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On the Fractional Derivative and Integral Operators

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1.1 Introduction

It is formulated using mathematical expressions to solve problems in engineering, science, and many other fields. These formulas are obtained and functions are formed to solve the problems under certain initial and boundary conditions. The resulting equation generally contains derivatives of fractional, first or higher order. Expressions containing such equations are solved by some known methods. While making these solutions, the known methods of classical analysis are not always sufficient. In this case, fractional calculus tools are activated. As many researchers know, the story of the fractional calculus that began with that letter in 1695 was answered by Leibniz [1, 2].

Although fractional calculus tools have been known and used in different fields for a long time, the theory of fractional differential equations has recently begun to be studied. Many important books have been written on this subject in the literature. The subject of fractional calculus can still be improved and is of great importance in helping other fields. In this section, chronologically fractional derivative and integral operators will be introduced and important properties of these operators will be given [3]. Firstly, the derivative of Grünwald–Letnikov developed with the classical derivative half will be given. Later, Riemann and Liouville developed the definition of fractional derivative of Grünwald–Letnikov and introduced a new operator to the literature [4]. This operator has had an important place for a long time as it is today. Later, in 1967, Caputo made a significant development in this regard and introduced an operator to be used until the early 2000s [5]. In 2015, Caputo and Fabrizio changed the kernel of the Caputo derivative definition to a definition. The kernel they use is very important in terms of singularity. It also gives good results in solving real world problems [6]. Finally, Atangana and Baleanu put forward a definition of both nonsingular and nonlocal. The

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core obtained by using the generalized form of the Mittag-Leffler function has an important place, although it has recently been discovered [7].

It is tried to explain the important features of all the operators given above by supporting them with definitions, theorems and lemmas. In the last chapter, the application of two important models such as Keller–Segel and Cancer Treatment to these derivatives is shown [8, 9].

1.2 Fractional Derivative and Integral Operators

In this section, fractional derivative and integral operators will be introduced chronologically. Important theorems and lemmas will be given about these operators.

1.2.1 Properties of the Grünwald–Letnikov Fractional Derivative and Integral

Let $y = f(t)$ is a continuous function. According to the definition, the first-order derivative of the function $f(t)$ is defined by

$$f'(t) = \frac{df}{dt} = \lim_{h\to 0} \frac{f(t) - f(t-h)}{h}. \tag{1.1}$$

The second-order derivative using Eq. (1.1), then

$$\begin{aligned} f''(t) = \frac{d^2f}{dt^2} &= \lim_{h\to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h}\right] \\ &= \lim_{h\to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \end{aligned} \tag{1.2}$$

Similarly, third-order derivative as:

$$f'''(t) = \frac{d^3f}{dt^3} = \lim_{h\to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3}. \tag{1.3}$$

When this situation is generalized, Eq. (1.14) is obtained

$$f^n(t) = \frac{d^nf}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^n \binom{n}{r} f(t - rh), \tag{1.4}$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \tag{1.5}$$

Let now examine the following expression generalizing the fractions (1.2)–(1.15):

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^k \binom{p}{r} f(t - rh), \tag{1.6}$$

where p is an arbitrary integer number, n is also integer, as above.

Obviously, for $p \leq n$ we have,

$$\lim_{h \to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p} \tag{1.7}$$

because in such a case, as follows from (1.15), all the coefficients in the numerator after $\binom{p}{p}$ are equal to 0.

Let us consider negative values of p. For convenience, let us denote

$$\begin{bmatrix} p \\ r \end{bmatrix} = \frac{p(p+1)\cdots(p+r-1)}{r!}. \tag{1.8}$$

Then we have

$$\binom{-p}{r} = \frac{-p(-p-1)\cdots(-p-r+1)}{r!} = (1-)^r \begin{bmatrix} p \\ r \end{bmatrix} \tag{1.9}$$

and replacing p in (1.6) with $-p$ we can write

$$f_h^{(-p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t - rh), \tag{1.10}$$

where p is a positive integer number.

If n is fixed, then $f_h^{(-p)}(t)$ tends to the uninteresting limit 0 as $h \to 0$. To arrive at a nonzero limit, we have to suppose that $n \to \infty$ as $h \to 0$. We can take $h = \frac{t-a}{n}$, where a is a real constant, and consider the limit value, either finite or infinite, of $f_h^{(-p)}(t)$, which we will denote as

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-p)}(t) = {}_a D_h^{(-p)}(t).$$

Here $D_h^{(-p)}(t)$ denotes, in fact, a certain operation performed on the function $f(t)$; a and t are the terminals – the limits relating to this operation.

Let us consider several particular cases.

For $p = 1$, we have

$$f_h^{(-1)}(t) = h \sum_{r=0}^{n} f(t - rh). \tag{1.11}$$

Taking into account that $t - nh = a$ and that the function $f(t)$ is assumed to be continuous, we conclude that

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-1)}(t) = {}_a D_t^{(-1)} f(t) = \int_0^{t-a} f(t - z) dt = \int_a^t f(\tau) d\tau. \tag{1.12}$$

Let us take $p = 2$. In this case

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{2, 3, \cdots, (2 + r - 1)}{r!} = r + 1$$

and we have

$$f_h^{(-2)}(t) = h \sum_{r=0}^{n} (rh) f(t - rh). \tag{1.13}$$

Denoting $t + h = y$, we can write

$$f_h^{(-2)}(t) = h \sum_{r=1}^{n+1} (rh) f(t - rh) \tag{1.14}$$

and taking $h \to 0$, we obtain

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-2)}(t) =_a D_t^{(-2)} f(t) = \int_0^{t-a} z f(t - z) dt = \int_a^t (t - \tau) f(\tau) d\tau, \tag{1.15}$$

because $y \to t$ as $h \to 0$. Relationships (11)–(15) suggest the following general expression:

$$_a D_t^{(-p)} f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} h^p \begin{bmatrix} p \\ r \end{bmatrix} f(t - rh) = \frac{1}{(p-1)!} \int_a^t (t - \tau)^{p-1} f(\tau) d\tau. \tag{1.16}$$

To prove the formula (1.16) by induction, we have to show that f holds for some p, then it holds also for $p + 1$.

Let us introduce the function

$$f_1(t) = \int_a^t f(\tau) d\tau, \tag{1.17}$$

which has the obvious property $f_1(a) = 0$, and consider

$$\begin{aligned} {}_a D_t^{(-p-1)} f(t) &= \lim_{\substack{h \to 0 \\ nh = t-a}} h^{p+1} \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f(t - rh) \\ &= \lim_{\substack{h \to 0 \\ nh = t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f_1(t - rh) \\ &\quad - \lim_{\substack{h \to 0 \\ nh = t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f_1(t - (r+1)h). \end{aligned} \tag{1.18}$$

Using (1.8), it is easy to verify that

$$\begin{bmatrix} p+1 \\ r \end{bmatrix} = \begin{bmatrix} p \\ r \end{bmatrix} + \begin{bmatrix} p+1 \\ r-1 \end{bmatrix}, \tag{1.19}$$

where we must put

$$\begin{bmatrix} p+1 \\ -1 \end{bmatrix} = 0.$$

Relationship (1.19) applied to the first sum in (1.18) and the replacement of r by $r-1$ in the second sum gives:

$$\begin{aligned} {}_aD_t^{(-p-1)}f(t) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f(t-rh) \\ &\quad + \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r-1 \end{bmatrix} f_1(t-rh) \\ &\quad - \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=1}^{n+1} \begin{bmatrix} p+1 \\ r-1 \end{bmatrix} f_1(t-rh) \\ &={}_a D_t^{(-p)} f_1(t) - \lim_{\substack{h\to 0\\ nh=t-a}} h^p \begin{bmatrix} p+1 \\ n \end{bmatrix} f(t-(n+1)h) \\ &={}_a D_t^{(-p)} f_1(t) - (t-a)^p \lim_{n\to\infty} \begin{bmatrix} p+1 \\ n \end{bmatrix} \tfrac{1}{n^p} f_1\left(a - \tfrac{t-a}{n}\right). \end{aligned} \tag{1.20}$$

It follows from the definition (1.16) of the function $f_1(t)$ that

$$\lim_{n\to\infty} f_1\left(a - \frac{t-a}{n}\right) = 0.$$

Taking into account the known limit

$$\lim_{n\to\infty} \begin{bmatrix} p+1 \\ n \end{bmatrix} \frac{1}{n^p} = \lim_{n\to\infty} \frac{(p+1)(p+2)\cdots(p+n)}{n^p n!} = \frac{1}{\Gamma(p+1)},$$

we obtain

$$\begin{aligned} {}_aD_t^{(-p-1)}f(t) &={}_a D_t^{(-p)} f_1(t) = \frac{1}{(p-1)!}\int_a^t (t-\tau)^{p-1} f_1(\tau)d\tau \\ &= -\frac{(t-\tau)^p f_1(\tau)}{p!}\Big|_{\tau=a}^{\tau=t} + \frac{1}{p!}\int_a^t (t-\tau)^p f(\tau)d\tau \\ &= \frac{1}{p!}\int_a^t (t-\tau)^p f(\tau)d\tau, \end{aligned} \tag{1.21}$$

which ends the proof of formula (1.16) by induction.

Now let us show that formula (1.16) is a representation of a p-fold integral. Integrating the relationship

$$\frac{d}{dt}({}_aD_t^{-p}f(t)) = \frac{1}{(p-2)!}\int_a^t (t-\tau)^{p-2} f(\tau)d\tau ={}_a D_t^{-p+1} f(t)$$

from a to t, we obtain:

$${}_aD_t^{-p}f(t) = \int_a^t ({}_aD_t^{-p+1}f(t))dt,$$

$${}_aD_t^{-p+1}f(t) = \int_a^t ({}_aD_t^{-p+2}f(t))dt, etc.,$$

and therefore

$$
\begin{aligned}
{}_aD_t^{-p}f(t) &= \int_a^t dt \int_a^t ({}_aD_t^{-p+2}f(t)) \\
&= \int_a^t dt \int_a^t dt \int_a^t ({}_aD_t^{-p+3}f(t))dt \\
&= \underbrace{\int_a^t dt \int_a^t dt \cdots \int_a^t}_{p \text{ times}} f(t)dt.
\end{aligned}
\tag{1.22}
$$

We see that the derivative of an integer order n (1.14) and the p-fold integral (1.16) of the continuous function $f(t)$ are particular cases of the general expression

$$
{}_aD_t^p f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh), \tag{1.23}
$$

which represent the derivative of order m if $p = m$ and the m-fold integral if $p = -m$ [4].

1.2.1.1 Integral of Arbitrary Order

Let us consider the case of $p < 0$. For convenience let us replace p by $-p$ in the expression (1.23). Then (1.23) takes the form

$$
{}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{p} \sum_{r=0}^{n} (-1)^r \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh), \tag{1.24}
$$

where, as above, the values of h and n relate as $nh = t - a$.

To prove the existence of the limit (1.24) and evaluate that limit, we need the following theorem [3].

Theorem 1.1 *Let us take a sequence β_k, $(k = 1, 2, \dots)$ and suppose that*

$$
\lim_{k\to\infty} \beta_k = 1; \tag{1.25}
$$

$$
\lim_{n\to\infty} v_{n,k} = 0 \quad \textit{for all } k, \tag{1.26}
$$

$$
\lim_{n\to\infty} \sum_{k=1}^{n} v_{n,k} = A \quad \textit{for all } k, \tag{1.27}
$$

$$
\sum_{k=1}^{n} |v_{n,k}| < K \quad \textit{for all } n. \tag{1.28}
$$

Then

$$
\lim_{n\to\infty} \sum_{k=1}^{n} v_{n,k}\beta_k = A. \tag{1.29}
$$

1.2.1.2 Derivatives of Arbitrary Order

Let us consider the case of $p > 0$. Our aim is, as above, to evaluate the limit [4]

$$ {}_aD_t^{-p}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) = \lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(p)}(t), \tag{1.30} $$

where

$$ f_h^{(p)}(t) = h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh). \tag{1.31} $$

To evaluate the limit (1.30), let us first transform the expression for $f_h^{(p)}(t)$ in the following way. Using the known property of the binomial coefficient

$$ \binom{p}{r} = \binom{p-1}{r} + \binom{p-1}{r-1}. \tag{1.32} $$

We can write

$$ \begin{aligned} f_h^{(p)}(t) &= h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p-1}{r} f(t-rh) \\ &\quad + h^{-p} \sum_{r=1}^{n} (-1)^r \binom{p-1}{r-1} f(t-rh) \\ &= h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p-1}{r} f(t-rh) \\ &\quad + h^{-p} \sum_{r=0}^{n-1} (-1)^{r+1} \binom{p-1}{r} f(t-(r+1)h) \\ &= (-1)^n \binom{p-1}{n} h^{-p} f(a) \\ &\quad + h^{-p} \sum_{r=0}^{n-1} (-1)^r \binom{p-1}{r} \Delta f(t-rh), \end{aligned} \tag{1.33} $$

where we denote

$$ \Delta f(t-rh) = f(t-rh) - f(t-(r+1)h). $$

Obviously, $\Delta f(t-rh)$ is a first-order backward difference of the function $f(\tau)$ at the point $\tau = t - rh$.

Applying the property (1.32) of the binomial coefficients repeatedly m times, we obtain starting from (1.33):

$$ \begin{aligned} f_h^{(p)}(t) &= (-1)^n \binom{p-1}{n} h^{-p} f(a) + (-1)^{n-1} \binom{p-2}{n-1} h^{-p} \Delta f(a+h) \\ &\quad + h^{-p} \sum_{r=0}^{n-2} (-1)^r \binom{p-2}{r} \Delta^2 f(t-rh) \\ &= (-1)^n \binom{p-1}{n} h^{-p} f(a) + (-1)^{n-1} \binom{p-2}{n-1} h^{-p} \Delta f(a+h) \end{aligned} $$

$$
\begin{aligned}
&+(-1)^{n-2}\binom{p-3}{n-3}h^{-p}\Delta^2 f(a+2h)\\
&+h^{-p}\sum_{r=0}^{n-3}(-1)^r\binom{p-3}{r}\Delta^3 f(t-rh)\\
&=\cdots\\
&=\sum_{r=0}^{m}(-1)^{n-k}\binom{p-k-1}{n-k}h^{-p}\Delta^k f(a+kh)\\
&+h^{-p}\sum_{r=0}^{n-m-1}(-1)^r\binom{p-m-1}{r}\Delta^{m+1}f(t-rh). \qquad (1.34)
\end{aligned}
$$

Let us evaluate the limit of the kth term in the first sum in (1.34):

$$
\begin{aligned}
&\lim_{\substack{h\to 0\\ nh=t-a}}(-1)^{n-k}\binom{p-k-1}{n-k}h^{-p}\Delta^k f(a+kh)\\
&=\lim_{\substack{h\to 0\\ nh=t-a}}(-1)^{n-k}\binom{p-k-1}{n-k}(n-k)^{p-k}\\
&\quad\times\left(\frac{n}{n-k}\right)^{p-k}(nh)^{-p+k}\frac{\Delta^k f(a+kh)}{h^k}\\
&=(t-a)^{-p+k}\lim_{n\to\infty}(-1)^{n-k}\binom{p-k-1}{n-k}(n-k)^{p-k}\\
&\quad\times\lim_{n\to\infty}\left(\frac{n}{n-k}\right)^{p-k}\times\lim_{h\to\infty}\frac{\Delta^k f(a+kh)}{h^k}\\
&=\frac{f^k(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)}.
\end{aligned}
\qquad (1.35)
$$

Using the property of the gamma function limits,
$\lim_{n\to\infty}(-1)^{n-k}\binom{p-k-1}{n-k}(n-k)^{p-k}$

$$
=\lim_{n\to\infty}\frac{(-p+k+1)(-p+k+2)\cdots(-p+n)}{(n-k)^{-p+k}(n-k)!}=\frac{1}{\Gamma(-p+k+1)}
$$

and

$$
\lim_{n\to\infty}\left(\frac{n}{n-k}\right)^{p-k}=1,
$$

$$
\lim_{h\to 0}\frac{\Delta^k f(a+kh)}{h^k}=f^k(a).
$$

Knowing the limit (1.35), we can easily write the limit of the first sum in (1.34). To evaluate the limit of the second sum in (1.34), let us write it in the form

$$\frac{1}{\Gamma(-p+m+1)}\sum_{r=0}^{n-m-1}(-1)^r\Gamma(-p+m+1)\binom{p-m-1}{r}r^{-m+p} \times h(rh)^{m-p}\frac{\Delta^{m+1}f(t-rh)}{h^{m+1}}. \tag{1.36}$$

Using the property of the gamma function limits, we verify that

$$\lim_{r\to\infty} = \lim_{r\to\infty}(-1)^r\Gamma(-p+m+1)\binom{p-m-1}{r}r^{-m+p} = 1. \tag{1.37}$$

In addition, if $m-p>-1$, then

$$\begin{aligned}\lim_{n\to\infty}\sum_{r=0}^{n-m-1} v_{n,r} &= \lim_{\substack{h\to 0\\ nh=t-a}}\sum_{r=0}^{n-m-1} h(rh)^{m-p}\frac{\Delta^{m+1}f(t-rh)}{h^{m+1}}\\ &= \int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau.\end{aligned} \tag{1.38}$$

Taking into account (1.37) and (1.38) and applying Theorem 1.1, we conclude that

$$\begin{aligned}&\lim_{\substack{h\to 0\\ nh=t-a}} h^{-p}\sum_{r=0}^{n-m-1}(-1)^r\binom{p-m-1}{r}\Delta^{m+1}f(t-rh)\\ &\quad= \frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau.\end{aligned} \tag{1.39}$$

Using (1.35) and (1.39), we finally obtain the limit (1.30):

$$\begin{aligned}{}_aD_t^p &= \lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(p)}(t)\\ &= \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)}\\ &\quad+\frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau.\end{aligned} \tag{1.40}$$

The formula (1.40) has been obtained under the assumption that the derivatives $f^{(k)}(t)$, $(k=1,2,\dots,m+1)$ are continuous in the closed interval $[a,t]$ and that m is an integer number satisfying the condition $m>p-1$. The smallest possible value for m is determined by the inequality

$$m<p<m+1.$$

1.2.2 Properties of Riemann–Liouville Fractional Derivative and Integral

Manipulation with the Grünwald–Letnikov fractional derivatives defined as a limit of a fractional-order backward difference is not convenient. The obtained expression (1.40) looks better because of the presence of the integral in it; but what about the nonintegral terms? The answer is simple and elegant: to consider the expression (1.40) as a particular case of the integro-differential expression

$$ {}_aD_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, \quad (m \le p \le m+1). \tag{1.41}$$

The expression (1.41) is the most widely known definition of the fractional derivative; it is usually called the Riemann–Liouville definition.

Obviously, the expression (1.40), which has been obtained for the Grünwald–Letnikov fractional derivative under the assumption that the function $f(t)$ must be $m+1$ times continuously differentiable, can be obtained from (1.41) under the same assumption by performing repeatedly integration by parts and differentiation. This gives

$$\begin{aligned} {}_aD_t^p &= \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\ &\quad + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)d\tau \\ &= {}_aD_t^p f(t), \quad (m \le p \le m+1). \end{aligned} \tag{1.42}$$

Therefore, if we consider a class of functions $f(t)$ having $m+1$ continuous derivatives for $t \ge 0$, then the Grünwald–Letnikov definition (1.30) (or, what is in this case the same, its integral form (1.40)) is equivalent to the Riemann–Liouville definition (1.41).

From the pure mathematical point of view, such a class of functions is narrow, however, this class of functions is very important for applications, because the character of the majority of dynamical processes is smooth enough and does not allow discontinuities. Understanding this fact is important for the proper use of the methods of the fractional calculus in applications, especially because of the fact that the Riemann–Liouville definition (1.41) provides an excellent opportunity to weaken the conditions on the function $f(t)$. Namely, it is enough to require the integrability of $f(t)$; then the integral (1.41) exists for $t > a$ and can be differentiated $m+1$ times. The weak conditions on the function $f(t)$ in (1.41) are necessary, for example, for obtaining the solution of the Abel integral equation.

Let us look at how the Riemann–Liouville definition (1.41) appears as the result of the unification of the notions of integer-order integration and differentiation [4].

1.2.2.1 Unification of Integer-Order Derivatives and Integrals

Let us suppose that the function $f(\tau)$ is continuous and integrable in every finite interval (a, t); the function $f(t)$ may have an integrable singularity of order $r < 1$ at the point $\tau = a$:

$$\lim_{\tau \to a} (\tau - a)^r f(t) = \text{const}(\neq 0). \tag{1.43}$$

Then the integral

$$f^{-1}(t) = \int_a^t f(\tau) d\tau \tag{1.44}$$

exists and has a finite value, namely equal to 0, as $t \to a$. Indeed, performing the substitution $\tau = a + y(t - a)$ and the denoting $\epsilon = t - a$, we obtain

$$\begin{aligned} \lim_{t \to a} f^{(-1)}(t) &= \lim_{t \to a} \int_a^t f(\tau) d\tau \\ &= \lim_{t \to a} (t - a) \int_0^1 f(a + y(t - a)) dy \\ &= \lim_{\epsilon \to 0} \epsilon^{1-r} \int_0^1 (\epsilon y)^r f(a + y\epsilon) y^{-r} dy = 0, \end{aligned} \tag{1.45}$$

because $r < 1$. Therefore, we can consider the twofold integral

$$\begin{aligned} f^{-2}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau) d\tau = \int_a^t f(\tau) d\tau \int_\tau^t d\tau_1 \\ &\int_a^t (t - \tau) f(\tau) d\tau. \end{aligned} \tag{1.46}$$

Integration of (1.46) gives the threefold integral of $f(\tau)$:

$$\begin{aligned} f^{-3}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} f(\tau_3) d\tau_3 \\ &= \int_a^t d\tau_1 \int_a^{\tau_1} (\tau_1 - \tau) f(\tau) d\tau \\ &= \tfrac{1}{2} \int_a^t (t - \tau)^2 f(\tau) d\tau \end{aligned} \tag{1.47}$$

and by induction in the general case, we have the Cauchy formula

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t - \tau)^{n-1} f(\tau) d\tau. \tag{1.48}$$

Let us suppose that $n \geq 1$ is fixed and take integer $k \geq 0$. Obviously, we will obtain

$$f^{(-k-n)}(t) = \frac{1}{\Gamma(n)} D^{-k} \int_a^t (t - \tau)^{n-1} f(\tau) d\tau, \tag{1.49}$$

where the symbol $D^{-k} (k \geq 0)$ denotes k iterated integrations.

On the other hand, for a fixed $n\ geq 1$ and integer $k \geq n$, the $(k - n)$-th derivative of the function $f(t)$ can be written as

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^k \int_a^t (t - \tau)^{n-1} f(\tau) d\tau, \tag{1.50}$$

where the symbol $D^k (k \geq 0)$ denotes k iterated differentiations.

We see that the formulas (1.49) and (1.50) can be considered as particular cases of one them, namely (1.50), in which $n(n \leq 1)$ is fixed and the symbol D^k means k integrations if $k \leq 0$ and k differentiations if $k > 0$. If $k = n-1, n-2, \ldots$, then the formula (1.50) gives iterated integrals of $f(t)$; for $k = n$, it gives the function $f(t)$; for $k = n+1, n+2, n+3, \ldots$, it gives derivatives of order $k - n = 1, 2, 3, \ldots$ of the function $f(t)$ [4].

1.2.2.2 Integrals of Arbitrary Order

To extend the notion of n-fold integration to noninteger values of n, we can start with the Cauchy formula (1.48) and replace the integer n in it by a real $p > 0$:

$$ {}_aD_t^{-p} = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. \tag{1.51}$$

In (1.48), the integer n must satisfy the condition $n \geq 1$; the corresponding for p is weaker: for the existence of the integral (1.51), we must have $p > 0$.

Moreover, under certain reasonable assumptions

$$ \lim_{p\to 0} D_t^{-p} f(t) = f(t), \tag{1.52}$$

so we can put

$$ {}_aD_t^0 f(t) = f(t). \tag{1.53}$$

The proof of the relationship (1.52) is very simple if $f(t)$ has continuous derivatives for $t \leq 0$. In such a case, integration by parts and the use of gamma property, it gives

$$ {}_aD_t^{-p} f(t) = \frac{(t-a)^p}{f}(a)\Gamma(p+1) + \frac{1}{\Gamma(p+1)} \int_a^t (t-\tau)^p f'(\tau) d\tau, $$

and we obtain

$$ \lim_{p\to 0} D_t^{-p} = f(a) + \int_a^t f'(\tau) d\tau = f(a) + (f(t - f(a))) = f(t). $$

If $f(t)$ is only continuous for $t \geq a$, then the proof (1.52) is somewhat longer. In such case, let us write ${}_aD_t^{(-p)} f(t)$ in the form,

$$ \begin{aligned} {}_aD_t^{(-p)} f(t) &= \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} (f(\tau) - f(t)) d\tau + \frac{f(t)}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} d\tau \\ &= \frac{1}{\Gamma(p)} \int_a^{t-\delta} (t-\tau)^{p-1} (f(\tau) - f(t) d\tau \\ &\quad + \frac{1}{\Gamma(p)} \int_{t-\delta}^t (t-\tau)^{p-1} (f(\tau) - f(t) d\tau + \frac{f(t)(t-a)^p}{\Gamma(p+1)}. \end{aligned} \tag{1.54}$$

Let us second part of the integral (1.54). Since $f(t)$ is continuous, for every $\delta > 0$ there exists $\epsilon > 0$ such that

$$|f(\tau) - f(t)| < \epsilon.$$

Then we have following estimate of the second part of the integral (1.54):

$$|I_2| < \frac{\epsilon}{\Gamma(p)} \int_{t-\delta}^{t} (t-\tau)^{p-1} d\tau < \frac{\epsilon \delta^p}{\Gamma(p+1)}, \tag{1.55}$$

and taking into account that $\epsilon \to 0$ and $\delta \to 0$, we obtain that for all $p \geq 0$

$$\lim_{\delta \to 0} |I_2| = 0. \tag{1.56}$$

Let us now take an arbitrary $\epsilon > 0$ and choose δ such that

$$|I_2| < \epsilon \tag{1.57}$$

for all $p \geq 0$. For this fixed δ, we obtain the following estimate first part of the integral (1.54):

$$|I_1| \leq \frac{M}{\Gamma(p)} \int_{a}^{t-\delta} (t-\tau)^{p-1} d\tau \leq \frac{M}{\Gamma(p+1)} (\delta^p - (t-a)^p), \tag{1.58}$$

from which it follows that, for fixed $\delta > 0$

$$\lim_{p \to 0} |I_1| = 0. \tag{1.59}$$

Considering

$$|{}_aD_t^{(-p)} f(t) - f(t)| \leq |I_1| + |I_2| + |f(t)| \times \left| \frac{(t-a)^p}{\Gamma(p+1)} - 1 \right|$$

and taking into account the limits (1.56) the estimate (1.57), we obtain

$$\limsup_{p \to 0} |{}_aD_t^{(-p)} f(t) - f(t)| \leq \epsilon,$$

where ϵ can be chosen as small as we wish. Therefore,

$$\limsup_{p \to 0} |{}_aD_t^{(-p)} f(t) - f(t)| = 0,$$

and (1.52) holds if $f(t)$ is continuous for $t \geq a$.

If $f(t)$ is continuous for $t \geq a$, then integration of arbitrary real order defined by (1.51) has the following important property:

$$D_t^{-p}(D_t^{-q} f(t)) = D_t^{-p-q} f(t). \tag{1.60}$$

Indeed, we have

$$
\begin{aligned}
D_t^{-p}(D_t^{-q}f(t)) &= \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1}D_\tau^{-p}d\tau \\
&= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t (t-\tau)^{q-1}d\tau \int_a^\tau (\tau-\xi)^{p-1}f(\xi)d\xi \\
&= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)d\xi \int_\xi^t (t-\tau)^{q-1}(\tau-\xi)^{p-1}d\tau \\
&= \frac{1}{\Gamma(p+q)}\int_a^t (t-\xi)^{p+q-1}f(\xi)d\xi \\
&= D_t^{-p-q}f(t).
\end{aligned} \tag{1.61}
$$

Obviously, we can interchange p and q, so we have

$$
D_t^{-p}(D_t^{-q}f(t)) = D_t^{-q}(D_t^{-p}f(t)) = D_t^{-p-q}f(t). \tag{1.62}
$$

One may note that the rule (1.62) is similar to the well-known property of integer-order derivatives:

$$
\frac{d^m}{dt^m}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^n}{dt^n}\left(\frac{d^m f(t)}{dt^m}\right) = \frac{d^{m+n}f(t)}{dt^{m+n}}. \tag{1.63}
$$

1.2.2.3 Derivatives of Arbitrary Order

The representation (1.50) for the derivative of an integer order $k-n$ provides an opportunity for extending the notion of differentiation to noninteger order. Namely, we can leave integer k and replace integer n with a real ν so that $k-\nu > 0$. This gives

$$
{}_aD_t^{k-\nu}f(t) = \frac{1}{\Gamma(\nu)}\frac{d^k}{dt^k}\int_{(a)}^t (t-\tau)^{\nu-1}f(\tau)d\tau, \quad (0 < \nu \le 1), \tag{1.64}
$$

where the only substantial restriction for $\nu > 0$, which is necessary for the convergence of the integral in (1.164). This restriction, however, can be without loss of generality, this can be easily shown with the help of the property (1.62) of the integrals of arbitrary real order and the definition (1.164).

Denoting $p = k-\nu$, we can write (1.164) as

$$
{}_aD_t^{p}f(t) = \frac{1}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_{(a)}^t (t-\tau)^{k-p-1}f(\tau)d\tau, \quad (k-1 \le p < k) \tag{1.65}
$$

or

$$
{}_aD_t^{p}f(t) = \frac{d^k}{dt^k}({}_aD_t^{-(k-p)}f(t)), \quad (k-1 \le p < k). \tag{1.66}
$$

If $p = k-1$, then we obtain a conventional integer-order derivative of order $k-1$:

$$
\begin{aligned}
{}_aD_t^{p}f(t) &= \frac{d^k}{dt^k}({}_aD_t^{-(k-(k-1))}f(t)) \\
&= \frac{d^k}{dt^k}({}_aD_t^{-1}f(t)) = f^{(k-1)}(t).
\end{aligned} \tag{1.67}
$$

Moreover, using (1.68) we see that for $p = k \geq 1$ and $t > a$

$$ {}_aD_t^p f(t) = \frac{d^k}{dt^k}({}_aD_t^0 f(t)) = \frac{d^k f(t)}{dt^k} = f^{(k)}(t), \tag{1.68}$$

which means that for $t > a$ the Riemann–Liouville fractional derivative (1.165) of order $p = k > 1$ coincides with the conventional derivative of order k.

Let us now consider some properties of the Riemann–Liouville fractional derivatives. The first and maybe the most important property of the Riemann–Liouville fractional derivative is that for $p > 0$ and $t > a$

$$ {}_aD_t^p({}_aD_t^{-p} f(t)) = f(t), \tag{1.69}$$

which means that the Riemann–Liouville fractional differentiation operator is a left inverse to the Riemann–Liouville fractional integration operator of the some order p.

To prove the property (1.69), let us consider the case of integer $p = n \geq 1$:

$$\begin{aligned} {}_aD_t^n({}_aD_t^{-n} f(t)) &= \frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau \\ &= \frac{d}{dt}\int_a^t f(\tau)d\tau = f(t). \end{aligned} \tag{1.70}$$

Taking now $k - 1 \leq p < k$ and using the composition rule (1.62) for the Riemann–Liouville fractional integrals, we can write

$$ {}_aD_t^{-k} f(t) = {}_aD_t^{-k-p}({}_aD_t^{-p} f(t)). \tag{1.71}$$

Therefore,

$$\begin{aligned} {}_aD_t^p({}_aD_t^{-p} f(t)) &= \frac{d^k}{dt^k}\{{}_aD_t^{-(k-p)}({}_aD_t^{-p} f(t))\} \\ &= \frac{d^k}{dt^k}\{{}_aD_t^{-p} f(t)\} = f(t), \end{aligned} \tag{1.72}$$

which ends the proof of the property (1.71).

As with conventional integer-order differentiation and integration, fractional differentiation and integration do not compute.

If the fractional derivative ${}_aD_t^p f(t)$, $k - 1 \leq p < k$, of a function $f(t)$ is integrable, then

$$ {}_aD_t^p({}_aD_t^{-p} f(t)) = f(t) - \sum_{j=1}^{k} [{}_aD_t^{p-j}]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(p-j+1)}. \tag{1.73}$$

Indeed, on the one hand we have

$$\begin{aligned} {}_aD_t^{-p}({}_aD_t^p f(t)) &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}({}_aD_\tau^p f(\tau))d\tau \\ &= \frac{d}{dt}\left\{\frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^p({}_aD_\tau^p f(\tau))d\tau\right\}. \end{aligned} \tag{1.74}$$

On the other hand, repeatedly integrating by parts and then using (1.72), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^p({}_aD_\tau^p f(\tau))d\tau \\
&= \frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^p \frac{d^k}{d\tau^k}({}_aD_\tau^{-(k-p)} f(\tau))d\tau \\
&= \frac{1}{\Gamma(p-k+1)}\int_a^t (t-\tau)^{p-k}({}_aD_\tau^{-(k-p)} f(\tau))d\tau \\
&\quad - \sum_{j=1}^{k}\left[\frac{d^{k-j}}{dt^{k-j}}{}_aD_t^{-(k-p)} f(t)\right]_{t=a}\frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \\
&= \frac{1}{\Gamma(p-k+1)}\int_a^t (t-\tau)^{p-k}({}_aD_\tau^{-(k-p)} f(\tau))d\tau \\
&\quad - \sum_{j=1}^{k}[{}_aD_t^{p-j} f(t)]_{t=a}\frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \\
&= {}_aD_t^{-(p-k+1)}({}_aD_t^{-(k-p)} f(t)) \\
&\quad - \sum_{j=1}^{k}[{}_aD_t^{p-j} f(t)]_{t=a}\frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \\
&= {}_aD_t^{-1} f(t) - \sum_{j=1}^{k}[{}_aD_t^{p-j} f(t)]_{t=a}\frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)}.
\end{aligned} \tag{1.75}$$

The existence of all terms in (1.75) follows from the integrability of ${}_aD_t^p f(t)$, because due to this condition the fractional derivatives ${}_aD_t^{p-j} f(t), (j = 1, 2, \dots, k)$ are all bounded at $t = a$.

Combining (1.74) and (1.75) ends the proof of the relationship (1.73). An important particular case must be mentioned. If $0 < p < 1$, then

$${}_aD_t^{-p}({}_aD_t^p f(t)) = f(t) - [{}_aD_t^{p-1}]_{t=a}\frac{(t-a)^{p-1}}{\Gamma(p)}. \tag{1.76}$$

The property (1.69) is a particular case of a more general property

$${}_aD_t^p({}_aD_t^{-p} f(t)) = {}_aD_t^{p-q} f(t), \tag{1.77}$$

where we assume that $f(t)$ is continuous and, if $p \geq q \geq 0$, that the derivative ${}_aD_t^{p-q} f(t)$ exists.

Two cases must be considered: $q \geq p \geq 0$ and $p > q \geq 0$.

If $q \geq p \geq 0$, then using the properties (1.69) and (1.77), we obtain

$$\begin{aligned}
{}_aD_t^p({}_aD_t^{-q} f(t)) &= {}_aD_t^p({}_aD_t^{-p}\,{}_aD_t^{-(q-p)} f(t)) \\
&= {}_aD_t^{-(q-p)} f(t) = {}_aD_t^{p-q} f(t).
\end{aligned} \tag{1.78}$$

Now let us consider the case $p > q \geq 0$. Let us denote by m and n integers such that $0 \leq m-1 \leq p < m$ and $0 \leq n \leq p-q < n$. Obviously, $n \leq m$. Then, using the definition (1.165) and the property (1.77), we obtain

$$
\begin{aligned}
{}_aD_t^p({}_aD_t^{-q}f(t)) &= \frac{d^m}{dt^m}\{{}_aD_t^{-(m-p)}({}_aD_t^{-q}f(t))\} \\
&= \frac{d^m}{dt^m}\{{}_aD_t^{p-q-m}f(t)\} \\
&= \frac{d^n}{dt^n}\{{}_aD_t^{p-q-n}f(t)\} = {}_aD_t^{p-q}f(t).
\end{aligned}
\tag{1.79}
$$

The above-mentioned property (1.73) is a particular case of the more

$$
{}_aD_t^{-p}({}_aD_t^{q}f(t)) = {}_aD_t^{q-p}f(t) - \sum_{j=1}^{k}[{}_aD_t^{q-j}]_{t=a}\frac{(t-a)^{p-j}}{\Gamma(p-j+1)}. \tag{1.80}
$$

To prove the formula (1.80), we first use property (1.77) (if $q \leq p$) or property (1.79) (if $q \geq p$) and then property (1.73). This gives

$$
\begin{aligned}
{}_aD_t^{-p}({}_aD_t^{q}f(t)) &= {}_aD_t^{q-p}\{{}_aD_t^{-q}({}_aD_t^{q}f(t))\} \\
&= {}_aD_t^{q-p}\left\{f(t) - \sum_{j=1}^{k}[{}_aD_t^{q-j}f(t)]_{t=a}\frac{(t-a)^{q-j}}{\Gamma(p-j+1)}\right\} \\
&= {}_aD_t^{q-p}f(t) - \sum_{j=1}^{k}[{}_aD_t^{q-j}f(t)]_{t=a}\frac{(t-a)^{p-j}}{\Gamma(p-j+1)},
\end{aligned}
\tag{1.81}
$$

where we used the known derivative of the power function [4].

1.3 Properties of Caputo Fractional Derivative and Integral

The definition (1.165) of the fractional differentiation of the Riemann–Liouville type played an important role in the development of the theory of fractional derivatives and integrals and for its applications in pure mathematics (solution of integer-order differential equations, definitions of new function classes, summation of series, etc.).

However, the demands of modern technology require a certain revision of the well-established pure mathematical approach. There have appeared a number of works, especially in the theory of viscoelasticity and in hereditary solid mechanics, where fractional derivatives are used for a better description of material properties. Mathematical modeling based on enhances rheological models naturally leads to differential equations of fractional order – and to the necessity of the formulation of initial conditions to such equations. Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a), f'(a)$, etc. Unfortunately, the Riemann–Liouville approach leads to initial conditions containing the limit values of the Riemann–Liouville fractional derivatives at the lower terminal $t = a$, for example

$$\begin{aligned}
&\lim_{t\to a}({}_aD_t^{\nu-1}f(t)) = b_1,\\
&\lim_{t\to a}({}_aD_t^{\nu-2}f(t)) = b_2,\\
&\qquad\vdots,\\
&\lim_{t\to a}({}_aD_t^{\nu-n}f(t)) = b_n,
\end{aligned} \tag{1.82}$$

where $b_k, k = 1, 2, \dots, n$ are given constants.

In spite of the fact that initial value problems with such initial conditions can be successfully solved mathematically (see, for example, solutions given in [10] arid in this book), their solutions are practically useless, because there is no known physical interpretation for such types of initial conditions.

Here we observe a conflict between the well-established and polished mathematical theory and practical needs.

A certain solution to this conflict was proposed by M. Caputo first in his paper [5] and two years later in his book [11], and recently (in Banach spaces) by El-Sayed [12, 13]. Caputo's definition can be written as

$${}_a^CD_t^\nu f(t) = \frac{1}{\Gamma(\nu-n)}\int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\nu+1-n}}, \qquad (n-1<\nu<n). \tag{1.83}$$

Under natural conditions on the function $f(t)$, for $\nu \to n$ the Caputo derivative becomes a conventional nth derivative of the function $f(t)$. Indeed, let us assume that $0 \le n-1 < \nu < n$ and that the function $f(t)$ has $n+1$ continuous bounded derivatives in $[a, T]$ for every $T > a$. Then

$$\begin{aligned}
{}_a^CD_t^\nu f(t) &= \lim_{a\to n}\left(\frac{f^n(a)(t-a)^{n-\nu}}{\Gamma(n-\nu+1)}\right.\\
&\qquad\left.+\frac{1}{\Gamma(n-\nu+1)}\int_a^t (t-\tau)^{n-\nu}f^{(n+1)}(\tau)d\tau\right)\\
&= f^n(a) + \int_a^t f^{(n+1)}(\tau)d\tau = f^{(n)}(t), \quad n = 1, 2, \dots
\end{aligned} \tag{1.84}$$

This says that, similarly to the Grünwald–Letnikov and the Riemann–Liouville approaches, the Caputo approach also provides an interpolation between integer-order derivatives.

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations, i.e. contain the limit values of integer-order derivatives of unknown functions at the lower terminal $t = a$.

To underline the difference in the form of the initial conditions which must accompany fractional differential equations in terms of the Riemann–Liouville and the Caputo derivatives, let us recall the corresponding Laplace transform formulas for the case $\nu = 0$.

The formula for the Laplace transform of the Riemann-Liouville fractional derivative is

$$\int_0^\infty \{{}^0D_t^\nu f(t)\}dt = p^\nu F(p) - \sum_{k=0}^{n-1} p^k \{{}^0D_t^{\nu-k-1} f(t)\}|_{t=0}, \quad (n-1 < \nu \le n), \tag{1.85}$$

whereas Caputo's formula, first obtained in [5], for the Laplace transform of the Caputo derivative is

$$\int_0^\infty e^{-pt}\{{}_C^0D_t^\nu f(t)\}dt = p^\nu F(p) - \sum_{k=0}^{n-1} p^{\nu-k-1} f^{(k)}(0), \quad (n-1 < \nu \le n). \tag{1.86}$$

We see that the Laplace transform of the Riemann-Liouville fractional derivative allows utilization of initial conditions of the type (1.97), which can cause problems with their physical interpretation. On the contrary, the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations.

The Laplace transform method is frequently used for solving applied problems. To choose the appropriate Laplace transform formula, it is very important to understand which type of definition of fractional derivative must be used.

Another difference between the Riemann–Liouville definition (1.165) and the Caputo definition (1.83) is that the Caputo derivative of a constant is 0, whereas in the cases of a finite value of the lower terminal a the Riemann–Liouville fractional derivative of a constant C is not equal to 0, but

$${}^0D_t^\nu C = \frac{Ct^{-\nu}}{\Gamma(1-\nu)}. \tag{1.87}$$

This fact led, for example, Ochmann and Makarov [14] to using the Riemann–Liouville definition with $a = -\infty$, because, on the one hand, from the physical point of view they need the fractional derivative of a constant equal to zero and on the other hand, formula (1.87) gives 0 if $a \to -\infty$. The physical meaning of this step is that the starting time of the physical process is set to $-\infty$. In such a case transient effects cannot be studied. However, taking $a = -\infty$ is the necessary abstraction for the consideration of the steady-state processes, for example for studying the response of the fractional-order dynamic system to the periodic input signal, wave propagation in viscoelastic materials, etc.

Putting $a = -\infty$ in both definitions and requiring reasonable behavior of $f(t)$ and its derivatives for $t \to -\infty$, we arrive at the same formula

$${}_{-\infty}D_t^\nu f(t) = {}_{-\infty}^{C} D_t^\nu f(t) = \frac{1}{\Gamma(n-\nu)} \int_{-\infty}^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\nu+1-n}}, \quad (n-1 < \nu < n), \tag{1.88}$$

which shows that for the study of steady-state dynamical processes the Riemann–Liouville definitions and the Caputo definitions must give the same results.

There is also another difference between the Riemann–Liouville and the Caputo approaches, which we would like to mention here and which seems to be important for applications. Namely, for the Caputo derivative, we have

$$ {}_a^C D_t^\nu ({}_a^C D_t^m f(t)) = {}_a^C D_t^{\nu+m} f(t), \quad (m = 0, 1, 2, \dots; \quad n-1 < \nu < n), \tag{1.89} $$

while for the Riemann–Liouville derivative,

$$ {}_a D_t^m ({}_a D_t^\nu f(t)) = {}_a D_t^{\nu+m} f(t), \quad (m = 0, 1, 2, \dots; \quad n-1 < \nu < n). \tag{1.90} $$

The interchange of the differentiation operators in formulas (1.89) and (1.90) is allowed under different conditions:

$$ {}_a^C D_t^\nu ({}_a^C D_t^m f(t)) = {}^C D_t^m ({}_a^C D_t^\nu f(t)) = {}_a^C D_t^{\nu+m} f(t), \; (m = 0, 1, 2, \dots; \; n-1 < \nu < n) \tag{1.91} $$

$$ f^{(s)}(0) = 0, \quad s = n, n+1, \dots, m $$

and

$$ {}_a D_t^m ({}_a D_t^\nu f(t)) = {}_a D_t^\nu ({}_a D_t^m f(t)) = {}_a D_t^{\nu+m} f(t), \; (m = 0, 1, 2, \dots; \; n-1 < \nu < n) \tag{1.92} $$

$$ f^{(s)}(0) = 0, \quad s = 0, 1, 2, \dots, m. $$

We see that contrary to the Riemann–Liouville approach, in the case of the Caputo derivative there are no restrictions on the values $f^{(s)}(0)$, $(s = 0, 1, \dots, n-1)$ [4].

1.4 Properties of the Caputo–Fabrizio Fractional Derivative and Integral

Let us recall the usual Caputo fractional time derivative (UFD$_t$) of order ν, given by [6]

$$ D_t^{(\nu)} f(t) = \frac{1}{\Gamma(1-\nu)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\nu} d\tau \tag{1.93} $$

with $\nu \in [0, 1]$ and $a \in (-\infty, t)$, $f \in H^1(a, b)$, $b > a$. By changing the kernel $(t-\tau)^\nu$ with the function $\exp(-\frac{\nu}{1-\nu} t)$ and $\frac{1}{\Gamma(1-\nu)}$ with $\frac{M(\nu)}{1-\nu}$, we obtain the following new definition of fractional time derivative NFD$_t$

$$ \mathfrak{D}_t^{(\nu)} f(t) = \frac{M(\nu)}{1-\nu} \int_a^t f'(\tau) \exp\left[-\frac{\nu(t-\tau)}{1-\nu}\right] d\tau, \tag{1.94} $$

where $M(\nu)$ is a normalization function such that $M(0) = M(1) = 1$. According to the definition (1.94), the NFD_t is zero when $f(t)$ is constant, as in the UFD_t, but, contrary to the UFD_t, the kernel does not have singularity for $t = \tau$.

The new NFD_t can also be applied to functions that do not belong to $H^1(a, b)$. Indeed, the definition (1.94) can be formulated also for $f \in L^1(-\infty, b)$ and for any $\nu \in [0, 1]$ as

$$\mathfrak{D}_t^{(\nu)} f(t) = \frac{\nu M(\nu)}{1-\nu} \int_{-\infty}^{t} (f(t) - f(\tau)) \exp\left[-\frac{\nu(t-\tau)}{1-\nu}\right] d\tau. \tag{1.95}$$

Now, it is worth to observe that if we put

$$\sigma = \frac{1-\nu}{\nu} \in [0, \infty], \quad \nu = \frac{1}{1+\sigma} \in [0, 1]$$

the definition (1.94) of NFD_t assumes the form

$$\tilde{\mathfrak{D}}_t^{(\sigma)} f(t) = \frac{N(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau, \tag{1.96}$$

where $\sigma \in [0, \infty]$ and $N(\sigma)$ is the corresponding normalization term of $M(\nu)$, such that $N(0) = N(\infty) = 1$. Moreover, because

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau \tag{1.97}$$

and for $\nu \to 1$, we have $\sigma \to 0$. Then,

$$\begin{aligned} \lim_{\nu \to 1} \mathfrak{D}_t^{(\nu)} f(t) &= \lim_{\nu \to 1} \tfrac{M(\nu)}{1-\nu} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{1-\nu}\right] d\tau \\ &= \lim_{\sigma \to 0} \tfrac{N(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = f'(t). \end{aligned} \tag{1.98}$$

Otherwise, when $\nu \to 0$, then $\sigma \to +\infty$. Hence,

$$\begin{aligned} \lim_{\nu \to 0} \mathfrak{D}_t^{(\nu)} f(t) &= \lim_{\nu \to 0} \tfrac{M(\nu)}{1-\nu} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{1-\nu}\right] d\tau \\ &= \lim_{\sigma \to +\infty} \tfrac{N(\sigma)}{\sigma} \int_a^t f'(\tau) \exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = f(t) - f(a). \end{aligned} \tag{1.99}$$

Theorem 1.2 *For NFD_t, if the function $f(t)$ is such that*

$$f^{(s)}(a) = 0, \quad s = 1, 2, \ldots, n$$

then, we have

$$\mathfrak{D}_t^{(n)}(\mathfrak{D}_t^{(\nu)} f(t)) = \mathfrak{D}_t^{(\nu)}(\mathfrak{D}_t^{(n)} f(t)). \tag{1.100}$$

Proof: We begin considering $n = 1$, then from definition (1.101) of $\mathfrak{D}_t^{(\nu+1)} f(t)$, we obtain

$$\mathfrak{D}_t^{(\nu)}(\mathfrak{D}_t^{(1)} f(t)) = \frac{M(\nu)}{1-\nu} \int_a^t f'(\tau) \exp\left[-\frac{\nu(t-\tau)}{1-\nu}\right] d\tau. \tag{1.101}$$

Hence, after an integration by parts and assuming $f'(a) = 0$, we have

$$\begin{aligned}\mathfrak{D}_t^{(\nu)}(\mathfrak{D}_t^{(1)}f(t)) &= \tfrac{M(\nu)}{1-\nu}\int_a^t \left(\frac{d}{d\tau}f'(\tau)\right)\exp\left[-\frac{\nu(t-\tau)}{1-\nu}\right]d\tau \\ &= \tfrac{M(\nu)}{1-\nu}\left[\int_a^t \frac{d}{d\tau}(f'(\tau))\exp\left(-\frac{\nu(t-\tau)}{1-\nu}\right)d\tau\right. \\ &\left.-\tfrac{\nu}{1-\nu}\int_a^t f'(\tau)\exp\left(-\frac{\nu(t-\tau)}{1-\nu}\right)d\tau\right] \\ &= \tfrac{M(\nu)}{1-\nu}\left[f'(t)-\tfrac{\nu}{1-\nu}\int_a^t f'(\tau)\exp\left(-\frac{\nu(t-\tau)}{1-\nu}\right)d\tau\right],\end{aligned} \tag{1.102}$$

otherwise

$$\begin{aligned}\mathfrak{D}_t^{(1)}(\mathfrak{D}_t^{(\nu)}f(t)) &= \tfrac{d}{dt}\left(\tfrac{M(\nu)}{1-\nu}\int_a^t f'(\tau)\exp\left[-\frac{\nu(t-\tau)}{1-\nu}\right]d\tau\right) \\ &= \tfrac{M(\nu)}{1-\nu}\left[f'(t)-\tfrac{\nu}{1-\nu}\int_a^t f'(\tau)\exp\left(-\frac{\nu(t-\tau)}{1-\nu}\right)d\tau\right].\end{aligned} \tag{1.103}$$

It is easy to generalize the proof for any $n > 1$ [6]. □

It is well known that Laplace transform plays an important role in the study of ordinary differential equations. In the case of this new fractional definition, it is also known (see [6]) that, for $0 < \nu < 1$,

$$\mathfrak{L}[{}^{CF}D_t^{\nu}f(t)](s) = \frac{(2-\nu)M(\nu)}{2(s+\nu(1-s))}(s\mathfrak{L}[f(t)](s)-f(0)), \quad s > 0, \tag{1.104}$$

where $\mathfrak{L}[g(t)]$ denotes the Laplace transform of function g. So, it is clear that if we work with Caputo–Fabrizio derivative, Laplace transform will also be a very useful tool [15].

After the notion of fractional derivative of order $0 < \nu < 1$, that of fractional integral of order $0 < \nu < 1$ becomes a natural requirement. In this section, we obtain the fractional integral associated to the Caputo–Fabrizio fractional derivative previously introduced. Let $0 < \nu < 1$. Consider now the following fractional differential equation,

$${}^{CF}D_t^{\nu}f(t) = u(t), \quad t \geq 0 \tag{1.105}$$

Using Laplace transform, we obtain:

$$\mathfrak{L}[{}^{CF}D_t^{\nu}f(t)](s) = \mathfrak{L}[u(t)](s), \quad s > 0. \tag{1.106}$$

That is, using (1.106), we have that

$$\frac{(2-\nu)M(\nu)}{2(s+\nu(1-s))}(s\mathfrak{L}[f(t)](s)-f(0)) = \mathfrak{L}[u(t)](s), \quad s > 0,$$

or equivalently,

$$\mathfrak{L}[f(t)](s) = \frac{1}{s}f(0) + \frac{2\nu}{s(2-\nu)M(\nu)}\mathfrak{L}[u(t)](s) + \frac{2(1-\nu)}{(2-\nu)M(\nu)}\mathfrak{L}[u(t)](s), \ s > 0.$$

Hence, using now well-known properties of inverse Laplace transform, we deduce that

$$f(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}u(t) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t u(s)ds + c, \quad t \geq 0, \tag{1.107}$$

where $c \in R$ is a constant and is also a solution of (1.107).

We can also rewrite fractional differential equation (1.107) as

$$\frac{(2-\nu)M(\nu)}{2(1-\nu)}\int_0^t \exp\left(-\frac{\nu}{1-\nu}(t-s)\right)f'(s)ds = u(t), \quad t \geq 0,$$

or equivalently,

$$\int_0^t \exp\left(\frac{\nu}{1-\nu}s\right)f'(s)ds = \frac{2(1-\nu)}{(2-\nu)M(\nu)}\exp\left(\frac{\nu}{1-\nu}t\right)u(t), \quad t \geq 0.$$

Differentiating both sides of the latter equation, we obtain that,

$$f'(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}\left(u'(t) + \frac{\nu}{1-\nu}u(t)\right), \quad t \geq 0.$$

Hence, integrating now from 0 to t, we deduce as in (1.109), that

$$f(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}[u(t) - u(0)] + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t u(s)ds + f(0), \quad t \geq 0.$$

Thus, as consequence, we expect that the fractional integral of Caputo–Fabrizio type must be defined as follows.

Definition 1.1 Let $0 < \nu < 1$. The fractional integral of order ν of a function f is defined by,

$$^{CF}I^{\nu}f(t) = \frac{2(1-\nu)}{(2-\nu)M(\nu)}u(t) + \frac{2\nu}{(2-\nu)M(\nu)}\int_0^t u(s)ds, \quad t \geq 0. \tag{1.108}$$

Definition 1.2 Let $0 < \nu < 1$. The fractional Caputo–Fabrizio derivative of order ν of a function f is given by,

$$^{CF}D^{\nu}_{\star}f(t) = \frac{1}{1-\nu}\int_0^t \exp\left(-\frac{\nu}{1-\nu}(t-s)\right)f'(s)ds, \quad t \geq 0. \tag{1.109}$$

Lemma 1.1 *Let $0 < \nu < 1$ and f be a solution of the following fractional differential equation,*

$$^{CF}D^{\nu}f(t) = 0, \quad t \geq 0. \tag{1.110}$$

Then, f is a constant function. The converse, as indicated in the Introduction, is also true [15].

Proof: From (1.109), we obtain that the solution of (1.112) must satisfy $f(t) = f(0)$ for all $t \geq 0$. Hence, it is clear that f must be a constant function. □

Proposition 1.1 *Let* $0 < \nu < 1$. *Then, the unique solution of the following initial value problem* [15]

$$^{CF}D^{\nu}f(t) = \sigma(t), \quad t \geq 0, \tag{1.111}$$

$$f(0) = f_0 \in R \tag{1.112}$$

is given by

$$f(t) = f_0 + a_{\nu}(\sigma(t) - \sigma(0)) + b_{\nu} I^1 \sigma(t), \quad t \geq 0, \tag{1.113}$$

where $I^1\sigma$ *denotes a primitive of* σ *and*

$$a_{\nu} = \frac{2(1-\nu)}{(2-\nu)M(\nu)}, \quad b_{\nu} = \frac{2\nu}{(2-\nu)M(\nu)}. \tag{1.114}$$

Proposition 1.2 *Let* $0 < \nu < 1$. *Then, initial value problem given by* [15]

$$^{CF}D^{\nu}f(t) = \lambda f(t) + u(t), \quad t \geq 0,$$

$$f(0) = f_0 \in R$$

has a unique solution for any $\lambda \in R$ [15].

1.5 Properties of the Atangana–Baleanu Fractional Derivative and Integral

We recall that the Mittag-Leffler function is the solution of the following fractional ordinary differential equation [16–18]:

$$\frac{d^{\nu}y}{dx^{\nu}} = ay, \quad 0 < \nu < 1. \tag{1.115}$$

The Mittag-Leffler function and its generalized versions are therefore considered as nonlocal functions. Let us consider the following generalized Mittag-Leffler function:

$$E_{\nu}(-t^{\nu}) = \sum_{k=0}^{\infty} \frac{(-t)^{\nu k}}{\Gamma(\nu k + 1)}. \tag{1.116}$$

The Taylor series of $\exp(-(t-y))$ at the point t is given by:

$$\exp(-a(t-y)) = \sum_{k=0}^{\infty} \frac{(-a(t-y))^k}{k!}. \tag{1.117}$$

If we chose $a = \frac{\nu}{1-\nu}$ and replace the above expression into Caputo–Fabrizio derivative, we conclude that

$$D_t^\nu(f(t)) = \frac{M(\nu)}{1-\nu} \sum_{k=0}^{\infty} \frac{(-a)^k}{k!} \int_b^t \frac{df(y)}{dy}(t-y)^k dy. \tag{1.118}$$

To solve the problem of nonlocality, we derive the following expression.

In Eq. (1.120), we replace $k!$ by $\Gamma(\nu k+1)$ also $(t-y)^k$ is replaced by $(t-y)^{\nu k}$ to obtain:

$$D_t^\nu(f(t)) = \frac{M(\nu)}{1-\nu} \sum_{k=0}^{\infty} \frac{(-a)^k}{\Gamma(\nu k+1)} \int_b^t \frac{df(y)}{dy}(t-y)^{\nu k} dy. \tag{1.119}$$

Thus, the following derivative is proposed.

Definition 1.3 Let $f \in H^1(a,b), b > a, \nu \in [0,1]$ then, the definition of the new fractional derivative is given as:

$${}_b^{ABC}D_t^\nu(f(t)) = \frac{B(\nu)}{1-\nu} \int_b^t f'(x) E_\nu\left[-\nu\frac{(t-x)^\nu}{1-\nu}\right] dx. \tag{1.120}$$

Of course $B(\nu)$ has the same properties as in Caputo and Fabrizio case. The above definition will be helpful to discuss real world problems, and it also will have a great advantage when using the Laplace transform to solve some physical problem with initial condition. However, when ν is 0 we do not recover the original function except when at the origin the function vanishes. To avoid this issue, we propose the following definition.

Definition 1.4 Let $f \in H^1(a,b), b > a, \nu \in [0,1]$ then, the definition of the new fractional derivative is given as:

$${}_b^{ABC}D_t^\nu(f(t)) = \frac{B(\nu)}{1-\nu} \frac{d}{dt} \int_b^t f(x) E_\nu\left[-\nu\frac{(t-x)^\nu}{1-\nu}\right] dx. \tag{1.121}$$

Equations (1.122) and (1.123) have a nonlocal kernel. Also in Eq. (1.122) when the function is constant, we get zero. We now show the relation between both derivatives with Laplace transform. By simple calculation, we conclude that

$$\mathfrak{L}[{}_0^{ABR}D_t^\nu f(t)](s) = \frac{B(\nu)}{1-\nu} \frac{s^\nu \mathfrak{L}\{f(t)\}(s)}{s^\nu + \frac{\nu}{1-\nu}} \tag{1.122}$$

and

$$\mathfrak{L}[{}_0^{ABC}D_t^\nu f(t)](s) = \frac{B(\nu)}{1-\nu} \frac{s^\nu \mathfrak{L}\{f(t)\}(s) - s^{\nu-1}f(0)}{s^\nu + \frac{\nu}{1-\nu}} \tag{1.123}$$

respectively.

The following theorem can therefore be established.

Theorem 1.3 *Let $f \in H^1(a,b), b > a, \nu \in [0,1]$ then, the following relation is obtained*

$$ {}_0^{ABC}D_t^{\nu}(f(t)) = {}_0^{ABR} D_t^{\nu}(f(t)) + H(t). \tag{1.124}$$

Proof: By using the definition (1.126) and the Laplace transform applied on both sides, we obtain easily the following result:

$$\mathfrak{L}[{}_0^{ABC}D_t^{\nu}f(t)](s) = \frac{B(\nu)}{1-\nu}\frac{s^{\nu}\mathfrak{L}\{f(t)\}(s)}{s^{\nu}+\frac{\nu}{1-\nu}} - \frac{s^{\nu-1}f(0)}{s^{\nu}+\frac{\nu}{1-\nu}}\frac{B(\nu)}{1-\nu}. \tag{1.125}$$

Following Eq. (1.124), we have

$$\mathfrak{L}[{}_0^{ABC}D_t^{\nu}f(t)](s) = \mathfrak{L}[{}_0^{ABR}D_t^{\nu}f(t)](s) - \frac{s^{\nu-1}f(0)}{s^{\nu}+\frac{\nu}{1-\nu}}\frac{B(\nu)}{1-\nu}. \tag{1.126}$$

Applying the inverse Laplace on both sides of Eq. (1.128), we obtain

$$\mathfrak{L}[{}_0^{ABC}D_t^{\nu}f(t)] = \mathfrak{L}[{}_0^{ABR}D_t^{\nu}f(t)] - \frac{B(\nu)}{1-\nu}f(0)E_{\nu}\left(-\frac{\nu}{1-\nu}t^{\nu}\right). \tag{1.127}$$

This completes the proof. □

Theorem 1.4 *Let f be a continuous function on a closed interval* $[a,b]$. *Then, the following inequality is obtained on* $[a,b]$

$$||{}_0^{ABC}D_t^{\nu}f(t)|| < \frac{B(\nu)}{1-\nu}K, \quad ||h(t)|| = \max_{a\le t\le b}|h(t)|. \tag{1.128}$$

Proof:

$$\begin{aligned} ||{}_0^{ABR}D_t^{\nu}f(t)|| &= \frac{B(\nu)}{1-\nu}\frac{d}{dt}\int_0^t f(x)E_{\nu}\left[-\nu\frac{(t-x)^{\nu}}{1-\nu}\right]dx \\ &< \frac{B(\nu)}{1-\nu}||\frac{d}{dt}\int_0^t f(x)dx|| = \frac{B(\nu)}{1-\nu}||f(x)||. \end{aligned}$$

Then taking K to be $||f(x)||$ the proof is completed. □

Theorem 1.5 *The A.B. derivative in Riemann and Caputo sense possess the Lipschitz condition, that is to say, for a given couple function f and h, the following inequalities can be established:*

$$||{}_0^{ABR}D_t^{\nu}f(t) - {}_0^{ABR} D_t^{\nu}h(t)|| \le H||f(t)-h(t)|| \tag{1.129}$$

and also

$$||{}_0^{ABC}D_t^{\nu}f(t) - {}_0^{ABC} D_t^{\nu}h(t)|| \le H||f(t)-h(t)||. \tag{1.130}$$

We present the proof of (1.131) *as the proof of* (1.132) *can be obtained similarly.*

Proof:

$$||{}_0^{ABR}D_t^\nu f(t) - {}_0^{ABR} D_t^\nu h(t)|| =$$

$$||\frac{B(\nu)}{1-\nu}\frac{d}{dt}\int_0^t f(x)E_\nu\left[-\nu\frac{(t-x)^\nu}{1-\nu}\right]dx - \frac{B(\nu)}{1-\nu}\frac{d}{dt}\int_0^t h(x)E_\nu\left[-\nu\frac{(t-x)^\nu}{1-\nu}\right]dx||$$

Using the Lipschitz condition of the first-order derivative, we can find a small positive constant such that:

$$||{}_0^{ABR}D_t^\nu f(t) - {}_0^{ABR} D_t^\nu h(t)|| < \frac{B(\nu)\theta_1}{1-\nu}E_\nu\left[-\nu\frac{t^\nu}{1-\nu}\right]||\int_0^t f(x)dx - \int_0^t h(x)dx|| \tag{1.131}$$

and then the following result is obtained:

$$\begin{aligned} ||{}_0^{ABR}D_t^\nu f(t) - {}_0^{ABR} D_t^\nu h(t)|| &< \frac{B(\nu)\theta_1}{1-\nu}E_\nu\left[-\nu\frac{t^\nu}{1-\nu}\right] \\ &||f(x)-h(x)||t = H||f(x)-h(x)||, \end{aligned} \tag{1.132}$$

which produces the requested result.

Let f be an n-times differentiable with natural number and $f^{(k)}(0) = 0,\ k = 1, 2, 3, \ldots, n$, then by inspection we obtain

$${}_0^{ABC}D_t^\nu\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^n}{dt^n}({}_0^{ABR}D_t^\nu f(t)). \tag{1.133}$$

Now, we can easily prove by taking the inverse Laplace transform and using the convolution theorem that the following time fractional ordinary differential equation:

$${}_0^{ABC}D_t^\nu(f(t)) = u(t) \tag{1.134}$$

has a unique solution, namely

$$f(t) = \frac{1-\nu}{B(\nu)}u(t) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t u(y)(t-y)^{\nu-1}dy.$$

□

Definition 1.5 The fractional integral associate to the new fractional derivative with nonlocal kernel is defined as:

$${}_a^{AB}I_t^\nu(f(t)) = \frac{1-\nu}{B(\nu)}u(t) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^t f(y)(t-y)^{\nu-1}dy.$$

When ν is zero, we recover the initial function and if also ν is 1, we obtain the ordinary integral [7].

1.6 Applications

1.6.1 Keller–Segel Model with Caputo Derivative

The best approach in terms of the technique to be employed to research on this topic is to visit the Keller and Segel model depicted in the ground-breaking paper (1970). It predates the formal structure Keller–Segel though it is probably the first edition. They [19, 20] presented the illustration of the aggregation behavior of cellular slime mold which they said is caused by instability. Oldham and Spainer also came up with four species important to the approach in the Keller and Segel [21].

One-dimensional Keller–Segel model is given by

$$\begin{cases} \rho_t = D\rho_{xx} - \chi(\rho a_x)_x, \\ a_t = D_a a_{xx} + h\rho - ka. \end{cases} \tag{1.135}$$

The parameters D, D_a, χ are constants. D and D_a are the diffusion coefficient and a, respectively, h and k are positive constants. The first term in the first equation in (1.135) involves a Laplacian, representing the random spatial motion of the cells. The second term models the chemotactic motion of the cells. In the second equation in (1.135), the first term represent diffusion of the chemoattractant. The second term models the production of the chemoattractant by the cells, and the third term represents linear decay. The initial conditions are $\rho(x, 0) = \rho_0(x)$ and $a(x, 0) = a_0(x)$ for the system. The system (1.135) with Caputo derivative is given as below

$$\begin{cases} {}_0^C D_t^{\nu} \rho = D\rho_{xx} - \chi(\rho a_x)_x, \\ {}_0^C D_t^{\nu} a = D_a a_{xx} + h\rho - ka. \end{cases} \tag{1.136}$$

1.6.1.1 Existence and Uniqueness Solutions

We will give in this chapter the existence and uniqueness of the solutions. We will also present the uniqueness of the positive solutions. Let us present every continuous functions $G = C[a, b]$ in the Banach space defined in the closed set $[a, b]$ and consider $Z = \{\rho, a \in G, \rho(x, t) \geq 0 \quad and \quad a(x, t) \geq 0, a \leq t \leq b\}$.

Definition 1.6 Let X be a Banach space with a cone H. H initiates a restricted order $\leq$ in E in the succeeding approach.

$$y \geq x \Longrightarrow y - x \in H.$$

Now applying the fractional integral in Eq. (1.136), we obtain the following,

$$\begin{cases} \rho(x,t) - \rho(x,0) = \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-r)^{\nu-1}[D\rho(x,t)_{xx} - \chi(\rho(x,t)a(x,t)_x)_x]dr, \\ a(x,t) - a(x,0) = \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-r)^{\nu-1}[D_a a(x,t)_{xx} + h\rho(x,t) - ka(x,t)]dr. \end{cases} \tag{1.137}$$

Now we can use system (1.137) to show the existence of Eq. (1.136). Necessary lemma for the existence of the solutions are given as Lemma 1.2. We now need to define an operator which $T : K \to K$.

$$\begin{cases} T\rho(x,t) = \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-r)^{\nu-1} s(x,r,\rho(x,r))dr, \\ Ta(x,t) = \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-r)^{\nu-1} s(x,r,a(x,r))dr. \end{cases} \tag{1.138}$$

To be dealt with more easily, let us consider below

$$\begin{cases} s(x,r,\rho) = D\rho_{xx} - \chi\rho_x a_{xx}, \\ s(x,r,a) = D_a a_{xx} + h\rho - ka. \end{cases} \tag{1.139}$$

Lemma 1.2 *The mapping $T : K \to K$ is completely continuous.*

Proof: Let $M \subset K$ be bounded. There exists a constants $l, m > 0$ such that $||p|| < l$, $||a|| < m$. Let,

$$L_1 = \max_{\substack{0 \le t \le 1 \\ 0 \le \rho \le l}} s(x,t,\rho(x,t)) \quad \text{and} \quad L_2 = \max_{\substack{0 \le t \le 1 \\ 0 \le a \le m}} s(x,t,a(x,t)),$$

$\forall \rho, a \in M$, we have

$$\begin{aligned} ||T\rho(x,t)|| &\le \frac{1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} ||s(x,r,\rho(x,r))||dr \\ &\le \frac{L_1}{\Gamma(\nu)} \int_0^t (t-r)^{\nu-1} dr \\ &= \frac{L_1}{\Gamma(\nu+1)} t^\nu. \end{aligned} \tag{1.140}$$

So that, we can write as below,

$$||T\rho|| \le \frac{L_1}{\Gamma(\nu+1)}.$$

Similarly,

$$
\begin{aligned}
||Ta(x,t)|| &\leq \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}||s(x,r,a(x,r))||dr \\
&\leq \frac{L_2}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}dr \\
&= \frac{L_2}{\Gamma(\nu+1)}t^{\nu}.
\end{aligned}
\tag{1.141}
$$

So that, we can write as below,

$$
||Ta|| \leq \frac{L_2}{\Gamma(\nu+1)}.
$$

Hence, $T(M)$ is bounded.

Now in the following part, we will consider $t_1 < t_2$ and $\rho(x,t), a(x,t) \in M$ and then for a given $\epsilon > 0$ if $|t_2 - t_1| < \delta$, we have

$$
\begin{aligned}
||T\rho(x,t_2) - T\rho(x,t_1)|| &= \frac{1}{\Gamma(\nu)}\int_0^{t_2} (t_2-r)^{\nu 1}||s(x,r,\rho(x,r))||dr \\
&\quad -\frac{1}{\Gamma(\nu)}\int_0^{t_1} (t_1-r)^{\nu-1}||s(x,r,\rho(x,r))dr|| \\
&= \frac{1}{\Gamma(\nu)}\int_0^{t_2} (t_2-r)^{\nu-1}||s(x,r,\rho(x,r))||dr \\
&\quad -\frac{1}{\Gamma(\nu)}\int_0^{t_2} (t_1-r)^{\nu-1}||s(x,r,\rho(x,r))||dr \\
&\quad -\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2} (t_1-r)^{\nu-1}||s(x,r,\rho(x,r))||dr \\
&\leq \frac{1}{\Gamma(\nu)}\int_0^{t_2} ||(t_2-r)^{\nu-1} - (t_1-r)^{\nu-1}||\ ||s(x,r,\rho(x,r))||dr \\
&\quad +\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_2} ||(t_1-r)^{\nu-1}||\ ||s(x,r,\rho(x,r))||dr \\
&\leq \frac{L_1}{\Gamma(\nu)}\int_0^{t_2} ((t_2-r)^{\nu-1} - (t_1-r)^{\nu-1})dr + \frac{L_1}{\Gamma(\nu)}\int_{t_1}^{t_2} (t_1-r)^{\nu-1}dr \\
&= \frac{L_1}{\Gamma(\nu)}\left(\int_0^{t_2} (t_2-r)^{\nu-1}dr - \int_0^{t_2} (t_1-r)^{\nu-1}dr + \int_{t_1}^{t_2} (t_1-r)^{\nu-1}dr\right) \\
&= \frac{L_1}{\Gamma(1+\nu)}(t_2^{\nu} + (t_1-t_2)^{\nu} - t_1^{\nu} + (t_1-t_2)^{\nu}) \\
&\leq \frac{2L_1}{\Gamma(1+\nu)}(t_1-t_2)^{\nu} + \frac{L_1}{\Gamma(1+\nu)}(t_1-t_1)^{\nu} \\
&= \frac{2L_1}{\Gamma(1+\nu)}(t_1-t_2)^{\nu}
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{2L_1}{\Gamma(1+\nu)}\delta^{\nu} \\
&= \epsilon. \qquad (1.142)
\end{aligned}
$$

It is clearly seen that, when the same steps are applied to the $a(x,t)$ function, we get same situation. Finally, $|T\rho(x,t_2) - T\rho(x,t_1)| \leq \epsilon$ and $|Ta(x,t_2) - Ta(x,t_1)| \leq \epsilon$ are satisfied, where $\delta = (\epsilon\Gamma(1+\nu/2L))^{1/\nu}$. Therefore, $T(M)$ is equicontinuous. So that $\overline{T(M)}$ is compact via the Arzela–Ascoli theorem. □

Theorem 1.6 *Let $S : [\rho_1, \rho_2] \times [0,\infty) \to [0,\infty)$, then $S(x,t)$ is nondecreasing for each t in $[\rho_1,\rho_2]$. There exists a positive constants ν_1 and ν_2 such that $B(n)\nu_1 \leq S(x,t,\nu_1)$, $B(n)\nu_2 \geq S(x,t,\nu_2)$, $0 \leq \nu_1(x,t) \leq \nu_2(x,t)$, $\rho_1 \leq t \leq \rho_2$. This means that the new equation has a positive solution.*

Proof: We only need to consider the fixed point for operator of T. With framework of Lemma 1.2, the considered operator $T : H \to H$ is completely continuous. Let us take two arbitrary ρ_1 and ρ_2,

$$
\begin{aligned}
T\rho_1(x,t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}s(x,r,\rho_1(x,r))dr \\
&\leq \frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}s(x,r,\rho_2(x,r))dr \qquad (1.143) \\
&= Tp_2(x,t).
\end{aligned}
$$

Hence T is a nondecreasing operator. So that the operator $T : \langle \nu_1, \nu_2\rangle \to \langle \nu_1,\nu_2\rangle$ is compact and continuous via Lemma 1.2. In that case, H is a normal cone of T. □

1.6.1.2 Uniqueness of Solution

The aim of this section is to prove the uniqueness of solutions to the system (1.136). So the uniqueness of the solution is presented as below,

$$
\begin{aligned}
||T\rho_1(x,t) - T\rho_2(x,t)|| &= ||\frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}(s(x,r,\rho_1(x,r)) \\
&\quad -s(x,r,\rho_2(x,r)))dr|| \\
&\leq \frac{1}{\Gamma(\nu)}C_1\int_0^t (t-r)^{\nu-1}||\rho_1(x,r)-\rho_2(x,r)||dr.
\end{aligned}
\qquad (1.144)
$$

So that,

$$
||T\rho_1(x,t) - T\rho_2(x,t)|| \leq \left\{\frac{C_1 t^{\nu}}{\Gamma(\nu+1)}\right\} ||\rho_1(x,r) - \rho_2(x,r)||.
$$

Similarly,

$$
\begin{aligned}
||Ta_1(x,t) - Ta_2(x,t)|| &= ||\frac{1}{\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}(s(x,r,a_1(x,r)) \\
&\quad -s(x,r,a_2(x,r)))dr|| \\
&\leq \frac{1}{\Gamma(\nu)}C_2\int_0^t (t-r)^{\nu-1}||a_1(x,r)-a_2(x,r)||dr.
\end{aligned}
\tag{1.145}
$$

So that,

$$
||Ta_1(x,t) - Ta_2(x,t)|| \leq \left\{\frac{C_2 t^\nu}{\Gamma(\nu+1)}\right\}||a_1(x,r)-a_2(x,r)||.
$$

Therefore, if the following conditions hold,

$$
\left\{\frac{C_1 t^\nu}{\Gamma(\nu+1)}\right\} < 1 \text{ and } \left\{\frac{C_2 t^\nu}{\Gamma(\nu+1)}\right\} < 1.
$$

Then mapping T is a contraction, which implies fixed point, and thus the model has a unique positive solution.

1.6.1.3 Keller–Segel Model with Atangana–Baleanu Derivative in Caputo Sense

We present in this section the existence and uniqueness of solutions of the Keller–Segel model using the Atangana–Baleanu derivative. Let $\Omega = (a,b)$ be an open and bounded subset of R^n. Let $\nu \in (0,1)$ and functions $\rho(x,t), a(x,t) \in H^1(\Omega) \times [0,T]$. Here $\rho(x,t)$ represent the concentration of the chemical substance and the function $a(x,t)$ represent concentration of amoebae. We apply the system (1.135) to the Atangana–Baleanu fractional derivative,

$$
\begin{cases}
{}_0^{ABC}D_t^{\nu}\rho = \sigma_1(x,t,\rho), \\
{}_0^{ABC}D_t^{\nu}a = \sigma_2(x,t,a),
\end{cases}
\tag{1.146}
$$

where

$$
\begin{cases}
\sigma_1(x,t,\rho) = D\rho_{xx} - \chi\rho_x a_{xx}, \\
\sigma_2(x,t,a) = D_a a_{xx} + h\rho - ka.
\end{cases}
\tag{1.147}
$$

Using the Atangana–Baleanu integral to (1.146), it yields

$$
\begin{cases}
\rho(x,t) = \rho(x,0) + \dfrac{1-\nu}{B(\nu)}\sigma_1(x,t,\rho(x,t)) \\
\quad +\dfrac{\nu}{B(\nu)\Gamma(\nu)}\displaystyle\int_0^t \sigma_1(x,r,\rho(x,r))(t-r)^{\nu-1}dr, \\
a(x,t) = a(x,0) + \dfrac{1-\nu}{B(\nu)}\sigma_2(x,t,a(x,t)) \\
\quad +\dfrac{\nu}{B(\nu)\Gamma(\nu)}\displaystyle\int_0^t \sigma_2(x,r,a(x,r))(t-r)^{\nu-1}dr,
\end{cases}
\tag{1.148}
$$

for all $t \in [0,T]$.

Theorem 1.7 *If the inequality* (1.149) *hold,* σ_1 *and* σ_2 *satisfy Lipschitz condition and contraction.*

$$0 < D\gamma_1^2 + \chi\gamma_2 || \frac{\partial^2 a(x,y)}{\partial x^2} || \leq 1. \tag{1.149}$$

Proof: We would like to start with the kernel σ_1.Let κ_1 and κ_2 are two functions, the following equation is written as:

$$\begin{aligned} &||\sigma_1(x,t,\kappa_1) - \sigma_1(x,t,\kappa_2)|| \\ &= ||D(\kappa_1(x,t)_{xx} - \kappa_2(x,t)_{xx}) - \chi(\kappa_1(x,t)_x - \kappa_2(x,t)_x)\frac{\partial^2 a(x,t)}{\partial x^2}||. \end{aligned}$$

When we convert the above equation via triangular inequality, we get

$$\begin{aligned} &||\sigma_1(x,t,\kappa_1) - \sigma_1(x,t,\kappa_2)|| \\ &\leq D||(\kappa_1(x,t)_{xx} - \kappa_2(x,t)_{xx})|| + \chi|| - (\kappa_1(x,t)_x - \kappa_2(x,t)_x)\frac{\partial^2 a(x,t)}{\partial x^2}||. \end{aligned}$$

Using the operator derivative, we can find two constants such as γ_1 and γ_2 :

$$\begin{cases} D||(\kappa_1(x,t)_{xx} - \kappa_2(x,t)_{xx})|| \leq D\gamma_1^2||\kappa_1(x,t) - \kappa_2(x,t)|| \\ \chi|| - (\kappa_1(x,t)_x - \kappa_2(x,t)_x)\frac{\partial^2 a(x,t)}{\partial x^2}|| \leq \chi\gamma_2||\frac{\partial^2 a(x,t)}{\partial x^2}|| \\ \quad ||(\kappa_1(x,t) - \kappa_2(x,t)||. \end{cases} \tag{1.150}$$

When we substitute Eq. (1.150) in below equation, we get:

$$||\sigma_1(x,t,\kappa_1) - \sigma_1(x,t,\kappa_2)|| \leq K||(\kappa_1(x,t) - \kappa_2(x,t)||, \tag{1.151}$$

where

$$K = \left(D\gamma_1^2 + \chi\gamma_2||\frac{\partial^2 a(x,t)}{\partial x^2}||\right).$$

Therefore, σ_1 satisfies the Lipschitz condition. Then we can say that it is a contraction. In the another case, the following inequality can be written because our kernel is linear,

$$\sigma_2(x,t,\nu_1) - \sigma_2(x,t,\nu_2) \leq (c\vartheta_1^2 + d)||\nu_1(x,t) - \nu_2(x,t)||.$$

Hence, the proof is complete. We can now show that the uniqueness of the solution. □

1.6.1.4 Uniqueness of Solution

The uniqueness solution for system (1.146) is presented as below. Let $\rho_1, \rho_2 \in H^1$ be two solutions of (1.146). Let $\rho = \rho_1 - \rho_2$, the following equation can be written as,

$$\rho = \frac{1-\nu}{B(\nu)}(\sigma_1(x,t,\rho_1(x,t)) - \sigma_1(x,t,\rho_2(x,t))) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (\sigma_1(x,r,\rho_1(x,r)) - \sigma_1(x,r,\rho_2(x,r)))dr.$$

If the norms of both sides are taken, by the Gronwall inequality (1.146),

$$\begin{aligned}||\rho|| &\le \frac{1-\nu}{B(\nu)}||\sigma_1(x,t,\rho_1(x,t)) - \sigma_1(x,t,\rho_2(x,t))|| \\ &+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t ||\sigma_1(x,r,\rho_1(x,r)) - \sigma_1(x,r,\rho_2(x,r))||dr \\ &\le K_1\int_0^t ||\sigma_1(x,t,\rho_1(x,t))||_{H^1}dr.\end{aligned}$$

Similarly, let $a_1, a_2 \in H^1$ be two solutions of (1.146). Let $a = a_1 - a_2$, the following equation can be written as,

$$\begin{aligned}||a|| &\le \frac{1-\nu}{B(\nu)}||\sigma_2(x,t,a_1(x,t)) - \sigma_2(x,t,a_2(x,t))|| \\ &+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t ||\sigma_2(x,r,a_1(x,r)) - \sigma_2(x,r,a_2(x,r))||dr \\ &\le K_2\int_0^t ||\sigma_2(x,t,a_1(x,t))||_{H^1}dr.\end{aligned} \tag{1.152}$$

Finally, the system (1.146) has a unique solution for the equations ρ and a.

1.6.2 Cancer Treatment Model with Caputo-Fabrizio Fractional Derivative

It is well known that cancer is one of the most common diseases causing deaths in the last century. Emerging in various parts of the human body, this disease becomes unresponsive to the treatment when it is intervened late. There is a great deal of research done for the treatment of cancer, which is the disease of the century. In this chapter, we are going to examine the existence and uniqueness of the cancer treatment model. By developing models for the treatment of cancer, it is aimed to contribute to the studies in this field.

It is assumed that healthy and cancer cells be located in the same area of the organism. Let $\rho(t)$ denotes the concentration of healthy cells, and $a(t)$ denotes the concentration of cancer cells. Then the model is given by [22]:

$$\begin{cases} \dfrac{d\rho(t)}{dt} = \alpha_1\rho\left(1 - \dfrac{\rho}{S_1}\right) - \beta_1\rho a - \epsilon D(t)\rho, \\ \dfrac{da(t)}{dt} = \alpha_2 a\left(1 - \dfrac{a}{S_2}\right) - \beta_2 a\rho - D(t)a, \end{cases} \tag{1.153}$$

where $D(t)$ is the strategy of the radiotherapy. It is supposed that $D(t) \equiv \gamma > 0$ when $t \in [nw, nw + L)$(treatment stage) and $D(t) \equiv 0$ when $t \in [nw + L, (n+1)w$ (no treatment stage) for all $n = 0, 1, 2, \dots$, where w is the radiation treatment time [22]. The system (1.153) with Caputo derivative is given as below,

$$\begin{cases} {}_0^{CF}D_t^{\,\nu}(\rho(t)) = \alpha_1 \rho \left(1 - \frac{\rho}{S_1}\right) - \beta_1 \rho a - \epsilon D(t)\rho, \\ {}_0^{CF}D_t^{\,\nu}(a(t)) = \alpha_2 a \left(1 - \frac{a}{S_2}\right) - \beta_2 a \rho - D(t) a. \end{cases} \tag{1.154}$$

1.6.2.1 Existence Solutions

We will give in this section the existence of the solutions for the cancer treatment model by radiotherapy. After that, we also will present the uniqueness of the positive solutions.

Now applying the fractional integral in Eq. (1.153), we obtain the following,

$$\begin{cases} \rho(t) - \rho_0(t) = \frac{2(1-\nu)}{2M(\nu) - \nu M(\nu)} \left[\alpha_1 \rho \left(1 - \frac{\rho}{S_1}\right) - \beta_1 \rho a - \epsilon D(t)\rho\right] \\ \quad + \frac{2\nu}{2M(\nu) - \nu M(\nu)} \int_0^t \left[\alpha_1 \rho \left(1 - \frac{\rho}{S_1}\right) - \beta_1 \rho a - \epsilon D(y)\rho\right] dy, \\ a(t) - a_0(t) = \frac{2(1-\nu)}{2M(\nu) - \nu M(\nu)} \left[\alpha_2 a \left(1 - \frac{a}{S_2}\right) - \beta_2 a \rho - D(t) a\right] \\ + \frac{2\nu}{2M(\nu) - \nu M(\nu)} \int_0^t \left[\alpha_2 a \left(1 - \frac{a}{S_2}\right) - \beta_2 a \rho - D(y) a\right] dy. \end{cases} \tag{1.155}$$

For simplicity, we choose our kernels as $s(t, \rho(t))$ and $s(t, a(t))$ as follows:

$$s(t, \rho(t)) = \alpha_1 \rho \left(1 - \frac{\rho}{S_1}\right) - \beta_1 \rho a - \epsilon D(t)\rho,$$

$$s(t, a(t)) = \alpha_2 a \left(1 - \frac{a}{S_2}\right) - \beta_2 a \rho - D(t) a.$$

First we need to be able to identify an operator. We will then show that this operator is compact. So that the operator which $T : H \to H$. Then we get,

$$\begin{cases} T\rho(t) = \frac{2(1-\nu)}{2M(\nu) - \nu M(\nu)} s(t, \rho(t)) + \frac{2\nu}{2M(\nu) - \nu M(\nu)} \int_0^t s(y, \rho(y)) dy, \\ Ta(t) = \frac{2(1-\nu)}{2M(\nu) - \nu M(\nu)} s(a, a(t)) + \frac{2\nu}{2M(\nu) - \nu M(\nu)} \int_0^t s(a, a(y)) dy. \end{cases} \tag{1.156}$$

Lemma 1.3 *The mapping $T : H \to H$ is completely continuous.*

Proof: Let $M \subset H$ be bounded. There exists a constants $l, m > 0$ such that $||\rho|| < l$ and $||a|| < m$. Let

$$L_1 = \max_{\substack{0 \le t \le 1 \\ 0 \le \rho \le l}} s(t, \rho(t)) \quad \text{and} \quad L_2 = \max_{\substack{0 \le t \le 1 \\ 0 \le a \le m}} s(t, a(t)),$$

$\forall \rho, a \in M$, we have

$$\begin{aligned}
|T\rho(t)| &= \left|\frac{2-2\nu}{2M(\nu)-\nu M(\nu)} s(t,\rho(t)) + \frac{2\nu}{2M(\nu)-\nu M(\nu)} \int_0^t s(y,\rho(y))dy\right| \\
&\le \left|\frac{2-2\nu}{2M(\nu)-\nu M(\nu)}\right| |s(t,\rho(t)) + |\frac{2\nu}{2M(\nu)-\nu M(\nu)}\right| \left|\int_0^t s(y,\rho(y))dy\right| \\
&\le \left[\frac{2-2\nu}{2M(\nu)-\nu M(\nu)} + \frac{2\nu}{2M(\nu)-\nu M(\nu)} c_1\right] |s(t,\rho(t))| \\
&\le \left[\frac{2-2\nu}{2M(\nu)-\nu M(\nu)} + \frac{2\nu}{2M(\nu)-\nu M(\nu)} c_1\right] |L_1 \\
||T\rho|| &\le \frac{2L_1}{2M(\nu)-\nu M(\nu)} [1-\nu+\nu c_1].
\end{aligned} \tag{1.157}$$

Similarly,

$$\begin{aligned}
|Ta(t)| &= \left|\frac{2-2\nu}{2M(\nu)-\nu M(\nu)} s(t,a(t)) + \frac{2\nu}{2M(\nu)-\nu M(\nu)} \int_0^t s(y,a(y))dy\right| \\
&\le \frac{2-2\nu}{2M(\nu)-\nu M(\nu)} |s(t,a(t)) + |\frac{2\nu}{2M(\nu)-\nu M(\nu)} \left|\int_0^t s(y,a(y))dy\right| \\
&\le \left[\frac{2-2\nu}{2M(\nu)-\nu M(\nu)} + \frac{2\nu}{2M(\nu)-\nu M(\nu)} c_2\right] |s(t,a(t))| \\
&\le \left[\frac{2-2\nu}{2M(\nu)-\nu M(\nu)} + \frac{2\nu}{2M(\nu)-\nu M(\nu)} c_2\right] |L_2 \\
||Ta|| &\le \frac{2L_2}{2M(\nu)-\nu M(\nu)} [1-\nu+\nu c_2].
\end{aligned} \tag{1.158}$$

Hence, $T(M)$ is bounded.

Now in the following part, we will consider $t_1 < t_2$ and $\rho(t), a(t) \in M$ and then for a given $\epsilon > 0$ if $|t_2 - t_1| < \delta$, we have

$$\begin{aligned}
||T\rho(t_2) - T\rho(t_1)|| \le &\left|\frac{2-2\nu}{2M(\nu)-\nu M(\nu)} (s(t_2,\rho(t_2)) - s(t_1,\rho(t_1)))\right| \\
&+ \left|\frac{2\nu}{2M(\nu)-\nu M(\nu)} \int_0^{t_2} s(y,\rho(y))dy\right.
\end{aligned}$$

$$-\frac{2\nu}{2M(\nu)-\nu M(\nu)}\int_0^{t_1} s(y,\rho(y))dy\Bigg|$$
$$\leq \frac{2-2\nu}{2M(\nu)-\nu M(\nu)}|s(t_2,\rho(t_2))-s(t_1,\rho(t_1))|$$
$$+\frac{2\nu}{2M(\nu)-\nu M(\nu)}L_1|s(t_2,\rho(t_2))-s(t_1,\rho(t_1))|. \tag{1.159}$$

Now we will investigate the following,

$$\begin{aligned}|s(t_2,\rho(t_2))-s(t_1,\rho(t_1))| &\leq \Bigg|\alpha_1(\rho(t_2)-\rho(t_1))\left(1-\frac{\rho(t_2)-\rho(t_1)}{S_1}\right)\\ &\quad -\beta_1(\rho(t_2)-\rho(t_1)a-\epsilon\gamma(\rho(t_2)-\rho(t_1)|\\ &\leq c_3|(\rho(t_2)-\rho(t_1)|-c_4|(\rho(t_2)-\rho(t_1)|\\ &\quad -c_5|(\rho(t_2)-\rho(t_1)|\\ &\leq (c_3-c_4-c_5)|(\rho(t_2)-\rho(t_1)|\\ &\leq C|t_2-t_1|.\end{aligned} \tag{1.160}$$

Now putting Eq. (1.159) and the integral part of Eq. (1.158) in Eq. (1.158), we get,

$$|T\rho(t_2)-T\rho(t_1)| \leq \frac{2\nu}{2M(\nu)-\nu M(\nu)}C|t_2-t_1|+\frac{2(1-\nu)}{2M(\nu)-\nu M(\nu)}L_1|t_2-t_1|, \tag{1.161}$$

$$\delta = \frac{\epsilon}{\frac{2\nu}{2M(\nu)-\nu M(\nu)}C+\frac{2(1-\nu)}{2M(\nu)-\nu M(\nu)}L_1}. \tag{1.162}$$

Such that $|T\rho(t_2)-T\rho(t_1)| \leq \epsilon$.

When we can acquire the following for the function a with same rules, we get

$$\delta = \frac{\epsilon}{\frac{2\nu}{2M(\nu)-\nu M(\nu)}G+\frac{2(1-\nu)}{2M(\nu)-\nu M(\nu)}L_2}. \tag{1.163}$$

Such that $|Ta(t_2)-Ta(t_1)| \leq \epsilon$ are satisfied. Therefore $T(M)$ is equicontinuous. So that $\overline{T(M)}$ is compact via the Arzela–Ascoli theorem. □

Theorem 1.8 *Let $N : [\rho_1,\rho_2]\times[0,\infty)\to[0,\infty)$, then $N(t,\cdot)$ is nondecreasing for each t in $[\rho_1,\rho_2]$. There exists positive constants v_1 and v_2. So that $B(n)v_1 \leq S(t,v_1)$, $B(n)v_2 \geq N(t,v_2)$, $0 \leq v_1(t) \leq v_2(t)$, $\rho_1 \leq t \leq \rho_2$. Thus, the equation has a positive solution.*

Proof: We only need to consider the fixed point for operator of T. It is considered that $T : H \to H$ is completely continuous. Let $\rho_1 \leq \rho_2$ and $a_1 \leq a_2$. Here four variables are arbitrary.

$$\begin{aligned}T\rho_1(t) = &\ \frac{2-2v}{2M(v)-vM(v)}|s(t,\rho_1(t))| + \frac{2v}{2M(v)-vM(v)}\int_0^t s(y,\rho_1(y))dy\\ &\le \frac{2-2v}{2M(v)-vM(v)}|s(t,\rho_2(t))| + \frac{2v}{2M(v)-vM(v)}\int_0^t s(y,\rho_2(y))dy\\ &\le T\rho_2(x,t)\end{aligned} \tag{1.164}$$

and

$$\begin{aligned}Ta_1(t) = &\ \frac{2-2v}{2M(v)-vM(v)}|s(t,a_1(t))| + \frac{2v}{2M(v)-vM(v)}\int_0^t s(y,a_1(y))dy\\ &\le \frac{2-2v}{2M(v)-vM(v)}|s(t,a_2(t))| + \frac{2v}{2M(v)-vM(v)}\int_0^t s(y,a_2(y))dy\\ &\le Ta_2(x,t).\end{aligned} \tag{1.165}$$

Hence, T is a nondecreasing operator. So that the operator $T : \langle v_1, v_2\rangle \rightarrow \langle v_1, v_2\rangle$ is compact and continuous via Lemma 1.2. In that case, H is a normal cone of T. □

1.6.2.2 Uniqueness Solutions

In the previous section, we proved using the fixed point theorem, that the coupled cancer treatment model with Caputo–Fabrizio time fractional derivative has an existing solution. The goal of this section is to show the uniqueness of solutions to the system (1.152) with the initial conditions. Let assume in addition that, we can find two special coupled solutions (ρ_1, ρ_2) and (a_1, a_2). So the uniqueness of the solution is presented as:

$$\begin{aligned}|T\rho_1(t) - T\rho_2(t)| &= \left|\frac{2-2v}{2M(v)-vM(v)}\right|(s(t,\rho_1(t) - s(t,\rho_2(t))\\ &\quad + \frac{2v}{2M(v)-vM(v)}\int_0^t (s(y,\rho_1(y)) - s(y,\rho_2(y)))dy|\\ &\le \frac{2-2v}{2M(v)-vM(v)}|(s(t,\rho_1(t) - s(t,\rho_2(t))|\\ &\quad + \frac{2v}{2M(v)-vM(v)}\int_0^t |(s(y,\rho_1(y)) - s(y,\rho_2(y)))dy|\\ &\le \frac{2-2v}{2M(v)-vM(v)}F_1|\rho_1(t) - \rho_2(t)|\\ &\quad + \frac{2v}{2M(v)-vM(v)}F_1|\rho_1(t) - \rho_2(t)|.\end{aligned} \tag{1.166}$$

So that we can write the above Eq. (165),

$$|T\rho_1(t) - T\rho_2(t)| \leq \left\{ \frac{2-2\nu}{2M(\nu)-\nu M(\nu)}F_1 + \frac{2\nu}{2M(\nu)-\nu M(\nu)}F_1 \right\} |\rho_1(t) - \rho_2(t)|.$$

Similarly,

$$\begin{aligned} |Ta_1(t) - Ta_2(t)| &= |\frac{2-2\nu}{2M(\nu)-\nu M(\nu)}|(s(t,\rho_1(t) - s(t,\rho_2(t))) \\ &\quad + \frac{2\nu}{2M(\nu)-\nu M(\nu)} \int_0^t (s(y,\rho_1(y)) - s(y,\rho_2(y)))dy| \\ &\leq \frac{2-2\nu}{2M(\nu)-\nu M(\nu)}F_2|a_1(t) - a_2(t)| \\ &\quad + \frac{2\nu}{2M(\nu)-\nu M(\nu)}F_2|a_1(t) - a_2(t)|. \end{aligned} \tag{1.167}$$

So that we can write the above Eq. (166),

$$||Ta_1(t) - Ta_2(t)|| \leq \left\{ \frac{2-2\nu}{2M(\nu)-\nu M(\nu)}F_2 + \frac{2\nu}{2M(\nu)-\nu M(\nu)}F_2 \right\} |a_1(t) - a_2(t)|.$$

Therefore, if the following conditions hold,

$$\left\{ \frac{2-2\nu}{2M(\nu)-\nu M(\nu)}F_1 + \frac{2\nu}{2M(\nu)-\nu M(\nu)}F_1 \right\} < 1 \text{ and}$$

$$\left\{ \frac{2-2\nu}{2M(\nu)-\nu M(\nu)}F_2 + \frac{2\nu}{2M(\nu)-\nu M(\nu)}F_2 \right\} < 1.$$

Then mapping T is a contraction. We can say that the model has a unique positive solution using fixed point theorem.

1.6.2.3 Conclusion

We first integrated this cancer treatment model with the new fractional derivative. After that, we found the existence solution of the cancer treatment model. Finally, we analyzed how the model is the uniqueness positive solution under which conditions. We tried to help the researcher working on cancer education with this work. When the results are examined, it has been shown that the fractional derivative gives important information about the process.

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