

# 1

## Basic Concepts, Challenges and Methods

The fluid dynamics world is inundated with thousands of books on the subject, volumes on theory, numerical and engineering niches to no end. Within the specialty of computational fluids, hundreds of thousands of papers have appeared within the past two decades. And in the subset dubbed “aerodynamics,” tens of thousands may be found authored by specialists from dozens of countries. This being the case, we will not offer still another “first principles” derivation of governing equations. We will cite relevant subjects and refer readers to readily available literature where excellent presentations are already available. But it will be the author’s responsibility to develop and critique significant areas of fluids research that deserve further investigation. And, just as important, introduce ambitious students to key ideas quickly and rigorously, in the least amount of time, with minimal formal course work but with objectivity and honest speculation – to prepare him to understand, contribute and write software to evaluate new ideas. To this end, we have developed a fast-paced presentation style combining “simple numerics” with modern ideas in aerodynamics. With these disclaimers said and done, we now begin discussions on many exciting subjects.

### 1.1 Governing Equations – An Unconventional Synopsis

The equations governing fluid motions are numerous, for example, as developed in excellent books by Batchelor (1967), Schlichting (2017), Yih (1969) and others. They cover constant density and compressible fluids; liquids and gases; inviscid and viscous motions; one, two and three dimensions; steady and unsteady flows; irrotational and rotational limits; and rectangular, polar, spherical and curvilinear coordinates. For the most part, we will deal with a special subset of these properties to develop the great majority of our ideas. In two dimensions, assuming Cartesian or rectangular coordinates, the momentum and mass conservation equations governing constant density, constant viscosity flows can be written concisely in the form

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$$\rho (\partial u/\partial t + u \partial u/\partial x + v \partial u/\partial y) = - \partial p/\partial x + \mu(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) \quad (1.1.1a)$$

$$\rho (\partial v/\partial t + u \partial v/\partial x + v \partial v/\partial y) = - \partial p/\partial y + \mu(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2) \quad (1.1.1b)$$

$$\partial u/\partial x + \partial v/\partial y = 0 \quad (1.1.1c)$$

These represent a highly simplified version of the Navier-Stokes equations. Generalizations of the above have appeared for special applications. For example, in high-speed aerodynamics, the density  $\rho$  is variable, and equations of state and energy conservation laws apply (we will describe some transonic applications in Chapters 2, 3 and 4). The viscosity  $\mu$  shown above is constant, but in gas dynamics, it may well be a function of temperature; in meteorology and oceanography, additional dependencies of pressure on properties like humidity and salinity will appear, implying more complicated mathematical descriptions and solutions. Sometimes the stress terms on the right are replaced by an anisotropic tensor; this author has developed models of fluid flow in petroleum reservoirs in a number of books (refer to “About the Author” for further publication information). For our purposes, it suffices to note how Equations 1.1.1a,b,c and similar high-order models (with high-order derivatives) require “Navier-Stokes solvers,” which are a challenge to develop, and computationally expensive and resource-intensive to run.

A simpler limit is found by eliminating  $\mu$  at the very outset, leading to what we call “Euler’s equations,” a low-order system, namely

$$\rho (\partial u/\partial t + u \partial u/\partial x + v \partial u/\partial y) = - \partial p/\partial x \quad (1.1.2a)$$

$$\rho (\partial v/\partial t + u \partial v/\partial x + v \partial v/\partial y) = - \partial p/\partial y \quad (1.1.2b)$$

$$\partial u/\partial x + \partial v/\partial y = 0 \quad (1.1.2c)$$

The above applies at constant density only and the great majority of applications appears in flows, for instance, with oncoming velocity shear. The reader may recall words of caution. For irrotational flows, Bernoulli’s equation “ $p/\rho + \frac{1}{2}(u^2 + v^2) = \text{constant}$ ” applies, where the constant, fixed for the entire flowfield, is known from upstream conditions. But for rotational flows, this “constant” is only so along a streamline; in fact, it varies from streamline to streamline. What happens when the flow about a jet engine is to be modeled? The external flow, uniform upstream, is irrotational and satisfies a simple Laplace and Bernoulli equation model; however, the flow behind the actuator disk, which imparts radial position-dependent work, is sheared and requires “Euler solvers” with complicated streamline tracking. Algorithm development combining potential with Euler solvers is no small task.

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Investigators have developed sophisticated Euler equation solvers requiring equally sophisticated users. And all because “potential flow solvers” for  $\phi_{xx} + \phi_{yy} = 0$  (or “ $\phi_{xx} + \phi_{rr} + 1/r \phi_r = 0$ ,” axisymmetrically) will not apply. Every student of fluid mechanics understands how potentials *only* apply to flows without shear. But what if potentials *did* apply? What if it were possible to solve  $\phi_{xx} + \phi_{rr} + (1/r - 2U_m'/U_m) \phi_r = 0$  valid for mean background flows with strong  $U_m(r)$  velocity profiles? Simple potential flow codes would, through minor modification, address new classes of important flow problems. In fact, the mathematical basis behind “superpotentials” is developed in Chapter 4 with examples.

Now let's digress and turn to “analysis problems” described by classic potential formulations, that is, solving  $\phi_{xx} + \phi_{yy} = 0$  subject to tangency conditions for the normal derivative  $\phi_y$  along  $y = 0$ , plus a requirement on a “potential jump”  $[\phi]$  related to Kutta's condition at the trailing edge. This formulation, which determines the surface pressure due to a prescribed geometry, as old as aerodynamics itself, has been solved straightforwardly in numerous ways: Glauert's series, panel methods, finite differences, finite elements and so on. But the complementary “inverse problem,” searching for the geometry that induces a prescribed pressure, is more subtle and also known as the “indirect” problem. And for good reason. Often, the above analysis solver is run over and over, varying all sorts of empirically defined parameters in endless ways, until some type of convergence is achieved. Is there a simple but “direct approach to indirect problems?”

The answer is, “Yes.” Enter the streamfunction, the “black sheep” of modern computational fluid-dynamics. We will show that the airfoil shape is described by the ordinate  $y(x) = -\psi(x,0)$  where  $\psi_{xx} + \psi_{yy} = 0$  is solved, subject to *normal* specifications for  $\psi_y(x,0) = -\frac{1}{2} U_\infty C_p(x)$  along  $y = 0$ , plus a requirement on a jump  $[\psi]$  related to the degree of trailing edge closure. In other words, given the surface pressure coefficient  $C_p(x)$ , the shape can be directly (meaning non-iteratively) solved using any potential flow algorithm for analysis problems already available!

Then again, pessimists might argue that the method is limited because it could not be extended to, say transonic supercritical problems. In developing our model, we drew upon Cauchy-Riemann conditions (from complex variables) which strictly apply to complementary equation pairs like  $\phi_{xx} + \phi_{yy} = 0$  and  $\psi_{xx} + \psi_{yy} = 0$ . And so, “no constant density assumption, no streamfunction inversion.” Correct? Incorrect. To solve the problem, we developed a completely rigorous “engineer's

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Cauchy-Riemann transform” that allowed us to create a compressible, mixed subsonic and supersonic extension of  $\psi_{xx} + \psi_{yy} = 0$  to solve inverse formulations in a single pass. Not quite a pure partial differential equation, but one with an integral coefficient that could be just as easily solved. So that’s another success story, where we’ve solved indirect problems directly, initially with a lots of speculation and then some luck.

Next consider compressible flow extensions of Equations 1.1.2a,b,c, which are interesting in a very different way. At steady cruise conditions, under the assumption of irrotationality, these equations (as we will show) lead to a potential flow model not unlike  $(\ ) \phi_{xx} + \phi_{yy} = 0$  where  $(\ )$  is somewhat tricky. At Mach zero, or flight speeds say 300 mph or less, this reduces to a purely subsonic (scaled) equation not unlike Laplace’s “ $\phi_{xx} + \phi_{yy} = 0$ .” Dozens of classical texts, conformal maps and singular integral equation methods are and have been available for decades. Near 550 mph or so, fluid particles accelerate so rapidly around leading edges that flows become locally supersonic. Most of the time, they terminate abruptly at shockwaves – where sudden discontinuous increases in pressure lead to losses and unstable wing oscillations. Such are typical of problems suggestive of a “sonic barrier” just several decades ago. But computational methods were non-existent until the 1970s, when Murman and Cole (1971) published a pioneering “type-dependent” numerical algorithm for mixed elliptic and hyperbolic equations. Their idea was simple: use “upwind differencing” for supersonic points and central for subsonic to proper account for domains of influence and dependence. The original scheme did not conserve mass, but later researchers would introduce “conservative schemes” and curvilinear grid refinements that seemed to suggest . . . well, end of story.

Just when the story was finally told, workers in the mid-1980s discovered that computed solutions could be non-unique. For a given set of flight conditions, more than a single solution existed! Was this a computational anomaly or physical reality? Was it related to buffeting and aerodynamic instability? Or was it an artifact inherent in Equations 1.1.2a,b,c, which while simpler than Equations 1.1.1a,b,c, were low-order and only partially descriptive of the physics? Non-uniqueness aside, the Murman-Cole scheme and its derivatives were not perfect. Iterations were required to “march” in the direction in which the supersonic flow evolved – which was, of course, unknown at the outset. This placed limitations on mesh generation flexibility, since coordinate lines must somehow align with the flow – but this was hopeless since

curvilinear grid definition usually bears no relationship to the physics (for now, anyway). And so, tedious local, point-by-point type-testing, often employing “rotated differencing” in more sophisticated software, would continue with only evolutionary or minor change.

Early on, this author had experimented with a “viscous transonic equation” of the form “ $\varepsilon\phi_{xxx} + ( ) \phi_{xx} + \phi_{yy} = 0$ .” where  $( )$  could be positive, negative or both. This work, described in Chapter 3, focused on transonic supercritical applications with embedded shockwaves. The idea was simple: this model, like Equations 1.1.1a,b,c, was high-order in the sense that the viscous shock structure of any evolving discontinuities could be modeled (here, the  $\varepsilon$  represents the longitudinal viscosity). We believed that, since the complete model was actually parabolic, the need for mixed subsonic and supersonic differencing was unnecessary. Moreover, “sweeping” need not proceed in the direction of the supersonic flow. A series of three papers published in the *AIAA Journal* documented our speculations and successes. At the time, we also speculated that since an “ $\varepsilon\phi_{xxx}$ ” term was included, then the model implicitly contained all of the requisite thermodynamic properties. In other words, the required conservation form and entropy conditions are self-contained in the viscous transonic model – thus, any nonlinear computed solution should be unique and completely determined. In fact, the role of high-order terms had been discussed in the classics *Supersonic Flow and Shock Waves* and *Linear and Nonlinear Waves* by Courant and Friedrichs (1948) and Whitham (1974). A short proof using the steady form of Burger’s equation  $u_t + uu_x = \varepsilon u_{xx}$  was given previously by the author and here in Chapter 3 – also, a uniqueness theorem for the unsteady equation was offered recently by Benea and Sadallah (2016).

So much for our very brief synopsis of the governing equations. Suffice it to say, to those who believe that computational fluids has ended with modern Navier-Stokes and Euler equation solvers, we believe that greater surprises await us. In this book, the author hopes to introduce new perspectives to interpret aerodynamics by explaining ideas rigorously but simply, by injecting healthy degrees of scientific speculation, and educating the reader in problems of importance to the industry. Invariably, every new approach involves numerical solution, and recognizing the unlikelihood that new students will have studied computational methods in depth, we offer a condensed presentation in Chapter 2 with readily understood but sparse Fortran. However, mathematics is truly essential, a focus that now guides our long journey.

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### 1.2 Fundamental “Analysis” or “Forward Modeling” Ideas

In this section, we will present fundamental ideas and develop basic analytical results that will be used to broaden our understanding of both analysis and inverse problems. Our discussion is comprehensive and self-contained, and takes a unique approach to numerical analysis that is intuitive and mathematically rigorous from a formulation perspective.

**Fundamental equations.** We start our presentation by requiring fluid “irrotationality,” this assumption thus precluding boundary layer, viscous flow and related non-ideal flow effects. If  $\mathbf{q}$  denotes the total velocity vector, this kinematic condition can be stated precisely in the form  $\nabla \times \mathbf{q} = 0$ . From vector analysis, we understand that it is possible to represent this velocity as the gradient of a “potential”  $\Phi$ , or here a “velocity potential,” that is, write  $\mathbf{q} = \nabla\Phi$ . If we further express steady, constant density, mass conservation in the form  $\nabla \cdot \mathbf{q} = 0$ , direct substitution shows that irrotational potential flows are governed by the classical Laplace equation  $\nabla^2 \Phi = 0$ . We will restrict our attention to planar flows in this section. For rectangular or “Cartesian”  $x$  and  $y$  coordinates, this can be rewritten as  $\Phi_{xx} + \Phi_{yy} = 0$ , while in cylindrical polar  $r$  and  $\theta$  coordinates, this takes the form  $\Phi_{rr} + 1/r \Phi_r + 1/r^2 \Phi_{\theta\theta} = 0$ . Either is transformable into the other, using the relationship “ $x = r \cos \theta$ ,  $y = r \sin \theta$ ,” or equivalently, “ $r = (x^2 + y^2)^{1/2}$ ,  $\theta = \tan^{-1} y/x$ .” Both representations will find important applications in our discussions.

**Small-disturbance results.** Although exact solutions for flows past a limited number of geometrically complicated bodies can be constructed from the theory of complex variables, often numerically, in practice, small-disturbance flows past thin airfoils approximately aligned with the rapid oncoming flow form the great majority of applications. For such problems, analytical and computational methods for the forward or analysis problem, in which pressure fields are sought when a geometry is specified, are well developed; in this section, basic developments are reviewed and studied in greater depth than is usual. This development serves multiple purposes. For one, we will later derive a direct or “forward like” inverse methodology that is discussed at much greater length in Chapter 4, drawing on our understanding of the analysis problem. Second, our constant density exposition sets the foundation for more advanced methods in mixed-type transonic flow simulation, treated in Chapter 2, and finally, the introductory work here leads us to the “viscous transonic” approaches developed in Chapter 3.

Importantly the relative simplicity behind constant density planar flow formulations allows us to explore in detail the properties of singularities related to source-like and vortex-like flows, which we will find very useful application in inverse techniques. We begin by considering small-disturbance flows in rectangular or Cartesian coordinates. Here it is customary to write the total velocity vector as  $\mathbf{q} = \Phi_x \mathbf{i} + \Phi_y \mathbf{j}$  where  $\Phi_x$  represents the horizontal speed “u” in the “x” direction having a unit vector  $\mathbf{i}$ , while  $\Phi_y$  denotes the vertical speed “v” in the “y” direction having a unit vector  $\mathbf{j}$ . Suppose a large horizontal speed exists, e.g., the wind blowing in a wind tunnel, or the relative speed experienced by an aircraft flying at cruise. Further, suppose that this speed greatly exceeds the disturbance velocities induced by the thin airfoil. We thus write  $\Phi = U_\infty x + \phi$  where  $\phi$  is the so-called “disturbance potential” to the constant speed  $U_\infty$  and require  $U_\infty \gg |\phi_x|$  and  $|\phi_y|$ . Substitution in  $\nabla^2 \Phi = 0$  shows that the disturbance potential likewise satisfies  $\nabla^2 \phi = 0$ . This equation is solved with auxiliary conditions, namely, flow tangency conditions at the airfoil surface, regularity conditions faraway at infinity and, as will be discussed in detail, a special Kutta condition *not* found in conventional expositions for Laplace solutions in heat transfer, electrostatics or petroleum reservoir flow.

We address airfoil surface kinematic conditions first. Now, the total horizontal speed is represented by  $\Phi_x = U + \phi_x$  while the vertical speed is  $\Phi_y = \phi_y$ . Because the airfoil surface is solid and impenetrable to flow, steadily moving fluid particles must flow tangent to it. That is, the ratio of the vertical to horizontal speed must equal the surface slope, writing,  $\phi_y / (U + \phi_x) = F'(x)$  where  $y = F(x)$  is the airfoil ordinate and prime denotes the horizontal derivative. We emphasize that this is evaluated at the surface, so that  $\phi_y(x, y(x)) / (U + \phi_x) = F'(x)$ . However, since  $U_\infty \gg |\phi_x|$  we consider a simpler expression along the horizontal axis itself, with  $\phi_y(x, \pm 0) / U_\infty \approx F'(x)$ . Far from the airfoil, we require that  $\nabla \phi \rightarrow 0$ .

In most non-aerospace applications, this boundary value problem formulation alone would suffice. If  $\phi$  had represented the steady-state temperature on a plate containing a portion of the slit (or thin hole)  $y = 0$ , the specification of the normal temperature gradient  $\phi_y$  together with regularity conditions would completely determine temperature to within a constant – if temperature were fixed at one additional location, the complete temperature field would be fully determined. But this is not so with inviscid aerodynamic analysis and we will see why shortly.

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**Thickness and camber formulations.** To develop the ideas suggested in above, it is convenient to understand that the airfoil ordinate  $y = F(x)$  actually consists of two functions,  $y = F^u(x)$  for the upper surface, and  $y = F^l(x)$  for the lower surface. A “camber line” function is introduced as the mean arithmetic position between upper and lower surfaces, that is,  $\frac{1}{2}(F^u + F^l) = F^c$ , while a “thickness function” is defined as half of the local airfoil thickness with  $\frac{1}{2}(F^u - F^l) = F^t$ . If we write  $F^u + F^l = 2F^c$  and  $F^u - F^l = 2F^t$ , addition and subtraction then lead to  $F^u = F^c + F^t$  and  $F^l = F^c - F^t$ . This suggests that we resolve the complete boundary value problem for  $y = F^{u,l}(x)$  into two simpler ones, namely,

$$\nabla^2 \phi^t = 0 \quad (1.2.1a)$$

$$\phi^t_{,y}(x, y = \pm 0) \approx \pm U_\infty dF^t(x)/dx \text{ along chord} \quad (1.2.1b)$$

$$\nabla \phi^t \rightarrow 0 \text{ as } x^2 + y^2 \rightarrow \infty \quad (1.2.1c)$$

and

$$\nabla^2 \phi^c = 0 \quad (1.2.2a)$$

$$\phi^c_{,y}(x, y = \pm 0) \approx U_\infty dF^c(x)/dx \text{ along chord} \quad (1.2.2b)$$

$$\nabla \phi^c \rightarrow 0 \text{ as } x^2 + y^2 \rightarrow \infty \quad (1.2.2c)$$

$$\textit{plus Kutta condition (to be discussed)} \quad (1.2.2d)$$

Notice that the normal derivative  $\phi_y$  reverses sign or “jumps” across the chord in Equation 1.2.1b, whereas in Equation 1.2.2b, it does not and is “continuous.” Once solutions to the foregoing problems are available, the total disturbance velocity potential is obtained by linear superposition, that is, calculated from  $\phi = \phi^t + \phi^c$  and substituted in  $\Phi = U + \phi$  to yield the complete potential. Differentiation yields velocities.

**Evaluation of pressure and lift.** Under the physical assumptions stated above, Bernoulli’s equation, which follows as a specific limit to the inviscid Euler equations, applies to the calculated flowfield. If we apply the foregoing small-disturbance assumptions, we have

$$\begin{aligned} P_\infty + \frac{1}{2} \rho U_\infty^2 &= P + \frac{1}{2} \rho |\mathbf{q}|^2 \\ &= P + \frac{1}{2} \rho (U_\infty^2 + 2U_\infty \phi_x + \phi_x^2 + \phi_y^2) \\ &\approx P + \frac{1}{2} \rho (U_\infty^2 + 2U_\infty \phi_x) \\ &= P + \frac{1}{2} \rho U_\infty^2 + \rho U_\infty \phi_x \end{aligned}$$

or

$$P \approx P_\infty - \rho U_\infty \phi_x \quad (1.2.3a)$$

that is, on combination with the definition of the pressure coefficient  $C_p$ , the well known formula

$$C_p = (P - P_\infty) / (\frac{1}{2} \rho U_\infty^2) \approx -2\phi_x / U_\infty \quad (1.2.3b)$$

Two observations will be important. First, consider  $P_\infty + \frac{1}{2} \rho U_\infty^2 = P + \frac{1}{2} \rho |\mathbf{q}|^2$  or  $P = P_\infty + \frac{1}{2} \rho U_\infty^2 - \frac{1}{2} \rho |\mathbf{q}|^2$ . At a stagnation point where  $|\mathbf{q}|^2 = 0$ , we have  $P = P_\infty + \frac{1}{2} \rho U_\infty^2$  so that  $C_p$  in Equation 1.2.3b physically takes on an absolute maximum of 1 (however, an improperly operated small disturbance algorithm may lead to results that exceed this). This provides an excellent check point for numerical analysis methods. Second, for the purposes of our inverse formulation later, it is important to note how, for *analysis* problems, the *normal* derivative  $\phi_y$  is first specified along the chord on  $y = 0$  while the *tangential* derivative  $\phi_x$  is later evaluated from  $\phi$  to calculate pressure from the solved potential.

**Point singularity representations.** We digress to discuss properties of “source” and “vortex” singularities which will prove useful to developing key ideas. Earlier we noted how a  $\Phi$  function satisfies Laplace’s equation, and gave both rectangular Cartesian and a cylindrical radial forms. The disturbance potential  $\phi$  likewise satisfies these relationships. In the latter polar coordinates, we have

$$\phi_{rr} + 1/r \phi_r + 1/r^2 \phi_{\theta\theta} = 0 \quad (1.2.4a)$$

$$\phi_{rr} + 1/r \phi_r = 0 \text{ if } \partial/\partial\theta = 0, \text{ leading to } \phi = A \log r + B \quad (1.2.4b)$$

$$\phi_{\theta\theta} = 0 \text{ if } \partial/\partial r = 0, \text{ leading to } \phi = C \theta + D \quad (1.2.4c)$$

Let us study two simplifications. In Equation 1.2.4b, we had set angular dependencies to zero, so that the solution for potential is the simple logarithmic function  $\phi = A \log r + B$ , where A and B are constants. It is important to observe that  $\phi$  is identical in *all* directions around the origin  $r = 0$ ; thus, it cannot be associated with lift, which has a preferred vertical direction. We emphasize that  $\phi$  is *single-valued* and does *not* depend on  $\theta$ . Because its velocity  $\mathbf{q} = \nabla\phi$  is identical in all directions and also varies like  $1/r \mathbf{e}_r$  (where  $\mathbf{e}_r$  is the unit vector in the radial direction), this  $\phi$  represents that due to a point source or sink.

On the other hand, if in Equation 1.2.4a we had set radial dependencies to zero, we would have the solution  $\phi = C \theta + D$ , where C and D are constants, and  $\theta = \tan^{-1} y/x$  is an arctangent function. The

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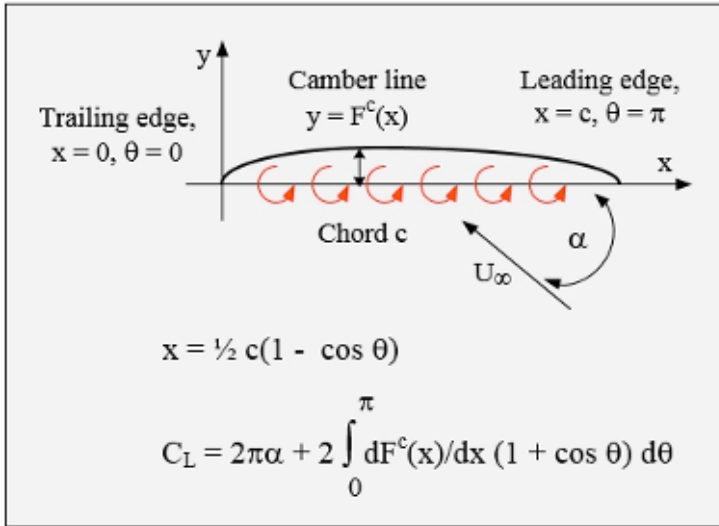
potential would then depend on angle; at any point,  $\theta$  can be represented by a given value, or that value, plus  $2\pi$ . Unlike the logarithmic potential, it is *double-valued* and *does* depend on  $\theta$ . What is this solution physically? Consider our solution  $\phi = C\theta + D$  with a positive value of  $C$ . Then the velocity  $\mathbf{q} = \nabla\phi = 1/r \partial\phi/\partial\theta \mathbf{e}_\theta$  (where  $\mathbf{e}_\theta$  is the unit vector in the angular direction) reduces to  $\mathbf{q} = C/r \mathbf{e}_\theta$  which, say, points to the right at the top and to the left at the bottom. Thus, at the top, the total velocity exceeds that of the freestream, while at the bottom, it is lower – this is just the description of vortex flow. High speeds at the top and lower ones at the bottom, via Bernoulli's equation, imply that pressure is lower at the top and higher at the bottom. In other words, vortices are associated with the singularities needed to model lift – again, they are multivalued potentials. Finally, note that the velocity  $\mathbf{q} = \nabla\phi$  varies like  $1/r$  and decays away from the airfoil. Vortices are associated with antisymmetric velocities and lifting effects, while sources model thickness, since they displace streamlines symmetrically, equally outwards at top and bottom.

Before studying thickness and camber flows in detail, we derive a formula useful in computational applications for lift calculation. The lift  $L$  acting on an airfoil having chord  $C$  and depth  $D$  into the page is given by  $L = \int (P^- - P^+) D dx$  where  $P^-$  and  $P^+$  are, respectively, pressures acting at the bottom and the top, and the integral is taken over the airfoil chord. If we now invoke Equation 1.2.3a, that is, the simplified Bernoulli equation  $P \approx P_\infty - \rho U_\infty \phi_x$ , we obtain  $L = \int (-\rho U_\infty \phi_x^- + \rho U_\infty \phi_x^+) D dx$  or the result  $L = \rho U_\infty D \int (\phi_x^+ - \phi_x^-) dx = \rho U_\infty D \int \partial[\phi]/\partial x dx$  where  $[\phi]$  is the “jump in potential” defined by  $[\phi] = \phi^+ - \phi^-$  due to vorticity effects. This result can be further simplified to give  $L = \rho U_\infty D \{[\phi]_{TE} - [\phi]_{LE}\}$  where “TE” and “LE” denote trailing and leading edge values. Later, we will explain why the leading edge term  $[\phi]_{LE}$  vanishes while  $[\phi]_{TE}$  does not. For now we can write  $L = (\frac{1}{2} \rho U_\infty^2) (2D [\phi]_{TE} / U_\infty)$ . From “ $\Phi = U_\infty x + \dots$ ” the units of potential are Length<sup>2</sup>/Time. Thus,  $2D [\phi]_{TE} / U_\infty$  has units of area, so  $L$  is consistent with “Force =  $\frac{1}{2} \rho U_\infty^2 \times \text{Area}$ .” The dimensionless lift coefficient is defined by  $C_L = L / (\frac{1}{2} \rho U_\infty^2 \times \text{Area})$  where  $\text{Area} = D \times C$ , so we have

$$L = \rho U_\infty D [\phi]_{TE} \quad (1.2.5a)$$

$$C_L = L / (\frac{1}{2} \rho U_\infty^2 DC) = 2 [\phi]_{TE} / (U_\infty C) \quad (1.2.5b)$$

Equations 1.2.5a and 1.2.5b were derived for use with the Laplace potential function solvers developed in Chapter 2 which are formulated in terms of jumps in potential  $[\phi]$ . We also indicate that the lift  $L$  is often expressed in the form  $L = \rho U_\infty \Gamma$  where  $\Gamma$  is known as the “circulation.” For completeness, the classical Glauert (1947) solution for lift coefficient is summarized in Figure 1.1.



**Figure 1.1.** Glauert camber solution for  $C_L$  in constant density flow.

The well known Glauert (1947) solution solves an integral equation formulation for the lifting problem in constant density flow using trigonometric series – it does not apply to transonic flows with shockwaves, although for nonzero subsonic Mach numbers, scaled solutions are available using the Prandtl-Glauert transformation (e.g., see Ashley and Landahl (1965)). Furthermore, the above solution applies to two-dimensional airfoils only; for three-dimensional problems, “lifting line” and “lifting surface” approaches apply. Detailed discussions are offered in the classic book *Aerodynamics of Wings and Bodies* due to Ashley and Landahl (1965). We cite the above solutions because they are useful for validating numerical solutions such as those developed in Chapter 2. We next discuss properties of singularity distributions because they are essential to developing our inverse methods, which take an approach uniquely different from existing methods.

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**Thickness formulation and properties.** We now consider the thickness problem in greater detail. For Equation 1.2.1b, we had indicated how velocities above and below the axis point in *opposite* directions. Thus, the boundary value problem in Equations 1.2.1a,b,c represents the thickness problem. On the other hand, Equation 1.2.2b shows velocities that are *identical in sign* above and below the chord, so that Equations 1.2.2a,b,c,d solve the camber problem.

We consider the thickness problem first using methods from singular integral equations. A closed form analytical solution can be obtained. Now the “log r” source solution derived previously, centered at the origin  $r = \sqrt{(x^2 + y^2)} = 0$ , solves Laplace’s equation. It follows that  $\log \sqrt{\{(x-\xi)^2 + y^2\}}$  centered at  $x = \xi$ ,  $y = 0$  also satisfies Laplace’s equation, where  $\xi$  represents only a shift in the choice of origin.

Now,  $\xi$  can be viewed as a general point source position over which the effects of numerous sources can be summed. But rather than examining multiple discrete point sources, we examine *continuous* line source distributions placed along a slit on  $y = 0$  to represent the thickness distribution. This is clearly the situation physically. We therefore consider the superposition

$$\phi(x,y) = \int f(\xi) \log \sqrt{\{(x-\xi)^2 + y^2\}} d\xi + H$$

This integral also satisfies Laplace’s equation for the potential, since the governing equation is linear. Integration limits extending over the airfoil chord are understood and excluded for clarity. This represents the solution for a continuously distributed line source along  $y = 0$  and along the chord, assumed consistently with small-disturbance theory, where  $H$  is an integration constant that we need not consider here (for example, in petroleum reservoir fracture flow in a finite circular field, Chin (2017) explicitly evaluates  $H$  when the farfield pore pressure is given at a finite distance). Here,  $H$  is zero and velocities vanish at infinity.

We are next interested in developing properties of the above integral and the “source strength”  $f(x)$ . Let us return to the expression for potential and differentiate it with respect to the vertical coordinate  $y$  normal to the chord.

$$\begin{aligned} \partial\phi(x,y)/\partial y &= \partial/\partial y \left\{ \int f(\xi) \log \sqrt{\{(x-\xi)^2 + y^2\}} d\xi + H \right\} \\ &= y \int f(\xi) / \{(x-\xi)^2 + y^2\} d\xi \end{aligned}$$

Following the limit process in Yih (1969), introduce the change of coordinates  $\eta = (\xi - x)/y$  so that

$$\frac{\partial\phi(x,y)}{\partial y} = \int_{\eta^-}^{\eta^+} f(\xi)/(1 + \eta^2) d\eta$$

Now for small positive  $y$ 's, we find that on using  $x = \xi - \eta y$ , that the vertical derivative satisfies

$$\frac{\partial\phi(x,0+)}{\partial y} = \int_{-\infty}^{+\infty} f(\xi)/(1 + \eta^2) d\eta = \pi f(x)$$

Similarly, for small negative  $y$ 's, we obtain  $\partial\phi(x,0-)/\partial y = -\pi f(x)$ . Hence,  $\partial\phi(x,0+)/\partial y - \partial\phi(x,0-)/\partial y = \pi f(x)$ . Our results also imply  $\partial\phi(x,0+)/\partial y = -\partial\phi(x,0-)/\partial y$ , that is, the vertical velocities on either side of the slit are antisymmetric, in agreement with Equation 1.2.1b. We emphasize that, from  $\phi(x,y) = \int f(\xi) \log \sqrt{\{(x-\xi)^2 + y^2\}} d\xi + H$ , the potential is an even function of  $y$ , that is,  $\phi(x,y)$  is symmetric with respect to  $y = 0$ . Also, as anticipated from the properties of the logarithm, the potential is a continuous function in space that does not jump. These provide two key check points for numerical calculations that are frequently used later. The superposition integral itself provides still another check point for evaluating computed behavior throughout  $x$  and  $y$  space. We had proved that  $\partial\phi(x,0+)/\partial y = \pi f(x)$ . From  $\phi'_y(x, y = \pm 0) \approx \pm U_\infty dF(x) / dx$  along chord, representing the tangency condition, we find that  $\pi f(x) = U_\infty dF(x)/dx$ . While  $f(x)$  can be related to thickness function slope, a blunted edge or infinite slope would invalidate thin airfoil models.

**Camber line properties.** Here we derive some properties associated with flows past camber only geometries. Previously, we showed why  $\phi = C \theta + D$  or  $\phi = \tan^{-1} y/x$  is a solution to Laplace's equation. Following the approach used previously, we might consider a point vortex at  $(x = \xi, y = 0)$  taken in the form  $\phi = \tan^{-1} y/(x - \xi)$ , or more generally as a *continuous* distribution of vortexes satisfying

$$\phi(x,y) = \int_{-1}^{+1} g(\xi) \tan^{-1} y/(x-\xi) d\xi + G$$

where the arbitrary constant is set to zero in order to satisfy regularity conditions at infinity. This solution also satisfies Laplace's equation by

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virtue of linear superposition and  $g(\xi)$  is an unknown function. Differentiation with respect to  $y$ , using standard formulas, yields

$$\frac{\partial \phi(x,y)}{\partial y} = \int_{-1}^{+1} g(\xi) \frac{\{(x-\xi)/\{(x-\xi)^2 + y^2\}\}}{d\xi}$$

If we evaluate this at  $y = 0$  and use  $\phi_y(x, y = \pm 0) \approx U_\infty dF^c(x)/dx$  from Equation 1.2.2b, we have

$$PV \int_{-1}^{+1} g(\xi) / (x-\xi) d\xi \approx U_\infty dF^c(x)/dx = -\alpha$$

This singular integral equation, with the Cauchy kernel  $1/(x-\xi)$ , governs the vortex strength  $g(\xi)$ . The PV indicates that the integral is improper and to be evaluated using a “principal value” limit defined in calculus.

Fortunately, we do not need to understand integral equation methods to solve the problem. Indeed, the general solution to the equation  $PV \int g(\xi)/(x-\xi)d\xi = -h(x)$  is

$$g(x) = - (1/\pi^2) \sqrt{\{(1-x)/(1+x)\}} PV \int \{h(\xi) \sqrt{(1+\xi)}/\{(\xi-x) \sqrt{(1-\xi)}\}} d\xi + \gamma / \sqrt{(1-x^2)}$$

where we have omitted the integration limits for clarity. This solution is derived and discussed in classical references (Mikhlin, 1964; Muskhelishvili, 2008; Carrier, Krook, and Pearson, 1966). Note that the  $\gamma / \sqrt{(1-x^2)}$  term represents the nonuniqueness associated with solutions to our singular integral equation, with the arbitrary constant  $\gamma$  related to the so-called “circulation” of a flow. Its specific value is determined by Kutta’s condition requiring smooth flow from the trailing edge.

What is the physical significance of vortex strength? If we differentiate our superposition integral with respect to  $x$ , it follows that

$$\frac{\partial \phi}{\partial x} = - \int_{-1}^{+1} g(\xi) \frac{y/\{(x-\xi)^2 + y^2\}}{d\xi}$$

This integral was studied earlier. In the limit  $y = 0$ , from earlier results,  $\partial \phi(x,0+)/\partial x = -\pi g(x)$  and  $\partial \phi(x,0-)/\partial x = +\pi g(x)$ . Since the velocity parallel to the camber line is proportional to  $\partial \phi / \partial x$ , the camber line is responsible for a discontinuity in the tangential velocity that is proportional to  $g(x)$ . The above show a net jump in the tangential derivative (i.e., velocity slip) of  $\partial \phi(x,0+)/\partial x - \partial \phi(x,0-)/\partial x = -2\pi g(x)$ .

### 1.3 Basic “Inverse” or “Indirect Modeling” Ideas

In Section 1.2 we formulated and studied the “analysis,” “forward” or “direct” problem, one in which the potential field, pressure distribution, pressure coefficient and total lift were sought when an airfoil geometry was prescribed. Solutions were direct or straightforward in that the formulations of Equations 1.2.1a,b,c or Equations 1.2.2a,b,c,d could be solved in a single pass (using a relaxation solver) without further work – this is now standard given the proliferation of Laplace equation solvers and the like. Again, each of these formulations require an iterative solution, but at least, this tedious process is pursued only once (or at worst twice for general non-symmetric geometries requiring both thickness and camber solutions). In summary, the prior analysis methods provide the surface pressure coefficient  $C_p = -2\phi_x / U_\infty$  once the airfoil ordinates  $y = F^{u,l}(x)$  are given. Surface pressures are useful for calculating lift and moment, in determining viscous drag using boundary layer methods, or in assessing the likelihood of flow separation and stall.

What if, however, we wanted the reverse: prescribe  $C_p(x)$  along a finite slit  $y = 0$  and calculate the shape  $y = F^{u,l}(x)$  that induces the given pressure? This is the so-called “inverse” or “indirect” problem – indirect because it is not obvious how one should proceed. Many procedures based on pure guess work have been published. For example, in the numerical approach of Carlson (1975), an approximate nose shape furnishes starting slope conditions for calculations carried out in an analysis mode and, over the remainder of the chord, tangential derivatives of  $\phi$  are used in design mode calculations. Intermediate results for geometry are monitored at different stages of the relaxation and, if the required degree of trailing edge closure is not fulfilled, the starting nose shape is modified until such is assured. There is nothing special about starting modifications at the nose – any other point might well be justified, and in fact, all other points are likely candidates.

One thus observes that, while potential function formulations may be useful in engineering practice, they do require considerable experience, expertise and intuition on the part of the designer. They invariably require a human decision on the choice of a free parameter indirectly related to trailing closure so that airfoils are not opened unrealistically: the “man-in-loop” requirement arises from the fact that monotonic changes to arbitrarily defined parameters generally do not correlate with monotonic changes to the degree of closure. But is there a better or more rational approach to solving inverse problems?

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To be sure, is it possible to develop a direct method to solve indirect or inverse problems in a single pass as we had solved analysis problems? The answer is, “*Yes.*” This subject is developed in Chapter 4 for airfoils, inlets, three-dimensional wings, and so on, for irrotational and rotational flows. In this section, we motivate the method with a simple example – a elementary but powerful application for which we also derive a complementary finite difference solver in Chapter 2. To do so, we begin with our disturbance potential equation  $\phi_{xx} + \phi_{yy} = 0$ , rewritten as

$$\partial(\phi_x)/\partial x + \partial(\phi_y)/\partial y = 0 \quad (1.3.1)$$

This so-called “conservation form” suggests that we might introduce a function  $\psi(x,y)$  such that

$$\phi_x = \psi_y \quad (1.3.2a)$$

and

$$\phi_y = -\psi_x \quad (1.3.2b)$$

which is nothing more than “ $0 = 0$ ” on substitution into Equation 1.3.1. Students of complex variables will recognize these as Cauchy-Riemann conditions, which we just derived using elementary methods. What are the mathematical properties of  $\psi$ ? Partial differentiation of the first equation with respect to  $y$  and the second with respect to  $x$ , and elimination of  $\phi_{xy}$  between the two, shows that  $\psi(x,y)$  satisfies

$$\psi_{xx} + \psi_{yy} = 0 \quad (1.3.3)$$

Observe that if we wished to specify the pressure coefficient along a finite slip  $y = 0$ , we would have, using Equations 1.2.3b and 1.3.2a, the transformation  $C_p = (P - P_\infty) / (\frac{1}{2} \rho U_\infty^2) \approx -2\phi_x / U_\infty = -2\psi_y / U_\infty$  or

$$\psi_y(x, y = 0) = -\frac{1}{2} U_\infty C_p(x) \quad (1.3.4)$$

Further, if our airfoil is finite in extent, any disturbances to the freestream that it induces must vanish faraway, with  $\nabla\psi \rightarrow 0$  as  $x^2 + y^2 \rightarrow \infty$ . Let us now collect our key results –

$$\psi_{xx} + \psi_{yy} = 0 \quad (1.3.5a)$$

$$\psi_y(x, y = 0) = -\frac{1}{2} U_\infty C_p(x) \quad (1.3.5b)$$

$$\nabla\psi \rightarrow 0 \text{ as } x^2 + y^2 \rightarrow \infty \quad (1.3.5c)$$

$$\textit{plus trailing edge } [\psi] \text{ or } [\psi_x] \text{ jump condition (to be discussed)} \quad (1.3.5d)$$

Once the solution to Equations 1.3.5a,b,c,d is available, the required airfoil geometry is easily calculated. Recall that the surface kinematic or tangency boundary condition derives from  $dy/dx \approx \phi_y/U_\infty$  or, on applying Equation 1.3.2b,  $dy/dx \approx -\psi_x/U_\infty$  which directly integrates to

$$y(x) = -\psi(x,0)/U_\infty + \text{constant} \tag{1.3.6}$$

upon evaluation along the slit  $y = 0$ . We can easily check our results for dimensional consistency and correctness. From  $\Phi = U_\infty x + \phi$ , the velocity potentials have units of  $(L/T)L$  or  $L^2/T$  (that is, length<sup>2</sup>/time). From Equations 1.3.2a,b, it is clear that  $\psi$  also has units of  $L^2/T$ . Now, the right side of Equation 1.3.6 therefore varies like  $(L^2/T)(T/L)$  which correctly has dimensions of length.

Some comments on mass flux or mass or volume flow rate are also worthwhile. Consider a section of space that is  $D$  deep into the page, and varying from a vertical location  $y$  to  $y^*$ . The horizontal mass flux  $M$  through any line connecting  $y$  and  $y^*$  is simply  $M = \int \phi_x D dy$  where the integral is taken over the interval  $y$  to  $y^*$ . Since Equation 1.3.2a requires that  $\phi_x = \psi_y$ , we can write this integral as  $M = \int \psi_y D dy$  or

$$M = D \{ \psi(x,y^*) - \psi(x,y) \} \tag{1.3.7}$$

In other words, the difference in the value of  $\psi$  between any two points in space is proportional to the mass flux between those points.

**New direct inverse formulation.** The foregoing equations completely define the boundary value problem and post-processing needed to solve the inverse problem in a *direct* manner! Again, Equation 1.3.5a is our familiar Laplace equation, Equation 1.3.5b specifies a convenient normal “y” derivative related to prescribed pressure, Equation 1.3.5c states the expected regularity condition, while a further prescribed jump in  $[\psi]$  or  $[\psi_x]$  furnishes a Kutta-type or circulatory statement required to render a solution to Laplace’s equation unique! The jump  $[\psi]$ , following Equation 1.3.7, is related to mass flow ejected from the trailing edge, that is, the degree to which it is opened. A jump in  $[\psi_x]$  (or the slope  $[\phi_y]$ ) would measure trailing included angle. In fact, the foregoing inverse formulation is *identical* to the forward analysis problems defined by Equations 1.2.1a,b,c and Equations 1.2.2a,b,c,d. Thus, computer algorithms developed to solve for velocity potentials can be used to solve inverse problems with minor change and only simple reinterpretation of the dependent variable! Approaches such as those in Carlson (1975) are not necessary and inverse problems can be solved in a

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single step. The function  $\psi$  is the so-called “streamfunction” well known to fluid-dynamicists, but until this author’s first publication of the above algorithm, the role of streamfunctions in inverse formulations had never been recognized in the context of inverse formulations.

In Chapter 2, we will introduce the finite difference method and its application to solving elliptic partial differential equations, first concentrating on the historically important analysis problem for the velocity potential. Then, we will show how simple modifications to one algorithm can solve a well known inverse problem quickly, efficiently and simply. But our discussion on inverse methods does not stop there. In fact, a major objective of this book focuses on more general inverse problems of greater difficulty, whose solutions have actually been stymied by impediments more semantic than mathematical.

Students of mathematics will recognize that potentials and streamfunctions, such as are introduced above, can be described much more elegantly than in the derivation underlying Equations 1.3.1 – 1.3.3. Usually, a background in the theory of complex variables is assumed, and authors indicate how real and imaginary parts of “analytical functions of  $z$ ,” where  $z = x + i y$  is complex, are “harmonic,” that is, they satisfy Laplace’s equation as  $\phi$  and  $\psi$  do. Equations 1.3.2a and 1.3.2b are the classical Cauchy-Riemann conditions, named after mathematicians Cauchy and Riemann who discovered the relationships in the nineteenth century, and  $\phi$  and  $\psi$  are said to be “harmonic conjugates.” An important application for  $\psi$  is streamline plotting, e.g., it can be shown that lines of constant  $\psi$  define fluid paths in steady flow – this property provides the main practical stimulus for solving Equation 1.3.5a. Other limited applications are also available, e.g., see the classical work of Milne-Thomson (1968) provides numerous derivations for exact solutions, but we will refrain from further discussion here.

When this author first discovered the practical utility in Equations 1.3.5a,b,c,d he immediately sought to extend the powerful new inverse methodology to other complicated problems in aerodynamics. However, several unfortunate “semantic traps” suggested that this was not possible. After all, the prevailing wisdom behind Cauchy-Riemann conditions and harmonic functions required governing equations of the precise form given by  $\phi_{xx} + \phi_{yy} = 0$ , a narrow subset of fluid-dynamics indeed. However, using the “engineer’s derivation” given in the development starting with conservation forms similar to Equation 1.3.1 proved to be very constructive. Soon, the axisymmetric equation  $\phi_{rr} + \phi_{\theta\theta} + 1/r \phi_r = 0$ ,

important to modeling flows through engine inlets and nacelles, with and without power addition, was transformed into streamfunction format for inverse analysis. Similarly, the small-disturbance transonic flow equation  $\{1 - M_\infty^2 - (\gamma + 1) M_\infty^2 \phi_x\} \phi_{xx} + \phi_{yy} = 0$ , by rewriting it in conservation form, led to an integro-differential equation for  $\psi$  that could be solved by existing mixed-type partial differential equation solvers with only a single coefficient modification.

The three-dimensional Laplace potential equation  $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$  used in wing theory was tackled next. Streamfunctions are not widely used in three-dimensional aerodynamics because, it turns out, the streamfunction is a vector potential requiring three components. Unlike two dimensional theory, where formulations employing two velocities can be replaced by simpler ones utilizing a single potential or streamfunction, no such advantage is found three-dimensionally for the streamfunction. For such applications, the author developed a rigorously defined “streamlike function”  $\psi$  satisfying a single scalar Laplace equation  $\psi_{xx} + \psi_{yy} + \psi_{zz} = 0$ . This powerful transformation suggested that existing three-dimensional analysis methods using the velocity potential, e.g., conventional lifting line and lifting surface approaches, could be used to solve three-dimension inverse problems for finite wings. Finally, several of these irrotational models were extended to rotational flows driven by the presence of strong oncoming shear. The approach started with streamfunction formulations which *do* apply to shear flow, recasting the governing equations to conservation form, and then introducing a “superpotential” which is governed by a simple extension of  $\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$  requiring only a redefinition of certain coefficients. The key contribution here allows us to extend existing potential flow codes to solve analysis problems in the presence of shear with only minor modification. The theoretical approach is shown to be completely consistent with the inviscid Euler equations. These methods are all described in Chapter 4.

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### 1.4 Literature Overview and Modeling Issues

The references listed below are recommended reading and directly support multiple chapters and sections developed in our book. With possible exceptions related to the integral equation literature, all are readily understood to audiences with some exposure to advanced math. Many of the cited works, not listed in any particular order, provide the background needed to extend and refine the ideas introduced later.

**Basic fluid mechanics.** As our work focuses on aerodynamics, basic fluid mechanics books with an aerospace yet fundamental emphasis are recommended reading. Some of these are, for instance,

- Ashley, H. and Landahl, M., *Aerodynamics of Wings and Bodies*, Addison-Wesley, Reading, Massachusetts, 1965. Well written text covering aerodynamic theory from subsonic to supersonic flow in multiple dimensions.
- Batchelor, G.K., *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, 1967. A well known author explains issues and subtleties in fluid flow with an emphasis on physical phenomena.
- Katz, J. and Plotkin, A., *Low Speed Aerodynamics, Second Edition*, Cambridge University Press, Cambridge, 2010. An excellent aerodynamics book covering a wide range of topics rigorously – a thorough discussion of modern panel methods for low speed flow.
- Milne-Thomson, L.M., *Theoretical Hydrodynamics*, Macmillan Company, New York, 1968. Although the presentation style is somewhat out-dated, this classic presents and develops numerous key results, ideas and exact solutions in hydrodynamics and aerodynamics very rigorously.
- Streeter, V.L., *Handbook of Fluid Dynamics*, McGraw-Hill, New York, 1961. A very well written book (not your usual “handbook”) covering a wide range of fluid mechanics problems and solutions.
- Tsien, H.S., *Collected Works of H. S. Tsien (1938-1956)*, Academic Press, New York, 2012. A “must read” classic.
- Yih, C.S., *Fluid Mechanics*, McGraw-Hill, New York, 1969. The author is a well known author who discusses a wide range of topics clearly and mathematically, summarizing major issues and results in an easily understood manner. Great place to learn applied math.

**Mathematics.** Many of our models focus on finite difference analyses, which require a solid background in numerical methods. The classic volume due to Carnahan, Luther and Wilkes (1969) provides a well written introduction (with excellent examples and Fortran code) that guides beginners to productive results quickly. The integral equation references below are useful to those developing source, vortex and other singularity superposition methods and provide detailed solutions for many common formulations. They are suitable for advanced students.

- Carnahan, B., Luther, H.A. and Wilkes, J.O., *Applied Numerical Methods*, John Wiley & Sons, New York, 1969.
- Carrier, G.F., Krook, M. and Pearson, C.E., *Functions of a Complex Variable: Theory and Technique*, McGraw-Hill, New York, 1966.
- Mikhlin, S.G., *Integral Equations*, International Series of Monographs on Pure and Applied Mathematics, Vol. 4, Pergamon Press., Oxford, 1964.
- Muskhelishvili, N.I., *Singular Integral Equations: Boundary Problems of Function Theory and Their Application to Mathematical Physics*, translation by J.R.M. Radok, Dover Publications, New York, 2008.
- Tricomi, F.G., *Integral Equations*, Interscience Publishers, London, 1957. An exact solution to “the airfoil equation” is presented.

**Jameson research.** The works of Jameson and his colleagues are well known to the aerospace community and are “must” reading. The references below represent a cross-section of key contributions and results. These identify main simulation issues, approaches and solutions, plus pitfalls likely to be encountered in transonic flow simulation.

- Jameson, A., “Iterative Solution of Transonic Flows over Airfoils and Wings, Including Flows at Mach 1,” *Communications on Pure and Applied Mathematics*, Vol. XXVII, 1974, pp. 283-309. Detailed discussion on transonic analysis methodology.
- Jameson, A., Schmidt, W. and Turkel, E., “Numerical Solution of the Euler Equations by Finite Volume Methods Using Runge-Kutta Time-Stepping Schemes,” *14th Fluid and Plasma Dynamics Conference*, Palo Alto, CA, 23-25 June 1981. Discussions on Euler equation method, arbitrary domains, convergence to steady state, structure of dissipation terms needed for stability.

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- Salas, M.D., Jameson, A. and Melnik, R.E., “A Comparative Study of the Nonuniqueness Problem of the Potential Equation,” AIAA Paper 83-1888, *6th Computational Fluid Dynamics Conference*, Danvers, Mass., July 13-15, 1983. Prior investigators had shown how multiple flowfield solutions are possible for given geometric and freestream input parameters, leading obviously to several disturbing questions. Are these solutions computational anomalies? If they exist, are they possibly unstable physically? Could they be related to buffeting? We will return to nonuniqueness ideas shortly.
- Jameson, A., “Essential Elements of Computational Algorithms for Aerodynamic Analysis and Design,” NASA/CR-97-206268, ICASE Report 97-68, Langley Research Center, National Aeronautics and Space Administration, Dec. 1997. From the Abstract – “This paper traces the development of computational fluid dynamics as a tool for aircraft design. It addresses the requirements for effective industrial use, and trade-offs between modeling accuracy and computational costs. Essential elements of algorithm design are discussed in detail, together with a unified approach to the design of shock capturing schemes. Finally, the paper discusses the use of techniques drawn from control theory to determine optimal aerodynamic shapes. In the future multidisciplinary analysis and optimization should be combined to provide an integrated design environment.”
- Jameson, A., “Transonic Flow Calculations,” MAE Report 1651, Princeton University, Dec. 2003. Mathematics and numerical issues.
- Jameson, A., “Chapter 11, Aerodynamics,” *Encyclopedia of Computational Mechanics, Volume 3: Computational Fluid Dynamics*, edited by Erwin Stein, Rene de Borst and Thomas J.R. Hughes, John Wiley & Sons, 2004. Write-up provides a good overall survey. Explains how boundary layer to first order increases effective thickness of the body for inviscid analysis. (Note, our own inverse modeling, where shape is sought which induces a prescribed surface pressure, incorporates displacement thickness modeling by allowing thick or opened trailing edges.) Chapter 11 emphasizes ‘simple’ checks, e.g., “Does the numerical solution of a symmetric profile at zero angle of attack preserve the symmetry, with no lift?” Paper gives brief survey of mathematical fluid flow models, conservative versus non-conservative transonic schemes, transonic type-differencing, over-relaxation and convergence acceleration.

- Jameson, A., “Inverse Problems in Aerodynamics and Control Theory,” *International Conference on Control, PDEs and Scientific Computing*, Beijing, China, September 10-13, 2004. Jameson notes, “Because a shape does not necessarily exist for an arbitrary pressure distribution the inverse problem may be ill posed if one tried directly to enforce a specified pressure as a boundary condition” and goes on to develop his method. Our small-disturbance method *always* produces geometric solutions, which may include surface cross-over; uniqueness is obtained by specifying trailing edge constraints.
- Jameson, A. and Ou, K., “Fifty Years of Transonic Aircraft Design,” *Progress in Aerospace Sciences*, Elsevier, 2011. A historical overview of aircraft design, most references from 1960s and 1970s. Description of full nonlinear potential equation solver with rotated differencing scheme extends Murman-Cole method. Described Euler equation solver, unstructured meshes. In one section on optimum shape design, several methods are reviewed. Formulations of (inverse) design methods for aerodynamic problems dates to a conformal mapping of Lighthill. For transonic flow, earliest design methods were based on the hodograph method. The complex characteristics method of Garabedian and Korn is cited, along with optimization procedures of Hicks and Henne to design transonic airfoils and wings. Pironneau used optimal control techniques for the design of shapes governed by elliptic equations. Subsequently, Jameson developed the use control theory, based on the solution of adjoint problems, and applied it (together with Reuther) to the design of aerodynamic shapes in transonic and supersonic flow governed by (nonlinear) potential flow and the Euler equations, and (with Martinelli and Pierce) to Navier–Stokes equation formulations.

**Solution non-uniqueness.** Nonunique solutions are commonplace in fluid mechanics, a classic example in the aerodynamic context illustrated by Kutta’s trailing edge constraint. However, until recently, they were unexpected in calculations of flowfields past aircraft bodies. A number of researches in this area are available, and while we do not pursue this topic in our own work, we provide for completeness an early reference (dating more than three decades ago) and one completed more recently in 2017.

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- Salas, M.D., Jameson, A. and Melnik, R.E., “A Comparative Study of the Nonuniqueness Problem of the Potential Equation,” AIAA Paper 83-1888, *6th Computational Fluid Dynamics Conference*, Danvers, Mass., July 13-15, 1983. This paper was briefly cited above under “Jameson research” topics. Here, the authors indicate, “additional evidence has been provided which supports the thesis of Ref. 1, that the nonuniqueness is a problem inherent to the conservative potential differential equation.” Further, “none of the multiple solutions obtained with the conservative potential formulation seems to be relevant to the physical problem. Rather, they seem to indicate a breakdown of the theory. It appears that to avoid the anomaly the conservative formulation must be abandoned.”  
*Our note:* This is unfortunate after years of industry effort focusing on conservative schemes. It may be that the present author’s use of the high-order “viscous transonic equation” in Chapter 3 may provide a more viable path to unique solutions since it is physically grounded. The paper just cited appeared in 1983; the next, due to Liu, Luo and Liu (2017), provides updates to a controversial subject.
- Liu, Y., Luo, S. and Liu, F., “Multiple Solutions and Stability of the Steady Transonic Small-Disturbance Equation,” *Theoretical and Applied Mechanics Letters*, Vol. 7, 292-300, 2017. From the Abstract: “Numerical solutions of the steady transonic small-disturbance (TSD) potential equation are computed using the conservative Murman–Cole scheme. Multiple solutions are discovered and mapped out for the Mach number range at zero angle of attack and the angle of attack range at Mach number 0.85 for the NACA 0012 airfoil. We present a linear stability analysis method by directly assembling and evaluating the Jacobian matrix of the nonlinear finite-difference equation of the TSD equation. The stability of all the discovered multiple solutions are then determined by the proposed eigen analysis. The relation of stability to convergence of the iterative method for solving the TSD equation is discussed. Computations and the stability analysis demonstrate the possibility of eliminating the multiple solutions and stabilizing the remaining unique solution by adding a sufficiently long splitter plate downstream the airfoil trailing edge. Finally, instability of the solution of the TSD equation is shown to be closely connected to the onset of transonic buffet by comparing with experimental data. From the first paragraph of paper: “The non-uniqueness of numerical

solutions of potential equations at transonic speeds has been found for three decades. Steinhoff and Jameson first reported multiple solutions for the full potential (FP) equation. Chen first reported the existence of multiple solutions of the steady transonic small-disturbance (TSD) equation using the nonconservative Murman–Cole scheme. Nixon also found multiple solutions using the TSD equation modified with vorticity and entropy corrections. Salas et al. did extensive study on multiple solutions of the FP equation. Jameson demonstrated that non-unique solutions of Euler equations can be obtained for certain airfoils. Hafez and Guo investigated the flow over airfoils with flat and wavy surface by solving the steady potential equations, the Euler equations, and the Navier–Stokes equations. They found that all of the equations can generate multiple solutions at zero angle of attack in certain Mach number ranges. Luo et al. showed that the multiple solutions of the transonic small transverse disturbance equation are independent of the difference schemes and iterative methods and found multiple solutions for a three dimensional wing.”

**Kutta condition.** While Kutta’s trailing edge condition has been long accepted in aerodynamics, recent work, particularly the research of Rienstra (1992) points to mathematical issues that are unresolved and previously unidentified. The commonly accepted model is summarized in our “Basic fluid mechanics” listing above, e.g., Glauert’s classic solution as explained in the books by Ashley and Landahl, Batchelor, Milne-Thomson and Yih. We have listed a recent papers Kutta’s model below.

We emphasize that solutions to potential flows are not unique and “Kutta type” constraints are required to define circulation levels enabling flows to leave the trailing edge smoothly. In general, the “right side” streamline can leave the airfoil anywhere. But in practice, a finite angle trailing edge requires a stagnation point, while for a cusped edge, upper and lower velocities must be aligned. Many published papers, however, do not provide implementation details, and it is often unclear what is being modeled or even if the baseline computations satisfy “ $C_L = 2\pi\alpha$ .”

- Rienstra, S.W., “A Note on the Kutta Condition in Glauert’s Solution of the Thin Airfoil Problem, IWDE Report; Vol. 91-04. Eindhoven: Technische Universiteit Eindhoven, Oct. 1991. Also, *Journal of Engineering Mathematics*, Vol. 26, pp. 61-69, 1992, Kluwer Academic Publishers, The Netherlands. Subtle aspects related to

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uniqueness of Kutta's condition in two and three-dimensional airfoils and wings and related numerical methods are discussed.

- Iannelli, G., Grillo, C. and Tulumello, L., "A Kutta Condition Enforcing BEM Technology for Airfoil Aerodynamics," *Transactions on Modelling and Simulation*, Vol. 15, WIT Press, 1997. Deals with correct procedures for Kutta condition enforcement and solution non-uniqueness.
- Mohebbi, F. and Sellier, M., "On the Kutta Condition in Potential Flow Over Airfoil," Volume 2014, Article ID 676912, *Journal of Aerodynamics*, Hindawi Publishing, April 2014. More on implementation, finite difference method and curvilinear coordinates.
- Radwan, S.F., "Critical Review of the Trailing Edge Condition in Steady and Unsteady Flow," Master of Science Thesis, Lehigh University, 1981. Good discussion on real viscous fluid effects.
- Golberg, M.A., "The Numerical Solution of Cauchy Integral Equations With Constant Coefficients," Vol. 9, 1985, pp. 127-151, *Journal of Integral Equations*, Elsevier Science Publishing.
- Tricomi, F.G., *Integral Equations*, Interscience Publishers, London, 1957. A classic reference on the "airfoil equation."

**Subsonic aerodynamics.** The references below cover a range of interests. For example, Drela and Youngren (2001) describe the very successful XFOIL airfoil analysis and design system developed at M.I.T., while Katz and Plotkin (2010) have authored perhaps the best low speed reference and panel method descriptions currently available. Of special importance are the classic works of van Dyke, who developed the "method of matched asymptotic expansions," at first used to develop "inner solutions" near the leading edges of airfoils where thin airfoil, small disturbance theory breaks down. The method has proven indispensable to many areas of engineering, and will, ultimately, prove to be useful in our own small disturbance streamfunction approach to aerodynamic inverse problems. van Dyke's leading edge corrections are, perhaps, less prominent nowadays, given the popularity of curvilinear grids (where local breakdown does not occur), but the work is useful in formulating local models that are convenient and rapid to operate. A readable description together with algorithms appears in this author's book *Managed Pressure Drilling* (see "About the Author").

- Drela, M. and Youngren, H., *XFOIL 6.94 User Guide*, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Mass., 2001.
- DeJongh, J., “Computation of Incompressible Potential Flow Over and Airfoil Using a High-Order Aerodynamic Panel Method Based on Circular Arc Panels,” Doctoral Thesis, Air Force Institute of Technology, Aug. 1982.
- Katz, J. and Plotkin, A., *Low Speed Aerodynamics, Second Edition*, Cambridge University Press, Cambridge, 2010. An excellent aerodynamics book covering a wide range of topics rigorously – a thorough discussion of modern panel methods for low speed flow.
- van Dyke, M.D., *Second-Order Subsonic Airfoil Section Theory and its Practical Application*, NACA TN 3390, National Advisory Committee for Aeronautics, March 1955.
- van Dyke, M.D., *Second Order Subsonic Airfoil Theory Including Edge Effects*, Report 1274, National Advisory Committee for Aeronautics, 1956.
- Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964.

**Transonic analysis.** This section cites the original Murman-Cole publication, together with papers describing other algorithms and developments of interest in transonic flow modeling.

- Murman, E.M. and Cole, J.D., “Calculation of Plane Steady Transonic Flows,” *AIAA Journal*, Vol. 9, No. 1, 1971, pp. 114-121. Paper describes the original type-dependent differencing method used to model mixed subsonic-supersonic flows.
- Mostrel, M.M., “On Some Numerical Schemes for Transonic Flow Problems,” *Mathematics of Computation*, Vol. 52, No. 186, April 1989, 587-613. New steady and unsteady shock capturing schemes are explained.
- Langley, M.J., “Numerical Methods for Two-Dimensional and Axisymmetric Transonic Flows,” C.P. Number 1376, Aeronautical Research Council, Her Majesty’s Stationery Office, London, Sept. 1973.
- Stahara, S.S., *Operational Manual for Two-Dimensional Transonic Code TSFOIL*, NASA Contractor Report 3064, National Aeronautics

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and Space Administration, Dec. 1978. An indispensable “must have” reference on the classic Murman-Cole method.

- Stahara, S.S. and Spreiter, J.R., “Research on Unsteady Transonic Flow Theory,” Neilsen Engineering and Research, NR-061-215, work supported by the Office of Naval Research, Contract N00014-73-C-0379, April 1977.
- Zheng, Z., “Survey of Transonic Flow Research in China,” Foreign Technology Division, Wright-Patterson Air Force Base, FTD-ID(RS)T-1077-82, Oct. 1982.
- Ballhaus, W.F. and Steger, J.L., “Implicit Approximate Factorization Schemes for the Low-Frequency Transonic Equation,” NASA Technical Memorandum No. 73082, National Aeronautics and Space Administration, November 1975.

**Supersonic flow.** While supersonic flowfields are routinely obtained computationally, the following classic references describe relevant theory useful, for example, in panel method simulator design. The methods described below, and extended by Boeing research staff, in fact, motivated the exact supersonic models in Chapter 3.

- Fraenkel, L.E., “The Theoretical Wave Drag of Some Bodies of Revolution,” Reports & Memoranda No. 2842, A.R.C. Technical Report, Aeronautical Research Council, Her Majesty’s Stationery Office, London, May 1951. Theoretical studies for given geometric shapes, highly mathematical.
- Hayes, W. D., *Linearized Supersonic Flow*, North American Aviation, Los Angeles, Calif., Report AL 222, 1947. A classic.
- Hayes, W.D., “Pseudo-transonic Similitude and First-Order Wave Structure,” *Journal of the Aeronautical Sciences*, Vol. 21, Nov. 1954, pp. 721-730.
- Heaslet, M.A. and Lomax, H., “The Use of Source-Sink and Doublet Distributions Extended to the Solution of Boundary Value Problems in Supersonic Flow,” NACA Rept. 900, 1948.
- Courant, R. and Friedrichs, K.O., *Supersonic Flow and Shock Waves*, Wiley Interscience, New York, 1948, pp. 134-138.
- Whitham, G.B., *Linear and Nonlinear Waves*, John Wiley and Sons, New York, 1974.

**Inverse models.** “Analysis,” “forward” or “direct models” solve for the pressure fields induced by prescribed aerodynamic shapes, while “inverse” or “indirect models” determine the geometric shapes that induce a specified pressure. Various approaches are available, and for completeness, these are listed below. Note that they all differ from the approach developed in this book. In our approach, we derive “direct methods” for aerodynamic inverse problems using a novel streamfunction formulation solved subject to a “Kutta type” trailing edge constraint. Strategies and results are outlined for various classes of problems in Chapters 2 and 4 of this book.

- AGARD Fluid Dynamics Panel, “Optimum Design Methods for Aerodynamics,” AGARD Report 803, *Advisory Group for Aerospace Research and Development, North Atlantic Treaty Organization*, November 1994. Reviewed different approaches, discusses foreign developments.
- Yiu, K.F.C., “Computational Methods for Aerodynamic Shape Design,” Vol. 20, No. 12, *Mathematics and Computer Modelling*, Elsevier Science, 1994, pp. 3-29. General review of various inverse approaches.
- Matsushima, K., Yamamichi, I. and Tokugawa, N., “Aerodynamic Design Method for Supersonic Slender Body Using an Inverse Problem,” *29<sup>th</sup> Congress of the International Council of the Aeronautical Sciences*, St. Petersburg, Russia, September 7-12, 2014.
- Soemarwoto, B., “The Variational Method for Aerodynamic Optimization Using the Navier Stokes Equations,” ICASE Report No. 97-71, National Aeronautics and Space Administration, Langley Research Center, Virginia, December 1997.
- Jameson, A., “A Sequel to Lighthill’s Early Work – Aerodynamic Inverse Design and Shape Optimization via Control Theory,” Florida State University Lecture Series, February 15-20, 2010.
- Qingsheng, L., Yuxin, Z., Yiling, Z. and Hongyang, L., “The Pressure Inverse Problem of Three-Dimensional Supersonic Flow,” *Acta Aerodynamic Sinica*, October 2016.
- Soemarwoto, B., “Robust Inverse Shape Design in Aerodynamics,” Vol. 1, Issue 2, *Inverse Problems in Engineering*, 1995.

### 30 Modern Aerodynamic Methods

- Skinner, S.N. and Zare-Behtash, H., “State-of-the-Art in Aerodynamic Shape Optimization Methods,” Vol. 62, *Applied Soft Computing*, Elsevier Science, 2018, pp. 933-962.
- Dulikravich, G.S., Dennis, B.H., Baker, D.P., Kennon, S.R., Orlande, H.R.B. and Colaco, M.J., “Inverse Problems in Aerodynamics, Heat Transfer, Elasticity and Materials Design,” *International Journal of Aeronautical and Space Sciences*, 13(4), 405–420 (2012).
- Ren, J., “Aerodynamic Shape Optimization by Multi-Fidelity Modeling and Manifold Mapping,” Master of Science Thesis, Iowa State University, 2016.
- Fincham, J.H.S. and Friswell, M.I., “Aerodynamic Optimisation of a Camber Morphing Aerofoil,” *Aerospace Science and Technology*, Elsevier Science, Vol. 43, 2015, pp. 245-255.
- Mukesh, R., Lingadurai, K. and Selvakumar, U., “Airfoil Shape Optimization Using Non-Traditional Optimization Technique and Its Validation,” Vol. 26, *Journal of King Saud University – Engineering Sciences*, 2014, pp. 191-197.
- Yang, Y.L., Tan, C.S. and Hawthorne, W.R., “Aerodynamic of Turbomachinery Blading in Three-Dimensional Flow: An Application to Radial Inflow Turbines,” *Journal of Turbomachinery* 115(3) · July 1992
- Yang, W., “Three-Dimensional Inverse Method for Aerodynamic Optimization in Compressor,” *6th International Conference on Pumps and Fans with Compressors and Wind Turbines*, IOP Publishing, *IOP Conf. Series: Materials Science and Engineering*, Vol. 52 (2013) 012008.
- Mohebbi, F. and Sellier, M., “Aerodynamic Optimal Shape Design Based on Body-Fitted Grid Generation,” *Mathematical Problems in Engineering*, Volume 2014, Article ID 505372, Hindawi Publishing.
- Jian, Z., Jing, T., Ming, Q., Youqi, D. and Xiaoquan, G., “Massively Parallel Numerical Simulation Program Design of Unsteady Full Annulus Turbomachine,” *Acta Aerodynamic Sinica*, Dec. 2017.
- Dulikravich, G.S., Martin, T.J. and Dennis, B.H., “Multidisciplinary Inverse Problems,” *Proceedings of Inverse Problems in Engineering: Theory and Practice*, 3rd International Conference on Inverse Problems in Engineering, Port Ludlow, WA, June 13-18, 1999.

- Liu, G.L. “A New Generation of Inverse Shape Design Problem in Aerodynamics and Aerothermoelasticity: Concepts, Theory and Methods,” *Aircraft Engineering and Aerospace Technology*, 2000, Vol. 72, Issue: 4, pp.334-344,
- Li, Y., Yang, D. and Feng, Z., “Inverse Problem in Aerodynamic Shape Design of Turbomachinery Blades,” *ASME Turbo Expo 2006: Power for Land, Sea, and Air, Volume 6: Turbomachinery, Parts A and B*, Barcelona, Spain, May 8–11, 2006, Conference Sponsors: ASME International Gas Turbine Institute, ISBN: 0-7918-4241-X, eISBN: 0-7918-3774-2
- Petrucci, D. and Filho, N.M., “A Fast Algorithm for Inverse Airfoil Design Using a Transpiration Model and an Improved Vortex Panel Method,” *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 2007, October-December 2007, Vol. XXIX, No. 4, pp. 354-365.
- Yang, J. and Wu, H., “The Development and Application of an Inviscid Inverse Method,” *Propulsion and Power Research*, May 2013, pp. 131-138.

**Engine and airframe integration.** While entire aircraft configurations can be simulated three-dimensionally on high speed computers, often including viscous effects and simple models of turbulence, and rapidly in just minutes, engine and airframe integration represents an art in the final analysis that serves as the sole judge as to the credibility of any computed results. The following papers provide a cross-section of integration approaches take for different aircraft types across different industrial settings. These researches document our own work described in Chapter 5.

- Jing, L., Zhenghong, G., Jiangtao, H. and Ke, Z., “Aerodynamic Design Optimization of Nacelle/Pylon Position on an Aircraft,” *Chinese Journal of Aeronautics*, April 2013, pp. 850-857.
- Rudnik, R., Rossow, C.C. and Geyr, H.F., “Numerical Simulation of Engine/Airframe Integration for High Bypass Engines,” *Aerospace Science and Technology*, Vol. 6, Issue 1, Jan. 2002, pp. 31-42.
- Knoth, F. and Breitsamter, C., “Aerodynamic Testing of Helicopter Side Intake Retrofit Modifications,” Vol. 4, Issue 3, *Aerospace* 2017.

## 32 Modern Aerodynamic Methods

- Takahashi, H., Munakata, T. and Sato, S., “Thrust Augmentation by Airframe-Integrated Linear-Spike Nozzle Concept for High-Speed Aircraft, Vol. 5, Issue 19, *Aerospace*, Feb. 2018.
- Staff, *Aerodynamic Engine/Airframe Integration for High Performance Aircraft and Missiles*, AGARD Conference Proceedings 498, Fluid Dynamics Panel, Advisory Group for Aerospace R & D, Fort Worth, Texas, Oct. 7-10, 1991.
- Hoheisel, H. and Geyr, H.F., “The Influence of Engine Thrust Behaviour on the Aerodynamics of Engine Airframe Integration,” *CEAS Aeronautical Journal*, April 2012, Vol. 3, Issue 1, pp. 79–92.
- Campomanes, C.S., “External Flow Effects in the Engine/Airframe Integration Testing Technique – A New Thrust/Drag Book-Keeping Approach at the German-Dutch Wind Tunnels,” M. Sc. Thesis, Aerospace Engineering, Delft University of Technology, Oct. 2017.
- Sanghi, V., Kumar, S.K., Sundararajan, V. and Sane, S.K., “Engine-Airframe Integration During Conceptual Design for Military Application,” *AIAA Journal of Aircraft*, Vol. 35, No. 3, May-June 1998, pp. 380-386.
- Heykena, C., Savoni, L., Friedrichs, J. and Rudnik, R., “Engine Airframe Integration Sensitivities for a STOL Commercial Aircraft Concept with Over-the-Wing Mounted UHBR-Turbofans,” GPPS-NA-2018-0109, Proceedings of Montreal 2018 Global Power and Propulsion Forum, Montreal, May 7-9, 2018.
- Isikveren, A.T., Seitz, A., Bijewitz, J., Hornung, M., Mirzoyan, A., Isyanov, A., Godard, J.L., Stückl, S. and Van Toor, J., “Recent Advances in Airframe-Propulsion Concepts with Distributed Propulsion,” *29th Congress of the International Council of the Aeronautical Sciences (ICAS 2014)*, SAINT-PETERSBOURG, Russia, Sept. 2014.
- Oates, G.C., *Aerothermodynamics of Gas Turbine and Rocket Propulsion*, Fifth Printing, American Institute of Aeronautics and Astronautics (AIAA), 1998.

### 1.5 References

Essential references are listed above. These and all chapter references appear in “Cumulative References” at the end of this book. Other author publications are found in “About the Author.”