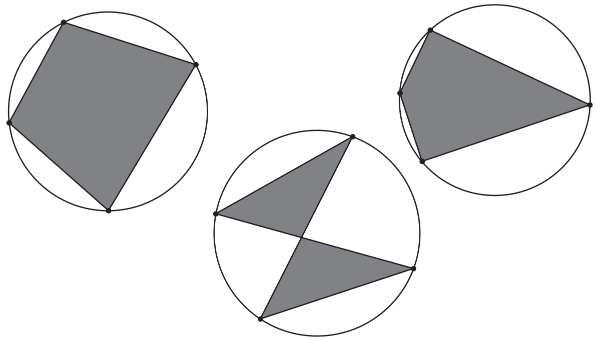
CHAPTER ONE



Using GeoGebra

Mathematics is too often viewed as a static list of facts and procedures. This is unfortunate since this perspective leaves no room for the excitement and satisfaction that can come from exploring mathematical ideas and discovering mathematical relationships. We hope to include this part of "doing mathematics" in your study of geometry. To this end, we hope to enhance your ability to visualize and reason about geometric ideas. GeoGebra®, a powerful geometry software package, will make it easier to see, literally, what happens to various geometric objects—points, lines, segments, circles, and so on—and the relationships among them as they are moved with respect to one another. As you work on the activities in this chapter, think about what it means to create a dynamic construction and how that differs from a static drawing. By the end of this chapter, we expect you will be familiar with some of the basic constructions available in GeoGebra.

As you begin your study of college geometry, you will be invited to look at many examples and explore many ideas using GeoGebra as a tool for your explorations. You will be asked to make a lot of observations and to formulate conjectures based on your observations. In general, a conjecture is a statement expressed in the form

If . . . [hypothesis] . . . **then** . . . [conclusion] . . . .

The *hypothesis* includes the assumptions you are making and the facts or conditions given in the problem. The *conclusion* is what you claim will always happen if the conditions named in your hypothesis hold. As you grow in mathematical maturity, you may be able to express your conjectures without using *If* . . . , *then* . . . , but for now we encourage you to use this format. Because the conclusion must follow from the hypothesis, writing your conjectures in this format makes your hypothesis explicit and clear—and this will help you develop clear and robust proofs. Once you have developed a proof for a conjecture—and tested it by having your colleagues critique it with you—you can call it a theorem.

We hope that engaging in the set of activities that open each chapter will pique your curiosity about some important ideas in geometry. You will get the most out of this course if you put serious effort into these explorations and think about the ideas that are introduced through these activities. Following the activities, we present a discussion about these important ideas. In the discussion, we answer many of the questions raised in the activities, so reading ahead in the text is always a good practice—but we hope that you will really engage in the explorations and activities before reading very far ahead.

The geometric constructions introduced in the activities of this chapter will be used throughout the course. So you should pay attention to the GeoGebra tools as you work on the activities. However, the focus at this point is on exploring certain geometric configurations and then making conjectures about what you observe. You will not be asked to prove anything in this chapter—but you should be asking yourself if you can explain what is going on. Ask yourself questions such as "What is happening here?" . . . "Why is it happening?" . . . "What properties of the figures might lead to the apparent results?" . . . .

Some of the interesting geometric ideas presented in this opening chapter will be investigated in greater depth in later chapters. While in some mathematics textbooks it is essential to thoroughly master every idea in the opening chapters before going on to later chapters, that is not the case for this book! Rather, in this opening chapter we hope to instill in you a bit of geometric curiosity and not to bog you down at the beginning of the course with a myriad of details. Enjoy the questions, and trust that in later chapters we will guide you gently toward the answers.

1.1 ACTIVITIES

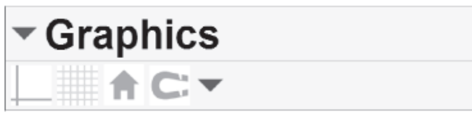
GETTING STARTED WITH GEOGEBRA

GeoGebra is an interactive tool for geometry that you can use to create very elaborate figures. Fortunately, even a beginner can do many interesting things right away. As the course progresses, you will become more and more sophisticated with GeoGebra. As with any good software tool, there are multiple ways of doing most tasks. Share the shortcuts you discover with each other. In a short time, you may even find that you can teach some useful tricks to your instructor!

In this opening chapter, you are invited to experiment with some of the capabilities of GeoGebra. Several versions of GeoGebra are available: Classic 5, GeoGebra Geometry online, and now Classic 6. There is even a GeoGebra app for cell phones! The activities presented in this text were developed using GeoGebra Classic 5, but any of these versions can be used. Be aware that the software tools for different versions may be in different locations on the screen from what is described here.

Let's get started. First, launch GeoGebra in a computer. At the top of the screen are the tools that GeoGebra has available. Look through the various options for these tools. If you click on a tool, you will see a menu of options available on that tool. In the first few chapters, we use only some of these tools, saving the more specialized ones for later.

Initially, you will see that the GeoGebra window is divided into two panes, one for Algebra and another for Graphics. Everything done in the Graphics pane will have a corresponding entry in the Algebra pane. This will be particularly useful for Chapter 5 on Analytic Geometry. For the moment, however, the Algebra pane will be used only occasionally. On the Graphics pane, there is a small triangle next to the word Graphics. Click on this to open the **Style Bar**. One of the buttons on the Style Bar will toggle the axes. For the first few chapters, you should use this button to hide the axes, for we will not need them.

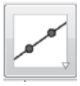


At the top of the window are the buttons for various tools, some of which you will use frequently. As you let the mouse move over these tools, you will see that each tool has a name, which is also the first of several options available on that tool. As you click on each tool, you will see a menu of different options available on that tool. It is worth discussing each of these tools individually. Some of the icons can be used to perform several different geometric constructions. Read through this discussion of the icons you see on the GeoGebra worksheet as you experiment with them.

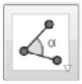
We will start with the **Move**, **Point**, **Line** and **Angle** tools:

Use the **Move** tool (the one with the Arrow) to select and move objects in the sketch. Click on any object—a point, a line, a segment, a circle, an arc—to select it; the object will be highlighted. Click on the object again to deselect it, or click on an open region (i.e., "white space") anywhere on the sketch to deselect all objects. To select more than one object on a PC, hold down the CTRL key and click on additional objects. (The keystroke commands will be different on a Mac computer.) You can drag selected objects across the sketch, or you can delete them. Also, the color and thickness of a selected object can be changed by using commands on the Style Bar. (Note: If you examine the menu attached to the Move tool, you will see some other options. For now, we will use only the basic Move arrow.)

The **Point** tool is next to the Move tool. Click on the Point tool to see a menu of options, including Intersect and Midpoint. Use the Point tool to construct points in the sketch. When you construct a point on another object such as a line, the new point will be restricted to stay on this object. This is useful for putting points on lines or circles. (The option for Point on Object works the same way.) If a point is constructed in an open region, then it is free to move over the entire plane.



The **Line** tool and its various options are used for constructing straight objects—segments, rays, or lines. If you hold down this button, you will see the seven options available. For this chapter, we focus on the Line and Segment options.



The **Angle** tool will allow you to construct angles and has options for measuring the size of an angle as well as the length of a segment and the area of a region. To measure an angle, choose the Angle tool. Then do one of the following things: either select the two rays (or segments) that define the angle or select three points that define the angle. You must be careful of the order in which you make these selections. GeoGebra will measure angles *counterclockwise* (the mathematically positive direction) from the first ray to the second ray. This will be relevant in Activity 1 and later.

Another useful tool is the **Text** tool. You will find this tool among the options on the **Slider** tool. You will be able to insert comments and answers to questions directly into your GeoGebra diagram by using the Text tool. (The Style Bar will allow you to change the size and color of your text if you wish.)

As you do the following activities, pay attention to anything that you observe as you construct each diagram. Use the Text tool to write your observations directly into the GeoGebra window. Try to think of explanations for the things you observe, and express your explanations in complete sentences.

Get into the habit of saving your work for each activity, as later work sometimes builds on earlier work. You will find it helpful to read ahead into the chapter as you work on these activities. Use a different GeoGebra window for each of the following activities.

1. Use the **Line|Segment** tool to construct a quadrilateral. Check to see if your sketch is robust by moving one of the vertices and seeing if the figure remains a quadrilateral.
   a. Use the **Midpoint or Center** option from the Point tool to construct the midpoints of the sides of the quadrilateral. Connect these midpoints in order. What kind of figure is formed? Complete the following sentence: "If the midpoints of an arbitrary quadrilateral are connected in order, the resulting midpoint quadrilateral will be . . . ." Drag the vertices of the original quadrilateral to help you decide whether your conjecture is always correct.
   b. Use the Angle tool to measure the angles of this midpoint quadrilateral. What do you observe about these angles?
   c. To measure the area of a polygon, it is necessary to construct its interior. The **Polygon** tool will do this. Choose the Polygon tool, then select the vertices—in the proper order—for the desired polygon, repeating the

first vertex to finish the construction. Measure the areas of the original quadrilateral and its midpoint quadrilateral. Notice that the area of the polygon is reported in the Algebra pane. You can also use the **Angle|Area** tool to measure the area.

What do you observe about these areas? Express your observation as a conjecture by completing the following sentence: "If the midpoints of an arbitrary quadrilateral are connected in clockwise order, the area of the resulting midpoint quadrilateral will be . . . ." Drag the vertices of the original quadrilateral. Does your conjecture continue to hold?

Use the Text tool to type your observations and conjectures into the worksheet. Select the Text tool and click anywhere in the diagram. This opens a text box. Your text can include mathematical symbols, chosen from a collection of menus. You can also include the value of a measurement by selecting the object from the list of objects in the text box. Once the text is complete, the Style Bar can be used to change its size, color, etc. A text box can be moved around the screen like any object.

GeoGebra automatically labels objects in a sketch. Labels move to stay with their objects. If you right-click on an object (on a PC) or double-click (on a Mac), you will see an option to rename the object. There will also be an option to **Show Label**; this same option will also hide the label.

The default setting for GeoGebra is to label every object, and you may find that all these labels begin to clutter the diagram. Clicking on the **Options** menu at the top of the GeoGebra window, you can select **Labeling|New Points Only**, and then **Save Settings** to control the proliferation of labels that appear in the diagram.

2. In a new window of GeoGebra, construct and label an arbitrary quadrilateral, $WXYZ$. Draw the diagonals WY and XZ. (The diagram will be easier to analyze if you use an option on the Style Bar to choose different colors for some of the lines.)
   a. Try to manipulate the quadrilateral so that the diagonals don't intersect each other. What do you observe about the shape of the quadrilateral?
   b. Write some conjectures about your observations to summarize what you notice by completing the following sentences:
      i. If the diagonals of a quadrilateral intersect each other, then the quadrilateral . . .
      ii. If the diagonals of a quadrilateral don't intersect each other, then . . .
   c. What if the diagonals bisect each other? (Two line segments are said to *bisect each other* if they intersect each other at their midpoints.) Can you construct a quadrilateral with this property? Write a conjecture to summarize what you learn in this exploration by completing the statement: "If the diagonals of a quadrilateral bisect each other, then . . . "

d. The *converse* of a conjecture is a statement that interchanges the role of the hypothesis (if-part) and the conclusion (then-part) of the statement. Write the converse of one or more of the conjectures you have formulated in this activity. Does the converse mean the same thing as the original statement?

3. Construct an arbitrary triangle, and measure its interior angles. Drag a vertex to check whether these measurements are always interior angles; if necessary, change the range of the angles by using an option on the Style Bar. You can also change the range of the angle measurements by a right-click on the angle, selecting **Object Properties|Basic**, and then choosing the desired range.

a. Calculate the sum of these angle measures. This is done in the **Input** line at the bottom of the window. There is a button marked with the Greek letter α; pressing this button opens a menu of symbols that can be inserted into the calculation. Look for the Greek letters that represent the angles of your triangle. The sum will appear in the Algebra pane. Complete the following sentence: "The sum of the measures of the interior angles of an arbitrary triangle is . . . " Drag the vertices of the triangle to help you decide whether your conjecture is always correct.

b. Repeat this experiment for an arbitrary quadrilateral, and write a conjecture about the angle sum of a quadrilateral. (As before, check that the angle being measured is always interior to the quadrilateral.) Drag the vertices to see whether your conjecture always holds.

c. Are your conjectures about the angle sum of triangles and the angle sum of quadrilaterals related to each other? Why or why not?

Use the **Circle with Center Through Point** tool to construct circles, just as you would with a mechanical compass. Click to select the center and move out to construct the circle. This will create a circle with a center point and a point on the circle. These two points define the circle; moving either of them will alter the circle.

4. Construct a circle with center point *C*. Then construct three points *P*, *Q*, and *R* on this circle. Make sure than none of these points are defining points of the circle.

a. Construct chords *PR*, *PQ*, and *QR*. Measure ∠*PQR*. The angle *PQR* is called an *inscribed angle* for the circle. Drag point *Q* around the circle, or right-click on *Q* to see the **Animation On** option. (A Mac computer will work differently.) What do you observe about the measure of ∠*PQR*? What happens when *Q* moves past point *R*? What happens if you change the radius of the circle? The angles that you are looking at as *Q* moves around the circumference of the circle are *angles subtended by the chord PR*. Express your observation in the form of a conjecture: "If *PR* is a fixed chord of a circle and *Q* is a point on that circle, then the measure of ∠*PQR* is . . . ."

b. Now construct radii *CP* and *CR*. Measure ∠*PCR*, which is called a *central angle* of the circle. There are two choices for this angle; you may wish to measure both. (Using different colors may make your diagram easier to analyze.) What do you observe about the measures of a central angle and the corresponding inscribed angle of a circle? Vary the points P, Q, and R by dragging them around the circle so that you examine many different angles. Does your observation still hold? Express your observation as a conjecture by completing the following sentence: "If $\angle PQR$ is an inscribed angle of a circle centered at C, then the measure of $\angle PQR$ will be ... [compared to] the measure of $\angle PCR$."

5. Construct a circle with diameter PR. Make sure you construct your circle in such a way that PR is certain to be the diameter, not merely a chord, of the circle.
   a. Construct a point Q on the circle, and measure $\angle PQR$. Move the point Q. What do you observe? What happens if Q moves past the point R?
   b. Make a conjecture about this situation. How is this conjecture related to the conjectures you made in Activity 4?

6. Construct an *equilateral triangle* using GeoGebra. (Don't just estimate; use circles and intersection points to guarantee that your triangle is equilateral.) This construction may require you to use objects that are not part of the final diagram. These objects can be hidden by locating the name of each object in the Algebra pane and clicking the button next to that name.
   a. Create a point P in the interior of the triangle. Construct line segments from P that are perpendicular to each side. Then hide the extra objects you used to do this. The lengths of the three segments will appear in the Algebra pane. Have GeoGebra calculate the sum.
   b. Drag P around your picture. Your segments should remain perpendicular to the three sides. What do you observe? Drag P onto one of the vertices of your triangle. What do you observe?
   c. Make a conjecture about this situation by completing the following sentence: "If P is a point interior to an equilateral triangle, then the sum of the lengths of the three perpendicular segments from P to the sides of the triangle is ...."
   d. Draw another triangle that is not equilateral, and repeat this experiment. What do you observe? Does anything different happen when the triangle is acute or obtuse?

7. A *rectangle* can be defined as a quadrilateral with four right angles.
   a. Construct a rectangle in GeoGebra. The **Perpendicular Line** tool will be useful to do this. Did your construction use the definition given here, or did it use some additional properties of a rectangle?
   b. Construct the diagonals of this rectangle, and notice how they intersect. The **Point|Intersect** tool can be helpful here. Express your observation as a conjecture by completing the following statement: "If a quadrilateral is a rectangle, then its diagonals ... "
   c. Write the converse of your conjecture. Explore whether or not the converse appears to be true.

8. A *square* can be defined as a regular quadrilateral. A polygon is said to be *regular* if all of its sides are the same length, and all of its angles have the same measure.
    a. Construct a square in GeoGebra. Did your construction use the definition given here, or did it use some additional properties of a square?
    b. Is every rectangle a square, or is every square a rectangle? Express your answer to this question as a conjecture by completing the following statement: "If a quadrilateral is a . . . , then it is a . . . "
    c. Write the converse of this conjecture. Does the converse mean the same thing as the original statement?
9. A quadrilateral is called *cyclic* if its four vertices lie on a common circle. Construct an example of this, and measure the four angles of your quadrilateral. What do you observe about the opposite angles? Express your observation as a conjecture. Does your conjecture still hold if you move the vertices? Use the observations you made in Activities 4 and 5 to explain why your conjecture is correct.

To complete our discussion of the tools available at the top of the GeoGebra screen, let's look at some additional options. On the Perpendicular Line tool, there are options for **Parallel Line**, **Perpendicular Bisector**, and **Angle Bisector**, among others. These will be useful later in the course. On the Polygon tool, there are options for **Regular Polygon** and **Rigid Polygon**. An equilateral triangle is an example of a regular polygon and so is a square. The Regular Polygon tool offers a shortcut method for the construction you created in Activity 6. On the other hand, the Rigid Polygon tool is used to construct polygons that, once created, do not change. These rigid polygons can be moved or turned, but will retain their original shape.

On the Circle tool, there are many additional options. The **Compass** tool lets you create a circle by selecting a segment or a pair of points to represent the radius, then selecting a location for the center of the desired circle. With the **Circle by Center and Radius**, you can construct a circle with any center and a radius given by a number value.

Under the Slider tool, there is the **Image** option. The Image tool lets you insert images into the GeoGebra window. These images can be manipulated like any other object in the window. Under the Move tool is the **Pen** option. The Pen will let you draw with your mouse. This drawing can be moved, and there are options on the Style Bar for it as well.

Other tools will be discussed as they are needed in later chapters. Feel free to experiment!

One more tip before we conclude these activities: Sometimes you will want to print the work you have done using GeoGebra. Go to the File menu and choose **Print Preview**. Here you can control the size of the printed image by adjusting the numbers in the boxes labeled *units* and *cm*. You can select Portrait or Landscape printing. Also, you can add a title and the author's name; these will appear on the printed copy.

EXPLORING AND CONJECTURING

SOME GEOGEBRA TIPS

A quadrilateral consists of four line segments connected at their endpoints to form a closed figure. For Activity 1, you can use the Line|Segment tool to construct one segment, then another segment connected to it, then a third segment, and finally, a fourth segment connected to both the first and third segments. Or you can use the Polygon tool. Click somewhere in the white space of your GeoGebra diagram to construct each of the four vertices, then click again on the first vertex to complete the polygon. With the Point|Midpoint or Center tool, you can either click on the endpoints of a segment or on the segment itself to construct the midpoint of each edge. The Segment tool then comes into play again as you construct the midpoint quadrilateral.

Using different colors can help focus your attention on certain portions of a figure as you experiment with a construction. For instance, select the four line segments that join the midpoints of the original quadrilateral. (Recall that by holding the CTRL key you can select multiple objects.) On the Style Bar, the **Color** menu gives you many choices. Pick a new color. Look again at the quadrilateral you constructed by connecting the midpoints. Because this midpoint quadrilateral is now in a contrasting color, it may be easier to see why it is interesting. Use the Move tool to drag one of the original vertices around the sketch and watch what happens.

The Text command can be used to type statements into your GeoGebra document. With the Symbols menu, you can insert a great variety of mathematical symbols and other symbols into your text. The Objects menu allows you to insert values from the document into the text. These values are dynamic; that is, as objects are moved around the diagram, the values for the objects in the text will be updated. After the Text box is created, selecting it with the Move arrow produces a selection of options in the Style Bar, such as text color, boldface, italics, and text size.

There are many other ways you might have done Activity 1. For instance, you might visualize a quadrilateral as four vertices connected by line segments. In this case, your construction would begin with four points. Then you would choose Segment and select these points two at a time. As you become familiar with the tools and the menus available, you will think of many approaches to any particular construction.

CONSTRUCTING → EXPLORING → CONJECTURING: INDUCTIVE REASONING

The process of observing and conjecturing is called *inductive reasoning*. By examining many examples, you can begin to see patterns and make educated guesses—conjectures—about what might be true. Looking at specific examples can suggest something that might be a general truth, and if that suggestion holds true for many examples, it is more likely to be true in a general setting. This process of exploring, experimenting, and conjecturing lies at the heart of scientific investigations.

GeoGebra is a wonderful tool for examining examples in geometry. Once a figure is constructed, it is easily varied by dragging objects around the sketch. Doing this makes it possible to explore many examples very quickly. Dragging objects in a GeoGebra construction can also help you see how the components of the construction are related to each other, which may suggest a reason why the conjecture is true.

Formulating a conjecture is an important step in the process of doing mathematics. The next step is justifying the conjecture—finding an explanation for why the conjecture holds. Once you have an explanation, it is important to communicate it to others in a detailed, logical presentation—a proof. The proof process relies on *deductive reasoning*. In the early chapters of this book, we will introduce the rules and methods of logic and deductive reasoning. As the course progresses, you will grow in your skill and confidence with developing valid geometric proofs.

LANGUAGE OF GEOMETRY

In the activities, we assumed that you already know some basic geometric vocabulary. We hope that you recognize most of the geometric terms used so far, even if you don't know their formal definitions. Formal definitions play an important role in mathematics, since the precision necessary in mathematics requires precision in the language. Unlike descriptive definitions often used in language arts, mathematical definitions are prescriptive. A good mathematical definition is complete and concise, using only terms that have been previously defined.

It is important to distinguish between a geometric object's definition and its properties. For instance, if a square is defined as a regular quadrilateral, then a property of a square would be that its interior angles are right angles. Similarly, in this text, we define a rectangle as a quadrilateral with four right angles. All the other features that we associate with a rectangle— opposite sides are equal and parallel, the diagonals bisect each other, and so on—can be proved as a consequence of that definition. In general, we can think of properties as consequences of a definition.

As you continue this course, it will be increasingly important to use geometric language carefully. It is important to have clear, concise definitions of the terms we are using in your own mind. Writing out a careful definition of a term is one way of clarifying its meaning for yourself; thus, you should practice writing careful definitions. Once you have written a definition, try to construct the figure you have defined, using only the properties you have stated in the definition. For example, a kite could be defined as a quadrilateral with two disjoint pairs of adjacent congruent sides. How would you use this definition to construct a kite?

A *polygon* is a closed figure in the plane formed by a set of line segments, each of which shares each endpoint with exactly one other line segment in the set.

Because it has many (*poly-*) angles (*-gons*), it is called a polygon. A *triangle* could be called a 3-gon, because it has three angles. It could also be called a *trilateral* because it has three sides (*laterals*). A *hexagon* has six angles. The corners of a polygon are called its *vertices* (singular, *vertex*). Does a polygon with *n* vertices always have *n* sides? Why or why not?

A *regular* polygon is a polygon whose angles all have the same measure, and whose sides are all the same length. So both a square and an equilateral triangle are regular, but a right triangle cannot be regular. (Why not?)

EXPLORATIONS, OBSERVATIONS, QUESTIONS

In several of the activities, you worked with various quadrilaterals. The most general *quadrilateral* is simply a polygon with four sides. The word quadrilateral comes from the Latin *quadrilaterus*, meaning four (*quadri-*) sided (*-laterus*). We could easily use the Greek roots and call a four-sided figure a *tetragon*, which means four (*tetra-*) angled (*-gon*). Within the family of quadrilaterals, there are some that have special names and specific characteristics such as rectangles, squares, parallelograms, and others. As you worked on Activity 1, did you observe that the quadrilateral formed when you connected the midpoints in order is a parallelogram? No matter what shape the original quadrilateral has—even if it is self-intersecting—its midpoint quadrilateral appears to be a parallelogram. In other words, if the midpoints of an arbitrary quadrilateral are connected in clockwise order, the resulting midpoint quadrilateral is a parallelogram.

A *parallelogram* is a quadrilateral whose opposite sides are parallel. Why does the construction you did in Activity 1 lead to a parallelogram? The opposite sides appear to be parallel, but how can you be sure that these segments really are parallel for any midpoint quadrilateral? While in this chapter we are not focused on explaining why, or proving that, our conjectures are true, it is a natural question to ask. In the next chapter, we will investigate these types of questions. If you are interested in trying to answer the question at this point, you might find it helpful to add line segments to create triangles that let you compare opposite sides of the midpoint quadrilateral.

What did you observe about the area of the midpoint quadrilateral compared to the area of the original quadrilateral? Your explorations may have led you to suspect that the area of the parallelogram is half the area of the original quadrilateral, at least in the case of non-self-intersecting quadrilaterals. In fact, Pierre Varignon established this as a theorem in 1731. What would you need to know in order to prove this result?

While working on Activities 1 and 2, you were faced with the question of what sorts of figures qualify as quadrilaterals. Certainly, there must be four sides. Are these sides allowed to cross each other? You can easily cause this to happen in your sketch by dragging a vertex. Should we allow this figure to be called a quadrilateral? Apply the definition of a polygon stated earlier: A quadrilateral is a closed figure in the plane formed by a set of four line segments, each of which shares each endpoint with exactly one other line segment in the set. Thus, the unusual-looking figures you observed while working on Activities 1 and 2 can be called quadrilaterals. Polygons whose sides intersect are called *self-intersecting* figures.

Even if we avoid self-intersecting quadrilaterals, there is still another issue raised in Activity 2. Think again about the figures you constructed while working on this activity. How are the quadrilaterals for which the diagonals don't intersect different from those for which the diagonals do intersect? Notice that the definition of quadrilateral allows us to call all of these figures "quadrilaterals" even though they fall into two distinctly different sets. Those quadrilaterals for which the diagonals intersect are *convex* quadrilaterals. Quadrilaterals and hexagons can be either convex or not convex. However, every triangle is convex. What about pentagons?

What does it mean for a polygon, or even a more general closed figure, to be convex? Try to state a clear definition for a convex figure. Your definition should allow you to decide whether any example of a polygon (or other closed figure in the plane) is convex, no matter what type it is. Experiment again with your sketch for Activity 2, and try to formulate a rule or strategy for determining whether a quadrilateral is convex. Does your rule work only for quadrilaterals, or does it work for other polygons (and other geometric figures) as well?

How did you construct the *equilateral triangle* you worked with in Activity 6? It is not enough to simply draw three line segments that look congruent. If your construction is *robust*, you will be able to select any of the vertices or edges of the triangle and move them around without losing the property that the figure is an equilateral triangle. Is your construction of an equilateral triangle robust? (*Hint*: The radii of any particular circle are all congruent. So if the base of your triangle is a radius, each of the sides can be a radius too, though maybe not for the same circle.) Once you get a robust construction that produces an equilateral triangle, ask yourself why that construction works. In other words, how can you be sure that the three sides are truly congruent? (If your construction is not a robust construction, go back now and redo Activity 6.)

As you worked on Activity 6, something interesting should have happened as you moved P around inside the triangle. What conjecture did you make about the sum of the lengths of the three segments you constructed? Viviani's Theorem states that this sum is always the same and, furthermore, that this sum is equal to an important value for this triangle. Does your conjecture say something similar?

When we talk about a point being *on* a circle, by the definition of circle, we mean that it is one of the points that is an equal distance from the fixed center point. Similarly, when we say that a point is *on* a triangle, we mean that it is on one of the segments that form the triangle, including the endpoints of the segments that are the vertices of the triangle. What does it mean for a point to be *interior* to a particular circle or triangle? This is a bit harder to define or explain. (Just to complicate things a bit, imagine that a large triangle is drawn on the surface of a sphere. Where is the interior now?) What does it mean for a point to be *exterior* to a triangle or circle?

Although in Activity 6 you were directed at first to put the point P interior to the equilateral triangle, GeoGebra might allow you to move P beyond the boundary of the triangle. (This depends on how you constructed the figure.) Does your conjecture still hold when P is on the triangle? Does it hold if P is outside the triangle? If not, it may be possible to modify your conjecture so that it still holds even when P is on or outside the triangle.

Did your conjecture continue to hold when you used an arbitrary triangle instead of an equilateral triangle? It appears that it does not, which may be an indication that the equal-length sides of the equilateral triangle play a key role in explaining why Viviani's Theorem holds true. While the sum of the lengths of the perpendicular segments is not constant for all points on the interior of an arbitrary triangle, it appears to remain fixed for some subset of points in the interior. Did you find such a set of points? The properties of these sets are not known to the authors, making this an intriguing exploration for the interested student.

You can take this activity in many directions. What happens for isosceles triangles? What if the figure is a square instead of an equilateral triangle? What about a pentagon? Perhaps you could investigate segments from P to each of the vertices instead of from P to the sides of the triangle. One of the exercises at the end of this chapter invites you to explore some of these questions. Let your imagination guide you to new discoveries.

The previous few paragraphs raised many questions! You may have answered some of these questions already as you worked on the activities. Many more questions will arise throughout this course. As questions arise in your explorations or in your reading, pause and try to answer them, or at least make sure you understand the questions and why we are asking them. As you work on the activities and read the discussions in this book, learn to ask your own questions: "Is that always true?" . . . "Why might it be true?" . . . "Will it still be true if I change some part of the problem?" . . . "Can I explain why it works?" . . . . Engaging in explorations and looking for explanations for the phenomena we observe is what "doing mathematics" is about.

THE FAMILY OF QUADRILATERALS

As you worked on Activities 1, 2, 7, 8, and 9, you encountered quite a few different quadrilaterals. Within the family of quadrilaterals, there are some special types—rectangle, square, kite, rhombus, trapezoid, parallelogram, and cyclic, to name a few. Try to write a one-sentence definition for each. If some of these names are not familiar to you, look up distinguishing characteristics to help you write these definitions.

When you write a definition for a geometric figure, try to use the smallest possible list of requirements to describe it. For example, in Activity 8 we defined a *square* as a regular quadrilateral. This means that a square has four sides (it's a quadrilateral), plus all of the sides have the same length and all of the angles are congruent (it's regular). However, it is sufficient to define a *square* as a quadrilateral with four equal angles and one pair of adjacent sides equal in length. From this shorter list of requirements, it is possible to prove the other features as properties of a square. Can you explain why knowing that a quadrilateral has four equal angles makes it possible to prove that the angles must be right angles? Similarly, in this book, we define a *rectangle* as a quadrilateral with four right angles. All the other features that we associate with a rectangle—opposite sides are equal and parallel, the diagonals bisect each other, and so on—can be proved using Euclid's postulates. We will examine these questions in greater depth beginning in the next chapter.

Every square is a rectangle; in other words, if a quadrilateral is a square, then it is a rectangle. The converse of this statement—if a quadrilateral is a rectangle, then it is a square—is not true. It is easy to sketch a figure of a rectangle that is not a square. Thus, we see that the converse of a statement does not mean the same thing as the original statement. This is an idea that we will return to again and again throughout this course.

There are many special types of quadrilaterals. Activity 9 introduces a type of quadrilateral that may be new to you. A quadrilateral is *cyclic* if its vertices lie on a common circle. Another way to say this is that a cyclic quadrilateral can be inscribed in a circle. Some familiar quadrilaterals, such as squares, are cyclic, but others are not necessarily cyclic. What did you observe about opposite angles in a cyclic quadrilateral? Does your observation still hold if the center of the circle is exterior to the quadrilateral—that is, if the quadrilateral lies entirely on one side of a diameter of the circle? You should be able to use your observations from Activity 4 to explain why your conjecture is correct. How is your conjecture different if the cyclic quadrilateral is self-intersecting?

A Venn diagram is a useful tool for picturing the relationships among various sets of figures. Consider some of the sets of figures we have been discussing in this chapter: parallelograms, quadrilaterals, rectangles, and triangles. In this situation, we are talking about sets of geometric figures that can be drawn in the plane, so we say that our *universe of discourse, U*, is all plane geometric figures. Let P denote the set of parallelograms, Q the set of quadrilaterals, R the set of rectangles, and T the set of triangles.

Figure 1.1 illustrates the relationships among these sets of figures. The large rectangle, labeled U, represents our universe of discourse. Within this universe, sketch a loop to represent each set. If two sets have things in common but each set also contains things that are not in the other set, draw the loops so that they overlap. If all of the objects in one set also belong to another set, we say that the smaller set is a *subset* of the larger set; draw the loop for the subset entirely within the loop for the other set. If two sets do not share any objects, we say that these sets are *disjoint*; in this case, draw the loops so that they do not overlap. For example,

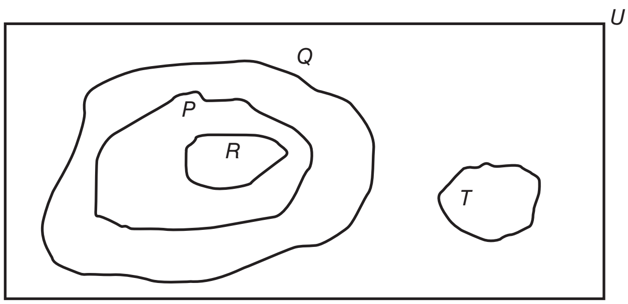


FIGURE 1.1
A Venn Diagram

in Figure 1.1, since every rectangle is a parallelogram, the set of rectangles, *R*, is drawn entirely within the set of parallelograms, *P*. Rectangles and parallelograms are subsets within the set of quadrilaterals, and we can see this in the way that the Venn diagram is drawn. Why is the set *T* off to the side by itself?

More sets can be added to this diagram. Let *S* denote the set of squares, and *K* the set of kites. A *kite* is a quadrilateral with two disjoint pairs of consecutive congruent sides. Since every square is a rectangle, the loop for squares must lie completely within the loop for rectangles. This illustrates the statement: If a quadrilateral is a square, then it is a rectangle. Since there are points in the loop for rectangles that are not in the loop for squares, the diagram also illustrates that the converse of this statement—if a quadrilateral is a rectangle, then it is a square—is not true. How would you add the loop for kites to this Venn diagram? You must decide if a kite could be a square or not. You must also decide whether all kites are parallelograms, or just some of them.

ANGLES INSCRIBED IN CIRCLES

A *circle* can be defined as a set of points that are equidistant from a fixed center point. Notice in this definition that the circle itself is only the set of points at a fixed distance from the center. This fixed distance from the center is the *radius* of the circle. Points that are closer to the center are not on the circle; rather, they are interior to the circle. Points whose distance from the center of the circle is greater than the radius are exterior to the circle.

If *PR* is a fixed chord of a circle and *Q* is any other point on the circle, then $\angle PQR$ is *subtended by the chord PR*. This angle is an *inscribed angle* of the circle. The angle *PCR*, where *C* is the center of the circle, is a *central angle* of the circle. As you worked on Activity 4, how did the measure of $\angle PQR$ change as you moved *Q* around the circle? What did you observe about the relationship between the central angle and the inscribed angles for a fixed chord *PR*? Your conjecture for this situation might say something like "If *PR* is a fixed chord of a circle with center *C* and *Q* is any point on the circle, then the measure of the central angle, $\angle PCR$, will be twice the measure of the inscribed angle, $\angle PQR$." Your conjecture should still be true if the central angle measures 180° (in which case *PR* will be a diameter of the circle). What happens when the central angle becomes greater than 180°? (You can force GeoGebra to display angles greater than 180° by an option on the Style Bar.) Isosceles triangles can be used to prove this conjecture.

If *PR* is a diameter of the circle, as in Activity 5, the central angle, $\angle PCR$, is formed by two opposite rays, $\overrightarrow{CP}$ and $\overrightarrow{CR}$. In this case, $\angle PCR$ is sometimes called a *straight angle*. What is its measure? An angle that subtends a diameter of a circle is said to be an *angle inscribed in a semicircle*. To be consistent with your conjecture from Activity 4—or if you've proved it, your theorem from that activity—the measure of an angle inscribed in a semicircle should be half the measure of the corresponding central angle. Is this what you observed?

These observations about inscribed and central angles will be useful throughout this course as you work on various problems involving circles.

RULES OF LOGIC

A *statement*, mathematical or otherwise, is a declarative sentence that can be evaluated as true or false. Not every sentence is a statement. For instance, a question or a command would not be a statement. On the other hand, every sentence in this paragraph is a statement. A *closed statement* is a sentence that is either true or false. An *open statement* contains a variable; once a value is given for the variable, the statement becomes either true or false. In Chapter 3, we will discuss another way that an open statement can be closed.

Many interesting statements are built up from simpler statements. The simpler statements are combined using the logical connectives *and* and *or*. For example,

- This figure has a right angle *and* two of its sides are congruent.
- Triangles have three vertices *and* every rectangle is a square.
- This figure is a rectangle *or* it is a rhombus.
- The diagonals of a parallelogram bisect each other, *but* the diagonals of a cyclic quadrilateral do not necessarily bisect each other.

To evaluate the truth value of compound statements like these, we first determine the truth value of each component of the statement, then combine these truth values using rules of logic. For example, the first statement has two components: "This figure has a right angle" is the first component and "two of its sides are congruent" is the second component. The word *figure* is a variable for the statement in each of these components. To evaluate this compound statement, we have to know which figure is being discussed. Then we have to decide whether each part of the statement is true or false for that particular figure. *Both* components of the statement must be true to make the compound statement "This figure has a right angle *and* two of its sides are congruent" be true. There are figures that make both parts of this statement true and hence make the entire statement true. Can you draw a few examples? Also draw a few non-examples. You should be able to draw figures that make the first component false, other figures that make the second component false, and finally figures that make both components false.

In the second statement, we know that triangles indeed have three vertices. However, some rectangles are not square. A compound statement of the form *P and Q* is true only if both parts of the statement are true. So the compound statement "Triangles have three vertices *and* every rectangle is a square" is false.

In the third sentence, we again have a statement with two components, but the components are connected by the word *or*. For this statement to be true, it is only necessary for one of the two components to be true. It can happen that both components are true, which also makes the statement true. Try to draw examples of figures that make the first component true but not the second; figures that make the second component true but not the first; and figures that make both components true. All of your examples will make the statement "This figure is a rectangle *or* it is a rhombus" true.

The fourth statement is correct English. How should we interpret it logically? There are two components, namely, “the diagonals of a parallelogram bisect each other” and “the diagonals of a cyclic quadrilateral do not necessarily bisect each other.” The difficulty is knowing how to treat the word *but* connecting these components. Typically, *but* is used to connect components that contrast in some way, as in the example statement. At its most basic level, however, *but* means *and*. Thus, we are deciding the truth value of “The diagonals of a parallelogram bisect each other, *and* the diagonals of a cyclic quadrilateral do not necessarily bisect each other.” Both components are true, so the statement is true.

There is another important logical operator—*not*. The logical connectives *and* and *or* discussed above are used to connect two statements, but the word *not* is applied to a single statement. This allows statements such as “This figure is *not* a rectangle.” The *not* in this statement reverses the truth value, so a rectangle makes the statement false while a trapezoid makes the statement true.

Let us summarize:

AND The compound statement *P and Q* is true only when both components are true. This is called *conjunction* and its mathematical symbol is $P \wedge Q$.

OR The compound statement *P or Q* is true when one or both components are true. This is called *disjunction* and its mathematical symbol is $P \vee Q$.

NOT The statement *not P* is true when P is false and it is false when P is true, thus *not* reverses the truth value of the statement. This is called *negation* and its mathematical symbol is $\neg P$. (Some authors use $\sim P$ to represent the negation of P. We will use the symbol $\sim$ to denote similar figures, and we will use $\neg$ for negation.)

Is it clear to you how to simplify $\neg\neg P$? Can you express the negation “It is not the case that ΔABC is not equilateral” more simply in clear English?

Here is another question about expressing a negation clearly and simply: How would you simplify $\neg(P \vee Q)$? Be careful! Try it for a specific statement: *It is not the case that this figure is a rectangle or it is a rhombus.* To make this negated statement true, its components must both be false. How could you say this more naturally?

1.3 EXERCISES

Give clear and complete answers to the following problems and questions. Remember to write in complete sentences and grammatically correct English. Include diagrams whenever appropriate.

1. A *rhombus* is a quadrilateral whose four sides all have the same length. Construct a rhombus. Is your construction based on this definition? List some additional properties of rhombi (or rhombuses).
2. A *parallelogram* is a quadrilateral whose opposite sides are parallel. Construct a parallelogram. Is your construction based on this definition? List some additional properties of parallelograms.
3. What are the relationships among a rhombus, a parallelogram, a rectangle, and a square?

4. Within the family of quadrilaterals, there are some special figures—rectangle, square, kite, rhombus, trapezoid, and parallelogram. Write a one-sentence definition for each of these terms. Try to write each definition using as small a list of defining characteristics as possible.

5. A *kite* is a quadrilateral with two disjoint pairs of consecutive sides congruent. Is it possible for a quadrilateral to be both a square and a kite? Are all squares kites? Add the sets of squares, S, and kites, K, to the Venn diagram of Figure 1.1, page 14.

6. In mathematics we talk about numerical values being equal if they are quantitatively the same. For example, $2 + 3 = 5$, because the sum $2 + 3$ represents the same quantity as the number 5. When two different geometric figures are the same size and shape, we say that they are *congruent* rather than equal figures. We say that two line segments are congruent if they have the same length. What might it mean for each of the following pairs of figures to be congruent?
   a. Two angles
   b. Two circles
   c. Two triangles (or other polygons)
   d. What is the difference between *congruence* and *equality*?

7. Triangles can be equilateral, isosceles, scalene, acute, right, and/or obtuse. Define each of these terms, and include a sketch of each. Describe how these different kinds of triangles are related.

8. Explain what it means for a point to be *on*, *interior to*, or *exterior to* a circle.

9. Teachers often use a disk to represent a circle. What do they mean by this and what kind of confusion could this create?

10. Some textbooks define a *trapezoid* as a quadrilateral with exactly one pair of parallel sides, while other textbooks say that a trapezoid has at least one pair of parallel sides.
    a. What are the consequences of choosing one of these definitions over the other? Which is correct?
    b. Is a parallelogram a trapezoid? Why or why not?

11. What is the converse of the statement: "If a quadrilateral is a square, then it is a rectangle"? Is this converse a true statement?

12. Rewrite the following statement as an if-then statement: The angles of an equilateral triangle are congruent. What is the converse of the statement you wrote? Is this converse a true statement?

13. A statement can be negated by putting "not" in front of the statement. For example, the negation of "Some cars are blue" is "Not (some cars are blue)." However, this is awkward and unclear. A more useful form of this negation is "No cars are blue." Note that in this case, the original statement is true, and its negation is false.
    a. Negate the following statement, and express the negation clearly in simple English: "All polygons are convex." Which statement is true, the original statement or its negation?
    b. Is it possible for a statement and its negation both to be true? Both to be false?

14. Define the property *convex* for quadrilaterals. Does your definition hold for convex polygons or convex geometric figures in general? If not, rewrite your definition so that it can be applied to any geometric figure in the plane.

15. If the diagonals of a quadrilateral bisect each other, could the quadrilateral be a parallelogram? A rhombus? A rectangle? A square? A cyclic quadrilateral? Construct diagrams in GeoGebra to support your answers.

16. If the diagonals of a quadrilateral are congruent, could the quadrilateral be a parallelogram? A rhombus? A rectangle? A square? A cyclic quadrilateral? Construct diagrams in GeoGebra to support your answers.

17. Does a polygon with n vertices always have n sides? Why or why not?

18. What are the common names of polygons with 2, 3, 4, . . . sides? What is the minimum number of sides for a polygon? What is the maximum number of sides?

19. Why can't a right triangle be regular?

20. Can a parallelogram be inscribed in a circle? In other words, can a single circle pass through all

four vertices of a given parallelogram? Why or why not?

21. Take another look at your investigation with the midpoint quadrilateral in Activity 1. Repeat that investigation using a triangle and its midpoint triangle.
   a. What is the ratio of the area of the midpoint triangle to the original triangle? Make a conjecture, and vary the vertices of the triangle. Does your conjecture continue to hold?
   b. Can you explain what is going on?

22. In Activity 6, you investigated the sum of the lengths of the perpendicular segments from the point P to the each of the sides of an equilateral triangle.
   a. Use GeoGebra to explore the sum of these lengths for some other polygons such as isosceles triangles, rectangles, and hexagons.
   b. Use GeoGebra to investigate the sum of the lengths of the segments from P to each of the vertices.

23. Identify which of the following sentences is a statement. If the sentence contains a variable, identify the variable. If possible, determine whether the sentence is *true* or *false*.
   a. The angle sum of a triangle is 360°.
   b. Is $\angle A$ in ΔABC a right angle?
   c. X is a quadrilateral.
   d. $a^2 + b^2 = c^2$.
   e. All rectangles are equiangular quadrilaterals.
   f. A kite is a quadrilateral.
   g. Triangles have three sides and rectangles have four right angles.
   h. The diagonals of a trapezoid bisect each other.
   i. Some cows are purple.
   j. Wow! That is a beautiful design!
   k. All right triangles are equilateral.
   l. Grizzly bears make affectionate pets.

24. Explain how to evaluate (find the truth value of) a statement of the form $P \vee Q$. Apply your evaluation method to the statement "ΔABC has a right angle *or* two of its sides are congruent."

25. Explain how to evaluate a statement of the form $P \wedge Q$. Apply your evaluation method to the statement "ΔABC has a right angle *and* two of its sides are congruent."

26. Set up a truth table for a statement of the form $P \vee Q$.

27. Set up a truth table for a statement of the form $P \wedge Q$.

28. For each of the following statements, sketch several figures that make the statement true; then sketch several figures that make the statement false. If some of these sketches are not possible, explain what is happening.
   a. This triangle has a right angle and two of its sides are congruent.
   b. This triangle is equilateral and it has a right angle.
   c. This quadrilateral is a rectangle or it is a rhombus.
   d. This quadrilateral has a right angle and its diagonals bisect each other.
   e. This quadrilateral is a kite and it is not convex.
   f. This figure is a pentagon and it is not convex.

29. How would you express the negation of a statement of the form $\neg(\neg P)$? Express your answer as a rule, and use this rule to negate the statement "It is not the case that $WXYZ$ is not a square."

30. How would you express the negation of a statement of the form $\neg(P \wedge Q)$? Express your answer as a rule, and use this rule to negate the statement "This figure is a rectangle and it is equilateral."

31. How would you express the negation of a statement of the form $\neg(P \vee Q)$? Express your answer as a rule, and use this rule to negate the statement "This figure is a rectangle or it is a rhombus."

Exercises 32–34 are especially for future teachers.

32. The Praxis Series™ Subject Assessments are tests designed by Educational Testing Service (ETS) to assess your knowledge of subject areas you plan to teach, and are used as part of the licensing procedure in many states.
   a. Is the Praxis test of *Mathematics Content Knowledge* required for teacher certification

in your state? Is the *Mathematics: Proofs, Models, and Problems* test required?

   b. Questions on the Praxis tests are of two types: *multiple choice* and *constructed response.* How is a constructed-response question different from an essay question? How are constructed-response questions graded?

33. In 2000, the National Council of Teachers of Mathematics (NCTM) published the *Principles and Standards for School Mathematics,* which presents a vision for school mathematics in the twenty-first century. One of the principles identified in this document is the Technology Principle. Find a copy of the Principles and Standards, and study the discussion of the Technology Principle [NCTM, 2000, 24–27].

   a. What are the specific recommendations of the NCTM regarding the use and integration of technology in the mathematics classroom?
   b. What opportunities does technology make possible in the mathematics classroom? What might be some benefits of learning and teaching in a technology-rich classroom?
   c. What cautions does the NCTM raise regarding the use of technology in the classroom?

34. Find the Common Core State Standards for Mathematics online: http://www.corestandards.org.

   a. What is the Common Core State Standards Initiative?
   b. What are the eight Standards for Mathematical Practice? How are the Standards for Mathematical Practice distinct from the Standards for Mathematical Content?
   c. Study the Introduction to the section on High School: Geometry. How can dynamic geometry tools, such as GeoGebra, be used to enhance student learning in the high school (or middle school, or elementary school) classroom?

Reflect on what you have learned in this chapter.

35. Review the main ideas of this chapter. Describe, in your own words, the concepts you have studied and what you have learned about them. What are the important ideas? How do they fit together? Which concepts were easy for you? Which were hard?

36. Reflect on the learning environment for this course. Describe aspects of the learning environment that helped you understand the main ideas in this chapter. Which activities did you like? Which stretched you beyond your personal comfort zone? Why?

1.4 CHAPTER OVERVIEW

One of the goals of this opening chapter was to introduce you to some of the basic constructions in GeoGebra. By now, you should be comfortable launching GeoGebra and beginning a new sketch. You should be familiar with many of the tools available; in particular, you should be comfortable using the Move, Point, Line, Circle, Text, and Angle tools. You should also have experimented with some of the options available on these tools.

There are options on the Style Bar that allow you to change the thickness, color, and style of points and lines. By a right-click on an object, you can open a menu with options to show or hide an object, to show or hide a label, and to animate a point on a line or circle. (Again, note that the commands will be different for a Mac computer.) The Algebra pane offers another method for hiding or showing objects and displays values of lengths, angles, and areas. The Input line lets you enter arithmetic expressions using numbers and/or measured quantities; this makes it possible to calculate such things as the angle sum of a polygon or the sum of the distances from a point to each side of a given triangle.

An important purpose of the activities in this chapter is to introduce you to the practice of engaging in explorations and making observations, which leads to making conjectures. Making conjectures based on your observations will lead you to develop theorems that you will learn to prove (or sometimes disprove) in the coming weeks of this course. In developing a proof, you will be basically answering such questions as "What happens in this sketch?" . . . "Can I expect it to happen this way again?" . . . "Can I explain why this is happening?" . . . .

We have introduced some mathematical language in this chapter, and we will build on this in future chapters. This mathematical language includes both the names of geometric objects (square, polygon, line segment, and so on), and language describing different kinds of mathematical statements (conjecture, statement, converse, and so on). We also have introduced the logical operators, *and* (denoted $A \wedge B$), *or* (denoted $P \vee Q$), and *not* (denoted $\neg S$), which can be helpful in reasoning about statements involving these operators.

Finally, we hope that we have piqued your curiosity about geometry. We are inviting you to join us on an intellectual journey. We hope that you will find the experience both fun and rewarding.