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Elastic Anisotropic Behavior of Composite Materials

1.1 Anisotropic Elasticity of Composite Materials

1.1.1 Fourth Rank Tensor Notation of Hooke's Law

Fiber composites consist of fibers with very high stiffness and strength that are embedded in a matrix of plastic. Fibers alone can absorb high tensile forces but cannot withstand bending or compression loads. In order to achieve a desired spectrum of properties that could not be achieved individually by each component, several material components are combined in a suitable form and spatial distribution. When plastics are combined with reinforcing materials, the aim is to achieve lightweight construction of highly stressed structural parts by increasing stiffness, hardness and strength. The main problem of material optimization lies in the inadequate or missing dependencies of such parameters as loading limit, fracture toughness and critical stress intensity factor from design variables (such as fiber diameter, fiber elasticity modulus and matrix and distance between fibers). The basic task is to obtain these dependencies in an analytical form.

The components of plastics can be relatively brittle (thermosetting reaction resins) or rather flexible (thermoplastics). Only through the combination of fibers and plastics and the firm connection of the plastic matrix to the fibers can high-strength components, such as aircraft and vehicle parts, be produced. For material laws of fiber-reinforced composites, the literature provides a broad knowledge base (e.g. Moser 1992; Chou 1990; Nettles 1994; Gibson 2016). In this chapter, the necessary information for modeling and optimization of structural components in the automotive powertrain and suspension will be provided.

The most general anisotropic form of linearly elastic constitutive relations is given by the generalized Hooke's law:

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \boldsymbol{\varepsilon} = \mathbf{S} \cdot \cdot \boldsymbol{\sigma}, \tag{1.1}$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \boldsymbol{\sigma} = \mathbf{C} \cdot \cdot \boldsymbol{\varepsilon}. \tag{1.2}$$

The tensor $\mathbf{S} = [S_{ijpq}]$ is the compliance tensor of rank four, $i, j,.. = 1,.., 3$. The summation convention is applied such that the repeated indices are implicitly summed over. The elasticity tensor of rank four is $\mathbf{C} = [C_{ijpq}]$. The relationship between the two tensors is:

$$C_{ijkl}S_{klpq} = I_{ijpq}, I_{ijpq} = (\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})/2, \mathbf{C} \cdot \cdot \mathbf{S} = \mathbf{I}. \tag{1.3}$$

Design and Analysis of Composite Structures for Automotive Applications: Chassis and Drivetrain, First Edition. Vladimir Kobelev.

Companion website: www.wiley.com/go/kobelev/automotive_suspensions

In (1.3), $\mathbf{I} = [I_{ijpq}]$ is the fourth rank identity tensor and δ_{jq} is the Kronecker symbol. The number of independent coefficients in the elasticity tensor varies depending on the grade of a material's symmetry. In general, it describes a tensor of rank four, which contains 81 different material specific elastic coefficients. The requirement that stress components occurring in the material are symmetric reduces the number of independent coefficients from a total of 81 to 36 due to the following correlation:

$$C_{ijkl} = C_{ijlk} = C_{jikl} = C_{jilk}. \tag{1.4}$$

1.1.2 Voigt's Matrix Notation of Hooke's Law

The generalized Hookes' law in Voigt's notation is a matrix equation in which the stiffness of the material is represented in the matrix with six rows and columns, and the stresses and strains in the column vectors with six components. According to Eq. (1.2), the generalized Hooke's law in Voigt's notation looks as follows:

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\varepsilon}, \text{or } \sigma_i = C_{ij}\varepsilon_j, \tag{1.5}$$

$$\boldsymbol{\varepsilon} = \mathbf{S} \cdot \boldsymbol{\sigma}, \text{or } \varepsilon_i = S_{ij}\sigma_j, \tag{1.6}$$

with $i, j = 1, .., 6$ and

$\boldsymbol{\sigma} = [\sigma_{11} = \sigma_1, \sigma_{22} = \sigma_2, \sigma_{33} = \sigma_3, \sigma_{13} = \sigma_4, \sigma_{32} = \sigma_5, \sigma_{21} = \sigma_6]^T$
Voigt's stress vector,
$\boldsymbol{\varepsilon} = [\varepsilon_{11} = \varepsilon_1, \varepsilon_{22} = \varepsilon_2, \varepsilon_{33} = \varepsilon_3, 2\varepsilon_{13} = \varepsilon_4, 2\varepsilon_{32} = \varepsilon_5, 2\varepsilon_{21} = \varepsilon_6]^T$
Voigt's strain vector,

C Voigt's elasticity matrix (6×6).

The elasticity matrix, which establishes the linear relationship between the stresses and distortions under a uniaxial load or stress, consists of 36 coefficients, 21 of which are independent of each other due to symmetry to the main diagonal. The structures of the elasticity and compliance matrices are:

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym. & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{pmatrix} \equiv \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{2212} \\ & & C_{3333} & C_{2223} & C_{3313} & C_{3312} \\ & & & C_{2323} & C_{2313} & C_{2312} \\ & sym. & & & C_{1313} & C_{1312} \\ & & & & & C_{1212} \end{pmatrix}, \tag{1.7}$$

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & sym. & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{pmatrix} \equiv \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 2S_{1123} & 2S_{1113} & 2S_{1112} \\ & S_{2222} & S_{2233} & 2S_{2223} & 2S_{2213} & 2S_{2212} \\ & & S_{3333} & 2S_{2223} & 2S_{3313} & 2S_{3312} \\ & & & 4S_{2323} & 4S_{2313} & 4S_{2312} \\ & sym. & & & 4S_{1313} & 4S_{1312} \\ & & & & & 4S_{1212} \end{pmatrix}. \tag{1.8}$$

There are different forms of direction-dependence or anisotropy of reinforced fiber composites.

In the most general case of elastic symmetry, there is one plane of symmetry. In this case, there are 13 elastic coefficients of each of the matrices (1.7) and (1.8). There is one additional relation between these coefficients, so 12 coefficients are independent. For example, if the symmetry plane is the plane 1–2, then the certain coefficients in both matrices (1.7) disappear:

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & C_{25} & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & sym. & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix} \equiv \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2222} & C_{2233} & 0 & C_{2213} & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2313} & 0 \\ & sym. & & & C_{1313} & 0 \\ & & & & & C_{1212} \end{pmatrix}, \tag{1.9}$$

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & S_{25} & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ & & & S_{44} & S_{45} & 0 \\ & sym. & & & S_{55} & 0 \\ & & & & & S_{66} \end{pmatrix} \equiv \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & 0 & 0 & 2S_{1112} \\ & S_{2222} & S_{2233} & 0 & 2S_{2213} & 2S_{2212} \\ & & S_{3333} & 0 & 0 & 2S_{3312} \\ & & & 4S_{2323} & 4S_{2313} & 0 \\ & sym. & & & 4S_{1313} & 0 \\ & & & & & 4S_{1212} \end{pmatrix}. \tag{1.10}$$

Due to the prevailing symmetries, however, the dominating case is the orthotropy. Here, there are three orthogonal planes of symmetry. If the intersection lines of the underlying symmetry planes are used as a coordinate system, the shear stresses and strains are completely decoupled in this case. If the material contains symmetries, the number of independent constants is abridged. Depending on the position and number of symmetry planes, different anisotropy cases are distinguished. For orthotropy only nine independent constants are required. For example, if the distances between fibers in unidirectional composite are distinct in two directions, the material will be orthotropic (Figure 1.1). The elasticity matrix in this case is as follows (Eqs. (2.107)–(2.108) and Vannucci 2018):

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & sym & & & S_{55} & 0 \\ & & & & & S_{66} \end{pmatrix} = \mathbf{C}^{-1}, \tag{1.11}$$

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{pmatrix},$$

$$\mathbf{C} \equiv \begin{pmatrix} \frac{S_{22}S_{33}-S_{23}^2}{S} & \frac{S_{13}S_{32}-S_{12}S_{33}}{S} & \frac{S_{12}S_{23}-S_{13}S_{22}}{S} & 0 & 0 & 0 \\ & \frac{S_{11}S_{33}-S_{13}^2}{S} & \frac{S_{21}S_{13}-S_{23}S_{11}}{S} & 0 & 0 & 0 \\ & & \frac{S_{11}S_{22}-S_{12}^2}{S} & 0 & 0 & 0 \\ & & & S_{44}^{-1} & 0 & 0 \\ & sym. & & & S_{55}^{-1} & 0 \\ & & & & & S_{66}^{-1} \end{pmatrix},$$

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{13}S_{23}S_{31}. \tag{1.12}$$

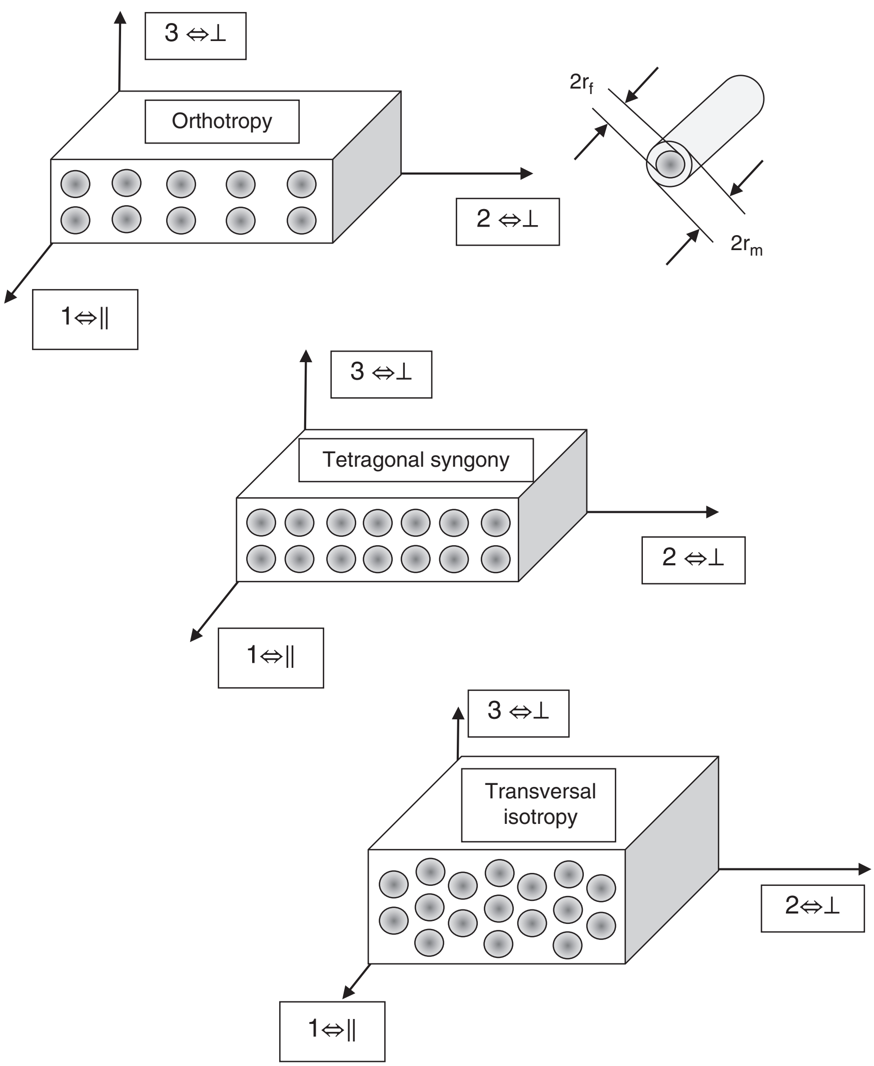

Figure 1.1 Coordinate system and elastic symmetry.

The special case of orthotropy is the tetragonal elastic syngony with six elastic constants:

$$\mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\ S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & sym & & & S_{44} & 0 \\ & & & & & S_{66} \end{pmatrix} = \mathbf{C}^{-1}, \mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & 0 \\ & & & & & C_{66} \end{pmatrix}. \quad (1.13)$$

Such symmetry appears in composites if the fibers arranged in a preferable orthogonal mosaic pattern with the equal distances between fibers in both directions (Figure 1.1).

The specific case of orthotropy is the transversal isotropy. Transversal isotropy presents when an infinite number of symmetries exist perpendicular to an isotropic plane. In the unidirectional material, the isotropic plane is normal to fiber direction. For example, the pattern of fibers allows threefold rotational symmetry. Each fiber after a rotation of 120° comes into the place of another fiber. Otherwise, if the fibers are chaotically arranged in the isotropic plane, directional dependence disappears as well. In these cases, the number of elastic constants reduces to five, because the additional constraint appears due to symmetry along axis 3:

$$C_{66} = C_{11} - C_{12}.$$

To complete the information given here, we will briefly discuss the elasticity tensor **C** for isotropic materials with no directional dependence. The isotropic material requires only the modulus of elasticity E and the contraction number ν:

$$S_{11} = 1/E, S_{12} = -\nu/E, S_{44} = 1/G = 2(1+\nu)/E.$$

1.1.3 Kelvin's Matrix Notation of Hooke's Law

Kelvin's notation was proposed by Kelvin (1856, 1878). The components of fourth rank elasticity tensors in three dimensions can be arranged into a matrix of six dimensions:

$$\hat{\boldsymbol{\sigma}} = \hat{\mathbf{C}} \cdot \hat{\boldsymbol{\varepsilon}}, \text{or } \hat{\sigma}_i = \hat{C}_{ij}\hat{\varepsilon}_j, \quad (1.14)$$

$$\hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\sigma}}, \text{or } \hat{\varepsilon}_j = \hat{S}_{ij}\hat{\sigma}_i, i,j = 1,..,6, \quad (1.15)$$

with

$\hat{\boldsymbol{\sigma}} = [\sigma_{11} = \hat{\sigma}_1, \sigma_{22} = \hat{\sigma}_2, \sigma_{33} = \hat{\sigma}_3, \sqrt{2}\sigma_{13} = \hat{\sigma}_4, \sqrt{2}\sigma_{32} = \hat{\sigma}_5, \sqrt{2}\sigma_{21} = \hat{\sigma}_6]^T$ Kelvin's stress vector,

$\hat{\boldsymbol{\varepsilon}} = [\varepsilon_{11} = \hat{\varepsilon}_1, \varepsilon_{22} = \hat{\varepsilon}_2, \varepsilon_{33} = \hat{\varepsilon}_3, \sqrt{2}\varepsilon_{13} = \hat{\varepsilon}_4, \sqrt{2}\varepsilon_{32} = \hat{\varepsilon}_5, \sqrt{2}\varepsilon_{21} = \hat{\varepsilon}_6]^T$ Kelvin's strain vector,

$\hat{\mathbf{C}}$ Kelvin's elasticity tensor (6×6):

$$\hat{\mathbf{C}} = \begin{pmatrix} \hat{C}_{11} & \hat{C}_{12} & \hat{C}_{13} & \hat{C}_{14} & \hat{C}_{15} & \hat{C}_{16} \\ & \hat{C}_{22} & \hat{C}_{23} & \hat{C}_{24} & \hat{C}_{25} & \hat{C}_{26} \\ & & \hat{C}_{33} & \hat{C}_{34} & \hat{C}_{35} & \hat{C}_{36} \\ & & & \hat{C}_{44} & \hat{C}_{45} & \hat{C}_{46} \\ & sym. & & & \hat{C}_{55} & \hat{C}_{56} \\ & & & & & \hat{C}_{66} \end{pmatrix},$$

$$\widehat{\mathbf{C}} \equiv \begin{pmatrix} C_{1111} & C_{1122} & C_{1133} & \sqrt{2}C_{1123} & \sqrt{2}C_{1113} & \sqrt{2}C_{1112} \\ & C_{2222} & C_{2233} & \sqrt{2}C_{2223} & \sqrt{2}C_{2213} & \sqrt{2}C_{2212} \\ & & C_{3333} & \sqrt{2}C_{2223} & \sqrt{2}C_{3313} & \sqrt{2}C_{3312} \\ & & & 2C_{2323} & 2C_{2313} & 2C_{2312} \\ & sym. & & & 2C_{1313} & 2C_{1312} \\ & & & & & 2C_{1212} \end{pmatrix}, \tag{1.16}$$

$\widehat{\mathbf{S}}$ Kelvin's compliance tensor (6×6):

$$\widehat{\mathbf{S}} = \begin{pmatrix} \widehat{S}_{11} & \widehat{S}_{12} & \widehat{S}_{13} & \widehat{S}_{14} & \widehat{S}_{15} & \widehat{S}_{16} \\ & \widehat{S}_{22} & \widehat{S}_{23} & \widehat{S}_{24} & \widehat{S}_{25} & \widehat{S}_{26} \\ & & \widehat{S}_{33} & \widehat{S}_{34} & \widehat{S}_{35} & \widehat{S}_{36} \\ & & & \widehat{S}_{44} & \widehat{S}_{45} & \widehat{S}_{46} \\ & sym. & & & \widehat{S}_{55} & \widehat{S}_{56} \\ & & & & & \widehat{S}_{66} \end{pmatrix} \equiv \begin{pmatrix} S_{1111} & S_{1122} & S_{1133} & \sqrt{2}S_{1123} & \sqrt{2}S_{1113} & \sqrt{2}S_{1112} \\ & S_{2222} & S_{2233} & \sqrt{2}S_{2223} & \sqrt{2}S_{2213} & \sqrt{2}S_{2212} \\ & & S_{3333} & \sqrt{2}S_{2223} & \sqrt{2}S_{3313} & \sqrt{2}S_{3312} \\ & & & 2S_{2323} & 2S_{2313} & 2S_{2312} \\ & sym. & & & 2S_{1313} & 2S_{1312} \\ & & & & & 2S_{1212} \end{pmatrix}. \tag{1.17}$$

The components of Kelvin's matrices of the elasticity and compliance will change under a change of coordinate system following the tensor transformation rule. The components of Kelvin's matrices change as the components of the second-rank tensor change and this notable feature is significant for optimization methods, as recognized in Mehrabadi and Cowin (1990).

The symmetry of Kelvin's matrices for orthotropic, tetragonal syngony and transversal isotropy is analogous to those of Voigt's matrices (Table 1.1).

Table 1.1 Coefficients of Voigt's and Kelvin's matrices for orthotropic materials.

	Kelvin's notation	Voigt's notation
$i = 1, 2, 3$	$\widehat{S}_{ii} = S_{ii} = \frac{1}{E_i}$, $\sigma_{11} = \widehat{\sigma}_1 = \sigma_1$, $\varepsilon_{11} = \widehat{\varepsilon}_1 = \varepsilon_1$, $\sigma_{22} = \widehat{\sigma}_2 = \sigma_2$, $\varepsilon_{22} = \widehat{\varepsilon}_2 = \varepsilon_2$, $\sigma_{33} = \widehat{\sigma}_3 = \sigma_3$, $\varepsilon_{33} = \widehat{\varepsilon}_3 = \varepsilon_3$,	
$i, j = 1, 2, 3$, $i \neq j$	$\widehat{S}_{ji} = S_{ji} = -\frac{\nu_{ij}}{E_i}$,	
$i, j = 1, 2, 3$, $i \neq j$, $k = 4, 5, 6$	$\widehat{S}_{kk} = \frac{1}{2G_{ij}}$ $\sqrt{2}\sigma_{13} = \widehat{\sigma}_4, \sqrt{2}\varepsilon_{13} = \widehat{\varepsilon}_4$, $\sqrt{2}\sigma_{32} = \widehat{\sigma}_5, \sqrt{2}\varepsilon_{32} = \widehat{\varepsilon}_5$, $\sqrt{2}\sigma_{21} = \widehat{\sigma}_6, \sqrt{2}\varepsilon_{21} = \widehat{\varepsilon}_6$.	$S_{kk} = \frac{1}{G_{ij}}$, $\sigma_{13} = \sigma_4, 2\varepsilon_{13} = \varepsilon_4$, $\sigma_{32} = \sigma_5, 2\varepsilon_{32} = \varepsilon_5$, $\sigma_{21} = \sigma_6, 2\varepsilon_{21} = \varepsilon_6$.

1.2 Unidirectional Fiber Bundle

1.2.1 Components of a Unidirectional Fiber Bundle

The anisotropy of a composite material can be defined as the difference in physical properties, such as the modulus and Poisson's ratio, when the material is loaded in different directions. The modern and comprehensive discussion of this topic is given by Vannucci (2018).

The properties of the composite material are particularly dependent on the proportion of fiber and matrix material in the end product. The fiber volume content V_f describes the ratio of the volume fraction of the fiber material to the total volume of the composite. The achievable fiber volume ratio and the fiber orientation are strongly dependent on the manufacturing process used.

The radius of the fibers r_f and the radius r_m of the surrounding cylinders made of matrix, or resin, material. The imaginary radius r_m is determined from the volume proportions of the fibers, such that the volume content of fiber and matrix materials to be preserved. Explicitly, the volume fractions of fibers V_f and matrix V_m must be directly proportional to the cross-sectional areas of fibers and matrix, respectively:

$$s_f/s_m = V_f/V_m, V_m = 1 - V_f, \tag{1.18}$$

$$s_f = \pi\, r_f^2, s_m = \pi(r_m^2 - r_f^2). \tag{1.19}$$

There is the following relation between the radii of the fibers r_f and the radii of the matrix cylinders r_m:

$$r_m = r_f/\sqrt{V_f}. \tag{1.20}$$

For the unidirectional layer, some assumptions for idealization are made that are required for the calculation of the mechanical properties. It is assumed that the fibers are ideally straight and parallel in one direction without interruption. Furthermore, the fibers are evenly distributed over the cross-section and adhere ideally to the matrix; that is, no shifts at the fiber-matrix interface occur during loading.

1.2.2 Elastic Properties of a Unidirectional Fiber Bundle

With knowledge of the special properties of fiber and matrix material and the structure of the unidirectional layer, it is easy to imagine that the unidirectional layer has anisotropic, that is, direction-dependent, mechanical properties. The stiffness of the unidirectional layer in the longitudinal direction of the fibers under tensile load is naturally many times higher than in the transverse direction. In the longitudinal direction of the fibers, the mechanical properties also depend on the direction of loading; that is, the strength properties are not as good under compressive loading as under tensile loading.

A unidirectional reinforced material with macroscopically homogeneous modules of elasticity is transversal isotropic. If such a fiber composite material also has an even distribution of reinforcing fibers, this results in a direction-independent modulus of elasticity normal to the fiber direction. For the transverse isotropy, five material constants are sufficient to characterize the stiffness behavior of the composite. With a unidirectional layer, the isotropic plane is normal to the longitudinal direction of the fiber; that is, all sections normal to the isotropic plane have the same properties. The fiber composite material is modeled from a series of elastic round cylinders of unlimited length, which are located in a cylindrical, tubular, elastic matrix (Figure 1.1).

For a complete description of the linear elasticity law for the unidirectional layer in space, only five independent constants are necessary. The engineering constants are (Figure 1.1):

E_L	longitudinal modulus of elasticity in the "1" direction,
E_T	transverse modulus of elasticity in the "2" and "3" directions,
G_{TL}	shear modulus in ("1–2") and ("1–3") planes,
G_{TT}	shear modulus in the ("2–3") plane,
ν_{TL}	Poisson's ratio in ("1–2") and ("1–3") planes,
ν_{TT}	Poisson's ratio in the ("2–3") plane.

The engineering constants are also known as basic elasticity values. The different experimental methods are available for determining the basic elasticity values. For our purposes, as well as for the introductory design of components, estimation of basic elasticity values could be done by means of analytical formulas as follows.

For a spatial unidirectional layer with known engineering constants, the law of elasticity is formulated with engineering constants; that is, the modulus of elasticity, shear modulus and transverse contraction numbers provide the compliance matrix in Voigt's notation:

$$\mathbf{S}^{(0)} = \begin{pmatrix} 1/E_L & -\nu_{TL}/E_L & -\nu_{TL}/E_L & & & \\ -\nu_{TL}/E_L & 1/E_T & -\nu_{TT}/E_T & & & \\ -\nu_{TL}/E_L & -\nu_{TT}/E_T & 1/E_T & & & \\ & & & 1/G_{TT} & & \\ & & & & 1/G_{TL} & \\ & & & & & 1/G_{TL} \end{pmatrix}. \tag{1.21}$$

The compliance matrix in Kelvin's notation reads:

$$\hat{\mathbf{S}}^{(0)} = \begin{pmatrix} 1/E_L & -\nu_{TL}/E_L & -\nu_{TL}/E_L & & & \\ -\nu_{TL}/E_L & 1/E_T & -\nu_{TT}/E_T & & & \\ -\nu_{TL}/E_L & -\nu_{TT}/E_T & 1/E_T & & & \\ & & & 1/2G_{TT} & & \\ & & & & 1/2G_{TL} & \\ & & & & & 1/2G_{TL} \end{pmatrix}. \tag{1.22}$$

1.2.3 Effective Elastic Constants of Unidirectional Composites

There are several approximate relationships between the modules of fibers and matrix and the homogenized, effective modules of elasticity of the composite material (Hill 1963; ECSS-E-HB-32-20 2011, Schürmann 2007; Younes et al. 2012). The formulas

Table 1.2 Effective modules of unidirectional composite material (Schürmann 2007).

Effective modules of a unidirectional composite material	
Modulus of elasticity	$E_L = E_f V_f + E_m V_m$
	$E_T = \frac{E_m}{1-\nu_m^2} \cdot \frac{1 + 0.85 \cdot V_f^{\ 2}}{V_m^{\ 1.25} + \frac{E_m}{E_{f.T}} \cdot \frac{V_f}{1-\nu_m^2}}$
Shear modulus	$G_{TL} = \frac{G_m(1 + 0.4\sqrt{V_f})}{V_m^{\ 1.45} + V_f G_m / G_{f.TL}}$
	$G_{TT} = \frac{E_T}{2 + 2\nu_{TT}}$
Poisson coefficient	$\nu_{\mathrm{TL}} = V_f \cdot \nu_{fTL} + V_m \cdot \nu_m$
	$\nu_{\mathrm{TT}} = \nu_{fTT} V_f + \nu_m V_m \frac{1 + \nu_m - \nu_{TL} E_m / E_L}{1 - \nu_m^{\ 2} + \nu_m \nu_{TL} E_m / E_L}$

Table 1.3 Input values for calculation of effective modules.

$\nu_{\mathrm{f.\,TL}}$	Transverse, longitudinal, transverse Poisson' ratio of fibers
ν_{m}	Poisson' ratio of matrix (resin)
$\nu_{f.\,TT}$	Transverse Poisson' ratio of fibers
$\nu_{m,\,eff}$	Effective matrix cross Poisson' ratio
ν_{LT}	Poisson's ratio
G_m	Shear modulus of the matrix (resin)
$G_{f.\,TL}$	Transverse longitudinal shear modulus of the fiber
$E_{f.\,T}$	Transverse modulus of elasticity of the fiber
E_f	Longitudinal modulus of elasticity of the fiber
E_m	Modulus of elasticity of the matrix (resin)
V_f	Fiber volume content
$V_m = 1 - V_f$	Matrix volume content

(Schürmann 2007) provide estimations of effective elastic constants, especially for a high volume concentration of fibers. These expressions are given in Tables 1.2 and 1.3.

For the fiber volume content of $V_f = 0.6$, the effective technical constants of the unidirectional layer:

$$\mathrm{E_L} = 45680MPa, \mathrm{E_T} = 13698MPa, \mathrm{G_{TT}} = 5101MPa,$$

$$\mathrm{G_{TL}} = 5384MPa, \nu_{\mathrm{TL}} = 0.272, \nu_{\mathrm{TT}} = 0.3427.$$

Table 1.4 Elastic constants and densities of matrix and fibers of UD fiberglass.

E_m	3200	E_f	74 000	MPa
G_m	1185	G_f	30 327	MPA
ν_{m}	0.35	ν_f	0.22	
ρ_{m}	1800	ρ_{f}	2000	$\mathrm{kg\,m^{-3}}$

For this example (Table 1.4), the elasticity and compliance matrices from Eqs. (1.10) and (1.12) of the unidirectional layer in Voigt's notation read:

$$\mathbf{S}^{(0)} = \begin{pmatrix} 21.89 & -5.95 & -5.95 & & & \\ -5.95 & 73.0 & -25.02 & & & \\ -5.95 & -25.02 & 73.0 & & & \\ & & & 196.04 & & \\ & & & & 185.7 & \\ & & & & & 185.7 \end{pmatrix} [10^{-12}\mathrm{Pa}^{-1}] \tag{1.23}$$

$$\mathbf{C}^{(0)} = \begin{pmatrix} 48987 & 6079 & 6079 & & & \\ 6079 & 16276 & 6074 & & & \\ 6079 & 6074 & 16276 & & & \\ & & & 5100 & & \\ & & & & 5384 & \\ & & & & & 5384 \end{pmatrix} [\mathrm{MPa}]. \tag{1.24}$$

1.3 Rotational Transformations of Material Laws, Stress and Strain

The layered view and the directional dependency of the mechanical properties of laminates means that when designing composite material components at least two coordinate systems must be distinguished; the component coordinate system xyz and the fiber coordinate system $x_f y_f z_f$ (Figure 1.2). The direction of a fiber coordinate system coincides with fiber direction, and the z- and z_f-axes are normal to the layer direction.

The direction in the longitudinal direction of a fiber of a unidirectional layer is marked with "x_f." The direction is transverse to the longitudinal direction of the fiber and the

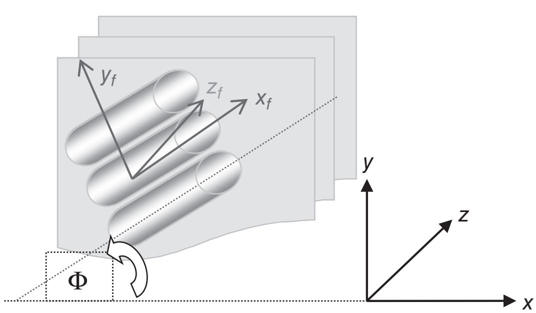

Figure 1.2 Coordinate system associated with fiber direction and rotated coordinate system of the structural component.

layer plane is described using the symbol "y_f." The third spatial direction, z, which is also transverse to the fiber direction but also perpendicular to the layer plane, has to be considered and is called the normal plane. The thickness direction of the laminate is the z-coordinate of the component coordinate system. The fiber angle Φ is defined as the mathematically positive angle between the x-axis and x_f-axes. The rotation of the component coordinate system around the z-axis is described by the rotation angle Φ. For further details, see Vannucci (2018).

1.3.1 Rotation of Fourth Rank Elasticity Tensors

The elasticity and compliance properties for a general anisotropic material are usually expressed in an arbitrarily chosen coordinate system. To exclude the orientational arbitrariness of a coordinate system, a special coordinate system based on a material's intrinsic orientation is needed. For example, the intrinsic coordinate system is associated with the direction of fibers of a unidirectional composite (layer coordinate system). The elasticity and compliance tensor in the layer (intrinsic) coordinate system possess the components:

$$\mathbf{C}^{(0)} = \mathbf{C}(\Phi = 0) = [c_{pqns}], \mathbf{S}^{(0)} = \mathbf{S}(\Phi = 0) = [s_{pqns}]. \quad (1.25)$$

The rotation coordinate system of an anisotropic material modifies the components of elasticity and compliance tensors. The tensors of the fourth rank $\mathbf{S}(\Phi)$ and $\mathbf{C}(\Phi)$ in the rotated coordinate system possess the components:

$$S_{ijkl} = S_{ijkl}(\Phi), C_{ijkl} = C_{ijkl}(\Phi), i, j, k, l = 1, 2, 3. \quad (1.26)$$

The components depend on the orientation of the principal axes of anisotropy, associated with the intrinsic layer coordinate system, relative to the axes of rotated coordinate system. The relations between the tensors in the intrinsic (1.25) and rotated coordinate system (1.26) are:

$$\begin{aligned}&\mathbf{S} = \mathbf{t} \cdot \mathbf{t} \cdot \mathbf{S}^{(0)} \cdot \mathbf{t}^T \cdot \mathbf{t}^T, \mathbf{C} = \mathbf{t} \cdot \mathbf{t} \cdot \mathbf{C}^{(0)} \cdot \mathbf{t}^T \cdot \mathbf{t}^T,\\ &S_{ijkl} = s_{pqns} t_{ip} t_{jq} t_{kn} t_{ls}, C_{ijkl} = c_{pqns} t_{ip} t_{jq} t_{kn} t_{ls}.\end{aligned} \quad (1.27)$$

The rotation matrix $\mathbf{t} = [t_{lk}]$ is orthogonal:

$$\mathbf{t} \cdot \mathbf{t}^T = \mathbf{I}, t_{sp} t_{pl} = \delta_{sl}. \quad (1.28)$$

For the rotation along the z-axis with the angle Φ:

$$\mathbf{t}(\varphi) = [t_{ij}] = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{with } c = \cos\Phi, s = \sin\Phi.$$

The angle Φ can differ from point to point of the structural element. The general rotations in space could be studied using the established methods (Altmann 1986).

1.3.2 Rotation of Elasticity Matrices in Voigt's Notation

It is not possible to perform the rotation of the elasticity matrices with one orthogonal rotation matrix, as required by the tensor transformation law. Two transformation matrices $\mathbf{T}_\sigma$, $\mathbf{T}_\varepsilon$ are required for the rotation of the elasticity matrices in Voigt's notation. These matrices are used also for rotation of stress and strain vectors:

$$\mathbf{C} = \mathbf{T}_\sigma \cdot \mathbf{C}^{(0)} \cdot \mathbf{T}_\sigma^T, \quad \mathbf{S} = \mathbf{T}_\varepsilon \cdot \mathbf{S}^{(0)} \cdot \mathbf{T}_\varepsilon^T, \tag{1.29}$$

In the Eq. (1.29) the following values are used:

$\mathbf{T}_\sigma$	σ-transformation matrix,
$\boldsymbol{\sigma} = \mathbf{T}_\sigma \cdot \boldsymbol{\sigma}^{(0)}$	stress vector in a rotated coordinate system,
$\boldsymbol{\sigma}^{(0)}$	stress vector in an intrinsic coordinate system,
$\mathbf{T}_\varepsilon$	ε-transformation matrix,
$\boldsymbol{\varepsilon} = \mathbf{T}_\varepsilon \cdot \boldsymbol{\varepsilon}^{(0)}$	strain vector in a rotated coordinate system,
$\boldsymbol{\varepsilon}^{(0)}$	strain vector in an intrinsic coordinate system,
$\mathbf{C} = [C_{ij}]$	elasticity matrix in a rotated coordinate system,
$\mathbf{C}^{(0)} = [c_{ij}]$	elasticity matrix in an intrinsic coordinate system,
$\mathbf{S} = [S_{ij}]$	compliance matrix in a rotated coordinate system,
$\mathbf{S}^{(0)} = [s_{ij}]$	compliance matrix in an intrinsic coordinate system.

The matrices $\mathbf{T}_\sigma$, $\mathbf{T}_\varepsilon$ in Eq. (1.29) obey the rules:

$$\mathbf{T}_\sigma{}^{-1} = \mathbf{T}_\varepsilon{}^{\mathrm{T}}, \mathbf{T}_\varepsilon{}^{-1} = \mathbf{T}_\sigma{}^{\mathrm{T}}.$$

Their components read:

$$\mathbf{T}_\sigma = \begin{pmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -sc & sc & 0 & 0 & 0 & c^2 - s^2 \end{pmatrix}, \tag{1.30}$$

$$\mathbf{T}_\varepsilon = \begin{pmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2sc & 2sc & 0 & 0 & 0 & c^2 - s^2 \end{pmatrix}. \tag{1.31}$$

We substitute of the relations

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}^{(0)T} \cdot \mathbf{T}_\sigma{}^T = \boldsymbol{\sigma}^{(0)T} \cdot \mathbf{T}_\varepsilon{}^{-1}, \boldsymbol{\sigma} = \mathbf{T}_\sigma \cdot \boldsymbol{\sigma}^{(0)} = (\mathbf{T}_\varepsilon{}^T)^{-1} \cdot \boldsymbol{\sigma}^{(0)}.$$

and (1.29) into the expression for elastic energy density $2W_\sigma = \boldsymbol{\sigma}^T \cdot \mathbf{S} \cdot \boldsymbol{\sigma}$. The substitution assures that the elastic energy density remains invariant during rotation:

$$\begin{cases} 2W_\sigma = \boldsymbol{\sigma}^T \cdot \mathbf{S} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma}^{(0)T} \cdot \mathbf{S}^{(0)} \cdot \boldsymbol{\sigma}^{(0)}, \\ 2W_\varepsilon = \boldsymbol{\varepsilon}^T \cdot \mathbf{C} \cdot \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)T} \cdot \mathbf{C}^{(0)} \cdot \boldsymbol{\varepsilon}^{(0)}. \end{cases} \tag{1.32}$$

The formulas for the components of compliance matrix $\mathbf{S}$ in rotated coordinate system in terms of the components of compliance matrices in intrinsic coordinate system $\mathbf{S}^{(0)}$ are:

$$
\begin{aligned}
S_{11} &= s_{11}c^4 + 2(s_{12} + s_{66})c^2s^2 + s_{22}s^4,\\
S_{22} &= s_{22}c^4 + 2(s_{12} + s_{66})c^2s^2 + s_{11}s^4\\
S_{12} &= s_{12}(c^4 + s^4) + (s_{11} + s_{22} - 4s_{66})c^2s^2,\\
S_{66} &= s_{66}(c^2 - s^2)^2 + (s_{11} + s_{22} - 2s_{12})c^2s^2,\\
S_{16} &= -cs(s_{22}c^2 - s_{11}s^2 - (s^2 - c^2)(s_{12} + 2s_{66})),\\
S_{26} &= -cs(s_{11}c^2 - s_{22}s^2 + (s^2 - c^2)(s_{12} + 2s_{66})).
\end{aligned}
\tag{1.33}
$$

1.3.3 Rotation of Elasticity Matrices in Kelvin's Notation

The rotation of the elasticity and compliance matrices in the Kelvin's notation uses the tensor transformation rule with the rotation matrix $\hat{\mathbf{T}}(\Phi)$ (Mehrabadi and Cowin 1990):

$$\hat{\mathbf{C}} = \hat{\mathbf{T}} \cdot \hat{\mathbf{C}}^{(0)} \cdot \hat{\mathbf{T}}^{-1}, \hat{\mathbf{S}} = \hat{\mathbf{T}} \cdot \hat{\mathbf{S}}^{(0)} \cdot \hat{\mathbf{T}}^{-1}. \tag{1.34}$$

The six-dimensional orthogonal rotation matrix rotates the components of the strain and stress vectors in Kelvin's notation as well:

$$\hat{\mathbf{T}}(\Phi) = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & \sqrt{2}sc \\ s^2 & c^2 & 0 & 0 & 0 & -\sqrt{2}sc \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -\sqrt{2}sc & \sqrt{2}sc & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \hat{\mathbf{T}}^{-1}(\Phi) = \hat{\mathbf{T}}^T(\Phi), \tag{1.35}$$

$\hat{\boldsymbol{\sigma}} = \hat{\mathbf{T}} \cdot \hat{\boldsymbol{\sigma}}^{(0)}$	stress vector in a rotated coordinate system,
$\boldsymbol{\sigma}^{(0)}$	stress vector in an intrinsic coordinate system,
$\hat{\boldsymbol{\varepsilon}} = \hat{\mathbf{T}} \cdot \hat{\boldsymbol{\varepsilon}}^{(0)}$	strain vector in a rotated coordinate system,
$\hat{\boldsymbol{\varepsilon}}^{(0)}$	strain vector in an intrinsic coordinate system,
$\hat{\mathbf{C}} = [\hat{C}_{ij}]$	elasticity matrix in a rotated coordinate system,
$\hat{\mathbf{C}}^{(0)} = [\hat{c}_{ij}]$	elasticity matrix in an intrinsic coordinate system,
$\hat{\mathbf{S}} = [\hat{S}_{ij}]$	compliance matrix in a rotated coordinate system,
$\hat{\mathbf{S}}^{(0)} = [\hat{s}_{ij}]$	compliance matrix in an intrinsic coordinate system.

The elastic energy density (1.32) remains invariant during rotation:

$$\begin{cases} 2W_\sigma = \hat{\boldsymbol{\sigma}}^T \cdot \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}^{(0)T} \cdot \hat{\mathbf{S}}^{(0)} \cdot \hat{\boldsymbol{\sigma}}^{(0)}, \\ 2W_\varepsilon = \hat{\boldsymbol{\varepsilon}}^T \cdot \hat{\mathbf{C}} \cdot \hat{\boldsymbol{\varepsilon}} = \hat{\boldsymbol{\varepsilon}}^{(0)T} \cdot \hat{\mathbf{C}}^{(0)} \cdot \hat{\boldsymbol{\varepsilon}}^{(0)}. \end{cases} \tag{1.36}$$

1.4 Elasticity Matrices for Laminated Plates

1.4.1 Voigt's Matrix Notation for Anisotropic Plates

Lightweight structures are typically thin-walled and flat and must transmit forces and torques from different directions. The fibers are therefore arranged according to load in order to make use of the advantages of the composite material. This is only possible by stacking different layers, which results in the so-called laminate or multi-layer composite. However, it is possible not only to stack unidirectional layers, but also to use individual layers whose fibers are at an angle of 0° or 90° to each other and are therefore referred to as cross-laminated layers.

With the information about the laminate structure, that is, layer thickness and layer sequence as well as the orientation of the layers, the constitutive law for laminates reads Vannucci (2018) and Toorani and Lakis (2000):

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}. \tag{1.37}$$

The elasticity matrix in Voigt's notation of a dimension of 6×6 determines the in-plane deformation and bending of the laminated plate. Its minors **A,B,D** of the third rank are:

$\mathbf{A} = \int_{-h/2}^{h/2} \mathbf{Q}(z) dz$	in-plane quadrant, square 3×3 matrix,
$\mathbf{B} = \int_{-h/2}^{h/2} \mathbf{Q}(z) z dz$	coupling quadrant, square 3×3 matrix,
$\mathbf{D} = \int_{-h/2}^{h/2} \mathbf{Q}(z) z^2 dz$	bending quadrant, square 3×3 matrix,
$\boldsymbol{\varepsilon}^T = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}]$	strain vector in Voigt's notation,
$\boldsymbol{\kappa}^T = [\kappa_{11}, \kappa_{22}, 2\kappa_{12}]$	curvature vector in Voigt's notation,
$\mathbf{N}^T = [N_{11}, N_{22}, N_{12}]$	in-plane force vector in Voigt's notation,
$\mathbf{M}^T = [M_{11}, M_{22}, M_{12}]$	bending moment vector in Voigt's notation.

This form of vector representation is common in engineering, although it requires different transformation matrices for all relevant elastic quantities, such as stress and strain vector and elasticity matrices.

For derivation, the stiffness relations for each singular layer are:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \mathbf{Q} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}, \mathbf{Q}(z) = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}. \tag{1.38}$$

The coefficients of the stiffness matrix (1.38) read:

$$Q_{ij} = Q_{ij}(z) = C_{ij} - C_{i3}C_{i3}/C_{33}, i, j = 1, 2, 6.$$

For a unidirectional layer, the reduced stiffness matrix is:

$$\mathbf{Q} = \begin{bmatrix} \dfrac{\mathrm{E_L}}{1 - \nu_{\mathrm{TL}} \cdot \nu_{LT}} & \dfrac{\nu_{LT} \cdot \mathrm{E_L}}{1 - \nu_{\mathrm{TL}} \cdot \nu_{LT}} & 0 \\ \dfrac{\nu_{\mathrm{LT}} \cdot \mathrm{E_L}}{1 - \nu_{\mathrm{TL}} \cdot \nu_{LT}} & \dfrac{\mathrm{E}_T}{1 - \nu_{\mathrm{TL}} \cdot \nu_{LT}} & 0 \\ 0 & 0 & \mathrm{G_{TL}} \end{bmatrix}.$$

The reactions and moments in Voigt's notation are expressed through the in-plane stress in singular layers:

$$\mathbf{N}^T = [N_{11}, N_{22}, N_{12}] = \int_{-h/2}^{h/2} [\sigma_{11}, \sigma_{22}, \sigma_{12}] dz, \tag{1.39}$$

$$\mathbf{M}^T = [M_{11}, M_{22}, M_{12}] = \int_{-h/2}^{h/2} z^2 [\sigma_{11}, \sigma_{22}, \sigma_{12}] dz. \tag{1.40}$$

The stiffness evaluation and rotation of anisotropy orientation are also appropriate to thin shell models.

1.4.2 Rotation of Matrices in Voigt's Notation

For the rotation of each minor in Voigt's matrix notation, two rotation matrices $\widetilde{\mathbf{T}}_\sigma$ and $\widetilde{\mathbf{T}}_\varepsilon$ are used:

$$\begin{aligned} \{\mathbf{A}, \mathbf{B}, \mathbf{D}\} &= \widetilde{\mathbf{T}}_\sigma \cdot \{\mathbf{A}^{(0)}, \mathbf{B}^{(0)}, \mathbf{D}^{(0)}\} \cdot \widetilde{\mathbf{T}}_\sigma^T, \\ \{\mathbf{A}^{-1}, \mathbf{B}^{-1}, \mathbf{D}^{-1}\} &= \widetilde{\mathbf{T}}_\varepsilon \cdot \{\mathbf{A}^{(0)-1}, \mathbf{B}^{(0)-1}, \mathbf{D}^{(0)-1}\} \cdot \widetilde{\mathbf{T}}_\varepsilon^T. \end{aligned} \tag{1.41}$$

$\widetilde{\mathbf{T}}_\sigma$	σ-transformation matrix for plates (3 × 3),
$\widetilde{\mathbf{T}}_\varepsilon$	ϵ-Transformation matrix for plates (3 × 3),
$\{\mathbf{A}, \mathbf{B}, \mathbf{D}\}$	minors of elasticity matrix in a rotated coordinate system,
$\{\mathbf{A}^{(0)}, \mathbf{B}^{(0)}, \mathbf{D}^{(0)}\}$	minors of elasticity matrix in an intrinsic coordinate system,
$\{\mathbf{A}^{-1}, \mathbf{B}^{-1}, \mathbf{D}^{-1}\}$	minors of compliance matrix in a rotated coordinate system,
$\{\mathbf{A}^{(0)^{-1}}, \mathbf{B}^{(0)^{-1}}, \mathbf{D}^{(0)^{-1}}\}$	minors of compliance matrix in an intrinsic coordinate system.

The rotation matrices obey the rules $\widetilde{\mathbf{T}}_\sigma^{-1} = \widetilde{\mathbf{T}}_\varepsilon^{\mathrm{T}}$, $\widetilde{\mathbf{T}}_\varepsilon^{-1} = \widetilde{\mathbf{T}}_\sigma^{\mathrm{T}}$ and read:

$$\widetilde{\mathbf{T}}_\sigma = \begin{pmatrix} c^2 & s^2 & 2cs \\ s^2 & s^2 & -2cs \\ -sc & sc & c^2 - s^2 \end{pmatrix}, \tag{1.42}$$

$$\widetilde{\mathbf{T}}_\varepsilon = \begin{pmatrix} c^2 & s^2 & cs \\ s^2 & s^2 & -cs \\ -2sc & 2sc & c^2 - s^2 \end{pmatrix}. \tag{1.43}$$

1.4.3 Kelvin's Matrix Notation for Anisotropic Plates

The constitutive law for laminates in Kelvin's notation reads (Vannucci 2018):

$$\begin{bmatrix} \widehat{\mathbf{N}} \\ \widehat{\mathbf{M}} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \\ \widehat{\mathbf{B}} & \widehat{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{\varepsilon}} \\ \widehat{\boldsymbol{\kappa}} \end{bmatrix}. \tag{1.44}$$

The elasticity matrix in Kelvin's notation of the second rank determines the in-plane deformation and bending. Its minors $\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{D}}$ of the third rank are:

$\widehat{\mathbf{A}} = \int_{-h/2}^{h/2} \widehat{\mathbf{Q}}(z)dz$	in-plane quadrant, square 3×3 matrix,
$\widehat{\mathbf{B}} = \int_{-h/2}^{h/2} \widehat{\mathbf{Q}}(z)zdz$	coupling quadrant, square 3×3 matrix,
$\widehat{\mathbf{D}} = \int_{-h/2}^{h/2} \widehat{\mathbf{Q}}(z)z^2dz$	bending quadrant, square 3×3 matrix,
$\widehat{\boldsymbol{\varepsilon}}^T = [\varepsilon_{11}, \varepsilon_{22}, \sqrt{2}\varepsilon_{12}]$	strain vector,
$\widehat{\boldsymbol{\kappa}}^T = [\kappa_{11}, \kappa_{22}, \sqrt{2}\kappa_{12}]$	curvature vector,
$\widehat{\mathbf{N}}^T = [N_{11}, N_{22}, \sqrt{2}N_{12}]$	in-plane force vector,
$\widehat{\mathbf{M}}^T = [M_{11}, M_{22}, \sqrt{2}M_{12}]$	bending moment vector.

Kelvin's notation was derived earlier (Kelvin 1856) than Voigt's (Voigt 1910); it uses the solitary orthogonal transformation matrix and obeys the tensor transformation law, but Kelvin's notation is not used habitually in engineering.

For derivation, the stiffness laws for singular layers are:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sqrt{2}\sigma_{12} \end{bmatrix} = \widehat{\mathbf{Q}} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \sqrt{2}\varepsilon_{12} \end{bmatrix}, \widehat{\mathbf{Q}}(z) = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16}/\sqrt{2} \\ Q_{12} & Q_{22} & Q_{26}/\sqrt{2} \\ Q_{16}/\sqrt{2} & Q_{26}/\sqrt{2} & Q_{66}/2 \end{bmatrix}. \tag{1.45}$$

The reactions and moments in Kelvin's notation are expressed through in-plane stress in singular layers:

$$\widehat{\mathbf{N}}^T = [N_{11}, N_{22}, \sqrt{2}N_{12}] = \int_{-h/2}^{h/2} [\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12}]dz, \tag{1.46}$$

$$\widehat{\mathbf{M}}^T = [M_{11}, M_{22}, \sqrt{2}M_{12}] = \int_{-h/2}^{h/2} z^2[\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12}]dz. \tag{1.47}$$

1.4.4 Rotation of Matrices in Kelvin's Notation

The rotation of the elasticity and compliance minors in Kelvin's notation (Vannucci 2018) is performed with the orthogonal transformation matrix $\widetilde{\mathbf{T}}$:

$$\begin{aligned} &\{\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{D}}\} = \widetilde{\mathbf{T}} \cdot \{\widehat{\mathbf{A}}^{(0)}, \widehat{\mathbf{B}}^{(0)}, \widehat{\mathbf{D}}^{(0)}\} \cdot \widetilde{\mathbf{T}}^{-1}, \\ &\{\widehat{\mathbf{A}}^{-1}, \widehat{\mathbf{B}}^{-1}, \widehat{\mathbf{D}}^{-1}\} = \widetilde{\mathbf{T}} \cdot \{\widehat{\mathbf{A}}^{(0)-1}, \widehat{\mathbf{B}}^{(0)-1}, \widehat{\mathbf{D}}^{(0)-1}\} \cdot \widetilde{\mathbf{T}}^{-1}. \end{aligned} \tag{1.48}$$

$\widetilde{\mathbf{T}}$	transformation matrix for plates tensor (3×3),
$\{\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{D}}\}$	minors of elasticity matrix in a rotated coordinate system,
$\{\widehat{\mathbf{A}}^{(0)}, \widehat{\mathbf{B}}^{(0)}, \widehat{\mathbf{D}}^{(0)}\}$	minors of elasticity matrix in an intrinsic coordinate system,
$\{\widehat{\mathbf{A}}^{-1}, \widehat{\mathbf{B}}^{-1}, \widehat{\mathbf{D}}^{-1}\}$	minors of compliance matrix in a rotated coordinate system,
$\{\widehat{\mathbf{A}}^{(0)^{-1}}, \widehat{\mathbf{B}}^{(0)^{-1}}, \widehat{\mathbf{D}}^{(0)^{-1}}\}$	minors of compliance matrix in an intrinsic coordinate system.

The three-dimensional orthogonal transformation matrix is used for rotation:

$$\widetilde{\mathbf{T}}(\theta) = \begin{pmatrix} c^2 & s^2 & \sqrt{2}sc \\ s^2 & c^2 & -\sqrt{2}sc \\ -\sqrt{2}sc & \sqrt{2}sc & c^2 - s^2 \end{pmatrix}, \widetilde{\mathbf{T}}^T = \widetilde{\mathbf{T}}^{-1}. \tag{1.49}$$

1.5 Coupling Effects of Anisotropic Laminates

1.5.1 Orthotropic Laminate Without Coupling

For common structural components, the usual aim is to eliminate coupling between the in-plane quadrant **A** and the bending quadrant **D.** Depending on how the quadrants are occupied, different coupling properties of the laminate can take place. If all components of matrix $\mathbf{Q}(z)$ are symmetric, $\mathbf{Q}(z)=\mathbf{Q}(-z)$, the coupling quadrant **B** disappears and they assume the diagonal form:

$$\mathbf{A}=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \mathbf{B}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{D}=\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}. \tag{1.50}$$

Moreover, the plate behaves as an orthotropic material; that is, the shear stress in the principal coordinates does not affect the directional stresses. Analogously, pure warping leads to no bending of a plate in principal coordinates. For example, this laminate structure occurs if the laminate type is $[0^{\circ}/90^{\circ}/90^{\circ}, 0^{\circ}]$ or $[0^{\circ}/\pm 45^{\circ}/../\pm 45^{\circ}, 0^{\circ}]$. This stacking leads to a fully symmetric structure with decoupling between in-plane deformations and bending.

1.5.2 Anisotropic Laminate Without Coupling

The coupling between bending and in-plane deformation also disappears if the matrices assume the form:

$$\mathbf{A}=\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \mathbf{B}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{D}=\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}. \tag{1.51}$$

However, the plate behaves as an anisotropic material; that is, the shear stress in the principal coordinates has an effect on the directional stresses. Analogously, pure warping causes bending of a plate in principal coordinates.

1.5.3 Anisotropic Laminate With Coupling

For some structural applications coupling is desired because this provides an additional grade of control. The smart adjustment of the additional control grade allows elimination of some undesired properties of a structure that can present in an uncoupled structure (Reddy and Miravete 1995).

The coupling between bending and in-plane deformation presents as (if the matrices are in the fully populated form):

$$\mathbf{A}=\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \mathbf{B}=\begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}, \mathbf{D}=\begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}. \tag{1.52}$$

The plate behaves as an anisotropic material; that is, the shear stress in the principal coordinates has an effect on the directional stresses. Pure warping causes bending of a plate in principal coordinates. Additionally, the bending and in-plane deformations are also coupled.

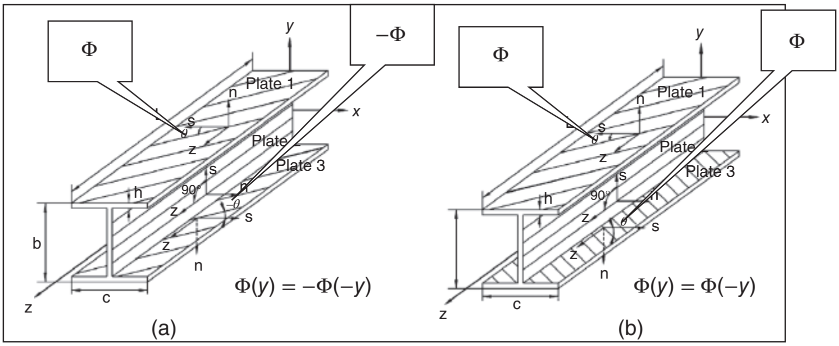

Figure 1.3 (a) Circumferentially asymmetric stiffness configuration (CAS), (b) circumferentially uniform stiffness configuration (CUS) (Librescu and Song 2006, Figure. 13.1.1) The local coordinate system *xyz* is associated with the twist-beam *z*-axis. Source: University of Siegen.

1.5.4 Coupling Effects in Laminated Thin-Walled Sections

The theory of an elastic coupling of the composite thin-walled beams with open cross-sections was developed by Chandra et al. (1990). This theory is based on the model by Vlasov and Gjelsvik, and includes the effects of transverse-shear deformation and coupling between laminate reactions and moments. Figure 1.3 shows the laminate structure for a flexure-torsion and tensile-torsion coupling for a fiber composite component with a rectangular cross-section. The beams that possess the circumferentially asymmetric stiffness (CAS) configuration (see Librescu and Song 2006; Rehfield and Atilgan 1989 and Vo and Lee 2008) are shown in Figure 1.3. This structural configuration is attainable when the ply-angle distributions are equal in the top and bottom walls as well as the lateral walls. The equations that describe coupling between bending and torsion are derived in Section 4.4.4 (Librescu and Song 2006). The circumferentially uniform stiffness (CUS) and CAS configurations are known as antisymmetric and symmetric configurations, respectively (Smith and Chopra 1991).

1.6 Conclusions

This chapter compiles the equations of the anisotropic elastic behavior of fiber composites. The common tensor and Voigt's vector notations are used for the derivation of elasticity equations of an anisotropic medium. Besides these, Kelvin's notations are also presented because this form of vector notation leads to a tensor-invariant representation and, consequently, is favorable for use in optimization problems.

For further reading, the following reference works about anisotropic elasticity and composite materials are recommended: Halpin (1992), Tsai (1968), Hill (1950), Barbero (1999), Hull and Clyne (1996), Altenbach et al. (2004), Tin (1996), Gay et al. (2013), Peters (1998), Kollár and Springer (2003), Tschoegl et al. (2002), Lakes and Wineman (2006), Hertz et al. (1981), Abramovich and Livshitz (1994), Kaiser (1999), Kaiser and Francescatti (1996), Kroker (2013), Dugas (2002), Kindervater (2008), Santiuste et al. (2008) and Foye (1972).

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