# 1

# Introduction

*There are two ways of spreading light:*
*to be the candle or the mirror that reflects it.*

Edith Wharton

This book presents an introduction to the Galerkin finite element method (FEM) as a powerful and general tool for approximating solution of differential equations. Our objective is twofold.

i) To present the main ordinary and partial differential equations (ODEs and PDEs) modeling different phenomena in science and engineering and introduce mathematical tools and environments for their analytic and numerical studies.
ii) To construct some common FEMs for approximate solutions of differential equations and analyze their well-posedness (existence, uniqueness, and stability of such approximate solutions) as well as the accuracy of the approximation.

In its final step, a finite element procedure yields a linear system of equations (LSEs) where the unknowns are the approximate values of the solution at certain nodes. Then, an approximate solution is constructed by adapting piecewise polynomials of certain degree to these, approximate, nodal values.

The entries of the coefficient matrix and the right-hand side of FEM's final LSEs consist of integrals which, e.g. for complex geometries or less smooth, and/or more complex, data, are not always easily computable. Therefore, numerical integration and quadrature rules are introduced to approximate such integrals. Furthermore, iteration procedures are included in order to efficiently compute the numerical solutions of such obtained LSEs.

*An Introduction to the Finite Element Method for Differential Equations,* First Edition.
Mohammad Asadzadeh.

Interpolation techniques are presented for both accurate polynomial approximations and also to derive basic a priori and a posteriori error estimates necessary to determine qualitative properties of the approximate solutions. That is to show how the approximate solution, in some adequate measuring environment, e.g. a certain norm, approaches the exact solution as the number of nodes, hence, the number of unknowns increases. For convenience, the frequently used classical inequalities, such as the *Cauchy–Schwarz'* and Poincare, likewise the *inverse and trace estimates*, that are of vital importance in error analysis and stability estimates, are introduced. In the theoretical abstraction, we demonstrate the fundamental solution approach based on Green's functions and prove the Riesz (Lax–Milgram) theorem which is essential in proving the existence of a unique solution for a minimization problem that in turn is equivalent both to a variational formulation as well as a corresponding boundary value problem (BVP).

Galerkin's method for solving a general differential equation is based on seeking an approximate solution, which is

1. Easy to differentiate and integrate
2. Spanned by a set of "nearly orthogonal" base functions in a finite-dimensional vector space.
3. Satisfies *Galerkin orthogonality* relation.
   Roughly speaking, this means *a closeness relation* in the sense that:
   I). In a priori case, the difference between the exact and approximate solution is orthogonal to the finite dimensional vector space of the approximate solution.
   II). In a posteriori case, the *residual* of the approximate solution (=the difference between the left- and right-hand side of an expression obtained from the differential equation where exact solution is replaced by the approximate solution) is orthogonal to the finite dimensional vector space of the approximate solution.

## 1.1 Preliminaries

In this section, we give a brief introduction to some key concepts in differential equations. A standard classification and some general properties are presented in Trinities below.

- A *differential equation* is a relation between an unknown function $u$ and its derivatives.
- If the differentiation in the equation is with respect to only one variable, e.g. $x \in \mathbb{R}$, $t \in \mathbb{R}^+$ (or $x_1$ in $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$), then the equation is called an *ordinary differential equation* (ODE).

**Example 1.1** As a simple example of an ODE, we mention the population dynamics model

$$\frac{du}{dt}(t) - \lambda u(t) = f(t), \quad t > 0. \tag{1.1.1}$$

If $f(t) \equiv 0$, then the equation is called *homogeneous*; otherwise, it is called *inhomogeneous*. For $\lambda > 0$, the homogeneous equation $\frac{du}{dt}(t) - \lambda u(t) = 0$ has an exponentially growing analytic solution given by $u(t) = u(0)e^{\lambda t}$, where $u(0)$ is the initial population. On the contrary, $\lambda < 0$ yields a population that vanishes (dies out) with time.

- The *order* of a differential equation is the order of the highest derivative of the function $u$ that appears in the equation.
- If the function $u$ depends on more than one variable, and the differential equation possesses derivatives with respect to at least two variables, then the differential equation is called a *partial differential equation* (PDE), e.g.

  $$u_t(x,t) - u_{xx}(x,t) = 0,$$

  is a *homogeneous* PDE of the second order, whereas for $f \neq 0$, the equations

  $$u_{xx}(x,y) + u_{yy}(x,y) = f(x,y),$$

  and

  $$u_t(x,y,t) - u_{xx}(x,y,t) - u_{yy}(x,y,t) + u_x(x,y,t) + u_y(x,y,t) = f(x,y,t),$$

  are *nonhomogeneous* PDEs of the second order.
- A solution to a differential equation is a function; (e.g. $u(x)$, $u(x,t)$, $u(x,y)$, or $u(x,y,t)$ above), which satisfies the corresponding differential equation.
- In general, the solution of a differential equation cannot be expressed in terms of elementary functions, and numerical methods are the only way to solve the differential equations through constructing *approximate solutions*. Then, the main questions are
  To what extent does the approximate solution preserve the physical properties of the exact solution, or satisfies a corresponding, discrete, version of the differential equation (*consistency*)?
  How sensitive is the solution to the change of the data (*stability*)?
  How close is the approximate solution to the exact solution (*convergence*)?
  Which are the adequate environments to measure this *closeness*?
- These are some of the questions that we want to deal within this text when approximating with the FEMs.
- A linear ODE of order $n$ has the general form:

  $$L(t,D)u = u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)u^{(k)}(t) = b(t),$$

where $D = d/dt$ denotes the derivative, with respect to $t$, and $u^{(k)} := D^k u$, with $D^k := \frac{d^k}{dt^k}$, $1 \le k \le n$ (the $k$-th order derivative). The corresponding *linear differential operator* is denoted by

$$L(t, D) = \frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t).$$

## 1.2 Trinities for Second-Order PDEs

Problems modeled by PDEs of the second order can be classified using, the so-called, trinities. Below we introduce basic ingredients of this concept. For detailed study see, e.g. [68].

**The usual three operators in PDEs of second order in $\mathbb{R}^n$.**

$$\text{Laplace operator } \Delta_n := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}, \tag{1.2.1}$$

$$\text{Diffusion operator } \frac{\partial}{\partial t} - \Delta_n, \tag{1.2.2}$$

$$\text{d'Alembert operator } \Box := \frac{\partial^2}{\partial t^2} - \Delta_n, \tag{1.2.3}$$

where we have the space variable $\mathbf{x} := (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$, the time variable $t \in \mathbb{R}^+$, and $\partial^2/\partial x_i^2$ denotes the second partial derivative with respect to $x_i$, $1 \le i \le n$. We also define a first-order operator, namely the *gradient operator* $\nabla_n$ which is the vector valued operator

$$\nabla_n := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right).$$

Often, the dimension is obvious from the context and therefore, usually, the subindex $n$ is suppressed and the operators $\Delta_n$ and $\nabla_n$ are simply denoted by $\Delta$ (or by $\nabla^2$) and $\nabla$, respectively.

**Classifying general second-order PDEs in two dimensions.**

I) **The constant coefficients case**
A second-order PDE in two dimensions, with constant coefficients, can be written in its general form as

$$Au_{xx}(x, y) + 2Bu_{xy}(x, y) + Cu_{yy}(x, y) + Du_x(x, y) + Eu_y(x, y) + Fu(x, y) + G = 0.$$

We introduce the *discriminant* $d = AC - B^2$: a quantity that specifies the role of the coefficients of the second-order terms, in determining the equation type in the sense that the equation is
*Elliptic*: if $d > 0$, *Parabolic*: if $d = 0$, and *Hyperbolic*: if $d < 0$.

**Example 1.2** Below are the classes of the most common, second-order differential equations (in any dimension) and examples of their forms in $2d$:

| Potential equation | Heat equation | Wave equation |
|---|---|---|
| $\Delta u = 0$ | $\frac{\partial u}{\partial t} - \Delta u = 0$ | $\frac{\partial^2 u}{\partial t^2} - \Delta u = 0$ |
| $u_{xx} + u_{yy} = 0$ | $u_t - u_{xx} = 0$ | $u_{tt} - u_{xx} = 0$ |
| $A = C = 1, B = 0$ | $B = C = 0, A = -1$ | $A = -1, B = 0, C = 1$ |
| $d = 1$ (elliptic) | $d = 0$ (parabolic) | $d = -1$ (hyperbolic). |

II) **The case of variable coefficients**
In the variable coefficients case, one can only have a local classification.

**Example 1.3** Consider the Tricomi equation of gas dynamics:

$$Lu(x, y) = yu_{xx} + u_{yy}.$$

Here, the coefficient $y$ is not a constant, and we have $A = y, \quad B = 0$, and $C = 1$. Hence, $d = AC - B^2 = y$ and consequently, the domain of ellipticity is $y > 0$, and so on, see Figure 1.1.

**The usual three types of problems in differential equations**.

1. Initial value problems (IVPs)
The simplest differential equation is $u'(x) = f(x)$ for $a < x \leq b$. But for any such $u$, also $(u(x) + c)' = f(x)$ for any constant $c$. To determine a unique solution, a specification of the initial value $u(a) = u_0$ is generally required. For example for $f(x) = 2x, \quad 0 < x \leq 1$, we have $u'(x) = 2x$ and the general solution is $u(x) = x^2 + c$. With an *initial value* of $u(0) = 0$, we get $u(0) = 0^2 + c = c = 0$. Hence,

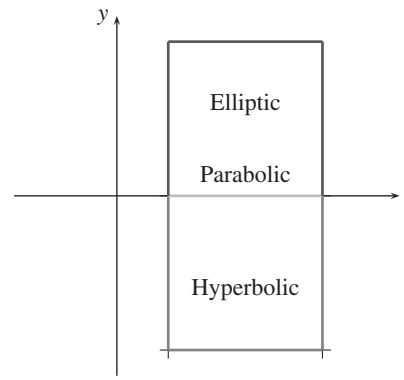


**Figure 1.1** Tricomi equation: an example of a variable coefficient classification.

the unique solution to this IVP is $u(x) = x^2$. Likewise, for a time-dependent differential equation of *second order* (two time derivatives), the initial values for $t = 0$, i.e. $u(x, 0)$ and $u_t(x, 0)$, are generally required. For a PDE such as the heat equation, the initial value can be a *function* of the space variable.

**Example 1.4** The wave equation, on the real line, augmented with the given initial data:

$$\begin{cases} u_{tt} - u_{xx} = 0, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & -\infty < x < \infty, \quad t = 0. \end{cases}$$

**Remark 1.1** Note that, here, for the unbounded spatial domain $(-\infty, \infty)$, it is required that $u(x, t) \to 0$ (or $\to u_\infty$ =constant) as $|x| \to \infty$. This corresponds to two boundary conditions.

2. Boundary value problems (BVP)

**Example 1.5** (A BVP in $\mathbb{R}$)
Consider the one-dimensional *stationary heat equation*:

$$-(a(x)u'(x))' = f(x), \quad \text{for} \quad 0 < x < 1.$$

In order to determine a solution $u(x)$ uniquely (see Remark 1.2), the differential equation is complemented by boundary conditions imposed at the boundary points $x = 0$ and $x = 1$; for example $u(0) = u_0$ and $u(1) = u_1$, where $u_0$ and $u_1$ are given real numbers.

**Example 1.6** (A BVP in $\mathbb{R}^n$).
The Laplace equation in $\mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$,

$$\begin{cases} \Delta_n u = \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dots + \dfrac{\partial^2 u}{\partial x_n^2} = 0, & \mathbf{x} \in \Omega \subset \mathbb{R}^n, \\ u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \partial\Omega \ \text{ (boundary of } \Omega). \end{cases}$$

**Remark 1.2** In general, in order to obtain a unique solution for a (partial) differential equation, one needs to supply physically adequate boundary data. For instance for the potential problem $u_{xx} + u_{yy} = f$, stated in a rectangular domain $(x, y) \in \Omega := (0, 1) \times (0, 1)$, to determine a unique solution we need to give two boundary conditions in the $x$-direction, i.e. the numerical values

for $u(0,y)$ and $u(1,y)$, and another two in the $y$-direction: the numerical values of $u(x,0)$ and $u(x,1)$. Whereas to determine a unique solution for the wave equation $u_{tt} - u_{xx} = 0$, it is necessary to supply two initial conditions in the time variable $t$, and two boundary conditions in the space variable $x$. We observe that in order to obtain a unique solution, we need to supply the same number of boundary (initial) conditions as the order of the differential equation in each spatial (time) variable. The general rule is that one should supply as many conditions as the highest order of the derivative in each variable. See also Remark 1.1, in the case of unbounded spatial domain.

3. Eigenvalue problems (EVPs)
Let $\mathbf{A}$ be a given square, say $n \times n$ matrix. The relation $\mathbf{Av} = \lambda\mathbf{v}, \quad \mathbf{v} \neq 0, \quad (\mathbf{v} \in \mathbb{R}^n)$, is a linear equation system, where $\lambda$ is an *eigenvalue* and $\mathbf{v}$ is an *eigenvector*. In the Example 1.7 below, we introduce the corresponding terminology for differential equations.

**Example 1.7** A differential equation describing a time-independent *vibrating string* is given by

$$-u''(x) = \lambda u(x), \qquad u(0) = u(\pi) = 0,$$

where $\lambda$ is an eigenvalue and $u(x)$ is an eigenfunction. $u(0) = 0$ and $u(\pi) = 0$ are boundary values.
To derive this equation, we consider the differential equation for a *time-dependent vibrating string*, with small displacement $u(x,t)$, which is fixed at the end points, viz.

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \quad t > 0, & u(x,0) = f(x), \ u_t(x,0) = g(x). \end{cases}$$

Then, using the separation of variables technique, this equation is split into two EVPs: Insert $u(x,t) = X(x)T(t) \neq 0$ (a nontrivial solution) into the differential equation to get

$$u_{tt}(x,t) - u_{xx}(x,t) = X(x)T''(t) - X''(x)T(t) = 0. \tag{1.2.4}$$

Dividing (1.2.4) by $X(x)T(t) \neq 0$ separates the variables so that we have

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda \quad \text{(a constant independent of } x \text{ and } t\text{)}. \tag{1.2.5}$$

Consequently we get two ODEs (two EVPs):

$$X''(x) = \lambda X(x), \qquad \text{and} \qquad T''(t) = \lambda T(t). \tag{1.2.6}$$

**The usual three types of boundary conditions**.

1. Dirichlet boundary condition
   Here, the solution is known at the boundary of the domain, as

$$u(\mathbf{x}, t) = f(\mathbf{x}), \quad \text{for } \mathbf{x} = (x_1, x_2, \dots, x_n) \in \partial\Omega, \quad t > 0.$$

   This is the case, for example, describing a fixed temperature at the boundary.
2. Neumann boundary condition
   In this case, the derivative of the solution in certain direction is given:

$$\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n} \cdot \mathbf{grad}(u) = \mathbf{n} \cdot \nabla u = f(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega,$$

   where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is, e.g. the *outward unit normal* to $\partial\Omega$ at $\mathbf{x} \in \partial\Omega$, and

$$\mathbf{grad}(u) = \nabla u = \left( \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right).$$

3. Robin boundary condition (a combination of 1 and 2),

$$\frac{\partial u}{\partial \mathbf{n}} + k \cdot u(\mathbf{x}, t) = f(\mathbf{x}), \quad k > 0, \quad \mathbf{x} = (x_1, x_2, \dots, x_n) \in \partial\Omega.$$

   In a homogeneous case, i.e. for $f(\mathbf{x}) \equiv 0$, Robin condition means that the heat flux $\frac{\partial u}{\partial \mathbf{n}}$ through the boundary is proportional to the temperature (in fact the temperature difference between the inside and outside) at the boundary.

**Example 1.8** In two dimensions, $\mathbf{n} = (n_x, n_y)$ with $|\mathbf{n}| = \sqrt{n_x^2 + n_y^2} = 1$ and hence, for $u = u(x, y)$, we have $\mathbf{n} \cdot \nabla u = n_x u_x + n_y u_y$, see Figure 1.2.

**Example 1.9** Let $u(x, y) = x^2 + y^2$. We determine the normal derivative of $u$ in (the assumed normal) direction $\mathbf{v} = (1, 1)$. The gradient of $u$ is the vector valued function $\nabla u = 2x \cdot \mathbf{e}_x + 2y \cdot \mathbf{e}_y$, where $\mathbf{e}_x = (1, 0)$ and $\mathbf{e}_y = (0, 1)$ are the

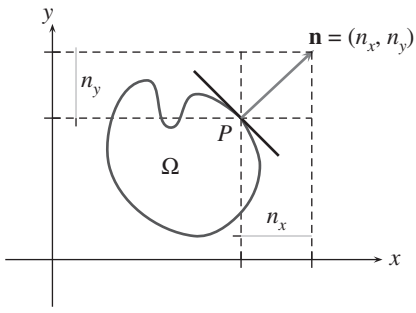


**Figure 1.2** Outward unit normal $\mathbf{n}$ at a point $P \in \partial\Omega$.

unit orthonormal basis in $\mathbb{R}^2$: $\mathbf{e}_x \cdot \mathbf{e}_x = \mathbf{e}_y \cdot \mathbf{e}_y = 1$ and $\mathbf{e}_x \cdot \mathbf{e}_y = \mathbf{e}_y \cdot \mathbf{e}_x = 0$. Note that $\mathbf{v} = \mathbf{e}_x + \mathbf{e}_y = (1, 1)$ is not a unit vector. The normalized $\hat{\mathbf{v}}$ is obtained as $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$, i.e.

$$\hat{\mathbf{v}} = \frac{\mathbf{e}_x + \mathbf{e}_y}{|\mathbf{e}_x + \mathbf{e}_y|} = \frac{(1,1)}{\sqrt{1^2 + 1^2}} = \frac{(1,1)}{\sqrt{2}}.$$

Thus, $\hat{\mathbf{v}} \cdot \nabla u(x, y) = \frac{\mathbf{e}_x + \mathbf{e}_y}{|\mathbf{e}_x + \mathbf{e}_y|} \cdot (2x \cdot \mathbf{e}_x + 2y \cdot \mathbf{e}_y)$, which gives

$$\hat{\mathbf{v}} \cdot \nabla u(x, y) = \frac{(1,1)}{\sqrt{2}} \cdot [2x(1, 0) + 2y(0, 1)] = \frac{(1,1)}{\sqrt{2}} \cdot (2x, 2y) = \frac{2x + 2y}{\sqrt{2}}.$$

**The usual three criteria to deal with**

1) In theory

   A given differential equation problem is called *well-posed* if the following three conditions hold true:

   I1. Existence: there exists at least one solution $u$.

   I2. Uniqueness: we have either one solution or no solutions at all.

   I3. Stability: the solution depends continuously on the data.

   *Note*: A property that concerns behavior, such as growth or decay, or perturbations of a solution as time increases is generally called a stability property.

2) In applications

   II1. Construction: exact design of the solution.

   II2. Regularity: how smooth is the found solution.

   II3. Approximation: when an exact construction is not possible.

**Three general approaches for solving differential equations**

1. Separation of variables method: The separation of variables technique reduces the (PDEs) to simpler EVPs (ODEs). This method is known as *Fourier method, or solution by eigenfunction expansion*.
2. Variational formulation method: Variational formulation or the multiplier method is a strategy for extracting information by multiplying a differential equation by suitable test functions and then integrating (e.g. like Fourier or Laplace transforms). This is also referred to as *The Energy Method*. A discrete version of this is the subject of our study.
3. Green's function method: Fundamental solutions, or solution of integral equations, which we have briefly addressed in the chapter of mathematical tools is the subject of an advanced PDE course.

## 1.3 PDEs in $\mathbb{R}^n$, Further Classifications

In this section, we extend the overture of the Sections 1.1 and 1.2 to higher dimensions and give definitions for linearity, nonlinearity, and superposition concepts.

We recall the common notation $\mathbb{R}^n$ for the real Euclidean spaces of dimension $n$ with the elements $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. In most of the applications, $n$ will be 1, 2, 3, or 4 and the variables $x_i,\ i = 1, 2, 3$ denote coordinates in space dimensions, whereas $x_4$ represents the time variable. In this case, we usually replace $(x_1, x_2, x_3, x_4)$ by the most common notation: $(x, y, z, t)$. Further, we shall use the subscript notation for the partial derivatives, viz.

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad \dot{u} = u_t = \frac{\partial u}{\partial t}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad \ddot{u} = \frac{\partial^2 u}{\partial t^2}, \quad \text{etc.}$$

A more general notation for partial derivatives of a sufficiently smooth function $u$ (see Definition 1.1 below) is given by

$$\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u,$$

where $\frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}},\ 1 \le i \le n$, denotes the partial derivative of order $\alpha_i$ with respect to the variable $x_i$, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multi-index of integers $\alpha_i \ge 0$ with $|\alpha| = \alpha_1 + \dots + \alpha_n$.

**Definition 1.1** A function $f$ of one real variable is said to be of class $C^{(k)}$ on an open interval $I$ if its derivatives $f', \dots, f^{(k)}$ exist and are continuous on $I$. A function $f$ of $n$ real variables is said to be of class $C^{(k)}$ on a set $S \subset \mathbb{R}^n$ if all of its partial derivatives of order $\le k$, i.e. $\partial^{|\alpha|} f / (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$ with the multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $|\alpha| \le k$, exist and are continuous on $S$.

As we mentioned, a key defining property of a PDE is that there are derivatives with respect to more than one independent variable and a PDE is a relation between an unknown function $u(x_1, \dots, x_n)$ and its partial derivatives:

$$F(x_1, \dots, x_n, u, u_{x_1}, u_{x_2}, \dots, u_{x_1 x_1}, \dots, \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \dots \partial x_l^{\alpha_i}, \dots) = 0. \tag{1.3.1}$$

**Example 1.10** The one-space dimensional, homogeneous, heat, and wave equations (here $x_1 = t$) are among the simplest PDEs:

$$u_t - u_{xx} = 0, \quad \text{and} \quad u_{tt} - u_{xx} = 0.$$

Other examples are

$$u_t + u_x - u_{xx} = 0, \quad \text{convection–diffusion equation}$$
$$y u_{xx} + u_{yy} = 0, \quad \text{Tricomi equation.}$$

The most general PDE of first order in two independent variables, $x$ and $y$ is of the form

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) =: F(x, y, u, u_x, u_y) = 0. \tag{1.3.2}$$

Likewise, the most general PDE of second order in two independent variables can be written as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0. \tag{1.3.3}$$

As stated in Remark 1.2, when Eqs. (1.3.1)–(1.3.3) are considered in bounded domains $\Omega \subset \mathbb{R}^2$, in order to obtain a *unique solution* one should supply *boundary conditions*: conditions at the boundary of the domain $\Omega$, denoted, e.g. by $B(u) = f$ or $B(u) = 0$ (as well as conditions for $t = 0$, initial conditions; denoted, e.g. by $I(u) = g$ or $I(u) = 0$; in the time-dependent cases), as in the theory of ODEs. $B$ and $I$ are expressions of $u$ and its partial derivatives, stated on the whole or a part of the boundary of $\Omega$ (or, in case of $I$, for $t = 0$), and are associated with the underlying PDE. Below we shall discuss the choice of relevant initial and boundary conditions for a PDE.

A *solution* of a PDE of type (1.3.1)–(1.3.3) is a function $u$ that identically satisfies the corresponding PDE, and the associated initial and boundary conditions, in some region of the variables $x_1, x_2, \ldots, x_n$, or $x, y$ (and $t$). Note that a solution of an equation of order $k$ has to be $k$ times differentiable. A function in $C^{(k)}$ that satisfies a PDE of order $k$ is called a *classical (or strong) solution* of the PDE. We sometimes also have to deal with solutions that are not classical. Such solutions are called *weak solutions*. In this note, in the variational formulation for FEMs, we actually deal with weak solutions. For a more thorough discussion on weak solutions, see Chapter 2 or any textbook in distribution theory.

**Definition 1.2** *Hadamard's criteria; compare with the three criteria in theory* A problem consisting of a PDE associated with boundary and/or initial conditions is called well-posed if it fulfills the following three criteria:

1. **Existence** The problem has a solution.
2. **Uniqueness** There is no more than one solution.
3. **Stability** A small change in the equation or in the side (initial and/or boundary) conditions gives rise to a small change in the solution.

If one or more of the conditions abovementioned does not hold, then we say that the problem is *ill-posed*. The fundamental theoretical question of PDEs is whether the problem consisting of the equation and its associated side conditions is well-posed. However, in certain engineering applications, we might encounter problems that are ill-posed. In practice, such problems are unsolvable. Therefore,

when we face an ill-posed problem, the first step should be to modify it appropriately in order to render it well-posed.

**Definition 1.3** An equation is called linear if in (1.3.1), $F$ is a linear function of the unknown function $u$ and its derivatives.

Thus, for example, the equation $e^{x^2y}u_x + x^7u_y + \cos(x^2 + y^2)u = y^3$ is a linear equation, while $u_x^2 + u_y^2 = 1$ is a nonlinear equation. The nonlinear equations are often further classified into subclasses according to the type of their nonlinearity. Generally, the nonlinearity is more pronounced when it appears in higher-order derivatives. For example, the following equations are both nonlinear

$$u_{xx} + u_{yy} = u^3 + u. \tag{1.3.4}$$

$$u_{xx} + u_{yy} = |\nabla u|^2 u. \tag{1.3.5}$$

Here $|\nabla u|$ denotes the norm of the gradient of $u$. While (1.3.5) is nonlinear, it is still linear as a function of the highest-order derivative (here $u_{xx}$ and $u_{yy}$). Such a nonlinearity is called *quasilinear*. On the other hand, in (1.3.4), the nonlinearity is only in the unknown solution $u$. Such equations are called *semilinear*.

## 1.4 Differential Operators, Superposition

*Differential and integral operators* are examples of mappings between function classes as $C^{(k)}(\Omega)$ where $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$. We denote by $\mathcal{L}[u]$ the operation of a mapping (operator) $\mathcal{L}$ on a function $u$.

**Definition 1.4** An operator $\mathcal{L}$ that satisfies

$$\mathcal{L}[\beta_1 u_1 + \beta_2 u_2] = \beta_1 \mathcal{L}[u_1] + \beta_2 \mathcal{L}[u_2], \quad \forall \beta_1, \beta_2 \in \mathbb{R}, \tag{1.4.1}$$

where $u_1$ and $u_2$ are functions, is called a linear operator. We may generalize (1.4.1) as

$$\mathcal{L}[\beta_1 u_1 + \cdots + \beta_k u_k] = \beta_1 \mathcal{L}[u_1] + \cdots + \beta_k \mathcal{L}[u_k], \quad \forall \beta_1, \ldots, \beta_k \in \mathbb{R}, \tag{1.4.2}$$

i.e. $\mathcal{L}$ maps any linear combination of $u_j$'s to corresponding linear combination of $\mathcal{L}[u_j]$'s.

For instance the *integral operator* $\mathcal{L}[f] = \int_a^b f(x)\, dx$ defined on the space of continuous functions on $[a, b]$ defines a linear operator from $C[a, b]$ into $\mathbb{R}$, which satisfies both (1.4.1) and (1.4.2).

A *linear partial differential operator* $\mathcal{L}$ that transforms a function $u$ of the variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$ into another function $\mathcal{L}u$ is given by

$$\mathcal{L}[u] = a(\mathbf{x})u + \sum_{i=1}^{n} b_i(\mathbf{x})\frac{\partial\, u}{\partial x_i} + \sum_{i,j=1}^{n} c_{ij}(\mathbf{x})\frac{\partial^2\, u}{\partial x_i \partial x_j} + \cdots \tag{1.4.3}$$

where $u$ represents any function in, say $C^{(\ell)}$, and the dots at the end indicate higher-order derivatives, but the sums contain only *finitely* many terms.

The term *linear* in the phrase *linear partial differential operator* refers to the following fundamental property: if $\mathcal{L}$ is given by (1.4.3) and $u_j,\ \ 1 \le j \le k$, are any set of functions possessing the requisite derivatives, and $\beta_j,\ 1 \le j \le k$ are any constants, then relation (1.4.2) is fulfilled. This is an immediate consequence of the fact that (1.4.1) and (1.4.2) are valid for $\mathcal{L}$ replaced with the derivative of any admissible order. A linear differential equation defines a linear differential operator: the equation can be expressed as $\mathcal{L}[u] = f$, where $\mathcal{L}$ is a linear operator and $f$ is a given function. The differential equation of the form $\mathcal{L}[u] = 0$ is called a *homogeneous equation*. For example, define the operator $\mathcal{L} = \partial^2/\partial x^2 - \partial^2/\partial y^2$. Then

$$\mathcal{L}[u] = u_{xx} - u_{yy} = 0,$$

is a homogeneous equation, while the equation

$$\mathcal{L}[u] = u_{xx} - u_{yy} = x,$$

is an example of an *inhomogeneous equation*. In a similar way, we may define another type of constraint for the PDEs that appears in many applications: the boundary conditions. In this regard, the linear boundary conditions are defined as operators $B$ satisfying

$$B(\beta_1 u_1 + \beta_2 u_2) = \beta_1 B(u_1) + \beta_2 B(u_2), \quad \forall \beta_1, \beta_2 \in \mathbb{R}, \tag{1.4.4}$$

at the boundary of a given domain $\Omega$.

Note that Laplace, heat, and wave equations are linear. Likewise, all the important boundary conditions (Dirichlet, Neumann, Robin) are linear.

**The Superposition Principle.** An important property of the linear operators is that if the functions $u_j,\ 1 \le j \le k$, satisfy the linear differential equations $\mathcal{L}[u_j] = f_j$, and the *linear boundary* conditions $B(u_j) = g_j$ for $j = 1, 2, \dots, k$, then any linear combination $v := \sum_{i=1}^{\ell} \beta_i u_i,\ \ \ell \le k$, satisfies the equation $\mathcal{L}[v] = \sum_{i=1}^{\ell} \beta_i f_i$ as well as the boundary condition $B(v) = \sum_{i=1}^{\ell} \beta_i g_i$. In particular, if each of the functions $u_i,\ 1 \le i \le \ell$, satisfies the homogeneous equation $\mathcal{L}[u] = 0$ and the homogeneous boundary condition $B(u) = 0$, then every linear combination of them satisfies that equation and boundary condition too. This property is called the *superposition principle*. It allows to construct complex solutions through combining simple solutions: suppose we want to determine all solutions of a

differential equation associated with a boundary condition, viz.

$$\mathcal{L}[u] = f, \qquad B(u) = g. \tag{1.4.5}$$

We consider the corresponding, *simpler homogeneous problem*:

$$\mathcal{L}[u] = 0, \qquad B(u) = 0. \tag{1.4.6}$$

Now, it suffices to find just one solution, say $v$ of the original problem (1.4.5). Then, for any solution $u$ of (1.4.5), $w = u - v$ satisfies (1.4.6), since $\mathcal{L}[w] = \mathcal{L}[u] - \mathcal{L}[v] = f - f = 0$ and $B(w) = B(u) - B(v) = g - g = 0$. Hence, we obtain a general solution of (1.4.5) by adding the general (homogeneous) solution $w$ of (1.4.6) to any particular solution of (1.4.5).

Following the same idea, one may apply superposition to split a problem involving several inhomogeneous terms into simpler ones each with a single inhomogeneous term. For instance, we may split (1.4.5) as

$$\mathcal{L}[u_1] = f, \qquad B(u_1) = 0,$$
$$\mathcal{L}[u_2] = 0, \qquad B(u_2) = g,$$

and then take $u = u_1 + u_2$.

The most important application of the superposition principle is in the homogeneous case: linear homogeneous differential equations satisfying homogeneous boundary conditions (which we repeat from above).

**The Superposition principle for the homogeneous case.** If the functions $u_j$, $1 \le j \le k$, satisfy (1.4.6): the linear differential equation $\mathcal{L}[u_j] = 0$ and the boundary conditions (linear) $B(u_j) = 0$ for $j = 1, 2, \dots, k$, then any linear combination $v := \sum_{i=1}^{\ell} \beta_i u_i$, $\ell \le k$, satisfies the same equation and boundary condition: (1.4.6).

### 1.4.1 Exercises

**Problem 1.1** Consider the problem

$$u_{xx} + u = 0, \quad x \in (0, \ell); \qquad u(0) = u(\ell) = 0.$$

Clearly, the function $u(x) \equiv 0$ is a solution. Is this solution unique? Does the answer depend on $\ell$?

**Problem 1.2** Consider the problem

$$u_{xx} + u_x = f(x), \quad x \in (0, \ell); \qquad u(0) = u'(0) = \frac{1}{2}[u'(\ell) + u(\ell)].$$

a) Is the solution unique? ($f$ is a given function).
b) Under what condition on $f$ a solution exists?

**Problem 1.3** Suppose $u_i,\ i = 1, 2, \dots, N$ are $N$ solutions of the linear differential equation $\mathcal{L}[u] = F$, where $F \neq 0$. Under what condition on the constant coefficients $c_i,\ i = 1, 2, \dots, N$ is the linear combination $\sum_{i=1}^{N} c_i u_i$ also a solution of this equation?

**Problem 1.4** Consider the nonlinear ODE $u_x = u(1 - u)$.

a) Show that $u_1(x) \equiv 1$ and $u_2(x) = 1 - 1/(1 + e^x)$ both are solutions, but $u_1 + u_2$ is not a solution.
b) For which value of $c_1$ is $c_1 u_1$ a solution? What about $c_2 u_2$?

**Problem 1.5** Show that each of the following equations has a solution of the form $u(x, y) = f(ax + by)$ for a proper choice of constants $a,\ b$. Find the constants for each example.

a) $u_x + 3u_y = 0$.
b) $3u_x - \pi u_y = 0$.
c) $2u_x + eu_y = 0$.

**Problem 1.6**

a) Consider the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$. Write the equation in the coordinates $s = x,\ t = x - y$.
b) Find the general solution of the equation.
c) Consider the equation $u_{xx} - 2u_{xy} + 5u_{yy} = 0$. Write it in the coordinates $s = x + y$ and $t = 2x$.

**Problem 1.7**

a) Show that for $n = 1, 2, 3, \dots,\quad u_n(x, y) = \sin(n\pi x)\sinh(n\pi y)$ satisfies

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(1, y) = u(x, 0) = 0.$$

b) Find a linear combination of $u_n$s that satisfies $u(x, 1) = \sin 2\pi x - \sin 3\pi x$.
c) Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(1, y) = 0, u(x, 0) = 2\sin \pi x,$$
$$u(x, 1) = \sin 2\pi x - \sin 3\pi x.$$

## 1.5 Some Equations of Mathematical Physics

Throughout the book, we focus on the study of some of the basic PDEs of mathematical physics. These equations involve differential operators as the *gradient* and *Laplacian*, acting on $C^{(2)}(\Omega),\quad \Omega \subseteq \mathbb{R}^n$:

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right), \qquad \nabla^2 u := \Delta_n = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \quad (1.5.1)$$

These equations describe fundamental physical phenomena as follows:

$$\begin{aligned} \nabla^2 u &= f(\mathbf{x}), && \text{The stationary diffusion} \\ u_t - k\nabla^2 u &= f(\mathbf{x}, t), && \text{The diffusion process} \\ u_{tt} - c^2\nabla^2 u &= f(\mathbf{x}, t), && \text{The wave propagation} \\ u_t + \beta \cdot \nabla u - \alpha\nabla^2 u &= f(\mathbf{x}, t), && \text{Convection–diffusion.} \end{aligned} \tag{1.5.2}$$

Here $f$ is a given function. In the special case, when $f \equiv 0$, i.e. when Eq. (1.5.2) are *homogeneous*, the first equation is called *the Laplace equation* and, being time-independent, has some special features: besides the fact that it describes the steady-state heat transfer and the standing wave equations (loosely speaking, the time-independent versions of the other two equations), the Laplace equation arises in describing several physical phenomena such as *electrostatic potential* in regions with no electric charge, *gravitational potential* in the regions with no mass distribution, in elasticity, etc.

### 1.5.1 The Poisson Equation

The *Poisson equation* is a BVP of the form

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega \subset \mathbb{R}^n, \quad n \geq 2 \\ u(x) = 0, & x \in \partial\Omega \qquad \text{(Dirichlet data).} \end{cases} \tag{1.5.3}$$

Below are some common phenomena modeled by the Poisson's equation.

- *Electrostatics*: To describe the components of the Maxwell's equations, associating the electric- and potential fields $E(x)$ and $\varphi(x)$ to the charge density $\rho$, roughly speaking, we have

$$\left.\begin{cases} \nabla \cdot E = \rho, \quad \text{in} \quad \Omega & \text{(Coulomb's law)} \\ \nabla \times E = 0, & \text{(Faraday's law)} \quad \Longrightarrow E = \nabla\varphi \end{cases}\right\} \Longrightarrow \Delta\varphi = \rho,$$

  and with a Dirichlet boundary condition $\varphi = g$ on $\partial\Omega$.
- *Fluid mechanics*: The velocity field $u$ of a rotation-free fluid flow satisfies $\nabla \times u = 0$ and hence, $u$ is a, so-called, gradient field: $u = \nabla\varphi$, with $\varphi$ being a scalar potential field. The rotation-free incompressible fluid flow satisfies $\nabla \cdot u = 0$, which yields the Laplace's equation $\Delta\varphi = 0$ for its potential. At a solid boundary, this problem will be associated with homogeneous Neumann boundary condition, due to the fact that in such a boundary, the normal velocity is zero.
- Statistical physics: The random motion of particles inside a container $\Omega$ until they hit the boundary is described by the probability $u(x)$ of a particle starting at the point $x \in \Omega$ winding up to stop at some point on $\partial\Omega$, where $u(x) = 1$ means that it is certain and $u(x) = 0$ means that it never happens. It turns out that $u$ solves the Laplace's equation $\Delta u = 0$, with discontinuity at the boundary: $u = 0$ on $\partial\Omega_0$ and $u = 1$ on $\partial\Omega_1$ where $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$. Poisson's equation is of vital

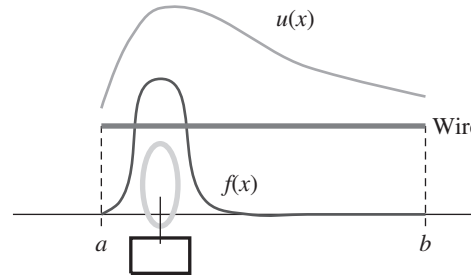


**Figure 1.3** A heat-conducting one-dimensional wire.

importance in describing, the coupling of nonlinearity aspects, in the system of gas kinetic/dynamic equations.

## 1.5.2 The Heat Equation

### 1.5.2.1 A Model Problem for the Stationary Heat Equation in 1*d*

Below we model the heat conduction in a thin heat-conducting wire stretched between the two endpoints of an interval $[a, b]$ that is subject to a heat source of intensity $f(x)$, as in Figure 1.3. We are interested in the stationary distribution of temperature $u(x)$ in the wire.

To this end, let $q(x)$ denote the heat flux in the direction of the positive $x$-axis in the wire $a < x < b$. Conservation of energy in the stationary case requires that the net heat through the endpoints of an arbitrary subinterval $(x_1, x_2)$ of $(a, b)$ is equal to the heat produced in $(x_1, x_2)$ per unit time:

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x)\, dx.$$

By the *Fundamental Theorem of Calculus*,

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} q'(x)\, dx,$$

Hence, we conclude that

$$\int_{x_1}^{x_2} q'(x)\, dx = \int_{x_1}^{x_2} f(x)\, dx.$$

Since $x_1$ and $x_2$ are arbitrary, assuming that the integrands are continuous, yields

$$q'(x) = f(x), \quad \text{for} \quad a < x < b, \tag{1.5.4}$$

which expresses conservation of energy in differential equation form. We need an additional equation that relates the heat flux $q$ to the temperature gradient $u'$ called a constitutive equation. The simplest *constitutive equation* for heat flow is *Fourier's law*:

$$q(x) = -c(x)u'(x), \tag{1.5.5}$$

which states that heat flows from warm regions to cold regions at a rate proportional to the temperature gradient $u'(x)$. The constant of proportionality is the *coefficient of heat conductivity* $c(x)$, which we assume to be a positive function in $[a, b]$. Combining (1.5.4) and (1.5.5) gives the *stationary heat equation in one dimension*:

$$-(c(x)u'(x))' = f(x), \quad \text{for} \quad a < x < b. \tag{1.5.6}$$

To define a solution $u$ uniquely, the differential equation is complemented by *boundary conditions* imposed at the boundary points $x = a$ and $x = b$. A common example is the homogeneous *Dirichlet conditions* $u(a) = u(b) = 0$, corresponding to keeping the temperature at zero at the endpoints of the wire. The result is a *two-point BVP*:

$$\begin{cases} -(c(x)u'(x))' = f(x), & \text{in} \quad (a, b), \\ u(a) = u(b) = 0. \end{cases} \tag{1.5.7}$$

The boundary condition $u(a) = 0$ may be replaced by $-c(a)u'(a) = q(a) = 0$, corresponding to prescribing zero heat flux, or insulating the wire, at $x = a$. Later, we also consider nonhomogeneous boundary conditions of the form $u(a) = u_a$ or $q(a) = g$ where $u_a$ and $g$ may be different from zero. For other types of boundary conditions, see Trinities (Section 1.2).

*The time-dependent heat equation* in (1.5.2) describes the diffusion of thermal energy in a homogeneous material, where $u = u(\mathbf{x}, t)$ is the temperature at a position $\mathbf{x}$ at time $t$ and $k = k(\mathbf{x})$ is called *thermal diffusivity* or *heat conductivity* (corresponding to $c(x)$ in (1.5.5)–(1.5.7)) of the material.

**Remark 1.3** The heat equation can be used to model the heat flow in solids and fluids, in the latter case, however, it does *not* take into account the convection phenomenon; and provides a reasonable model only if phenomena such as macroscopic currents in the fluid are not present (or negligible). Further, the heat equation is not a fundamental law of physics, and it does not give reliable answers at very low or very high temperatures.

Since temperature is related to heat, which is a form of energy, the basic idea in deriving the heat equation is to use the *law of conservation of energy*. Below we derive the general form of the heat equation in arbitrary dimension.

#### 1.5.2.2 Fourier's Law of Heat Conduction, Derivation of the Heat Equation

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a fixed spatial domain with boundary $\partial\Omega$. The rate of change of thermal energy with respect to time in $\Omega$ is equal to the net flow of energy across the boundary of $\Omega$ plus the rate at which heat is generated within $\Omega$.

Let $u(\mathbf{x}, t)$ denote the temperature at the position $\mathbf{x} = (x, y, z) \in \Omega$ and at time $t$. We assume that the solid is at rest and that it is rigid, so that the only energy present is thermal energy and the density $\rho(\mathbf{x})$ is independent of the time $t$ and temperature $u$. Let $\mathcal{E}$ denote the energy per unit mass. Then the amount of thermal energy in $\Omega$ is given by

$$\int_\Omega \rho\ \mathcal{E}\ d\mathbf{x},$$

and the time rate (time derivative) of change of thermal energy in $\Omega$ is:

$$\frac{d}{dt}\int_\Omega \rho\ \mathcal{E}\ d\mathbf{x} = \int_\Omega \rho\ \frac{\partial \mathcal{E}}{\partial t}\ d\mathbf{x}.$$

Let $\mathbf{q} = (q_x, q_y, q_z)$ denote the heat flux vector and $\mathbf{n} = (n_x, n_y, n_z)$ denote the outward unit normal to the boundary $\partial\Omega$, at the point $\mathbf{x} \in \partial\Omega$. Then $\mathbf{q} \cdot \mathbf{n}$ represents the flow of heat per unit cross-sectional area per unit time crossing a surface element. Thus,

$$-\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\ dS$$

is the amount of heat per unit time flowing into $\Omega$ across the boundary $\partial\Omega$. Here, $dS$ represents the element of surface area. The minus sign reflects the fact that if more heat flows out of the domain $\Omega$ than in, the energy in $\Omega$ decreases. Finally, in general, the heat production is determined by external sources that are independent of the temperature. In some cases, (such as an air conditioner controlled by a thermostat), it depends on temperature itself, but not on its derivatives. Hence, in the presence of a source (or sink), we denote the corresponding rate at which heat is produced per unit volume by $f = f(\mathbf{x}, t, u)$ so that the source term becomes

$$\int_\Omega f(\mathbf{x}, t, u)\ d\mathbf{x}.$$

Now, the law of conservation of energy takes the form

$$\int_\Omega \rho\ \mathcal{E}_t\ d\mathbf{x} + \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n}\ dS = \int_\Omega f(\mathbf{x}, t, u)\ d\mathbf{x}. \tag{1.5.8}$$

Applying the Gauss divergence theorem to the integral over $\partial\Omega$, we get

$$\int_\Omega (\rho\ \mathcal{E}_t + \nabla \cdot \mathbf{q} - f)\ d\mathbf{x} = 0, \tag{1.5.9}$$

where $\nabla$ denotes the divergence operator. In the sequel, we shall use the following simple result:

**Lemma 1.1** *Let $h$ be a continuous function satisfying $\int_\Omega h(\boldsymbol{x})\ d\boldsymbol{x} = 0$ for every domain $\Omega \subset \mathbb{R}^d$. Then $h \equiv 0$.*

*Proof:* Let us assume to the contrary that there exists a point $\mathbf{x}_0 \in \mathbb{R}^d$ where $h(\mathbf{x}_0) \neq 0$. Assume without loss of generality that $h(\mathbf{x}_0) > 0$. Since $h$ is continuous, there exists a domain (maybe very small) $\Omega_0$, containing $\mathbf{x}_0$, and an $\varepsilon > 0$, such that $h(\mathbf{x}) > \varepsilon$, for all $\mathbf{x} \in \Omega_0 \subset \Omega$. Therefore, we have $\int_{\Omega_0} h(\mathbf{x})\, d\mathbf{x} > \varepsilon \mathrm{Vol}(\Omega_0) > 0$, which contradicts the assumption.

From (1.5.9), using Lemma 1.1, we conclude that

$$\rho\, \mathcal{E}_t = -\nabla \cdot \mathbf{q} + f. \tag{1.5.10}$$

This is the basic form of our heat conduction law. The functions $\mathcal{E}$ and $q$ are unknown and additional information of an empirical nature is needed to determine the equation for the temperature $u$. First, for many materials, over a fairly wide but not too large temperature range, the function $\mathcal{E} = \mathcal{E}(u)$ depends nearly linearly on $u$, so that

$$\mathcal{E}_t = \lambda u_t. \tag{1.5.11}$$

Here, $\lambda$, called the *specific heat*, is assumed to be constant. Next, we relate the temperature $u$ to the heat flux $\mathbf{q}$. Here, we use *Fourier's law* but, first, to be specific, we describe the simple facts supporting Fourier's law:

i. Heat flows from regions of high temperature to regions of low temperature.
ii. The rate of heat flux is small or large accordingly as temperature changes between neighboring regions are small or large.

To describe these quantitative properties of heat flux, we postulate a linear relationship between the rate of heat flux and the rate of temperature change. Recall that if $\mathbf{x}$ is a point in the heat-conducting medium and $\mathbf{n}$ is a unit vector specifying a direction at $\mathbf{x}$, then the rate of heat flow at $\mathbf{x}$ in the direction $\mathbf{n}$ is $\mathbf{q} \cdot \mathbf{n}$ and the rate of change of the temperature is $\partial u/\partial \mathbf{n} = \nabla u \cdot \mathbf{n}$, the directional derivative of the temperature. Since $\mathbf{q} \cdot \mathbf{n} > 0$ requires $\nabla u \cdot \mathbf{n} < 0$, and vice versa (from calculus the direction of maximal growth of a function is given by its gradient), our linear relation takes the form $\mathbf{q} \cdot \mathbf{n} = -\kappa \nabla u \cdot \mathbf{n}$, with $\kappa = \kappa(\mathbf{x}) > 0$. Since $\mathbf{n}$ specifies any direction at $\mathbf{x}$, this is equivalent to the assumption

$$\mathbf{q} = -\kappa \nabla u, \tag{1.5.12}$$

which is *Fourier's law*. The positive function $\kappa$ is called the *heat conduction (or Fourier) coefficient*. Let now $\sigma = \kappa/\lambda\rho$ and $F = f/\lambda\rho$ and insert (1.5.11) and (1.5.12) into (1.5.10) to get the final form of the heat equation:

$$u_t = \nabla \cdot (\sigma \nabla u) + F. \tag{1.5.13}$$

The quantity $\sigma$ is referred to as the *thermal diffusivity (or diffusion) coefficient*. If we assume that $\sigma$ is constant, then the final form of the heat equation would be

$$u_t = \sigma \nabla^2 u + F, \quad \text{or} \quad u_t = \sigma\, \Delta u + F. \tag{1.5.14}$$

### 1.5.3 The Wave Equation

The third equation in (1.5.2) is the wave equation: $u_{tt} - c^2\nabla^2 u = F$. Here, $u$ represents a wave traveling through an $n$-dimensional medium; $c$ is the speed of propagation of the wave in the medium and $u(\mathbf{x}, t)$ is the amplitude of the wave at position $\mathbf{x}$ and time $t$. The wave equation provides a mathematical model for a number of problems involving different physical processes as, e.g. in the following examples (i)–(vi):

(i) Vibration of a stretched string, such as a violin string (one-dimensional).
(ii) Vibration of a column of air, such as a clarinet (one-dimensional).
(iii) Vibration of a stretched membrane, such as a drumhead (two-dimensional).
(iv) Waves in an incompressible fluid, such as water (two-dimensional).
(v) Sound waves in air or other elastic media (three-dimensional).
(vi) Electromagnetic waves, such as light waves and radio waves (three-dimensional).

Note that in (i), (iii) and (iv), $u$ represents the transverse displacement of the string, membrane, or fluid surface; in (ii) and (v), $u$ represents the longitudinal displacement of the air; and in (vi), $u$ is any of the components of the electromagnetic field. For detailed discussions and a derivation of the equations modeling (i)–(vi), see, e.g. Folland [62], Strauss [129], and Taylor [134]. We should point out, however, that in most cases, the derivation involves making some simplifying assumptions. Hence, the wave equation gives only an approximate description of the actual physical process, and the validity of the approximation will depend on whether certain physical conditions are satisfied. For instance, in example (i), the vibration should be small enough so that the string is not stretched beyond its limits of elasticity. In example (vi), it follows from Maxwell's equations, the fundamental equations of electromagnetism, that the wave equation is satisfied *exactly* in regions containing no electrical charges or current, which of course cannot be guaranteed under normal physical circumstances and can only be approximately justified in the real world. So an attempt to derive the wave equation corresponding to each of these examples from physical principles is beyond the scope of these notes. Nevertheless, to give an idea, below we shall derive the wave equation for a vibrating string.

#### 1.5.3.1 The Vibrating String, Derivation of the Wave Equation in 1*d*

Consider a perfectly elastic and flexible string stretched along the segment $[0, L]$ of the $x$-axis, moving perpendicular to its equilibrium position. Let $\rho_0(x)$ denote the density of the string in the equilibrium position and $\rho(x, t)$ the density at time $t$. In an arbitrary small interval $[x, x + dx]$, the mass will satisfy, see Figure 1.4.

$$m = \int_x^{x+dx} \rho_0(x)\, dx = \int_x^{x+dx} \rho(x,t)\sqrt{1+u_x^2}\, dx. \tag{1.5.15}$$

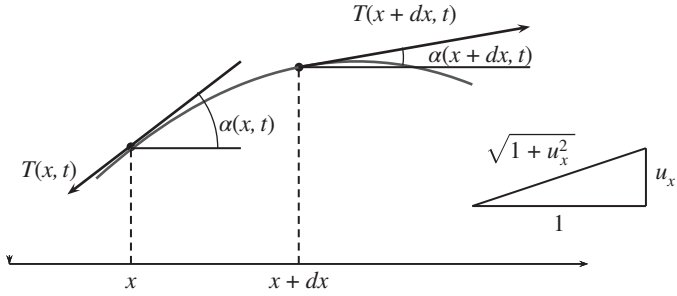


**Figure 1.4** A vibrating string.

Thus, using Lemma 1.1, (1.5.15) gives the conservation of mass:

$$\rho_0(x) = \rho(x,t)\sqrt{1+u_x^2}. \tag{1.5.16}$$

Now we use the tensions $T(x,t)$ and $T(x+dx,t)$, at the endpoints of an element of the string and determine the forces acting on the small interval $[x, x+dx]$. Since we assumed that the string moves only vertically, the forces in the horizontal direction should be in balance: i.e.

$$T(x+dx,t)\cos\alpha(x+dx,t) - T(x,t)\cos\alpha(x,t) = 0. \tag{1.5.17}$$

Dividing (1.5.17) by $dx$ and letting $dx \to 0$, we thus obtain

$$\frac{\partial}{\partial x}(T(x,t)\cos\alpha(x,t)) = 0, \tag{1.5.18}$$

Hence,

$$T(x,t)\cos\alpha(x,t) = \tau(t), \tag{1.5.19}$$

where $\tau(t) > 0$ because it is the magnitude of the horizontal component of the tension.

On the other hand, the vertical motion is determined by the fact that the time rate of change of linear momentum is given by the sum of the forces acting in the vertical direction. Hence, using (1.5.16), the momentum of the small element $[x, x+dx]$ is given by

$$\int_x^{x+dx} \rho_0(x) u_t \, dx = \int_x^{x+dx} \rho(x,t)\sqrt{1+u_x^2}\, u_t \, dx, \tag{1.5.20}$$

with the time rate of change:

$$\frac{d}{dt}\int_x^{x+dx} \rho_0 u_t \, dx = \int_x^{x+dx} \rho_0 u_{tt} \, dx. \tag{1.5.21}$$

There are two kinds of forces acting on the segment $[x, x+dx]$ of the string: (i) the forces due to tension that keep the string taut and whose horizontal components

are in balance, and (ii) the forces acting along the whole length of the string, such as weight. Thus, using (1.5.19), the net tension force acting on the ends of the string element $[x, x + dx]$ is

$$\begin{aligned} F_\tau &:= T(x + dx, t) \sin \alpha(x + dx, t) - T(x, t) \sin \alpha(x, t) \\ &= \tau \left( \frac{\sin \alpha(x + dx, t)}{\cos \alpha(x + dx, t)} - \frac{\sin \alpha(x, t)}{\cos \alpha(x, t)} \right) \\ &= \tau(\tan \alpha(x + dx, t) - \tan \alpha(x, t)) \\ &= \tau(u_x(x + dx, t) - u_x(x, t)). \end{aligned} \tag{1.5.22}$$

Further, the weight of the string acting downward is

$$F_g = -\int \rho\, g\, dS = -\int_x^{x+dx} \rho\, g \sqrt{1 + u_x^2}\, dx = -\int_x^{x+dx} \rho_0 g\, dx. \tag{1.5.23}$$

Next, for an external load, with density $f(x, t)$, acting on the string (e.g. when a violin string is bowed), we have

$$F_f = \int \rho\, f\, dS = \int_x^{x+dx} \rho_0 f(x, t)\, dx. \tag{1.5.24}$$

Finally, one should model the friction forces acting on the string segment. We shall assume a linear law of friction of the form:

$$F_{\text{fr}} := -\int \sigma \rho u_t\, dS = -\int_x^{x+dx} \sigma \rho \sqrt{1 + u_x^2}\, u_t\, dx = -\int_x^{x+dx} \sigma \rho_0 u_t\, dx. \tag{1.5.25}$$

Now applying Newton's second law yields

$$\begin{aligned} \int_x^{x+dx} \rho_0 u_{tt}\, dx = F_\tau + F_g + F_f + F_{\text{fr}} &= \tau[u_x(x + dx, t) - u_x(x, t)] \\ &\quad - \int_x^{x+dx} \sigma \rho_0 u_t\, dx + \int_x^{x+dx} \rho_0 (f - g)\, dx. \end{aligned} \tag{1.5.26}$$

Dividing (1.5.26) by $dx$ and letting $dx \to 0$, we obtain the equation

$$\rho_0 u_{tt} = \tau u_{xx} - \sigma \rho_0 u_t + \rho_0 (f - g). \tag{1.5.27}$$

Letting $c^2 = \tau/\rho_0$ and $F = f - g$, we end up with the following concise form:

$$u_{tt} + \sigma u_t = c^2 u_{xx} + F. \tag{1.5.28}$$

Equation (1.5.28) describes the vibration of the considered string once it is set into motion. The *smallness* assumption here results in a single linear equation for $u$. Due to the presence of the friction term $\sigma u_t$, Eq. (1.5.28) is often referred to as the

*damped one-dimensional wave equation*. If friction is negligible, then we can let $\sigma = 0$ and get the *inhomogeneous wave equation*

$$u_{tt} = c^2 u_{xx} + F. \tag{1.5.29}$$

In the absence of external forces and when the weight of the string is negligible, we may take $F \equiv 0$ to get the *one-dimensional wave equation*:

$$u_{tt} = c^2 u_{xx}. \tag{1.5.30}$$

Note that since $u$ has the unit of length $\ell$, $u_{tt}$ has the unit of acceleration and $u_{xx}$ the unit of $\ell^{-1}$, hence, $c$ has the unit of velocity.

### 1.5.4 Exercises

**Problem 1.8** Show that $u(x, y) = \log(x^2 + y^2)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$ for $(x, y) \neq (0, 0)$.

**Problem 1.9** Show that $u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ satisfies Laplace's equation $u_{xx} + u_{yy} + u_{zz} = 0$, for $(x, y, z) \neq (0, 0, 0)$.

**Problem 1.10** Show that $u(r, \theta) = Br^n \sin(n\theta)$ satisfies the Laplace equation in polar coordinates:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

**Problem 1.11** Verify that

$$u = \frac{-2y}{x^2 + y^2 + 2x + 1}, \qquad v = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 2x + 1}$$

both satisfy the Laplace equation, and sketch the curves $u =$ constant and $v =$ constant. Show that

$$u + iv = \frac{i(z - 1)}{z + 1}, \quad \text{where } z = x + iy.$$

**Problem 1.12** Show that $u(x, t) = t^{-1/2} \exp(-x^2/4kt)$ satisfies the heat equation $u_t = k u_{xx}$, for $t > 0$.

**Problem 1.13** Show that $u(x, y, t) = t^{-1} \exp[-(x^2 + y^2)/4kt]$ satisfies the heat equation $u_t = k(u_{xx} + u_{yy})$, for $t > 0$.

**Problem 1.14** The spherically symmetric form of the heat conduction equation is given by

$$u_{rr} + \frac{2}{r}u_r = \frac{1}{\kappa}u_t.$$

Show that $v = ru$ satisfies the standard one-dimensional heat equation.

**Problem 1.15** Show that the equation

$$\theta_t = \kappa\theta_{xx} - h(\theta - \theta_0)$$

can be reduced to the standard heat conduction equation by writing $u = e^{ht}(\theta - \theta_0)$. How do you interpret the term $h(\theta - \theta_0)$?

**Problem 1.16** Use the substitution $\xi = x - vt$, $\eta = t$ to transform the one-dimensional convection–diffusion equation

$$u_t = ku_{xx} - vu_x,$$

into a heat equation for $\tilde{u}(\xi, \eta) = u(\xi + v\eta, \eta)$.

**Problem 1.17** If $f \in C[0, 1]$, let $u(x, t)$ satisfy

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, \quad t > 0, \\ u(0, t) = u(1, t) = 0, & t \geq 0, \\ u(x, 0) = f(x), & 0 \leq x \leq 1. \end{cases}$$

Derive the identity $2u(u_t - u_{xx}) = (u^2)_t - (2uu_x)_x + 2u_x^2$.

**Problem 1.18** Find the possible values of $a$ and $b$ in the expression $u = \cos at \sin bx$, such that it satisfies the wave equation $u_{tt} = c^2u_{xx}$.

**Problem 1.19** Taking $u = f(x + \alpha t)$, where $f$ is any function, find the values of $\alpha$ that will ensure $u$ satisfies the wave equation $u_{tt} = c^2u_{xx}$.

**Problem 1.20** The spherically symmetric version of the wave equation $u_{tt} = c^2u_{xx}$ takes the form

$$u_{tt} = c^2(u_{rr} + 2u_r/r).$$

Show, by putting $v = ru$, that it has a solution of the form

$$v = f(ct - r) + g(ct + r).$$

**Problem 1.21** Let $\xi = x - ct$ and $\eta = x + ct$. Use the chain rule to show that

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}.$$

**Problem 1.22** Show that the solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x),$$

satisfies *d'Alembert's* formula:

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy.$$