicular to l.
(c) Graph the equations in parts (a) and (b).

51. An empty champagne bottle is tossed from a hot-air balloon. Its upward velocity is measured every second and recorded in Table 1.37.

(a) Describe the motion of the bottle in words. What do negative values of v represent?
(b) Find a formula for v in terms of t.
(c) Explain the physical significance of the slope of your formula.
(d) Explain the physical significance of the t-axis and v-axis intercepts.

Table 1.37

t (sec)	0	1	2	3	4	5
v (ft/sec)	40	8	−24	−56	−88	−120

52. Table 1.38 shows the pressure P, in torr, at a depth of h meters below a lake.

(a) Explain how you can tell that the relationship between P and h is linear, and find a formula for P as a function of h.
(b) Give practical interpretations of the slope and the vertical intercept of the function from part (a).
(c) At what depth is the pressure equal to twice the pressure at the surface of the lake?

Table 1.38

h (meters)	6	8	10	12	14
P (torr)	1201	1348	1495	1642	1789

53. The starting temperature of a 100-gram sample of cobalt metal is 20° Celsius. For each 10 calories of heat energy applied to the sample, its temperature increases by 1° Celsius.

(a) Fill in Table 1.39 with values for the temperature $T = f(h)$ of the sample, where h is the number of calories applied.
(b) Find a formula for $T = f(h)$.
(c) How many calories must be applied to the sample to raise its temperature to 95°C?

Table 1.39

h (calories)	0	20	40	60
T (°C)				

In Problems 54–56, use Table 1.40, which gives the cost, $C(n)$, of producing a good as a linear function of n, the number of units produced.

Table 1.40

n (units)	100	125	150	175
$C(n)$ (dollars)	11000	11125	11250	11375

54. Evaluate the following expressions. Give economic interpretations for each.

(a) $C(175)$ **(b)** $C(175) - C(150)$
(c) $\dfrac{C(175) - C(150)}{175 - 150}$

55. Estimate $C(0)$. What is the economic significance of this value?

56. The *fixed cost* of production is the cost incurred before any goods are produced. The *unit cost* is the cost of producing an additional unit. Find a formula for $C(n)$ in terms of n, given that

$$\text{Total cost} = \text{Fixed cost} + \text{Unit cost} \cdot \text{Number of units}$$

57. John wants to buy a dozen rolls. The local bakery sells sesame and poppy-seed rolls for the same price.

(a) Make a table of all the possible combinations of rolls if he buys a dozen, where s is the number of sesame seed rolls and p is the number of poppy-seed rolls.
(b) Find a formula for p as a function of s.
(c) Graph this function.

58. In a college meal plan you pay a membership fee; then all your meals are at a fixed price per meal.

(a) If 90 meals cost \$1005 and 140 meals cost \$1205, write a linear function that describes the cost of a meal plan, C, in terms of the number of meals, n.
(b) What is the cost per meal and what is the membership fee?
(c) Find the cost for 120 meals.
(d) Find n in terms of C.
(e) Use part (d) to determine the maximum number of meals you can buy on a budget of \$1285.

59. The solid waste generated each year in the cities of the US is increasing.[23] The solid waste generated, in millions of tons, was 88.1 in 1960 and 258.5 in 2014. The trend appears linear during this time.

(a) Construct a formula for the amount of municipal solid waste generated in the US by finding the equation of the line through these two points.
(b) Use this formula to predict the amount of municipal solid waste generated in the US, in millions of tons, in the year 2020.

60. The demand for gasoline can be modeled as a linear function of price. If the price of gasoline is $p = \$3.10$ per gallon, the quantity demanded in a fixed period is $q = 65$ gallons. If the price rises to \$3.50 per gallon, the quantity demanded falls to 45 gallons in that period.

(a) Find a formula for q in terms of p.
(b) Explain the economic significance of the slope of your formula.
(c) Explain the economic significance of the q-axis and p-axis intercepts.

61. Find a formula for the linear function $h(t)$ whose graph intersects the graph of $j(t) = 30(0.2)^t$ at $t = -2$ and $t = 1$.

[23] www.epa.gov/smm/advancing-sustainable-materials-management-facts-and-figures, accessed April 22, 2017.

62. Find the equation of line l in Figure 1.45.

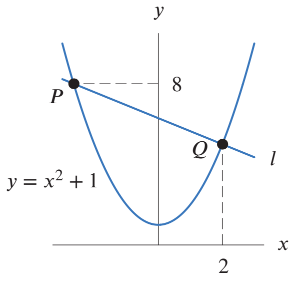


Figure 1.45

63. Find the equation of the line l, shown in Figure 1.46, if its slope is $m = 4$.

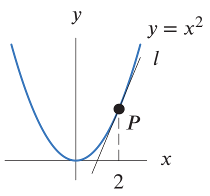


Figure 1.46

64. Find an equation for the line l in Figure 1.47 in terms of the constant A and values of the function f.

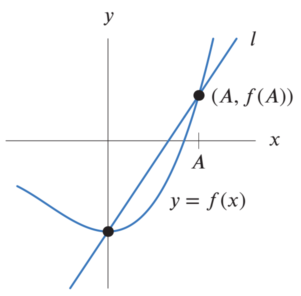


Figure 1.47

65. A dose-response function can be used to describe the increase in risk associated with the increase in exposure to various hazards. For example, the risk of contracting lung cancer depends, among other things, on the number of cigarettes a person smokes per day. This risk can be described by a linear dose-response function. For example, it is known that smoking 10 cigarettes per day increases a person's probability of contracting lung cancer by a factor of 25, while smoking 20 cigarettes a day increases the probability by a factor of 50.

(a) Find a formula for $i(x)$, the increase in the probability of contracting lung cancer for a person who smokes x cigarettes per day as compared to a nonsmoker.
(b) Evaluate $i(0)$.
(c) Interpret the slope of the function i.

66. Wire is sold by gauge size, where the diameter of the wire is a decreasing linear function of gauge. Gauge 2 wire has a diameter of 0.2656 inches and gauge 8 wire has a diameter of 0.1719 inches. Find the diameter for wires of gauge 12.5 and gauge 0. What values of the gauge do not make sense in this model?

67. You can type four pages in 50 minutes and nine pages in an hour and forty minutes.

(a) Find a linear function for the number of pages typed, p, as a function of time, t. If time is measured in minutes, what values of t make sense in this example?
(b) How many pages can be typed in two hours?
(c) Interpret the slope of the function in practical terms.
(d) Use the result in part (a) to solve for time as a function of the number of pages typed.
(e) How long does it take to type a 15-page paper?
(f) Write a short paragraph explaining why it is useful to know both of the formulas obtained in part (a) and part (d).

In Problems 68–69, write the functions in slope-intercept form. Identify the values of b and m.

68. $v(s) = \pi x^2 - 3xr - 4rs - s\sqrt{x}$

69. $w(r) = \pi x^2 - 3xr - 4rs - s\sqrt{x}$

70. The development time, t, of an organism is the number of days required for the organism to mature, and the development rate is defined as $r = 1/t$. In cold-blooded organisms such as insects, the development rate depends on temperature: the colder it is, the longer the organism takes to develop. For such organisms, the degree-day model[24] assumes that the development rate r is a linear function of temperature H (in °C):

$$r = b + kH.$$

(a) According to the degree-day model, there is a minimum temperature H_{min} below which an organism never matures. Find a formula for H_{min} in terms of the constants b and k.

[24] Information drawn from a web site created by Dr. Alexei A. Sharov at the Virginia Polytechnic Institute, http://www.ento.vt.edu/s̃harov/PopEcol/popecol.html, accessed April 27, 2017.

(b) Define S as $S = (H - H_{\text{min}})t$, where S is the number of degree-days. That is, S is the number of days t times the number of degrees between H and H_{min}. Use the formula for r to show that S is a constant. In other words, find a formula for S that does not involve H. Your formula will involve k.

(c) A certain organism requires $t = 25$ days to develop at a constant temperature of $H = 20$°C and has $H_{\text{min}} = 15$°C. Using the fact that S is a constant, how many days does it take for this organism to develop at a temperature of 25°C?

(d) In part (c) we assumed that the temperature H is constant throughout development. If the temperature varies from day to day, the number of degree-days can be accumulated until they total S, at which point the organism completes development. For instance, suppose on the first day the temperature is $H = 20$°C and that on the next day it is $H = 22$°C. Then for these first two days

Total number of degree days

$$= (20 - 15) \cdot 1 + (22 - 15) \cdot 1 = 12.$$

Based on Table 1.41, on what day does the organism reach maturity?

Table 1.41

Day	1	2	3	4	5	6	7	8	9	10	11	12
H (°C)	20	22	27	28	27	31	29	30	28	25	24	26

71. (Continuation of Problem 70.) Table 1.42 gives the development time t (in days) for an insect as a function of temperature H (in °C).

(a) Find a linear formula for r, the development rate, in terms of H.

(b) Find the value of S, the number of degree-days required for the organism to mature.

Table 1.42

H, °C	20	22	24	26	28	30
t, days	14.3	12.5	11.1	10.0	9.1	8.3

1.5 GRAPHS AND MODELS WITH LINEAR FUNCTIONS AND INEQUALITIES

The slope-intercept form for a linear function is $y = b + mx$, where b is the y-intercept and m is the slope. The parameters b and m can be used to compare linear functions.

Example 1 With time, t, in years, the populations of four towns, P_A, P_B, P_C and P_D, are given by the functions:

$$P_A = 20{,}000 + 1600t, \quad P_B = 50{,}000 - 300t, \quad P_C = 650t + 45{,}000, \quad P_D = 15{,}000(1.07)^t.$$

(a) Which populations are represented by linear functions?

(b) Describe in words what each linear model tells you about that town's population. Of the towns that grow linearly, which town starts out with the most people? Which is growing fastest?

Solution (a) The populations of towns A, B, and C are represented by linear functions because they are written in the form $P = b + mt$. Town D's population does not grow linearly since its formula, $P_D = 15{,}000(1.07)^t$, cannot be expressed in the form $P_D = b + mt$.

(b) For town A, we have

$$P_A = \underbrace{20{,}000}_{b} + \underbrace{1600}_{m} \cdot t,$$

so $b = 20{,}000$ and $m = 1600$. This means that in year $t = 0$, town A has 20,000 people. It grows by 1600 people per year.

For town B, we have

$$P_B = \underbrace{50{,}000}_{b} + \underbrace{(-300)}_{m} \cdot t,$$

so $b = 50{,}000$ and $m = -300$. This means that town B starts with 50,000 people. The negative slope indicates that the population is decreasing at the rate of 300 people per year.

For town C, we have

$$P_C = \underbrace{45{,}000}_{b} + \underbrace{650}_{m} \cdot t,$$

so $b = 45{,}000$ and $m = 650$. This means that town C begins with 45,000 people and grows by 650 people per year.

Town B starts out with the most people, 50,000, but town A, with a rate of change of 1600 people per year, grows the fastest of the three towns A, B, C.

Interpretation of the Parameters of a Line

Changing the values of b and m in the equation $y = b + mx$ gives different lines. We summarize the interpretation of the parameters b and m:

In the line with equation $y = b + mx$:

- The slope, m, is the change in y corresponding to an increase of 1 unit in x.
  - m is measured in units of y per unit of x
- The y-intercept, b, is the value of y when $x = 0$.
  - b is measured in units of y

Example 2 (a) Graph the three linear functions P_A, P_B, P_C from Example 1 and show how to identify the values of b and m from the graph.

(b) Graph P_D from Example 1 and explain how the graph shows P_D is not a linear function.

Solution (a) Figure 1.48 gives graphs of the three functions:

$$P_A = 20{,}000 + 1600t, \qquad P_B = 50{,}000 - 300t, \quad \text{and} \quad P_C = 45{,}000 + 650t.$$

The values of b identified in Example 1 tell us the vertical intercepts. Figure 1.48 shows that the graph of P_A crosses the P-axis at $P = 20{,}000$, the graph of P_B crosses at $P = 50{,}000$, and the graph of P_C crosses at $P = 45{,}000$.

Notice that the graphs of P_A and P_C are both climbing and that P_A climbs faster than P_C. This corresponds to the fact that the slopes of these two functions are positive ($m = 1600$ for P_A and $m = 650$ for P_C) and the slope of P_A is larger than the slope of P_C.

The graph of P_B falls when read from left to right, indicating that population decreases over time. This corresponds to the fact that the slope of P_C is negative ($m = -300$).

(b) Figure 1.49 gives a graph of P_D. Since it is not a line, P_D is not a linear function.

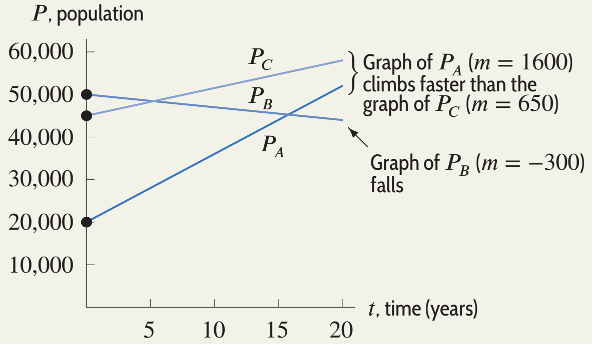


Figure 1.48: Graphs of three linear functions, P_A, P_B, and P_C, showing starting values and rates of climb

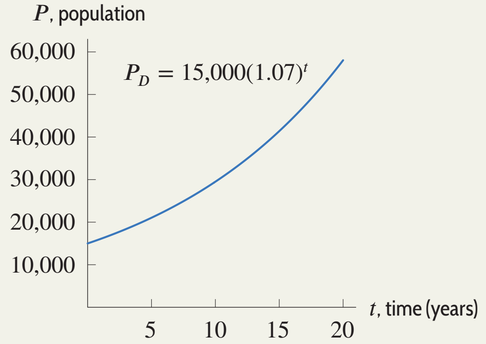


Figure 1.49: Graph of $P_D = 15{,}000(1.07)^t$ is not a line

Intersection of Two Lines

The point (x, y) at which two lines intersect satisfies the equations for both lines. Thus, to find this point, we solve the equations simultaneously.[25] When modeling real phenomena, the point of intersection often has a practical meaning.

Example 3 The cost, in dollars, of renting a car for a day from three different rental agencies and driving it d miles is given by the following functions:

$$C_1 = 0.50d, \qquad C_2 = 30 + 0.20d, \qquad C_3 = 50 + 0.10d.$$

(a) For each agency, describe the rental agreement in words and graph the cost function. Use one set of axes for all three functions.
(b) For what driving distance does Agency 1 cost the same as Agency 2? If you drive 50 miles or less, which agency is cheapest?
(c) How do you determine which agency provides the cheapest car rental for any given driving distance?

Solution (a) See Figure 1.50. Agency 1 charges \$0.50 per mile driven. Agency 2 charges \$30 plus \$0.20 per mile driven. Agency 3 charges \$50 plus \$0.10 per mile driven.

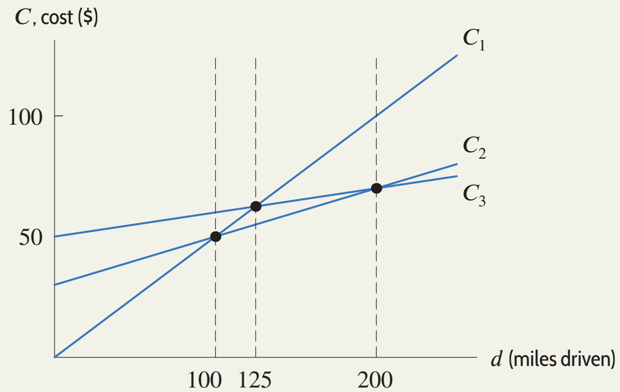


Figure 1.50: Cost of driving a car d miles when renting from three different agencies. The cheapest agency corresponds to the lowest graph for a given d value

(b) The costs for Agency 1 and Agency 2 are equal when the graphs of their cost functions intersect. To find the intersection point, we set the cost functions equal, $C_1 = C_2$, and solve for d:

$$\begin{aligned} 0.50d &= 30 + 0.20d \\ 0.50d - 0.20d &= 30 \\ 0.30d &= 30 \\ d &= 100. \end{aligned}$$

So, if we drive 100 miles, the rental cost is the same for Agencies 1 and 2, as Figure 1.50 shows.

Figure 1.50 also shows that the graph of C_1 is below the graph of C_2 for $d < 100$. This means that it is cheaper to rent from Agency 1 than from Agency 2 if we drive less than 100 miles. For $d < 100$, the graph of C_1 is also below the graph of C_3, so renting from Agency 1 is the cheapest option if we drive 50 miles or less.

(c) Which agency is cheapest depends on how far we drive. If we do not drive far, Agency 1 is cheapest because it charges only for miles driven and has no other fees. If we drive a long way, Agency 3 is cheapest (even though it charges \$50 up front) because it has the lowest per-mile rate.

[25] The algebra in this section is reviewed in the Skills Refresher on the book companion site at www.wiley.com/college/hughes-hallett.

Figure 1.50 and part (b) show that for d up to 100 miles, Agency 1 costs the least.

For d between 100 and about 200 miles, Figure 1.50 suggests the values of C_2 are less than the values of C_1 and C_3. To find the exact value of d where C_2 and C_3 intersect, we solve the equation $C_2 = C_3$:

$$\begin{aligned} 30 + 0.20d &= 50 + 0.10d \\ 0.10d &= 20 \\ d &= 200. \end{aligned}$$

Thus Agency 2 is cheapest when we drive between 100 and 200 miles.

For d more than 200 miles, the values of C_3 are less than the values of C_1 and C_2, meaning that, as expected, Agency 3 is cheapest for long distances.

Notice that the point where C_1 and C_3 intersect is not important for deciding which agency is the cheapest: C_2 still has the lowest cost value where C_1 and C_3 intersect.

Linear Inequalities

We now see how to solve problems using linear inequalities rather than linear equations.[26]

Example 4 The temperature at the base of a mountain is 90°F and drops by 5°F for every thousand-foot increase in elevation. On what part of the mountain is the temperature cooler than 40°F?

Solution Since the temperature, T, drops at a constant rate as the elevation, h, increases, if T is in degrees Fahrenheit and h is in thousands of feet from the base of the mountain, we have

$$T = 90 - 5h.$$

We want to know for what values of h we have $90 - 5h < 40$.

One way to solve the inequality is to subtract 40 from each side of the inequality to obtain

$$50 - 5h < 0,$$

and then add $5h$ to each side to get

$$50 < 5h.$$

Dividing each side by 5 gives

$$10 < h.$$

This means that the temperature is below 40° at points more than 10,000 feet above the base of the mountain.

A second approach is to subtract 90 from each side of $90 - 5h < 40$ to get

$$-5h < -50.$$

At this point, we divide each side of the inequality by -5. Since -5 is negative, this division switches the direction of the inequality, giving

$$h > 10.$$

(Note that if we forget to switch the direction of the inequality, we get the incorrect solution $h < 10$, suggesting that the temperature is cooler than 40°F below the 10,000-foot line, rather than above it. However, the temperature is cooler near the top of the mountain, not near the bottom.)

[26]Methods for solving linear inequalities are included in the Skills Refresher on the book companion site at www.wiley.com/college/hughes-hallett.

Example 5 When people consume more calories than required, they gain weight; when they consume fewer calories than required, they lose weight. For an active 20-year-old, six-foot-tall man weighing w pounds, the Harris-Benedict equation[27] states that the number of calories, C, needed daily to maintain weight is $C = 1321 + 9.44w$.

(a) The points satisfying $C = 1321 + 9.44w$ lie on a line. If the w-axis is horizontal, the points satisfying
$$C > 1321 + 9.44w \qquad \text{or} \qquad C < 1321 + 9.44w$$
lie above or below the line. Which of these two inequalities does the point $w = 170$, $C = 3500$ satisfy? Is the point above or below the line? How about the point $w = 200$, $C = 3000$?

(b) Explain what your answers to part (a) mean in the context of weight and caloric intake.

(c) Graph the set of all points that satisfy the inequality $C < 1321 + 9.44w$. What does this region represent in terms of calories and weight?

(d) Sketch the region where $C \leq 3000$ and $C > 1321 + 9.44w$. What does this region represent in terms of calories and weight?

Solution (a) To determine which inequality is satisfied, we substitute the value of w into the expression $1321 + 9.44w$ and compare the result with the value of C.

If $w = 170$, $C = 3500$, substituting w gives $1321 + 9.44 \cdot 170 = 2925.8$. Since $C > 2925.8$, we have
$$C > 1321 + 9.44w. \qquad \text{Thus, the point } (170, 3500) \text{ is above the line.}$$

If $w = 200$, $C = 3000$, substituting w gives $1321 + 9.44 \cdot 200 = 3209$. Since $C < 3209$, we have
$$C < 1321 + 9.44w. \qquad \text{Thus, the point } (200, 3000) \text{ is below the line.}$$

(b) A man weighing 170 pounds who eats 3500 calories per day is eating more calories than needed, so he will gain weight. On the other hand, a man weighing 200 pounds who eats 3000 calories daily will lose weight, because he is eating fewer calories than needed.

(c) Figure 1.51 shows the line $C = 1321 + 9.44w$, the point $(170, 3500)$ above the line and the point $(200, 3000)$ below the line.

The points satisfying $C < 1321 + 9.44w$ are those whose C-coordinates are lower than $1321 + 9.44w$; that is, all the points below the line. The region represents all of the combinations of weights and caloric intakes at which an active, 6-foot-tall, 20-year-old man loses weight.

Note that the line itself is not included in the region, since on this line C is equal to, not less than, $1321 + 9.44w$. We use a dashed line to indicate this.

(d) The region, shaded in Figure 1.52, shows the combinations of weights and calories—up to 3000 calories—for which an active 6-foot, 20-year-old man gains weight.

[27] A.M. Roza and H.M. Shizgal, "The Harris-Benedict Equation Reevaluated", *American Journal of Clinical Nutrition* 40, No. 1 (1984), 168–182.

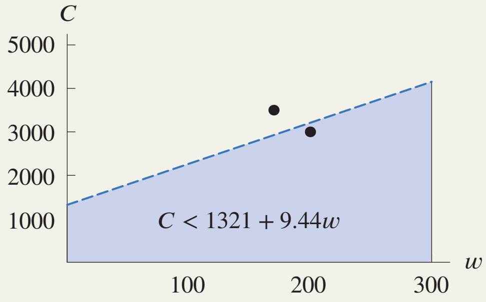


Figure 1.51: Shaded region is solution to $C < 1321 + 9.44w$

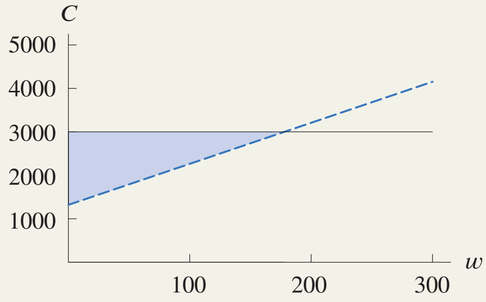


Figure 1.52: Shaded region is $C \leq 3000$ and $C > 1321 + 9.44w$

Summary for Section 1.5

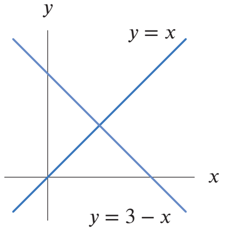
- The interpretation of the parameters of a linear function $y = b + mx$
  - **Slope, m**: The change in y corresponding to an increase of 1 unit in x
  - **Vertical intercept, b**: Value of y when $x = 0$; initial value of y; starting value of y
- **Intersection of lines**: Coordinates of the point satisfying the simultaneous equations.
- **Linear inequalities**: Interpreting and graphing as regions in the plane.

Exercises and Problems for Section 1.5

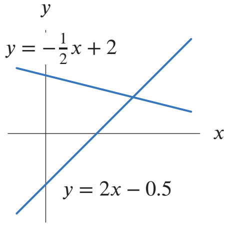
Skill Refresher

In Exercises **S1–S2**, determine the point of intersection.

S1.



S2.



In Exercises **S3–S8**, solve the equations simultaneously if possible.

S3. $\begin{cases} x + y = 3 \\ y = 5 \end{cases}$

S4. $\begin{cases} x + y = 3 \\ x - y = 5 \end{cases}$

S5. $\begin{cases} x + y = 2 \\ 2x + 2y = 7 \end{cases}$

S6. $\begin{cases} y = x - 3 \\ 2y - 2x = -6 \end{cases}$

S7. $\begin{cases} 2x - y = 10 \\ x + 2y = 15 \end{cases}$

S8. $\begin{cases} 2(x + y) = 3 \\ x = y + 3(x - 5) \end{cases}$

In Exercises **S9–S13**, find all values of x which satisfy the given inequality.

S9. $5x + 1 > 2x + 10$

S10. $6x - 3 \leq 3x + 2$

S11. $4x + 1 \geq 2x - 8$

S12. $x - 5 > 4x - 14$

S13. $2x + 5 < 7x - 3$

EXERCISES

1. Figure 1.53 gives lines A, B, C, D, and E. Without a calculator, match each line to f, g, h, u or v:

$f(x) = 20 + 2x$
$g(x) = 20 + 4x$
$h(x) = 2x - 30$
$u(x) = 60 - x$
$v(x) = 60 - 2x$

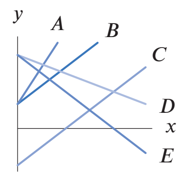


Figure 1.53

2. Without a calculator, match the following functions to the lines in Figure 1.54:

$f(x) = 5 + 2x$
$g(x) = -5 + 2x$
$h(x) = 5 + 3x$
$j(x) = 5 - 2x$
$k(x) = 5 - 3x$

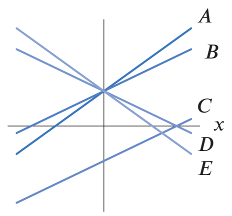


Figure 1.54

In Exercises 3–5, give the units of the slope of the linear function.

3. The amount, A, of water flowing over a dam, in gallons, is a linear function of time, t, in minutes.

4. The distance, D, of an object from a point, in feet, is a linear function of time, t, in seconds.

5. The cost, C (in dollars), to heat a house is a linear function of the number of days, d.

In Exercises 6–9, determine the coordinates of the point of intersection of the two lines.

6. $\begin{cases} y = 5 + x \\ y = 9 - 3x \end{cases}$

7. $\begin{cases} y = 2 - 5x \\ y = 8 - 2x \end{cases}$

8. $\begin{cases} y = 7x + 1 \\ y = 2x + 9 \end{cases}$

9. $\begin{cases} y = 2x + 10 \\ y = -3x - 7 \end{cases}$

In Exercises 10–13, write a formula for V, the volume in gallons of water in a barrel, as a function of time, t in hours from now.

10. There are 30 gallons in the barrel now and the volume is increasing at 5 gallons per hour.

11. There are 45 gallons in the barrel now and the volume is decreasing by 3 gallons per hour.

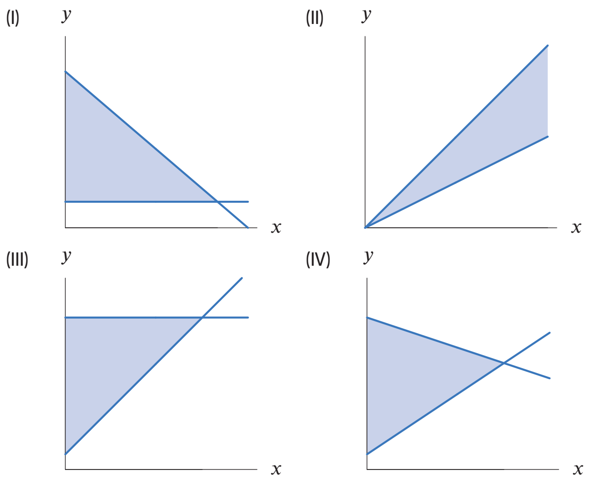
12. The volume is going down by 8 gallons per hour, and there are 32 gallons in the barrel now.

13. The volume is going up by 10 gallons per hour, and there are 4 gallons in the barrel now.

14. Without a calculator, match the systems of inequalities (a)–(d) to the graphs (I)–(IV). Give reasons.

(a) $y \geq x$ and $y \leq 2x$ **(b)** $y \geq 1+2x$ and $y \leq 10-x$

(c) $y \geq 2$ and $y \leq 12-2x$ **(d)** $y \geq 3x+1$ and $y \leq 10$



PROBLEMS

15. Sketch five different functions in the family $y = -2-ax$ for $a < 0$.

16. Estimate the slope of the line in Figure 1.55 and find an approximate equation for the line.

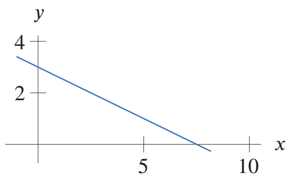


Figure 1.55

17. The cost of a Frigbox refrigerator is \$950, and it depreciates \$50 each year. The cost of an Arctic Air refrigerator is \$1200, and it depreciates \$100 per year.

(a) If a Frigbox and an Arctic Air are bought at the same time, when do the two refrigerators have equal value?

(b) If both refrigerators continue to depreciate at the same rates, what happens to the values of the refrigerators in 20 years' time? What does this mean?

18. You need to rent a car and compare the charges of three different companies. Company A charges 20 cents per mile plus \$20 per day. Company B charges 10 cents per mile plus \$35 per day. Company C charges \$70 per day with no mileage charge.

(a) Find formulas for the cost of driving cars rented from companies A, B, and C, in terms of x, the distance driven in miles in one day.

(b) Graph the costs for each company for $0 \leq x \leq 500$. Put all three graphs on the same set of axes.

(c) What do the slope and the vertical intercept tell you in this situation?

(d) Use the graph in part (b) to find under what circumstances company A is the cheapest. What about Company B? Company C? Explain why your results make sense.

Solve Problems 19–22 by setting up a linear inequality.

19. A dog requires 1300 calories per day. A cup of dry dog food has 350 calories. The dog has already eaten a can of wet food, containing 400 calories. How many cups of dry food, n, can it eat before exceeding its daily caloric limit?

20. Blood alcohol concentration (BAC) is often used as a measure of intoxication.[28] For a 180-pound man, the BAC increases linearly by 0.02% for every drink taken in a one-hour period. If at the start of the hour he has a BAC of $B = 0.01\%$, how many drinks, n, can he have during the hour while remaining below the legal limit of 0.08%?

21. For a 180-pound man, blood alcohol concentration (BAC) decreases linearly by 0.015% every hour if he does not consume any further drinks.[29] If at the start of the hour he has a BAC of $B = 0.13\%$, how many hours, t, must he wait before his BAC falls below the legal limit of 0.08%?

22. A car's fuel economy is 30 mpg. A warning light is displayed if the remaining distance that can be driven before the car runs out of gas drops below 50 miles. Initially the car has 10 gallons of gas. How many gallons, g, can be used before the warning light comes on?

23. Broadway Cabs and City Cars are two competing taxi companies in a small town. The cost in dollars of a cab ride of d miles is $C_1 = 2.5 + 2.6d$ for Broadway Cabs and $C_2 = 4 + 2.45d$ for City Cars.

(a) Describe in words the fare breakdown for each company.
(b) Determine the conditions under which Broadway Cabs is cheaper than City Cars.
(c) You need a ride to the airport, which is 12 miles from your house. To get the lowest fare, which company should you call and how much will it cost?

24. Assume A, B, C are constants with $A \neq 0$, $B \neq 0$. Consider the equation

$$Ax + By = C.$$

(a) Show that $y = f(x)$ is linear. State the slope and the x- and y-intercepts of $f(x)$.
(b) Graph $y = f(x)$, labeling the x- and y-intercepts in terms of A, B, and C, assuming

(i) $A > 0, B > 0, C > 0$
(ii) $A > 0, B > 0, C < 0$
(iii) $A > 0, B < 0, C > 0$

25. Fill in the missing coordinates for the points in the following figures.

(a) The triangle in Figure 1.56.
(b) The parallelogram in Figure 1.57.

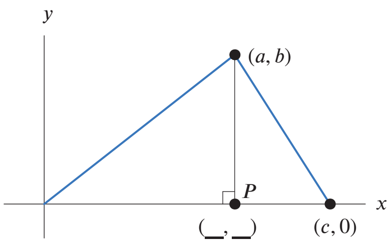


Figure 1.56

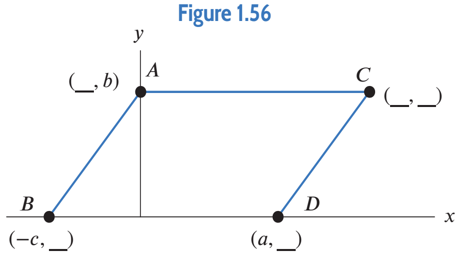


Figure 1.57

26. Fill in the missing coordinates in Figure 1.58. Write an equation for the line connecting the two points.

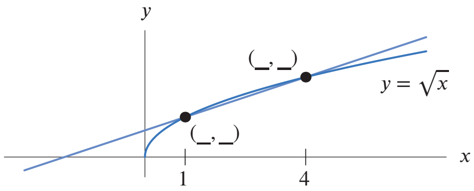


Figure 1.58

27. Using the window $-10 \le x \le 10, -10 \le y \le 10$, graph $y = x$, $y = 10x$, $y = 100x$, and $y = 1000x$.

(a) Explain what happens to the graphs of the lines as the slopes become large.
(b) Write an equation of a line that passes through the origin and is horizontal.

28. Graph $y = x + 1$, $y = x + 10$, and $y = x + 100$ in the window $-10 \le x \le 10, -10 \le y \le 10$.

(a) Explain what happens to the graph of a line, $y = b + mx$, as b becomes large.
(b) Write a linear equation whose graph cannot be seen in the window $-10 \le x \le 10, -10 \le y \le 10$ because all its y-values are less than the y-values shown.

29. The graphical interpretation of the slope is that it shows steepness. Using a calculator or a computer, graph the function $y = 2x - 3$ in the following windows:

(a) $-10 \le x \le 10$ by $-10 \le y \le 10$

[28] http://en.wikipedia.org/wiki/Blood_alcohol_content#cite_ref-4, accessed April 22, 2017.
[29] http://en.wikipedia.org/wiki/Blood_alcohol_content#cite_ref-4, accessed April 22, 2017.

(b) $-10 \le x \le 10$ by $-100 \le y \le 100$
(c) $-10 \le x \le 10$ by $-1000 \le y \le 1000$
(d) Write a sentence about how steepness is related to the window being used.

30. A circle of radius 2 is centered at the origin and goes through the point $(-1, \sqrt{3})$.

(a) Find an equation for the line through the origin and the point $(-1, \sqrt{3})$.
(b) Find an equation for the tangent line to the circle at $(-1, \sqrt{3})$. [Hint: A tangent line is perpendicular to the radius at the point of tangency.]

31. Find an equation for the altitude through point A of the triangle ABC, where A is $(-4, 5)$, B is $(-3, 2)$, and C is $(9, 8)$. [Hint: The altitude of a triangle is perpendicular to the base.]

32. Two lines are given by $y = b_1 + m_1 x$ and $y = b_2 + m_2 x$, where b_1, b_2, m_1, and m_2 are constants.

(a) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have no points in common?
(b) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have all points in common?
(c) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have exactly one point in common?
(d) What conditions are imposed on b_1, b_2, m_1, and m_2 if the two lines have exactly two points in common?

In Problems 33–34, what is true about the constant β in the following linear equation if its graph has the given property?

$$y = \frac{x}{\beta - 3} + \frac{1}{6 - \beta}.$$

33. Positive slope, positive y-intercept.

34. Perpendicular to the line $y = (\beta - 7)x - 3$.

35. A theater manager graphed weekly profits as a function of the number of patrons and found that the relationship was linear. One week the profit was \$11,328 when 1324 patrons attended. Another week 1529 patrons produced a profit of \$13,275.50.

(a) Find a formula for weekly profit, y, as a function of the number of patrons, x.
(b) Interpret the slope and the y-intercept.
(c) What is the break-even point (the number of patrons for which there is zero profit)?
(d) Find a formula for the number of patrons as a function of profit.
(e) If the weekly profit was \$17,759.50, how many patrons attended the theater?

36. In economics, the *demand* for a product is the amount of that product that consumers are willing to buy at a given price. The quantity demanded of a product usually decreases if the price of that product increases. Suppose that a company believes there is a linear relationship between the demand for its product and its price. The company knows that when the price of its product was \$3 per unit, the quantity demanded weekly was 500 units, and that when the unit price was raised to \$4, the quantity demanded weekly dropped to 300 units. Let D represent the quantity demanded weekly at a unit price of p dollars.

(a) Calculate D when $p = 5$. Interpret your result.
(b) Find a formula for D in terms of p.
(c) The company raises the price of the good and the new quantity demanded weekly is 50 units. What is the new price?
(d) Give an economic interpretation of the slope of the function you found in part (b).
(e) Find D when $p = 0$. Find p when $D = 0$. Give economic interpretations of both these results.

37. In economics, the *supply* of a product is the quantity of that product suppliers are willing to provide at a given price. In theory, the quantity supplied of a product increases if the price of that product increases. Suppose that there is a linear relationship between the quantity supplied, S, of the product described in Problem 36 and its price, p. The quantity supplied weekly is 100 when the price is \$2 and the quantity supplied rises by 50 units when the price rises by \$0.50.

(a) Find a formula for S in terms of p.
(b) Interpret the slope of your formula in economic terms.
(c) Is there a price below which suppliers will not provide this product?
(d) The *market clearing price* is the price at which supply equals demand. According to theory, the free-market price of a product is its market clearing price. Using the demand function from Problem 36, find the market clearing price for this product.

38. When economists graph demand or supply equations, they place quantity on the horizontal axis and price on the vertical axis.

(a) On the same set of axes, graph the demand and supply equations you found in Problems 36 and 37, with price on the vertical axis.
(b) Indicate how you could estimate the market clearing price from your graph.

39. The cholesterol ratio[30] is used by many doctors to measure the quantity of the total cholesterol T in the body,

[30] http://www.heart.org/HEARTORG/Conditions/Cholesterol/AboutCholesterol/What-Your-Cholesterol-Levels-Mean_UCM_305562_Article.jsp, accessed April 22, 2017.

measured in milligrams per deciliter of blood (mg/dl), relative to the quantity of good cholesterol G, also measured in mg/dl. In order to minimize risk of cholesterol-related illnesses, the good cholesterol level should always exceed $T/5$ but never be less than $G = 40$.

(a) Which of the following pairs of T and G values satisfy the inequality $G > T/5$?

(i) $T = 100, G = 38$ (ii) $T = 220, G = 42$
(iii) $T = 240, G = 60$

(b) Which of the pairs in part (a) satisfy the inequality $G > 40$?

(c) What do your answers to parts (a) and (b) mean in terms of cholesterol levels?

(d) With T on the horizontal axis, graph all the points that satisfy both the inequalities $G > T/5$ and $G > 40$. What do the points in this region represent?

1.6 FITTING LINEAR FUNCTIONS TO DATA

When data are collected in the laboratory or the field, they are often subject to experimental error. Even if there is an underlying linear relationship between two quantities, real data may not fit this relationship perfectly. However, we may still be able to use a linear function to analyze the data.

Laboratory Data: The Viscosity of Motor Oil

The viscosity of a liquid, or its resistance to flow, depends on the liquid's temperature. Pancake syrup is a familiar example: straight from the refrigerator, it pours very slowly. When warmed on the stove, its viscosity decreases and it becomes quite runny.

The viscosity of motor oil is a measure of its effectiveness as a lubricant in the engine of a car. Thus, the effect of engine temperature is an important determinant of motor-oil performance. Table 1.43 gives the viscosity, v, of motor oil as measured in the lab at different temperatures, T.

Table 1.43 *The measured viscosity, v, of motor oil as a function of the temperature, T*

T, temperature (°F)	v, viscosity (lbs·sec/in^2)
160	28
170	26
180	24
190	21
200	16
210	13
220	11
230	9

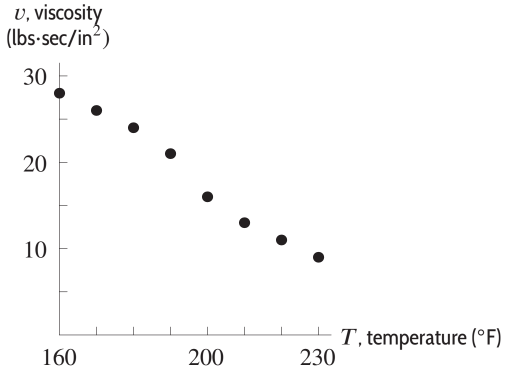


Figure 1.59: The viscosity data from Table 1.43

The *scatter plot* of the data in Figure 1.59 shows that the viscosity of motor oil decreases, approximately linearly, as its temperature rises. To find a formula relating viscosity and temperature, we fit a line to these data points.

Fitting the best line to a set of data is called *linear regression*. One way to fit a line is to draw a line "by eye." Alternatively, many computer programs and calculators compute regression lines. Figure 1.60 shows the data from Table 1.43 together with the computed regression line,

$$v = 75.6 - 0.293T.$$

Notice that the data points do not lie exactly on the regression line, although it fits the data well.

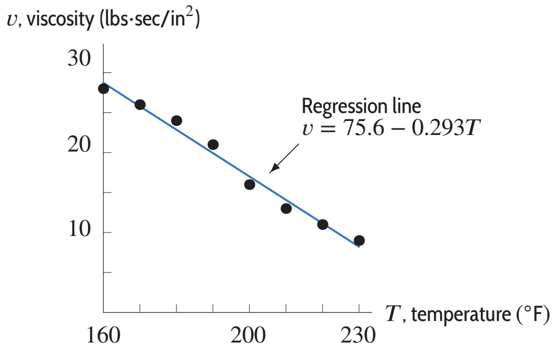


Figure 1.60: A graph of the viscosity data from Table 1.43, together with a regression line (provided by a calculator)

Interpolation and Extrapolation

The formula for viscosity can be used to make predictions. Suppose we want to know the viscosity of motor oil at $T = 196$°F. The formula gives

$$v = 75.6 - 0.293 \cdot 196 \approx 18.2 \text{ lb} \cdot \text{sec/in}^2.$$

To see that this is a reasonable estimate, compare with Table 1.43. At 190°F, the measured viscosity was 21, and at 200°F, it was 16; the predicted viscosity of 18.2 is between 16 and 21. See Figure 1.61. Of course, if we measured the viscosity at $T = 196$°F in the lab, we might not get exactly 18.2.

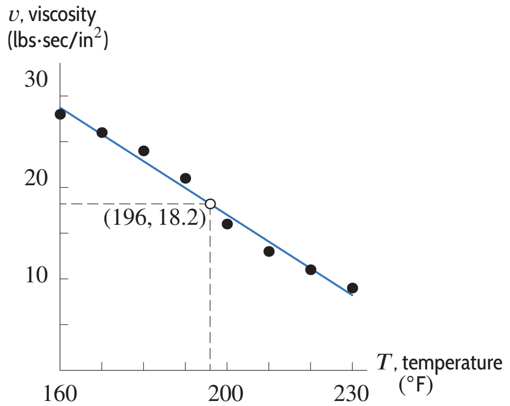


Figure 1.61: Regression line used to predict the viscosity at 196°F

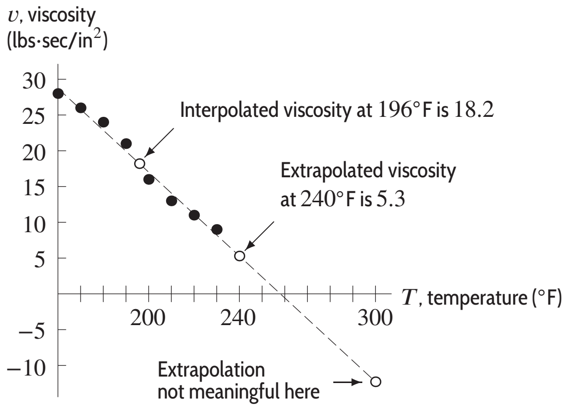


Figure 1.62: The data from Table 1.43 together with the predicted viscosity at $T = 196$°F, $T = 240$°F, and $T = 300$°F

Since the temperature $T = 196$°F is between two temperatures for which v is known (190°F and 200°F), the estimate of 18.2 is said to be an *interpolation*. If instead we estimate the value of v at a temperature outside the values for T in Table 1.43, our estimate is called an *extrapolation*.

Example 1 Predict the viscosity of motor oil at 240°F and at 300°F.

Solution At $T = 240$°F, the formula for the regression line predicts that the viscosity of motor oil is

$$v = 75.6 - 0.293 \cdot 240 = 5.3 \text{ lb} \cdot \text{sec/in}^2.$$

This is reasonable. Figure 1.62 shows that the predicted point—represented by an open circle on the graph—is consistent with the trend in the data points from Table 1.43.

On the other hand, at $T = 300$°F the regression-line formula gives

$$v = 75.6 - 0.293 \cdot 300 = -12.3 \text{ lb} \cdot \text{sec/in}^2.$$

This is unreasonable because viscosity cannot be negative. To understand what went wrong, notice that in Figure 1.62, the open circle representing the point (300, −12.3) is far from the plotted data points. By making a prediction at 300°F, we have assumed—incorrectly—that the trend observed in laboratory data extended as far as 300°F.

In general, interpolation tends to be more reliable than extrapolation because we are making a prediction on an interval we already know something about instead of making a prediction beyond the limits of our knowledge.

How Regression Works

How does a calculator or computer decide which line fits the data best? We assume that the value of y is related to the value of x, although other factors could influence y as well. Thus, we assume that we can pick the value of x exactly but that the value of y may be only partially determined by this x-value.

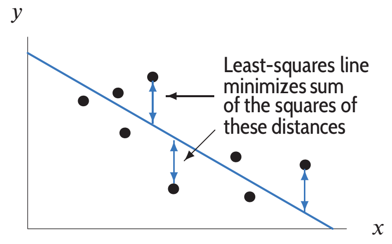


Figure 1.63: A given set of data and the corresponding least-squares regression line

One way to fit a line to the data is shown in Figure 1.63. The line shown was chosen to minimize the sum of the squares of the vertical distances between the data points and the line. Such a line is called a *least-squares line*. There are formulas which a calculator or computer uses to calculate the slope, m, and the y-intercept, b, of the least-squares line.

Correlation

When a computer or calculator calculates a regression line, it also gives a *correlation coefficient*, r. This number lies between -1 and $+1$ and measures how well a particular regression line fits the data. If $r = 1$, the data lie exactly on a line of positive slope. If $r = -1$, the data lie exactly on a line of negative slope. If r is close to 0, the data may be completely scattered, or there may be a nonlinear relationship between the variables. (See Figure 1.64.)

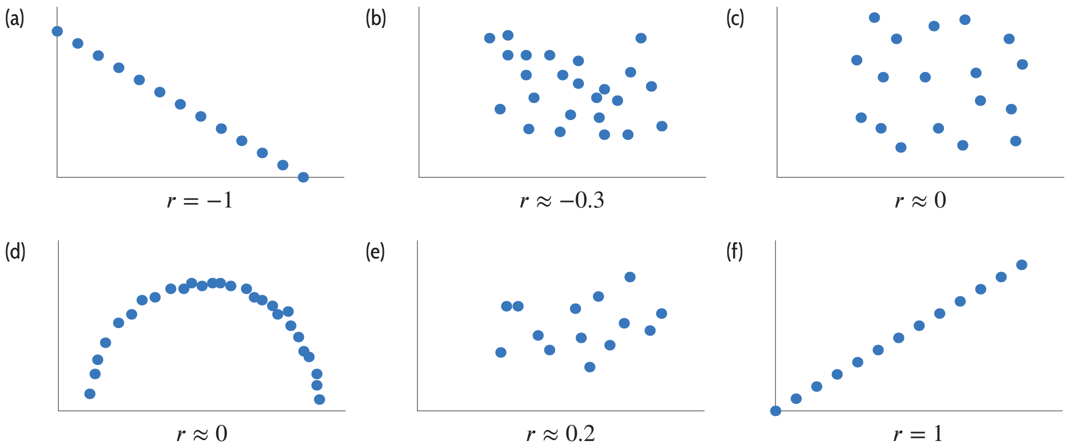


Figure 1.64: Various data sets and correlation coefficients

Example 2 The correlation coefficient for the viscosity data in Table 1.43 on page 49 is $r \approx -0.99$. The fact that r is negative tells us that the regression line has negative slope. The fact that r is close to -1 tells us that the regression line fits the data well.

The Differences Between Relation, Correlation, and Causation

It is important to understand that a high correlation (either positive or negative) between two quantities does *not* imply causation. For example, there is a high correlation between children's reading level and shoe size.[31] However, large feet do not cause a child to read better (or vice versa). Larger feet and improved reading ability are both a consequence of growing older.

Notice also that a correlation of 0 does not imply that there is no relationship between x and y. For example, in Figure 1.64(d) there is a relationship between x and y-values, while Figure 1.64(c) exhibits no apparent relationship. Both data sets have a correlation coefficient of $r \approx 0$. Thus a correlation of $r = 0$ usually implies there is no linear relationship between x and y, but this does not mean there is no relationship at all.

Summary for Section 1.6

Fitting Lines to Data

- **Regression Line**: Line that is the best fit to the data found using technology
- **Prediction**: Using the least-squares line, we can predict values of the dependent variable.
- **Correlation coefficient**, r: Measures closeness of fit of the line to the data
    - $-1 \le r \le 1$
    - If $r = \pm 1$, data is exactly on a line. Value of r near 0 means data is not well modeled by a line.
    - Sign of r is sign of slope.
- **Interpolation**: Estimating y-values between known x-values.
- **Extrapolation**: Estimating y-values outside known x-values. May lead to inaccurate estimates.

Exercises and Problems for Section 1.6

Skill Refresher

In Exercises **S1–S3**, to plot the data described, draw the axes you would use and label them with the dependent variable on the vertical axis.

S1. Your wages each year, which are based on the number of hours you worked in the year.

S2. The deer population on different dates.

S3. The distance from a park determines the average price per square foot of houses.

In Exercises **S4–S7**, for the linear relationship $y = a + bx$,

(a) What do x and y represent?

(b) What is the value of a and what are the units of a?

(c) What is the value of b and what are the units of b?

S4. The cost of a house in thousands of dollars in a town depends on the size of the house, in ft^2, and is given by $y = 125 + 0.15x$.

S5. The value of a laptop, in dollars, depends on the age of the laptop, in years, and is given by $y = 750 - 150x$.

S6. The average temperature, in degrees Fahrenheit, depends on the height, in feet, above sea level and is given by $y = 59 - 0.00356x$.

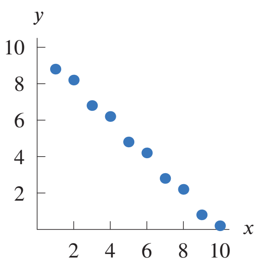
S7. The Body Mass Index, in kilograms per meter squared (kg/m^2), of a man of height 1.8 meters depends on the man's weight, in kilograms, and is given by $y = \dfrac{x}{3.24}$.

[31] D. Freedman, R. Pisani, R. Purves, A. Adhikari, *Statistics*, 2nd edition (New York: W.W. Norton, 1991), p. 142.

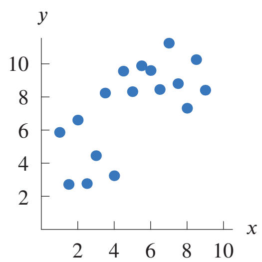
EXERCISES

In Exercises 1–6, is the given value of r reasonable? Give a reason for your answer.

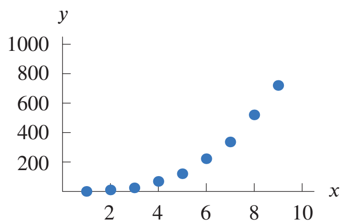
1. $r = 1$



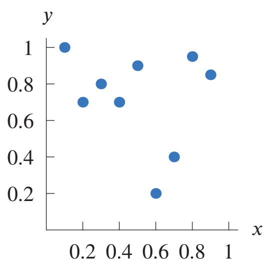
2. $r = 0.7$



3. $r = 0.9$



4. $r = -0.9$



5. $r = 0.9$

x	1	2	3	4	5
y	3.8	3.2	1.8	1.2	−0.2

6. $r = 0.3$

x	1	2	3	4	5
y	3.477	5.531	14.88	5.924	8.049

7. Match the r values with scatter plots in Figure 1.65.

$$r = -0.98, \quad r = -0.5, \quad r = -0.25,$$

$$r = 0, \quad r = 0.7, \quad r = 1.$$

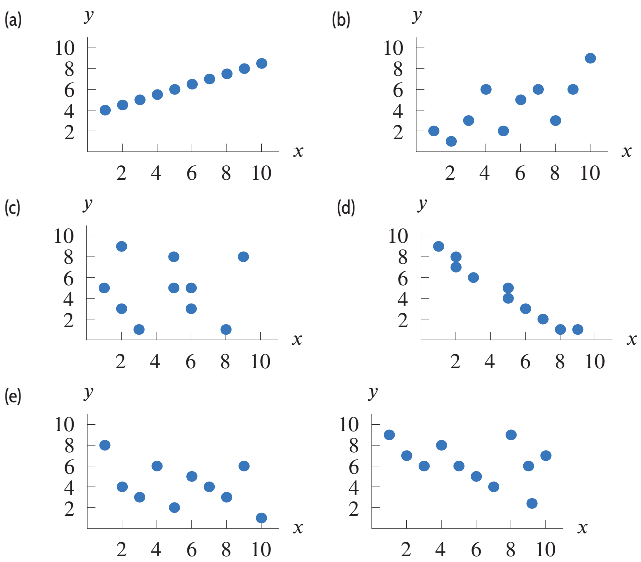


Figure 1.65

8. Table 1.44 shows the number of calories burned per minute by a person walking at 3 mph.

(a) Make a scatter plot of this data.
(b) Draw a regression line by eye.
(c) Roughly estimate the correlation coefficient by eye.

Table 1.44

Body weight (lb)	100	120	150	170	200	220
Calories	2.7	3.2	4.0	4.6	5.4	5.9

PROBLEMS

9. The manager of a small nature park found the regression line for the weekly profits, p in dollars, as a function of the number of visitors, n, was $p = 4n - 2200$.

(a) How much profit is predicted if 700 people visit the park in a week?
(b) Explain the meaning of the slope of the regression line in the context of this problem. Include units.
(c) Explain the meaning of the vertical intercept of the regression line in the context of this problem. Include units.
(d) How many weekly visitors does the park need if it is not to lose money?

10. An ecologist tracked 290 topi that were born in 2004. The number of topi, T, living each subsequent year is recorded in Table 1.45.

(a) Make a scatter plot of this data. Let $t = 0$ represent 2004.
(b) Draw by eye a line of good fit and estimate its equation. (Round the coefficients to integers.)
(c) Use a calculator or computer to find the equation of the least squares line. (Round the coefficients to integers.)
(d) Interpret the slope and each intercept of the line.
(e) Interpret the correlation between the year and the number of topi born in 2004 that are still alive.

Table 1.45

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012
Topi	290	288	268	206	140	90	64	44	8

11. Students in a chemistry lab are measuring the volume and mass of samples of distilled water and salt water. The results are in Tables 1.46 and 1.47.

(a) Make scatter plots of the data, with volume on the horizontal axis.
(b) Find the lines of best fit for each data set and graph them together with the scatter plot.
(c) What are the units of the slopes of the lines? Interpret the slopes in the context of the problem.
(d) What do the slopes tell you about the difference between distilled water and salt water?

Table 1.46 *Distilled water*

Volume (ml)	20	40	60	80	100
Mass (grams)	19	41	58	82	97

Table 1.47 *Salt water*

Volume (ml)	20	40	60	80	100
Mass (grams)	22	44.5	63	89	109

12. The rate of oxygen consumption, in microliters per gram per hour, for Colorado potato beetles increases with temperature.[32] See Table 1.48.

(a) Make a scatter plot of this data.
(b) Draw an estimated regression line by eye.
(c) Use a calculator or computer to find the equation of the regression line. Round constants in the equation to the nearest integer.
(d) Interpret the slope and each intercept of the regression equation.
(e) Interpret the correlation between temperature and oxygen rate.

Table 1.48

°C	10	15	20	25	30
Oxygen consumption rate	90	125	200	300	375

13. Table 1.49 gives the data on hand strength collected from college freshman using a grip meter.

(a) Make a scatter plot of these data treating the strength of the preferred hand as the independent variable.
(b) Draw a line on your scatter plot that is a good fit for these data and use it to find an approximate equation for the regression line.
(c) Using a graphing calculator or computer, find the equation of the least squares line.
(d) What would the predicted grip strength in the nonpreferred hand be for a student with a preferred hand strength of 37?
(e) Discuss interpolation and extrapolation using specific examples in relation to this regression line.
(f) Discuss why r, the correlation coefficient, is both positive and close to 1.
(g) Why do the points tend to cluster into two groups on your scatter plot?

Table 1.49 *Hand strength for 20 students in kilograms*

Preferred	28	27	45	20	40	47	28	54	52	21
Nonpreferred	24	26	43	22	40	45	26	46	46	22
Preferred	53	52	49	45	39	26	25	32	30	32
Nonpreferred	47	47	41	44	33	20	27	30	29	29

[32] Adapted from R. E. Rickefs and G. L. Miller, *Ecology,* 4th ed. (New York: W. H. Freeman, 1999).

14. Table 1.50 shows men's and women's world records for swimming distances from 50 meters to 1500 meters.[33]

(a) What values would you add to Table 1.50 to represent the time taken by both men and women to swim 0 meters?

(b) Plot men's time against distance, with time t in seconds on the vertical axis and distance d in meters on the horizontal axis. It is claimed that a straight line models this behavior well. What is the equation for that line? What does its slope represent? On the same graph, plot women's time against distance and find the equation of the straight line that models this behavior well. Is this line steeper or flatter than the men's line? What does that mean in terms of swimming? What are the values of the vertical intercepts? Do these values have a practical interpretation?

(c) On another graph plot the women's times against the men's times, with women's times, w, on the vertical axis and men's times, m, on the horizontal axis. It should look linear. How could you have predicted this linearity from the equations you found in part (b)? What is the slope of this line and how can it be interpreted? A newspaper reporter claims that the women's records are about 6% slower than the men's. Do the facts support this statement? What is the value of the vertical intercept? Does this value have a practical interpretation?

Table 1.50 *Men's and women's world swimming records*

Distance (m)	50	100	200	400	800	1500
Men (sec)	20.91	46.91	102.00	220.07	452.12	871.02
Women (sec)	23.73	52.06	112.98	236.46	484.79	925.48

STRENGTHEN YOUR UNDERSTANDING

Are the statements in Problems 1–54 true or false? Give an explanation for your answer.

1. $Q = f(t)$ means Q is equal to f times t.

2. A function must be defined by a formula.

3. If $P = f(x)$ then P is called the dependent variable.

4. Independent variables are always denoted by the letter x or t.

5. It is possible for two quantities to be related and yet neither be a function of the other.

6. A function is a rule that takes certain values as inputs and assigns to each input value exactly one output value.

7. It is possible for a table of values to represent a function.

8. If Q is a function of P, then P is a function of Q.

9. The graph of a circle is not the graph of a function.

10. If $n = f(A)$ is the number of angels that can dance on the head of a pin whose area is A square millimeters, then $f(10) = 100$ tells us that 10 angels can dance on the head of a pin whose area is 100 square millimeters.

11. Average speed can be computed by dividing the distance traveled by the time elapsed.

12. The average rate of change of a function Q with respect to t over an interval can be symbolically represented as $\dfrac{\Delta t}{\Delta Q}$.

13. If $y = f(x)$ and as x increases, y increases, then f is an increasing function.

14. If f is a decreasing function, then the average rate of change of f on any interval is negative.

15. The average rate of change of a function over an interval is the slope of a line connecting two points of the graph of the function.

16. The average rate of change of $y = 3x-4$ between $x = 2$ and $x = 6$ is 7.

17. The average rate of change of $f(x) = 10 - x^2$ between $x = 1$ and $x = 2$ is the ratio $\dfrac{10 - 2^2 - 10 - 1^2}{2 - 1}$.

18. If $y = x^2$ then the slope of the line connecting the point $(2, 4)$ to the point $(3, 9)$ is the same as the slope of the line connecting the point $(-2, 4)$ to the point $(-3, 9)$.

19. A linear function can have different rates of change over different intervals.

20. The graph of a linear function is a straight line.

21. If a line has the equation $3x + 2y = 7$, then the slope of the line is 3.

22. A table of values represents a linear function if

$$\frac{\text{Change in output}}{\text{Change in input}} = \text{Constant}.$$

23. If a linear function is decreasing, then its slope is negative.

[33] http://www.fina.org/sites/default/files/wr_50m_mar_20_2017.pdf, accessed April 22, 2017.

24. If $y = f(x)$ is linear and its slope is negative, then in the expression $\frac{\Delta y}{\Delta x}$ either Δx or Δy is negative, but not both.

25. A linear function can have a slope that is zero.

26. If a line has slope 2 and y-intercept -3, then its equation may be written $y = -3x + 2$.

27. The line $3x + 5y = 7$ has slope $3/5$.

28. A line that goes through the point $(-2, 3)$ and whose slope is 4 has the equation $y = 4x + 5$.

29. The line $4x + 3y = 52$ intersects the x-axis at $x = 13$.

30. If $f(x) = -2x + 7$ then $f(2) = 3$.

31. The line that passes through the points $(1, 2)$ and $(4, -10)$ has slope 4.

32. The linear equation $y - 5 = 4(x + 1)$ is equivalent to the equation $y = 4x + 6$.

33. The line $y - 4 = -2(x + 3)$ goes through the point $(4, -3)$.

34. The line whose equation is $y = 3 - 7x$ has slope -7.

35. The line $y = -5x + 8$ intersects the y-axis at $y = 8$.

36. The equation $y = -2 - \frac{2}{3}x$ represents a linear function.

37. The lines $y = 8 - 3x$ and $-2x + 16y = 8$ both cross the y-axis at $y = 8$.

38. The graph of $f(x) = 6$ is a line whose slope is six.

39. The lines $y = -\frac{4}{5}x + 7$ and $4x - 5y = 8$ are parallel.

40. The lines $y = 7 + 9x$ and $y - 4 = -\frac{1}{9}(x + 5)$ are perpendicular.

41. The lines $y = -2x + 5$ and $y = 6x - 3$ intersect at the point $(1, 3)$.

42. If two lines never intersect then their slopes are equal.

43. The equation of a line parallel to the y-axis could be $y = -\frac{3}{4}$.

44. A line parallel to the x-axis has slope zero.

45. The slope of a vertical line is undefined.

46. Fitting the best line to a set of data is called linear regression.

47. The process of estimating a value within the range for which we have data is called interpolation.

48. Extrapolation tends to be more reliable than interpolation.

49. If two quantities have a high correlation then one quantity causes the other.

50. If the correlation coefficient is zero, there is not a relationship between the two quantities.

51. A correlation coefficient can have a value of $-\frac{3}{7}$.

52. A value of a correlation coefficient is always between negative and positive 1.

53. A correlation coefficient of one indicates that all the data points lie on a straight line.

54. A regression line is also referred to as a least squares line.