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Introduction

1.1 Historical Setting

The mathematical theory of fluid flows in porous media has a distinguished history. Most of this theory ultimately rests on Henry Darcy's 1856 engineering study [43], summarized in Section 3.1, of the water supplies in Dijon, France. A year after the publication of this meticulous and seminal work, Jules Dupuit [49], a giant among early groundwater scientists, recognized that Darcy's findings implied a differential equation. This observation proved to be crucial. For the next 75 years or so, the subject grew to encompass problems in multiple space dimensions—hence **partial differential equations** (**PDEs**)—with major contributions emerging mainly from the groundwater hydrology community. Pioneers included Joseph Boussinesq [25, 26], Philipp Forchheimer [53, 54], Charles S. Slichter [136], Edgar Buckingham [30], and Lorenzo A. Richards [129].

Interest in the mathematics of porous-medium flows blossomed as oil production increased in economic importance during the early twentieth century. Prominent in the early petroleum engineering literature in this area are works by P.G. Nutting [110], Morris Muskat and his collaborators [104–107, 159, 160], and Miles C. Leverett and his collaborators [29, 95–97]. Between 1930 and 1960, mathematicians, groundwater hydrologists, petroleum engineers, and geoscientists made tremendous progress in understanding the PDEs that govern underground fluid flows.

Today, mathematical models of porous-medium flow encompass linear and nonlinear PDEs of all major types, as well as systems involving PDEs having different types. The analysis of these equations and their numerical approximations requires an increasing level of mathematical and computational sophistication, and the models themselves have become essential design tools in the management of underground fluid resources.

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From a philosophical perspective, credit for these advances belongs to scientists and engineers who clung tenaciously—often in the face of skepticism on the part of more "practically" oriented colleagues—to two premises. The first is that the key to effective modeling resides in careful mathematical reasoning. While this premise seems platitudinous, at any moment in history some practitioners believe that their science is too inherently messy to justify fastidious mathematics. On the contrary, the need for painstaking logical inferences from premises and hypotheses is arguably never greater than when the data are complicated, confusing, or hard to obtain.

The second premise is more subtle: In the absence of good data, sound mathematical models are essential. Far from outstripping the data, mathematical models tell us what data we really need. Moreover, they tell us what qualitative properties we can expect in predictions arising from a given input data set. They also reveal how properties of the data, such as its spatial variability and uncertainty, affect the models' predictive capabilities. If the required data cannot in principle be acquired, if the qualitative properties of the model conflict with the empirical evidence, or if the model cannot, in principle, provide stable predictions in the face of heterogeneity and uncertainty, then we must admit that our understanding is incomplete.

1.2 Partial Differential Equations (PDEs)

Most realistic models of fluid flows in porous media use PDEs, "the natural dialect of continuum science" [62], written at scales appropriate for bench- or field-scale observations. In practical applications, these equations are complicated. They are posed on geometrically irregular, multidimensional domains; they often have highly variable coefficients; they can involve coupled systems of equations; in many applications they are nonlinear. For these reasons, we must often replace the exact PDEs by arithmetic approximations that one can solve using electronic machines.

The practical need for computational methods notwithstanding, a grasp of the analytic aspects of the PDEs remains an important asset for any porous-medium modeler. What types of initial and boundary conditions yield well-posed problems? Do the solutions obey *a priori* bounds based on the initial or boundary data? Do the numerical approximations respect these bounds? Does the PDE tend to smooth or preserve numerically problematic sharp fronts as time advances? Do shocks form from continuous initial data?

In the first half of the twentieth century, pioneering numerical analysts Richard Courant, Kurt Friedrichs, Hans Lewy, and John von Neumann—all immigrants to the United States—recognized that one cannot successfully "arithmetize analysis" [23] without understanding the differential equations. Designing stable, convergent, accurate, and efficient approximations to PDEs requires mathematical insight into the equations being approximated. A visionary 1947 consulting report [152] by von Neumann, developing the first petroleum reservoir simulator designed for a computer, illustrates this principle.

This book aims to promote this type of insight. We examine PDE-based models of porous-medium flows in geometries and settings simple enough to admit analysis without numerical approximations but realistic enough to reveal important structures.

From a mathematical perspective, the study of fluid flows in porous media offers fertile ground for inquiry into PDEs more generally. In particular, this book employs many broadly applicable concepts in the theory of PDEs, including:

1. Mass and momentum balance laws
2. Variational principles
3. Fundamental solutions
4. The principle of superposition
5. Similarity methods
6. Stability analysis
7. The method of characteristics and jump conditions.

Where possible, the narrative introduces these topics in the simplest possible settings before applying them to more complicated problems.

Topic 1, covered in Chapter 2, deserves comment. Few PDE texts at this level discuss balance laws in the detail pursued here. However, it is hard to build intuition about porous-medium flows without knowing the principles from which they arise. The balance laws furnish those principles. On the other hand, a completely rigorous study of balance laws for fluids flowing in porous media would require a monograph-length treatment in its own right. Chapter 2 reflects an attempt to weigh the importance of fundamental principles against the need for a concise explanation of how the governing PDEs emerge from basic laws of physics. The references offer suggestions for deeper inquiry.

We frequently refer to PDEs according to a classification system inherited from the algebra of quadratic equations. The utility of this system becomes more apparent as one becomes more familiar with examples. For now, it suffices to review the system for second-order PDEs in two independent variables having the form

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{1.1}$$

Here, a, b, and c are functions of the independent variables x and y, which we can replace with x and t in time-dependent problems; $u(x, y)$ is the unknown solution; and F denotes a function of five variables that describes the lower-order terms in the PDE.

The highest-order terms determine the classification. The **discriminant** of Eq. (1.1) is $\Delta = b^2 - 4ac$, which is a function of (x, y). Equation (1.1) is

- **hyperbolic** at any point of the (x, y)-plane where $\Delta(x, y) > 0$;
- **parabolic** at any point of the (x, y)-plane where $\Delta(x, y) = 0$;
- **elliptic** at any point of the (x, y)-plane where $\Delta(x, y) < 0$.

Extending this terminology, we say that a first-order PDE of the form

$$\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = F(x, y, u)$$

is hyperbolic at any point (x, y) where $a(x, y) \neq 0$.

Exercise 1.1 *Verify the following classifications, where c and D are real-valued with $D > 0$:*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(one-dimensional wave equation)} \quad \textit{hyperbolic},$$

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0 \qquad \text{(one-dimensional heat equation)} \quad \textit{parabolic},$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{(two-dimensional Laplace equation)} \quad \textit{elliptic}.$$

Mathematicians associate the wave equation with time-dependent processes that exhibit wave-like behavior, the heat equation with time-dependent processes that exhibit diffusive behavior, and the Laplace equation with steady-state processes. These associations arise from applications, some of which this book explores, reinforced by theoretical analyses of the three exemplars in Exercise 1.1. For more information about the classification of PDEs, see [65, Section 2-6].

1.3 Dimensions and Units

In contrast to most texts on pure mathematics, in this book **physical dimensions** play an important role. We adopt the basic physical quantities length, mass, and time, having physical dimensions L, M, and T, respectively. All other physical quantities encountered in this book—except for one case involving temperature in Chapter 7—are derived quantities, having physical dimensions that are products of powers of L, M, and T.

For example, the physical dimension of force F arises from Newton's second law $F = ma$, where m denotes mass and a denotes acceleration:

$$\dim(F) = \dim(ma) = \dim(m) \cdot \dim(a) = \mathrm{M} \cdot \mathrm{LT}^{-2}.$$

Analyzing the physical dimensions of quantities that arise in physical laws can yield surprisingly powerful mathematical results. Subsequent chapters exploit this concept many times.

Physical laws such as $F = ma$ require a way to assign numerical values to the physical quantities involved. We do this by comparison with standards, a process called **measurement**. For example, to assign a numerical value to the length of an object, we compare it to a length to which we have assigned a numerical value by fiat. A choice of standards for measuring L, M, and T, applied consistently for all occurrences of length, mass, and time, defines a system of **units**. Changing the system of units typically changes the numerical values that we measure, the exception being **dimensionless** quantities, which have dimension 1.

Where practical, this book uses the *Système Internationale* (SI) as the preferred system of units. The current standards for time, length, and mass in the SI are as follows:

- *Time*: One second (s) is the duration of 9 192 631 770 periods of the radiation emitted by the transition between the two hyperfine levels of the ground state of cesium-133. This period of time is approximately 1/86 400 of one Earth day.
- *Length*: One meter (m) is the distance traveled in a vacuum by light in 1/299 792 458 s. This distance is approximately 10^{-7} times the distance from the Earth's geographic north pole to the equator along a great circle.
- *Mass*: One kilogram (kg) is the mass required to fix the value of the Planck constant as $6.62607015 \times 10^{-34}$ kg m^2 s^{-1}, given the definition of one second and 1 m. This mass is approximately that of 10^{-3} m^3 (1 liter) of water at room temperature and pressure.

In some cases, non-SI units are more convenient for measuring physical quantities that arise in the bench- or field-scale study of fluid flows in porous media. When these cases arise, we give the factor that enables conversion to SI units. The fact that scientists and engineers prefer non-SI units in some instances highlights the inherently subjective nature of units: Humans tend to prefer standards that yield numerical values not far from 1 in our everyday experience. One advantage of using dimensionless quantities—a technique employed frequently in this book—is that we avoid this subjectivity.

1.4 Limitations in Scope

Three limitations in scope are worth noting. First, we treat only isothermal flows in porous media, that is, flows at constant temperature. This restriction conveniently allows us to ignore the energy balance equation in deriving governing

PDEs. On the other hand, it also eliminates several types of flows that have important applications, including flows in geothermal reservoirs and thermal methods of enhanced oil recovery, such as steam flooding.

Also glaringly absent from the table of contents is the topic of flows in fractured porous media. Geoscientists correctly point out that most geologic porous media possess fractures, which exert significant influences on fluid flows. Yet the mathematics of flow in fractured porous media remains poorly delineated, owing not so much to the absence of mathematical models (see [21] for a recent overview and [8, 15, 86, 153] for prominent examples) but, more importantly, to the observation that fractures exist at many scales of observation. In some underground formations, one must know something about the geometry of individual fractures to model fluid flows accurately. In these settings, the modeler's challenge is to represent the discrete fracture system (or statistical realizations) on tractably coarse computational grids. In other geologic settings, it suffices to treat the pore network and the fracture network as overlapping porosity systems, and the challenge is to model how fluids move within *and* between them. This spectrum of modeling approaches deserves a monograph of its own.

Also missing from the topics covered here is a discussion of fluid flows in extremely flow-resistant media, often but debatably referred to as nanodarcy flows but more properly characterized as **non-Darcy flows**. Flows of this type have increased in practical importance during the past two decades, owing especially to vastly improved technologies for producing natural gas from shale formations when hydrocarbon commodity prices justify the costs. The physics here are complex, involving gas–rock interactions in interstices whose typical diameters approach the mean free path of the gas molecules. None of the classical macroscopic transport models—such as Darcy's law or Fick's law of diffusion—suffices by itself to capture these phenomena [37, 81]. One can hope that further advances in our understanding of these flows, analogous to the advances described above for classical Darcy flows, will yield more settled mathematical models in years to come.