1

Review of the Laplace Transform

Our emphasis in this book is on constant parameter linear time invariant dynamic systems. Laplace transform theory is most useful when applied to analysis of linear systems, and this transformation allows the analyst to reduce differential equation problems to algebraic problems. The theory discussed later will be utilized in both the dynamic system analysis and control system design portions of the text. It is demonstrated in this chapter that linear, time-invariant dynamic equations can be transformed into rational functions. Many important input (test) signals can be transformed in this way as well.

1.1 The Laplace Transform Concept

One of the most important problems in dynamic systems analysis is to predict the outputs of a physical system when the inputs are known. We can then ask the following question, Does a transformation scheme exist to help in the tasks of dynamic system analysis? The answer is yes. In fact, there are several transformations that could be used in addressing this problem. For example, the Fourier transformation can be used; however, the domain for the Fourier transformation is "limited." That is, too many important types of inputs do not belong to the domain of the Fourier transform. On the other hand, the Laplace transform has a very broad domain, and the existence of the Laplace transform tables aid the execution of the steps in the indirect path shown in Figure 1.1.

1.2 Singularity Functions

Before we can take up the definition of the Laplace transform, it is necessary to consider those functions of time which have a discontinuity at some instants. Especially important is some understanding of what we mean by an impulse function or the so called Dirac-δ function.

Our first definitions provide a terminology for values of a discontinuous function at either side of the instant of discontinuity.

Definition 1.1 The initial condition, denoted $y(0^-)$, is the limit from the left at $t = 0$:

$$y(0^-) = \lim_{t \to 0^-} y(t) \tag{1.1}$$

Definition 1.2 The initial value, denoted $y(0^+)$, is the limit from the right at $t = 0$:

$$y(0^+) = \lim_{t \to 0^+} y(t) \tag{1.2}$$

Notice that the restrictions are strict inequalities. In other words, the initial condition is the limiting value of the function just before a possible jump discontinuity at $t = 0$ and the initial value is the limiting value of the function just after a possible jump discontinuity at $t = 0$.

Robust Control: Youla Parameterization Approach, First Edition. Farhad Assadian and Kevin R. Mallon.

Companion Website: www.wiley.com/go/assadian/robust

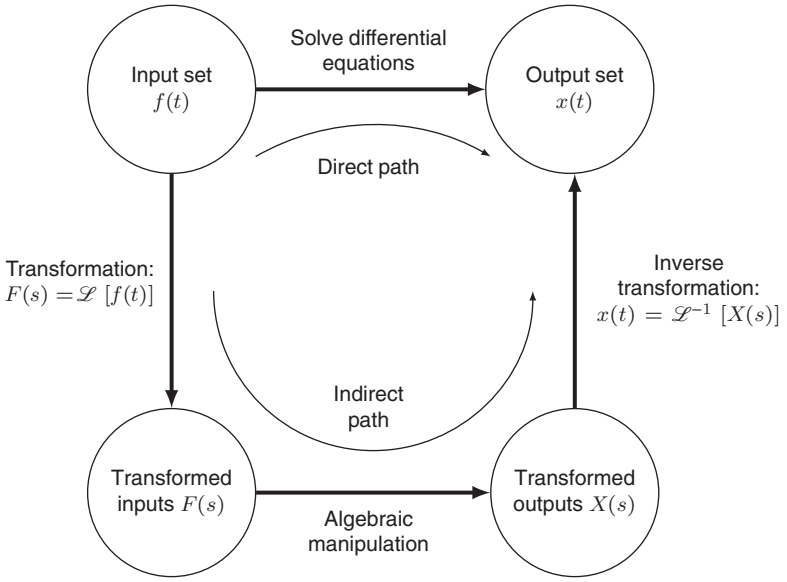

Figure 1.1 Laplace transform process.

Example 1.1 We consider the initial condition and initial value of a unit step function,

$$u(t) = 1(t) \tag{1.3}$$

$$u(0^-) = 0,\ u(0^+) = 1,\ \mathscr{L}[u(t)] = \frac{1}{s} \tag{1.4}$$

1.2.1 Definition of the Impulse Function

The impulse (or Dirac Delta) function $\delta(t)$ is defined such that

$$\delta(t) = 0 \text{ for } t \neq 0 \tag{1.5}$$

$$\int_{t_0}^{t_f} \delta(t-\tau)y(t)\mathrm{d}t = \begin{cases} y(\tau), & t_0 < \tau \leq t_f \\ 0, & \text{otherwise} \end{cases} \quad \text{for any integrable } y(t) \tag{1.6}$$

An important question is why the limits of this integration are chosen such that $t_0 < \tau \leq t_f$. Suppose we write things more symmetrically: This means that the boundary on τ is $t_0 \leq \tau \leq t_f$. Then

$$\begin{aligned} y(\tau) &= \int_{t_0}^{t_f} \delta(t-\tau)y(t)\mathrm{d}t \\ &= \int_{t_0}^{\tau} \delta(t-\tau)y(t)\mathrm{d}t + \int_{\tau}^{t_f} \delta(t-\tau)y(t)\mathrm{d}t \\ y(\tau) &= y(\tau) + y(\tau) = 2y(\tau) \end{aligned} \tag{1.7}$$

A contradiction! Similarly, suppose the boundary on τ is $t_0 < \tau < t_f$. Then

$$\begin{aligned} y(\tau) &= \int_{t_0}^{t_f} \delta(t-\tau)y(t)\mathrm{d}t \\ &= \int_{t_0}^{\tau} \delta(t-\tau)y(t)\mathrm{d}t + \int_{\tau}^{t_f} \delta(t-\tau)y(t)\mathrm{d}t \\ y(\tau) &= 0 + 0 = 0 \end{aligned} \tag{1.8}$$

So, we must have $t_0 < \tau \leq t_f$ or $t_0 \leq \tau < t_f$.

Another important point is even with the right definition of the boundary on τ, the definition of the impulse function given in Eq. (1.6) is inconsistent with the definition of the Riemann integral, the definition used in introduction to calculus. Without covering the details of this inconsistency (as this is outside the scope of this text), this issue is briefly discussed in Section 1.2.2.

1.2.2 The Impulse Function and the Riemann Integral

The impulse function is inconsistent with the definition of the Riemann integral (illustrated in Figures 1.2 and 1.3).

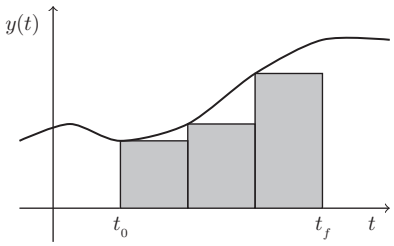

Figure 1.2 Riemann integral lower sum, S_L.

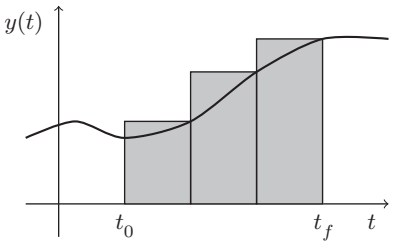

Figure 1.3 Riemann integral upper sum, S_U.

The Riemann integral (with lower sum S_L and upper sum S_U) is the common limit as the partitioning is refined. For the impulse function with $t_0 < \tau \leq t_f$, however, the lower limit is defined as $S_L \equiv 0$ while the upper limit is nonzero. The Riemann integral simply does not work. There are several ways to resolve this mathematical difficulty, the simplest is to use the Stieltjes definition of the integral [6].

1.2.3 The General Definition of Singularity Functions

The kth order derivative of singularity function $\delta^{(k)}(t)$ is defined such that:

$$\delta^{(k)}(t) \text{ for } t \neq 0 \tag{1.9}$$

$$\int_{t_0}^{t_f} \delta^{(k)}(t-\tau)y(t)\mathrm{d}t = \begin{cases} (-1)^k \frac{\mathrm{d}^k y(\tau)}{\mathrm{d}t^k}, & t_0 < \tau \leq t_f \\ 0, & \text{otherwise} \end{cases} \tag{1.10}$$

Once again, it can be shown that this definition is inconsistent with Riemann's definition of integral.

1.2.3.1 "Graphs" of Some Singularity Functions

The main reason for the persistent use of the singularity functions in the control-system literature, in spite of the mathematical problems, is the simple fact that they are useful theoretical devices. In addition, we are able to manufacture signals in the laboratory or on the computer which seem to be endowed with all of the theoretical properties of the singularity functions. A good way to think of all this is that the physical notion of the impulse is not what creates the difficulty; rather, it is the notion of what we mean by the integral that lies at the root of the theoretical problems.

The laboratory impulse is illustrated as dash lines overlaid on the function, $\rho(t)$ in Figure 1.4. (Notice that this pulse-like function ends at $t = \tau$.) An apparent mathematical justification for this claim derives from use of the mean-value theorem provided by the following equation,

$$\int_a^b y(t)\mathrm{d}t = y(\zeta)(b-a),\ a \leq \zeta \leq b \tag{1.11}$$

That is, imagine multiplying this pulse by $y(t)$ and integrating. It then seems to follow from the mean-value theorem,

$$\int_{t_0}^{t_f} \rho(t)y(t)\mathrm{d}t = \int_{\tau-\epsilon}^{\tau} \rho(t)y(t)\mathrm{d}t = \frac{1}{\epsilon} \cdot y(\zeta) \cdot \epsilon = y(\zeta),\ \tau - \epsilon \leq \zeta \leq \tau. \tag{1.12}$$

If we continue this line of reasoning, we can imagine laboratory versions of the higher order singularity functions. For example, a finite difference approximation for the first singularity function. To "construct" $\delta^{(1)}(\tau) \Rightarrow -\frac{\mathrm{d}y}{\mathrm{d}t}(\tau)$, refer to the finite difference approximation:

$$\frac{\mathrm{d}y(\tau)}{\mathrm{d}t} \approx \frac{y(\tau) - y(\tau - \epsilon)}{\epsilon} \tag{1.13}$$

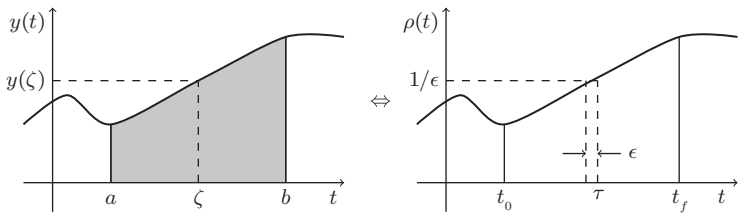

Figure 1.4 Mean value theorem development of singularity functions.

The following example provides an illustration of an impulse function and a finite difference approximation for the first singularity function and their associated dynamic responses.

Example 1.2 We compare the impulse and doublet response for a moving mass. Consider applying a force $F(t)$ to the mass shown in Figure 1.5.

First, let $F(t)$ be an approximate impulse or doublet function, as shown in Figures 1.6 and 1.7. In these cases, ϵ is considered to be a very small (but finite) number.

The approximate impulse function is zero except for a length of time ϵ starting at $t = -\epsilon$, where the magnitude is $1/\epsilon$. The velocity response is a ramp between $t = -\epsilon$ and $t = 0$, ending at a magnitude of 1. In turn, the displacement response is a parabola between $t = -\epsilon$ and $t = 0$ and a ramp with slope equal to 1 for $t > 0$.

The approximate doublet is zero except for a length of time 2ϵ starting at $t = -2\epsilon$. For the first half of the duration, the approximate doublet has a magnitude of $1/\epsilon^2$. For the second half, it has a magnitude of $-1/\epsilon^2$. The velocity response is a positive ramp from $t = -2\epsilon$ to $t = -\epsilon$ and a negative ramp from $t = -\epsilon$ to $t = 0$, ending with a magnitude of $1/\epsilon$. Further integration gives the displacement response, a parabolic increase from $t = -2\epsilon$ to $t = -\epsilon$ and parabolic decrease from $t = -\epsilon$ to $t = 0$, such that the magnitude at $t = -\epsilon$ is $1/2$ and the magnitude at $t = 0$ is 1.

Next, consider the exact response to an impulse or doublet force input, as shown in Figures 1.8 and 1.9. As $\epsilon \to 0$, the approximate impulse and doublet functions approach true impulse and doublet functions and integrate into the more familiar step and ramp functions.

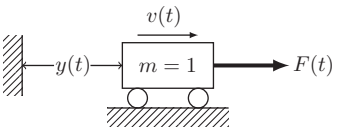

Figure 1.5 An mass with an applied force.

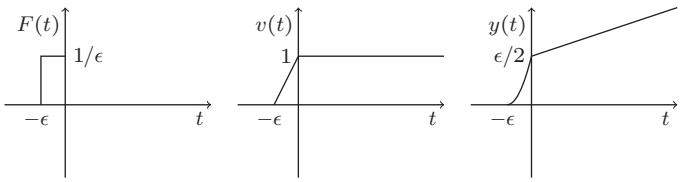

Figure 1.6 Impulse approximate response.

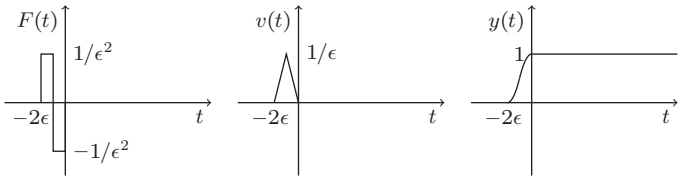

Figure 1.7 Doublet approximate response.

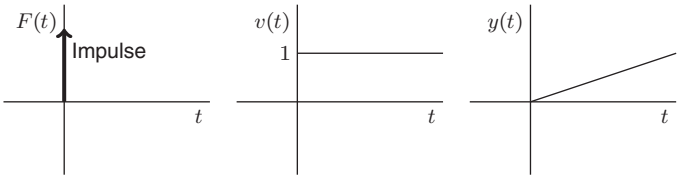

Figure 1.8 Impulse exact response ($\epsilon \to 0$).

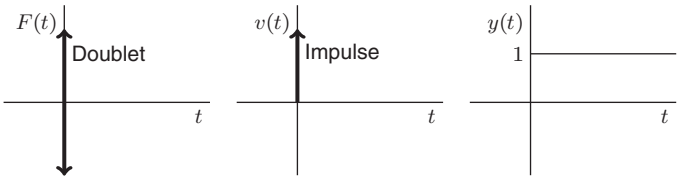

Figure 1.9 Doublet exact response ($\epsilon \to 0$).

Similarly, we can construct a finite difference approximation for the second derivative and then infer a laboratory version of the third singularity function. In a similar way, we can approximate all the singularity functions.

1.3 The Laplace Transform

1.3.1 Definition of the Laplace Transform

The Laplace transform converts a function from the time domain with variable t, to the Laplace domain with variable s (the Laplace domain is sometimes referred to as the s-domain). The Laplace variable s is a complex number and can be written in terms of its real part σ and imaginary part ω.

$$s = \sigma + j\omega \tag{1.14}$$

The choice for s is arbitrary but proves to be very convenient in the application of dynamic systems theory [7]. The Laplace transform of $y(t)$ is denoted by $\hat{y}(s)$ or $\mathscr{L}[y(t)]$ and is defined as [8]

$$\mathscr{L}[y(t)] = \int_{0^-}^{\infty} e^{-st} y(t) \mathrm{d}t \tag{1.15}$$

Notice the lower limit of 0^-, which reads "slightly before zero." The Fourier transform is defined by

$$\mathscr{F}[y(t)] = \int_{-\infty}^{\infty} e^{-j\omega t} y(t) \mathrm{d}t \tag{1.16}$$

The Laplace transformation domain [9] is given as follows: for a function to be Laplace transformable, $f(t)$ must be

1. Sectionally continuous; i.e. $f(t)$ may have only a finite number of discontinuities in any finite interval; and
2. Of exponential order; i.e. for some real $\sigma_0, A > 0$, and $t_1 > 0, Ae^{-\sigma_0 t} \geq |f(t)|$ for $t_1 > t$. The region of the s plane for which $Re(s) > \sigma_0$ is called the "region of convergence."

These conditions are sufficient conditions for the existence of the integral in Eq. (1.15).

Example 1.3 We consider the Laplace transform of the unit step function, illustrated in Figure 1.10.

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{1.17}$$

The unit step function, $u(t)$, can be written as $1(t)$.

$$\hat{u}(s) = \int_{0^-}^{\infty} e^{-st} 1(t) \mathrm{d}t$$

$$= \int_0^\infty e^{-st}\mathrm{d}t$$
$$= \frac{-e^{-st}}{s}\Big|_0^\infty$$
$$= +\frac{1}{s} \text{ (if } Re\ s > 0, \text{ where } Re\ s = \sigma) \tag{1.18}$$

Plotting $\hat{u}(s)$ vs. s cannot be done in a very convenient manner, as was done with $u(t)$ vs. t, because s is complex. However, it is very useful to plot the critical points of $\hat{u}(s)$, i.e. points where $\hat{u}(s)$ is infinite (called poles) and is zero (called zeros), on the s plane. For the unit step function, $\hat{u}(s)$ has a single pole at origin as illustrated in pole–zero diagram of Figure 1.11.

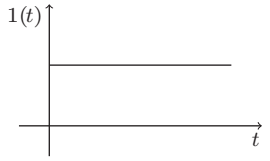

Figure 1.10 Unit step time domain.

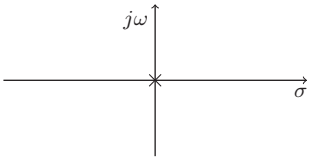

Figure 1.11 Unit step s-domain.

1.3.2 Laplace Transform Properties

In this section, we discuss the properties of the Laplace transform by series of propositions with their associated proofs. This should aid in quickly deriving Laplace transforms for sets of functions.

Proposition 1.1 *The Laplace transform is linear, i.e.*

$$\mathscr{L}[c_1 y(t) + c_2 g(t)] = c_1 \mathscr{L}[y(t)] + c_2 \mathscr{L}[g(t)] \tag{1.19}$$

for any c_1, c_2, $y(t)$, $g(t)$.

Proof: Linearity of the integration.

$$\int [c_1 f(t) + c_2 h(t)]\mathrm{d}t = c_1 \int f(t)\mathrm{d}t + c_2 \int h(t)\mathrm{d}t,$$

Then, per the definition of the Laplace transform in Eq. (1.15), the Laplace transform is linear. ■

Proposition 1.2 *The Laplace transform is nonmultiplicative, i.e.*

$$\mathscr{L}[y(t)g(t)] \neq \mathscr{L}[y(t)]\mathscr{L}[g(t)] \tag{1.20}$$

due to the properties of integration.

Proof:

$$\int f(t)h(t)\mathrm{d}t \neq \int f(t)\mathrm{d}t \int h(t)\mathrm{d}t$$

Therefore, per the definition of the Laplace transform in Eq. (1.15), the Laplace transform cannot be multiplicative. ■

1.3.3 Shifting the Laplace Transform

The definition of the Laplace transform is manipulated to yield results that are useful for obtaining the transforms of an important set of functions and for manipulating and interpreting mathematical models of physical systems.

Proposition 1.3 (Complex Shift of the Laplace Transform): *If* $\hat{y}(s) = \mathscr{L}[y(t)]$, *then*

$$\mathscr{L}[e^{-\sigma t}y(t)] = \hat{y}(s+\sigma) \tag{1.21}$$

Proof:

$$\begin{aligned}\mathscr{L}[e^{-\sigma t}y(t)] &= \int_{0^-}^{\infty} e^{-st}e^{-\sigma t}y(t)\mathrm{d}t \\ &= \int_{0^-}^{\infty} e^{-(s+\sigma)t}y(t)\mathrm{d}t \\ &= \hat{y}(s+\sigma)\end{aligned}$$

Therefore, multiplying $y(t)$ by $e^{-\sigma t}$ shifts $\hat{y}(s)$ by σ. ■

This result shares a symmetry with the following proposition.

Proposition 1.4 (Time Shift of the Laplace Transform): *If* $\hat{y}(s) = \mathscr{L}[y(t)]$ *and* $y(t) \equiv 0$ *for* $t < 0^-$, *then*

$$\mathscr{L}[y(t-\tau)] = e^{-\tau s}\hat{y}(s) \tag{1.22}$$

Proof: Let $\zeta = t - \tau$, so

$$\begin{aligned}\mathscr{L}[y(t-\tau)] &= \int_{0^-}^{\infty} e^{-st}y(t-\tau)\mathrm{d}t \\ &= \int_{-\tau^-}^{\infty} e^{-s(\zeta+\tau)}y(\zeta)\mathrm{d}\zeta \\ &= e^{-s\tau}\int_{-\tau^-}^{\infty} e^{-s\zeta}y(\zeta)\mathrm{d}\zeta \\ &\quad -\tau^- = 0^- \text{ because } y(t) \equiv 0 \text{ for } t < 0^-, \text{ so} \\ &= e^{-s\tau}\int_{0^-}^{\infty} e^{-s\zeta}y(\zeta)\mathrm{d}\zeta \\ &= e^{-s\tau}\hat{y}(s)\end{aligned}$$

Therefore, multiplying $\hat{y}(s)$ by $e^{-\tau s}$ shifts $y(t)$ by τ. ■

Example 1.4 Consider the function illustrated in Figure 1.12

$$y(t) = e^{-\sigma t}1(t) \tag{1.23}$$

The critical points of this function are plotted in Figure 1.13.

$$\mathscr{L}[1(t)] = \frac{1}{s}, \text{ so } \mathscr{L}[e^{-\sigma t}1(t)] = \frac{1}{s+\sigma} \tag{1.24}$$

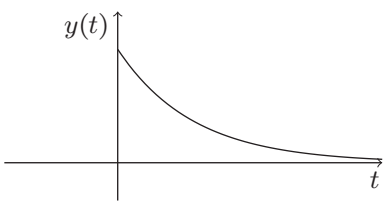

Figure 1.12 Decaying exponential time response.

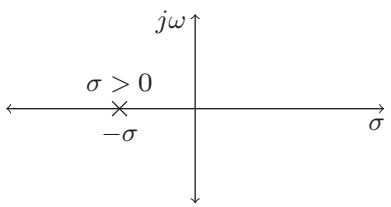

Figure 1.13 Decaying exponential time domain.

Example 1.5 Consider the function

$$y(t) = \cos(\omega t)1(t) \tag{1.25}$$

As illustrated in Figure 1.14, if z is complex, then by definition $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$. Suppose $z = j\theta$:

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} + \cdots \tag{1.26}$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta),\ \ e^{-j\theta} = \cos(\theta) - j\sin(\theta) \tag{1.27}$$

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{1.28}$$

Therefore,

$$\begin{aligned}\mathscr{L}[\cos(\omega t)1(t)] &= \frac{1}{2}\mathscr{L}[e^{j\omega t}1(t)] + \frac{1}{2}\mathscr{L}[e^{-j\omega t}1(t)] \\ &= \frac{1}{2}\frac{1}{s - j\omega} + \frac{1}{2}\frac{1}{s + j\omega} \\ &= \frac{s}{s^2 + \omega^2}\end{aligned} \tag{1.29}$$

The critical points of this function are plotted in Figure 1.15.

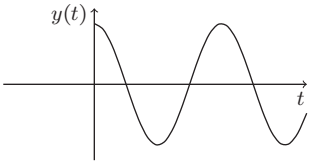

Figure 1.14 Cosine function time response.

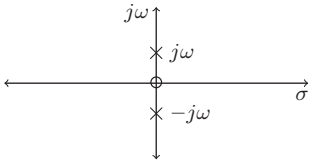

Figure 1.15 Cosine function s-domain.

Example 1.6 Consider the function

$$y(t) = e^{-\sigma t}\cos(\omega t)1(t) \tag{1.30}$$

As illustrated in Figure 1.16,

$$\mathscr{L}[e^{-\sigma t}\cos(\omega t)1(t)] = \frac{s + \sigma}{(s + \sigma)^2 + \omega^2} \tag{1.31}$$

The critical points of this function are plotted in Figure 1.17.

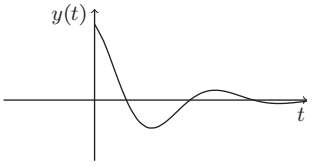

Figure 1.16 Decaying cosine time response.

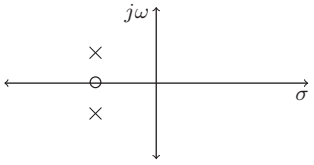

Figure 1.17 Decaying cosine s-domain.

1.3.4 Laplace Transform Derivatives

The next two propositions are also symmetrical.

Proposition 1.5 (Derivative of a Transformed Function): *If* $\hat{y}(s) = \mathscr{L}[y(t)]$, *then*

$$\mathscr{L}[ty(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}\hat{y}(s) \tag{1.32}$$

Proof:

$$-\frac{\mathrm{d}}{\mathrm{d}s}\hat{y}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\int_{0^-}^{\infty} e^{-st}y(t)\mathrm{d}t$$
$$= \int_{0^-}^{\infty} e^{-st}(ty(t))\mathrm{d}t = \mathscr{L}[ty(t)]$$

Therefore, when a function $y(t)$ is multiplied by t, the Laplace transform of the resulting output is the negative derivative in s of $\mathscr{L}[y(t)]$. ■

Example 1.7 Consider the ramp function, $t(1(t))$, as shown in Figure 1.18.

$$\mathscr{L}[t(1(t))] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{s}\right] = \frac{1}{s^2} \tag{1.33}$$

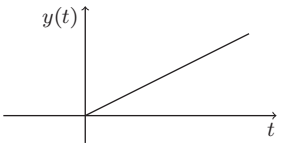

Figure 1.18 Ramp input.

Apply this proposition in an iterative fashion to establish

$$\mathscr{L}[t^n y(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\hat{y}(s) \tag{1.34}$$

Proposition 1.6 (Transformation of a Derivative): *If $\hat{y}(s) = \mathscr{L}[y(t)]$, then the Laplace transform of the derivative is related to the initial condition by*

$$\mathscr{L}\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right] = s\hat{y}(s) - y(0^-) \tag{1.35}$$

Proof: Reminder: Integration by parts.

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{1.36}$$

Then,

$$\mathscr{L}\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right] = \int_{0^-}^{\infty} e^{-st}\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t$$

Integration by parts:

$$u = e^{-st},\ \mathrm{d}u = -se^{-st}\mathrm{d}t,\ v = y(t), \mathrm{d}v = \left(\frac{\mathrm{d}y(t)}{\mathrm{d}t}\right)\mathrm{d}t$$

$$= e^{-st}y(t)\Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} -se^{-st}y(t)\mathrm{d}t$$

knowing that

$$e^{-st}y(t)\Big|_{0^-}^{\infty} = 0 - y(0^-)$$

so

$$\mathscr{L}\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right] = s\hat{y}(s) - y(0^-)$$

Therefore, when a function $y(t)$ is differentiated with respect to time, the Laplace transform of the resulting output is s times $\mathscr{L}[y(t)]$ minus the initial condition $y(0^-)$. ■

Applying this result in an iterative way establishes a corollary to this proposition.

Corollary 1.1

$$\mathscr{L}\left[\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n}\right] = s^n \hat{y}(s) - \sum_{k=1}^{n} s^{n-k} \frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}} y(0^-) \tag{1.37}$$

Example 1.8 Consider a force applied to a mass, as shown in Figure 1.19.
Newton's law: $m\frac{d^2y}{dt^2} = u(t)$, where $u(t) = F(t)$

$$\mathscr{L}\left[\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right] = s^2\hat{y}(s) - sy(0^-) - \frac{\mathrm{d}}{\mathrm{d}t}(y(0^-)) \tag{1.38}$$

$$m[s^2\hat{y}(s) - sy(0^-) - \frac{\mathrm{d}y}{\mathrm{d}t}(0^-)] = \hat{u}(s) \tag{1.39}$$

$$\hat{y}(s) = \underbrace{\frac{sy(0^-) + \frac{\mathrm{d}y}{\mathrm{d}t}(0^-)}{s^2}}_{\text{I.C. response}} + \underbrace{\frac{1}{ms^2}\hat{u}(s)}_{\text{forced response}} \tag{1.40}$$

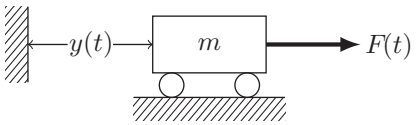

Figure 1.19 A mass with an applied force.

When all initial conditions are zero, we are left with the transfer function, $G(s)$, from input $\hat{u}$ to output $\hat{y}$.

$$G(s) = \frac{\hat{y}}{\hat{u}}(s) = \frac{1}{ms^2} \tag{1.41}$$

The block diagram and critical points of this system are illustrated in Figures 1.20 and 1.21, respectively.

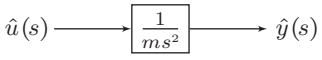

Figure 1.20 Forced mass block diagram.

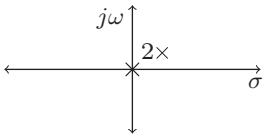

Figure 1.21 Poles of the forced mass.

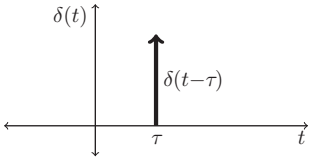

Figure 1.22 Impulse function.

1.3.5 Transforms of Singularity Functions

To this point, our example transforms have been rational functions. Here, we determine the transforms of the singularity functions that are rational functions (polynomials actually) for the special case of $\tau = 0$. As it turns out, it is this value of

τ that is used most often in our discussions. A simple application of Eq. (1.15) and the definition of an impulse function of the last section yields the transform

$$\mathscr{L}[\delta(t-\tau)] = \int_{0^-}^{\infty} \delta(t-\tau)e^{-st}\mathrm{d}t = e^{-s\tau} \tag{1.42}$$

where the impulse $\delta(t-\tau)$ is illustrated in Figure 1.22. Note: If $\tau = 0$, then $\mathscr{L}[\delta(t)] = 1$. By linearity,

$$\mathscr{L}^{-1}[R_0] = R_0\delta(t) \tag{1.43}$$

More generally, the application of Eq. (1.15) and the previous definition of an impulse function of the last section results in

$$\mathscr{L}[\delta^{(k)}(t-\tau)] = s^k e^{-s\tau}, \quad \mathscr{L}[\delta^{(k)}(t)] = s^k \tag{1.44}$$

1.4 Inverse Laplace Transform

At this point, it is clear that the differential equations of linear, time invariant, systems can be converted to polynomials and rational functions where the Laplace transform is effected; the Laplace transform can be used to solve differential equations by purely algebraic means. Nearly all that is needed to complete the solution of ordinary differential equations with constant coefficients by Laplace transformation is contained in the propositions and examples of previous sections. The final ingredient is a method for obtaining the inverse transform, $y(t) = \mathscr{L}^{-1}[\hat{y}(s)]$. One can certainly utilize the Cauchy integral formula, which is discussed in Appendix A, for this purpose. Other, more mundane but useful methods are developed in this section.

1.4.1 Inverse Laplace Transformation by Heaviside Expansion

In this section, we shall use the simpler procedure of partial fraction or Heaviside expansion to obtain an inverse transformation. That is to say, we exploit the properties of rational functions as follows:

Suppose we have a transfer function, $G(s)$ and want $g(t)$. Also suppose that $G(s)$ is strictly proper, where the number of poles, m, is greater than number of zeros, n, $m > n$. Write any strictly proper transfer function as

$$G(s) = \frac{q_{n-1}s^{n-1} + q_{n-2}s^{n-2} + \cdots + q_1 s + q_0}{(s+p_1)(s+p_2)\cdots(s+p_m)} \tag{1.45}$$

The problem of converting this form with a Heaviside expansion is divided into subcases according to whether or not the poles, p_k, are repeated.

1.4.1.1 Distinct Poles

Where the p_k are distinct, it is possible to make the Heaviside expansion as

$$G(s) = \frac{R_1}{s+p_1} + \frac{R_2}{s+p_2} + \cdots + \frac{R_n}{s+p_n} \tag{1.46}$$

The R_k are called residues. Multiply this by $(s+p_k)$ and note that result must apply to all values of s. In particular, we can set $s = -p_k$ to obtain the formula

$$R_k = (s+p_k)G(s)\big|_{s=-p_k} \tag{1.47}$$

The residuals can then be independently inverted.

$$g(t) = [R_1 e^{-p_1 t} + R_2 e^{-p_2 t} + \cdots + R_n e^{-p_n t}]1(t) \tag{1.48}$$

1.4.1.2 Distinct Poles with $G(s)$ Being Proper

When $G(s)$ is proper, the degree of the numerator is equal to the degree of the denominator. If $G(s)$ is proper, then it can be written as

$$G(s) = R_0 + \frac{R_1}{s+p_1} + \frac{R_2}{s+p_2} + \cdots + \frac{R_n}{s+p_n}, \text{ where } R_0 = G(\infty) \tag{1.49}$$

$$g(t) = R_0\delta(t) + [R_1 e^{-p_1 t} + R_2 e^{-p_2 t} + \cdots + R_n e^{-p_n t}]1(t) \tag{1.50}$$

Remember that $g(t) = \mathscr{L}^{-1}[G(s)]$.

1.4.1.3 Repeated Poles

Suppose one of the poles of $G(s)$, call it $-p_1$, is repeated a times,

$$G(s) = \frac{q_{n-1}s^{n-1} + q_{n-2}s^{n-2} + \cdots + q_1 s + q_0}{(s+p_1)^a (s+p_2)\cdots(s+p_{n-a+1})} \tag{1.51}$$

The Heaviside expansion is modified in the following way,

$$G(s) = \frac{R_{11}}{s+p_1} + \frac{R_{12}}{(s+p_1)^2} + \cdots + \frac{R_{1a}}{(s+p_1)^a} + \frac{R_2}{s+p_2} + \cdots + \frac{R_{n-a+1}}{s+p_{n-a+1}} \tag{1.52}$$

It is easy to verify that the residues associated with linear denominator terms are calculated as in Eq. (1.47). For a single repeated pole ($a = 2$), it is easily verified that

$$R_{12} = \frac{\mathrm{d}}{\mathrm{d}s}[(s+p_1)^a G(s)]\Big|_{s=-p_1} \tag{1.53}$$

In addition, for values of $a - l \geq 2$, we find that

$$R_{1l} = \frac{1}{(a-l)!}\frac{\mathrm{d}^{a-l}}{\mathrm{d}s^{a-l}}[(s+p_1)^a G(s)]\Big|_{s=-p_1} \tag{1.54}$$

Then $g(t)$ can be written as

$$\begin{aligned} g(t) = \big[&R_{11}e^{-p_1 t} + R_{12}te^{-p_1 t} + \cdots \\ &+ R_{1a}t^a e^{-p_1 t} + R_2 e^{p_2 t} + \cdots \\ &+ R_{n-a+1}e^{-p(n-a+1)t}\big]\, 1(t) \end{aligned} \tag{1.55}$$

The examples below illustrate the utility of Heaviside expansion in inverse Laplace transformation.

Example 1.9 Consider the transfer function (distinct poles – proper)

$$G(s) = \frac{s^2 + 5s + 6}{s^2 + 3s + 2} \tag{1.56}$$

This is the case where $G(s)$ is proper. First, carry out any pole/zero cancellations.

$$G(s) = \frac{\cancel{(s+2)}(s+3)}{(s+1)\cancel{(s+2)}} = \frac{s+3}{s+1} \tag{1.57}$$

Therefore,

$$G(s) = \frac{s+3}{s+1} = R_0 + \frac{R_1}{s+p_1} = R_0 + \frac{R_1}{s+1} \tag{1.58}$$

According to Eq. (1.49),

$$R_0 = G(\infty) = 1 \tag{1.59}$$

And according to Eq. (1.47),

$$R_1 = (s+1)G(s)\big|_{s=-1} \tag{1.60}$$

yields $R_1 = 2$. Therefore, the partial fraction expansion of $G(s)$ is

$$G(s) = 1 + \frac{2}{s+1} \tag{1.61}$$

and therefore,

$$g(t) = \delta(t) + [2e^{-t}]1(t) \tag{1.62}$$

Example 1.10 Consider the transfer function (distinct poles, strictly proper)

$$G(s) = \frac{4s^2 + 13s + 12}{s(s^2 + 5s + 6)} \tag{1.63}$$

First, factor $G(s)$.

$$G(s) = \frac{4s^2 + 13s + 12}{s(s + 2)(s + 3)} \tag{1.64}$$

Therefore, the partial fraction expansion is of the form

$$G(s) = \frac{R_1}{s + p_1} + \frac{R_2}{s + p_2} + \frac{R_3}{s + p_3} = \frac{R_1}{s} + \frac{R_2}{s + 2} + \frac{R_3}{s + 3} \tag{1.65}$$

According to Eq. (1.47)

$$R_k = (s + p_k)G(s)\big|_{s = -p_k} \tag{1.66}$$

This yields

$$\begin{aligned} R_1 &= sG(s)|_{s=0} = 2 \\ R_2 &= (s + 2)G(s)|_{s = -2} = -1 \\ R_3 &= (s + 3)G(s)|_{s = -3} = 3 \end{aligned} \tag{1.67}$$

Therefore, the partial fraction expansion of $G(s)$ is

$$G(s) = \frac{2}{s} - \frac{1}{s + 2} + \frac{3}{s + 3} \tag{1.68}$$

and

$$g(t) = [2 - e^{-2t} + 3e^{-3t}]1(t) \tag{1.69}$$

Example 1.11 Consider the transfer function (repeated poles)

$$G(s) = \frac{4s^3 + 16s^2 + 23s + 13}{(s + 1)^3(s + 2)} \tag{1.70}$$

The partial fraction expansion is of the form

$$G(s) = \frac{R_{11}}{(s + 1)} + \frac{R_{12}}{(s + 1)^2} + \frac{R_{13}}{(s + 1)^3} + \frac{R_2}{(s + 2)} \tag{1.71}$$

According to Eqs. (1.53) and (1.54)

$$\begin{aligned} R_{11} &= \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}s^2}[(s + 1)^3 G(s)]\Big|_{s = -1} = 3 \\ R_{12} &= \frac{\mathrm{d}}{\mathrm{d}s}[(s + 1)^3 G(s)]\Big|_{s = -1} = 1 \\ R_{13} &= [(s + 1)^3 G(s)]\Big|_{s = -1} = 2 \\ R_2 &= [(s + 2)G(s)]|_{s = -2} = 1 \end{aligned} \tag{1.72}$$

Therefore, the partial fraction expansion of $G(s)$ is

$$G(s) = \frac{3}{(s + 1)} + \frac{1}{(s + 1)^2} + \frac{2}{(s + 1)^3} + \frac{1}{(s + 2)} \tag{1.73}$$

and

$$g(t) = [(3 + t + 2t^2)e^{-t} + e^{-2t}]1(t) \tag{1.74}$$

1.5 The Transfer Function and the State Space Representations (State Equations)

The true value of the Laplace transform is that it provides an important, mathematical characterization of a dynamic system, the transfer function. In this text, for control system design with Youla parameterization, we discover that the transfer function is more valuable than the underlying state space representation. Of course the state equations are the usual product of dynamic system analysis and as discussed later, the transfer function is obtained from the state equations. The state equations tell us much that is important about the dynamics of any system and are used here to draw-out crucial mathematical properties of the transfer function and the transfer function approach to feedback design.

1.5.1 The Transfer Function

Our picture of the dynamic system is that of Figure 1.23. We shall use "u" to denote input and "y" to denote output throughout the text. In continuous-time application, the transfer function is the ratio of the transformed output to the transformed input $G(s) = \frac{\hat{y}}{\hat{u}}$ and all initial conditions are zero.

One can obtain this ratio anywhere the transforms exist. In some cases, the term "transfer function" is utilized to mean some input–output characteristic with direct physical measures rather than the "s-plane" transform. In these cases, the transfer function depends upon the particular input. In the case of the linear, time-invariant (LTI) system, the transfer function is completely specified by the parameters of the system and is independent of the input.

This independence of the transfer function for the input, for the LTI systems, is easily established if one considers the differential equation describing a dynamic system. Since such an equation transforms to a polynomial operator times the output equal to a polynomial times the input, the ratio of output to input is a rational function. Clearly, this rational function is independent of the particular input.

We have shown through several examples, one can always derive the equation of motion (the differential equation) and then write the transfer function by inspection. That is almost the way things go, but not quite.

1.5.2 The State Equations

Rather than a single, high-order equation, it is usually more convenient to write a system of simultaneous equations to describe the dynamics of a physical system. In mechanical engineering, for example, Hamilton's principle leads to a system of first-order equations. In general, the dynamics of a system can be written in the matrix form.[1]

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}\underline{u} \\ \underline{y} &= \underline{c}^T\underline{x} + D\underline{u} \end{aligned} \tag{1.75}$$

The first equation is called the dynamic equation, while the second is referred to as the message equation. The combination is called the *state variable model*.[2] A great deal of computer software is based upon the assumption that the state equations are an important part of the user's natural language.

The vector $\underline{x}$ represents a set of intermediate variables (momenta and generalized displacements in the context of Hamilton's principle) needed to complete the description of the relation of input to output. These intermediates are called *state variables*. If one takes the input–output point-of-view, as is often the case in control studies, the identities of the state variables are immaterial.

The state equations are represented in the block diagram of Figure 1.24.

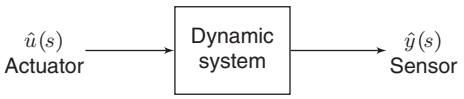

Figure 1.23 Block diagram of a dynamic system.

1 The matrix notation used here is conventional: column matrices ("vectors") are represented by lower case and rectangular matrices are represented by upper case. The "T" superscript represents the matrix transpose so c^T represents a row matrix.

2 Kalman called the combination a state-variable realization. His explanation is that an "analog" computer circuit could be directly wired from these equations and such a circuit is a physical synthesis or "realization" of the equations. The analog computer is antiquated at this date so the modern student may have difficulty interpreting these ideas.

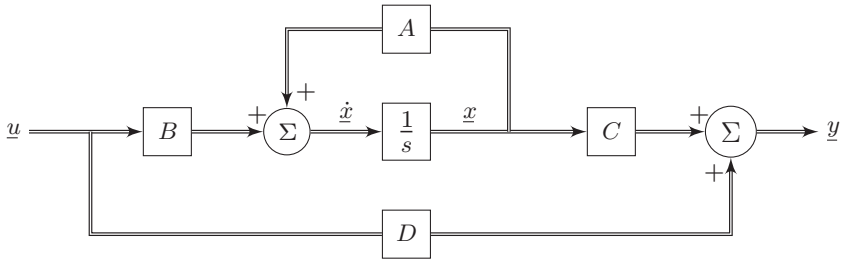

Figure 1.24 State-space block diagram.

The transfer function is related to state variable models after some preparation. For one thing, we must decide what we mean by the Laplace transform of a vector or a matrix. We let the Laplace transform of a vector be a vector of Laplace transforms.

$$\mathscr{L}[\underline{x}(t)] = \mathscr{L}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \triangleq \begin{bmatrix} \mathscr{L}[x_1(t)] \\ \mathscr{L}[x_2(t)] \\ \vdots \\ \mathscr{L}[x_n(t)] \end{bmatrix} \tag{1.76}$$

The kth component of the Laplace transform of the vector is the Laplace transform of the kth component. The Laplace transform of a matrix is defined in a similar fashion. The linearity property has implications for the Laplace transform of matrix products. Notice that by linearity, $\mathscr{L}[A\underline{x}(t)] = A\mathscr{L}[\underline{x}(t)]$. In fact, we can use the linearity property to obtain many matrix-transform relations similar to the scalar counterparts.

The state variable model of (1.75) can be transformed into the Laplace domain.

$$\begin{aligned} s\hat{\underline{x}} - \underline{x}(0^-) &= A\hat{\underline{x}} + \underline{b}\hat{u} \\ \hat{\underline{y}} &= \underline{c}^T\hat{\underline{x}} + D\hat{u} \end{aligned} \tag{1.77}$$

These equations are manipulated to obtain

$$\hat{\underline{y}} = \left[\underline{c}^T(sI - A)^{-1}\right]\hat{\underline{x}}(0^-) + \left[\underline{c}^T(sI - A)^{-1}\underline{b} + D\right]\hat{u} \tag{1.78}$$

where for any variable v, $\hat{v} = \mathscr{L}[v(t)]$.

It is here that we can justify the name "state." In order to obtain the output, $\hat{y}(s)$, we must know the input, $\hat{u}(s)$ (that is, $u(t)$ for $t \geq 0$) and the initial conditions, $\underline{x}(0^-)$. These initial conditions contain all the information that is required about what had happened before the input was applied; they completely describe the current state of the system.

In order to obtain a connection with the transfer function, set $\underline{x}(0^-) = \underline{0}$ and knowing that $G(s) = \frac{\hat{y}}{\hat{u}}$, we obtain

$$G(s) = \underline{c}^T(sI - A)^{-1}\underline{b} + D \tag{1.79}$$

1.5.3 Transfer Function Properties

Consider a rational function

$$G(s) = \frac{\hat{y}}{\hat{u}} = \frac{\text{num}}{\text{den}} \tag{1.80}$$

Definition 1.3 $G(s)$ is proper if deg(num) = deg(den).

Definition 1.4 $G(s)$ is strictly proper if deg(num) < deg(den).

Definition 1.5 $G(s)$ is improper otherwise (i.e. if deg(num) > deg(den)).

Recall the formula for matrix inversion.

$$M^{-1} = \frac{\text{adj } M}{\det M} \tag{1.81}$$

We can then write

$$\underset{n\times n}{(sI - A)^{-1}} = \frac{\text{adj } (sI - A)}{\det (sI - A)} = \frac{\text{poly. with deg} \leq n-1}{\text{poly. with deg} = n} \tag{1.82}$$

where n is the dimension of matrix A. Then we can write $G(s)$ as

$$\begin{aligned}\frac{\hat{y}}{\hat{u}} = G(s) &= \underline{c}^T(sI - A)^{-1}\underline{b} + D \\ &= \frac{\underline{c}^T \text{adj } (sI - A)\underline{b}}{\det (sI - A)} + D \\ &= \frac{\underline{c}^T \text{adj } (sI - A)\underline{b} + D \det (sI - A)}{\det (sI - A)}\end{aligned} \tag{1.83}$$

An element of the adjugate matrix, adj $(sI - A)$, is formed by crossing out a single row and a single column and applying the determinant operation. This means that the elements of adj $(sI - A)$ are polynomials of degree no greater than $n - 1$. Using this observation, we can say the followings about $G(s)$:

- Strictly proper if $D = 0$.
- Proper if $D \neq 0$.
- Never improper.

The question arises whether any transfer function is improper: Is there another formulation of the dynamic equations such that such a thing is possible? One does see such formulations from time to time, but one should be skeptical on physical grounds. The Paley–Wiener theorem [10] establishes that an improper transfer function is physically impossible. In Section 1.5.4 of this chapter, we argue in a more heuristic way that the transfer function of a physical system is not improper.

1.5.4 Poles and Zeros of a Transfer Function

Given the state equations below, the transfer function $G(s)$ can be written as

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} + \underline{b}u \\ \underline{y} &= c^T\underline{x} + \underline{D}u\end{aligned} \tag{1.84}$$

$$G(s) = \frac{\underline{c}^T \text{adj } (sI - A)\underline{b} + D \det (sI - A)}{\det (sI - A)} = \frac{\eta(s)}{\Delta(s)}, \quad \deg \eta \leq \deg \Delta \tag{1.85}$$

Poles of $G(s)$ are roots of $\det (sI - A) = 0$ and depend only upon the internal connections of the system; that is, the poles depend only upon the A-matrix. The zeros of $G(s)$ are the roots of $\underline{c}^T \text{adj } (sI - A)\underline{b} + D \det (sI - A) = 0$. In contrast to poles, zeros also depend upon the way that the system connected to its environment (characterized by $\underline{c}^T$ and $\underline{b}$). The physical interpretation of these state matrices is provided in Table 1.1.

Let $G(s)$ be the transfer function from $\hat{u}(s)$ to $\hat{y}(s)$ without any repeated poles,

$$G(s) = \frac{\eta(s)}{\Delta(s)} = \frac{(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}, \quad m \leq n \tag{1.86}$$

then, as shown in the previous sections, the output response $\hat{y}(s)$ is

$$\hat{y}(s) = G(s)\hat{u}(s) = \cdots + \frac{R_k}{s + p_k} + \cdots \tag{1.87}$$

Table 1.1 Matrix form parameters.

Matrix	Function
A	State interaction
B	Connection to actuator ("environment")
C^T	Connection to sensor ("environment")
D	Direct connection from actuator to sensor

where,

$$\mathscr{L}\left[\frac{R_k}{s+p_k}\right]^{-1} = R_k e^{-p_k t} \tag{1.88}$$

In this context, a pole and zero cancellation could be problematic in feedback design, and it will be discussed in the following sections. The example below introduces the idea of a pole and zero cancellation.

Example 1.12 Pole cancellation. Consider a system with dynamics

$$\hat{y}(s) = \frac{(s+1)}{(s+1)(s+2)}\hat{u}(s) \tag{1.89}$$

Partial fraction expansion without carrying out pole–zero cancellation will result in a term with denominator $(s+1)$ and numerator 0.

$$\hat{y}(s) = \frac{(s+1)}{(s+1)(s+2)}\hat{u}(s) = \cdots + \frac{0}{s+1} + \cdots \tag{1.90}$$

Inverse Laplace and retransform:

$$\ddot{y} + 3\dot{y} + 2 = \dot{u} + u \tag{1.91}$$

$$s^2\hat{y}(s) - \dot{y}(0^-) - sy(0^-) + 3s\hat{y}(s) - 3y(0^-) + 2\hat{y}(s) = s\hat{u} + 1\hat{u} \tag{1.92}$$

$$(s^2 + 3s + 2)\hat{y}(s) = (s+1)\hat{u}(s) + \dot{y}(0^-) + sy(0^-) + 3y(0^-) \tag{1.93}$$

$$\hat{y}(s) = \frac{(s+1)}{(s+1)(s+2)}\hat{u} + \frac{[y(0^-)]s + [3y(0^-) + \dot{y}(0^-)]}{(s+1)(s+2)} \tag{1.94}$$

$$\hat{y}(s) = \cdots + \frac{R}{s+1} + \cdots, \quad \text{where } R \neq 0 \tag{1.95}$$

The pole is canceled from the forced response, but not the initial condition response. This is called a "hidden mode."

1.5.5 Physical Realizability

It is important to obtain results that indicate the range of possibilities for real, physical transfer functions. The most satisfactory results are obtained by imposing a constraint of "cause and effect": the output of a causal physical system cannot anticipate the input. We claim that these causal physical systems are proper, and we ask the following question, Can we build a system for which $G(s)$ is improper? Is an improper transfer function realizable?

The simplest improper $G(s)$ is illustrated in Figure 1.25, this system is a simple differentiator.

Let $u(0^-) = 0, \hat{y}(s) = s\hat{u}(s), G(s) = \frac{\hat{y}}{\hat{u}}(s)$. More generally, suppose $G(s) = \frac{\eta(s)}{\Delta(s)}$, where $\deg \eta \geq \deg \Delta + 1$. We can expand $G(s)$ such that

$$G(s) = R_0 + R_{-1}s + \cdots + \underbrace{G'(s)}_{\text{strictly proper}} \tag{1.96}$$

as illustrated in Figure 1.26.

But, what is the definition of a derivative? The derivative at time t is the common limit (if it exists) of the slope from the right and left.

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0^-} \frac{y(t) - y(t-\Delta t)}{\Delta t} = \lim_{\Delta t \to 0^+} \frac{y(t+\Delta t) - y(t)}{\Delta t} \tag{1.97}$$

This is shown in Figure 1.27.

We can determine the limit from the left, $\Delta t \to 0^-$ (from past values), but it is impossible to measure the limit from the right $\Delta t \to 0^+$ (and therefore the common limit), because no device cannot predict the future.

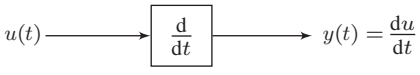

Figure 1.25 An improper system.

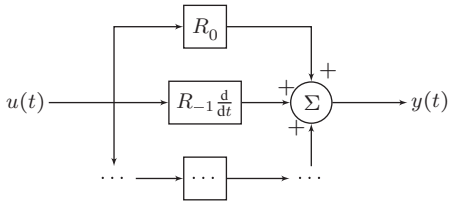

Figure 1.26 An expansion of $G(s)$ into proper and improper components.

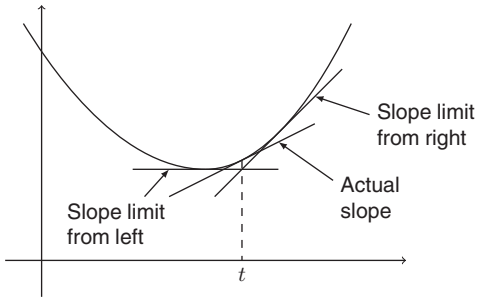

Figure 1.27 Derivative common limit.

Instead, we can implement some delay, e.g. $\frac{s}{\tau s+1}$, (making the $G(s)$ proper), in order to approximately measure the derivative. This delay results in using only the past values to approximate the computation of a differential operator.

It should be noted that some limit processes, such as integration, do not require a device that can foresee the future, as discussed later.

One can ask, is integration physically realizable? Integration, as shown in Figure 1.28, is the common limit (if it exists) as partitioning is refined Figure 1.29, shown in Figure 1.29, is refined. Information about future states is not required for integration, so integration is realizable.

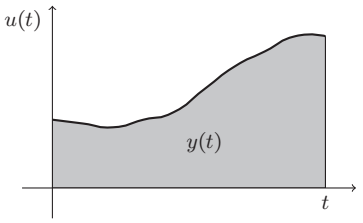

Figure 1.28 Integral of $u(t)$.

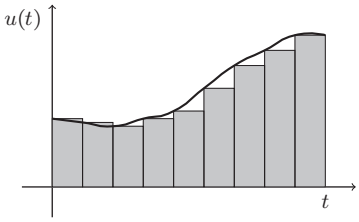

Figure 1.29 The integral is the common limit of the upper and lower sum.

The integral operator is shown in Figure 1.30.

The Laplace transform of this operator is given by

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = u(t) \tag{1.98}$$

$$\text{Transform: } s\hat{y}(s) = \hat{u}(s) \tag{1.99}$$

$$G(s) = \frac{\hat{y}}{\hat{u}} = \frac{1}{s} \tag{1.100}$$

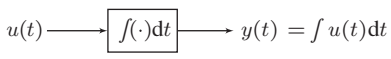

Figure 1.30 Integration block diagram.

A consideration of frequency response allows one to anticipate the more rigorous result laid down by the Paley–Wiener theorem [10], which is outside the scope of this text. Intuitively, it seems impossible for a physical system to have increasing gain with increasing frequency: If this sort of thing were possible, the power requirements would be enormous. This leads to speculate that *the degree of the numerator of a transfer function of a physically realizable dynamic system is never greater than the degree of the denominator*.

Notice that we satisfy this constraint wherever formula (1.79) is used: State equations (1.75) meet expectations for physical realizability. Something negative can be said about those formulations of the state equations from which improper transfer functions are calculated.

The same sort of reasoning leads some to argue that a physical system must be strictly proper (similar to our integral operator), where the D-matrix is zero, since it is hard to imagine that a real system can pass a very high frequency input without some attenuation. That is fair enough, but this conclusion is not supported by the more rigorous realizability theory based upon causality principles and complex analysis.

1.6 Problems

1.1 Find the Laplace transform of the two functions of time illustrated in Figure 1.31a,b. Hint: You can write an expression for these signals using step and delayed steps.

1.2 Obtain the Laplace transform of

$$y(t) = te^{-t}\cos(2t)$$

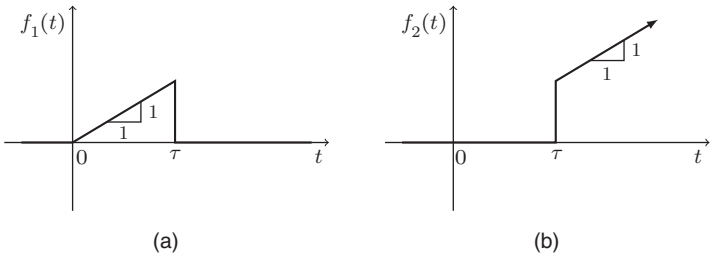

Figure 1.31 Time domain functions for Problem 1.1. (a) Truncated ramp. (b) Delayed ramp.

1.3 Find the inverse Laplace transforms of

(a) $\frac{7s-6}{s^2-s-6}$

(b) $\frac{2s^2+5}{s^2+3s+2}$

(c) $\frac{6(s+34)}{s(s^2+10s+34)}$

(d) $\frac{8s+10}{(s+1)(s+2)^3}$

1.4 Consider the following state-space equation,

$$\dot{x} = Ax + \underline{b}u$$

$$y = \underline{c}^T x$$

where

$$A = \begin{pmatrix} 0 & 1/J \\ 0 & -b/J \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underline{c}^T = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad D = 0$$

Find the transfer function $G(s)$ where $G(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$.

1.5 For the system shown in Figure 1.32, the elements are designed so that

$$\frac{\mathrm{d}c}{\mathrm{d}t} = r(t) - c(t-\tau)$$

The unavoidable delay τ occurs because the measuring device must be placed downstream of the roller. Use the time-shifting property to find the transfer function $G(s) = \frac{C(s)}{R(s)}$.

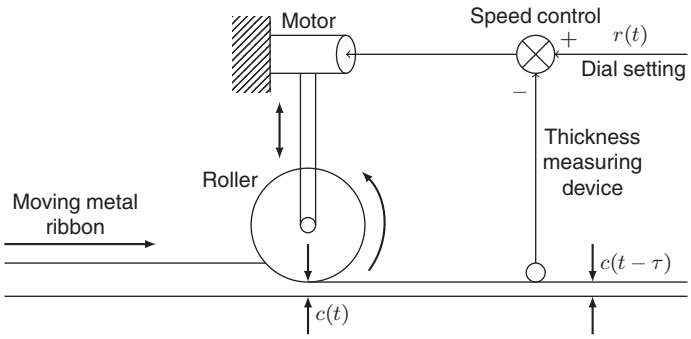

Figure 1.32 Controlled metal-rolling device.

1.6 Apply the relevant propositions and proofs discussed in the class in a iterative fashion to establish the following corollaries:

(a)

$$\mathscr{L}[t^n y(t)] = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\hat{y}(s)$$

(b)

$$\mathscr{L}\left[\frac{\mathrm{d}^n}{\mathrm{d}t^n}y(t)\right] = s^n\hat{y}(s) - \sum_{k=1}^{n} s^{n-k}\frac{\mathrm{d}^{k-1}}{\mathrm{d}t^{k-1}}y(0^-)$$

1.7 Consider the following state-space equation,

$$\dot{x} = Ax + \underline{b}u_1$$

$$y = \underline{c}^T x$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{m} \\ 0 & -\frac{b}{m} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{b}{m} \\ -\frac{b^2}{m} \end{bmatrix}, \quad \underline{c}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Find the transfer function $G(s)$, where

$$G(s) = \frac{\hat{y}}{\hat{u}_1}(s)$$

Explain how u_1 has to change so that the transfer function $G(s)$ will be equal to

$$G(s) = \frac{-\frac{1}{m}}{s\left(s + \frac{b}{m}\right)}$$

1.8 Newton's second law for the submersible shown is

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F \tag{1.101}$$

where F is regulated by pumping water in or out of tanks. The relationship between F and the pumping rate, $u(t)$, is

$$\frac{\mathrm{d}F}{\mathrm{d}t} = k_B u \tag{1.102}$$

(a) Find the transfer function $\frac{\hat{y}}{\hat{u}}(s)$. Indicate the pole–zero diagram.
(b) Actually, if the water is expelled or brought in from the bottom of the craft, then we get a little added thrust, so the second law becomes

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F + k_T u \tag{1.103}$$

What is the transfer function $\frac{\hat{y}}{\hat{u}}(s)$ in this case? Indicate the pole–zero diagram.
(c) If, instead, the water is expelled or brought in from the top of the craft, then the second law becomes

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F - k_T u \tag{1.104}$$

What is the transfer function $\frac{\hat{y}}{\hat{u}}(s)$ in this case? Indicate the pole–zero diagram.