CHAPTER 1

FUNCTIONAL ANALYSIS

Functional analysis is the branch of mathematics that deals with spaces of functions and the transformation properties of functions between function spaces in terms of operators. Since these operators could be differential or integral, it makes functional analysis extremely useful in the study of differential and integral equations. Since the function and space concepts could be used to represent many different things, functional analysis has found a wide range of applications in science and engineering. It is also at the very foundation of numerical simulation. The most rudimentary concept of functional analysis is the definition of a function, which is basically a rule or a mapping that relates the members of one set of objects to the members of another set. In this chapter, we discuss the basic properties of functions like continuity, limit, convergence, inverse, differentiation, integration, etc.

1.1 CONCEPT OF FUNCTION

We start with a quick review of the basic concepts of **set theory**. Let S be a set of objects of any kind: points, numbers, functions, vectors, etc. When s

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is an element of the set S, we show it as $s \in S$. For finite sets, we may define S by listing its elements as $S = \{s_1, s_2, \ldots, s_n\}$. For infinite sets, S is usually defined by a phrase describing the condition to be a member of the set, for example, $S = \{\text{All points on the sphere of radius } R\}$. When there is no room for confusion, we may also write an infinite set like the set of all odd numbers as $S = \{1, 3, 5, \ldots\}$. When each member of a set A is also a member of set B, we say that A is a **subset** of B and write $A \subset B$. The phrase B covers or contains A is also used. The **union** of two sets, $A \cup B$, consists of the elements of both A and B. The **intersection** of two sets, A and B, is defined as $A \cap B = \{\text{All elements common to } A \text{ and } B\}$. When two sets have no common element, their intersection is called the **null set** or the **empty set**, which is usually shown by ϕ. The **neighborhood** of a point (x_1, y_1) in the xy-plane is the set of all points (x, y) inside a circle centered at (x_1, y_1) with the radius δ: $(x - x_1)^2 + (y - y_1)^2 < \delta^2$. An **open set** is defined as the set of points with neighborhoods entirely within the set. The interior of a circle defined by $x^2 + y^2 < 1$ is an open set. A **boundary point** is a point whose every neighborhood contains at least one point in the set and at least one point that does not belong to the set. The boundary of $x^2 + y^2 < 1$ is the set of points on the circumference, that is, $x^2 + y^2 = 1$. An open set plus its boundary is a **closed set**.

A **function** f is in general a rule or a relation that uniquely associates members of one set A with the members of another set B. The concept of function is essentially the same as that of **mapping**, which in general is so broad that it allows mathematicians to work with them without any resemblance to the simple class of functions with numerical values. The set A that f acts upon is called the **domain**, and the set B composed of the elements that f can produce is called the **range**. For **single-valued** functions, the common notation used is

$$f : x \to f(x).$$

Here, f stands for the function or the mapping that acts upon a single number x, which is an element of the domain, and produces $f(x)$, which is an element of the range. In general, f refers to the function itself, and $f(x)$ refers to the value it returns. However, in practice, $f(x)$ is also used to refer to the function itself. In this chapter, we basically concern ourselves with functions that take numerical values as $f(x)$, where the **argument** x is called the **independent variable**. We usually define a new variable y as $y = f(x)$, which is called the **dependent variable**.

Functions with multiple variables, that is, **multivariate** functions, can also be defined. For example, for each point (x, y) in some region of the xy-plane, we may assign a unique real number $f(x, y)$ according to the rule $f : (x, y) \to f(x, y)$. We now say that $f(x, y)$ is a function of two independent variables as x and y. In applications, $f(x, y)$ may represent physical properties like the temperature or the density distribution of a flat disc with negligible thickness. Definition of a function can be extended to cases with several independent variables as $f(x_1, \ldots, x_n)$, where n stands for the number of independent variables.

The term function is also used for the objects that associate more than one element in the domain to a single element in the range. Such objects are called **multiple-to-one** relations. For example,

$f(x, y) = 2xy + x^2$:	single-valued or one-to-one,
$f(x) = \sin x$:	many-to-one,
$f(x, y) = x + x^2$:	single-valued,
$f(x) = x^2,\ x \neq 0$:	two-to-one,
$f(x, y) = \sin xy$:	many-to-one.

Sometimes the term "function" is also used for relations that map a single point in its domain to multiple points in its range. As we shall discuss in Chapters 9 and 10, such functions are called **multivalued functions**, which are predominantly encountered in **complex analysis**.

1.2 CONTINUITY AND LIMITS

Similar to its usage in everyday language, the word continuity in mathematics also implies the absence of abrupt changes. In astrophysics, pressure and density distributions inside a solid neutron star are represented by continuous functions of the radial position as $P(r)$ and $\rho(r)$, respectively. This means that small changes in the radial position inside the star also result in small changes in the pressure and density. At the surface R, where the star meets the outside vacuum, pressure has to be continuous. Otherwise, there will be a net force on the surface layer, which will violate the static equilibrium condition. In this regard, in static neutron star models, pressure has to be a monotonic decreasing function of r, which smoothly drops to zero at the surface, that is, $P(R) = 0$. On the other hand, the density at the surface can change abruptly from a finite value to zero. This is also in line with our everyday experiences, where solid objects have sharp contours marked by density discontinuities. For gaseous stars, both pressure and density have to vanish continuously at the surface. In constructing physical models, deciding on which parameters are going to be taken as continuous at the boundaries requires physical reasoning and some insight. Usually, a collection of rules that have to be obeyed at the boundaries is called the **junction conditions** or the **boundary conditions**.

We are now ready to give a formal definition of continuity as follows:

Continuity: A numerically valued function $f(x)$ defined in some domain D is said to be continuous at the point $x_0 \in D$, if for any positive number $\varepsilon > 0$, there is a neighborhood N about x_0 such that $|f(x) - f(x_0)| < \varepsilon$ for every point common to both N and D, that is, $N \cap D$. If the function $f(x)$ is continuous at every point of D, we say it is continuous in D.

We finally quote the following theorems without proof [1, 2]:

Theorem 1.1. Let $f(x)$ be a continuous function at x, and let $\{x_n\}$ be a sequence of points in the domain of $f(x)$ with the limit $x_n \to x$ as $n \to \infty$, then the following is true:

$$\lim_{n\to\infty} f(x_n) \to f(x). \tag{1.1}$$

Theorem 1.2. For a function $f(x)$ defined in D, if the limit $f(x_n) \to f(x)$ as $n \to \infty$ exists whenever $x_n \in D$ and

$$\lim_{n\to\infty} x_n \to x \in D, \tag{1.2}$$

then the function $f(x)$ is continuous at x. For the limit $f(x_n) \to f(x)$ as $n \to \infty$ to exist, it is sufficient to show that the right and the left limits agree, that is,

$$\lim_{\varepsilon\to 0} f(x-\varepsilon) = \lim_{\varepsilon\to 0} f(x+\varepsilon), \tag{1.3}$$

$$f(x^-) = f(x^+) = f(x). \tag{1.4}$$

In practice, the second theorem is more useful in showing that a given function is continuous. If a function is discontinuous at a finite number of points in its interval of definition $[x_a, x_b]$, it is called **piecewise continuous**.

Generalization of these theorems to multivariate functions is easily accomplished by taking x to represent a point in a space with n independent variables as $x = (x_1, x_2, \ldots, x_n)$. However, with more than one independent variable, one has to be careful. Consider the simple function

$$F(x) = \frac{x^2 - y^2}{x^2 + y^2}, \tag{1.5}$$

which is finite at the origin. However, depending on the direction of approach to the origin, $f(x, y)$ takes different values:

$$\lim_{(x,y)\to(0,0)} f(x,y) \to 0 \text{ if we approach along the } y = x \text{ line,}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \to 0 \text{ if we approach along the } x \text{ axis,}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \to -1 \text{ if we approach along the } y \text{ axis.}$$

Hence, the limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, and the function $f(x, y)$ is not continuous at the origin.

Basic properties of limits, which we give for functions with two variables, also hold for a general multivariate function: Let $u = f(x, y)$ and $v = g(x, y)$ be two functions defined in the domain D of the xy-plane.

Limit: If the limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = g_0 \tag{1.6}$$

exist, then we can write

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y) + g(x,y)] = f_0 + g_0, \tag{1.7}$$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \cdot g(x,y)] = f_0 \cdot g_0, \tag{1.8}$$

$$\lim_{(x,y)\to(x_0,y_0)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{f_0}{g_0}, \quad g_0 \neq 0. \tag{1.9}$$

If the functions $f(x,y)$ and $g(x,y)$ are continuous at (x_0,y_0), then the functions

$$f(x,y) + g(x,y),\ f(x,y)g(x,y),\ \text{and}\ \frac{f(x,y)}{g(x,y)} \tag{1.10}$$

are also continuous at (x_0,y_0), provided that in the last case, $g(x,y)$ is different from zero at (x_0,y_0).

Let $F(u,v)$ be a continuous function defined in some domain D_0 of the uv-plane, and let $F(f(x,y),g(x,y))$ be defined for (x,y) in D. Then, if (f_0,g_0) is in D_0, we can write

$$\lim_{(x,y)\to(x_0,y_0)} F(f(x,y),g(x,y)) = F(f_0,g_0). \tag{1.11}$$

If $f(x,y)$ and $g(x,y)$ are continuous at (x_0,y_0), then so is $F(f(x,y),g(x,y))$. In evaluating limits of functions that can be expressed as ratios, L'Hôpital's rule is very useful.

L'Hôpital's rule: Let f and g be differentiable functions in the interval $a \leq x < b$ with $g'(x) \neq 0$, where the upper limit b could be finite or infinite. If f and g have the limits

$$\lim_{x\to b} f(x) = 0 \quad \text{and} \quad \lim_{x\to b} g(x) = 0, \tag{1.12}$$

or

$$\lim_{x\to b} f(x) = \infty \quad \text{and} \quad \lim_{x\to b} g(x) = \infty, \tag{1.13}$$

and if the limit

$$\boxed{\lim_{x\to b} \frac{f'(x)}{g'(x)} = L} \tag{1.14}$$

exists, where L could be zero or infinity, then

$$\boxed{\lim_{x\to b} \frac{f(x)}{g(x)} = L.} \tag{1.15}$$

1.3 PARTIAL DIFFERENTIATION

A necessary and sufficient condition for the derivative of $f(x)$ to exist at x_0 is that the left, $f'_-(x_0)$, and the right, $f'_+(x_0)$, derivatives exist and be equal (Figure 1.1), that is,

$$f'_+(x_0) = f'_-(x_0), \tag{1.16}$$

where

$$f'_+(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \tag{1.17}$$

$$f'_-(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}. \tag{1.18}$$

When the derivative exists, we always mean a finite derivative. If $f(x)$ has derivative at x_0, it means that it is continuous at that point. When the derivative of $f(x)$ exists at every point in the interval (a, b), we say that $f(x)$ is differentiable in (a, b) and write its derivative as

$$\frac{df(x)}{dx} \text{ or } f'(x). \tag{1.19}$$

Geometrically, derivative at a point is the **slope** of the tangent line at that point:

$$\tan\theta = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}. \tag{1.20}$$

When a function depends on two variables, $z = f(x, y)$, the partial derivative with respect to x at (x_0, y_0) is defined as the limit

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x} \tag{1.21}$$

and we show it as in one of the following forms:

$$\frac{\partial f}{\partial x}(x_0, y_0),\ f_x(x_0, y_0),\ \frac{\partial z}{\partial x}(x_0, y_0), \text{or} \left(\frac{\partial f}{\partial x}\right)_0. \tag{1.22}$$

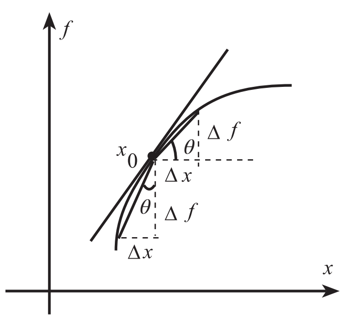

Figure 1.1 Derivative is the slope of the tangent line.

Similarly, the partial derivative with respect to y at (x_0, y_0) is defined as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}. \tag{1.23}$$

A geometric interpretation of the partial derivative is that the section of the surface $z = f(x, y)$ with the plane $y = y_0$ is the curve $z = f(x, y_0)$; hence, the partial derivative $f_x(x_0, y_0)$ is the slope of the tangent line (Figure 1.2) to $z = f(x, y_0)$ at (x_0, y_0). Similarly, the partial derivative $f_y(x_0, y_0)$ is the slope of the tangent line to the curve $z = f(x_0, y)$ at (x_0, y_0). For a multivariate function, the partial derivative with respect to the ith independent variable is defined as

$$\frac{\partial f(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{\Delta x_i}. \tag{1.24}$$

For a given $f(x, y)$, the partial derivatives f_x and f_y are functions of x and y, and they also have partial derivatives which are written as

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right], \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right], \tag{1.25}$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right], \quad f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]. \tag{1.26}$$

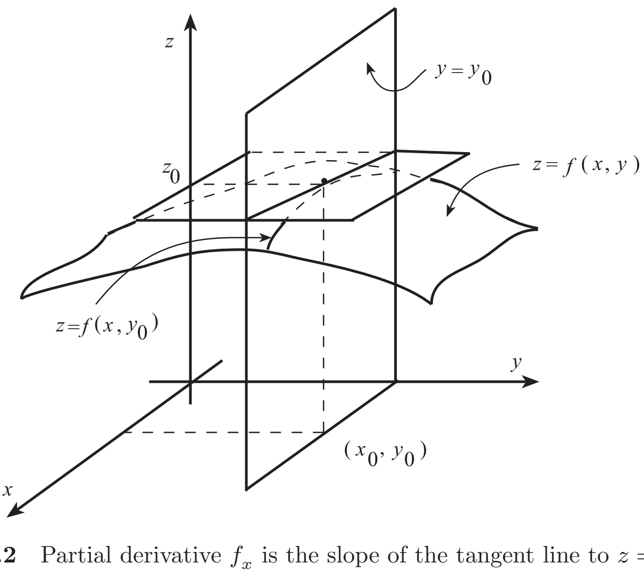

Figure 1.2 Partial derivative f_x is the slope of the tangent line to $z = f(x, y_0)$.

When f_{xy} and f_{yx} are continuous at (x_0, y_0), then the relation $f_{xy} = f_{yx}$ holds at (x_0, y_0). Under similar conditions, this result can be extended to cases with more than two independent variables and to higher order mixed partial derivatives.

1.4 TOTAL DIFFERENTIAL

When a function depends on two or more variables, we have seen that the limit at a point may depend on the direction of approach. Hence, it is important that we introduce a nondirectional derivative for functions with several variables. Given the function

$$f(x, y, z) = xz - y^2, \tag{1.27}$$

for a displacement of $\Delta r = (\Delta x, \Delta y, \Delta z)$, we can write its new value as

$$f(r + \Delta r) = (x + \Delta x)(z + \Delta z) - (y + \Delta y)^2 \tag{1.28}$$

$$= xz + x\Delta z + z\Delta x + \Delta x\Delta z - y^2 - 2y\Delta y - (\Delta y)^2 \tag{1.29}$$

$$= (xz - y^2) + (z\Delta x - 2y\Delta y + x\Delta z) + \Delta x\Delta z - (\Delta y)^2, \tag{1.30}$$

where r stands for the point (x, y, z), and Δr is the displacement $(\Delta x, \Delta y, \Delta z)$. For small Δr, the change in $f(x, y, z)$ to first order can be written as

$$\Delta f \simeq f(r + \Delta r) - f(r) = (xz - y^2) + (z\Delta x - 2y\Delta y + x\Delta z) - (xz - y^2), \tag{1.31}$$

$$\Delta f \simeq z\Delta x - 2y\Delta y + x\Delta z. \tag{1.32}$$

Considering that the first-order partial derivatives of f are given as

$$\frac{\partial f}{\partial x} = z, \quad \frac{\partial f}{\partial y} = -2y, \quad \frac{\partial f}{\partial z} = x. \tag{1.33}$$

Equation (1.32) is nothing but

$$\Delta f \simeq \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z. \tag{1.34}$$

In general, if a function $f(x, y, z)$ is differentiable at (x, y, z) in some domain D with the partial derivatives

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \tag{1.35}$$

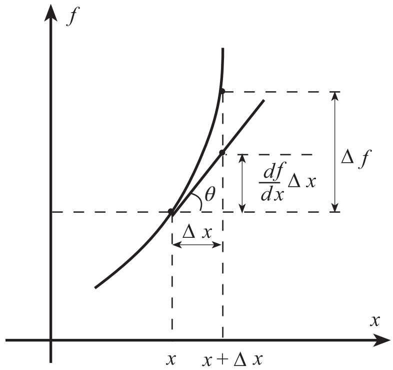

Figure 1.3 Total differential gives a local approximation to the change in a function.

then the change in $f(x, y, z)$ in D to first order in $(\Delta x, \Delta y, \Delta z)$ can be written as

$$\boxed{\Delta f \simeq \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z.} \tag{1.36}$$

In the limit as $\Delta r \to 0$, we can write Eq. (1.36) as

$$\boxed{df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}\,dz,} \tag{1.37}$$

which is called the **total differential** of $f(x, y, z)$. In the case of a function with one variable, $f(x)$, the differential reduces to

$$\Delta f \simeq \frac{df}{dx}\Delta x, \tag{1.38}$$

which gives the local approximation to the change in the function at the point x via the value of the tangent line (Figure 1.3) at that point. The smaller the value of Δx, the better the approximation. In cases with several independent variables, Δf is naturally approximated by using the tangent plane at that point.

1.5 TAYLOR SERIES

The Taylor series of a function about x_0, when it exists, is given as

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(x - x_0)^n = a_0 + a_1(x - x_0) + \frac{a_2}{2!}(x - x_0)^2 + \cdots. \tag{1.39}$$

To evaluate the coefficients, we differentiate repeatedly and set $x = x_0$ to find

$$\begin{aligned} f(x_0) &= a_0, \\ f'(x_0) &= a_1, \\ f''(x_0) &= a_2, \\ &\vdots \\ f^{(n)}(x_0) &= a_n, \end{aligned} \tag{1.40}$$

where

$$f^{(n)}(x_0) = \left(\frac{d^n f}{dx^n}\right)_{x_0} \tag{1.41}$$

and the zeroth derivative is defined as the function itself, that is,

$$f^{(0)}(x) = f(x). \tag{1.42}$$

Hence, the **Taylor series** of a function with a **single variable** is written as

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f}{dx^n}\right)_{x_0} (x - x_0)^n.} \tag{1.43}$$

This formula assumes that $f(x)$ is infinitely differentiable in an open domain including x_0. Functions that are equal to their **Taylor series** in the neighborhood of any point x_0 in their domain are called **analytic functions**. Taylor series about $x_0 = 0$ are called **Maclaurin series**.

Using the Taylor series, we can approximate a given differentiable function in the neighborhood of x_0 to orders beyond the linear term in Eq. (1.38). For example, to second order we obtain

$$f(x_0 + \Delta x) \simeq f(x_0) + \left(\frac{df}{dx}\right)_{x_0} \Delta x + \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x_0} (\Delta x)^2, \tag{1.44}$$

$$f(x_0 + \Delta x) - f(x_0) \simeq \left(\frac{df}{dx}\right)_{x_0} \Delta x + \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x_0} (\Delta x)^2, \tag{1.45}$$

$$\Delta^{(2)} f(x_0) = \left(\frac{df}{dx}\right)_{x_0} \Delta x + \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x_0} (\Delta x)^2. \tag{1.46}$$

Since x_0 is any point in the open domain that the Taylor series exists, we can drop the subscript in x_0 and write

$$\Delta^{(2)} f(x) = \frac{df}{dx}\Delta x + \frac{1}{2}\frac{d^2 f}{dx^2}(\Delta x)^2, \tag{1.47}$$

where $\Delta^{(2)} f$ denotes the differential of f to the second order. Higher order differentials are obtained similarly.

The **Taylor series** of a function depending on several independent variables is also possible under similar conditions, and in the case of **two independent variables**, it is given as

$$\boxed{f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[(x-x_0)\frac{\partial}{dx} + (y-y_0)\frac{\partial}{dy}\right]^n f(x_0,y_0),} \tag{1.48}$$

where the derivatives are to be evaluated at (x_0, y_0). For functions with two independent variables and to second order in the neighborhood of (x, y), Eq. (1.48) gives

$$\begin{aligned} f(x+\Delta x, y+\Delta y) \simeq f(x,y) &+ \left[\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right] \\ &+ \frac{1}{2}\left[\Delta x\frac{\partial}{dx} + \Delta x\frac{\partial}{dy}\right]^2 f(x,y), \end{aligned} \tag{1.49}$$

which yields the differential, $\Delta^{(2)} f(x,y) = f(x+\Delta x, y+\Delta y) - f(x,y)$, as

$$\begin{aligned} \Delta^{(2)} f(x,y) = &\left[\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right] \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta y)^2 \end{aligned} \tag{1.50}$$

$$= \Delta^{(1)} f(x,y) + \left[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right]. \tag{1.51}$$

For the higher order terms, note how the powers in Eq. (1.48) are expanded. Generalization to n independent variables is obvious.

Example 1.1. ***Partial derivatives:*** Consider the function

$$z(x,y) = xy^2 + e^x. \tag{1.52}$$

Partial derivatives are written as

$$\frac{\partial z}{\partial x} = y^2 + e^x, \tag{1.53}$$

$$\frac{\partial z}{\partial y} = 2xy, \tag{1.54}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = e^x, \tag{1.55}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = 2x, \tag{1.56}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = 2y, \tag{1.57}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = 2y. \tag{1.58}$$

Example 1.2. Taylor series*:* Using the partial derivatives obtained in the previous example, we can write the first two terms of the Taylor series [Eq. (1.48)] of $z = xy^2 + e^x$ about the point $(0, 1)$. First, the required derivatives at $(0, 1)$ are evaluated as

$$z(0, 1) = 1, \tag{1.59}$$

$$\left(\frac{\partial z}{\partial x}\right)_{(0,1)} = 2, \tag{1.60}$$

$$\left(\frac{\partial z}{\partial y}\right)_{(0,1)} = 0, \tag{1.61}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\right]_{(0,1)} = \left(\frac{\partial^2 z}{\partial x^2}\right)_{(0,1)} = 1, \tag{1.62}$$

$$\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]_{(0,1)} = \left(\frac{\partial^2 z}{\partial y^2}\right)_{(0,1)} = 0, \tag{1.63}$$

$$\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\right]_{(0,1)} = \left(\frac{\partial^2 z}{\partial y \partial x}\right)_{(0,1)} = 2, \tag{1.64}$$

$$\left[\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\right]_{(0,1)} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{(0,1)} = 2. \tag{1.65}$$

Using these derivatives, we can write the first two terms of the Taylor series about the point $(0, 1)$ as

$$z(x, y) = z(0, 1) + \left(\frac{\partial z}{\partial x}\right)_0 x + \left(\frac{\partial z}{\partial y}\right)_0 (y - 1) + \frac{1}{2}\left(\frac{\partial^2 z}{\partial x^2}\right)_0 x^2 + \left(\frac{\partial^2 z}{\partial x \partial y}\right)_0 x(y - 1) + \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)_0 (y - 1)^2 + \cdots \tag{1.66}$$

$$= 1 + 2x + 0(y - 1) + \frac{1}{2}x^2 + 2x(y - 1) + \frac{1}{2}0(y - 1)^2 + \cdots \tag{1.67}$$

$$= 1 + 2x + \frac{1}{2}x^2 + 2x(y-1) + \cdots \,, \tag{1.68}$$

where the subscript 0 indicates that the derivatives are to be evaluated at the point $(0,1)$. To find $\Delta^{(2)}z(0,1)$, which is good to the second order, we first write

$$\Delta^{(1)}z(0,1) = \left(\frac{\partial z}{\partial x}\right)_0 \Delta x + \left(\frac{\partial z}{\partial y}\right)_0 \Delta y = 2\Delta x \tag{1.69}$$

and then obtain

$$\Delta^{(2)}z(0,1) = \Delta^{(1)}z(0,1)$$

$$+ \frac{1}{2}\left(\frac{\partial^2 z}{\partial x^2}\right)_0 (\Delta x)^2 + \left(\frac{\partial^2 z}{\partial x \partial y}\right)_0 \Delta x \, \Delta y + \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)_0 (\Delta y)^2 \tag{1.70}$$

$$= 2\Delta x + \frac{1}{2}(\Delta x)^2 + 2\Delta x \, \Delta y. \tag{1.71}$$

1.6 MAXIMA AND MINIMA OF FUNCTIONS

We are frequently interested in the maximum or the minimum values that a function $f(x)$ attains in a closed domain $[a, b]$. The **absolute maximum** M_1 is the value of the function at some point x_0, if the inequality $M_1 = f(x_0) \geq f(x)$ holds for all x in $[a, b]$. An **absolute minimum** is also defined similarly. In general, we can quote the following theorem (Figure 1.4):

Theorem 1.3. If a function $f(x)$ is continuous in the closed interval $[a, b]$, then it possesses an absolute maximum M_1 and an absolute minimum M_2 in that interval.

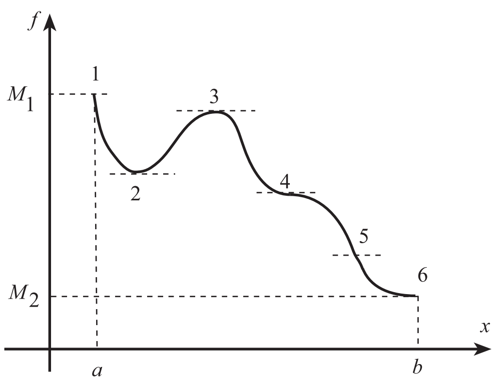

Figure 1.4 Maximum and minimum points of a function.

Proof of this theorem requires a rather detailed analysis of the real number system. On the other hand, we are usually interested in the **extremum** values, that is, the **local maximum** or the **minimum** values of a function. Operationally, we can determine whether a given point x_0 corresponds to an extremum or not by looking at the change or the variation in the function in the neighborhood of x_0. The total differential introduced in the previous sections is just the tool needed for this. We have seen that in one dimension, we can write the first, $\Delta f^{(1)}$, the second, $\Delta^{(2)} f$, and the third, $\Delta^{(3)} f$, differentials of a function with single independent variable as

$$\Delta^{(1)} f(x_0) = \left(\frac{df}{dx}\right)_{x_0} \Delta x, \tag{1.72}$$

$$\Delta^{(2)} f(x_0) = \left(\frac{df}{dx}\right)_{x_0} \Delta x + \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x_0} (\Delta x)^2, \tag{1.73}$$

$$\Delta^{(3)} f(x_0) = \left(\frac{df}{dx}\right)_{x_0} \Delta x + \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x_0} (\Delta x)^2 + \frac{1}{3!}\left(\frac{d^3 f}{dx^3}\right)_{x_0} (\Delta x)^3. \tag{1.74}$$

Extremum points are defined as the points where the first differential vanishes, which means

$$\left(\frac{df}{dx}\right)_{x_0} = 0. \tag{1.75}$$

In other words, the tangent line at an extremum point is horizontal (Figure 1.5a,b). In order to decide whether an extremum point corresponds to a local maximum or minimum, we look at the second differential:

$$\Delta^{(2)} f(x_0) = \frac{1}{2}\left(\frac{d^2 f}{dx^2}\right)_{x_0} (\Delta x)^2. \tag{1.76}$$

For a local maximum, the function decreases for small displacements about the extremum point (Figure 1.5a), which implies $\Delta^{(2)} f(x_0) < 0$. For a local minimum, a similar argument yields $\Delta^{(2)} f(x_0) > 0$. Thus, we obtain the following criteria:

$$\boxed{\left(\frac{df}{dx}\right)_{x_0} = 0 \text{ and } \left(\frac{d^2 f}{dx^2}\right)_{x_0} < 0 \quad \text{for a local maximum}} \tag{1.77}$$

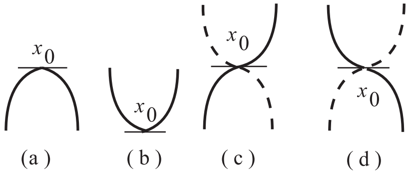

Figure 1.5 Analysis of critical points.

and

$$\boxed{\left(\frac{df}{dx}\right)_{x_0} = 0 \text{ and } \left(\frac{d^2 f}{dx^2}\right)_{x_0} > 0 \quad \text{for a local minimum.}} \tag{1.78}$$

In cases where the second derivative also vanishes, we look at the third differential $\Delta^{(3)} f(x_0)$. We now say that we have an **inflection point**; and depending on the sign of the third differential, we have either the third or the fourth shape in Figure 1.5.

Consider the function $f(x) = x^3$, where the first derivative, $f'(x) = 3x^2$, vanishes at $x_0 = 0$. However, the second derivative, $f''(x) = 6x$, also vanishes there, thus making $x_0 = 0$ a point of inflection. From the third differential:

$$\Delta^{(3)} f(x_0) = \frac{1}{3!}\left(\frac{d^3 f}{dx^3}\right)_{x_0} (\Delta x)^3 = \frac{1}{3!} 6(\Delta x)^3, \tag{1.79}$$

we see that $\Delta^{(3)} f(x_0) > 0$ for $\Delta x > 0$ and $\Delta^{(3)} f(x_0) < 0$ for $\Delta x < 0$. Thus, we choose the third shape in Figure 1.5 and plot $f(x) = x^3$ as in Figure 1.6. Points where the first derivative of a function vanishes are called the **critical points**.

Usually the **potential** in one-dimensional conservative systems can be represented by a (scalar) function $V(x)$. Negative of the derivative of the potential gives the x component of the force on the system:

$$F_x(x) = -\frac{dV}{dx}. \tag{1.80}$$

Thus, the critical points of a potential function $V(x)$ correspond to the points where the net force on the system is zero. In other words, the critical points are the points where the system is in **equilibrium**. Whether an equilibrium is stable or unstable depends on whether the critical point is a minimum or a maximum, respectively.

Analysis of the extrema of functions depending on more than one variable follows the same line of reasoning. However, since we can now approach the critical point from different directions, one has to be careful. Consider a continuous

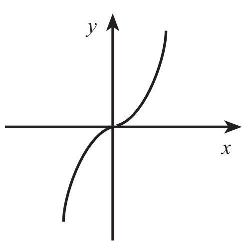

Figure 1.6 Plot of $y(x) = x^3$.

function $z = f(x, y)$ defined in some domain D. We say this function has a local maximum at (x_0, y_0) if the inequality

$$f(x, y) \leq f(x_0, y_0) \tag{1.81}$$

is satisfied for all points in some neighborhood of (x_0, y_0) and to have a local minimum if the inequality

$$f(x, y) \geq f(x_0, y_0) \tag{1.82}$$

is satisfied. In the following argument, we assume that all the necessary partial derivatives exist. Critical points are now defined as the points where the **first differential** $\Delta^{(1)} f(x, y)$ vanishes:

$$\Delta^{(1)} f(x, y) = \left[\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y\right] = 0. \tag{1.83}$$

Since the displacements Δx and Δy are arbitrary, the only way to satisfy this equation is to have both partial derivatives, f_x and f_y, vanish. Hence at the critical point (x_0, y_0), shown with the subscript 0, one has

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_0 = 0, \quad \left(\frac{\partial f}{\partial y}\right)_0 = 0.} \tag{1.84}$$

To study the nature of these critical points, we again look at the **second differential** $\Delta^{(2)} f(x_0, y_0)$, which is now given as

$$\Delta^{(2)} f(x_0, y_0) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_0 (\Delta x)^2 + \left(\frac{\partial^2 f}{\partial x \partial y}\right)_0 \Delta x \, \Delta y + \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)_0 (\Delta y)^2. \tag{1.85}$$

For a local maximum, the second differential $\Delta^{(2)} f(x_0, y_0)$ has to be negative and for a local minimum positive. Since we can approach the point (x_0, y_0) from different directions, we substitute (Figure 1.7)

$$\Delta x = \Delta s \cos\theta \quad \text{and} \quad \Delta y = \Delta s \sin\theta \tag{1.86}$$

to write Eq. (1.85) as

$$\Delta^{(2)} f(x_0, y_0) = \frac{1}{2}[A\cos^2\theta + 2B\cos\theta \, \sin\theta + C\sin^2\theta](\Delta s)^2, \tag{1.87}$$

where we have defined

$$A = \left(\frac{\partial^2 f}{\partial x^2}\right)_0, \quad B = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_0, \quad C = \left(\frac{\partial^2 f}{\partial y^2}\right)_0. \tag{1.88}$$

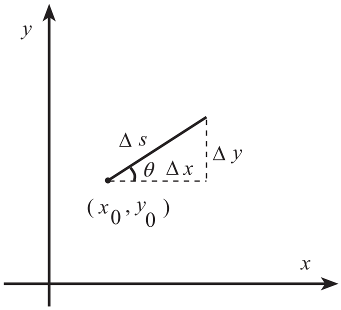

Figure 1.7 Definition of Δs.

Now the analysis of the nature of the critical points reduces to investigating the sign of $\Delta^{(2)}f(x_0, y_0)$ [Eq. (1.87)]. We present the final result as a theorem [2].

Theorem 1.4. Let $z = f(x, y)$ and its first and second partial derivatives be continuous in a domain D, and let (x_0, y_0) be a point in D, where the partial derivatives $\left(\frac{\partial z}{\partial x}\right)_0$ and $\left(\frac{\partial z}{\partial y}\right)_0$ vanish. Then, we have the following cases:

I. For $B^2 - AC < 0$ and $A + C < 0$, we have a local maximum at (x_0, y_0).
II. For $B^2 - AC < 0$ and $A + C > 0$, we have a local minimum at (x_0, y_0).
III. For $B^2 - AC > 0$, we have a saddle point at (x_0, y_0).
IV. For $B^2 - AC = 0$, the nature of the critical point is undetermined.

When $B^2 - AC > 0$ at (x_0, y_0), we have what is called a **saddle point**, where for some directions $\Delta^{(2)}f(x_0, y_0)$ is positive and negative for the others. When $B^2 - AC = 0$, for some directions $\Delta^{(2)}f(x_0, y_0)$ will be zero, hence one must look at higher order derivatives to study the nature of the critical point. When A, B, and C are all zero, then $\Delta^{(2)}f(x_0, y_0)$ also vanishes. Hence, we need to investigate the sign of $\Delta^{(3)}f(x_0, y_0)$.

1.7 EXTREMA OF FUNCTIONS WITH CONDITIONS

A problem of significance is finding the critical points of functions while satisfying one or more conditions. Consider finding the extremums of

$$w = f(x, y, z) \tag{1.89}$$

while satisfying the conditions

$$g_1(x, y, z) = 0 \tag{1.90}$$

and

$$g_2(x, y, z) = 0. \tag{1.91}$$

In principle, the two conditions define two surfaces, the intersection of which can be expressed as

$$x = x, \quad y = y(x), \quad z = z(x), \tag{1.92}$$

where we have used the variable x as a parameter. We can now substitute this parametric equation into $w = f(x, y, z)$ and write it entirely in terms of x as

$$w(x) = f(x, y(x), z(x)). \tag{1.93}$$

The extremum points can now be found by the technique discussed in the previous section. Geometrically, this problem corresponds to finding the extremum points of $w = f(x, y, z)$ on the curve defined by the intersection of $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$. Unfortunately, this method rarely works to yield a solution analytically. Instead, we introduce the following method: At a critical point, we have seen that the change in w to first order in the differentials Δx, Δy, and Δz is zero:

$$\Delta w = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z = 0. \tag{1.94}$$

We also write the differentials of $g_1(x, y, z)$ and $g_2(x, y, z)$ as

$$\frac{\partial g_1}{\partial x}\Delta x + \frac{\partial g_1}{\partial y}\Delta y + \frac{\partial g_1}{\partial z}\Delta z = 0, \tag{1.95}$$

$$\frac{\partial g_2}{\partial x}\Delta x + \frac{\partial g_2}{\partial y}\Delta y + \frac{\partial g_2}{\partial z}\Delta z = 0. \tag{1.96}$$

We now multiply Eq. (1.95) with λ_1 and Eq. (1.96) with λ_2 and add to Eq. (1.94) to write

$$\left(\frac{\partial f}{\partial x} + \lambda_1\frac{\partial g_1}{\partial x} + \lambda_2\frac{\partial g_2}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial y} + \lambda_1\frac{\partial g_1}{\partial y} + \lambda_2\frac{\partial g_2}{\partial y}\right)\Delta y + \left(\frac{\partial f}{\partial z} + \lambda_1\frac{\partial g_1}{\partial z} + \lambda_2\frac{\partial g_2}{\partial z}\right)\Delta z = 0. \tag{1.97}$$

Because of the given conditions in Eqs. (1.90) and (1.91), Δx, Δy, and Δz are not independent. Hence, their coefficients in Eq. (1.94) cannot be set to zero directly. However, the values of λ_1 and λ_2, which are called the **Lagrange undetermined multipliers**, can be chosen so that the coefficients of Δx, Δy, and Δz are all zero in Eq. (1.97):

$$\left(\frac{\partial f}{\partial x} + \lambda_1\frac{\partial g_1}{\partial x} + \lambda_2\frac{\partial g_2}{\partial x}\right) = 0, \tag{1.98}$$

$$\left(\frac{\partial f}{\partial y} + \lambda_1\frac{\partial g_1}{\partial y} + \lambda_2\frac{\partial g_2}{\partial y}\right) = 0, \tag{1.99}$$

$$\left(\frac{\partial f}{\partial z}+\lambda_1\frac{\partial g_1}{\partial z}+\lambda_2\frac{\partial g_2}{\partial z}\right)=0. \tag{1.100}$$

Along with the two conditions, $g_1(x,y,z)=0$ and $g_2(x,y,z)=0$, these three equations are to be solved for the five unknowns, x, y, z, λ_1, and λ_2. The values that λ_1 and λ_2 assume are used to obtain the x, y, and z values needed, which correspond to the locations of the critical points. Analysis of the critical points now proceeds as before. Note that this method is quite general, and as long as the required derivatives exist and the conditions are compatible, it can be used with any number of conditions.

Example 1.3. ***Extremum problems:*** We now find the dimensions of a rectangular swimming pool with fixed volume V_0 and minimal area of its base and sides. If we denote the dimensions of its base with x and y and its height with z, the fixed volume is $V_0=xyz$, and the total area of the base and the sides is

$$a=xy+2xz+2yz. \tag{1.101}$$

Using the condition of fixed volume, we write a as a function of x and y as

$$a=xy+\frac{2V_0}{y}+\frac{2V_0}{x}. \tag{1.102}$$

Now, the critical points of a are determined from the equations

$$\frac{\partial a}{\partial x}=0,\quad \frac{\partial a}{\partial y}=0, \tag{1.103}$$

which give the following two equations:

$$y-\frac{2V_0}{x^2}=0, \tag{1.104}$$

$$x-\frac{2V_0}{y^2}=0 \tag{1.105}$$

or

$$yx^2-2V_0=0, \tag{1.106}$$

$$xy^2-2V_0=0. \tag{1.107}$$

If we subtract Eq. (1.107) from Eq. (1.106), we obtain $y=x$, which when substituted back into Eq. (1.106) gives the critical dimensions

$$x=(2V_0)^{1/3}, \tag{1.108}$$

$$y=(2V_0)^{1/3}, \tag{1.109}$$

$$z=\left(\frac{V_0}{4}\right)^{1/3}, \tag{1.110}$$

where the final dimension is obtained from $V_0 = xyz$. To assure ourselves that this corresponds to a minimum, we evaluate the second-order derivatives at the critical point,

$$A = \frac{\partial^2 a}{\partial x^2} = \frac{4V_0}{x^3} = \frac{4V_0}{2V_0} = 2, \tag{1.111}$$

$$B = \frac{\partial^2 a}{\partial x \partial y} = 1, \tag{1.112}$$

$$C = \frac{\partial^2 a}{\partial y^2} = \frac{4V_0}{y^3} = \frac{4V_0}{2V_0} = 2, \tag{1.113}$$

and find

$$B^2 - AC = 1 - 4 = -3 < 0 \quad \text{and} \quad A + C = 2 + 2 = 4 > 0. \tag{1.114}$$

Thus, the critical dimensions we have obtained [Eqs. (1.108)–(1.110)] are indeed for a minimum by Theorem 1.4.

Example 1.4. ***Lagrange undetermined multipliers:*** We now solve the aforementioned problem by using the method of Lagrange undetermined multipliers. The equation to be minimized is now

$$f(x, y, z) = xy + 2xz + 2yz \tag{1.115}$$

with the condition

$$g(x, y, z) = V_0 - xyz = 0. \tag{1.116}$$

The equations to be solved are obtained from Eqs. (1.98)–(1.100) as

$$y + 2z - yz\lambda = 0, \tag{1.117}$$

$$x + 2z - xz\lambda = 0, \tag{1.118}$$

$$2x + 2y - \lambda xy = 0. \tag{1.119}$$

Along with $V_0 = xyz$, these give four equations to be solved for the critical dimensions x, y, z, and λ. Multiplying the first equation by x and the second one by y and then subtracting gives

$$x = y. \tag{1.120}$$

Substituting this into the third equation [Eq. (1.119)] gives the value of the Lagrange undetermined multiplier as $\lambda = 4/x$, which when substituted into Eqs. (1.117)–(1.119) gives

$$xy + 2xz - 4yz = 0, \tag{1.121}$$

$$x + 2z - 4z = 0, \tag{1.122}$$

$$2x + 2y - 4y = 0. \tag{1.123}$$

Using the condition $V_0 = xyz$ and Eq. (1.120), these three equations [Eqs. (1.121)–(1.123)] can be solved easily to yield the critical dimensions in terms of V_0 as

$$x = (2V_0)^{1/3}, \tag{1.124}$$

$$y = (2V_0)^{1/3}, \tag{1.125}$$

$$z = \left(\frac{V_0}{4}\right)^{1/3}. \tag{1.126}$$

Analysis of the critical point is done as in the previous example by using Theorem 1.4.

1.8 DERIVATIVES AND DIFFERENTIALS OF COMPOSITE FUNCTIONS

In what follows, we assume that the functions are defined in their appropriate domains and have continuous first partial derivatives.

Chain rule: If $z = f(x, y)$ and $x = x(t)$, $y = y(t)$, then

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.} \tag{1.127}$$

Similarly, if $z = f(x, y)$, $x = g(u, v)$, and $y = h(u, v)$, then

$$\frac{\partial z}{du} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \tag{1.128}$$

$$\frac{\partial z}{dv} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{1.129}$$

A better notation to use is

$$\left(\frac{\partial z}{du}\right)_v = \left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v, \tag{1.130}$$

$$\left(\frac{\partial z}{dv}\right)_u = \left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial v}\right)_u. \tag{1.131}$$

This notation is particularly useful in thermodynamics, where z may also be expressed with another choice of variables, such as

$$z = f(x, y) \tag{1.132}$$

$$= g(x, w) \tag{1.133}$$

$$= h(u, y). \tag{1.134}$$

Hence, when we write the derivative $\frac{\partial z}{\partial x}$, we have to clarify whether we are in the (x, y) or the (x, w) space by writing

$$\left(\frac{\partial z}{\partial x}\right)_y \text{ or } \left(\frac{\partial z}{\partial x}\right)_w. \tag{1.135}$$

These formulas can be extended to any number of variables. Using Eq. (1.127), we can write the differential dz as

$$dz = \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right) dt = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy. \tag{1.136}$$

We now treat x, y, and z as functions of (u, v) and write the differential dz as

$$dz = \frac{\partial z}{\partial u}\, du + \frac{\partial z}{\partial v}\, dv \tag{1.137}$$

$$= \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right) du + \left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right) dv \tag{1.138}$$

$$= \left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial x}{\partial u}\, du + \frac{\partial x}{\partial v}\, dv\right) + \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial u}\, du + \frac{\partial y}{\partial v}\, dv\right). \tag{1.139}$$

Since x and y are also functions of u and v, we have the differentials

$$dx = \frac{\partial x}{\partial u}\, du + \frac{\partial x}{\partial v}\, dv \tag{1.140}$$

and

$$dy = \frac{\partial y}{\partial u}\, du + \frac{\partial y}{\partial v}\, dv, \tag{1.141}$$

which allow us to write Eq. (1.139) as

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy. \tag{1.142}$$

This result can be extended to any number of variables. In other words, any equation in differentials that is true in one set of independent variables is also true for another choice of variables.

1.9 IMPLICIT FUNCTION THEOREM

A function given as

$$F(x, y, z) = 0 \tag{1.143}$$

can be used to describe several functions of the following forms:

$$z = f(x, y), \; y = g(x, z), \text{etc.} \tag{1.144}$$

For example,

$$x^2 + y^2 + z^2 - 9 = 0 \tag{1.145}$$

can be used to define the function

$$z = \sqrt{9 - x^2 - y^2}, \tag{1.146}$$

or

$$z = -\sqrt{9 - x^2 - y^2}, \tag{1.147}$$

both of which are defined in the domain $x^2 + y^2 + z^2 \leq 9$. We say these functions are **implicitly** defined by Eq. (1.145). In order to be able to define a differentiable function, $z = f(x, y)$, by the implicit function $F(x, y, z) = 0$, the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ should exist in some domain so that we can write the differential

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy. \tag{1.148}$$

Using the implicit function $F(x, y, z) = 0$, we write

$$F_x\, dx + F_y\, dy + F_z\, dz = 0 \tag{1.149}$$

and

$$dz = -\frac{F_x}{F_z}\, dx - \frac{F_y}{F_z}\, dy, \tag{1.150}$$

where

$$F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_z = \frac{\partial F}{\partial z}. \tag{1.151}$$

Comparing the two differentials [Eqs. (1.148) and (1.150)], we obtain the partial derivatives

$$\frac{\partial f}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial f}{\partial y} = -\frac{F_y}{F_z}. \tag{1.152}$$

Hence, granted that $F_z \neq 0$, we can use the implicit function $F(x, y, z) = 0$ to define a function of the form $z = f(x, y)$.

We now consider a more complicated case, in which we have two implicit functions:

$$F(x, y, z, w) = 0, \tag{1.153}$$

$$G(x, y, z, w) = 0. \tag{1.154}$$

Using these two equations in terms of four variables, we can solve, in principle, for two of the variables in terms of the remaining two as

$$z = f(x, y), \tag{1.155}$$

$$w = g(x, y). \tag{1.156}$$

For $f(x, y)$ and $g(x, y)$ to be differentiable, certain conditions must be met by $F(x, y, z, w)$ and $G(x, y, z, w)$. First, we write the differentials

$$F_x\, dx + F_y\, dy + F_z\, dz + F_w\, dw = 0, \tag{1.157}$$

$$G_x\, dx + G_y\, dy + G_z\, dz + G_w\, dw = 0 \tag{1.158}$$

and rearrange them as

$$F_z\, dz + F_w\, dw = -F_x\, dx - F_y\, dy, \tag{1.159}$$

$$G_z\, dz + G_w\, dw = -G_x\, dx - G_y\, dy. \tag{1.160}$$

We now have a system of two linear equations for the differentials dz and dw to be solved simultaneously. We can either solve by elimination or use determinants and the Cramer's rule to write

$$dz = \frac{\begin{vmatrix} -F_x\, dx - F_y\, dy & F_w \\ -G_x\, dx - G_y\, dy & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}, \tag{1.161}$$

$$dw = \frac{\begin{vmatrix} F_z & -F_x\, dx - F_y\, dy \\ G_z & -G_x\, dx - G_y\, dy \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}. \tag{1.162}$$

Using the properties of determinants, we can write these as

$$dz = -\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dx - \frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dy, \tag{1.163}$$

$$dw = -\frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dx - \frac{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} dy. \tag{1.164}$$

For differentiable functions, $z = f(x, y)$ and $w = g(x, y)$, with existing first-order partial derivatives, we can write

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy, \tag{1.165}$$

$$dw = \frac{\partial g}{\partial x}\, dx + \frac{\partial g}{\partial y}\, dy. \tag{1.166}$$

Comparing with Eqs. (1.163) and (1.164), we obtain the partial derivatives:

$$\frac{\partial f}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,w)}}{\frac{\partial(F,G)}{\partial(z,w)}}, \quad \frac{\partial f}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,w)}}{\frac{\partial(F,G)}{\partial(z,w)}}, \tag{1.167}$$

$$\frac{\partial g}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(z,x)}}{\frac{\partial(F,G)}{\partial(z,w)}}, \quad \frac{\partial g}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(z,y)}}{\frac{\partial(F,G)}{\partial(z,w)}}, \tag{1.168}$$

where the determinants:

$$\frac{\partial(F,G)}{\partial(x,w)} = \begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}, \quad \frac{\partial(F,G)}{\partial(z,w)} = \begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}, \ldots \tag{1.169}$$

are called the **Jacobi determinants**. In summary, given two implicit equations:

$$F(x, y, z, w) = 0 \quad \text{and} \quad G(x, y, z, w) = 0, \tag{1.170}$$

we can define two differentiable functions

$$z = f(x, y) \quad \text{and} \quad w = g(x, y) \tag{1.171}$$

with the partial derivatives given as in Eqs. (1.167) and (1.168), provided that the **Jacobian:**

$$\frac{\partial(F,G)}{\partial(z,w)} = \begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}, \tag{1.172}$$

is different from zero in the domain of definition.

This useful technique can be generalized to a set of m equations in $n+m$ number of unknowns:

$$\begin{aligned} F_1(y_1,\ldots,y_m,x_1,\ldots,x_n) &= 0, \\ &\vdots \\ F_m(y_1,\ldots,y_m,x_1,\ldots,x_n) &= 0. \end{aligned} \tag{1.173}$$

We look for m differentiable functions in terms of n variables as

$$\begin{aligned} y_1 &= y_1(x_1,\ldots,x_n), \\ &\vdots \\ y_m &= y_m(x_1,\ldots,x_n). \end{aligned} \tag{1.174}$$

We write the following differentials:

$$\begin{aligned} F_{1y_1}\,dy_1 + \cdots + F_{1y_m}\,dy_m &= -F_{1x_1}\,dx_1 - \cdots - F_{1x_n}\,dx_n, \\ &\vdots \\ F_{my_1}\,dy_1 + \cdots + F_{my_m}\,dy_m &= -F_{mx_1}\,dx_1 - \cdots - F_{mx_n}\,dx_n \end{aligned} \tag{1.175}$$

and obtain a set of m linear equations to be solved for the m differentials dy_i, $i = 1,\ldots,m$, of the dependent variables. Using Cramer's rule, we can solve for dy_i if and only if the determinant of the coefficients is different from zero:

$$\begin{vmatrix} F_{1y_1} & \cdots & F_{1y_m} \\ \vdots & \ddots & \vdots \\ F_{my_1} & \cdots & F_{my_m} \end{vmatrix} = \frac{\partial(F_1,\ldots,F_m)}{\partial(y_1,\ldots,y_m)} \neq 0. \tag{1.176}$$

To obtain closed expressions for the partial derivatives $\partial y_1/\partial x_j$, we take partial derivatives of Eq. (1.173) to write

$$\begin{aligned} F_{1y_1}\frac{\partial y_1}{\partial x_j} + \cdots + F_{1y_m}\frac{\partial y_m}{\partial x_j} &= -F_{1x_j}, \\ &\vdots \\ F_{my_1}\frac{\partial y_1}{\partial x_j} + \cdots + F_{my_m}\frac{\partial y_m}{\partial x_j} &= -F_{mx_j}, \end{aligned} \tag{1.177}$$

which gives the solution for $\partial y_1/\partial x_j$ as

$$\frac{\partial y_1}{\partial x_j} = -\frac{\begin{vmatrix} F_{1x_j} & F_{1y_2} & \cdots & F_{1y_m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{mx_j} & F_{my_2} & \cdots & F_{my_m} \end{vmatrix}}{\begin{vmatrix} F_{1y_1} & F_{1y_2} & \cdots & F_{1y_m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{my_1} & F_{my_2} & \cdots & F_{my_m} \end{vmatrix}} = -\frac{\dfrac{\partial(F_1, F_2, \ldots, F_m)}{\partial(x_j, y_2, \ldots, y_m)}}{\dfrac{\partial(F_1, F_2, \ldots, F_m)}{\partial(y_1, y_2, \ldots, y_m)}} \tag{1.178}$$

and similar expressions for the other partial derivatives can be obtained. In general, granted that the Jacobi determinant does not vanish:

$$\frac{\partial(F_1, F_2, \ldots, F_m)}{\partial(y_1, y_2, \ldots, y_m)} \neq 0,$$

we can obtain the partial derivatives $\partial y_i/\partial x_j$ as

$$\frac{\partial y_i}{\partial x_j} = -\frac{\dfrac{\partial(F_1, F_2, \ldots, F_m)}{\partial(y_1, \ldots, y_{i-1}, x_j, y_{i+1}, \cdots, y_m)}}{\dfrac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}}, \tag{1.179}$$

where $i = 1, \ldots, m$ and $j = 1, \ldots, n$. We conclude this section by stating the implicit function theorem [2]:

Implicit function theorem: Let the functions

$$F_i(y_1, \ldots, y_m, x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, m, \tag{1.180}$$

be defined in the neighborhood of the point

$$P_0 = (y_{01}, \ldots, y_{0m}, x_{01}, \ldots, x_{0n}) \tag{1.181}$$

with continuous first-order partial derivatives existing in this neighborhood. If

$$\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)} \neq 0 \quad \text{at } P_0, \tag{1.182}$$

then in an appropriate neighborhood of P_0, there is a unique set of continuous functions

$$y_i = f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, m, \tag{1.183}$$

with continuous partial derivatives,

$$\frac{\partial y_i}{\partial x_j} = -\frac{\dfrac{\partial(F_1, F_2, \ldots, F_m)}{\partial(y_1, \ldots, y_{i-1}, x_j, y_{i+1}, \cdots, y_m)}}{\dfrac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}}, \tag{1.184}$$

where $i = 1, \ldots, m$ and $j = 1, \ldots n$, such that

$$y_{0i} = f_i(x_{01}, \ldots, x_{0n}), \quad i = 1, \ldots, m, \tag{1.185}$$

and

$$F_i(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n), x_1, \ldots, x_n) = 0, \quad i = 1, \ldots, m, \tag{1.186}$$

in the neighborhood of P_0. Note that if the Jacobi determinant [Eq. (1.182)] is zero at the point of interest, then we search for a different set of dependent variables to avoid the difficulty.

1.10 INVERSE FUNCTIONS

A pair of functions,

$$x = f(u, v), \tag{1.187}$$

$$y = g(u, v), \tag{1.188}$$

can be considered as a **mapping** from the xy space to the uv space. Under certain conditions, this maps a certain domain D_{xy} in the xy space to a certain domain D_{uv} in the uv space on a one-to-one basis. Under such conditions, an inverse mapping should also exist. However, analytically it may not always be possible to find the **inverse mapping** or the functions

$$u = u(x, y), \tag{1.189}$$

$$v = v(x, y). \tag{1.190}$$

In such cases, we may consider Eqs. (1.187) and (1.188) as implicit functions and write them as

$$F_1(x, y, u, v) = f(u, v) - x = 0, \tag{1.191}$$

$$F_2(x, y, u, v) = g(u, v) - y = 0. \tag{1.192}$$

We can now use Eq. (1.178) with $y_1 = u$, $y_2 = v$ and $x_1 = x$, $x_2 = y$ to write the partial derivatives of the inverse functions as

$$\frac{\partial u}{\partial x} = \frac{\dfrac{\partial(F_1, F_2)}{\partial(x, v)}}{\dfrac{\partial(F_1, F_2)}{\partial(u, v)}} = -\frac{\begin{vmatrix} -1 & \partial f/\partial v \\ 0 & \partial g/\partial v \end{vmatrix}}{\begin{vmatrix} \partial f/\partial u & \partial f/\partial v \\ \partial g/\partial u & \partial g/\partial v \end{vmatrix}} \tag{1.193}$$

$$= \frac{\partial g}{\partial v} \Big/ \left[\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v} \right]. \tag{1.194}$$

Similarly, the other partial derivatives can be obtained. As seen, the inverse function or the inverse mapping is well defined only when the **Jacobi determinant** J is different from zero:

$$\boxed{J = \frac{\partial(f,g)}{\partial(u,v)} = \left[\frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial g}{\partial u}\frac{\partial f}{\partial v}\right] \neq 0,} \tag{1.195}$$

where J is also called the **Jacobian of the mapping**. We will return to this point when we discuss coordinate transformations in Chapter 3. Note that the Jacobian of the inverse mapping is $1/J$. In other words,

$$\frac{\partial(f,g)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(f,g)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \tag{1.196}$$

Example 1.5. ***Change of independent variable:*** We now transform the Laplace equation:

$$\frac{\partial^2 z(x,y)}{\partial x^2} + \frac{\partial^2 z(x,y)}{\partial y^2} = 0, \tag{1.197}$$

into polar coordinates, that is, to a new set of independent variables defined by the equations

$$x = r\cos\phi, \quad y = r\sin\phi, \tag{1.198}$$

where $r \in (0,\infty)$ and $\phi \in [0,2\pi]$. We first write the partial derivatives of $z = z(x,y)$:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}, \tag{1.199}$$

$$\frac{\partial z}{\partial \phi} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \phi}, \tag{1.200}$$

which lead to

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\phi + \frac{\partial z}{\partial y}\sin\phi, \tag{1.201}$$

$$\frac{\partial z}{\partial \phi} = \frac{\partial z}{\partial x}(-r\sin\phi) + \frac{\partial z}{\partial y}(r\cos\phi). \tag{1.202}$$

Solving for $\partial z/\partial x$ and $\partial z/\partial y$, we obtain

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\phi - \frac{\partial z}{\partial \phi}\frac{1}{r}\sin\phi, \tag{1.203}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\sin\phi + \frac{\partial z}{\partial \phi}\frac{1}{r}\cos\phi. \tag{1.204}$$

We now repeat this process with $\partial z/\partial x$ to obtain the second derivative $\partial^2 z/\partial x^2$ as

$$\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial x^2} = \frac{\partial\left[\frac{\partial z}{\partial x}\right]}{\partial r}\cos\phi - \frac{\partial\left[\frac{\partial z}{\partial x}\right]}{\partial \phi}\frac{1}{r}\sin\phi \tag{1.205}$$

$$= \cos\phi\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial r}\cos\phi - \frac{\partial z}{\partial \phi}\frac{1}{r}\sin\phi\right]$$
$$-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\left[\frac{\partial z}{\partial r}\cos\phi - \frac{\partial z}{\partial \phi}\frac{1}{r}\sin\phi\right] \tag{1.206}$$

$$= \frac{\partial^2 z}{\partial r^2}\cos^2\phi - \frac{\partial^2 z}{\partial r\partial\phi}\frac{2}{r}\cos\phi\sin\phi + \frac{\partial^2 z}{\partial\phi^2}\frac{1}{r^2}\sin^2\phi$$
$$+\frac{1}{r}\frac{\partial z}{\partial r}\sin^2\phi + \frac{\partial z}{\partial\phi}\frac{2}{r^2}\sin\phi\ \cos\phi. \tag{1.207}$$

A similar procedure for $\partial z/\partial y$ yields $\partial^2 z/\partial y^2$:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2}\sin^2\phi + \frac{\partial^2 z}{\partial r\partial\phi}\frac{2}{r}\sin\phi\ \cos\phi + \frac{\partial^2 z}{\partial\phi^2}\frac{1}{r^2}\cos^2\phi$$
$$+\frac{1}{r}\frac{\partial z}{\partial r}\cos^2\phi - \frac{\partial z}{\partial\phi}\frac{2}{r^2}\sin\phi\ \cos\phi. \tag{1.208}$$

Adding Eqs. (1.207) and (1.208), we obtain the transformed equation:

$$\frac{\partial^2 z(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial z(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z(r,\theta)}{\partial\phi^2} = 0. \tag{1.209}$$

Since the Jacobian of the mapping is different from zero, that is,

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\phi & \sin\phi \\ -r\sin\phi & r\cos\phi \end{vmatrix} = r, \quad r \neq 0, \tag{1.210}$$

the inverse mapping exists:

$$r = \sqrt{x^2+y^2}, \quad \phi = \tan^{-1}\frac{y}{x}. \tag{1.211}$$

1.11 INTEGRAL CALCULUS AND THE DEFINITE INTEGRAL

Let $f(x)$ be a continuous function in the interval $[x_a, x_b]$. By choosing $(n-1)$ points in this interval, $x_1, x_2, \ldots, x_{n-1}$, we can subdivide it into n subintervals, $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, which are not necessarily all equal in length. From Theorem 1.3, we know that $f(x)$ assumes a maximum, M, and a minimum, m, in $[x_a, x_b]$. Let M_i represent the maximum and m_i the minimum values that $f(x)$ assumes

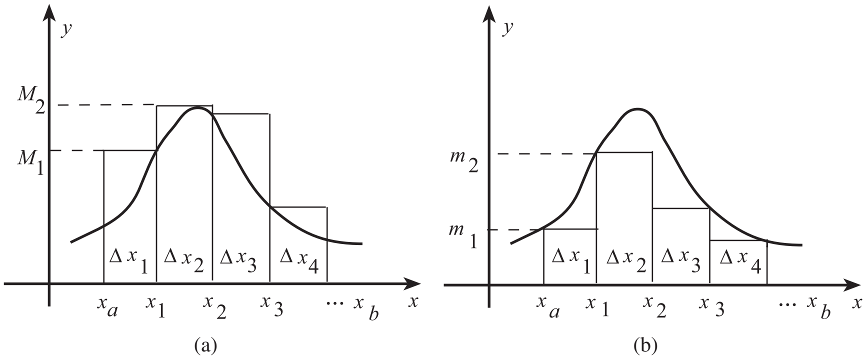

Figure 1.8 Upper (a) and lower (b) Darboux sums.

in Δx_i. We now denote a particular subdivision by d and write the sum of the rectangles shown in Figure 1.8a as

$$S(d) = \sum_{i=1}^{n} M_i \Delta x_i \tag{1.212}$$

and in Figure 1.8b as

$$s(d) = \sum_{i=1}^{n} m_i \Delta x_i. \tag{1.213}$$

The sums $S(d)$ and $s(d)$ are called the upper and the lower **Darboux sums**, respectively. Naturally, their values depend on the subdivision d. We pick the smallest of all $S(d)$ and call it the upper integral of $f(x)$ in $[x_a, x_b]$:

$$\overline{\int_{x_a}^{x_b}} f(x)dx. \tag{1.214}$$

Similarly, the largest of all $s(d)$ is called the lower integral of $f(x)$ in $[x_a, x_b]$:

$$\underline{\int_{x_a}^{x_b}} f(x)dx. \tag{1.215}$$

When these two integrals are equal, we say the definite integral of $f(x)$ in the interval $[x_a, x_b]$ exists and we write

$$\boxed{\overline{\int_{x_a}^{x_b}} f(x)dx = \underline{\int_{x_a}^{x_b}} f(x)dx = \int_{x_a}^{x_b} f(x)dx.} \tag{1.216}$$

This definition of integral is also called the **Riemann integral**, and the function $f(x)$ is called the **integrand**.

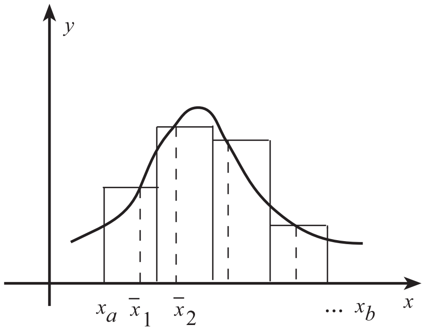

Figure 1.9 Riemann integral.

Darboux sums are not very practical to work with. Instead, for a particular subdivision, we write the sum

$$\sigma(d) = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x_k, \tag{1.217}$$

where $\overline{x}_k$ is an arbitrary point in Δx_k (Figure 1.9). It is clear that the inequality

$$s(d) \leq \sigma(d) \leq S(d) \tag{1.218}$$

is satisfied. For a given subdivision the largest value of Δx_i is called the **norm** of d, which we will denote as $n(d)$.

1.12 RIEMANN INTEGRAL

We now give the basic definition of the Riemann integral as follows:

Definition 1.1. Given a sequence of subdivisions $d_1, d_2, \ldots$ of the interval $[x_a, x_b]$ such that the sequence of norms $n(d_1), n(d_2), \ldots$ has the limit

$$\lim_{k\to\infty} n(d_k) \to 0 \tag{1.219}$$

and if $f(x)$ is integrable in $[x_a, x_b]$, then the Riemann integral is defined as

$$\int_{x_a}^{x_b} f(x)dx = \lim_{k\to\infty} \sigma(d_k), \tag{1.220}$$

where

$$\lim_{k\to\infty} S(d_k) = \lim_{k\to\infty} s(d_k) = \lim_{k\to\infty} \sigma(d_k). \tag{1.221}$$

Theorem 1.5. For the existence of the **Riemann integral**:

$$\boxed{\int_{x_a}^{x_b} f(x)dx,}$$

where x_a and x_b are finite numbers, it is sufficient to satisfy one of the following conditions:

(i) $f(x)$ is continuous in $[x_a, x_b]$.

(ii) $f(x)$ is bounded and piecewise continuous in $[x_a, x_b]$.

From these definitions, we can deduce the following properties of the Riemann integral [1]:

I. If $f_1(x)$ and $f_2(x)$ are integrable in $[x_a, x_b]$, then their sum is also integrable, and we can write

$$\int_{x_a}^{x_b} [f_1(x) + f_2(x)]dx = \int_{x_a}^{x_b} f_1(x)dx + \int_{x_a}^{x_b} f_2(x)dx. \tag{1.222}$$

II. If $f(x)$ is integrable in $[x_a, x_b]$, then the following are true:

$$\int_{x_a}^{x_b} \alpha f(x)dx = \alpha \int_{x_a}^{x_b} f(x)dx, \quad \alpha \text{is a constant}, \tag{1.223}$$

$$\left| \int_{x_a}^{x_b} f(x)dx \right| \leq \int_{x_a}^{x_b} |f(x)|dx, \tag{1.224}$$

$$\left| \int_{x_a}^{x_b} f(x)dx \right| \leq M(x_b - x_a) \quad \text{if } |f(x)| \leq M \text{in } [x_a, x_b], \tag{1.225}$$

$$\int_{x_a}^{x_b} f(x)dx = -\int_{x_b}^{x_a} f(x)dx. \tag{1.226}$$

III. If $f(x)$ is continuous and $f(x) \geq 0$ in $[x_a, x_b]$, then

$$\int_{x_a}^{x_b} f(x)dx = 0 \tag{1.227}$$

means $f(x) \equiv 0$.

IV. The **average** or the **mean**, $\langle f \rangle$, of $f(x)$ in the interval $[x_a, x_b]$ is defined as

$$\boxed{\langle f \rangle = \frac{1}{x_b - x_a} \int_{x_a}^{x_b} f(x)dx.} \tag{1.228}$$

If $f(x)$ is continuous, then there exist at least one point $x^* \in [x_a, x_b]$ such that

$$\boxed{\int_{x_a}^{x_b} f(x)dx = f(x^*)(x_b - x_a).} \tag{1.229}$$

This is also called the **mean value theorem** or **Rolle's theorem**.

V. If $f(x)$ is integrable in $[x_a, x_b]$ and if $x_a < x_c < x_b$, then

$$\int_{x_a}^{x_b} f(x)dx = \int_{x_a}^{x_c} f(x)dx + \int_{x_c}^{x_b} f(x)dx. \tag{1.230}$$

VI. If $f(x) \geq g(x)$ in $[x_a, x_b]$, then

$$\int_{x_a}^{x_b} f(x)dx \geq \int_{x_a}^{x_b} g(x)dx. \tag{1.231}$$

VII. Fundamental theorem of calculus: If $f(x)$ is continuous in $[x_a, x_b]$, then the function

$$F(x) = \int_{x_a}^{x} f(t)dt \tag{1.232}$$

is also a continuous function of x in $[x_a, x_b]$. The function $F(x)$ is differentiable for every point in $[x_a, x_b]$ and its derivative at x is $f(x)$:

$$\boxed{\frac{dF}{dx} = \frac{d}{dx}\int_{x_a}^{x} f(t)dt = f(x).} \tag{1.233}$$

$F(x)$ is called the **primitive** or the **antiderivative** of $f(x)$. Given a primitive, $F(x)$, then

$$F(x) + C_0,$$

where C_0 is a constant, is also a primitive. If a primitive is known for $f(x)$ in $[x_a, x_b]$, we can write

$$\int_{x_a}^{x_b} f(x)dx = \int_{x_a}^{x_b} \frac{dF}{dx}\, dx \tag{1.234}$$

$$= F(x)|_{x_a}^{x_b} \tag{1.235}$$

$$= F(x_b) - F(x_a). \tag{1.236}$$

When the region of integration is not specified, we write the **indefinite integral**:

$$\int f(x)dx = F(x) + C, \tag{1.237}$$

where C is an arbitrary constant and $F(x)$ is any function the derivative of which is $f(x)$.

VIII. If $f(x)$ is continuous and $f(x) \geq 0$ in $[x_a, x_b]$, then **geometrically** the integral

$$\int_{x_a}^{x_b} f(x)dx \tag{1.238}$$

is the **area** under $f(x)$ between x_a and x_b.

IX. A very useful inequality in deciding whether a given integral is convergent or not is the **Schwarz inequality**:

$$\boxed{\left[\int_{x_a}^{x_b} f(x)g(x)dx\right]^2 \leq \int_{x_a}^{x_b} f^2(x)dx \int_{x_a}^{x_b} g^2(x)dx.} \tag{1.239}$$

X. One of the most commonly used techniques in integral calculus is the **integration by parts**:

$$\boxed{\int_{x_a}^{x_b} vu'\ dx = [uv]_{x_a}^{x_b} - \int_{x_a}^{x_b} uv'\ dx} \tag{1.240}$$

or

$$\boxed{\int_{x_a}^{x_b} v\ du = [uv]_{x_a}^{x_b} - \int_{x_a}^{x_b} u\ dv,} \tag{1.241}$$

where the derivatives u' and v' and u and v are continuous in $[x_a, x_b]$.

XI. In general, the following inequality holds:

$$\boxed{\int_{x_a}^{x_b} f(x)dx \leq \int_{x_a}^{x_b} |f(x)|dx,} \tag{1.242}$$

that is, if the integral $\int_{x_a}^{x_b} |f(x)|dx$ converges, then the integral $\int_{x_a}^{x_b} f(x)\ dx$ also converges. A convergent integral, $\int_{x_a}^{x_b} f(x)\ dx$, is said to be **absolutely convergent**, if $\int_{x_a}^{x_b} |f(x)|dx$ also converges. Integrals that converge but do not converge absolutely are called **conditionally convergent**.

1.13 IMPROPER INTEGRALS

We introduced Riemann integrals for bounded functions with finite intervals. Improper integrals are basically their extension to cases with infinite range and to functions that are not necessarily bounded.

Definition 1.2. Consider the integral

$$\int_a^c f(x)dx, \tag{1.243}$$

which exists in the Riemann sense in the interval $[a, c]$, where $a < c < b$. If the limit

$$\lim_{c \to b^-} \int_a^c f(x)dx \to A \tag{1.244}$$

exists, where the function $f(x)$ could be unbounded in the left neighborhood of b, then we say the integral $\int_a^b f(x)\,dx$ exists, or converges, and write

$$\int_a^b f(x)dx = A. \tag{1.245}$$

Example 1.6. ***Improper integrals:*** Consider the improper integral

$$I_1 = \int_0^1 \frac{x\,dx}{(1-x)^{1/2}}, \tag{1.246}$$

where the integrand, $x/(1-x)^{1/2}$, is unbounded at the end point $x = 1$. We write I_1 as the limit

$$I_1 = \lim_{c \to 1^-} \int_0^c \frac{x\,dx}{(1-x)^{1/2}} = \lim_{c \to 1^-} \left[\frac{2(1-x)^{3/2}}{3} - 2(1-x)^{1/2} \right]_0^c \tag{1.247}$$

$$= \lim_{c \to 1^-} \left[\frac{2}{3}(1-c)^{3/2} - 2(1-c)^{1/2} - \frac{2}{3} + 2 \right] = \frac{4}{3}, \tag{1.248}$$

thereby obtaining the value of I_1 as 4/3. We now consider the integral

$$I_2 = \int_0^1 \frac{dx}{(1-x)}, \tag{1.249}$$

which does not exist since

$$I_2 = \lim_{c \to 1^-} \int_0^c \frac{dx}{(1-x)} = \lim_{c \to 1^-} [-\ln(1-x)]_0^c \tag{1.250}$$

$$= \lim_{c \to 1^-} [-\ln(1-c)] \to \infty. \tag{1.251}$$

In this case, we say the integral does not exist, or it is divergent, and for its value, we write $+\infty$.

A parallel argument is given if the integral $\int_c^b f(x)\,dx$ exists in the interval $[c, b]$, where $a < c < b$. We now write the limit

$$I = \lim_{c \to a^+} \int_c^b f(x)dx, \tag{1.252}$$

where $f(x)$ could be unbounded in the right neighborhood of a. If the limit

$$\lim_{c \to a^+} \int_c^b f(x)dx \to B \tag{1.253}$$

exists, we write

$$\int_a^b f(x)dx = B. \tag{1.254}$$

We now present another useful result from integral calculus:

Theorem 1.6. Let c be a point in the interval (a, b), and let $f(x)$ be integrable in the intervals $[a, a']$ and $[b', b]$, where $a < a' < c < b' < b$. Furthermore, $f(x)$ could be unbounded in the neighborhood of c. Then, the integral $I = \int_a^b f(x)\, dx$ exists if the integrals

$$I_1 = \int_a^c f(x)dx \tag{1.255}$$

and

$$I_2 = \int_c^b f(x)dx \tag{1.256}$$

both exist and when they exist, their sum is equal to I:

$$I = I_1 + I_2. \tag{1.257}$$

If either I_1 or I_2 diverges, then I also diverges.

Example 1.7. ***Improper integrals:*** Consider $I = \int_{-1}^3 dx/x$:

$$I = \int_{-1}^0 \frac{dx}{x} + \int_0^3 \frac{dx}{x}, \tag{1.258}$$

which converges provided that the integrals on the right-hand side converge. However, they both diverge:

$$\int_{-1}^0 \frac{dx}{x} = \lim_{c\to 0^-} \int_{-1}^c \frac{dx}{x} = \lim_{c\to 0^-} [\ln|x|]_{-1}^c = \lim_{c\to 0^-} \ln|c| \to -\infty \tag{1.259}$$

and similarly,

$$\int_0^3 \frac{dx}{x} \to \lim_{c\to 0^+} \int_c^3 \frac{dx}{x} = \lim_{c\to 0^+} [\ln|x|]_c^3 \to \ln 3 - \lim_{c\to 0^+} \ln|c| \to +\infty, \tag{1.260}$$

hence their sum also diverges.

When the range of the integral is infinite, we use the following results: If $f(x)$ is integrable in $[a, b]$ and the limit

$$\lim_{b\to\infty} \int_a^b f(x)dx \to A \tag{1.261}$$

exists, we can write

$$\int_a^\infty f(x)dx = A. \tag{1.262}$$

Similarly, we define the integral

$$\int_{-\infty}^b f(x)dx = B. \tag{1.263}$$

If the integrals

$$I_1 = \int_a^\infty f(x)dx \tag{1.264}$$

and

$$I_2 = \int_{-\infty}^a f(x)dx \tag{1.265}$$

both exist, then we can write

$$\int_{-\infty}^\infty f(x)dx = I_1 + I_2. \tag{1.266}$$

1.14 CAUCHY PRINCIPAL VALUE INTEGRALS

Since the integrals in Example 1.7, $I_1 = \int_{-1}^0 dx/x$ and $I_2 = \int_0^3 dx/x$, both diverge, we used Theorem 1.6 to conclude that their sum, $I = I_1 + I_2$, is also divergent. However, notice that I_1 diverges as $\lim_{c\to 0^-} \ln|c| \to -\infty$, while I_2 diverges as $\lim_{c\to 0^+}(-\ln|c|) \to +\infty$. In other words, if we consider the two integrals together, the two divergences offset each other, thus yielding a finite result for the value of the integral as

$$I = \int_{-1}^3 \frac{dx}{x} = \lim_{c\to 0^-} [\ln|x|]_{-1}^c + \lim_{c\to 0^+} [\ln|x|]_c^3 \tag{1.267}$$

$$= \lim_{c\to 0^-} \ln|c| - \ln 1 + \ln 3 - \lim_{c\to 0^+} \ln|c| \to \ln 3 \tag{1.268}$$

$$= \ln 3. \tag{1.269}$$

The problem with $\int_{-1}^3 dx/x$ is that the integrand, $1/x$, diverges at the origin. However, at all the other points in the range $[-1, 3]$, it is finite. In Riemann integrals (Theorem 1.6), divergence of either I_1 or I_2 is sufficient to conclude that the integral I does not exist. However, as in the aforementioned case, sometimes by considering the two integrals, I_1 and I_2, together, one may obtain a finite result. This is called taking the **Cauchy principal value** of the integral. Since it corresponds to a **modification** of the Riemann definition of integral, it has to be mentioned explicitly that we are taking the Cauchy principal value as

$$PV\int_{-1}^{3}\frac{dx}{x}=\ln 3. \tag{1.270}$$

Another example is the integral

$$I=\int_{-\infty}^{\infty}x^3\,dx, \tag{1.271}$$

which is divergent in the ordinary sense, since

$$\int_{0}^{\infty}x^3\,dx=\lim_{a\to\infty}\int_{0}^{a}x^3\,dx=\lim_{a\to\infty}\frac{a^4}{4}\to\infty. \tag{1.272}$$

However, if we take its Cauchy principal value, we obtain

$$PV\int_{-\infty}^{\infty}x^3\,dx=\lim_{a\to\infty}\left[\int_{-a}^{0}x^3\,dx+\int_{0}^{a}x^3\,dx\right] \tag{1.273}$$

$$=\lim_{a\to\infty}\left[\frac{a^4}{4}-\frac{a^4}{4}\right]=0. \tag{1.274}$$

Example 1.8. ***Cauchy principal value:*** Considering the following integral:

$$I=\int_{-\infty}^{\infty}\frac{(1+x)dx}{1+x^2}, \tag{1.275}$$

which we write as

$$I=\lim_{c\to\infty}\left[\int_{-c}^{0}\frac{(1+x)dx}{1+x^2}+\int_{0}^{c}\frac{(1+x)dx}{1+x^2}\right]. \tag{1.276}$$

For finite c, we evaluate the second integral:

$$\int_{0}^{c}\frac{(1+x)dx}{1+x^2}=\tan^{-1}x+\frac{1}{2}\log(1+x^2)\Big|_{0}^{c} \tag{1.277}$$

$$=\tan^{-1}c+\frac{1}{2}\log(1+c^2), \tag{1.278}$$

which in the limit as $c\to\infty$ diverges as $\left[\tan^{-1}c+\frac{1}{2}\log(1+c^2)\right]\to\infty$. Hence, the integral I also diverges in the Riemann sense by Theorem 1.6. However, since the first integral also diverges, but this time as

$$\lim_{c\to\infty}\int_{-c}^{0}\frac{(1+x)dx}{1+x^2}\to\lim_{c\to\infty}\left[-\tan^{-1}(-c)-\frac{1}{2}\log(1+c^2)\right], \tag{1.279}$$

we consider the two integrals [Eq. (1.276)] together to obtain the Cauchy principal value of I as

$$PV \int_{-\infty}^{\infty} \frac{(1+x)dx}{1+x^2} = \pi. \tag{1.280}$$

1.15 INTEGRALS INVOLVING A PARAMETER

Integrals given in terms of a parameter play an important role in applications. In particular, integrals involving a parameter and with infinite range are of considerable significance. In this regard, we quote three useful theorems:

Theorem 1.7. If there exists a positive function $Q(x)$ satisfying the inequality $|f(\alpha,x)| \leq Q(x)$ for all $\alpha \in [\alpha_1,\alpha_2]$, and if $\int_a^\infty Q(x)dx$ is convergent, then the integral

$$g(\alpha) = \int_a^\infty f(\alpha,x)dx \tag{1.281}$$

is **uniformly convergent** in the interval $[\alpha_1,\alpha_2]$. This is also called the **Weierstrass M-test** for uniform convergence. If an integral, $\int_a^\infty f(\alpha,x)dx$, is uniformly convergent in $[\alpha_1,\alpha_2]$, then for any given $\varepsilon > 0$, there exists a number c_0 depending on ε but independent of α such that $\left|\int_c^\infty f(\alpha,x)dx\right| < \varepsilon$ for all $c > c_0 > a$.

Example 1.9. ***Uniform convergence:*** Consider the integral

$$I = \int_0^\infty e^{-\alpha x} \sin x \, dx, \tag{1.282}$$

which is uniformly convergent for $\alpha \in [\varepsilon,\infty)$ for every $\varepsilon > 0$. To show this, we choose $Q(x)$ as $e^{-\varepsilon x}$ so that

$$|e^{-\alpha x} \sin x| \leq e^{-\varepsilon x} \tag{1.283}$$

is true for all $\alpha \geq \varepsilon$. Uniform convergence of I follows, since the integral

$$\int_0^\infty e^{-\varepsilon x} \, dx \tag{1.284}$$

is convergent. Note that by using integration by parts twice, we can evaluate the integral as

$$\int_0^\infty e^{-\alpha x} \sin x \, dx = \frac{1}{1+\alpha^2}, \quad \alpha > 0. \tag{1.285}$$

The case where $\alpha = 0$ may be excluded, since the integral $\int_0^\infty \sin x \, dx$ does not converge at all.

Theorem 1.8. Let $f(\alpha, x)$ and $\partial f(\alpha, x)/\partial\alpha$ be continuous for all $\alpha \in [\alpha_1, \alpha_2]$ and $x \in [a, \infty)$. If the integral

$$g(\alpha) = \int_a^\infty f(\alpha, x)dx \tag{1.286}$$

exists for all $\alpha \in [\alpha_1, \alpha_2]$ and if the integral

$$\int_a^\infty \frac{\partial f(\alpha, x)}{\partial\alpha}\, dx \tag{1.287}$$

is uniformly convergent for all $\alpha \in [\alpha_1, \alpha_2]$, then $g(\alpha)$ is differentiable in $[\alpha_1, \alpha_2]$ (at α_1 from the right and at α_2 from the left) with the derivative

$$\frac{dg}{d\alpha} = \int_a^b \frac{\partial f(\alpha, x)}{\partial\alpha}\, dx. \tag{1.288}$$

In other words, we can interchange the order of differentiation with respect to α and integration with respect to x as

$$\boxed{\frac{d}{d\alpha}\int_a^b f(\alpha, x)dx = \int_a^b \frac{\partial f(\alpha, x)}{\partial\alpha}\, dx.} \tag{1.289}$$

This is also called the **Leibnitz's rule** [2].

Theorem 1.9. Let $f(\alpha, x)$ be continuous for all $\alpha \in [\alpha_1, \alpha_2]$ and $x \in [a, \infty)$. Also let the integral

$$g(\alpha) = \int_a^\infty f(\alpha, x)dx \tag{1.290}$$

be uniformly convergent for all $\alpha \in [\alpha_1, \alpha_2]$. Then,

(a) $g(\alpha)$ is continuous in $[\alpha_1, \alpha_2]$ (at α_1 from the right and at α_2 from the left).

(b) The relation

$$\int_{\alpha_1}^{\alpha} g(\alpha')d\alpha' = \int_a^\infty \left[\int_{\alpha_1}^{\alpha} f(x, \alpha')\, d\alpha'\right] dx, \tag{1.291}$$

that is,

$$\boxed{\int_{\alpha_1}^{\alpha}\left[\int_a^\infty f(\alpha', x)dx\right] d\alpha' = \int_a^\infty \left[\int_{\alpha_1}^{\alpha} f(x, \alpha')d\alpha'\right] dx,} \tag{1.292}$$

is true for all $\alpha \in [\alpha_1, \alpha_2]$. In other words, the order of the integrals with respect to x and α' can be interchanged. Note that in case (a), the interval for α does not have to be finite.

Remark: In the aforementioned theorems, if the limits of integration are finite but the function $f(\alpha, x)$ or its partial derivative $\partial f(\alpha, x)/\partial \alpha$ is not bounded in the neighborhood of the segment defined by $x = b$ and $\alpha \in [\alpha_1, \alpha_2]$, we say that the integral

$$g(\alpha) = \int_a^b f(\alpha, x)dx \tag{1.293}$$

is uniformly convergent for all $\alpha \in [\alpha_1, \alpha_2]$, if for every $\epsilon > 0$, we can find a $\delta_0 > 0$ independent of α such that the inequality

$$\left| \int_{b-\delta}^b f(\alpha, x)dx \right| < \epsilon \tag{1.294}$$

is true for all $\delta \in [0, \delta_0]$. We can now apply the aforementioned theorems with the upper limit ∞ in the integrals replaced by b and the domain $x \in [a, \infty)$ by $x \in [a, b]$.

Example 1.10. ***Integrals depending on a parameter:*** Given the integral

$$g(\alpha) = \int_0^\infty \frac{\sin \alpha x}{x} \, dx = \frac{\pi}{2}, \quad \alpha \neq 0, \tag{1.295}$$

we differentiate with respect to α to write

$$\frac{\partial}{\partial \alpha} \int_0^\infty \frac{\sin \alpha x}{x} \, dx = \frac{\partial}{\partial \alpha} \left(\frac{\pi}{2} \right) = 0. \tag{1.296}$$

However, this is not correct. The integral on the right-hand side of

$$\int_0^\infty \frac{\partial}{\partial \alpha} \left[\frac{\sin \alpha x}{x} \right] dx = \int_0^\infty \cos \alpha x \, dx \tag{1.297}$$

does not exist, since the limit

$$\lim_{b \to \infty} \int_0^b \cos \alpha x \, dx = \lim_{b \to \infty} \left. \frac{\sin \alpha x}{\alpha} \right|_0^b = \lim_{b \to \infty} \frac{1}{\alpha} \sin \alpha b \tag{1.298}$$

does not exist. Hence, the differentiation $dg/d\alpha$ is not justified (Theorem 1.8). On the other hand, given the integral

$$\int_0^{\pi/2} \frac{dx}{\alpha^2 \cos^2 x + \sin^2 x} = \frac{\pi}{2\alpha}, \quad \alpha > 0, \tag{1.299}$$

we can write

$$\frac{\partial}{\partial \alpha} \int_0^{\pi/2} \frac{dx}{\alpha^2 \cos^2 x + \sin^2 x} = \frac{d}{d\alpha} \left(\frac{\pi}{2\alpha} \right) \tag{1.300}$$

to obtain the integral

$$-\int_0^{\pi/2} \frac{2\alpha\cos^2 x\, dx}{(\alpha^2\cos^2 x + \sin^2 x)^2} = -\frac{\pi}{2\alpha^2}. \tag{1.301}$$

Example 1.11. ***Integrals depending on a parameter:*** Consider

$$f(\alpha, x) = \begin{cases} \dfrac{e^{-\alpha x}\sin x}{x}, & x \neq 0, \\ 1, & x = 0, \end{cases} \tag{1.302}$$

which is continuous for all x and α. Since

$$\frac{\partial f(\alpha, x)}{\partial \alpha} = -e^{-\alpha x}\sin x, \tag{1.303}$$

which is also continuous for all x and α, and the integral

$$\int_0^{\infty} \frac{\partial f(\alpha, x)}{\partial \alpha}\, dx = \int_0^{\infty} e^{-\alpha x}\ \sin x\, dx \tag{1.304}$$

converges uniformly for all $\alpha > 0$ (Example 1.9), using Theorem 1.8, we conclude that

$$g(\alpha) = \int_0^{\infty} e^{-\alpha x}\frac{\sin x}{x}\, dx, \quad \alpha > 0, \tag{1.305}$$

exists and can be differentiated to write

$$g'(\alpha) = \frac{\partial}{\partial \alpha}\int_0^{\infty} e^{-\alpha x}\frac{\sin x}{x}\, dx = \int_0^{\infty} \frac{\partial}{\partial \alpha}\left[e^{-\alpha x}\frac{\sin x}{x}\right] dx \tag{1.306}$$

$$= -\int_0^{\infty} e^{-\alpha x}\ \sin x\, dx = -\frac{1}{1+\alpha^2}, \tag{1.307}$$

where we have used the result in Eq. (1.285). We now use Theorem 1.9 to integrate $g'(\alpha)$ [Eqs. (1.306) and (1.307)], which is continuous for all $\alpha > 0$ to obtain

$$\int_0^{\infty} g'(\alpha) d\alpha = -\int_0^{\infty} \frac{1}{1+\alpha^2}\, d\alpha = -\tan^{-1}|_0^{\infty} = -\frac{\pi}{2}. \tag{1.308}$$

However, we can also write

$$\int_0^{\infty} g'(\alpha) d\alpha = -\int_0^{\infty}\left[\int_0^{\infty} e^{-\alpha x}\sin x\, dx\right] d\alpha = -\int_0^{\infty}\left[\int_0^{\infty} e^{-\alpha x}\sin x\, d\alpha\right] dx \tag{1.309}$$

$$= -\int_0^{\infty}\left[-\frac{e^{-\alpha x}\sin x}{x}\right]_0^{\infty} dx = -\int_0^{\infty} \frac{\sin x}{x} dx, \quad \alpha > 0, \tag{1.310}$$

which along with Eq. (1.308) yields the definite integral

$$\boxed{\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2.} \tag{1.311}$$

1.16 LIMITS OF INTEGRATION DEPENDING ON A PARAMETER

Let $A(x)$ and $B(x)$ be two continuous functions with continuous derivatives in $[x_1, x_2]$, with $B(x) > A(x)$. Also let $f(t,x)$ and $\partial f(t,x)/\partial x$ be continuous in the region defined by $[x_1, x_2]$ and $[x_1 = A(x), x_2 = B(x)]$. We can now write the integral

$$\left[\int_{A(x)}^{B(x)} f(t,x)dt\right] = F(B(x), A(x), x) \tag{1.312}$$

and its partial derivative with respect to x as

$$\frac{d}{dx}\left[\int_{A(x)}^{B(x)} f(t,x)dt\right] = \frac{\partial F}{\partial B}\frac{dB}{dx} + \frac{\partial F}{\partial A}\frac{dA}{dx} + \frac{\partial F}{\partial x}. \tag{1.313}$$

Using the relations [Eq. (1.233)]:

$$\frac{\partial}{\partial v}\int_u^v f(t)dt = f(v), \tag{1.314}$$

$$\frac{\partial}{\partial u}\int_u^v f(t)dt = -f(u), \tag{1.315}$$

we can write

$$\frac{\partial F}{\partial B} = f(B,x), \quad \frac{\partial F}{\partial A} = -f(A,x). \tag{1.316}$$

We can also write

$$\frac{\partial}{\partial x}\left[\int_{A(x)}^{B(x)} f(t,x)dt\right] = \int_{A(x)}^{B(x)} \frac{\partial f(t,x)}{\partial x}\, dt, \tag{1.317}$$

thus obtaining the useful formula

$$\boxed{\frac{d}{dx}\left[\int_{A(x)}^{B(x)} f(t,x)dt\right] = \int_{A(x)}^{B(x)} \frac{\partial f(t,x)}{\partial x}dt + \frac{dB}{dx}f(B(x),x) - \frac{dA}{dx}f(A(x),x).} \tag{1.318}$$

1.17 DOUBLE INTEGRALS

Consider a continuous and bounded function $f(x, y)$ defined in a closed region R of the xy-plane. It is important that R be bounded, that is, we can enclose it with a circle of sufficiently large radius. We subdivide R into rectangles by drawing parallels to the x and the y axes (Figure 1.10). We choose only the rectangles in R and numerate them from 1 to n. Area of the ith rectangle is shown as ΔA_i, and the largest of the diagonals, h, is called the **norm** of the mesh. We now form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i, \tag{1.319}$$

where, as in the one-dimensional integrals, (x_i^*, y_i^*) is a point arbitrarily chosen in the ith rectangle. If the sum converges to a limit as $h \to 0$, we define the double integral as the limit

$$\lim_{h\to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta A_i \to \int\int_R f(x, y)dx\ dy. \tag{1.320}$$

When the region R can be described by the inequalities

$$y_1(x) \le y \le y_2(x), \quad x_1 \le x \le x_2, \tag{1.321}$$

or

$$x_1(y) \le x \le x_2(y), \quad y_1 \le y \le y_2, \tag{1.322}$$

where $y_1(x)$, $y_2(x)$ and $x_1(y)$, $x_2(y)$ are continuous functions (Figure 1.11), we can write the double integral for the first case as the iterated integral

$$I = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} f(x, y)dy \right] dx. \tag{1.323}$$

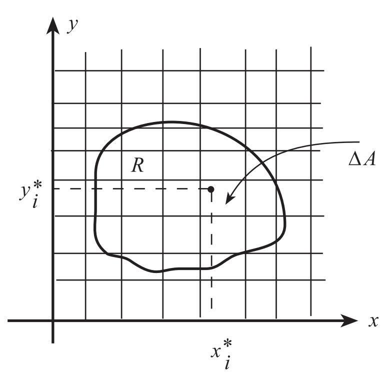

Figure 1.10 The double integral.

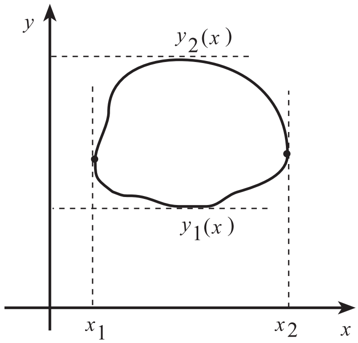

Figure 1.11 Ranges in the iterated integrals.

The definite integral inside the square brackets will yield a function $F(x)$, which reduces I to a one-dimensional definite integral:

$$\int_{x_1}^{x_2} F(x)dx. \tag{1.324}$$

A similar argument can be given for the second case [Eq. (1.322)]. We now present these results in terms of a theorem:

Theorem 1.10. If $f(x, y)$ is continuous and bounded in a closed interval described by the region [1, 2]

$$y_1(x) \leq y \leq y_2(x), \quad x_1 \leq x \leq x_2, \tag{1.325}$$

then

$$\int_{y_1(x)}^{y_2(x)} f(x, y)dy \tag{1.326}$$

is a continuous function of x and

$$\boxed{\int\int_R f(x, y)dx\ dy = \int_{x_1}^{x_2} \left[\int_{y_1(x)}^{y_2(x)} f(x, y)dy\right] dx.} \tag{1.327}$$

Similarly, if R is described by

$$x_1(y) \leq x \leq x_2(y), \quad y_1 \leq y \leq y_2, \tag{1.328}$$

then we can write

$$\boxed{\int\int_R f(x, y)dx\ dy = \int_{y_1}^{y_2} \left[\int_{x_1(y)}^{x_2(y)} f(x, y)dx\right] dy.} \tag{1.329}$$

1.18 PROPERTIES OF DOUBLE INTEGRALS

We can summarize the basic properties of double integrals, which are essentially same as the definite integrals of functions with single variable as follows:

I.

$$\int\int_R [f(x,y)+g(x,y)]dx\,dy = \int\int_R f(x,y)dx\,dy + \int\int_R g(x,y)dx\,dy, \tag{1.330}$$

$$\int\int cf(x,y)dx\,dy = c\int\int_R f(x,y)dx\,dy,\ c\text{is a constant}, \tag{1.331}$$

$$\int\int_R f(x,y)dx\,dy = \int\int_{R_1} f(x,y)dx\,dy + \int\int_{R_2} f(x,y)dx\,dy, \tag{1.332}$$

where R is composed of R_1 and R_2, which overlap only at the boundary.

II. There exists a point (x_1, y_1) in R such that

$$\int\int_R f(x,y)dx\,dy = A_R f(x_1,y_1), \tag{1.333}$$

where A_R is the area of R. The value $f(x_1, y_1)$ is also the **mean value** $\langle f\rangle_R$ of the function in the region R:

$$\boxed{\langle f\rangle_R = \frac{1}{A_R}\int\int_R f(x,y)dx\,dy.} \tag{1.334}$$

III.

$$\left|\int\int_R f(x,y)dx\,dy\right| \le M_R\cdot A_R, \tag{1.335}$$

where M_R is the absolute maximum in R:

$$|f(x,y)| \le M_R \tag{1.336}$$

and A_R is the area of R.

IV. Uses of double integrals: If we set $f(x,y)=1$ in $\int\int_R f(x,y)\,dx\,dy$, the double integral corresponds to the area A_R of the region R:

$$\boxed{\int\int_R dx\,dy = A_R.} \tag{1.337}$$

For $f(x,y)\ge 0$, we can interpret the double integral as the volume between the surface $z=f(x,y)$ and the region R in the xy-plane. If we interpret $f(x,y)$ as the mass density of a flat object lying on the xy-plane covering the region R, the double integral

$$M = \int\int_R f(x,y)dx\ dy \tag{1.338}$$

gives its total mass M.

1.19 TRIPLE AND MULTIPLE INTEGRALS

The methods and the results developed for the double integrals can easily be extended to the triple and multiple integrals:

$$\iiint_R f(x,y,z)dx\ dy\ dz,\quad \iiiint_R f(x,y,z,w)dx\ dy\ dz\ dw, \text{etc.} \tag{1.339}$$

Following the arguments given for the single and the double integrals, for a continuous and bounded function $f(x,y,z)$ in a bounded region R defined by

$$z_1(x,y) \le z \le z_2(x,y),\quad y_1(x) \le y \le y_2(x),\quad x_1 \le x \le x_2, \tag{1.340}$$

we can define the triple integral

$$\boxed{\iiint_R f(x,y,z)dx\ dy\ dz = \int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)dz\right]dy\right]dx.} \tag{1.341}$$

An obvious application of the triple integral is when $f(x,y,z) = 1$, which gives the volume V_R of the region R:

$$\boxed{\iiint_R dx\ dy\ dz = V_R.} \tag{1.342}$$

In physical applications, total amount of mass, charge, etc., with the density $\rho(x,y,z)$ are given as the triple integral

$$\iiint_R \rho(x,y,z)dx\ dy\ dz. \tag{1.343}$$

The **average** or the **mean** value of a function $f(x,y,z)$ in the region R with the volume V_R is defined as

$$\boxed{\langle f\rangle_R = \frac{1}{V_R}\iiint_R f(x,y,z)dx\ dy\ dz.} \tag{1.344}$$

Example 1.12. ***Volume between two surfaces:*** To find the volume between the cone $z = \sqrt{x^2 + y^2}$ and the paraboloid $z = x^2 + y^2$, we first write the triple integral

$$V = \int_0^1 \int_0^1 \left[\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz \right] dx\, dy = \int_0^1 \int_0^1 \left[\sqrt{x^2 + y^2} - x^2 - y^2 \right] dx\, dy. \tag{1.345}$$

We now use plane polar coordinates to write this as

$$V = \int_0^1 \int_0^{2\pi} (\rho - \rho^2)\rho\, d\phi\, d\rho = 2\pi \int_0^1 (\rho - \rho^2)\rho\, d\rho = 2\pi \left[\frac{\rho^3}{3} - \frac{\rho^4}{4} \right]_0^1 = \frac{\pi}{6}. \tag{1.346}$$

REFERENCES

1. Apostol, T.M. (1971). *Mathematical Analysis*. Reading, MA: Addison-Wesley, fourth printing.
2. Kaplan, W. (1984). *Advanced Calculus*, 3e. Reading, MA: Addison-Wesley.

PROBLEMS

1. Determine the critical points as well as the absolute maximum and minimum of the functions

$$\begin{aligned} &\text{(i)} \quad y = \ln x,\ 0 < x \leq 2, \\ &\text{(ii)} \quad y = x/(1 + 2x^2), \\ &\text{(iii)} \quad y = x^3 + 2x^2 + 1, -2 < x < 1. \end{aligned}$$

2. Determine the critical points of the functions

$$\begin{aligned} &\text{(i)} \quad z = x^3 - 6xy^2 + y^3, \\ &\text{(ii)} \quad z = 1 + x^2 + y^2, \\ &\text{(iii)} \quad z = x^2 - 4xy - y^2. \end{aligned}$$

   and test for maximum or minimum.

3. Find the maximum and minimum points of $z = x^2 + 24xy + 8y^2$ subject to the condition $x^2 + y^2 = 25$.
4. Find the critical points of $w = x + y$ subject to $x^2 + y^2 + z^2 = 1$ and identify whether they are maximum or minimum.

5. Express the partial differential equation

$$\frac{\partial^2 \Psi(\overrightarrow{r})}{\partial x^2} + \frac{\partial^2 \Psi(\overrightarrow{r})}{\partial y^2} + \frac{\partial^2 \Psi(\overrightarrow{r})}{\partial z^2} = 0,$$

in spherical coordinates (r, θ, ϕ) defined by the equations

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta,$$

where $r \in [0, \infty)$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$. Next, first show that the inverse transformation exists and then find it.

6. Given the mapping

$$x = u^2 - v^2,$$
$$y = 2uv.$$

(i) Write the Jacobian.

(ii) Evaluate the derivatives $\left(\frac{\partial u}{\partial y}\right)_x$ and $\left(\frac{\partial v}{\partial y}\right)_x$.

7. Find $\left(\frac{\partial u}{\partial x}\right)_y$ and $\left(\frac{\partial u}{\partial y}\right)_x$ for

$$e^u + xu - yv - 1 = 0,$$
$$e^v - xv + yu - 2 = 0.$$

8. Given the transformation functions

$$x = x(u, v),$$
$$y = y(u, v),$$

show that the inverse transformations

$$u = u(x, y),$$
$$v = v(x, y)$$

satisfy

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial v}, \quad \frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v}, \quad \frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial u}, \quad \frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial u},$$

where $J = \frac{\partial(x, y)}{\partial(u, v)}$. Apply your result to Problem 6.

9. Given the transformation functions

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

with the Jacobian $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)}$, show that the inverse transformation functions have the derivatives

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial(y, z)}{\partial(v, w)}, \quad \frac{\partial u}{\partial y} = \frac{1}{J}\frac{\partial(z, x)}{\partial(v, w)}, \quad \frac{\partial u}{\partial z} = \frac{1}{J}\frac{\partial(x, y)}{\partial(v, w)},$$
$$\frac{\partial v}{\partial x} = \frac{1}{J}\frac{\partial(y, z)}{\partial(w, u)}, \quad \frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial(z, x)}{\partial(w, u)}, \quad \frac{\partial v}{\partial z} = \frac{1}{J}\frac{\partial(x, y)}{\partial(w, u)},$$
$$\frac{\partial w}{\partial x} = \frac{1}{J}\frac{\partial(y, z)}{\partial(u, v)}, \quad \frac{\partial w}{\partial y} = \frac{1}{J}\frac{\partial(z, x)}{\partial(u, v)}, \quad \frac{\partial w}{\partial z} = \frac{1}{J}\frac{\partial(x, y)}{\partial(u, v)}.$$

Verify your result in Problem 5.

10. In a one-dimensional conservative system, potential energy can be represented by a (scalar) function $V(x)$, where the negative of the derivative of the potential gives the x component of the force: $F_x(x) = -dV/dx$. With the aid of a sketch, analyze the forces on a conservative system when it is displaced away from equilibrium by a small amount.

11. In one-dimensional potential problems, show that near equilibrium potential can be approximated by the harmonic oscillator potential

$$V(x) = \frac{1}{2}k(x - x_0)^2,$$

where k is a constant and x_0 is the equilibrium point. What is k?

12. Expand $z(x, y) = x^3 \sin y + y^2 \cos x$ in Taylor series up to third order about the origin.

13. If $x = x(u, v)$ and $y = y(u, v)$, then show the following:

$$\text{(i)} \quad \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y = \left(\frac{\partial y}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x,$$
$$\text{(ii)} \quad \left(\frac{\partial x}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v.$$

14. Show the integrals

$$\text{(i)} \quad \int_0^\infty \frac{\sin x \cos x}{x}\, dx = \frac{\pi}{4},$$

$$\text{(ii)} \quad \int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$$

Hint: Use $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

15. Evaluate the improper integrals:

$$\text{(i)} \quad \int_0^1 \frac{dx}{\sqrt{1-x^2}},$$

$$\text{(ii)} \quad \int_0^{1/2} \frac{dx}{\sqrt{x}(1-2x)}.$$

16. First show the following:

$$\text{(i)} \quad \int_1^\infty \frac{dx}{x^p} \text{ converges if and only if } p > 1,$$

$$\text{(ii)} \quad \int_0^1 \frac{dx}{x^p} \text{ converges if and only if } p < 1,$$

$$\text{(iii)} \quad \int_0^c \frac{dx}{|c-x|^p} \text{ converges if and only if } p < 1$$

and then check the convergence of

$$\text{(i)} \quad \int_0^\infty \frac{dx}{\sqrt{2x+x^3}},$$

$$\text{(ii)} \quad \int_0^\infty \frac{dx}{\sqrt{x+2x^2}}.$$

17. Check the convergence of the integral

$$\int_0^1 \frac{x^2\, dx}{(1-x^2)^{1/2}(2x^3+1)}.$$

18. Show that the following integral is convergent by using integration by parts:

$$\int_1^\infty \frac{\sin x}{x}\, dx.$$

19. Using the integral

$$I(a,b) = \int_0^{\pi/2} \frac{dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi}{2ab}, \quad a > 0,\ b > 0,$$

where a and b are two parameters, show the integral

$$\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2} = \frac{\pi}{4ab}\left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

20. Determine the α values for which the following integrals are uniformly convergent:

$$\text{(i)} \quad \int_0^\infty \frac{\cos x\alpha}{1+x^2}\, dx,$$

$$\text{(ii)} \quad \int_0^\infty \frac{1}{x^2+\alpha^2}\, dx.$$

21. Can the order of integration be interchanged in the following integral (explain):

$$I = \int_0^1 \left[\int_0^1 \frac{x-\alpha}{(x+\alpha)^3}\, dx\right] d\alpha.$$

22. Use the result

$$g(\alpha) = \int_0^\infty \frac{\sin x\alpha}{x(x^2+1)}\, dx = \frac{\pi}{2}(1-e^{-\alpha}), \quad \alpha > 0,$$

to deduce the following integrals:

$$\text{(i)} \quad \int_0^\infty \frac{\sin x\alpha}{x(x^2+c^2)}\, dx = \frac{\pi}{2c^2}(1-e^{-c\alpha}), \quad c > 0,$$

$$\text{(ii)} \quad \int_0^\infty \frac{\cos x\alpha}{(x^2+c^2)} dx = \frac{\pi e^{-c\alpha}}{2c}, \quad c > 0.$$

23. Evaluate the following double integral over the triangle with the vertices $(-1, 0)$, $(0, 1)$, and $(2, 0)$:

$$I = \int\int 2y\, dx\, dy.$$

24. Evaluate I over the triangle with the vertices $(0, 0)$, $(1, 1)$, and $(1, 3)$, where

$$I = \int\int xy\, dx\, dy.$$

25. Evaluate the integral

$$\int_{x=0}^{2} \int_{y=0}^{x^2} xy\, dy\, dx.$$

26. First evaluate the integral

$$\int_{x=0}^{2}\int_{y=0}^{x^2}(x^2+2y^2)xy\ dy\ dx$$

and then repeat the integration over the same region but with the x integral taken first.

27. Test the following integral for convergence:

$$\iiint_{x^2+y^2+z^2\leq 1}\ln(x^2+y^2+z^2)dx\ dy\ dz.$$

28. Evaluate the integrals

$$\text{(i)}\quad \int_{z=0}^{2}\int_{x=z}^{2}\int_{y=6x}^{z}dy\ dx\ dz,$$

$$\text{(ii)}\quad \int_{y=-2}^{2}\int_{z=1}^{2}\int_{x=y+z}^{2y+z}y\ dx\ dz\ dy.$$