CHAPTER 1

Rewards for Problem-Based Approach: Range, Rigor, and Resilience

Range Ignites Curiosity

As educators, we understand the importance of depth and breadth in learning. For beginning piano players, listening to a concert pianist perform can ignite curiosity and inspire them to practice more. In mathematics, there seems to be a reticence to hear the symphony for fear that it will be too much, too soon, and by limiting the range, we limit curiosity and growth.

Our AwesomeMath Enrichment programs are filled with students whose schools placed a ceiling on their mathematics education and they are seeking outside resources. Sometimes, this is because the school is concerned that if students are accelerated too quickly, they won't have the maturity to truly understand what is being taught. Other times, the school is concerned that if students move two or three levels ahead, by the time they are seniors they will have run out of classes to take. Another common reason our parents provide is that the school thinks it will be *too much* information for the student at too young an age and will spoil their performance in future classes. With a problem-based curriculum, there is no ceiling on learning and there is ample depth and breadth of subjects to keep students challenged throughout their lifetime. The real danger of not giving students adequate challenge and range to satiate their curiosity is that they will turn off on mathematics and learning altogether. Students need to hear the beauty and art that is mathematics to kindle their joy of learning.

Why Is a Collaborative Problem-Based Approach Worthwhile?

Community Outreach

Dr. Emily Herzig: Collaboration in the classroom has many benefits. Research has demonstrated that active learning improves performance on exams, and the effect is especially large for disadvantaged students. Currently, education is not equitably accessible to all students, with students from underserved populations and first-generation college students in particular facing additional obstacles to entering, navigating, and excelling in higher education. Thus, collaborative learning in the classroom could be key to closing the achievement gap and allowing capable but underprepared students to reach greater success in math.

Furthermore, a collaborative and problem-based approach gives younger students a more accurate impression of what higher-level math entails. Students too often carry the belief that success in math is based in rote memorization and drilling problems. While those skills are certainly useful for efficiently carrying out the basic mechanics of solving problems, it is equally important that students are able to formulate and interpret more complex problems, and work with their colleagues to develop and execute problem-solving strategies. Arguably, this process is also what makes math such an enticing subject. A focus on collaborative problem solving is a great way to attract students to and prepare them for careers in math.

Even the terms used for learning piano and learning mathematics are different: Students ***play*** piano and ***work*** on math problems. There needs to be a fundamental shift in approach and exposure to a range of problems that are harder and more interesting so that students can see where math can take them. So much of math education today is about *waiting*:

- Wait until high school, and then what you've been learning in middle school will be useful.
- Wait until college, and then what you've been learning in high school will be useful.
- Wait until you learn topic x before you can see the beauty of topic y or z or beyond.
- Wait until you learn a subject, like geometry, in isolation before you have the ability to learn how it connects and contributes to other areas such as algebra, engineering, art, science, etc.

And so on. . . .

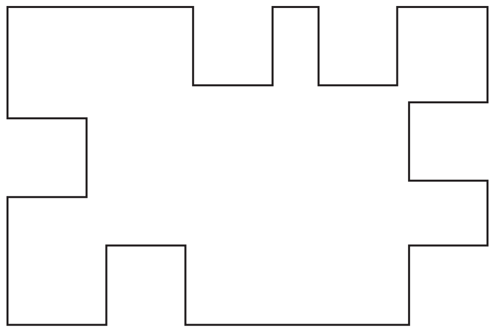
> *A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.*
>
> *G.H. Hardy*

Students need to *work together* to weave a pattern of ideas with what they know and can add more as their knowledge base grows. To fully appreciate the full tapestry of mathematics is not by adding one color of thread at a time, but weaving a picture with various thread colors within a group of learners who are just as excited by the beauty as you are.

Collaborative problem-based learning holds the key to unlocking abilities and mathematical growth. The collaboration nurtures interpersonal skills, leadership traits, and conflict resolution, while providing interesting problems gives students a common goal to work toward where they can connect and share ideas. Students can see the fun and beauty in mathematics, start to play, and see where math can take them long term.

A fun problem that works well in a group is the following:

The diagram shows a polygon made by removing six 2×2 squares from the sides of an 8×12 rectangle. Find the perimeter of this polygon.



The answer is 60. The square removed from the lower-right corner of the rectangle does not change the perimeter of the polygon, but when each of the other five squares is removed, the perimeter is increased by 4. Thus, the perimeter of the polygon is $2 \times 8 + 2 \times 12 + 5 \times 4 = 16 + 24 + 20 = 60$.[1]

Students are allowed to be transported by a musical symphony before understanding individual notes, so why not provide the symphony of mathematics and introduce students to its wonders and challenges? Parents and educators will read books beyond a student's personal reading level so that they can hear the richness of language and be exposed to more intricate sentence structure. There are so many wonderful places an enriching math education can take you.

When educating young math students, you can let them know that they are the captains of their ship, but as their navigator, you can guide them to really interesting destinations and expose them to a wider range of mathematics.

Following are some ideas for increasing range in a mathematics program:

Logic problems. Thinking through these problems primes the brain for mathematical reasoning.

Combinatorics. This discrete mathematics field involves counting and probability.

The mathematics of science. Solving, for example, real-world physics problems adds range and connectivity across academic disciplines.

Mind benders and puzzles. These are a fun way to introduce mathematical concepts, such as magic squares.

Game theory. Utilize mathematical modeling to understand the strategies employed by rational game players (decision makers).

Computational linguistics. Apply computational analysis to language and speech for linguistic phenomena.

And so much more!

There is a world of mathematics to discover, and the playful pursuit will ignite their interest and provide them with the introspection to know their strengths and passions. Context while explaining mathematical concepts is also important, so it's not just range within the topic of mathematics but range outside the topic as well. Knowing the history and story of what they are learning makes a huge difference. Mathematical discoveries were made to solve real-life problems, and if students learn the story, they are more connected with the material.

For example, students will often ask, "When will I ever really need to use algebra?" It's easy to give them the F.U.D. answer (a marketing term that means Fear, Uncertainty, and Doubt to nudge consumers into decisions): "If you don't learn algebra, you won't do well on the SAT, and then you won't get into a good college." That's a lousy answer, and unfortunately is the message heard by many students, either directly or indirectly.

When running a mathematics club, a student asked this exact question, even though her mother had majored in math when she was in college. In response, she was assigned a one- to two-page paper and oral report on the importance of algebra, which, as you can imagine, wasn't initially well received. The paper needed to include:

1. Research the story of algebra (why, how, and by whom was it created).
2. Describe where algebra is used in life (why, how, and when it is necessary today).
3. Explore how algebra would be helpful in her own life.

She was given the resources to get started, and then she knocked this assignment out of the ballpark. Her understanding deepened and her love for the topic, in turn, grew.

Below is an algebra problem that students can also use logic to figure out. These problems help engage the class and get them thinking!

I have two hourglass clocks filled with sand. One empties in 4 minutes, the other in 7 minutes. How can I use them to measure exactly 10 minutes?

Solution: Start both clocks at the same time. When the 4 minute timer runs out, start it again. When the 7 minute timer runs out, turn the 4 minute time over and it will run for the required extra 3 minutes.
(*Math Leads for Mathletes,* Book 2, page 48)

Providing a *range* in topics and connecting mathematics with other disciplines, such as history, will ignite students' *curiosity* to dig deeper and find the beauty and relevance of the mathematics they learn.

Studying ancient number systems is a great way to show the progression of numbers from being an adjective, e.g., a one-to-one correspondence as in one sheep per tally mark, to being a noun, e.g., thinking of numbers in the abstract. Students are a microcosm of what took ancient civilizations thousands of years to understand.

These systems are also a great way to learn about base systems beyond base 10. Babylonians used a hexigesimal system (base 60) and the Mayans used a vigesimal (base 20) system, probably because in the warmer climate, they used their fingers and toes for counting. It's an easier mental leap to begin to learn binary systems (base 2) that are integral in today's digital world.

When exploring a range of topics, this is not something to be done alone. Exploration should, of course, be a shared experience. Students need to present what they've learned to the class so that they can all grow together and gain the confidence for bigger challenges.

Rigor Taps Critical Thinking

With the high-pressure stakes of standardized testing, rigor is starting to take on a negative connotation. It's been used to *pump and dump*, namely, shove as much information as possible into a student so that they, in turn, can dump that information on to a bubble sheet of multiple-choice answers.

In this book, we refer to rigor as a way to tap critical thinking so that students can have meaningful experiences and try novel approaches to solve problems. That means presenting the student with *problems* instead of *exercises*. Math education must be more than a series of easily obtained answers (exercises) (e.g., find the perimeter of a square, training students to do what a computer can do better). Problem solving goes much deeper and taps into what makes us human, namely multiple creative approaches with a string of steps to solving challenging and interesting problems.

Scaffolding problems based on difficulty adds rigor and progression for students of various skill levels so as to reach a wider range of kids. Below are some example problems from the book *Math Leads for Mathletes,* Book 2, centered on least common multiple (lcm). You can always start with exercises as a warm-up, but then ramp up the challenge to more thought-provoking problems.

PROBLEMS

1. What is the least common multiple of 6, 8, 24, and 30?
2. What is the least common multiple of 585 and 10 985?

3. Four cargo ships left a port at noon, January 2, 2010. The first ship returns to this port every 4 weeks, the second every 8 weeks, the third every 12 weeks, and the fourth every 16 weeks. When did all four ships meet again at the port?
4. It is given that the number of spots on a Dalmatian is less than 20. Also, the number of spots is divisible by 3. Furthermore, when the number of spots is divided by the number of legs, the remainder is 3. Finally, the number of spots leaves a remainder of 6 when divided by 9. Find the number of spots on the Dalmatian.
5. I am thinking of a number. The least common multiple of my number and 9 is 45. What could my number be?

SOLUTIONS

1. The lcm (6, 8, 24, 30) = $2^{(3\times3\times5)}$ = 120.
2. The lcm (585, 10 985) = $3^{(2\times5)} \times 13^3$ = 98 865.
3. The lcm (4, 8, 12, 16) = $2^{(4\times3)}$ = 48. Hence, all ships will meet again at the port in 48 weeks' time, on December 4, 2010.
4. The lcm (3, 4) = 12. Adding the remainder 3 gives 15. We find lcm (3, 9) = 9. Adding the remainder 6 gives 15 as well. The lcm (3, 4, 9) = 36; therefore, other possible solutions are $15 + 36 = 51$, $15 + 2 \bullet 36 = 87$, $15 + 3 \bullet 36, \ldots$, but 15 is the only positive solution less than 20.
5. The prime factorization of $45 = 3^2 \bullet 5$. Therefore, the number could be 5, 15, 45 because lcm (5, 9) = lcm (15, 9) = lcm (45, 9) = 45.

Teaching through challenging problems requires rigor on both the part of the educator and the student, but also requires maintaining a balance between providing these problems and learning foundational concepts. If students can be exposed to the tough problems in mathematics early on, this rigorous training will make other challenges easier to face. Just as it's easier to jog a mile if you've been training for harder races, when you learn the critical thinking that comes along with problem solving, then every other topic that requires math (and most do, in one way or another, e.g., economics) will be easier to understand.

Rigor is both the result of critical thinking and creativity as well as the discipline to learn the foundational concepts necessary to be successful. This rigor is easier to manage when you have a collaborative team (the teacher and fellow students) offering support and encouragement along the way. This leads to the next reward: resilience.

Resilience Is Born Through Creativity

Making the simple complicated is commonplace; making the complicated simple, awesomely simple, that's creativity.

Charles Mingus

Problem solving requires creativity to reach a solution, as there isn't always just one clear-cut approach. Creativity requires bravery and resilience. People feel that creativity just happens, when in actuality it is the product of trial and error – requiring the resilience to persevere. Pushing through failures to the other side is a worthy goal because the benefits are so high, such as developing the resilience to take on harder and harder challenges.

In competitive chess, coaches will say you need to lose 10000 games to become a grand master, so whenever a student loses a game or makes a mistake with a problem, *celebrate!* You are on the road to mastery, and the resilience of making mistakes will take you there.

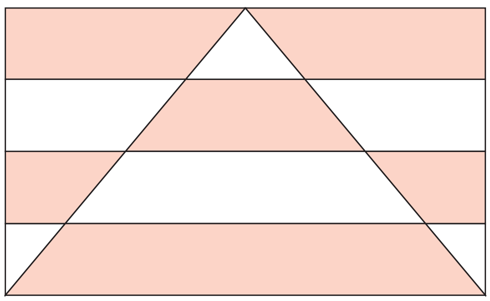
Life is not a zero-sum game (in game theory, this is the mathematical representation that one player's gain is equivalent to another player's loss). Mistakes make us better and help to develop a growth mindset when viewed in the proper light.

Students, through rigor and range, grow their courage for trying new things as well as hone their creativity and resilience through constraints. How are creativity and constraints related? Math, in and of itself, can be a beautiful and creative pursuit – elegant proofs, creative problem solving, and being connected to a global community where ideas can flourish. That creativity, of course, operates within the constraints of mathematical reasoning and rigorous proofs.

There are myriad problems where you have to jump in headfirst and creatively play before a solution will present itself to you. They take time, patience, and resilience to continue and chip away at them, and that process can be the most exciting part.

Following is a problem that requires a lot of thought and needs a well-conceived plan of attack, building resilience while also providing the thrill of solving!

The diagram here shows a 12 by 20 rectangle, split into four stripes of equal widths, all surrounding an isosceles triangle. Find the area of the shaded region.



The answer is 150. The triangle has base 20 and altitude 12, so its area is $(20 \times 12)/2 = 120$. Each strip has a width that is ¼ the height of the entire triangle, so the small triangle in the top strip has area $120 \times (\frac{1}{4})^2 = 15/2$. Similarly, the area of the triangle within the second strip has area $120 \times (\frac{1}{2})^2 - 120 \times (\frac{1}{4})^2 = 45/2$. The area of the triangle within the third strip has area $120 \times (\frac{3}{4})^2 - 120 \times (\frac{1}{2})^2 = 75/2$, and the area of the triangle within the fourth strip has area $120 - 120 \times (\frac{3}{4})^2 = 105/2$. Each strip is ¼ the width of the entire rectangle, so each has area $\frac{1}{4} \times 12 \times 20 = 60$. It follows that the sum of the shaded areas within the four strips is $(60 - 15/2) + 45/2 + (60 - 75/2) + 105/2 = 150$.[2]

Because of these constraints, people will reach the conclusion that mathematics is dry and formulaic, but anyone that has ever delved deeper beyond the typical classroom approach to math understands how these constraints free the mind and lead to amazing discoveries. Certainly, math isn't the only area where constraints can be creative. One only has to listen to minimalistic music, read poetry, or see modern art for examples of limiting choices to create simple and elegant forms. Many interesting and delectable meals have been invented by people who are constrained by budget, choices, or geography.

Notes

1. Purple Comet Math Meet*!* 2019 contest, http://purplecomet.org.
2. Purple Comet*!* Math Meet contest 2019.