1

Fundamentals of Electrical Machines

1.1 Preliminary Remarks

The study of the construction of electrical machines, along with the principles and laws on which they operate, gives the knowledge to predict behaviour and characteristics of the machines. Therefore, the study of electric machines requires an understanding of basic electromagnetic fields. The purpose of this chapter is to give some familiarity with the principles and laws that help understand the principle of Electric Machinery.

1.2 Basic Laws of Electrical Engineering

1.2.1 Ohm's Law

The current through an ideal *resistance* element is directly proportional to the voltage drop across it, as shown in Figure 1.1.

Ideal resistance element here refers to the element without any parasitic inductance or capacitance.

The relationship between voltage and current in an ideal resistor is given in Eq. (1.1) in which v is in volts, i is in amps, and the constant of proportionality is resistance R measured in ohms (Ω).This simple formula is known as Ohm's law in honour of the German physicist, Georg Ohm, whose original experiments led to this incredibly useful and important relationship.

$$v = Ri \tag{1.1}$$

Notice that voltage v is measured across the resistor. That is, it is the voltage at point A with respect to the voltage at point B. When the current is in the direction shown, the voltage at A with respect to B is positive, so it is quite common to say that there is a *voltage drop* across the resistor.

An equivalent relationship for a resistor is given in Eq. (1.2), where the current is given in terms of voltage and the proportionality constant is conductance G with units of Siemens (S). In older literature, the unit of conductance was mhos.

$$i = Gv \tag{1.2}$$

Electrical Machine Fundamentals with Numerical Simulation using MATLAB/SIMULINK, First Edition.
Atif Iqbal, Shaikh Moinoddin, and Bhimireddy Prathap Reddy.

Companion website: www.wiley.com/go/iqbal/electricalmachine

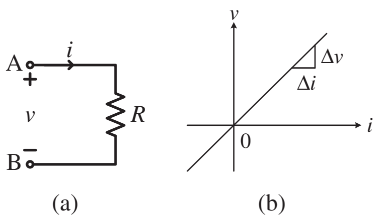


Figure 1.1 (a) An ideal resistor symbol. (b) voltage–current relationship.

Magnetic field problems involving components such as current coils, ferromagnetic cores, and air gaps can be solved as magnetic circuits according to the analogous behaviour of the magnetic quantities to the corresponding electric quantities in an electric circuit.

1.2.2 Generalization of Ohm's Law

Generalization of Ohm's law means expressing the law in terms of **E** (electric field in V/m) and **J** (current density in Amp/m^2), rather than V and I. Consider a length l of a conductor of uniform cross-sectional area A through which a current I flows. In general, the electrical resistance of the conductor is proportional to its length, l, and inversely proportional to its cross-sectional area, A. Thus, resistance can be expressed as

$$R = \rho\frac{l}{A} = \frac{1}{\sigma}\frac{l}{A} \tag{1.3}$$

Here, the constants ρ and σ are called the *resistivity* and *conductivity* of the conducting medium, respectively, and are measured in units of ohm-meters and (ohm-meter)$^{-1}$. Therefore, Ohm's law is stated as

$$V = \rho\frac{l}{A}I \tag{1.4}$$

But, $I/A = \mathrm{J_z}$ (assuming that the conductor is aligned along the z-axis) and $V/l = E_z$, hence, Eq. (1.4) modifies to

$$E_z = \rho J_z \tag{1.5}$$

Equation (1.5) can be generalized as

$$\mathbf{E} = \rho\mathbf{J} \tag{1.6a}$$

or

$$\mathbf{J} = \sigma\mathbf{E} \tag{1.6b}$$

where σ is the conductivity of the medium. Equations (1.6a) and (1.6b) are the vector form of Ohm's law.

1.2.2.1 Derivation of Eq. (1.6)

Let a unit volume of conductor contains n number of free electrons. In the presence of electric field **E**, the free electrons accelerate with drift velocity

$$\mathbf{v}_d = -\frac{q\tau}{m_e}\mathbf{E} \tag{1.7}$$

where q, τ, and m_e are charged in coulomb, relaxation time in seconds and mass of electron respectively. The total charge of electrons $(-nq)$ is moving with velocity $(\mathbf{v}_d)$. Therefore, the current density

$$\mathbf{J} = (-nq)\,\mathbf{v}_d = \frac{nq^2\tau}{m_e}\mathbf{E} = \sigma\mathbf{E} \tag{1.8}$$

Thus comparing Eqs. (1.6b) and (1.8), the conductivity can be defined as

$$\sigma = \frac{1}{\rho} = \frac{ne^2\tau}{m_e} \tag{1.9}$$

Let P be the power dissipated per unit volume inside the conducting medium. P is given as

$$\mathrm{P} = \mathbf{j}\cdot\mathbf{E} = \rho\,\mathrm{j}^2\left(\left(\mathrm{Amp/m^2}\right).(\mathrm{V/m}) = \mathrm{Volt.Amp/m^3} = \mathrm{Watt/m^3}\right) \tag{1.10}$$

1.2.3 Ohm's Law for Magnetic Circuits

This law is also called as Hopkinson's law, after *John Hopkinson*, but was formulated earlier by *Henry Augustus Rowland* in 1873. It states that

$$\mathfrak{F} = \phi\mathfrak{R} \tag{1.11}$$

where $\mathfrak{F}$, ϕ, and $\mathfrak{R}$ are magneto motive force (MMF) in Amp-turn (A.t), flux in Wb and reluctance in A.t/Wb, respectively.

1.2.4 Kirchhoff's Laws for Magnetic Circuits

As in electric circuits, Kirchhoff's laws can be applied in magnetic circuits because Kirchhoff's voltage and Kirchhoff's current laws are analogous to Ampere's law and Gauss Law, respectively.

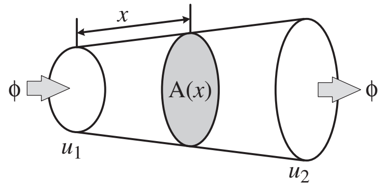
Magnetic circuits obey other laws of electric circuits such as reluctances in series $\left(\mathfrak{R}_{total} = \mathfrak{R}_1 + \mathfrak{R}_2 + \mathfrak{R}_3 + \ldots\right)$ and parallel $\left(\frac{1}{\mathfrak{R}_{total}} = \frac{1}{\mathfrak{R}_1} + \frac{1}{\mathfrak{R}_2} + \frac{1}{\mathfrak{R}_3} + \ldots\right)$

A space containing a quasi-stationary electric or magnetic field may be partitioned into *flux tubes:* geometrical figures in which all lines of flux are perpendicular to their bases and no lines of flux cut their sides; see Figure 1.2.

Lines of the equal magnetic scalar potential, u, are perpendicular to lines of flux, ϕ; therefore, the bases of a flux tube are equipotential planes. The magnetic scalar potential difference between the bases equals the MMF drop. Generally, the ratio of potential difference at the ends of the flux tube that contains no current and the flux through it, is a function of flux tube geometry and the characteristics of the medium. Mathematically, this ratio is equal to:

$$R = \int_0^l \frac{dx}{c\,(x)\,A\,(x)} \tag{1.12}$$

Figure 1.2 A flux tube in a field [1].



where l is the total flux tube length, A is its cross-sectional area, and c is a function of material properties. The quantity R is defined as reluctance in magnetic fields and is analogous to resistance in electric fields and the inverse of capacitance in electrostatic fields. R is a function of field quantities and also a function of geometry. The quantity c is equal to the flux tube material permeability, μ, for magnetic fields. For electric fields, c is equal to the flux tube conductivity, σ, while for electrostatic fields c is the permittivity, ϵ. For the three field types, one can write:

$$R_{magnetic} = \int_0^l \frac{dx}{\mu(x) A(x)} = Reluctance \tag{1.13}$$

$$R_{electrostatic} = \int_0^l \frac{dx}{\varepsilon(x) A(x)} = Inverse\ of\ capacitance \tag{1.14}$$

$$R_{electric} = \int_0^l \frac{dx}{\sigma(x) A(x)} = Resistance \tag{1.15}$$

The analogy between the three types of fields is almost complete, except in one detail: to maintain a certain energy level in a magnetic and an electrostatic field, no support from outside the field is needed. On the contrary, in an electric field, there must be a current from a source to cover losses. Electrostatic and magnetic fields can store energy, whereas in an electric field, the complete energy is irreversibly converted into heat. If all quantities in a field are constant, then stored energy in a magnetic or an electrostatic field is constant. Under the same circumstances, the energy lost in an electric field increases in proportion to time. A comparison of the field is tabulated in Table 1.1.

By comparing expressions for stored energy in electrostatic and magnetic fields, one can answer the question of why the magnetic field is chosen as an electromechanical energy conversion medium. Assuming the same level of field strength in both types of fields in free space, one can store μ_o/ϵ_o, or approximately 1.42×10^5, more energy per volume in a magnetic than in an electrostatic field. Stored energy per volume is representative of electro-mechanical energy conversion capability and, therefore, the choice of the magnetic field as a medium is obvious.

Table 1.1 Analogies between different types of fields [1].

	Type of field		
	Magnetic	**Electrostatic**	**Electric**
Material Constant	μ	ϵ	Σ
Flux or current density	$\overline{B}$	$\overline{D}$	$\overline{J}$
Field strength	$\overline{H}$	$\overline{E}$	$\overline{E}$
Reluctance, capacitance, resistance	$R_m = \int_0^l \frac{dx}{\mu(x)A(x)} = \frac{N^2}{L}$	$R_{es} = \int_0^l \frac{dx}{\varepsilon(x)A(x)} = \frac{1}{C}$	$R_{el} = \int_0^l \frac{dx}{\sigma(x)A(x)}$
Potential difference	$\mathfrak{F} = \int_0^l \overline{H} \cdot d\bar{l}$	$V = \int_0^l \overline{E} \cdot d\bar{l}$	$V = \int_0^l \overline{E} \cdot d\bar{l}$
Flux, charge, current	$\phi = \int \overline{B} \cdot d\overline{A} = \frac{\mathfrak{F}}{R_m}$	$Q = \int \overline{D} \cdot d\overline{A} = \frac{V}{R_{es}} = VC$	$I = \int \overline{J} \cdot d\overline{A} = \frac{V}{R_{el}}$
Energy	$\int \mathfrak{F} d\phi$	$\int VdQ$	$\int IVdt$

Further comparison between electric and magnetic fields shows that magnetic circuits almost always operate in a saturated, i.e. nonlinear mode, while most elements in electric circuits are linear. There is also a large difference in the ratio of material constants between non-conducting and conducting media in magnetic and electric fields. The electrical conductivity of air is typically more than six orders of magnitude less than the conductivity of a current-carrying medium while, in a magnetic field, the permeability of air is hardly ever less than four orders of magnitude smaller than that of the flux-carrying medium. The ratio of air to medium permeability increases as the magnetic medium becomes more saturated. Therefore, in a magnetic field, one must take into account the non-negligible part of the flux that goes through the air, a phenomenon that is unknown in an electric field.

1.2.5 Lorentz Force Law

Electric and magnetic fields are set up by moving charged particles or current flow. This is described by Maxwell's equations, but the force acting on the moving charge q in the presence of magnetic fields can be found using Lorentz force law. It can be stated as

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \tag{1.16}$$

or

$$F = qvB \sin\theta \tag{1.17}$$

where q is the charge of particles moving with a velocity of $\mathbf{v}$ in a magnetic field whose density is $\mathbf{B}$. The magnitude of the force is equal to the product of velocity, flux density, and sine of the angle between $\mathbf{v}$ and $\mathbf{B}$. The direction of the force can be determined by right corkscrew rule as shown in Figure 1.3. It can be seen that the direction of force is perpendicular to the v-B plane. One Newton of force will be exerted when a charge of 1 coulomb is moving with a velocity of 1 m/s in a magnetic field having a flux density of 1 T with an angle between velocity and flux density of 90°.

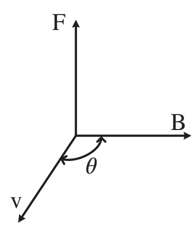


Figure 1.3 Force F, Flux density B, and velocity v vectors 90° displaced.

Equation (1.16) can be written in an incremental form as

$$d\mathbf{F} = dq\mathbf{v} \times \mathbf{B} = dq\frac{d\mathbf{l}}{dt} \times \mathbf{B} = \frac{dq}{dt}\mathbf{l} \times \mathbf{B} = id\mathbf{l} \times \mathbf{B} \tag{1.18}$$

Equation (1.18) shows that the moving charges constitute a line current and only have mathematical significance because, practically, $id\mathbf{l}$ itself does not exist. So, the force can be found by integrating Eq. (1.18).

$$\mathbf{F} = \oint id\mathbf{l} \times \mathbf{B} \tag{1.19}$$

Equations (1.18) and (1.19) are very useful in the analysis and design of electric motors. If the angle between $d\mathbf{l}$ and $\mathbf{B}$ is 90°, then Eq. (1.19) becomes

$$F = ilB \tag{1.20}$$

One can observe the following points when charged particles of various types are moving in a magnetic field:

(i) The magnetic force F is proportional to charge q and velocity v of the particle.
(ii) No force will be experienced by the conductor when it moves parallel to the field.
(iii) The direction of force is perpendicular to both **v** and **B** assuming that the angle between them is not equal to zero.
(iv) The direction of the magnetic force will reverse if the nature of the charge (positive or negative) is reversed.
(v) The magnetic force F is also proportional to $\sin\theta$ where θ is the angle between **v** and **B**.

The differences between Electric ($F_e = qE$) and magnetic forces are:

(i) The direction of the electric force is in the direction of the electric field but the direction of the magnetic force is perpendicular to the magnetic field (Lorentz Force law).
(ii) The electric force acts on a stationary or moving charged particle whereas the magnetic force acts when the charge is moving in a magnetic field but not in parallel to the magnetic field.
(iii) There is work done by the electric force by displacing the charged particle whereas there is no work done by the magnetic force by displacing the charged particle because the angle between the direction of movement of the particle and the magnetic force is 90°.

1.2.6 Biot-Savart Law

Biot and Savart (pronounced 'Bee-oh' and 'Suh-var') law states that magnetic field intensity (dH) at any point P in free space is proportional to the product of differential current segment idl and sine of the angle between the line segment and the line joining the segment and the point; and inversely proportional to the squire of the distance between the line segment and the point as shown in Figure 1.4. The direction of dH can be found using the right-handed corkscrew rule (Figure 1.5a) or right-hand rule (Figure 1.5b). The directions of dH at different points are shown in Figure 1.5c.

$$dH \propto \frac{idl \sin\alpha}{R^2}$$

where R is the distance between the point and the segment and α is the angle between dl and R.

$$dH = k\frac{idl \sin\alpha}{R^2} \tag{1.21}$$

Here k is constant of proportionality which equals to ¼π in SI units.

$$k = \frac{1}{4\pi}$$

$$dH = \frac{1}{4\pi}\frac{idl \sin\alpha}{R^2} \tag{1.22}$$

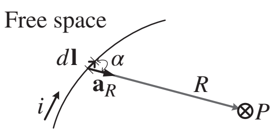


Figure 1.4 Magnetic field intensity $d\mathbf{H}$, directed into the page, due to $id\mathbf{l}$.

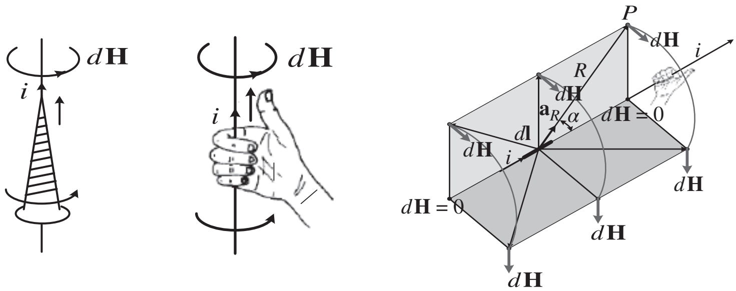


Figure 1.5 (a) Right-handed corkscrew rule (b) Right-hand rule (c) Directions of *dH* at various points.

The equation above can be represented in vector cross-product form

$$d\mathbf{H} = \frac{id\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \frac{id\mathbf{l} \times \mathbf{R}}{4\pi R^3} \tag{1.23}$$

where $d\mathbf{l}$ and $\mathbf{R}$ are differential line vector and vector from $d\mathbf{l}$ to point P respectively; $R = |\mathbf{R}|$ and the unit vector $\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|}$.

Equation (1.23) can also be written for magnetic flux density as

$$d\mathbf{B} = \mu_o \frac{id\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} = \mu_o \frac{id\mathbf{l} \times \mathbf{R}}{4\pi R^3} \tag{1.24}$$

Integrating Eq. (1.24), one gets the magnitude of **B** which is constant around the circle, with a radius of *R*, centring the conductor when the current flowing through a long conductor as

$$B = \frac{\mu_o i}{2\pi R} \tag{1.25}$$

Example 1.1 ***Magnetic Field Surrounding a Thin, Straight Conductor***

Consider a thin, straight wire of finite length carrying a constant current *I* and placed along the *x*-axis as shown in Figure E1.1 [2]. Determine the magnitude and direction of the magnetic field at point *P* due to this current.

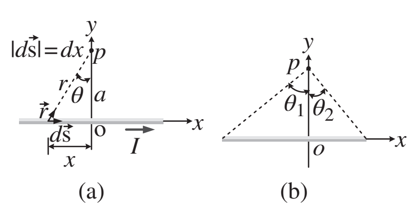


Figure E1.1 (Example 1.1) (a) A thin, straight wire carrying a current *I*. (b) The angles θ_1 and θ_2 used for determining the net field.

Solution

From the Biot–Savart law, we expect that the magnitude of the field is proportional to the current in the wire and decreases as the distance '*a*' from the wire to point *P* increases. We also expect the field to depend on the angles θ_1 and θ_2 in Figure E1.1b. We place the origin at *O* and let point *P* be along the positive *y*-axis, with $\hat{y}$ being a unitvector pointing out of the page.

We are asked to find the magnetic field due to a simple current distribution, so this example is a typical problem for which the Biot–Savart law is appropriate. We must find the field contribution from a small element of current and then integrate over the current distribution.

Let's start by considering a length element $d\vec{s}$l located a distance r from P. The direction of the magnetic field at point P due to the current in this element is out of the page because of $d\vec{s} \times \vec{r}$ is out of the page. In fact, because *all* current elements $Id\vec{s}$ lie in the plane of the page, they all produce a magnetic field directed out of the page at point P. Therefore, the direction of the magnetic field at point P is out of the page and we need only find the magnitude of the field.

Evaluate the cross product in the Biot–Savart law:

$$d\vec{s} \times \vec{r} = \left|d\vec{s} \times \vec{r}\right| \hat{y} = \left[dx \sin\left(\frac{\pi}{2} - \theta\right)\right] \hat{y} = (dx \cos\theta)\, \hat{y}$$

Substitute for Eq. (1.24):

$$d\vec{B} = (dB)\,\hat{y} = \frac{\mu_o I}{4\pi} \frac{dx \cos\theta}{r^2} \hat{y} \tag{1.26}$$

From the geometry in Figure E1.1a, express r in terms of θ:

$$r = \frac{a}{\cos\theta} \tag{1.27}$$

Notice that $\tan\theta = -\frac{x}{a}$ from the right triangle in Figure E1.1a (the negative sign is necessary because $d\vec{s}$ is located at a negative value of x) and solve for x:$x = -a\tan\theta$

Find the differential dx:

$$dx = -a\sec^2\theta d\theta = -\frac{ad\theta}{\cos^2\theta} \tag{1.28}$$

Substitute Eqs. (1.27) and (1.28) into the expression for the z component of the field from Eq. (1.26):

$$dB = -\frac{\mu_o I}{4\pi}\left(-\frac{ad\theta}{\cos^2\theta}\right)\left(\frac{1}{\left(\frac{a}{\cos\theta}\right)^2}\right)\cos\theta = -\frac{\mu_o I}{4\pi a}\cos\theta d\theta \tag{1.29}$$

Integrate Eq. (1.29) overall length elements on the wire, where the subtending angles range from θ_1 to θ_2 as defined in Figure E1.1b:

$$B = -\frac{\mu_o I}{4\pi a}\int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{\mu_o I}{4\pi a}\left(\sin\theta_1 - \sin\theta_2\right) \tag{1.30}$$

We can use this result to find the magnitude of the magnetic field of *any* straight current-carrying wire if we know the geometry and hence the angles θ_1 and θ_2. Consider the special case of an infinitely long, straight wire. If the wire in Figure E1.1b becomes infinitely long, we see that $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = -\frac{\pi}{2}$ for length elements ranging between positions $x = -\infty$ and $x = +\infty$. Because $\left(\sin\theta_1 - \sin\theta_2\right) = \left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = 2$,

Equation (1.30) becomes

$$B = \frac{\mu_o I}{2\pi a} \tag{1.31}$$

Equations (1.30) and (1.31) both show that the magnitude of the magnetic field is proportional to the current, and decreases with increasing distance from the wire, as expected. Equation (1.31) has the same mathematical form as the expression for the magnitude of the electric field due to a long-charged wire.

Example 1.2 A long thin conductor is carrying a current of 1.5 A. Calculate the magnetic field strength at a distance of 20 cm from the wire.

Solution
The magnetic field is given by

$$B = \frac{\mu_o I}{2\pi r} = \frac{\left(4\pi \times 10^{-7}\ T.m/A\right)(1.5\,A)}{2\pi\,(0.20\,m)} = 1.5 \times 10^{-6}\ T$$

Example 1.3 The magnitude of magnetic field of a large tornado was measured as 14 Nano T, pointing north when the Tornado was 10 km east of the measurement observatory. Calculate the current carried up or down the funnel of the Tornado. The vortex can be modelled as a long, straight conductor carrying a current.

Solution
Model the tornado as a long, straight, vertical conductor and imagine grasping it with the right hand so the fingers point northward on the western side of the tornado (that is, at the observatory's location). The thumb is directed downward, meaning that the conventional current is downward. The magnitude of the current is found from $B = \mu_o I/2\pi r$ as

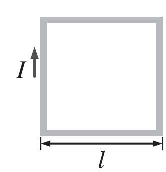
$$I = \frac{2\pi r B}{\mu_o} = \frac{2\pi\left(10 \times 10^{3}\ m\right)\left(1.40 \times 10^{-8}\ T\right)}{4\pi \times 10^{-7}\ T.m/A} = 700\,A$$

Thus, the current is 700 A, downward.

Example 1.4 **A square conducting loop is formed by a wire of sides** $l = 0.4$ m that carries a current $I = 5$ A as shown in Figure E1.4a.

(a) Compute the magnitude and direction of the magnetic field at the centre of the conducting loop.
(b) Compute magnitude and direction of the magnetic field at the centre, if the loop is formed in a circular shape with same current.

Figure E1.4a Square-conducting loop for Example 1.4.



Solution
(a) Use Eq. (1.30) for the field produced by each side of the square.

$$B = \frac{\mu_o I}{4\pi a}\left(\sin\theta_1 - \sin\theta_2\right)$$

Where $\theta_1 = 45^o$, $\theta_2 = -45^o$, and $a = \frac{l}{2}$

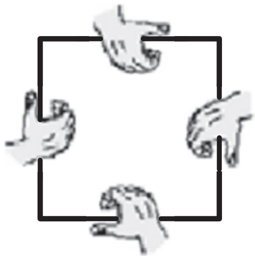


Figure E1.4b Square-conducting loop for Example 1.4.

Each side produces a field into the page. The four sides altogether produce

$$B_{centre} = 4B = 4\frac{\mu_o I}{4\pi a}\left(\sin\theta_1 - \sin\theta_2\right) = \frac{\mu_o I}{\pi\frac{l}{2}}\left[\sin 45^o - \sin\left(-45^o\right)\right] = \frac{2\mu_o I}{\pi l}\left[\frac{2}{\sqrt{2}}\right]$$

$$= \frac{2\sqrt{2}\mu_o I}{\pi l} = \frac{2\sqrt{2}\left(4\pi\times 10^{-7}\ T.m/A\right)(5.0\,A)}{\pi\,(0.4\,m)} = \sqrt{2}\times 10^{-5}\ T$$

$$= 14.15\ \mu T \text{ into the page}$$

(b) For a single circular turn with $4l = 2\pi R$, using equation $B = \frac{\mu_o I}{2R}$ for B at centre of the turn having radius of R,

$$B = \frac{\mu_o I}{2R} = \frac{\mu_o \pi I}{4l} = \frac{\pi\left(4\pi\times 10^{-7}\ T.m/A\right)(5.0\,A)}{4\,(0.4\,m)} = 12.33\ \mu T \text{ into the page}$$

Example 1.5 In Niels Bohr's hydrogen atom model, an electron circles the proton at a distance of 5.29×10^{-11} m with a speed of $2.19\times 10^{6}\ m/s$. Calculate the strength of the magnetic field this motion produces at the proton.

Solution

Treat the magnetic field as that produced in the centre of a ring of radius R carrying current I: from equation for the field is $B = \frac{\mu_o I}{2R}$.

The current due to the electron is

$$I = \frac{\Delta q}{\Delta t} = \frac{e}{2\pi\frac{R}{v}} = \frac{ev}{2\pi R}$$

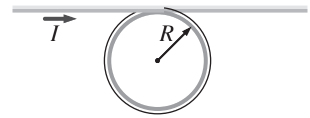
so, the magnetic field is

$$B = \frac{\mu_o I}{2R} = \frac{\mu_o}{2R}\left(\frac{ev}{2\pi R}\right) = \frac{\mu_o}{4\pi}\frac{ev}{R^2} = \left(\frac{\left(4\pi\times 10^{-7}\ T.m/A\right)}{4\pi}\right)$$

$$\times\frac{\left(1.6\times 10^{-19}\ C\right)\left(2.19\times 10^{6}\ m/s\right)}{\left(5.29\times 10^{-11}\ m\right)^2} = 12.5\ T$$

Example 1.6 A conducting loop is formed in circular form and two straight conductors as shown in Figure E1.6. The radius of circular loop $R = 10.0$ cm. The wire lies in the plane of the paper and carries a current $I = 2.00$ A. Find the magnetic field at the centre of the loop.

Figure E1.6 Circular-conducting wire for Example 1.6.



Solution

We can think of the total magnetic field as the superposition of the field due to the long, straight wire, having a magnitude $\frac{\mu_o I}{2\pi R}$ and directed into the page, and the field due to the circular loop, having a magnitude $\frac{\mu_o I}{2R}$ and directed to the page. The resultant magnetic field is:

$$\vec{B} = \left(1 + \frac{1}{\pi}\right)\frac{\mu_o I}{2R} = \left(1 + \frac{1}{\pi}\right)\frac{\left(4\pi \times 10^{-7}\, T.\frac{m}{A}\right)(2.0\, A)}{2\,(0.10\, m)} = 5.52 \times 10^{-6}$$

$$= 4\,\pi T\, \textit{into the page}$$

Example 1.7 The conducting loop of Figure E1.6 is of radius R with two long, straight sections. The wire lies in the plane of the paper and carries a current I. (a) What is the direction of the magnetic field at the centre of the loop? (b) Find an expression for the magnitude of the magnetic field at the centre of the loop.

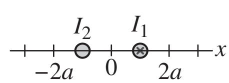
Solution

We can think of the total magnetic field as the superposition of the field due to the long, straight wire, having a magnitude $\frac{\mu_o I}{2\pi R}$ and directed into the page, and the field due to the circular loop, having a magnitude $\frac{\mu_o I}{2R}$ and directed to the page. The resultant magnetic field is:

$$\vec{B} = \left(1 + \frac{1}{\pi}\right)\frac{\mu_o I}{2R} \text{ directed into the page}$$

Example 1.8 Consider two long, straight, parallel conductors that carry currents directed perpendicular to the page as shown in Figure E1.8. Conductor 1 carries a current I_1 into the page (in the negative z-direction) and passes through the x-axis at $x = +a$. Conductor 2 passes through the x-axis at $x = -a$ and carries an unknown current I_2. The total magnetic field at the origin, due to the current-carrying conductors, has a magnitude of $2\mu_o I_1/(2\pi a)$. The current I_2 can have either of two possible values. (a) Determine the value of I_2 with the smaller magnitude, stating it in terms of I_1 and also find its direction. (b) Determine the other possible value of I_2. [2]

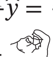
Figure E1.8 Current-carrying conductors for Example 1.8.



Solution

Wire 1 creates at the origin magnetic field:

$$\vec{B}_1 = \frac{\mu_o I}{2\pi r}\text{right-hand rule} = \frac{\mu_o I_1}{2\pi a}$$  $$= \frac{\mu_o I_1}{2\pi a}\hat{y}$$

(a) If the total field at the origin is $\frac{2\mu_o I_1}{2\pi a}\hat{y} = \frac{\mu_o I_1}{2\pi a}\hat{y} + \vec{B}_2$ then the second wire must create a field according to $\vec{B}_2 = \frac{\mu_o I_1}{2\pi a}\hat{y} = \frac{\mu_o I_2}{2\pi(2a)}$  Then $I_2 = 2I_1$ out of paper

(b) The other possibility is $\vec{B}_1 + \vec{B}_2 = \frac{2\mu_o I_1}{2\pi a}\left(-\hat{y}\right) = \frac{\mu_o I_1}{2\pi a}\hat{y} + \vec{B}_2$. Then

$$\vec{B}_2 = \frac{3\mu_o I_1}{2\pi a}\left(-\hat{y}\right) = \frac{\mu_o I_2}{2\pi\,(2a)}$$ and $I_2 = 6I_1$ into the paper.

Example 1.9 An infinitely long wire carrying a current I is in the form of a right angle, as shown in Figure E1.9 Calculate the magnetic field at point P, located a distance x from the bend of the wire. [2]

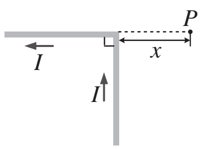


Figure E1.9 Current-carrying conductors for Example 1.9.

Solution

The vertical section of wire constitutes one half of an infinitely long, straight wire at distance x from P, so it creates a field equal to

$$\vec{B} = \frac{1}{2}\left(\frac{\mu_o I}{2\pi x}\right)$$

Hold your right hand with extended thumb in the direction of the current; the field is away from you, into the paper. For each bit of the horizontal section of wire $d\vec{s}$ is to the left and $\hat{r}$ is to the right, so $d\vec{s} \times \hat{r} = 0$. The horizontal current produces zero fields at P. Thus,

$$\vec{B} = \frac{\mu_o I}{4\pi x} \text{ into the paper}$$

Example 1.10 A right-angle bend made at the middle in a long, straight conductor carrying a current I, is shown in Figure E1.10. The bend forms an arc of a circle of radius r as shown in the Figure E1.10 Compute the magnetic field at point P, at the centre of the arc.

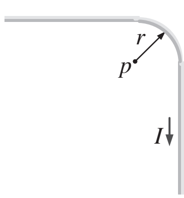


Figure E1.10 Current-carrying conductors for Example 1.8.

Solution

Every element of current creates a magnetic field in the same direction, into the page, at the centre of the arc. The upper straight portion creates one-half of the field that an infinitely long, straight wire would create. The curved portion creates one-quarter of the field that a circular loop produces at its centre. The lower straight segment also creates a field $\frac{1}{2}\left(\frac{\mu_o I}{2\pi r}\right)$

The total field is

$$\vec{B} = \left(\frac{1}{2}\frac{\mu_o I}{2\pi r} + \frac{1}{4}\frac{\mu_o I}{2r} + \frac{1}{2}\frac{\mu_o I}{2\pi r}\right) = \frac{\mu_o I}{2r}\left(\frac{1}{\pi} + \frac{1}{4}\right) = \left(\frac{0.28415\mu_o I}{r}\right) \textit{ into the page}$$

Example 1.11 Choose a flat, circular current loop of radius R carrying a current I. Take the x-axis to be along the axis of the loop, with the origin at the loop's centre. Plot a graph of the ratio of the magnitude of the magnetic field at coordinate x to that at the origin for $x = 0$ to $x = 5R$. *Use Matlab code to solve this problem*

Solution

Along the axis of a circular loop of radius R,

$$B = \frac{\mu_o I R^2}{\left(R^2 + x^2\right)^{\frac{3}{2}}}$$

Or

$$\frac{B}{B_o} = \frac{1}{\left(1 + \left(\frac{x}{R}\right)^2\right)^{\frac{3}{2}}}$$

Where

$$B_o = \frac{\mu_o I}{2R}$$

$\frac{x}{R}$	$\frac{B}{B_o}$
0.0	1
1.0	0.3535
2.0	0.0894
3.0	0.0316
4.0	0.0142
5.0	0.00754

MATLAB code

```
x=[0.0 1.0 2.0 3.0 4.0 5.0];
y=[1 0.3535 0.0894 0.0316 0.0142 0.00754];
xi = linspace(min(x), max(x), 150);   % Evenly-Spaced Interpolation Vector
yi = interp1(x, y, xi, 'spline', 'extrap');
figure(1)
plot(x, y, 'bp')
hold on
plot(xi, yi, '-r')
hold off
grid
xlabel('x/R')
ylabel('B/B_o')
legend('Original Data', 'Interpolation', 'Location', 'NE')
```

The plot is shown in Figure E1.11:

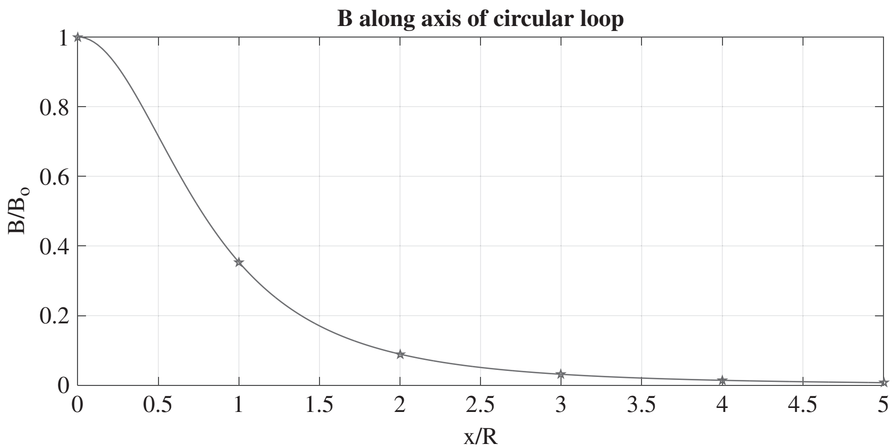


Figure E1.11 Output of Matlab program.

Example 1.12 A current loop is formed as shown in Figure E1.12a that produces a magnetic field at point P, which is at the centre of the arc. If the arc made an angle of $\theta = 45°$ and the radius of the arc is 0.5 m, calculate the magnitude and direction of the field produced at P if the current is 2.0 A?

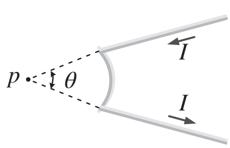


Figure E1.12a Current loop for Example 1.12.

Solution

We use the Biot-Savart law. For bits of wire along the straight-line sections, $\vec{ds}$ is at 0° or 180° to $\hat{r}$, so $\vec{ds} \times \hat{r} = 0$. Thus, only the curved section of wire contributes to $\vec{B}$ at P. Hence, $\vec{ds}$ is tangent to the arc and $\hat{r}$ is radially inward; so $\vec{ds} \times \hat{r} = |\vec{ds}|\, 1 \sin 90^o \otimes = |\vec{ds}| \otimes$. All points along the curve are the same distance $r = 0.600$ m from the field point, so

$$B = \int_{all\ curves} |d\vec{B}| = \int \frac{\mu_o I}{4\pi} \frac{|\vec{ds} \times \hat{r}|}{r^2} = \frac{\mu_o I}{4\pi r^2} \int |\vec{ds}| = \frac{\mu_o I}{4\pi r^2} s$$

Where s is the arc length of the curved wire

$$s = r\theta = (0.5\, m)\left(45^o\right)\left(\frac{2\pi}{360}\right) = 0.392\, m$$

Then

$$B = \frac{\mu_o I}{4\pi r^2} s = \frac{\left(4\pi \times 10^{-7}\ T.m/A\right)(2.0\, A)(0.392\, m)}{4\pi (0.5\, m)^2} = 313.6\, nT \text{ into the page}$$

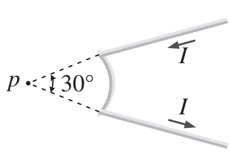


Figure E1.12b Solution for the Figure E1.12a.

Example 1.13 Current of 30 A passes through a long, straight conductor that lies in the left of the x-axis as shown in Figure E1.13. Current of 50 A passes through the second, long wire to the right along the line ($y = 0.280$ m, $z = 0$). (a) Compute the location where the total magnetic field is equal to zero (b) Consider a charge of -2.00 μC moving at a velocity of $150\hat{x}$ Mm/s along the line ($y = 0.100$ m, $z = 0$). Determine the vector magnetic force acting on the charge particle.

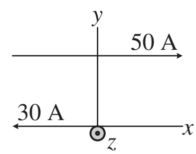


Figure E1.13 Current-carrying conductors for Example 1.13.

Solution

(a) Above the pair of wires, the field out of the page of the 50.0-A current will be stronger than the $(-\hat{z})$ field of the 30.0-A current, so they cannot add to zero. Between the wires, both produce fields into the page. They can only add to zero below the wires, at coordinate $y = -|y|$. Here the total field is

$\vec{B} = \frac{\mu_o I}{2\pi r}$ ☞ $+ \frac{\mu_o I}{2\pi r}$ ☜

$$0 = \frac{\mu_o}{2\pi}\left[\frac{50.0\,A}{|y| + 0.28\,m}\left(-\hat{z}\right) + \frac{30.0\,A}{|y|}\left(\hat{z}\right)\right]$$

$$50.0\,|y| = 30.0\,(|y| + 0.28\,m)$$

$$50.0\,(-y) = 30.0\,(-y + 0.28\,m)$$

$$-20.0y = 30.0\,(0.28\,m)$$

$$y = -0.420\,m$$

(b) At y = 0.100 m the total field is

$\vec{B} = \frac{\mu_o I}{2\pi r}$ ☞ $+ \frac{\mu_o I}{2\pi r}$ ☜

$$\vec{B} = \frac{\mu_o I_1}{2\pi r_1}\left(-\hat{z}\right) + \frac{\mu_o I_2}{2\pi r_2}\left(-\hat{z}\right) = \frac{\mu_o}{2\pi}\left[\frac{I_1}{r_1}\left(-\hat{z}\right) + \frac{I_2}{r_2}\left(-\hat{z}\right)\right] = \frac{\left(4\pi \times 10^{-7}\,T.m/A\right)}{2\pi}$$

$$\times\left[\frac{50.0\,A}{(0.28 - 0.10)\;m}\left(-\hat{z}\right) + \frac{30.0\,A}{0.10\,m}\left(-\hat{z}\right)\right] = 1.16 \times 10^{-4}\left(-\hat{z}\right)\;T$$

The force on the particle is

$$\vec{F} = q\vec{v} \times \vec{B} = \left(-2 \times 10^{-6}\;C\right)\left(150 \times 10^{-6}\;m/s\right)\left(\hat{x}\right) \times \left(1.16 \times 10^{-4}\frac{N.s}{C.m}\right)\left(-\hat{z}\right)$$

$$= 3.47 \times 10^{-2}\left(-\hat{y}\right)\;N$$

Note: Force (F) = Newton (N) = kg.m/s^2 and Magnetic Field (B) = T = kg/A.s^2, hence

$$T = \frac{N}{kg.m/s^2}\,\frac{kg}{\frac{C}{s}s^2} = \frac{N.s}{C.m}$$

Example 1.14 Three long, straight conductors are placed as shown in Figure E1.14a and they carry a current of 3 A each. The currents are coming out of the page.

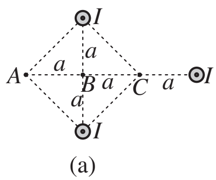


Figure E1.14a Current-carrying conductors for Example 1.14.

Given $a = 1.0$ cm, Compute the magnitude and direction of the magnetic field at (a) point A, (b) point B, and (c) point C.

Solution

Label the wires 1, 2, and 3 as shown in Sol. Figure E1.14b and let the magnetic field created by the currents in these wires be $\vec{B}_1, \vec{B}_2, \vec{B}_3$ respectively.

(a) At point A: $\left|\vec{B}_1\right| = \left|\vec{B}_2\right| = \frac{\mu_o I}{2\pi\left(a\sqrt{2}\right)}$, and $\left|\vec{B}_3\right| = \frac{\mu_o I}{2\pi(3a)}$

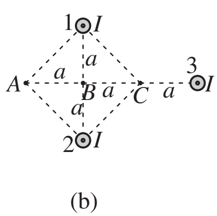


Figure E1.14b Solution for Example E1.14a.

The directions of these fields are shown in Sol. Figure E1.14c.

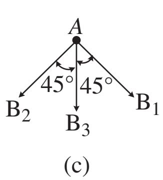


Figure E1.14c Solution for Example E1.14b.

Observe that the horizontal components of $\vec{B}_1$ and $\vec{B}_2$ cancel while their vertical components both add onto $\vec{B}_3$. Therefore, the net field at point A is

$$B_A = B_1 \cos 45^o + B_2 \cos 45^o + B_3 = \frac{\mu_o I}{2\pi a}\left[\frac{2}{\sqrt{2}}\cos 45^o + \frac{1}{3}\right]$$

$$= \frac{\left(4\pi \times 10^{-7}\ T.m/A\right)(3.0\ A)}{2\pi\left(1.0 \times 10^{-2}\ m\right)}\left[\frac{2}{\sqrt{2}}\cos 45^o + \frac{1}{3}\right]$$

$$= 79.95\ \mu T \text{ towards bottom of the page}$$

(b) At point B:$\vec{B}_1$ and $\vec{B}_2$ cancel, leaving

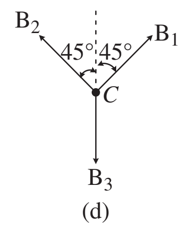
$$B_B = B_3 = \frac{\mu_o I}{2\pi\,(2a)} = \frac{\left(4\pi \times 10^{-7}\ T.m/A\right)(3.0\,A)}{4\pi\,\left(1.0 \times 10^{-2}\ m\right)}$$

$$= 30.0\ \mu T \text{ towards bottom of the page}$$

(c) At point C:$\left|\vec{B}_1\right| = \left|\vec{B}_2\right| = \frac{\mu_o I}{2\pi\left(a\sqrt{2}\right)}$, and $\left|\vec{B}_3\right| = \frac{\mu_o I}{2\pi(a)}$ with the directions shown in Sol. Figure E1.14d. Again, the horizontal components of $\vec{B}_1$ and $\vec{B}_2$ cancel. The vertical components both oppose $\vec{B}_3$ giving

$$B_C = B_1 \cos 45^o + B_2 \cos 45^o - B_3 = \frac{\mu_o I}{2\pi a}\left[\frac{2}{\sqrt{2}}\cos 45^o - 1\right] = 0$$

Figure E1.14d Solution for Example E1.14c.



1.2.7 Ampere Circuital Law

Consider a long conductor carrying current i and, according to Biot-Sarvat law, sets up a magnetic field **H** or **B** around it at a radius of R. Ampere's law is derived using Biot-Savart law. The *line integral* of **B** around a closed path is denoted by

$$\oint \mathbf{B} \cdot d\mathbf{l} \tag{1.32}$$

To evaluate the product $\mathbf{B} \cdot d\mathbf{l}$ consider a small length element $d\mathbf{l}$ on the circular path and sum the products for all elements over the closed circular path. Along this path, the vectors $d\mathbf{l}$ and **B** are parallel at each point (Figure 1.6c), so $\mathbf{B} \cdot d\mathbf{l} = Bdl$. And also, the magnitude of **B** is constant on this circle and is given by Eq. (1.25). Therefore, the sum of the products Bdl over the closed path, which is equivalent to the line integral of $\mathbf{B} \cdot d\mathbf{l}$ is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint B_{\parallel} \cdot dl = B \oint dl = \frac{\mu_o i}{2\pi R}(2\pi R) = \mu_o i$$

where $B_{\parallel}$ is the component of **B** which is parallel to $d\mathbf{l}$
or

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_o i \tag{1.33}$$

In free space $\mathbf{B} = \mu_o \mathbf{H}$

$$\oint \mathbf{H} \cdot d\mathbf{l} = i \tag{1.34}$$

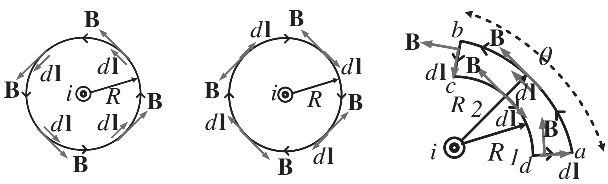


Figure 1.6 The current coming out of page (a) counter clockwise $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_o i$. Here, the angle between **B** and $d\mathbf{l}$ is 0° (b) clockwise $\oint \mathbf{B} \cdot d\mathbf{l} = -\mu_o i$. Here, the angle is 180° (c) Contour *abcda* not enclosing the current $\oint \mathbf{B} \cdot d\mathbf{l} = 0$ because fields along curves get cancelled and field at *da* and *bc* are zero since the angle is 90°.

Concerning Figure 1.6, one can show that the *enclosed current 'i'* in Eq. (1.34) is equal to the integral of the current density **J** over any surface bounded by the closed path *C*.

$$\oint_C \mathrm{H} \cdot d\mathrm{l} = \int_S \mathrm{J} \cdot d\mathrm{S} \tag{1.35}$$

where **J** and **S** are the current density (A/m^2) and surface area (m^2) vectors respectively.

The path need not always be circular. It can take any shape but it should enclose the current line. Equations (1.34) and (1.35) are applicable for static magnetic fields. It can be said that *the line integral of* **H** *around a closed path is equal to the total current flowing through the surface bounded by that path*.

Applying Maxwell's equation for the time-varying electric field, Ampere's circuital law can be modified that the MMF around a closed path C is equal to the sum of the current enclosed by that path due to actual flow of charges and the displacement current due to the time rate of increase of the electric flux (or displacement flux) enclosed by that path; that is

$$\oint_C \mathrm{H} \cdot d\mathrm{l} = \int_S \mathrm{J} \cdot d\mathrm{S} + \frac{d}{dt}\int_S \mathrm{D} \cdot d\mathrm{S} \tag{1.36}$$

where **D** is electric flux density (C/m^2), as shown, for example, in Figure 1.7

The first part of Eq. (1.36) is called conduction or convection current which flows through wires or leads of a capacitor and the second part is called displacement current which flows through a dielectric between palates of a capacitor under time-varying conditions.

In Figure 1.7a, we see that the total current entering the hemisphere through the base plate is equal to the total current leaving the hemisphere through the spherical surface. In Figure 1.7b, the base plate from the hemisphere is removed. The closed path C is formed by

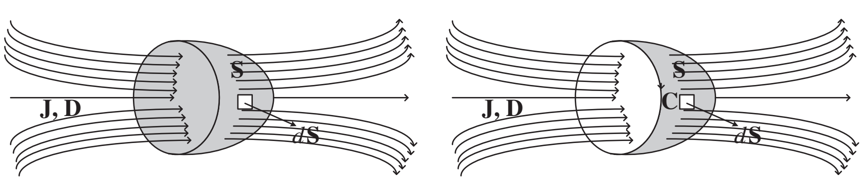


Figure 1.7 Illustration of Ampere's circuital law. (a) Closed hemisphere shell (b) Closed hemisphere shell but base plate removed; edge rim forms a closed path C.

the circular rim of the hemisphere. It can be seen in Figure 1.7b, the current enclosed by C is equal to the current flowing through the surface S bounded by C.

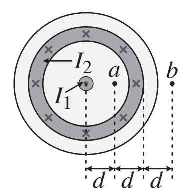
Example 1.15 Niobium metal becomes a superconductor when cooled below 9 K. Its superconductivity is destroyed when the surface magnetic field exceeds 0.100 T. In the absence of any external magnetic field, determine the maximum current a 2.00-mm-diameter niobium wire can carry and remain superconducting [2].

Solution

From $\oint \bar{B} \cdot d\bar{l} = \mu_o I \rightarrow I = \frac{2\pi rB}{\mu_o} = \frac{2\pi(1.0\times10^{-3}m)(0.1\ T)}{(4\pi\times10^{-7}T\cdot m/s)} = 500\ A$

Example 1.16 A coaxial cable is shown in Figure E1.16 where the central conductor is surrounded by a rubber layer, an outer conductor, and another rubber layer. The current in the inner conductor is $I_1 = 2.00$ A coming out of the page and the current in the outer conductor is $I_2 = 5.00$ A going into the page. Assuming the distance $d = 1.00$ mm, determine the magnitude and direction of the magnetic field at (a) point a and (b) point b.

Figure E1.16 Coaxial cable of Example 1.16.



Solution

(a) From Ampère's law, the magnetic field at point a is given by $B_a = \frac{\mu_o I_a}{2\pi r_a}$, where I_a is the net current through the area of the circle of radius r_a. In this case, $I_a = 1.00$ A out of the page (the current in the inner conductor), so

$$B_a = \frac{\mu_o I_a}{2\pi r_a} = \frac{(4\pi\times10^{-7}T\cdot m/s)\,(2.0\,A)}{2\pi\,(1.0\times10^{-3}m)} = 400\ \mu T \text{ towards the top of page}$$

(b) Similarly, at point b: $B_b = \frac{\mu_o I_b}{2\pi r_b}$, where I_b is the net current through the area of the circle having radius r_b. Taking out of the page as positive, $I_b = 2.00\text{ A} - 5.00\text{ A} = -3.00$ A, or $I_b = 3.00$ A into the page. Therefore

$$B_b = \frac{\mu_o I_b}{2\pi r_b} = \frac{(4\pi\times10^{-7}T\cdot m/s)\,(3.0\,A)}{2\pi\,(3.0\times10^{-3}m)} = 199.5\ \mu T \text{ towards the bottom of page}$$

Example 1.17 A toroidal shape magnetic coils is used inside a fusion reactor having an inner radius of 0.700 m and an outer radius of 1.30 m. The magnetic coil has 1000 turns that carries a current of 15.0 kA. Determine the magnitude of the magnetic field inside the toroid along (a) the inner radius and (b) the outer radius.

Solution

(a) $B_{inner} = \frac{\mu_o NI}{2\pi r} = \frac{(4\pi\times10^{-7}T\cdot m/s)(900)(15\times10^3 A)}{2\pi(0.7\ m)} = 3.86\ T$

(b) $B_{outer} = \frac{\mu_o NI}{2\pi r} = \frac{(4\pi\times10^{-7}T\cdot m/s)(900)(15\times10^3 A)}{2\pi(1.3\ m)} = 2.07\ T$

Example 1.18 A packed bundle of 100 long, straight, insulated wires forms a cylinder of radius $R = 0.500$ cm (Figure E1.18). If each wire carries 2.00 A, what are (a) the magnitude and (b) the direction of the magnetic force per unit length acting on a wire located 0.200 cm from the centre of the bundle? (c) Would a wire on the outer edge of the bundle experience a force greater or smaller than the value calculated in parts (a) and (b)? Give a qualitative argument for your answer [2].

Solution

By Ampère's law, the field at the position of the wire at distance r from the centre is due to the fraction of the other 99 wires that lie within the radius r.

$$\oint \overline{B} \cdot d\overline{s} = \mu_o I \rightarrow B(2\pi r) = \mu_o \left[99I\left(\frac{\pi r^2}{\pi R^2}\right)\right] \rightarrow B = \frac{\mu_o(99I)}{2\pi r}\left(\frac{r^2}{R^2}\right) = \frac{\mu_o(99I)}{2\pi R}\left(\frac{r}{R}\right)$$

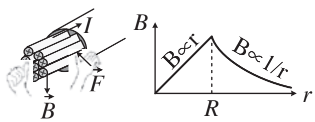


Figure E1.18 Cable and flux density vs radius of Example 1.18.

The field is proportional to r, as shown in Sol. Figure E1.18. This field points tangent to a circle of radius r and exerts a force $\overline{F} = I\overline{l} \times \overline{B}$ on the wire towards the centre of the bundle. The magnitude of the force is

$$\frac{F}{l} = IB\sin\theta = I\left[\frac{\mu_o(99I)}{2\pi R}\left(\frac{r}{R}\right)\right]\sin 90^o = \frac{\mu_o(99)I^2}{2\pi R}\left(\frac{r}{R}\right)$$

$$= \frac{\left(4\pi \times 10^{-7}T \cdot m/A\right)(99)(2.00\,A)^2}{2\pi\left(0.5 \times 10^{-2}m\right)}\left(\frac{0.2}{0.5}\right) = 6.34 \times 10^{-2}N/m$$

(a) 6.34×10^{-3} N/m
(b) Referring to the figure, the field is clockwise, so at the position of the wire, the field is downwards, and the force is inward towards the centre of the bundle.
(c) $B \propto r$, so B is greatest at the outside of the bundle. Since each wire carries the same current, F is greatest at the outer surface.

1.2.8 Faraday's Law

Faraday's law states that voltage will be induced across a loop when there is a rate of change of flux. If the circuit is closed, the induced current will flow. For N number of loops or turns, the law can be stated in mathematic form as

$$\varepsilon = -N\frac{d\phi}{dt} \tag{1.37a}$$

The negative sign is because of Lenz's law which states that the induced current must be directed so that the magnetic field it produces opposes the change in the external magnetic flux. There will be opposition not only to the change of flux but also opposition to the movement of coil or magnets. Figure 1.8 explains Lenz's law.

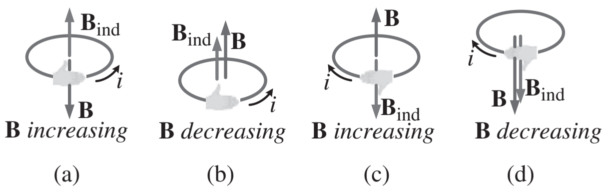


Figure 1.8 Induced current (*i*) induces field ($\mathbf{B}_{ind}$) which opposes the change. Fingers point induced current (*i*) direction while thumb points the induced field ($\mathbf{B}_{ind}$)(a) increasing **B** induces *i* with $\mathbf{B}_{ind}$ *opposes the change*, (b) decreasing **B** induces *i* with $\mathbf{B}_{ind}$ *opposes the change* (c) increasing **B** induces *i* with $\mathbf{B}_{ind}$ *opposes the change* (d) decreasing **B** induces *i* with $\mathbf{B}_{ind}$ *opposes the change.*

Faraday's law can be re-written as (considering only the magnitude, without direction);

$$\varepsilon = \frac{d\,(N\phi)}{dt}, \quad \text{and } L = \frac{N\phi}{i}$$

Therefore, $N\phi = Li$

$$\varepsilon = \frac{d\,(N\phi)}{dt} = \frac{d\,(Li)}{dt} = L\frac{di}{dt}$$

$$\varepsilon = L\frac{di}{dt} \tag{1.37b}$$

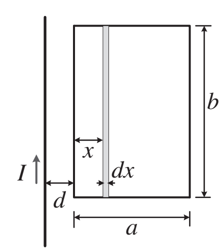
Generally, one can change the magnetic flux through a coil by the following means:

a. Change the magnitude **B** of the magnetic field with time within the coil.
b. Change the total area of the coil with time (for example by expanding the coil).
c. Change the portion of that area that lies within the magnetic field (for example, by sliding it into or out of the field).
d. Change the angle between the direction of the magnetic field **B** and normal to the plane of the coil. This can be achieved by rotating the coil in the magnetic field.
e. Any combination of the above will affect the change in the flux.

Emf can also be induced by moving the coil or conductor in a stationary magnetic field as well as by moving the conductor in a changing magnitude of the field with time. This induced emf is called *motional* emf.

Example 1.19 A long, straight wire carries a current I. A rectangular loop with two sides parallel to the straight wire has sides a and b, with its near side a distance d from the straight wire, as shown in Figure E1.19. (a) Compute the magnetic flux through the rectangular loop. (Hint: Calculate the flux through a strip of area $dA = b\,dx$ and integrate from $x = d$ to $x = d + a$.) (b) Evaluate your answer for $a = 5$ cm, $b = 10$ cm, $d = 2$ cm, and $I = 20$ A [3].

Figure E1.19 Rectangular loop of Example 1.19.



Solution

We can use the hint to set up the element of area dA and express the flux $d\phi_m$ through it and then carry out the details of the integration to express ϕ_m.

(a) Express the flux through the strip of area dA:	$d\phi_m = BdA$ where dA = bdx.
Express B at a distance x from a long, straight wire:	$B = \frac{\mu_o}{2\pi}\frac{I}{r} = \frac{\mu_o}{4\pi}\frac{2I}{x} = \frac{\mu_o}{2\pi}\frac{I}{x}$
Substitute to obtain:	$d\phi_m = \frac{\mu_o}{2\pi}\frac{I}{x}bdx = \frac{\mu_o Ib}{2\pi}\frac{dx}{x}$
Integrate from x = d to x = d + a:	$\phi_m = \frac{\mu_o Ib}{2\pi}\int_d^{d+a}\frac{dx}{x} = \frac{\mu_o Ib}{2\pi}\ln\left(\frac{d+a}{d}\right)$
(b) Substitute numerical values and evaluate ϕ_m:	$\phi_m = \frac{\mu_o Ib}{2\pi}\ln\left(\frac{d+a}{d}\right)$ $= \frac{(4\pi\times10^{-7}\,N/A^2)\,(20\,A)\,(0.1\,m)}{2\pi}\ln\left(\frac{(2+5)\,cm}{2\,cm}\right)$ $= 5.01\times10^{-7}\,Wb$

Example 1.20 A rectangular coil in the plane of the page has dimensions a and b. A long wire that carries a current I is placed directly above the coil (Figure E1.20). (a) Obtain an expression for the magnetic flux through the coil as a function of x for $0 \leq x \leq 2b$. (b) For what value of x, is the flux through the coil a maximum? For what value of x, is the flux a minimum? [3]

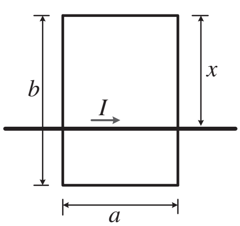


Figure E1.20 Rectangular loop of Example 1.19.

Solution

We can use its definition to express the flux through the rectangular region and Ampere's law to relate the magnetic field to the current in the wire and the position of the long, straight wire.

(a) Note that for 0 ≤ x ≤ b, B is symmetric about the wire, into the paper for the region below the wire and out of the paper for the region above the wire. Thus, for area $2(b-x)a$:	$\phi_{m\_net} = 0$
To find the flux through the remaining area of the rectangle, express the flux through a strip of area dA:	$d\phi_m = BdA$ where $dA = adx$.
Using Ampere's law, express B at a distance x from a long, straight wire:	$B = \frac{\mu_o}{2\pi}\frac{I}{r} = \frac{\mu_o}{4\pi}\frac{2I}{x} = \frac{\mu_o}{2\pi}\frac{I}{x}$

Substitute to obtain:	$d\phi_m = \frac{\mu_o}{2\pi}\frac{I}{x}adx = \frac{\mu_o Ia}{2\pi}\frac{dx}{x}$
For $0 \le x \le b$, integrate from $x' = b - x$ to $x' = x$:	$\phi_m\,(0 \le x \le b) = \frac{\mu_o Ia}{2\pi}\int_{b-x}^{x}\frac{dx'}{x'} = \frac{\mu_o Ia}{2\pi}\ln\left(\frac{x}{b-x}\right)$
For $x \ge b$, integrate from $x' = x$ to $x' = x + b$:	$\phi_m\,(x \le b) = \frac{\mu_o Ia}{2\pi}\int_{x}^{x+b}\frac{dx'}{x'} = \frac{\mu_o Ia}{2\pi}\ln\left(\frac{x+b}{x}\right)$
(b) From the expressions derived in (a), we see that $\phi_m \to \infty$ as:	$x \to 0$
The flux is a minimum ($\phi_m = 0$) for:	$x = \frac{1}{2}b$ as expected from symmetry.

Example 1.21 A uniform magnetic field B is established perpendicular to the plane of a loop of radius 5 cm, resistance 0.4 Ω, and negligible self-inductance. The magnitude of $\vec{B}$ is increasing at a rate of 40 mT/s. Find (a) the induced emf $\mathcal{E}$ in the loop, (b) the induced current in the loop, and (c) the rate of joule heating in the loop. [3]

Solution

We can find the induced emf by applying Faraday's law to the loop. The application of Ohm's law will yield the induced current in the loop and we can find the rate of joule heating using $P = I^2R$.

(a) Apply Faraday's law to express the induced emf in the loop in terms of the rate of change of the magnetic field:	$\lvert\mathcal{E}\rvert = \frac{d\phi_m}{dt} = \frac{d}{dt}(AB) = A\frac{dB}{dt} = \pi r^2\frac{dB}{dt}$
Substitute numerical values and evaluate ε:	$\lvert\mathcal{E}\rvert = \pi(0.05\,m)^2\,(40\,\mathrm{m}T/s) = 0.314\,\mathrm{m}V$
(b) Using Ohm's law, relate the induced current to the induced voltage and the resistance of the loop and evaluate I:	$I = \frac{\mathcal{E}}{R} = \frac{0.314\,mV}{0.4\,\Omega} = 0.785\,\mathrm{m}A$
(c) Express the rate at which power is dissipated in a conductor in terms of the induced current and the resistance of the loop and evaluate P:	$P = I^2R = (0.785\,\mathrm{m}A)^2(0.4\,\Omega) = 0.247\mu\mathrm{W}$

Example 1.22 The flux through a loop is given by $\phi_\mathrm{m} = (t^2 - 4\,t) \times 0^{-1}$ Wb, where t is in seconds. (a) Find the induced emf $\mathcal{E}$ as a function of time. (b) Find both ϕ_m and ε at $t = 0$, $t = 2$ s, $t = 4$ s, and $t = 6$ s [3].

Solution

Given ϕ_m as a function of time, we can use Faraday's law to express $\mathcal{E}$ as a function of time.

(a) Apply Faraday's law to express the induced emf in the loop in terms of the rate of change of the magnetic field:	$\mathcal{E} = -\frac{d\phi_m}{dt} = -\frac{d}{dt}\left[\left(t^2 - 4t\right) \times 10^{-1}\mathrm{Wb}\right] = -(2t - 4) \times \frac{10^{-1}\mathrm{Wb}}{\mathrm{s}} = -(0.2t - 0.4)\,\mathrm{V}$
(b) Evaluate ϕ_m at $t = 0$:	$\phi_m(0s) = [0^2 - 4(0)] \times 10^{-1}\mathrm{Wb} = 0$
Evaluate $\mathcal{E}$ at $t = 0$:	$\mathcal{E}\,(0s) = -[0.2\,(0) - 0.4]\,V = 0.4\,V$

Proceed as above to complete the table to the right:

t	ϕ_m	$\mathcal{E}$
(s)	(Wb)	(V)
0	0	0.4
2	−0.4	0
4	0	−0.4
6	1.2	−0.8

Example 1.23 (a) For the flux given in Example 1.22, sketch graphs of ϕ_m and ε versus t. (b) At what time is the flux at its minimum? What is the emf at this time? (c) At what times is the flux zero? What is the emf at these times? [3]

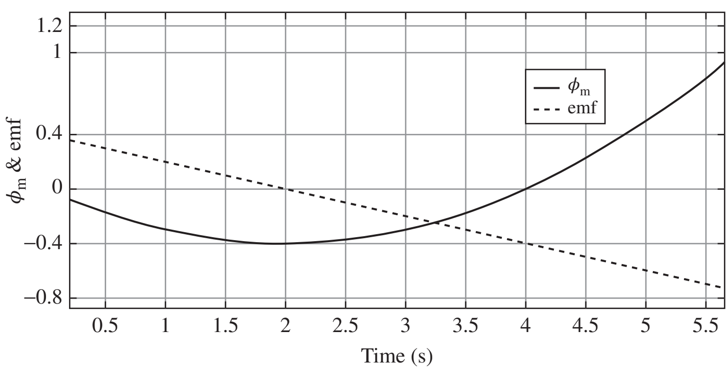


Figure E1.23 The flux and the induced emf as functions of time, t.

Solution

We can find the time at which the flux is at its minimum by looking for the lowest point on the graph of $\mathcal{E}$ versus t and the emf at this time by determining the value of V at this time from the graph. We can interpret the graphs to find the times at which the flux is zero and the corresponding values of the emf.

(a) The flux, ϕ_m, and the induced emf, $\mathcal{E}$, are shown in Figure E1.23 as functions of t in the following graph. The solid curve represents ϕ_m, the dashed curve represents $\mathcal{E}$.
(b) Referring to the graph, we see that the flux is a minimum at t = 2 s and that $\mathcal{E} = 0$ V at this instant.
(c) The flux is zero at t = 0 and t = 4 s. At these times, $\mathcal{E}$=0.4 V and − 0.4 V, respectively.

1.2.8.1 Motional emf

A conductor of length l as shown in Figure 1.9 moves through a uniform magnetic field **B** directed into the page. It is assumed that the conductor moves in a direction perpendicular to the field **B** with constant velocity using some external agent. A magnetic force $\mathbf{F}_B$ acts

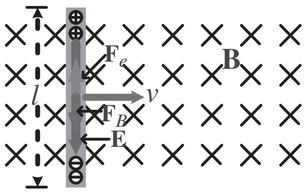


Figure 1.9 A conductor of length l moves with a velocity v through a uniform magnetic field B whose direction is perpendicular to v. Electrons are accumulated at the lower end of the conductor and equal and opposite charges become accumulated at the upper end of the conductor because of magnetic force F_B. This establishes an electric field E in the conductor. In steady-state, the electric and magnetic forces on the electrons in the wire are balanced.

upon the electrons in the conductor which is given as $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$ whose direction is along the length l, perpendicular to both $\mathbf{v}$ and $\mathbf{B}$ (Eq. (1.16)). Electrons are accumulated at the lower end of the conductor and equal and opposite charges are accumulated at the upper end of the conductor because of magnetic force $\mathbf{F}_B$. The accumulation of charges continues until the downward magnetic field $\mathbf{F}_B$ $(= qvB)$ and upward electric force $\mathbf{F}_e(= qE)$ become balanced. This establishes an electric field $\mathbf{E}$ in the conductor. At equilibrium, the forces on the electrons become balanced.

$$qE = qvB \quad or \quad E = vB \tag{1.38}$$

The potential difference between any two points in the electric field is given as

$$V_{AB} = \Delta V = V_B - V_A = -\int_A^B E \cdot dl = -El$$

or

$$|\Delta V| = V_B - V_A = El \tag{1.39}$$

Therefore, for the equilibrium condition

$$\Delta V = El = Blv \tag{1.40}$$

In Figure 1.9, the upper end of the conductor is at a higher potential than the lower end. Hence, the potential difference between two ends of a conductor is established when the conductor moves in a uniform magnetic field. If the movement of the conductor is reversed, then the potential difference polarity is also reversed.

A conducting bar of length l is forced to move along two conducting rails with a velocity $\mathbf{v}$ by applying a force $\mathbf{F}_{app}$ in a uniform magnetic field $\mathbf{B}$, whose direction is into the page, as shown in Figure 1.10a. Assume that the bar and the rails have negligible resistance. A resistance R is connected between the rails to complete the circuit. Free charges in the bar move towards the ends of the bar which sets up an electric potential difference between the ends. The potential difference makes the current i flow in the resistance. The bar moves a distance of x at that instant. The area covered during that time is xl. The total flux (ϕ) equals to Blx Webers.

According to Faraday's law

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -\frac{d}{dt}(Blx) = -Bl\frac{dx}{dt}$$

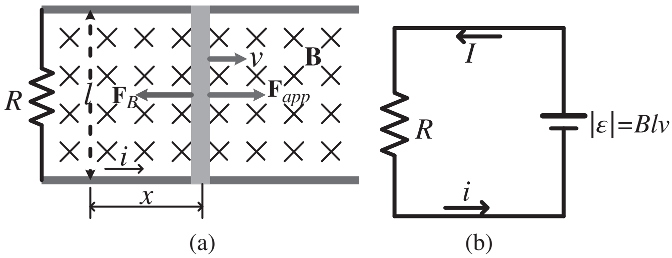


Figure 1.10 (a) A conducting bar has negligible resistance. Due to an applied force F_{app} the bar slides with a velocity v along two conducting rails. The magnetic force F_B opposes the motion, and the induced current *i* flows through the resistance R from the upper end to the lower end. (b) The equivalent circuit diagram.

Since

$$\frac{dx}{dt} = v$$

Therefore, motional emf is

$$\mathcal{E} = -Blv \tag{1.41}$$

Because the resistance of the circuit is R, the magnitude of the induced current is

$$i = \frac{|\mathcal{E}|}{R} = \frac{Blv}{R} \tag{1.42}$$

The equivalent circuit diagram is shown in Figure 1.10b.

Equation (1.41) can also be derived using the principle of conservation of energy. When the bar is moved by the force F_{app} with a velocity v, then at equilibrium, F_{app} experiences the equal and opposite magnetic force F_B.

Using Eq. (1.20)

$$F_B = -F_{app} = ilB$$

The power delivered by the magnetic force is

$$P = \left(F_B\right)(v) = (ilB)(v) = \left[\frac{Blv}{R}(lB)\right](v) = \frac{B^2l^2v^2}{R} = \frac{\mathcal{E}^2}{R} \tag{1.43}$$

Equation (1.38) shows that power is the rate of energy delivered to the resistance R.

Example 1.24 In Figure E1.24, $R = 6.00\,\Omega$, $l = 1.20\,\text{m}$, and a uniform 2.50-T magnetic field is directed into the page. (a) Find the speed of the bar to induce a current of 0.5 A in the resistor. (b) Find applied force to obtain a speed of 2 m/s to induce 0.5 A of current. (c) Find the rate of energy delivered to the resistor.

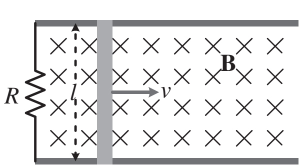


Figure E1.24 Conductor moving in the presence of magnetic field.

Solution

(a) From Eq. (1.40),

$$i = \frac{Blv}{R}$$

or speed is

$$v = \frac{iR}{Bl} = \frac{(0.5\,A)\,(6.0\,\Omega)}{(2.5\,T)\,(1.2\,m)} = 1\,m/s$$

(b)

$$F_B = \left|F_{app}\right| = (i)\,(lB) = \left(\frac{Blv}{R}\right)(lB) = \frac{B^2l^2v}{R} = \frac{(2.5)^2(1.2)^2 \times 2}{6} = 3.0\,\text{N}$$

(c) Rate of energy means power delivered to the resistor

$$P = \frac{B^2l^2v^2}{R} = \frac{(2.5)^2(1.2)^2 \times (2)^2}{6} = 6.0\,Watts$$

Alternatively

$$P = F_Bv = 3.0\,\text{N} \times 2.0\frac{\text{m}}{\text{s}} = 6.0\ \text{W}atts$$

Example 1.25 An electrical conductor (or rod) of l m in length and mass of m kg is made to move by a force F on two frictionless, horizontally placed rails at a speed of v m/s in a magnetic field B which is directed into the page as shown in Figure E1.24. A resistance of $R\ \Omega$ is connected between the conducting rails. Establish a relationship for the distance x m it travels in terms of R, B, l, m, and v.

Solution

Let the initial speed of the rod be v m/s and at a later time of t seconds the speed is u m/s.

The motional emf is given by

$$\mathcal{E} = Blv$$

Induced current is

$$i = \frac{\mathcal{E}}{R} = \frac{Blv}{R}$$

The magnetic force is opposite to applied mechanical force (= ma)

$$F = -ilB = -\frac{B^2l^2v}{R} = ma = m\frac{dv}{dt}$$

Separating the variables

$$-\frac{B^2l^2}{mR}dt = \frac{dv}{v}$$

Integrating both sides

$$\int_0^t \left(-\frac{B^2l^2}{mR}\right) dt = \int_v^u \frac{dv}{v}$$

Gives

$$-\frac{B^2l^2}{mR}\,(t-0) = ln\,(u) - ln\,(v) = ln\left(\frac{u}{v}\right)$$

Or

$$\frac{u}{v} = e^{-\frac{B^2l^2}{mR}t}$$

Or

$$u = ve^{-\frac{B^2l^2}{mR}t} = \frac{dx}{dt}$$

The distance travelled is given by

$$dx = ve^{-\frac{B^2l^2}{mR}t}dt$$

Integrating both sides with limits of x is 0 to $x_{\max}$ and that of time t is 0 to ∞.

$$\int_0^{x_{max}} dx = \int_0^{\infty} ve^{-\frac{B^2l^2}{mR}t}dt$$

$$[x]_0^{x_{max}} = v\int_0^{\infty} e^{-\frac{B^2l^2}{mR}t}dt = v\left[\frac{1}{\left(-\frac{B^2l^2}{mR}\right)}e^{-\frac{B^2l^2}{mR}t}\right]_0^{\infty}$$

Or

$$x_{max} - 0 = v\left(-\frac{mR}{B^2l^2}\right)\left(e^{-\infty} - e^0\right)$$

Or

$$x_{max} = \left(-\frac{vmR}{B^2l^2}\right)(0-1) = \frac{vmR}{B^2l^2}$$

Alternative Method: applying Newton's second law

$$-\frac{B^2l^2v}{R} = \left(-\frac{B^2l^2}{R}\right)\frac{dx}{dt} = m\frac{dv}{dt}$$

or

$$\left(-\frac{B^2l^2}{R}\right)dx = mdv$$

Integrating both sides from initial to the stopping point gives

$$\int_0^{x_{max}} \left(-\frac{B^2l^2}{R}\right)dx = \int_v^0 mdv$$

Or

$$\left(-\frac{B^2l^2}{R}\right)[x]_0^{x_{max}} = m\,[v]_v^0$$

Or

$$\left(-\frac{B^2l^2}{R}\right)\left(x_{max} - 0\right) = m\,(0 - v)$$

Therefore

$$x_{max} = \frac{vmR}{B^2l^2}$$

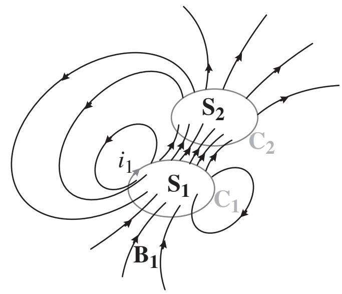


Figure 1.11 Magnetic coupling between two conducting loops.

1.2.9 Flux Linkages and Induced Voltages

Flux linkage is an extension of magnetic flux. Flux linkage is equivalent to the total flux passing through the surface formed by a coil and it is determined by knowing the value of flux (ϕ) and the number of turns as the following relation expresses.

$$\lambda = \int \mathbf{B} \cdot d\mathbf{S} = \int \mathcal{E} dt = N\phi \tag{1.44}$$

The closed loops C_1 and C_2 are placed near to each other whose surface area S_1 and S_2 respectively. C_1 carries current i_1 as shown in Figure 1.11. C_1 produces flux $\mathbf{B}_1$ because of the flow of current i_1. Part of the total flux passes through the surface S_2 bounded by C_2. Let that flux be ϕ_{12}. The magnetic flux ϕ_{12} is produced because of current i_1 flowing in C_1 is given by

$$\phi_{12} = \int_{S_2} \mathrm{B} \cdot d\mathrm{S} \tag{1.45}$$

Consider Figure 1.12 which has two turns. The magnetic flux ϕ_{12} links surface S_{21} and S_{22}. The total surface area will be $S_2 = S_{21} + S_{22}$. We see that the same magnetic flux passes through S_2 twice in the same direction. So, the magnetic flux linking C_2 must be $2\phi_{12}$. Hence, it can be said that the magnetic flux linkage is the product of ϕ_{12} and the number of turns.

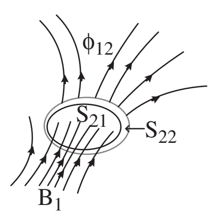


Figure 1.12 Flux linkage with a loop of 2 turns.

If C_2 has N_2 turns, then the magnetic flux linkage is given as

$$\lambda_{12} = N_2\phi_{12} \tag{1.46}$$

Here, we assumed the notation of subscript 12 as linking of flux due to current flow in C_1 to C_2. Hence, in general, the flux linkage is stated as

$$\lambda = N\phi \tag{1.47}$$

1.2.10 Induced Voltages

Induces voltage as per Faraday's law is given by

$$\mathcal{E} = \frac{d\lambda}{dt} = -N\frac{d\phi}{dt} \tag{1.48}$$

or by neglecting the negative sign, flux linkage is given by

$$\lambda = \int \left(N \frac{d\phi}{dt} \right) dt = \int N d\phi = N\phi \tag{1.49}$$

This states that Faraday's law i.e. rate of change of flux linkage will induce an electromotive force (emf).

1.2.11 Induced Electric Fields

Electric fields are induced by both static charges and changing magnetic fields. Both electric fields exert a force equal to $q_o E$. But there is a difference between them. When a positive test charge moved from point A to point B in an electric field **E**, electric force $q_o E$ is exerted on the test charge, and then we say work is done by the conservative force. The work done is given by

$$W = -\int \mathbf{F}_e \cdot d\mathbf{l} = -q_o \int \mathbf{E} \cdot d\mathbf{l} \tag{1.50}$$

The potential difference between point A and point B is

$$\Delta V = V_B - V_A = \frac{W}{q_o} = -\int_A^B \mathbf{E} \cdot d\mathbf{l} \tag{1.51}$$

But when the test charge moves from a particular point and comes back to the same point, then we say there is no work is done by Electric force (F_e). Therefore, the closed integral will be equal to zero.

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.52}$$

Now, consider a copper ring of radius r is immersed in a gradually increasing magnetic field of radius R whose direction is into the page as shown in Figure 1.13a, an emf (ε) is induced which makes current i to flow in the ring in a counter clockwise direction as per Lenz's law. The current set up an electric field in the same direction. As stated earlier, the

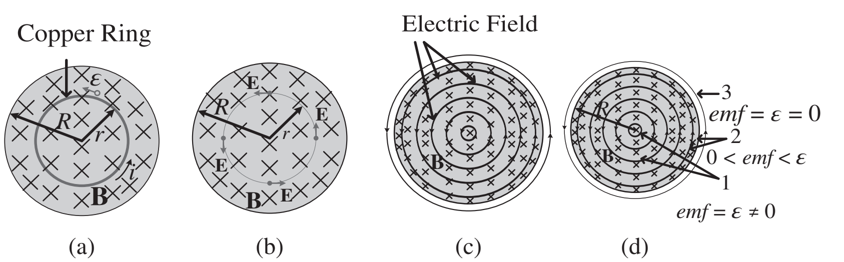


Figure 1.13 (a) When the magnetic field is changing at a uniform rate, then there will be inducement of current which flows in the copper ring of radius r. (b) Even if no ring is in the changing magnetic field, there is an induced electric field E. The four points are shown. (c) The electric fields are induced in every circular imaginary ring. (d) Three similar closed paths that enclose identical areas. Equal EMFs are induced around paths of region 1, which lie entirely within the region of changing the magnetic field. A smaller emf is induced around path 2, which only partially lies in that region. No net emf is induced around path 3, which lies entirely outside the magnetic field [4].

electric field is induced when there is a change of flux even if there is no current flowing. This has been shown in Figures 1.13b,c in that no copper ring is placed in the magnetic field which is changing with time. This induced electric field originates from a point and returns to the same point, or in other words, it has a closed path whereas the electric field, due to static charges, is radial i.e. it originates from a positive charge and ends up a negative charge. Hence, the close line integral of the Electric field due to static charges is zero (Eq. (1.50)). The electric field **E**, induced due to changing magnetic field **B** with time, induces EMFs. The imaginary ring outside B induces zero ems, rings near outer edge induces non zero emf but less than the inner rings. This is shown in Figure 1.13d.

1.2.12 Reformulation of Faraday's Law

In Figure 1.13b, a work W has done on a particle q_o in moving around the circular path is equal to εq_o, where ε is the induced emf. Another way of expressing work W done is

$$W = \int \mathbf{F} \cdot d\mathbf{l} = \left(q_o E\right)(2\pi r) \tag{1.53}$$

where $q_o E$ is the magnitude of the force acting on the charge and $2\pi r$ is the distance over which that force acts.

Or the Eq. (1.46) can be written as

$$W = \left(q_o E\right)(2\pi r) = q_o \mathcal{E}$$

or

$$\mathcal{E} = 2\pi r E \tag{1.54}$$

Rewriting Eq. (1.53)

$$W = \int \mathbf{F} \cdot d\mathbf{l} = q_o \oint \mathbf{E} \cdot d\mathbf{l} = q_o \mathcal{E}$$

or

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} \tag{1.55}$$

The integral in Eq. (1.54) becomes Eq. (1.53) when Figure 1.13b is considered.

According to Faraday's law or rewriting Eq. (1.37a)

$$\mathcal{E} = -\frac{d\phi}{dt}$$

Therefore, reformulated Faraday's law equation shall be

$$\oint_C \mathrm{E} \cdot d\mathrm{l} = -\frac{d\varphi}{dt} = -\frac{d}{dt}\int_S \mathrm{B} \cdot d\mathrm{S} \tag{1.56}$$

where S is the surface area enclosed by C as shown in Figure 1.14

Faraday's law can be stated as a time-varying magnetic field that gives rise to an electric field. Specifically, the electromotive force around a closed path C is equal to the negative of the time rate of increase of the magnetic flux enclosed by that path.

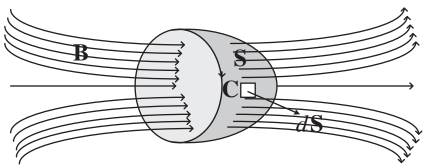


Figure 1.14 Illustration of Faraday's law.

Hence, Faraday's law can be applied to any closed path. In Figure 1.13d, it is shown that when there is a change in a magnetic field, then EMFs are induced: the inner most region '1' experiences inducement of emf = ε, outside magnetic field region '3' emf = 0, and region '2' near the outer edge of the changing magnetic field emf < ε.

Example 1.26 A conducting wire moves in a plane perpendicular to magnetic field of 40 mT. The length of the wire is 50 cm and the speed of the wire is 10 m/s. Determine (a) the force exerted on an electron in the wire, (b) the electrostatic field $\vec{E}$ in the wire, and (c) the potential difference produced between the ends of the wire.

Solution
We can apply the equation for the force on a charged particle moving in a magnetic field to find the magnetic force acting on an electron in the rod. We can use $\vec{E} = \vec{v} \times \vec{B}$ to find E and $V = El$, where l is the length of the rod, to find the potential difference between its ends.

(*a*) Relate the magnetic force on an electron in the rod to the speed of the rod, the electronic charge, and the magnetic field in which the rod is moving:	$\vec{F} = q\vec{v} \times \vec{B}$ and $F = qvB\sin\theta$
Substitute numerical values and evaluate F:	$F = (1.6 \times 10^{-19}\text{C})(10\text{ m/s})(0.04\text{ T})\sin 90^o = 6.4 \times 10^{-20}$
(*b*) Express the electrostatic field $\vec{E}$in the rod in terms of the magnetic field $\vec{B}$	$\vec{E} = \vec{v} \times \vec{B}$ and $E = vB\sin\theta$
Substitute numerical values and evaluate B:	$E = (10\text{ m/s})(0.04\text{ T})\sin 90^o = 0.4\text{ V/m}$
(*c*) Relate the potential difference between the ends of the rod to its length l and the electric field E:	$V = El$
Substitute numerical values and evaluate V:	$V = (0.4\text{ V/m})(0.5\text{ m}) = 0.20\text{ V}$

Example 1.27 Find the speed of the rod in Example 1.26 if the potential difference between the ends is 6 V.

Solution
We can use $\vec{E} = \vec{v} \times \vec{B}$ to relate the speed of the rod to the electric field in the rod and magnetic field in which it is moving and $V = El$ to relate the electric field in the rod to the potential difference between its ends.

Express the electrostatic field $\vec{E}$ in the rod in terms of the magnetic field $\vec{B}$ and solve for v:	$\vec{E} = \vec{v} \times \vec{B}$ and $v = \dfrac{E}{B \sin\theta}$
Relate the potential difference between the ends of the rod to its length l and the electric field E and solve for E:	$V = El \Rightarrow E = \frac{V}{l}$
Substitute for E to obtain:	$v = \dfrac{V}{Bl \sin\theta}$
Substitute numerical values and evaluate v:	$v = \dfrac{6\,\text{V}}{(0.04\,\text{T})\,(0.5\,\text{m})} = 300\,\text{m/s}$

Example 1.28 Given a conducting rod of 50 cm length moving at velocity of 10 m/s in the presence of a magnetic field of 1 T as shown in Figure E1.28. The resistive load R = 1 Ω Calculate (a) the induced emf in the circuit, (b) the current in the circuit, and (c) the force needed to move the conductor with constant velocity assuming negligible friction (d) the power input by the force found in Part (c), and (e) the rate of joule heat production I^2R.

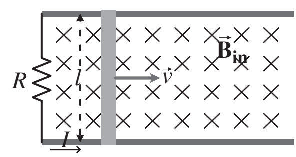


Figure E1.28 A conductor moving in the presence of magnetic field.

Solution

Because the speed of the rod is constant, an external force must action the rod to counter the magnetic force acting on the induced current. We can use the motional-emf equation $\mathcal{E} = vBl$ to evaluate the induced emf, Ohm's law to find the current in the circuit, Newton's 2nd law to find the force needed to move the rod with constant velocity, and $P = Fv$ to find the power input by the force.

(*a*) Relate the induced emf in the circuit to the speed of the rod, the magnetic field, and the length of the rod:	$\mathcal{E} = vBl = (10\,\text{m/s})\,(1.0\text{T})\,(0.5\,\text{m}) = 5.0\,\text{V}$
(*b*) Using Ohm's law, relate the current in the circuit to the induced emf and the resistance of the circuit:	$I = \dfrac{\mathcal{E}}{R} = \dfrac{5.0\,\text{V}}{1\,\Omega} = 5.0\,\text{A}$ Note that, because the rod is moving to the right, the flux in the region defined by the rod, the rails, and the resistor is increasing. Hence, by Lenz's law, the current must be counter clockwise if its magnetic field is to counter this increase in flux.
(*c*) Because the rod is moving with constant velocity, the net force acting on it must be zero. Apply Newton's second law to relate F to the magnetic force F_m:	$\sum F_x = F - F_m = 0$ and $F = F_m = BIl = (1.0\,\text{T})(5.0\,\text{A})(0.5\,\text{m}) = 2.5\,\text{N}$
(*d*) Express the power input by the force in terms of the force and the velocity of the rod:	$P = Fv = (2.5\,\text{N})(10\,\text{m/s}) = 25\,\text{W}$
(*e*) The rate of Joule heat production is given by:	$P = I^2R = (5\,\text{A})^2(1\,\Omega) = 25\,\text{W}$

Example 1.29 A 10 cm by 5 cm rectangular loop with resistance 2.5 Ω is pulled through a region of uniform magnetic field B = 1.7 T (Figure E1.29a) with constant speed $v = 2.4$ cm/s. The front of the loop enters the region of the magnetic field at time $t = 0$. (a) Find and graph the flux through the loop as a function of time. (b) Find and graph the induced emf and the current in the loop as a function of time. Neglect any self-inductance of the loop and extend your graphs from $t = 0$ to $t = 16$ s. [3]

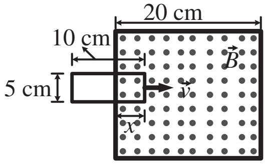


Figure E1.29a Rectangular Loop for Example 1.29.

Solution

We'll need to determine how long it takes for the loop to completely enter the region in which there is a magnetic field, how long it is in the region, and how long it takes to leave the region. Once we know these times, we can use its definition to express the magnetic flux as a function of time. We can use Faraday's law to find the induced emf as a function of time.

(a) Find the time required for the loop to enter the region where there is a uniform magnetic field:	$t = \frac{l_{\text{side of loop}}}{v} = \frac{10\ \text{cm}}{2.4\ \text{cm/s}} = 4.17\ \text{s}$
Letting w represent the width of the loop, express and evaluate ϕ_m for $0 < t < 4.17$ s:	$\phi_m = NBA = NBwvt$ $= (1.7\ \text{T})(0.05\ \text{m})(0.024\ \text{m/s})t$ $= (2.04\ m\text{Wb/s})t$
Find the time during which the loop is fully in the region where there is a uniform magnetic field:	$t = \frac{l_{\text{side of loop}}}{v} = \frac{20\ \text{cm}}{2.4\ \text{cm/s}} = 8.33\ \text{s}$ i.e., the loop will begin to exit the region when $t = 8.33$ s.
Express ϕ_m for $4.17\ \text{s} < t < 8.33$ s:	$\phi_m = NBA = NBlw$ $= (1.7\ \text{T})(0.1\ \text{m})(0.05\ \text{m})$ $= (8.50\ m\text{Wb/s})$
The left-end of the loop will exit the field when $t = \frac{30\ \text{cm}}{2.4\ \text{cm/s}} = 12.5\ s$. Express ϕ_m for $8.33\ \text{s} < t < 12.5$ s:	$\phi_m = mt + b$ where m is the slope of the line and b is the ϕ_m-intercept.
For $t = 8.33$ s and $\phi_m = 8.50$ mWb:	$8.50\ m\text{W/s} = m\,(8.33\ \text{s}) + b$
For $t = 12.5$ s and $\phi_m = 0$:	$0 = m\,(12.5\ \text{s}) + b$
Solve Eqs. (1) and (2) simultaneously to obtain $m = -(2.04\ m\text{Wb/s})$ and $b = 25.5\ m\text{Wb}$:	$\phi_m = mt + b = -(2.04\ m\text{Wb/s})t + 25.5\ m\text{Wb}$
The loop will be completely out of the magnetic field when $t > 12.5$ s and:	$\phi_m = 0$

The following graph of (t) vs $\phi_m$was plotted using a Matlab program, shown in Figure E1.29b.

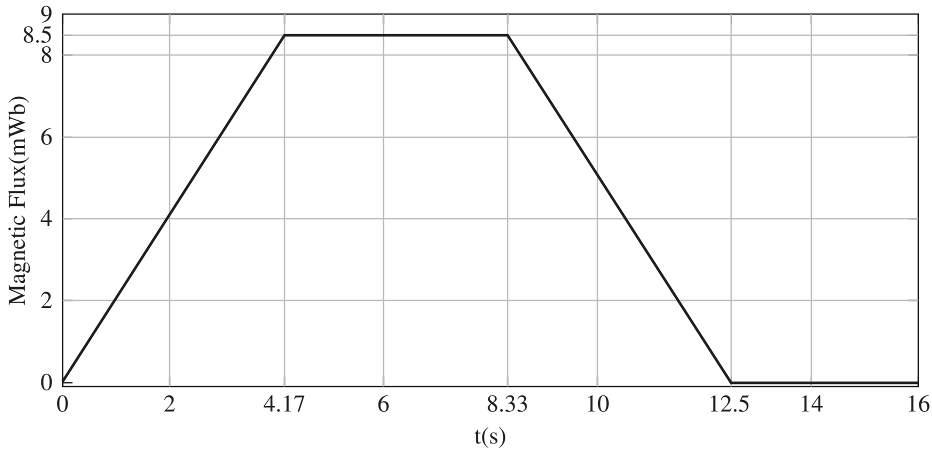


Figure E1.29b Graph of time (t) vs flux (ϕ_m).

(b) Using Faraday's law, relate the induced emf to the magnetic flux:	$\mathcal{E} = -\frac{d\phi_m}{dt}$
During the interval $0 < t < 4.17$ s:	$\mathcal{E} = -\frac{d}{dt}\left[(2.04\ m\text{Wb/s})\ t\right] = -2.04\ \text{mV}$
During the interval $4.17\ \text{s} < t < 8.33$ s:	$\mathcal{E} = -\frac{d}{dt}\left[8.50\ m\text{Wb/s}\right] = 0$
During the interval $8.33\ \text{s} < t < 12.5$ s:	$\mathcal{E} = -\frac{d}{dt}\left[-(2.04\ m\text{Wb/s})\ t + 25.5\ m\text{Wb}\right] = 2.04\ \text{mV}$
For $t > 12.5$ s:	$\mathcal{E} = 0$

The following graph of (t) vs ε was plotted using a Matlab program in Figure E1.29c.

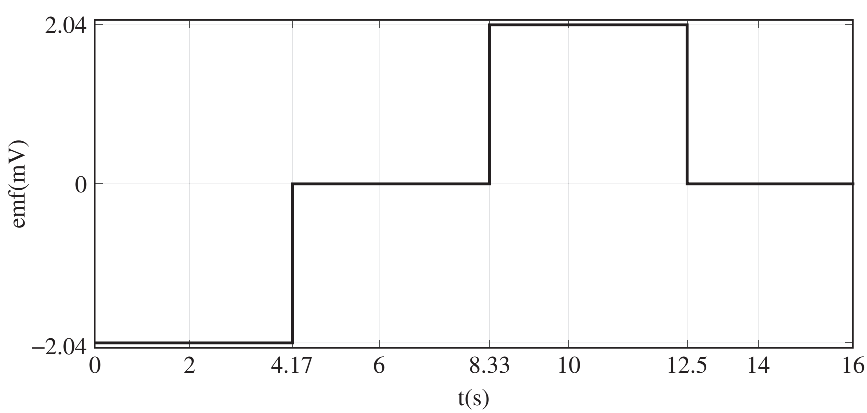


Figure E1.29c Graph of time (t) vs emf (ε).

Example 1.30 A conductor of length 15 cm lies parallel to the y-axis and oscillates in the x-direction with displacement given by $x = (2\text{ cm})\cos 120\pi t$. If a uniform magnetic field of magnitude 1.0 T is in the z-direction. Calculate induced emf in the conductor

Solution

The rod is executing simple harmonic motion in the xy plane, i.e. in a plane perpendicular to the magnetic field. The emf induced in the rod is a consequence of its motion in this magnetic field and is given by $|\mathcal{E}| = vBl$. Because we're given the position of the oscillator as a function of time, we can differentiate this expression to obtain v.

Express the motional emf in terms of v, B, and l:	$|\mathcal{E}| = vBl = Bl\dfrac{dx}{dt}$
Evaluate dx/dt:	$\dfrac{dx}{dt} = \dfrac{d}{dt}[(2\text{ cm})\ \cos 120\pi t]$ $= -(2\text{ cm})\left(120\text{ s}^{-1}\right)\pi \sin 120\pi t$ $= -(7.54\text{ m/s})\sin 120\pi t$
Substitute numerical values and evaluate $|\varepsilon|$:	$= -(1.0\text{ T})(0.15\text{ m})(7.54\text{ m/s})\sin 120\pi t$ $= -(1.133\text{ V})\sin 120\pi t$

Example 1.31 A conducting rod with a resistance R moving across the horizontal rails which have negligible resistance as shown in Figure E1.31. A battery of emf $\mathcal{E}_b$ and negligible internal resistance is connected across the horizontal rail between points a and b so that the current in the rod is flowing downward. The rod is placed at rest at $t = 0$. (a) Find the force on the rod as a function of the speed v. (b) Show that the rod moves at a terminal speed and find an expression for it. (c) What is the current when the rod will approach its terminal speed?

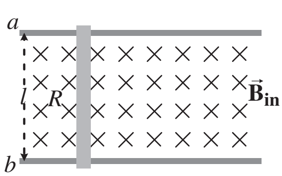


Figure E1.31 Conductor moving in a magnetic field, Example 1.31.

Solution

Let m be the mass of the rod and F be the net force acting on it due to the current in it. We can obtain the equation of motion of the rod by applying Newton's second law to relate its acceleration to B, I, and l. The net emf that drives I in this circuit is the emf of the battery minus the emf induced in the rod as a result of its motion.

(a) Letting the direction of motion of the rod be the positive x-direction, apply $\sum F_x = ma_x$ to the rod:	$BIl = m\dfrac{dv}{dt}$ where $I = \dfrac{\mathcal{E}_b - Blv}{R}$
Substitute to obtain:	$\dfrac{dv}{dt} = \dfrac{Bl}{mR}\left(\mathcal{E}_b - Blv\right)$
(b) Express the condition on dv/dt when the rod has achieved its terminal speed:	$\dfrac{Bl}{mR}\left(\mathcal{E}_b - Blv_t\right) = 0$
Solve for v_t to obtain:	$v_t = \dfrac{\mathcal{E}_b}{Bl}$
(c) Substitute v_t for v in Eq. (2) to obtain:	$I = \dfrac{\mathcal{E}_b - Bl\frac{\mathcal{E}_b}{Bl}}{R} = 0$

Example 1.32 A conductor of mass m moves along frictionless conducting rails in a region of the static uniform magnetic field $\vec{B}$ directed into the page (Figure E1.32). An external agent is pushing the conductor, maintaining its motion to the right at constant speed v_o. At time $t = 0$, the external force acting on the conductor is removed and the conductor continues forward, being slowed by the magnetic force. Calculate the speed v of the rod as a function of time. Also, find the total distance travelled by the rod, find the total energy dissipated in the resistance and show that it is equal to mv_o^2.

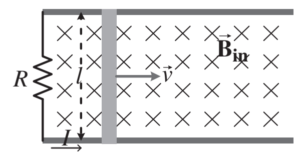


Figure E1.32 Conductor moving in a magnetic field, Example 1.32.

Solution

The speed of the rod changes because a magnetic force acts on the induced current. The motion of the rod through a magnetic field induces an emf $\mathcal{E} = Blv$ and, therefore, a current in the rod, $I = \frac{\mathcal{E}}{R}$. This causes a magnetic force to act on the rod, $F = IBl$. With the force known, we apply Newton's second law to find the speed as a function of time. Take the positive x-direction as being to the right.

1. Apply Newton's second law to the rod:	$F_x = ma_x = m\dfrac{dv}{dt}$
2. The force exerted on the rod is the magnetic force, which is proportional to the current and in the negative x-direction, as shown in Figure E1.32	$F_x = -IBl$
3. The current equals the motional emf divided by the resistance of the rod:	$I = \dfrac{\mathcal{E}}{R} = \dfrac{Blv}{R}$
4. Combining these results, we find the magnitude of the magnetic force exerted on the rod:	$F_x = -IBl = -\dfrac{Blv}{R}Bl = -\dfrac{B^2l^2v}{R}$
5. Newton's second law then gives:	$-\dfrac{B^2l^2v}{R} = m\dfrac{dv}{dt}$
6. Separate the variables, then integrate the velocity from v_o to v_f and integrate the time from 0 to t_f:	$\dfrac{dv}{v} = -\dfrac{B^2l^2}{mR}dt$ $\displaystyle\int_{v_0}^{v_f}\dfrac{dv}{v}$ $= -\dfrac{B^2l^2}{mR}\displaystyle\int_0^{t_f}dt$ $\ln\left(\frac{v_f}{v_0}\right) = -\dfrac{B^2l^2}{mR}t_f$
7. Let $v = v_f$ and $t = t_f$, then solve for v:	$v = v_0e^{-\frac{t}{\tau}}$, where $\tau = \dfrac{mR}{B^2l^2}$
8. In step 7 it is shown that the speed of the rod is given by $v = v_0e^{-(B^2l^2/mR)t}$. We can write v as dx/dt, separate the variables and integrate to find the total distance travelled by the rod.	
Apply the result from step 7 to obtain:	$\dfrac{dx}{dt} = v_0e^{-Ct}$ where $C = \dfrac{B^2l^2}{mR}$

Separate variables and integrate x' from 0 to x and t' from 0 to ∞:	$\int_0^x dx' = v_0 \int_0^\infty e^{-Ct} dt$
Evaluate the integrals to obtain:	$x = \frac{v_0}{C}$
Substitute for C and simplify:	$x = \frac{mRv_0}{B^2 l^2}$
9. In step 7, it is shown that the speed of the rod is given by $= v_0 e^{-(B^2l^2/mR)t}$. We can use the definition of power and the expression for a motional emf to express the power dissipated in the resistance in terms of B, l, v, and R. We can then separate the variables and integrate overall time to show that the total energy dissipated is equal to the initial kinetic energy of the rod.	
Express the power dissipated in terms of $\mathcal{E}$ and R:	$P = \frac{\mathcal{E}^2}{R}$
Express $\mathcal{E}$ as a function of B, l, and v:	$\mathcal{E} = Blv$ where $v = v_0 e^{-(B^2l^2/mR)t}$
Substitute to obtain:	$P = \frac{(Blv)^2}{R}$
The total energy dissipated as the rod comes to rest is obtained by integrating $dE = Pdt$:	$E = \int_0^\infty \frac{(Blv)^2}{R} dt = \int_0^\infty \frac{\left(Blv_0 e^{-(B^2l^2/mR)t}\right)^2}{R} dt = \frac{B^2l^2v_0^2}{R}\int_0^\infty e^{-2(B^2l^2/mR)t} dt$
Evaluate the integral (by changing variables to $u = -\frac{2B^2l^2}{mR}$) to obtain:	$E = \frac{B^2l^2v_0^2}{R}\left(\frac{mR}{2B^2l^2}\right) = \frac{1}{2}mv_0^2$

REMARKS If the force were constant, the rod's speed would decrease linearly with time. However, because the force is proportional to the rod's speed, as found in step 4, the force is large initially but the force decreases as the speed decreases. In principle, the rod never stops moving. Even so, the rod travels only a finite distance.

The general equation for motional emf is

$$\mathcal{E} = \oint_C \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l} = -\frac{d\phi_m}{dt}$$

where v is the velocity of the wire at the element $d\vec{l}$. The integral is taken at an instant in time.

1.3 Inductance

An inductor (symbol: ) is a solenoid wound on a core having N turns. When current i flows through, it produces a magnetic field. The inductance (L) of an inductor is defined as:

$$L = \frac{N\phi}{i} \tag{1.57}$$

The product $N\phi$ is called flux linkage. So, the inductance of an inductor is defined as the flux linkage per unit of current. SI unit of inductance is *henry* (H) which is equal to tesla-m^2/A.

1.3.1 Application of Ampere's Law to Find B in a Solenoid

Let us consider a solenoid of Figure 1.15 which carries current *i*. The direction of current and the magnetic flux is shown as per right-hand thumb rule. Select a rectangular Amperian path *abcda*. According to Ampere's law

$$\oint_C \mathrm{B} \cdot d\mathrm{l} = \mu_o i_{encl}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B} \cdot d\mathbf{l} + \int_b^c \mathbf{B} \cdot d\mathbf{l} + \int_c^d \mathbf{B} \cdot d\mathbf{l} + \int_d^a \mathbf{B} \cdot d\mathbf{l} + Bh + 0 + 0 + 0$$
$$= Bh = \mu_o i_{encl}$$

Here, the net current enclosed by the Amperian path i_{encl} is not the solenoid current *i* because the windings pass more than once through the enclosed path. Let *l* be the length of the solenoid having *N* number of turns. Let $n(=N/l)$ be the number of turns per unit length. Therefore, the number of turns in the enclosed path will be equal to *nh*. Enclosed current in terms of solenoid current will be

$$i_{encl} = number\ of\ turns\ in\ the\ enclosed\ path \times i = nhi$$

From Ampere's circuital law

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_o i_{encl} = \mu_o nhi = Bh$$

or

$$B = \mu_o ni \tag{1.58}$$

Equation (1.58) is derived for long solenoid but it is also useful for the short solenoid to find magnetic flux at the inner part of the solenoid. Flux does not depend upon the diameter of the solenoid.

Now flux in the solenoid is

$$\phi = BA$$

where A is the cross-sectional area of the solenoid.

Using Eq. (1.52) and taking $n = N/l$, Eq. (1.44) for inductance (in *henries*) becomes

$$L = \frac{N\phi}{i} = \frac{NBA}{i} = \frac{N\left(\mu_o ni\right)A}{i} = \frac{\mu_o NNA}{l} = \frac{\mu_o AN^2}{l} = \mu_o An^2 l \tag{1.59}$$

or

Inductance per unit length (H/m)

$$\frac{L}{l} = \frac{N\left(\mu_o ni\right)A}{i} = \mu_o An^2 \tag{1.60}$$

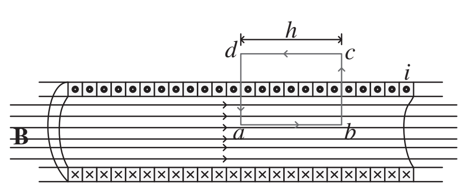


Figure 1.15 Application of Ampere's law to a section of a long ideal solenoid carrying a current *i*. The Amperian path is the rectangle *abcda*.

1.3.2 Magnetic Field of a Toroid

A toroid is shown in Figure 1.16a. Figure 1.16b shows the horizontal cross section of the toroid. The magnetic field inside the hollow portion of the toroid is found by applying Ampere's Circuital Law. From Eq. (1.39) one gets

$$(B)\,(2\pi r) = \mu_o Ni$$

where i is the current in the winding and N is the total number of turns. The current i is positive for those windings enclosed by the Amperian path. From the above equation, one can have

$$B = \frac{\mu_o Ni}{2\pi r}\ \text{(toroid)}. \tag{1.61}$$

Unlike a solenoid, B is not constant over the cross section of the toroid. It is assumed that the toroid is an ideal one, i.e. $B = 0$ outside the toroid. The direction of the magnetic field can be found using the right-hand thumb rule i.e. when the toroid is grasped, the fingers point in the direction of current and the thumb points in the direction of the magnetic field.

1.3.3 The Inductance of Circular Air-Cored Toroid

Magnetic field density **B** is given by

$$\mathbf{B} = B_\phi \mathbf{a}_\phi = \frac{\mu_o Ni}{2\pi r}\mathbf{a}_\phi$$

From Figure 1.17

$$r = r_o - \rho\cos\phi$$

Using a cylindrical coordinate system, flux is

$$\phi = \int_S \mathbf{B}\cdot\mathbf{A} = \int_0^b \int_0^{2\pi} \frac{\mu_o Ni}{2\pi r}\mathbf{a}_\phi \cdot (\rho d\rho d\phi)\,\mathbf{a}_\phi = \frac{\mu_o Ni}{2\pi}\int_0^b \int_0^{2\pi} \frac{\rho d\rho d\phi}{r_o - \rho\cos\phi}$$

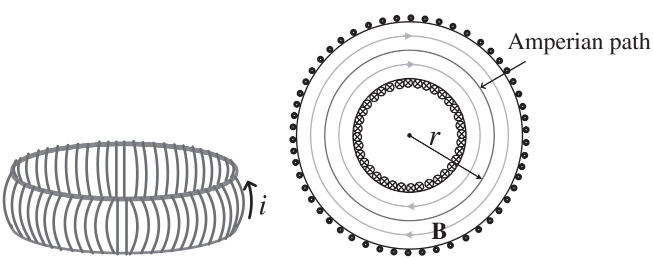


Figure 1.16 (*a*) A toroid carrying a current *i*. (*b*) A horizontal cross-section of the toroid. The interior magnetic field (inside the bracelet-shaped tube) can be found by applying Ampere's law with the Amperian loop shown [4].

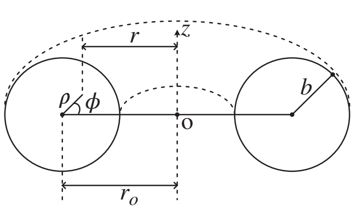


Figure 1.17 Vertical cross-section of the air-cored circular toroid.

or

$$\phi = \frac{\mu_o Ni}{2\pi}\int_0^b\int_0^{2\pi}\frac{\rho d\rho d\phi}{r_o - \rho\cos\phi} = \frac{\mu_o Ni}{2\pi}\int_0^b \rho d\rho\int_0^{2\pi}\frac{d\phi}{r_o - \rho\cos\phi}$$

$$= \frac{\mu_o Ni}{2\pi}\int_0^b \rho d\rho\left[I_1\right] \tag{1.62}$$

Where $I_1 = \int_0^{2\pi}\frac{d\phi}{r_o - \rho\cos\phi}$

Applying property of definite integration

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

If

$$f(x) = f(2a - x)$$

Here $f(\phi) = \frac{1}{r_o - \rho\cos\phi}$ and $f(2\pi - \phi) = \frac{1}{r_o - \rho\cos(2\pi - \phi)} = \frac{1}{r_o - \rho\cos\phi} = f(\phi)$

$$I_1 = \int_0^{2\pi}\frac{d\phi}{r_o - \rho\cos\phi} = 2\int_0^{\pi}\frac{d\phi}{r_o - \rho\cos\phi}$$

Let $t = \tan\left(\frac{\phi}{2}\right)$ so that $dt = \frac{1}{2}\sec^2\left(\frac{\phi}{2}\right)d\phi = \frac{1}{2}\left(1 + \tan^2\left(\frac{\phi}{2}\right)\right)d\phi = \frac{1}{2}\left(1 + t^2\right)$ or $d\phi = \frac{2dt}{(1+t^2)}$

The limits of integration are converted from ϕ to t as:

ϕ	0	π
$t = \tan\left(\frac{\phi}{2}\right)$	$t = 0$	$t = \infty$

Also $\cos\phi = \frac{1 - \tan^2\left(\frac{\phi}{2}\right)}{1 + \tan^2\left(\frac{\phi}{2}\right)} = \frac{1 - t^2}{1 + t^2}$

$$I_1 = 2\int_0^{\pi}\frac{d\phi}{r_o - \rho\cos\phi} = 2\int_0^{\infty}\frac{\frac{2dt}{(1+t^2)}}{r_o - \rho\left(\frac{1-t^2}{1+t^2}\right)} = 4\int_0^{\infty}\frac{\frac{dt}{(1+t^2)}}{\frac{(1+t^2)r_o - \rho(1-t^2)}{(1+t^2)}}$$

$$= 4\int_0^{\infty}\frac{dt}{\left(r_o + \rho\right)t^2 + \left(r_o - \rho\right)} = \frac{4}{\left(r_o + \rho\right)}\int_0^{\infty}\frac{dt}{t^2 + \left(\sqrt{\frac{(r_o - \rho)}{(r_o + \rho)}}\right)^2}$$

$$= \frac{4}{\left(r_o + \rho\right)}\cdot\frac{1}{\sqrt{\frac{(r_o - \rho)}{(r_o + \rho)}}}\left[\tan^{-1} t\right]_0^{\infty} = \frac{4}{\left(r_o + \rho\right)}\frac{\sqrt{\left(r_o + \rho\right)}}{\sqrt{\left(r_o - \rho\right)}}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right]$$

$$= \frac{4}{\sqrt{\left(r_o + \rho\right)}}\cdot\frac{1}{\sqrt{\left(r_o - \rho\right)}}\cdot\left[\frac{\pi}{2} - 0\right] = \frac{2\pi}{\sqrt{{r_o}^2 - \rho^2}} \tag{1.63}$$

Now substituting the value of $I1$ from (1.63) in (1.62), we get

$$\phi = \frac{\mu_o Ni}{2\pi}\int_0^b \rho d\rho\left[I_1\right] = \frac{\mu_o Ni}{2\pi}\int_0^b \rho d\rho\left[\frac{2\pi}{\sqrt{{r_o}^2 - \rho^2}}\right] = \mu_o Ni\int_0^b\frac{\rho d\rho}{\sqrt{{r_o}^2 - \rho^2}}$$

Let $u = r_o^{\,2} - \rho^2$ so that $du = -2\rho d\rho$ or $\rho d\rho = -\frac{du}{2}$
The limits of integration are converted from ρ to u as:

ρ	0	b
$u = r_o^{\,2} - \rho^2$	$u = r_o^{\,2}$	$u = r_o^{\,2} - b^2$

$$\phi = \mu_o Ni \int_0^b \frac{\rho d\rho}{\sqrt{r_o^{\,2} - \rho^2}} = \mu_o Ni \int_{r_o^{\,2}}^{r_o^{\,2}-b^2} \frac{-\frac{du}{2}}{\sqrt{u}} = -\frac{\mu_o Ni}{2} \int_{r_o^{\,2}}^{r_o^{\,2}-b^2} \frac{du}{\sqrt{u}} = -\frac{\mu_o Ni}{2} \left[\frac{\sqrt{u}}{\frac{1}{2}} \right]_{r_o^{\,2}}^{r_o^{\,2}-b^2}$$

$$= -\mu_o Ni \left[\sqrt{r_o^{\,2} - b^2} - \sqrt{r_o^{\,2}} \right] = \mu_o Ni \left(r_o - \sqrt{r_o^{\,2} - b^2} \right) \tag{1.64}$$

Therefore, the inductance of air-cored circular toroid will be

$$L = \frac{N\phi}{i} = \frac{N\left[\mu_o Ni \left(r_o - \sqrt{r_o^{\,2} - b^2} \right)\right]}{i} = \mu_o N^2 \left(r_o - \sqrt{r_o^{\,2} - b^2} \right) \tag{1.65}$$

If $r_o \gg b$, $B_\phi \cong \frac{\mu_o Ni}{2\pi r_o}$ (constant)

$$\phi = B_\phi . S = B_\phi \left(\pi b^2\right) \cong \frac{\mu_o Ni}{2\pi r_o} \left(\pi b^2\right) \cong \frac{\mu_o Nib^2}{2r_o}$$

The inductance of thin toroid will be

$$L = \frac{N\phi}{i} \cong \frac{N}{i} \left(\frac{\mu_o Nib^2}{2r_o} \right) \cong \frac{\mu_o N^2 b^2}{2r_o} \tag{1.66}$$

Equations (1.59), (1.60), (1.65), and (1.66) show that inductance depends upon the physical dimensions of the structure.

Example 1.33 A coil with a self-inductance of 10H carries a current of 5 A that is changing at a rate of 200 A/s. Determine (a) the magnetic flux through the coil and (b) the induced emf in the coil.

Solution

We can use $\phi_m = LI$ and the dependence of I on t to find the magnetic flux through the coil. We can apply Faraday's law to find the induced emf in the coil.

(a) Use the definition of self-inductance to express ϕ_m:	$\phi_m = LI$
Express I as a function of time:	$I = 5\text{A} + (200\ \text{A/s})t$
Substitute to obtain:	$\phi_m = L[5\text{A} + (200\ \text{A/s})t]$
Substitute numerical values and express ϕ_m:	$\phi_m = (10\text{H})[5\text{A} + (200\ \text{A/s})t]$ $= 50\text{Wb} + (2000\ \text{H. A/s})t$
(b) Use Faraday's law to relate $\mathcal{E}$, L, and dI/dt:	$\mathcal{E} = -L\frac{dI}{dt}$
Substitute numerical values and evaluate $\mathcal{E}$:	$\mathcal{E} = -(10\text{H})\left(200\frac{\text{A}}{\text{s}}\right) = -2.0\text{kV}$

Example 1.34 A coil with self-inductance L carries a current I, given by $I = I_o \sin(2\pi ft)$. Determine the relationship between the flux ϕ_m and the self-induced emf as functions of time.

Solution

We can apply $\phi_m = LI$ to find ϕ_m and Faraday's law to find the self-induced emf as functions of time.

Use the definition of self-inductance to express ϕ_m:	$\phi_m = LI = LI_o \sin 2\pi ft$

The graph of the flux ϕ_m as a function of time shown in figure E1.34a was plotted using a Matlab program. The maximum value of the flux is LI_o and we have chosen $2\pi f = 1$ rad/s.

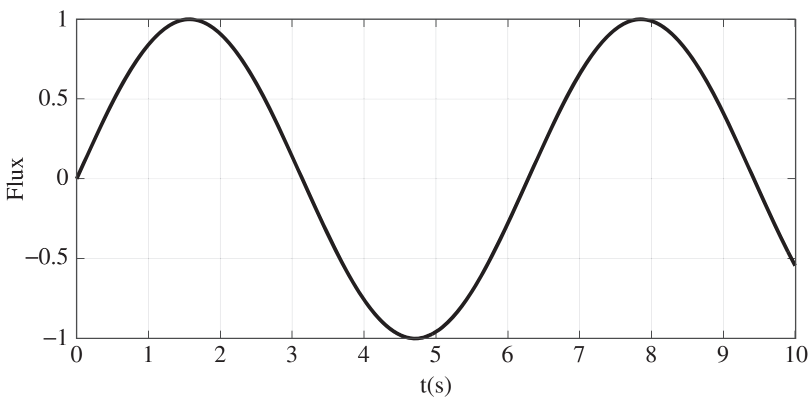


Figure E1.34a Graph of the flux ϕ_m as a function of time.

Apply Faraday's law to relate $\mathcal{E}$, L, and dI/dt:	$\mathcal{E} = -L\frac{dI}{dt} = -L\frac{d}{dt}\left[I_o \sin 2\pi ft\right] = -2\pi fLI_o \cos 2\pi ft$

The graph of the emf $\mathcal{E}$ as a function of time shown in figure E1.34b was plotted using a Matlab program. The maximum value of the induced emf is $2\pi fLI_o$ and we have chosen $2\pi f = 1$ rad/s.

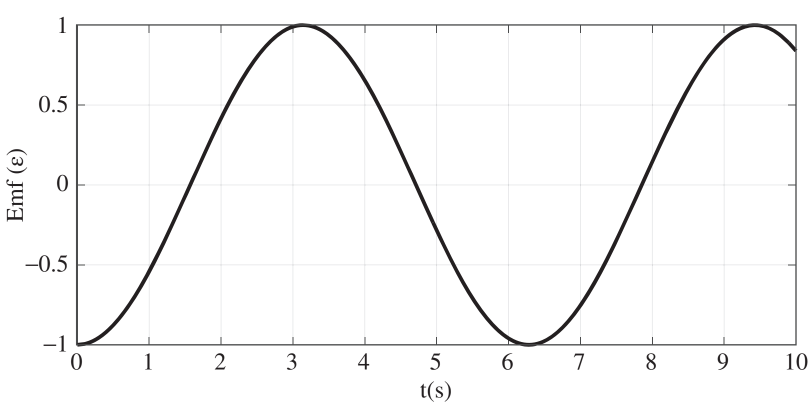


Figure E1.34b Graph of the emf $\mathcal{E}$ as a function of time.

Example 1.35 A solenoid has a length of 25 cm, a radius of 1 cm, 500 turns, and carries a current of 5-A. Determine (*a*) the flux density *B* on the axis at the centre of the solenoid; (b) the flux through the solenoid, assuming flux density *B* to be uniform; (*c*) the self-inductance of the solenoid; and (*d*) the induced emf in the solenoid when the current changes at 100 A/s.

Solution

We can use $B = \mu_o nI$ to find the magnetic field on the axis at the centre of the solenoid and the definition of magnetic flux to evaluate ϕ_m. We can use the definition of magnetic flux in terms of *L* and *I* to find the self-inductance of the solenoid. Finally, we can use Faraday's law to find the induced emf in the solenoid when the current changes at 100 A/s.

(*a*) Apply the expression for *B* inside a long solenoid to express and evaluate *B*:	$B = \mu_o nI = \left(4\pi \times 10^{-7}\text{N/A}^2\right)\left(\frac{400}{0.25\text{ m}}\right)(5\text{A}) = 10.05\ m\text{T}$
(*b*) Apply the definition of magnetic flux to obtain:	$\phi_m = NBA = 500(6.03\ m\text{T})\pi(0.01\text{ m})^2 = 9.475 \times 10^{-4}\text{Wb}$
(*c*) Relate the self-inductance of the solenoid to the magnetic flux through it and it's current:	$L = \frac{\phi_m}{I} = \frac{7.58 \times 10^{-4}\text{Wb}}{5\text{A}} = 0.1516\text{mH}$
(*d*) Apply Faraday's law to obtain:	$\mathcal{E} = -L\frac{dI}{dt} = -(0.253\text{ mH})\left(100\frac{\text{A}}{\text{s}}\right) = -25.3\text{mV}$

Example 1.36 Two solenoids of radii 2 and 5 cm are coaxial. They are each 25 cm long and have 300 turns and 5000 turns, respectively. Calculate their mutual inductance

Solution

We can find the mutual inductance of the two coaxial solenoids using

$$M_{2,1} = \frac{\phi_{m2}}{I_1} = \mu_o n_2 n_1 l\pi r^2$$

Substitute numerical values and evaluate $M_{2,1}$:

$$M_{2,1} = \left(4\pi \times 10^{-7}\text{N/A}^2\right)\left(\frac{300}{0.25\text{ m}}\right)\left(\frac{1000}{0.25\text{ m}}\right)(0.25\text{ m})\,\pi(0.02\text{ m})^2 = 0.945\text{mH}$$

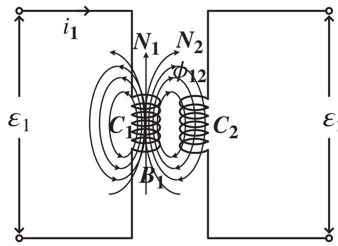
1.3.4 Mutual Inductance

Consider, Figure 1.18 where part of or perhaps the whole of magnetic flux density $\mathbf{B}_1$ induced due to time-varying current i_1 flowing in coil C_1 links with coil C_2. Let the part of that magnetic flux be ϕ_{12}. The flux linkage shall be $\lambda_{12} = N_2\phi_{12}$. Here, we say the two loops are magnetically coupled through mutual flux. Hence, mutual inductance between two coils $C_1$and $C_2$is defined as

$$M_{12} = \frac{\lambda_{12}}{i_1} = \frac{N_2}{i_1}\int_{S_2} \mathbf{B}_1 \cdot d\mathbf{S} \tag{1.67}$$

Ampere's circuital law $\nabla \times \mathbf{H} = \mathbf{J}$ can be written for a medium of μ permeability as $\nabla \times \frac{\mathbf{B}}{\mu} = \mathbf{J}$. It shows that magnetic field density $\mathbf{B}_1$ is proportional to current i_1 flowing in coil C_1. The permeability μ is independent of i_1 and $\mathbf{B}_1$ and its direction. Hence, the flux

Figure 1.18 Mutually coupled coils.



linkage $\lambda_{12} = N_2\phi_{12}$ is proportional to i_1 where N_2 is the number of turns of coil C_2. The constant of proportionality is called mutual inductance; it can be seen in Eq. (1.67).

The magnetic flux passing through each turn of the coil C_2 is

$$\phi_{12} = \mu_o n_1 i_1 \pi r_2^2 \tag{1.68}$$

where r_2, and $n_1 \left(= \frac{N_1}{l_1}\right)$ are the mean radius, number of turns per unit length of coils C_2 and C_1 respectively.

Therefore, the total flux linking the coil C_2 is

$$N_2\phi_{12} = \mu_o \pi r_2^2 n_2 l_2 n_1 i_1 = M_{12} i_1$$

where $N_2 (= n_2 l_2 =$ turns per unit length $\times$ length of coil C_2) is the number of turns in the coil C_2.

$$M_{12} = \mu_o \pi r_2^2 n_1 n_2 l_2 \tag{1.69}$$

Equation (1.69) shows that the mutual inductance depends upon the physical data of the coils and the medium in which they are placed.

Similarly, when C_2 is excited with current i_2, the mutual inductance M_{21} is given by

$$M_{21} = \mu_o \pi r_1^2 n_1 n_2 l_1 \tag{1.70}$$

where r_1 and l_1 are mean radius and length of coil C_1.

If $r_1 = r_2 = r$ *and* $l_1 = l_2 = l$, then

$$M_{12} = M_{21} = M = \mu_o \pi r^2 n_1 n_2 l \tag{1.71}$$

Using Eq. (1.59), self-inductances of coils C1 and C2 can be written as

$$L_1 = \mu_o \pi r^2 n_1^2 l \tag{1.72}$$

$$L_2 = \mu_o \pi r^2 n_2^2 l \tag{1.73}$$

From Eqs. (1.72) and (1.73)

$$M = \sqrt{\left(\mu_o \pi r^2 n_1^2 l\right)\left(\mu_o \pi r^2 n_2^2 l\right)} = \sqrt{L_1 L_2} = \mu_o \pi r^2 n_1 n_2 l \tag{1.74}$$

It should be noted that Eq. (1.67) depends on the assumption that *all* of the flux produced by coil C_1 passes through the coil C_2. But practically, some of the flux leaks out, so that the mutual inductance is somewhat less than that given in Eq. (1.74). The mutual inductance can be written as

$$M = k\sqrt{L_1 L_2} \tag{1.75}$$

where the constant k is called the *coefficient of coupling* and lies in the range $0 \le k \le 1$.

Example 1.37 Derive an expression for the mutual force between two coaxial circular coils of radii R_1, R_2, turn N_1, N_2 and currents I_1, 1_2 respectively, if the distance between their centres is L, as shown in Figure E1.37.

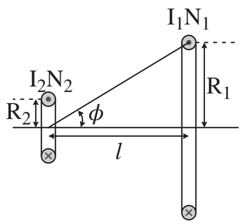


Figure E1.37 Coaxial circular coils of Example 1.37.

Assume the smaller coil is so situated that the flux, set up by the larger coil, completely envelops it and can be treated as uniform over the area of the smaller coil, together with the fact that the turns of each coil are co-planar. Applying the Biot-Savart law to the larger coil, to find the magnetic strength of the field on the co-axis, it can be seen that

$$H_1 = \frac{I_1 N_1 \sin \phi R_1}{2\left(R_1^2 + l^2\right)}$$

and since

$$\sin \phi = \frac{R_1}{\sqrt{\left(R_1^2 + l^2\right)}}$$

$$H_1 = \frac{I_1 N_1 R_1^2}{2\left(R_1^2 + l^2\right)^{\frac{3}{2}}}$$

thus, flux density B_1

$$B_1 = \mu_o H_1 = \frac{\mu_o I_1 N_1 R_1^2}{2\left(R_1^2 + l^2\right)^{\frac{3}{2}}}$$

Hence

flux linkage the small coil $\phi_2 = \pi R_2^2 B_1 = \frac{\pi R_2^2 \mu_o I_1 N_1 R_1^2}{2\left(R_1^2+l^2\right)^{\frac{3}{2}}}$

flux linkage with coil 2 $= \phi_2 N_2 = \frac{\pi R_2^2 \mu_o I_1 N_1 R_1^2 N_2}{2\left(R_1^2+l^2\right)^{\frac{3}{2}}}$

The mutual inductance between the coils is

$$M_{12} = \frac{\text{flux linkage with coil 2}}{\text{current of coil 1}} = \frac{\pi R_2^2 \mu_o I_1 N_1 R_1^2 N_2}{2\left(R_1^2 + l^2\right)^{\frac{3}{2}} I_1} = \frac{\mu_o \pi N_1 N_2 R_1^2 R_2^2}{2\left(R_1^2 + l^2\right)^{\frac{3}{2}}}$$

Since the potential energy of one circuit in the field of another is given by $M_{12}I_1I_2$, the force between the coils $= I_1 I_2 \frac{dM_{12}}{dl}$

$$F = \frac{3\mu_o \pi N_1 N_2 R_1^2 R_2^2 I_1 I_2 l}{2\left(R_1^2 + l^2\right)^{\frac{5}{2}}}$$

1.4 Energy

Let a coil having an inductance of L Henry with the resistance of R Ω connected to a voltage source of V volts, resulting in a flow of current *i* amperes as shown in Figure 1.19. The KVL equation for the circuit is

$$V = L\frac{di}{dt} + Ri \tag{1.76}$$

To find the power supplied by the source to the circuit, multiply both sides of Eq. (1.76) by *i*, and the following is obtained

$$Vi = Li\frac{di}{dt} + Ri^2 \tag{1.77}$$

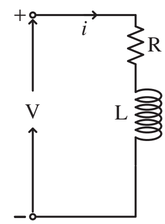


Figure 1.19
RL circuit.

Equation (1.77) shows the work done per unit time by the source. The total work is done by the battery in raising the current in the circuit from zero at time $t = 0$ to i_T at time $t = T$ is

$$W = \int_0^T Vidt \tag{1.78}$$

Using Eq. (1.77), in Eq. (1.78), we get

$$W = \int_0^T \left(Li\frac{di}{dt} + Ri^2\right) dt$$

or

$$W = \frac{1}{2}Li_T^2 + R\int_0^T i^2 dt \tag{1.79}$$

The first part of Eq. (1.79) shows the energy stored in the inductor which can be used once the battery is disconnected and the second part is the energy dissipated in the resistance of the coil but this energy is irreversible or it is the loss of energy.

$$W_{stored} = \frac{1}{2}Li^2 \tag{1.80}$$

where L is the self-inductance. We know from Eqs. (1.72), and (1.73) that

$$L = \mu_o n^2 \pi r^2 l \tag{1.81}$$

where n is the number of turns per unit length of the solenoid, r the radius, and l the length. The field inside the solenoid is uniform, with a magnitude

$$B = \mu_o ni \tag{1.82}$$

and is zero outside the solenoid. Equation (1.80) can be rewritten

$$W = \frac{B^2}{2\mu_o}V_{vol} \tag{1.83}$$

where $V_{vol} = \pi r^2 l$ is the volume of the solenoid. The above formula strongly suggests that a magnetic field possesses an energy density

$$U_B = \frac{B^2}{2\mu_o} \tag{1.84}$$

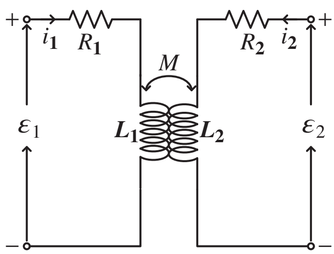


Figure 1.20 Two magnetically coupled coils.

Now, in Figure 1.20, two coils are wound one over the other and are connected to individual sources. The KVL equations are

$$\mathcal{E}_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + M\frac{di_2}{dt} \tag{1.85a}$$

$$\mathcal{E}_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + M\frac{di_1}{dt} \tag{1.85b}$$

The energy supplied by the two sources in increasing the currents in the two circuits, from zero at time 0, to i_1 and i_2 at time T, respectively, is

$$W = \int_0^T (\mathcal{E}_1 i_1 + \mathcal{E}_2 i_2)\, dt = \int_0^T (R_1 i_1^2 + R_2 i_2^2) + \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M\int_0^T \left(i_1 \frac{di_2}{dt} + i_2 \frac{di_1}{dt} \right) dt \tag{1.86}$$

Thus,

$$W = \int_0^T (R_1 i_1^2 + R_2 i_2^2) + \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2 \tag{1.87}$$

Hence, the total magnetic energy stored in the two coils is

$$W_{stored} = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2 \tag{1.88}$$

The sign of mutual inductance M will be positive if the currents i_1 and i_2 flow in such a direction that the magnetic fields produced help each other. Then, the energy stored increases and whereas the sign of mutual inductance M will be negative if their directions are such that fields oppose each other, then, the stored energy decreases. The stored energy cannot be negative.

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2 \geq 0 \tag{1.89}$$

or

$$\frac{1}{2}\left(i_1\sqrt{L_1} + i_2\sqrt{L_2} \right)^2 - i_1 i_2 \left(\sqrt{L_1 L_2} - M \right) \geq 0 \tag{1.90}$$

If we assume that $i_1 i_2 < 0$, then

$$M \leq \sqrt{L_1 L_2} \tag{1.91}$$

Equation (1.91) is the same as the Eq. (1.75).

1.5 Overview of Electric Machines

Electric Machines are the electromechanical energy conversion devices used to transform mechanical energy to electrical energy and vice-versa. A variety of electric machines are used in residential, commercial and industrial applications. Electric machines are classified into several types depending upon different parameters. Here some classifications are explained.

An electric Machine is a general term used for both motors and generators. Hence, the first classification is based on the fact that if the electrical power is converted to mechanical power (motors) or mechanical power is converted to electrical power (generator), as shown in Figure 1.21.

An electric machine is classified by the way it processes energy, stationary or rotating as shown in Figure 1.22. Stationary electric machines are 'Transformers'. Transformers are electric machines that transform electric energy from one circuit through to the magnetic field by changing the voltage and current levels without changing the frequency of the input and output signals. The most commonly used transformer has two windings coupled by a magnetic core. The source is connected to one winding called the 'primary winding' and a load is connected to the second winding, called the 'secondary winding'. The number of turns in each winding decides the voltage transfer ratio. Transformers are used in a huge number of applications from small power supplies (small appliances, mobile phone

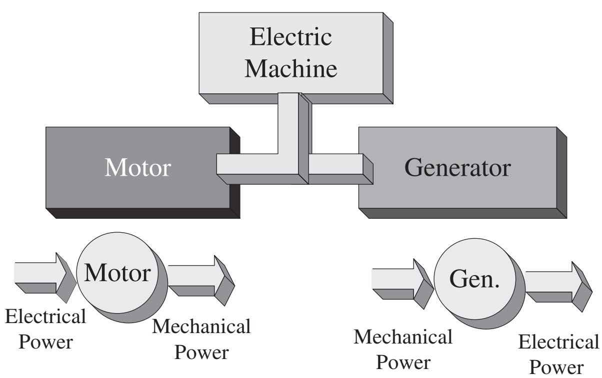
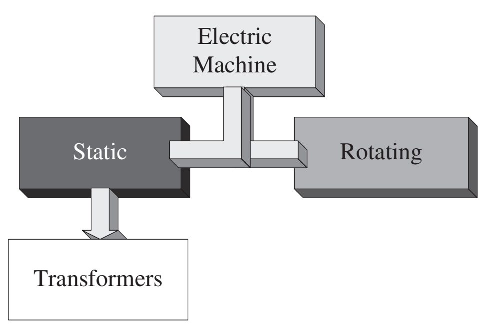


Figure 1.21 Electric machine classification based on their function.

Figure 1.22 Electric Machine classification, static or rotating.



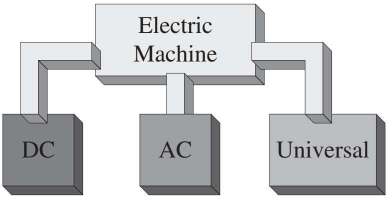


Figure 1.23 Electric Machine classification based on types of supply.

chargers, laptop chargers, home appliances) to big power stations and power transmission and distribution networks. A rotating machine has two parts; stator and rotor. One part is stationary, called the stator and other part is rotating called the 'rotor'. Stator and rotor are separated by a small air gap. The length of air-gap is kept as small as mechanically possible. The smaller the air-gap length, the smaller the magnetizing current drawn by the machine to establish the working flux will be.

Another classification of electric machine is based on the type of electric supplies it works on as shown in Figure 1.23. Hence a machine can be DC or AC. However, there exists a special machine that operates with both AC and DC, called a universal machine. Universal motors are most commonly used in Home Appliances for high-speed applications.

A classification of the DC machine is given in Figure 1.24, where all different possible types are listed. A PM machine has a permanent magnet for producing main flux. Thus, the field system is uncontrollable. However, there is space saving and no field losses, thus PM machines offer higher power density and greater efficiency. Major advantages of a DC machine include its simple and flexible control, especially shunt and separately excited machines where flux and torque can be independently controlled by the field current and

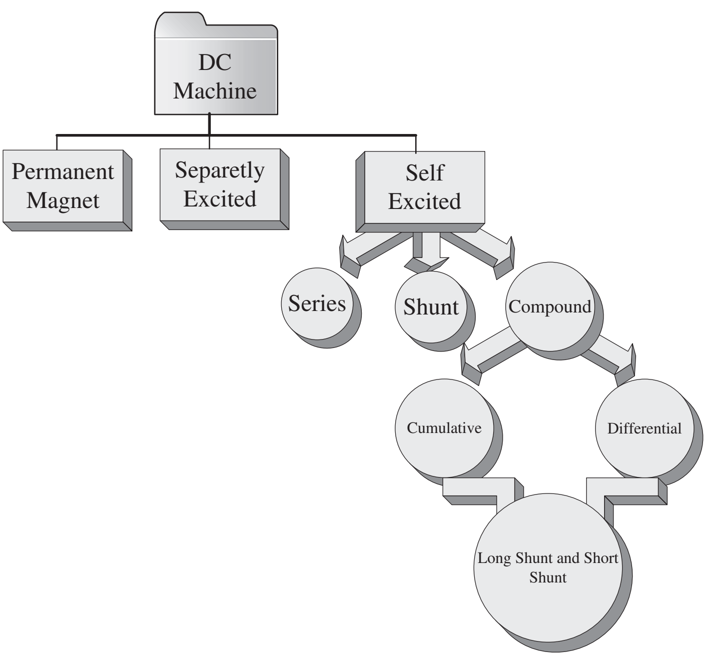


Figure 1.24 Classification of DC machines.

the armature current, respectively. The major problem with a DC machine is due to commutator and brushes. Due to commutator, a ripple in power occurs and it also limits the speed of the machine. Brushes cause friction and electromagnetic interference. Brushes wear out quickly hence, regular maintenance is required. Further, the size of the DC machine is comparatively bigger.

An AC machine classification is given in Figure 1.25. The major types are Induction and Synchronous. They are further classified according to the number of phases; single-phase, three-phase and multi-phase (more than three-phase), this classification is not shown in the figure. Machine refers to both the motor and generator. However, in conventional power stations (Thermal power plant, Hydroelectric power plant and Nuclear Power Plant) all the generators used are of the Synchronous type. This is because a synchronous machine

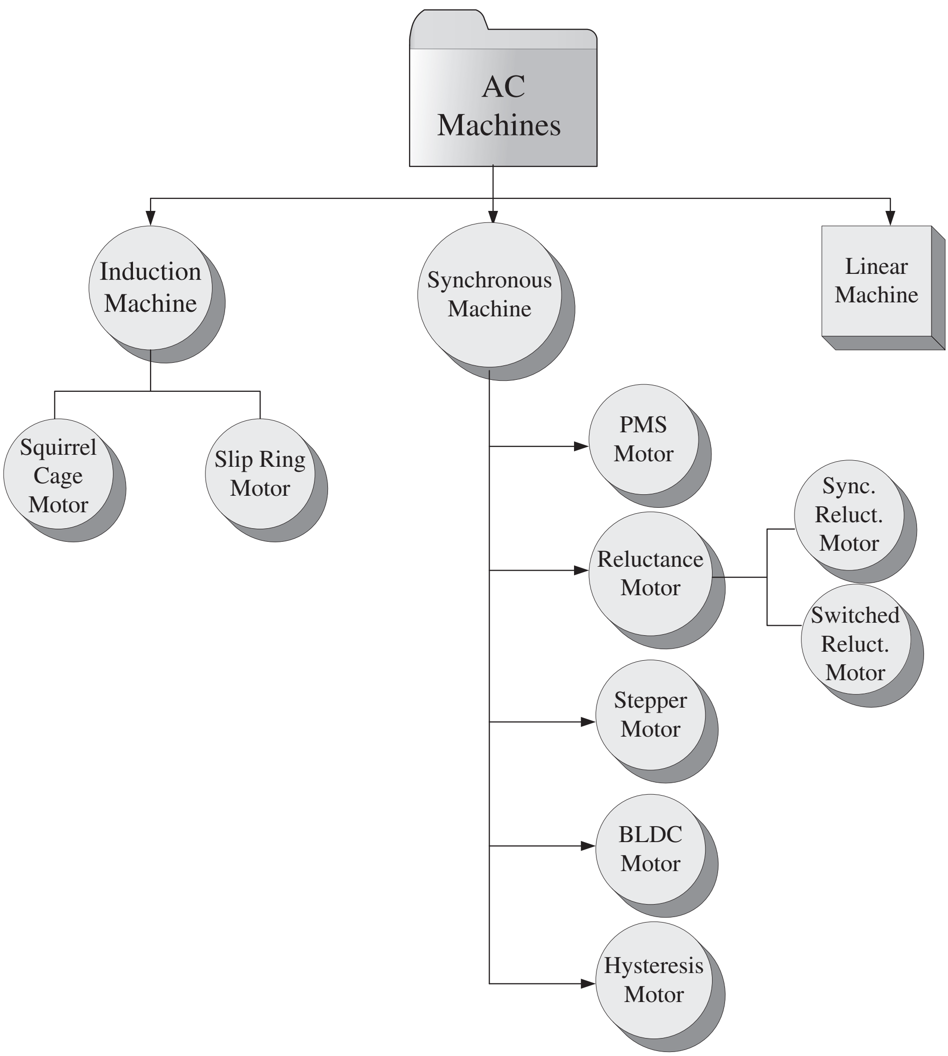


Figure 1.25 AC machine classification.

when operated at constant speed, gives a constant frequency voltage and current output. Induction generators are used mostly used in wind energy generation systems where the output frequency is variable and is controlled to be fixed using power electronic converters.

Three-phase induction motors are the most commonly used motors in industrial, commercial, and domestic applications. In domestic applications, mostly single-phase induction motors are used. Induction motors are called the workhorse of industry. Induction motors are reliable and rugged motors without any commutator or brushes. They are used for continuous operation in all types of environment. The principle of operation is based on Faraday's law of electromagnetic induction, hence the name. There are two types of induction machines based on their rotor constructions. If the rotor is a cage type where copper bars are embedded with short circuited end-rings without any winding, it is called a 'Squirrel cage' induction machine. The stator has three-phase balanced, either concentrated or distributed, windings. If the rotor has three-phase winding, it is called a 'wound rotor' or 'slip ring' type induction machine. In an induction machine, the field produced, due to stator current and rotor current, rotates at a synchronous speed (Ns = 120f/P, f-supply frequency and P is the number of poles of winding) while the rotor rotates at a speed lower than the synchronous and hence, this is called an asynchronous machine. Three-induction motors are self-starting while single-phase induction motors are not-self-starting. When the rotor of a wound rotor type induction machine is supplied by a power converter with variable frequency and the stator is connected to the grid, it is known as a 'Double Fed Induction Generator, DFIG'. It is mostly used in a Wind Power Generation system.

Synchronous machines are mostly used as a generator. Synchronous motors are only used for high-power applications in the MW range. As the name suggests, this is a constant speed machine that always rotates at a synchronous value (Ns = 120f/P). The stator carries AC armature winding (three-phase distributed) and the rotor has DC field winding. The rotor is supplied by an external DC source via the slip ring and brush arrangement. The rotor can have two shapes; cylindrical and salient pole. Cylindrical rotor machines are used for high-speed (thermal and nuclear power plants) and salient pole is used for low-speed (hydroelectric power plant) applications.

The cross-sectional view of the DC, induction and synchronous machines are given in Figure 1.26. The stator and rotor structures are clearly seen [5].

If the rotor winding is replaced by Permanent Magnets, the machine is called a Permanent Magnet Synchronous machine (PMSM) or sinusoidal PMSM. Use of PM offers several advantages, for example, no rotor losses (hence higher efficiency), faster response time as the electromechanical time constant is reduced. The PM material used for the rotor are; ferrite, aluminium-nickel-cobalt (Al-Ni-Co), samarium-cobalt (Sm-Co) and neodymium-iron-born (Nd-Fe-B). The most preferable PM material is Nd-Fe-B since it has high remanence which measures the strength of the magnetic field, high coercivity which denotes the resistance to becoming demagnetized and large energy product ($B.H_{max}$) which represents the density of magnetic energy. Permanent magnet, if used in the outer surface of the rotor, is called a surface mount PM machine, if PM is used inside the rotor surface it is called an inset PM machine and if PM is embedded inside the rotor, it is called an interior PM machine as shown in Figure 1.27.

The back emf of PMSM is sinusoidal in nature. The applied current is also sinusoidal. If the back emf is made trapezoidal, the machine becomes a Brushless DC Machine (BLDC).

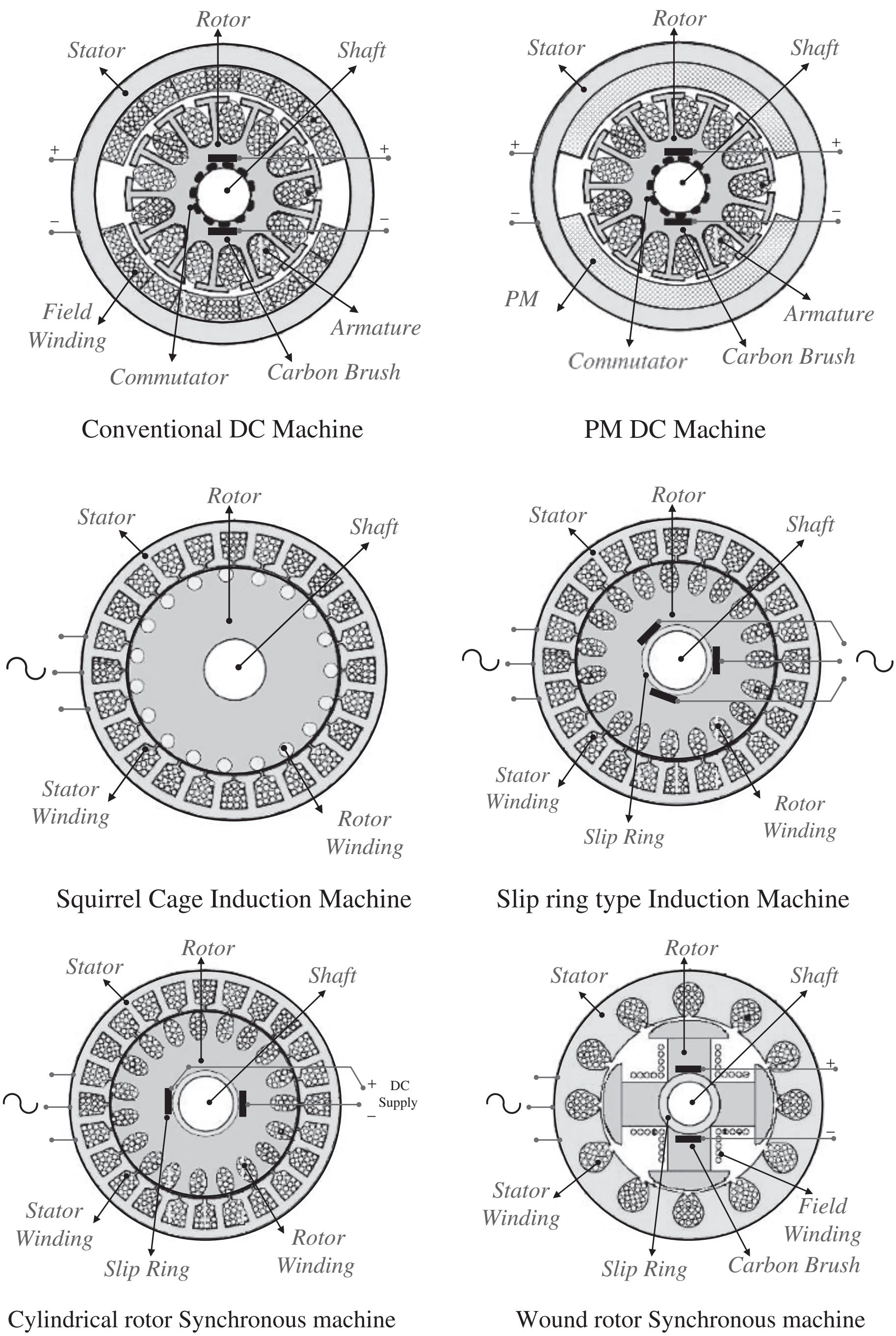


Figure 1.26 Cross-sectional view of different machines.

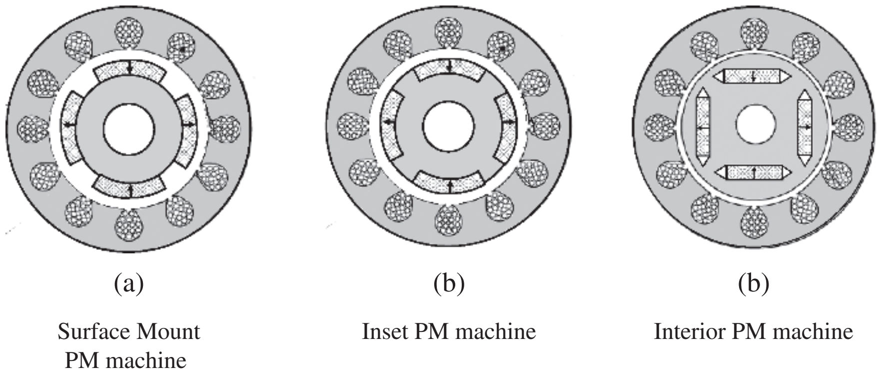


Figure 1.27 Sinusoidal PMSM machines.

The applied current is a quasi-rectangular waveform. For motoring operations, the supplied current is made positive for positive emf and negative for negative emf. Hence, their product is always positive. BLDC are ac synchronous machines with permanent magnets on rotor and trapezoidal back emf shape. The stator poles (projected type) are supplied by current to produce magnetic poles that attract/repel the rotor permanent magnets. The stator poles are energized in a proper sequence to produce a continuous motion. The cross-sectional view of a three-phase BLDC machine is shown in Figure 1.28.

The synchronous reluctance machine operates on the principle of reluctance torque. The torque is produced due to saliency in the rotor. The stator has a slotted structure with three-phase distributed winding placed inside the slots. The rotor has no windings and no permanent magnets. The number of poles on stator and rotors are same. The cross-sectional view of a synchronous reluctance machine is shown in Figure 1.29.

The switched reluctance machine has salient structure on both stator and rotor. The number of poles on stator and rotor are not integer multiples. This is done to avoid all stator and rotor poles aligning simultaneously. Under all aligned conditions, no torque is produced and the rotor is locked to the stator. Generally, the numbers of stator poles are higher than the rotor poles. The number of stator and rotor poles are 6/4, 8/6, 12/10, etc. The two windings on two opposite stator poles are connected in series or parallel to form one supply

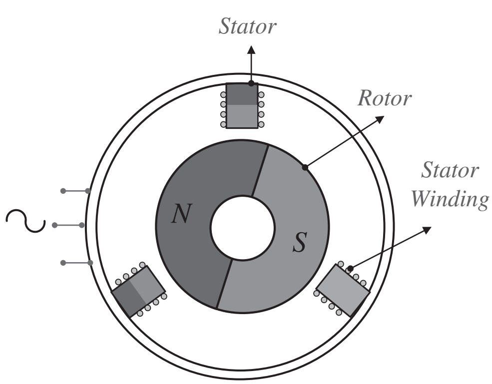


Figure 1.28 BLDC machine.

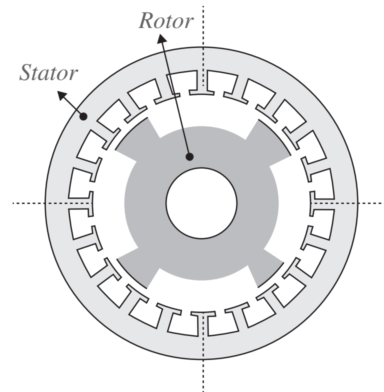


Figure 1.29 Synchronous reluctance machine.

phase. As such, all the supply phases are independent from one another. When a stator pole is equidistant from the two adjacent rotor poles, it is known as a 'fully unaligned position'. This is the position of maximum reluctance. When two (or more) stator poles are fully facing two (or more) rotor poles, it is called a 'fully aligned position' which is a position of minimum reluctance.

When a stator pole is excited, torque is produced in such a direction that the rotor moves toward minimum reluctance. Thus, the nearest rotor pole is pulled from the unaligned position into alignment position with the stator poles (lower reluctance position). Thus, by appropriately switching the phases, the machine rotates continuously in one direction. The major advantages of the SRM are their simple construction, high-power density and great efficiency. Since the rotor has no winding and no magnet, it is light in weight and there are no copper losses. Nowadays, SRM is mostly used in Electric Vehicle applications. Cross-sectional view of a 6/4 SRM machine is shown in Figure 1.30. The aligned and non-aligned positions in an 8/6 SRM machine is shown in Figure 1.31. SRM machines are rugged, reliable, have high-power density and require a simple unidirectional power converter. SRM requires rotor position sensor, has higher torque ripple with noisy machine operation. The current shape is non-sinusoidal. The machine construction is simple; however, the control is complex.

A stepper motor is not a continuously rotating machine but rather it rotates in steps. It rotates by a specific number of degrees. A train of input current pulses are supplied and

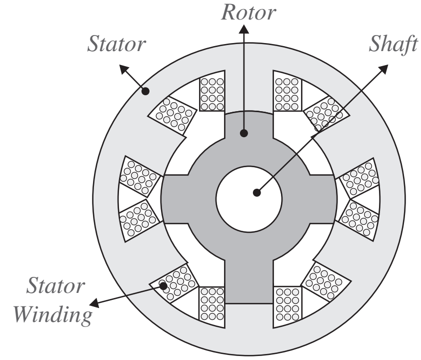


Figure 1.30 A 6/4 SRM motor.

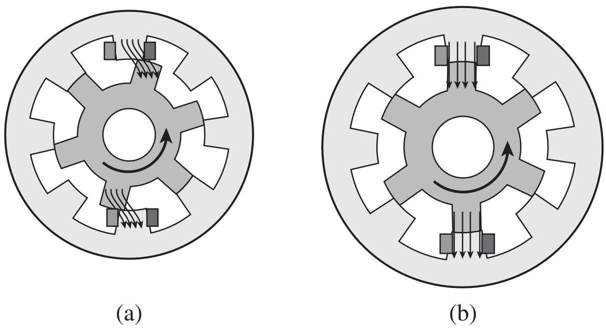


Figure 1.31 SRM, (a) unaligned, (b) aligned position.

the motor moves one step for each input current pulse. Typical steps sizes are 2°, 2.5°, 5°, 7.5°, and 15°. It is a digitally controlled control whose speed or steps are determined by the frequency of the current pulses. The machine does not need any position sensor or feedback system. There are two types of stepper motors namely; variable-reluctance type and permanent-magnet type.

Linear Motors gives linear or translational motion. Linear motor is obtained by simply cutting the round machine and unrolled onto a flat surface as shown in Figure 1.32. The stator and rotor in a linear motor is termed as primary and secondary, respectively. Any type of rotating machine can be converted into a linear motor version. In general, induction and synchronous linear motors are most commonly used. In a linear induction motor (LIM), the squirrel cage rotor is replaced by a cylinder of conductor (generally made of Aluminium) that encloses the rotor magnetic core. When three-phase supply is given to the primary, travelling flux wave is produced as shown in Figure 1.33. This flux links with the secondary, induces current in the secondary and produces a thrust (linear force). The air-gap length in an LIM is greater than rotatory IM. The typical value of air gap in an LIM is 15–30 mm. Larger air-gap length needs a larger magnetizing current and the power factor is poor. LIM operate at a greater slip value and hence losses are high. The efficacy of LIM is low. LIM can be both single-sided and double-sided. In double-sided, there are two primaries enclosing the secondary to enhance the efficiency of the machine as shown

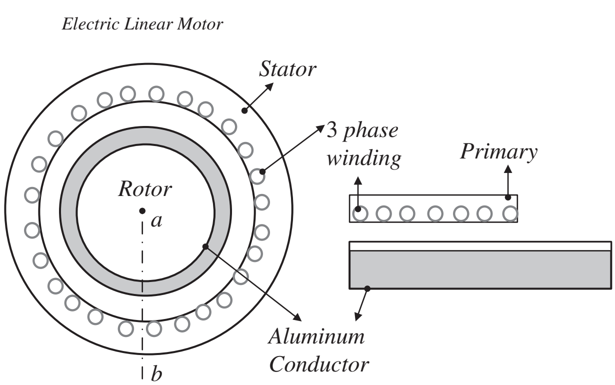


Figure 1.32 Linear Induction motor.

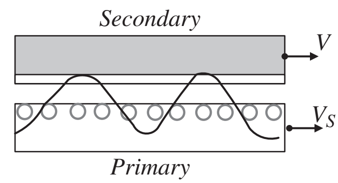


Figure 1.33 Linear Induction motor showing travelling wave.

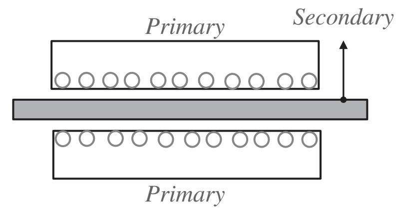


Figure 1.34 Double-sided Linear Induction motor.

in Figure 1.34. Major applications include the automatic sliding doors in trains, elevator etc., conveyor belts, high-speed trains, material handling, in automobile manufacturing industries etc. Conversion of a rotatory PM machine to Linear PM machine is shown in Figure 1.35.

Hysteresis motors are special synchronous motors that rely on hysteresis properties to produce torque. The stator is similar to induction and synchronous machines with either single- or three-phase winding. The rotor is smooth without any slots or any winding. It consists of hard magnetic material such as cobalt-steel wrapped around a non-magnetic material such as aluminium. The inside rotor material is chosen to reduce the rotor weight as shown in Figure 1.36.

When stator is supplied by current, a rotating magnetic field is produced that links with the rotor. The rotor is hard magnetic material with high retentivity. The rotor flux will lag the stator flux due to hysteresis property (retentivity). The angle between the stator and rotor flux causes torque production. Thus, the torque in a round rotor machine is given by

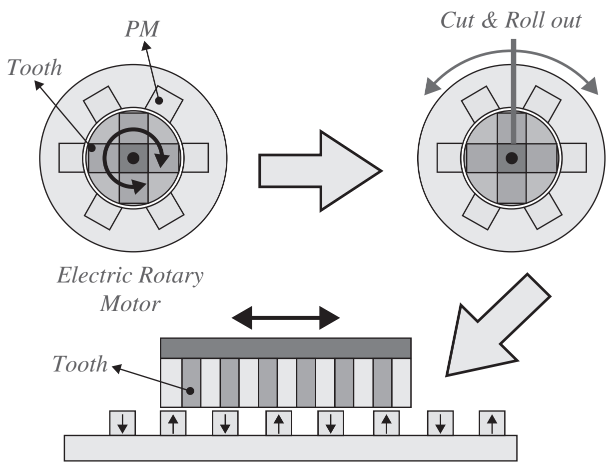


Figure 1.35 Linear PM synchronous machine.

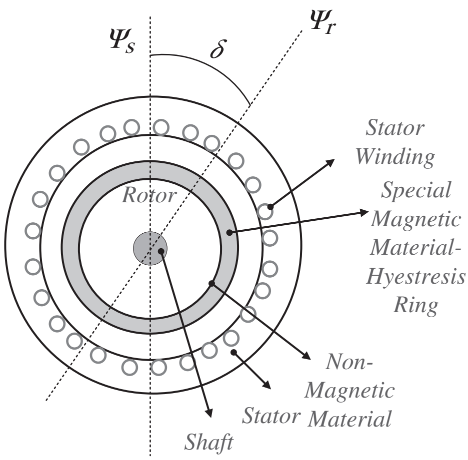


Figure 1.36 Hysteresis motor.

the relation;

$$T = K\psi_s\psi_r \sin(\delta) \tag{1.92}$$

The higher the value of δ, the higher the amount of torque will be. The angle δ depends upon the magnetic material property. The value of δ will be high if the hysteresis loop has a higher area. The rotor is smooth and hence is a noiseless operation. The major applications areas are timers, electric clocks and Magnetic tape recorders.

1.6 Summary

This chapter gives a foundation in electrical and magnetic systems covering the basic laws that govern their operation. All fundamental laws including Ohms law, Biot-Savart law, Ampere's law are discussed in detail with many solved numerical problems for clear understanding. The basics of Magnetic circuit are also included and which are further discussed in Chapter 2. Basic electrical components such as Resistance and Inductance are elaborated with numerical examples. Magnetically coupled circuits are also taken up for discussion. Principles and overview of electrical machines are given in the last section of the chapter.

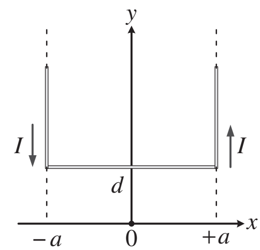
Problems

Problems with The Biot–Savart Law

1.1 In a long, straight, vertical lightning stroke, electrons move downward and positive ions move upward and constitute a current of magnitude 20.0 kA. At a location 50.0 m east of the middle of the stroke, a free electron drifts through the air towards the west with a speed of 300 m/s. (a) Make a sketch showing the various vectors involved. Ignore the effect of the Earth's magnetic field. (b) Find the vector force the lightning stroke exerts on the electron. (c) Find the radius of the electron's path. (d) Is it a good approximation to model the electron as moving in a uniform field? Explain your answer. (e) If it does not collide with any obstacles, how many revolutions will the electron complete during the 60.0-ms duration of the lightning stroke? [2]

1.2 Determine the magnetic field (in terms of I, a, and d) at the origin due to the current loop in Figure P1.2. The loop extends to infinity above figure [2].

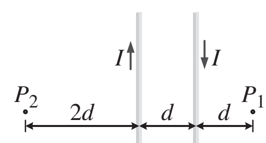
Figure P1.2



1.3 A wire carrying a current I is bent into the shape of an equilateral triangle of side L. (a) Find the magnitude of the magnetic field at the centre of the triangle. (b) At a point halfway between the centre and any vertex, is the field stronger or weaker than at the centre? Give a qualitative argument for your answer [2].

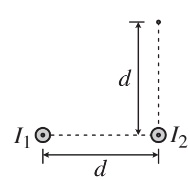
1.4 The two wires shown in Figure P1.4 are separated by $d = 10.0$ cm and carry currents of $I = 5.00$ A in opposite directions. Find the magnitude and direction of the net magnetic field (a) at a point midway between the wires; (b) at point P_1, 10.0 cm to the right of the wire on the right; and (c) at point P_2, $2d = 20.0$ cm to the left of the wire on the left. [2].

Figure P1.4



1.5 Two long, parallel wires carry currents of $I_1 = 3.00$ A and $I_2 = 5.00$ A in the directions indicated in Figure P1.5. (a) Find the magnitude and direction of the magnetic field at a point midway between the wires. (b) Find the magnitude and direction of the magnetic field at point P, located $d = 20.0$ cm above the wire carrying the 5.00-A current. [2].

Figure P1.5



1.6 A current $I = 1.00$ A circulates in a round thin-wire loop of radius $R = 100$ mm. Find the magnetic induction
(a) at the centre of the loop;
(b) at the point lying on the axis of the loop at a distance $x = 100$ mm from its centre [6].

1.7 A current I flows along a thin wire, shaped as a regular polygon with n sides, which can be inscribed into a circle of radius R. Find the magnetic induction at the centre of the polygon. Analyse the obtained expression at $n \to \infty$ [6].

1.8 Find the magnetic induction at the centre of a rectangular wire frame whose diagonal is equal to $d = 16$ cm and the angle between the diagonals is equal to $\phi = 30°$; the current flowing in the frame equals $I = 5.0$ A [6].

1.9 A current $I = 5.0$ A flows along a thin wire shaped as shown in Figure P1.9. The radius of a curved part of the wire is equal to $R = 120$ mm, the angle $2\phi = 90°$. Find the magnetic induction of the field at point *0* [6].

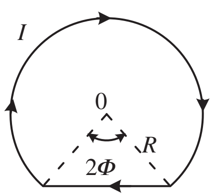


Figure P1.9

1.10 Find the magnetic induction of the field at point *0* of a loop with current I, whose shape is illustrated
(a) in Figure P1.10a, the radii a and b, as well as the angle ϕ are known;
(b) in Figure P1.10b, the radius a, and the side b are known [6].

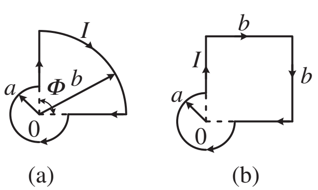


Figure P1.10

1.11 A current I flows along a lengthy thin-walled tube of radius R with the longitudinal slit of width h. Find the induction of the magnetic field inside the tube under the condition $h << R$ [6].

1.12 A current I flows in a long straight wire with a cross-section having the form of a thin half-ring of radius R (Figure P1.12). Find the induction of the magnetic field at point *0* [6].

Figure P1.12

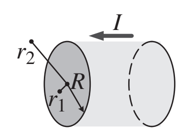
1.13 Inside a long straight uniform wire of round cross-section, there is a long round cylindrical cavity whose axis is parallel to the axis of the wire and displaced from the latter by a distance ***l***. A direct current of density j flows along the wire. Find the magnetic induction inside the cavity. Consider, in particular, the case $l = 0$ [6].

Problems with Ampere's law

1.14 The magnetic field created by a large current passing through plasma (ionized gas) can force current-carrying particles together. This *pinch effect* has been used in designing fusion reactors. It can be demonstrated by making an empty aluminium can carry a large current parallel to its axis. Let R represent the radius of the can and I the current, uniformly distributed over the can's curved wall. Determine the magnetic field (a) just inside the wall and (b) just outside. (c) Determine the pressure on the wall [2].

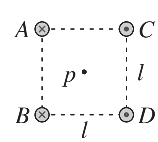
1.15 A long, cylindrical conductor of radius R carries a current I as shown in Figure P1.15. The current density J, however, is not uniform over the cross-section of the conductor but rather is a function of the radius according to $J = br$, where b is a constant. Find an expression for the magnetic field magnitude B (a) at a distance $r_1 < R$ and (b) at a distance $r_2 > R$, measured from the centre of the conductor [2].

Figure P1.15



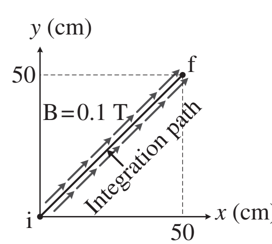
1.16 Four long, parallel conductors carry equal currents of $I = 5.00$ A. Figure P1.16 is an end view of the conductors. The current direction is into the page at points A and B and out of the page at points C and D. Calculate (a) the magnitude and (b) the direction of the magnetic field at point P, located at the centre of the square of edge length $l = 0.200$ m [2].

Figure P1.16



1.17 What is the line integral of $\overline{B}$ between points i and f in Figure P1.17? [7]

Figure P1.17



1.18 What is the line integral of $\overline{B}$ between points i and f in Figure P1.18? [7]

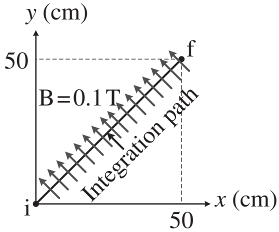


Figure P1.18 For problem 1.18

1.19 The value of the line integral of $\overline{B}$ around the closed path in Figure P1.19 is 3.77×10^{-6} Tm. What is I_3? [7]

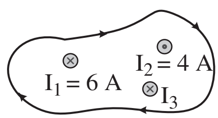


Figure P1.19

1.20 The value of the line integral around the closed path in Figure P1.20 is 1.38×10^{-5} Tm. What are the directions (in or out of the page) and the magnitude of I_3 [7].

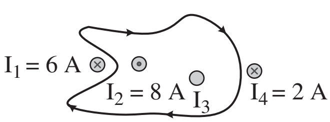


Figure P1.20

Induced EMF and Faraday's law

1.21 A solenoid of length 25 cm and radius 0.8 cm with 400 turns, is in an external magnetic field of 600 G that makes an angle of 50° with the axis of the solenoid. (a) Find the magnetic flux through the solenoid. (b) Find the magnitude of the emf induced in the solenoid if the external magnetic field is reduced to zero in 1.4 seconds [3].

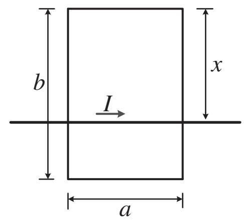
1.22 A 100-turn circular coil has a diameter of 2 cm and a resistance of 50 Ω. The plane of the coil is perpendicular to a uniform magnetic field of magnitude 1 T. The direction of the field is suddenly reversed. (a) Find the total charge that passes through the coil. If the reversal takes 0.1 seconds, find (b) the average current in the coil and (c) the average emf in the coil. [3]

1.23 At the equator, a 1000-turn coil with a cross-sectional area of 300 cm^2 and a resistance of 15 Ω is aligned with its plane perpendicular to the earth's magnetic field of 0.7 G. If the coil is flipped over, how much charge flows through the coil? [3]

1.24 A circular coil of 300 turns and a radius of 5 cm is connected to a current integrator. The total resistance of the circuit is 20 Ω. The plane of the coil is originally aligned perpendicular to the earth's magnetic field at some point. When the coil is rotated through 90°, the charge that passes through the current integrator is measured to be 9.4 μC. Calculate the magnitude of the earth's magnetic field at that point [3].

1.25 A rectangular coil in the plane of the page has dimensions a and b. A long wire that carries a current I is placed directly above the coil (Figure P1.25). The wire is placed at $x = b/4$. (a) Obtain an expression for the emf induced in the coil if the current varies with time according to $I = 2t$. (b) If a = 1.5 m and b = 2.5 m, what should the resistance of the coil be so that the induced current is 0.1 A? What is the direction of this current? [3]

Figure P1.25

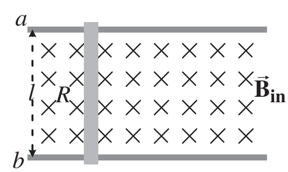


1.26 Repeat Problem 1.25 if the wire is placed at $x = b/3$. [3]

Problems on Motional EMF

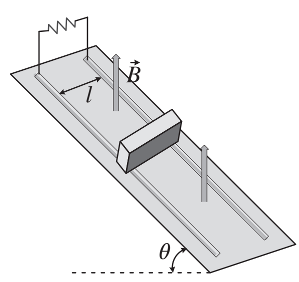
1.27 In Figure P1.27, the rod has a resistance R and the rails have negligible resistance. A capacitor with charge Q_o and capacitance C is connected between points a and b so that the current in the rod is downward. The rod is placed at rest at $t = 0$. (a) Write the equation of motion for the rod on the rails. (b) Show that the terminal speed of the rod down the rails is related to the final charge on the capacitor [3].

Figure P1.27



1.28 In Figure P1.28, a conducting rod of mass m and negligible resistance is free to slide without friction along two parallel rails of negligible resistance separated by a distance l and connected by a resistance R. The rails are attached to a long, inclined plane that makes an angle θ with the horizontal. There is a magnetic field B directed upward. (a) Show that there is a retarding force directed up the incline given by $F = \frac{B^2 l^2 v \cos^2\theta}{R}$. (b) Show that the terminal speed of the rod is $v_t = \frac{mgR \sin\theta}{B^2 l^2 \cos^2\theta}$ [3].

Figure P1.28



1.29 A square loop of conducting wire (area A) is pulled out of a region of constant, very high magnetic field B that is directed perpendicular to the plane of the wire. Half of the wire is in the field and half of the wire is out of the field when the wire is pulled out. A constant force F is exerted on the wire to pull the wire out. The wire is pulled out in time t. All else being equal, if the force were doubled, approximately how long would it take to pull the wire out? (*a*) t (*b*) $t/\sqrt{2}$ (*c*) $t/2$ (*d*) $t/4$ [3]

1.30 If, instead of doubling the force the resistance of the wire in Problem 1.29 was halved (all else being equal), what would the new time be? (*a*) t (*b*) $2t$ (*c*) $t/2$ (*d*) $t/\sqrt{2}$ [3]

1.31 A wire lies along the z-axis and carries current $I = 20$ A in the positive z-direction. A small conducting sphere of radius $r = 2$ cm is initially at rest on the y-axis at a distance $h = 45$ m above the wire. The sphere is dropped at time $t = 0$. (*a*) What is the electric field at the centre of the sphere at $t = 3$ s? Assume that the only magnetic field is the magnetic field produced by the wire. (b) What is the voltage across the sphere at $t = 3$ s? [3]

1.32 In Figure P1.32, let $\theta = 30°$; $m = 0.4$ kg, $l = 15$ m, and $R = 2\,\Omega$. The rod starts from rest at the top of the inclined plane at $t = 0$. The rails have negligible resistance. There is a constant, vertically directed magnetic field of magnitude $B = 1.2$ T. (*a*) Find the emf induced in the rod as a function of its velocity down the rails. (*b*) Write Newton's law of motion for the rod; show that the rod will approach a terminal speed and determine its value [3].

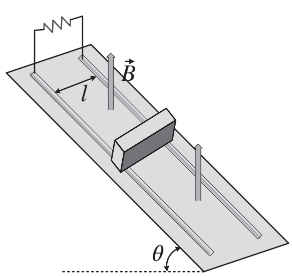
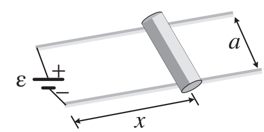


Figure P1.32

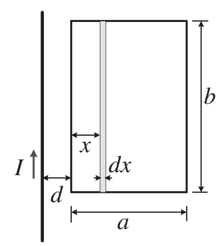
1.33 A solid conducting cylinder of radius 0.1 m and mass 4 kg rests on horizontal conducting rails (Figure P1.33). The rails, separated by a distance $a = 0.4$ m, have a rough surface, so the cylinder rolls rather than slides. A 12-V battery is connected to the rails as shown. The only significant resistance in the circuit is the contact resistance of $6\,\Omega$ between the cylinder and rails. The system is in a uniform vertical magnetic field. The cylinder is initially at rest next to the battery. (*a*) What must the magnitude be and the direction of $\vec{B}$ so that the cylinder has an initial acceleration of 0.1 m/s^2 to the right? (*b*) Find the force on the cylinder as a function of its speed v. (*c*) Find the terminal velocity of the cylinder. (*d*) What is the kinetic energy of the cylinder when it has reached its terminal velocity? (Neglect the magnetic field due to the current in the battery-rails-cylinder loop, and assume that the current density in the cylinder is uniform.) [3]

Figure P1.33



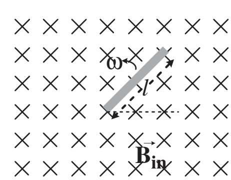
1.34 The loop in Figure P1.34 moves away from the wire with a constant speed v. At time $t = 0$, the left side of the loop is a distance d from the long straight wire. (a) Compute the emf in the loop by computing the motional emf in each segment of the loop that is parallel to the long wire. Explain why you can neglect the emf in the segments that are perpendicular to the wire. (b) Compute the emf in the loop by first computing the flux through the loop as a function of time and then using $\mathcal{E} = -\frac{d\phi_m}{dt}$. Compare your answer with that obtained in Part (a). Given: $a = 5\,\text{cm}$, $b = 10\,\text{cm}$, $d = 2\,\text{cm}$, and $I = 20$ A [3].

Figure P1.34



1.35 A conducting rod of length l rotates at a constant angular velocity about one end, in a plane perpendicular to a uniform magnetic field B (Figure P1.35). (a) Show that the magnetic force on a body whose charge is q at a distance r from the pivot is $Bqr\omega$. (b) Show that the potential difference between the ends of the rod is $V = \frac{1}{2}B\omega l^2$ (c) Draw any radial line in the plane from which to measure $\theta = \omega t$. Show that the area of the pie-shaped region between the reference line and the rod is $A = \frac{1}{2}l^2\theta$. Compute the flux through this area, and show that $\mathcal{E} = \frac{1}{2}B\omega l^2$ follows when Faraday's law is applied to this area [3].

Figure P1.35



Problems with Inductance

1.36 A long insulated wire with a resistance of 18 Ω/m is to be used to construct a resistor. First, the wire is bent in half and then the doubled wire is wound in a cylindrical form, as shown in Figure P1.36. The diameter of the cylindrical form is 2 cm, its length is 25 cm, and the total length of wire is 9 m. Find the resistance and inductance of this wire-wound resistor. [3]

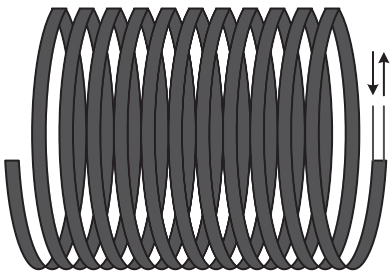


Figure P1.36

1.37 In Figure P1.37, circuit 2 has a total resistance of 300 Ω. A total charge of 2×10^{-4} C flows through the galvanometer in circuit 2 when switch S in circuit 1 is closed. After a long time, the current in circuit 1 is 5 A. What is the mutual inductance between the two coils? [3]

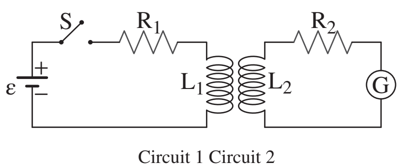


Figure P1.37

1.38 Show that the inductance of a toroid of rectangular cross-section, as shown in Figure P1.38, is given by $L = \frac{\mu_o NH \ln(b/a)}{2\pi}$, where N is the total number of turns, a is the inside radius, b is the outside radius, and H is the height of the toroid [3].

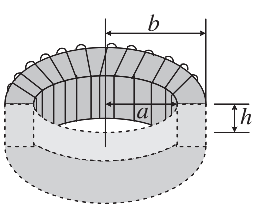


Figure P1.38

1.39 Using the result of Figure P1.38, calculate the self-inductance of an inductor wound from 10 cm of wire with a diameter of 1 mm into a coil with radius $R = 0.25$ cm [3].

1.40 A toroid of mean radius 25 cm and circular cross-section of radius 2 cm is wound with a superconducting wire of length 1000 m that carries a current of 400 A. (a) What is the number of turns on the coil? (b) What is the magnetic field at the mean radius? (c) Assuming that B is constant over the area of the coil, calculate the magnetic energy density and the total energy stored in the toroid [3].

References

1. Ostovic, V. (ed.) (1989). *Dynamics of Saturated Electric Machines*. Springer-Verlag.
2. Serway, R.A. and Jewett, J.W. Jr. (1996). *Physics for Scientists and Engineers with Modern Physics*, 9e. Boston, MA: Brooks/Cole.
3. Tipler, P. and Mosca, G. *Physics for Scientists and Engineers*, 5e. W. H. Freeman and Company.
4. Walker, J.(eds. J. Walker, D. Halliday and R. Resnick). *Fundamentals of Physics*, 10e. Wiley.
5. Chau, K.T., Li, W.L., and Lee, C.H.T. (2012). Challenges and Opportunities of Electric Machines for Renewable Energy (invited paper). In: *Progress In Electromagnetics Research B*, vol. 42, 45–74.
6. Irodov, I. (1979). *Problems in General Physics*. Mir Publishers of Moscow.
7. Knight, R.D. *Physics for Scientists and Engineers: A Strategic Approach*, 2e. Addison Wesley (book), Benjamin Cummings(solution manual).