

Chapter 1

Introduction

1.1 Basic Equations

Maxwell Equations

Electromagnetic problems involving fields and sources are governed by the Maxwell equations, medium equations and boundary conditions. For time-harmonic fields and sources, with time dependence $\exp(j\omega t)$, the Maxwell equations, in the form of Heaviside [19] and the formalism of Gibbs [13], can be expressed as

$$\nabla \times \mathbf{E} + j\omega \mathbf{B} = -\mathbf{J}_m \quad (1.1)$$

$$\nabla \times \mathbf{H} - j\omega \mathbf{D} = \mathbf{J}_e. \quad (1.2)$$

Here, $\mathbf{J}_e$ and $\mathbf{J}_m$ denote the respective electric and magnetic current density vectors while $\mathbf{E}, \mathbf{H}, \mathbf{D}$ and $\mathbf{B}$ represent the respective electric field, magnetic field, electric flux-density and magnetic flux-density vectors. All of these vectors may have complex components.

Medium Conditions

Conditions between the field vectors in a bianisotropic medium can be expressed in the form [29, 35]

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \bar{\bar{\epsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad (1.3)$$

where $\bar{\bar{\epsilon}}, \bar{\bar{\xi}}, \bar{\bar{\zeta}}$ and $\bar{\bar{\mu}}$ are medium dyadics. In terms of a given vector basis, any dyadic can be represented in terms of a 3×3 matrix involving nine scalar components (for rules of dyadic algebra, see Appendix C).

Here we mainly consider problems involving the simple-isotropic medium ('free space, vacuum'), for which the medium conditions are reduced to

$$\mathbf{D} = \epsilon_o \mathbf{E}, \quad \mathbf{B} = \mu_o \mathbf{H}. \quad (1.4)$$

The wave impedance η_o , expressing the ratio of the electric and magnetic field magnitudes in the simple-isotropic medium, is defined by

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}}. \quad (1.5)$$

Boundary Conditions

Boundary conditions form an essential part of formulating electromagnetic problems. By a boundary we mean a surface on which secondary electromagnetic sources are induced by the primary fields so that the fields beyond the surface vanish. A boundary surface is different from an interface of two media because the fields beyond the interface are not necessarily zero. From the mathematical point of view, boundary conditions are required to ensure existence and uniqueness of solutions for a particular problem. From the engineering point of view, two aspects of boundary conditions can be separated which may be called analytic and synthetic.

An *analytic* aspect of boundary conditions is encountered when a given physical problem requires mathematical analysis. Due to natural complications, a given structure must often be approximated by certain boundary conditions to find a numerical solution [21, 88]. As an example, solving radio wave propagation over ground requires that the ground be approximated by an impedance boundary or, if well-enough conducting, by the perfect electric conductor (PEC). Thus, in such a case, to analyze the effect of a given structure, we replace it by some approximate boundary conditions.

In contrast, a *synthetic* aspect emerges when we wish to realize given boundary conditions by some physical structure. As an example, when designing mobile phones with submerged antennas, the concept of perfect magnetic conductor (PMC) boundary has been suggested to solve the problem of efficient radiation [92]. In this case, the problem becomes a synthesis, how to find a physical structure to realize the PMC conditions.

In this book we are concerned about the synthetic aspect by studying different types of boundary conditions, their effect on electromagnetic fields, and possible realizations by interfaces of media defined by medium parameters. More practical physical realizations by (meta)materials are, however, beyond the topic of the book. Let us start with three most basic boundary conditions, properties of which will be considered in due course.

- Perfect Electric Conductor (PEC)

$$\mathbf{n} \times \mathbf{E} = 0 \quad (1.6)$$

- Perfect Magnetic Conductor (PMC)

$$\mathbf{n} \times \mathbf{H} = 0, \quad (1.7)$$

- Perfect Electromagnetic Conductor (PEMC) [45]

$$\mathbf{n} \times (\mathbf{H} + M\mathbf{E}) = 0 \quad (1.8)$$

The PEMC, involving a parameter M (the PEMC admittance), is a generalization of both the PEC and the PMC. For $M = 0$, the PEMC reduces to the PMC, and for $1/M = 0$, it reduces to the PEC.

1.2 Duality Transformation

A given electromagnetic problem can be transformed to another one in terms of duality transformation which does not change the geometry of the problem, but the sources, fields, medium and boundary parameters are transformed to have other values, in general. The concept of duality was evidently unknown to Maxwell, because he presented his equations in a very nonsymmetric form, in terms of 20 scalar field and potential quantities [79]. The concept was introduced by Heaviside in 1886 [19].

Duality transformation is based on the apparent symmetry of the Maxwell equations (1.1), (1.2), written more compactly as

$$\nabla \times \begin{pmatrix} \mathbf{E} \\ -\mathbf{H} \end{pmatrix} + j\omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = - \begin{pmatrix} \mathbf{J}_m \\ \mathbf{J}_e \end{pmatrix}. \quad (1.9)$$

In fact, for the simple change of symbols $\mathbf{E} \leftrightarrow -\mathbf{H}$, $\mathbf{B} \leftrightarrow \mathbf{D}$ and $\mathbf{J}_e \leftrightarrow \mathbf{J}_m$, the pair of equations (1.9) is invariant. More generally, the same property can be expressed in terms of the duality transformation $(\mathbf{E}, \mathbf{H}) \rightarrow (\mathbf{E}_d, \mathbf{H}_d)$, defined by [35]

$$\begin{pmatrix} \mathbf{E}_d \\ \eta_o \mathbf{H}_d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta_o \mathbf{H} \end{pmatrix}. \quad (1.10)$$

The wave impedance η_o has been included to obtain dimensionless transformation parameters $A \cdots D$. They are assumed to satisfy

$$AD - BC = 1, \quad (1.11)$$

whence the inverse transformation exists and has the form

$$\begin{pmatrix} \mathbf{E} \\ \eta_o \mathbf{H} \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \begin{pmatrix} \mathbf{E}_d \\ \eta_o \mathbf{H}_d \end{pmatrix}. \quad (1.12)$$

Applying (1.10) to (1.1) and (1.2), the associated transformation rules can be expressed as

$$\begin{pmatrix} \eta_o \mathbf{D}_d \\ \mathbf{B}_d \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \eta_o \mathbf{D} \\ \mathbf{B} \end{pmatrix} \quad (1.13)$$

$$\begin{pmatrix} \eta_o \mathbf{J}_{ed} \\ \mathbf{J}_{md} \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} \eta_o \mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix} \quad (1.14)$$

One can show that the PEMC boundary conditions (1.8) are transformed to

$$\mathbf{n} \times (\mathbf{H}_d + M_d \mathbf{E}_d) = 0, \quad (1.15)$$

with the transformed PEMC admittance satisfying

$$M_d \eta_o = -\frac{C - DM\eta_o}{A - BM\eta_o}. \quad (1.16)$$

From this it follows that both PEC and PMC boundaries are transformed to PEMC boundaries with $M_d \eta_o = -D/B$ and $M_d \eta_o = -C/A$, respectively. Also, any given PEMC boundary can be transformed to PEC and PMC boundaries when the transformation parameters are chosen to satisfy the respective restrictions $A/B = M\eta_o$ and $C/D = M\eta_o$.

In the general case, the dyadic parameters of the electromagnetic medium will be changed when the fields are subject to the duality transformation (1.10). For a bianisotropic medium defined by conditions of the form (1.3), the transformed medium dyadics can be shown to obey the relations [42]

$$\begin{pmatrix} \bar{\bar{\epsilon}}_d \\ \bar{\bar{\xi}}_d \\ \bar{\bar{\zeta}}_d \\ \bar{\bar{\mu}}_d \end{pmatrix} = \begin{pmatrix} D^2 & -CD/\eta_o & -CD/\eta_o & C^2/\eta_o^2 \\ -BD\eta_o & AD & BC & -AC/\eta_o \\ -BD\eta_o & BC & AD & -AC/\eta_o \\ B^2\eta_o^2 & -AB\eta_o & -AB\eta_o & A^2 \end{pmatrix} \begin{pmatrix} \bar{\bar{\epsilon}} \\ \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} \\ \bar{\bar{\mu}} \end{pmatrix}. \quad (1.17)$$

Requiring a transformation in which the simple isotropic medium (μ_o, ϵ_o) is invariant leads to a choice of the form

$$A = D = \cos \varphi, \quad B = -C = \sin \varphi, \quad (1.18)$$

where φ is a free transformation parameter. The resulting transformation matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \quad (1.19)$$

can be recognized as a 2D rotation matrix. The corresponding transformation rule for the medium dyadics (1.17) becomes in this case

$$\begin{pmatrix} \bar{\bar{\epsilon}}_d/\epsilon_o \\ \bar{\bar{\xi}}_d/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\zeta}}_d/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\mu}}_d/\mu_o \end{pmatrix} = \mathcal{Q}(\varphi) \begin{pmatrix} \bar{\bar{\epsilon}}/\epsilon_o \\ \bar{\bar{\xi}}/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\zeta}}/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\mu}}/\mu_o \end{pmatrix}, \quad (1.20)$$

with the 4×4 matrix $\mathcal{Q}(\varphi)$ defined by

$$\mathcal{Q}(\varphi) = \begin{pmatrix} \cos^2 \varphi & \sin \varphi \cos \varphi & \sin \varphi \cos \varphi & \sin^2 \varphi \\ -\sin \varphi \cos \varphi & \cos^2 \varphi & -\sin^2 \varphi & \sin \varphi \cos \varphi \\ -\sin \varphi \cos \varphi & -\sin^2 \varphi & \cos^2 \varphi & \sin \varphi \cos \varphi \\ \sin^2 \varphi & -\sin \varphi \cos \varphi & -\sin \varphi \cos \varphi & \cos^2 \varphi \end{pmatrix}. \quad (1.21)$$

For the simple-isotropic medium with $\bar{\epsilon} = \epsilon_o \bar{\mathbf{I}}$, $\bar{\xi} = \bar{\zeta} = 0$ and $\bar{\mu} = \mu_o \bar{\mathbf{I}}$, from (1.20) we obtain $\bar{\epsilon}_d = \epsilon_o \bar{\mathbf{I}}$, $\bar{\xi}_d = \bar{\zeta}_d = 0$ and $\bar{\mu}_d = \mu_o \bar{\mathbf{I}}$, as required. One can further show that the matrix $\mathcal{Q}(\varphi)$ satisfies

$$\mathcal{Q}(\varphi_1)\mathcal{Q}(\varphi_2) = \mathcal{Q}(\varphi_1 + \varphi_2), \quad (1.22)$$

$$\det \mathcal{Q}(\varphi) = 1, \quad (1.23)$$

$$\mathcal{Q}^{-1}(\varphi) = \mathcal{Q}^T(\varphi) = \mathcal{Q}(-\varphi), \quad (1.24)$$

i.e., it is an orthogonal matrix. The proof is left as an exercise.

Expressing the PEMC admittance parameter M in terms of another parameter φ as

$$M\eta_o = \tan \vartheta, \quad (1.25)$$

the transformation rule (1.16) takes the simple form

$$M_d\eta_o = \frac{M\eta_o + \tan \varphi}{1 - M\eta_o \tan \varphi} = \tan(\vartheta + \varphi). \quad (1.26)$$

Thus, for the choice $\varphi = -\vartheta$ of the duality parameter, the PEMC boundary is transformed to the PMC boundary ($M_d = 0$), while for $\varphi = \pi/2 - \vartheta$, it is transformed to the PEC boundary ($1/M_d = 0$).

1.3 Plane Waves

Basic Conditions

Let us consider time-harmonic plane waves with $\exp(j\omega t)$ dependence in a simple isotropic medium, in front of a boundary surface defined by $\mathbf{n} \cdot \mathbf{r} = 0$. For simplicity we assume that the surface is a plane, i.e., that $\mathbf{n}$ is a constant unit vector. The electric and magnetic field components of waves incident to the boundary are expressed by

$$\mathbf{E}^i(\mathbf{r}) = \mathbf{E}^i \exp(-j\mathbf{k}^i \cdot \mathbf{r}), \quad (1.27)$$

$$\mathbf{H}^i(\mathbf{r}) = \mathbf{H}^i \exp(-j\mathbf{k}^i \cdot \mathbf{r}), \quad (1.28)$$

and the reflected fields are

$$\mathbf{E}^r(\mathbf{r}) = \mathbf{E}^r \exp(-j\mathbf{k}^r \cdot \mathbf{r}), \quad (1.29)$$

$$\mathbf{H}^r(\mathbf{r}) = \mathbf{H}^r \exp(-j\mathbf{k}^r \cdot \mathbf{r}). \quad (1.30)$$

To satisfy the Maxwell equations, the two wave vectors must satisfy the dispersion equation as

$$\mathbf{k}^i \cdot \mathbf{k}^i = \mathbf{k}^r \cdot \mathbf{k}^r = k_o^2 = \omega^2 \mu_o \epsilon_o. \quad (1.31)$$

If the boundary condition is linear, the $\mathbf{k}^i$ and $\mathbf{k}^r$ vectors must have the same components tangential to the boundary surface, i.e., they can be represented as

$$\mathbf{k}^i = \mathbf{k}_t - k_n \mathbf{n}, \quad \mathbf{k}^r = \mathbf{k}_t + k_n \mathbf{n}. \quad (1.32)$$

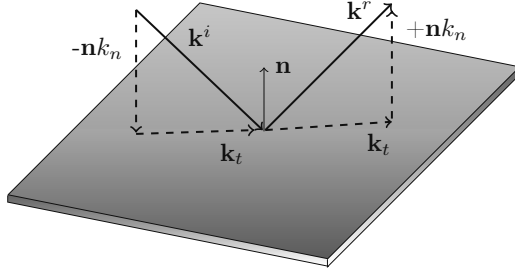


Figure 1.1: The incident and reflected plane waves have the same wave-vector components $\mathbf{k}_t$ tangential to the boundary surface.

A class of boundaries with conditions deviating from linear ones is considered in Chapter 6.

Substituting the plane-wave fields in the Maxwell equations (1.1) and (1.2), equations relating the electric and magnetic fields are obtained as

$$\mathbf{k}^i \times \mathbf{E}^i = k_o \eta_o \mathbf{H}^i, \quad (1.33)$$

$$\mathbf{k}^i \times \eta_o \mathbf{H}^i = -k_o \mathbf{E}^i, \quad (1.34)$$

and

$$\mathbf{k}^r \times \mathbf{E}^r = k_o \eta_o \mathbf{H}^r, \quad (1.35)$$

$$\mathbf{k}^r \times \eta_o \mathbf{H}^r = -k_o \mathbf{E}^r. \quad (1.36)$$

The field components normal to the boundary can be expressed in terms of their tangential components from the orthogonality conditions as

$$\mathbf{k}^i \cdot \mathbf{E}^i = 0, \quad \Rightarrow \quad \mathbf{n} \cdot \mathbf{E}^i = \frac{1}{k_n} \mathbf{k}_t \cdot \mathbf{E}_t^i, \quad (1.37)$$

$$\mathbf{k}^r \cdot \mathbf{E}^r = 0, \quad \Rightarrow \quad \mathbf{n} \cdot \mathbf{E}^r = -\frac{1}{k_n} \mathbf{k}_t \cdot \mathbf{E}_t^r, \quad (1.38)$$

valid for $k_n \neq 0$. Similar conditions are valid for the magnetic fields.

Field Relations

From (1.33) and (1.34) we obtain the following relations between the tangential components of the incident fields:

$$\mathbf{n} \cdot \mathbf{k}_t \times \mathbf{E}_t^i = \frac{k_o}{k_n} \mathbf{k}_t \cdot \eta_o \mathbf{H}_t^i, \quad (1.39)$$

$$\mathbf{n} \cdot \mathbf{k}_t \times \eta_o \mathbf{H}_t^i = -\frac{k_o}{k_n} \mathbf{k}_t \cdot \mathbf{E}_t^i. \quad (1.40)$$

Assuming also $\mathbf{k}_t \cdot \mathbf{k}_t = k_t^2 \neq 0$, the vectors $\mathbf{k}_t$ and $\mathbf{n} \times \mathbf{k}_t$ serve as a 2D basis, in terms of which we can expand the 2D unit dyadic as

$$\bar{\bar{\mathbf{l}}}_t = \bar{\bar{\mathbf{l}}} - \mathbf{nn} = \frac{1}{k_t^2}(\mathbf{k}_t \mathbf{k}_t + (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \times \mathbf{k}_t)). \quad (1.41)$$

The relation between the tangential electric and magnetic fields can be expanded as

$$\begin{aligned} \mathbf{E}_t^i &= \frac{1}{k_t^2}(\mathbf{k}_t(\mathbf{k}_t \cdot \mathbf{E}_t^i) + (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \times \mathbf{k}_t \cdot \mathbf{E}_t^i)) \\ &= \frac{1}{k_t^2}\left(-\frac{k_n}{k_o}\mathbf{k}_t(\mathbf{n} \times \mathbf{k}_t) + \frac{k_o}{k_n}(\mathbf{n} \times \mathbf{k}_t)\mathbf{k}_t\right) \cdot \eta_o \mathbf{H}_t^i \\ &= \frac{1}{k_o k_n} \mathbf{n} \times (\mathbf{k}_t \mathbf{k}_t + k_n^2 \bar{\bar{\mathbf{l}}}_t) \cdot \eta_o \mathbf{H}_t^i. \end{aligned} \quad (1.42)$$

The dyadic in this relation is of importance in the analysis of plane waves. Denoted by

$$\bar{\bar{\mathbf{J}}}_t = \frac{1}{k_o k_n} \mathbf{n} \times (\mathbf{k}_t \mathbf{k}_t + k_n^2 \bar{\bar{\mathbf{l}}}_t), \quad (1.43)$$

it can be shown to satisfy the properties

$$\text{tr} \bar{\bar{\mathbf{J}}}_t = 0, \quad (1.44)$$

$$\bar{\bar{\mathbf{J}}}_t^2 = -\bar{\bar{\mathbf{l}}}_t, \quad (1.45)$$

$$\bar{\bar{\mathbf{J}}}_t^{-1} = -\bar{\bar{\mathbf{J}}}_t, \quad (1.46)$$

derivations of which are left as exercises.

The dyadic $\bar{\bar{\mathbf{J}}}_t$ allows one to write relations between the tangential electric and magnetic fields of a plane wave in compact form as

$$\mathbf{E}_t^i = \bar{\bar{\mathbf{J}}}_t \cdot \eta_o \mathbf{H}_t^i \quad \eta_o \mathbf{H}_t^i = -\bar{\bar{\mathbf{J}}}_t \cdot \mathbf{E}_t^i. \quad (1.47)$$

For the reflected fields, similar expressions are valid when replacing k_n by $-k_n$:

$$\mathbf{E}_t^r = -\bar{\bar{\mathbf{J}}}_t \cdot \eta_o \mathbf{H}_t^r, \quad \eta_o \mathbf{H}_t^r = \bar{\bar{\mathbf{J}}}_t \cdot \mathbf{E}_t^r. \quad (1.48)$$

The expressions (1.47) and (1.48) will be needed in the analysis of reflections from various boundaries. In this, we can also make use of the following set of relations:

$$\mathbf{k}_t \cdot \bar{\bar{\mathbf{J}}}_t = -\frac{k_n}{k_o} \mathbf{n} \times \mathbf{k}_t, \quad \bar{\bar{\mathbf{J}}}_t \cdot \mathbf{k}_t = \frac{k_o}{k_n} \mathbf{n} \times \mathbf{k}_t, \quad (1.49)$$

$$(\mathbf{n} \times \mathbf{k}_t) \cdot \bar{\bar{\mathbf{J}}}_t = \frac{k_o}{k_n} \mathbf{k}_t, \quad \bar{\bar{\mathbf{J}}}_t \cdot (\mathbf{n} \times \mathbf{k}_t) = -\frac{k_n}{k_o} \mathbf{k}_t, \quad (1.50)$$

and,

$$\bar{\bar{\mathbf{J}}}_t^{(2)} = \frac{1}{2} \bar{\bar{\mathbf{J}}}_t \times \bar{\bar{\mathbf{J}}}_t = \mathbf{nn}, \quad \det_t \bar{\bar{\mathbf{J}}}_t = \text{tr} \bar{\bar{\mathbf{J}}}_t^{(2)} = 1. \quad (1.51)$$

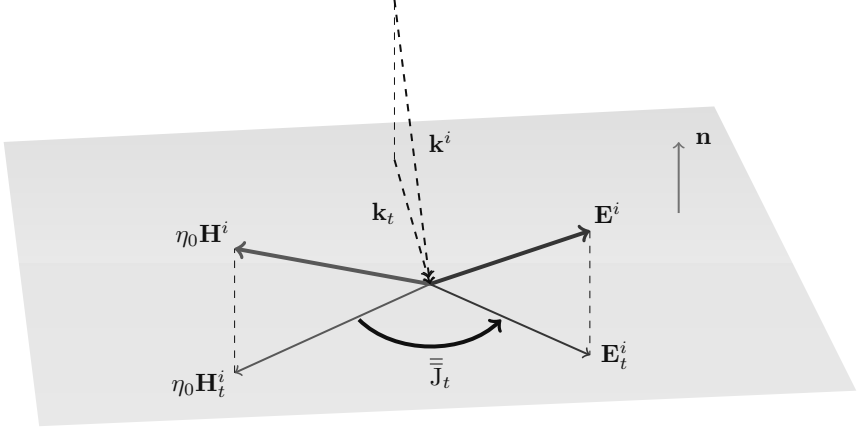


Figure 1.2: The dyadic $\bar{\bar{J}}_t$ maps the tangential component of the magnetic field of an incident plane wave to the tangential component of the electric field. The vectors $\mathbf{E}^i$, $\mathbf{H}^i$ and $\mathbf{k}^i$ are orthogonal to one another.

The dyadic $\bar{\bar{J}}_t$ and the antisymmetric dyadic $\mathbf{n} \times \bar{\bar{I}}$ share the property $(\mathbf{n} \times \bar{\bar{I}})^2 = \bar{\bar{J}}_t^2 = -\bar{\bar{I}}_t$. While $\mathbf{n} \times \bar{\bar{I}}$ rotates vectors on the tangent plane by $\pi/2$, the dyadic $\bar{\bar{J}}_t$ "rotates" the tangential part of the electric field to the tangential part of the magnetic field of the same plane wave. The $\bar{\bar{J}}_t$ dyadic is determined by $\mathbf{k}_t$, the tangential part of the wave vector in the simple-isotropic medium.

1.4 TE/TM Decomposition

It turns out that the field vectors of a plane wave, propagating with the wave vector $\mathbf{k} = \mathbf{n}k_n + \mathbf{k}_t$, can be expressed in terms of their normal components $\mathbf{n} \cdot \mathbf{E}$ and $\mathbf{n} \cdot \mathbf{H}$ as

$$\mathbf{E} = \frac{1}{k_t^2} ((k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}) + k_o (\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \eta_o \mathbf{H})) \quad (1.52)$$

$$\eta_o \mathbf{H} = \frac{1}{k_t^2} ((k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \eta_o \mathbf{H}) - k_o (\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \mathbf{E})). \quad (1.53)$$

Details of the derivation are left as an exercise. Applying (1.52) and (1.53), any plane wave can be decomposed in two parts as

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{TE} \\ \mathbf{H}_{TE} \end{pmatrix} + \begin{pmatrix} \mathbf{E}_{TM} \\ \mathbf{H}_{TM} \end{pmatrix}, \quad (1.54)$$

with

$$\mathbf{E}_{TE} = \frac{k_o}{k_t^2}(\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \eta_o \mathbf{H}), \quad (1.55)$$

$$\mathbf{H}_{TE} = \frac{1}{k_t^2}(k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{H}), \quad (1.56)$$

$$\mathbf{E}_{TM} = \frac{1}{k_t^2}(k_t^2 \mathbf{n} - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}), \quad (1.57)$$

$$\eta_o \mathbf{H}_{TM} = -\frac{k_o}{k_t^2}(\mathbf{n} \times \mathbf{k})(\mathbf{n} \cdot \mathbf{E}). \quad (1.58)$$

The partial fields are called tranverse electric (TE) and transverse magnetic (TM) because the two parts are restricted by the respective conditions

$$\mathbf{n} \cdot \mathbf{E}_{TE} = 0, \quad \mathbf{n} \cdot \mathbf{H}_{TM} = 0. \quad (1.59)$$

The decomposed fields satisfy the Maxwell equations separately. To check this, we can expand

$$\nabla \times \mathbf{E}_{TE} = -j\mathbf{k} \times \mathbf{E}^{TE} \quad (1.60)$$

$$= -j \frac{k_o \eta_o}{k_t^2} \mathbf{k} \times (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{H}) \quad (1.61)$$

$$= -j \frac{k_o \eta_o}{k_t^2} (\mathbf{n} k_t^2 - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{H}) \quad (1.62)$$

$$= -j k_o \eta_o \mathbf{H}^{TE}, \quad (1.63)$$

$$\nabla \times \mathbf{H}_{TM} = -j\mathbf{k} \times \mathbf{H}^{TM} \quad (1.64)$$

$$= j \frac{k_o}{\eta_o k_t^2} \mathbf{k} \times (\mathbf{n} \times \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}) \quad (1.65)$$

$$= j \frac{k_o}{\eta_o k_t^2} (\mathbf{n} k_t^2 - k_n \mathbf{k}_t)(\mathbf{n} \cdot \mathbf{E}) \quad (1.66)$$

$$= j \frac{k_o}{\eta_o} \mathbf{E}_{TM}. \quad (1.67)$$

The decomposition is unique for $\mathbf{k}_t \neq 0$. For $\mathbf{k}_t = 0$ the expansions (1.52) and (1.53) are not applicable. In this case the TE/TM decomposition is still possible but not unique. Actually, for $\mathbf{k}_t = 0$ the plane-wave has TEM polarization, $\mathbf{n} \cdot \mathbf{E} = \mathbf{n} \cdot \mathbf{H} = 0$.

Because the TE/TM decomposition of a plane wave does not depend on the choice of the $\mathbf{k}$ vector, it is valid for a sum of plane waves and, ultimately, for an integral of plane waves in the space of $\mathbf{k}$ vectors which may have complex values. Thus, any fields which can be expressed as an integral of plane waves can be decomposed in two independent TE and TM parts with respect to a given $\mathbf{n}$ vector.

1.5 Problems

- 1.1** For a given duality transformation, find a PEMC admittance which is invariant in the transformation. Apply the result to the special transformation (1.19).
- 1.2** Derive the transformation rule (1.17) for the medium dyadics and show that the dyadic $\bar{\bar{\xi}} - \bar{\bar{\zeta}}$ is invariant in the transformation.
- 1.3** Derive the transformation rule (1.20) from (1.17).
- 1.4** Show that the matrix $\mathcal{Q}(\varphi)$ in (1.21) is orthogonal, i.e., that it satisfies
- $$\mathcal{Q}^{-1}(\varphi) = \mathcal{Q}^T(\varphi) = \mathcal{Q}(-\varphi).$$
- 1.5** Show that it is possible to find a duality transformation which maps two given PEMC admittances M_1 and M_2 so that $M_{1d} = -M_{2d} = M_d$ and leaves the simple-isotropic medium invariant. Find the relation between M_1 , M_2 and M_d .
- 1.6** Derive the properties (1.45), (1.46) and (1.51) of the dyadic $\bar{\bar{\mathbf{J}}}_t$.
- 1.7** Derive the decomposition rules (1.52) and (1.53).
- 1.8** Applying (1.20), find another expression for the duality transformation of the medium dyadics in the form

$$\begin{pmatrix} \bar{\bar{\epsilon}}_d/\epsilon_o - \bar{\bar{\mu}}_d/\mu_o \\ (\bar{\bar{\xi}}_d - \bar{\bar{\zeta}}_d)/\sqrt{\mu_o\epsilon_o} \\ (\bar{\bar{\xi}}_d + \bar{\bar{\zeta}}_d)/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\epsilon}}_d/\epsilon_o + \bar{\bar{\mu}}_d/\mu_o \end{pmatrix} = \mathcal{Q}'(\varphi) \begin{pmatrix} \bar{\bar{\epsilon}}/\epsilon_o - \bar{\bar{\mu}}/\mu_o \\ (\bar{\bar{\xi}} - \bar{\bar{\zeta}})/\sqrt{\mu_o\epsilon_o} \\ (\bar{\bar{\xi}} + \bar{\bar{\zeta}})/\sqrt{\mu_o\epsilon_o} \\ \bar{\bar{\epsilon}}/\epsilon_o + \bar{\bar{\mu}}/\mu_o \end{pmatrix}$$

in terms of a matrix $\mathcal{Q}'(\varphi)$. Which dyadics appear invariant in the transformation?