
Introduction to Momentum Transfer

Momentum transfer in a fluid involves the study of the motion of fluids and the forces that produce these motions. From Newton's second law of motion it is known that force is directly related to the time rate of change of momentum of a system. Excluding action-at-a-distance forces, such as gravity, the forces acting on a fluid, such as those resulting from pressure and shear stress, may be shown to be the result of microscopic (molecular) transfer of momentum. Thus, the subject under consideration, which is historically fluid mechanics, may equally be termed momentum transfer.

The history of fluid mechanics shows the skillful blending of the nineteenth- and twentieth-century analytical work in hydrodynamics with the empirical knowledge in hydraulics that man has collected over the ages. The mating of these separately developed disciplines was started by Ludwig Prandtl in 1904 with his boundary-layer theory, which was verified by experiment. Modern fluid mechanics, or momentum transfer, is both analytical and experimental.

Each area of study has its phraseology and nomenclature. Momentum transfer being typical, the basic definitions and concepts will be introduced in order to provide a basis for communication.

► 1.1

FLUIDS AND THE CONTINUUM

A fluid is defined as a substance that deforms continuously under the action of a shear stress. An important consequence of this definition is that when a fluid is at rest, there can be no shear stresses. Both liquids and gases are fluids. Some substances such as glass are technically classified as fluids. However, the rate of deformation in glass at normal temperatures is so small as to make its consideration as a fluid impractical.

Concept of a Continuum Fluids, like all matter, are composed of molecules whose numbers stagger the imagination. In a cubic inch of air at room conditions there are some 10^{20} molecules. Any theory that would predict the individual motions of these many molecules would be extremely complex, far beyond our present abilities.

Most engineering work is concerned with the macroscopic or bulk behavior of a fluid rather than with the microscopic or molecular behavior. In most cases it is convenient to think of a fluid as a continuous distribution of matter, or a *continuum*. There are, of course, certain instances in which the concept of a continuum is not valid. Consider, for example, the number

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of molecules in a small volume of a gas at rest. If the volume were taken small enough, the number of molecules per unit volume would be time-dependent for the microscopic volume even though the macroscopic volume had a constant number of molecules in it. The concept of a continuum would be valid only for the latter case. The validity of the continuum approach is seen to be dependent upon the type of information desired rather than the nature of the fluid. The treatment of fluids as continua is valid whenever the smallest fluid volume of interest contains a sufficient number of molecules to make statistical averages meaningful. The macroscopic properties of a continuum are considered to vary smoothly (continuously) from point to point in the fluid. Our immediate task is to define these properties at a point.

▶ 1.2

PROPERTIES AT A POINT

When a fluid is in motion, the quantities associated with the state and the motion of the fluid will vary from point to point. The definition of some fluid variables at a point is presented below.

Density at a Point The density of a fluid is defined as the mass per unit volume. Under flow conditions, particularly in gases, the density may vary greatly throughout the fluid. The density, ρ , at a particular point in the fluid is defined as

$$\rho = \lim_{\Delta V \rightarrow \delta V} \frac{\Delta m}{\Delta V}$$

where Δm is the mass contained in a volume ΔV , and δV is the smallest volume surrounding the point for which statistical averages are meaningful. The limit is shown in Figure 1.1.

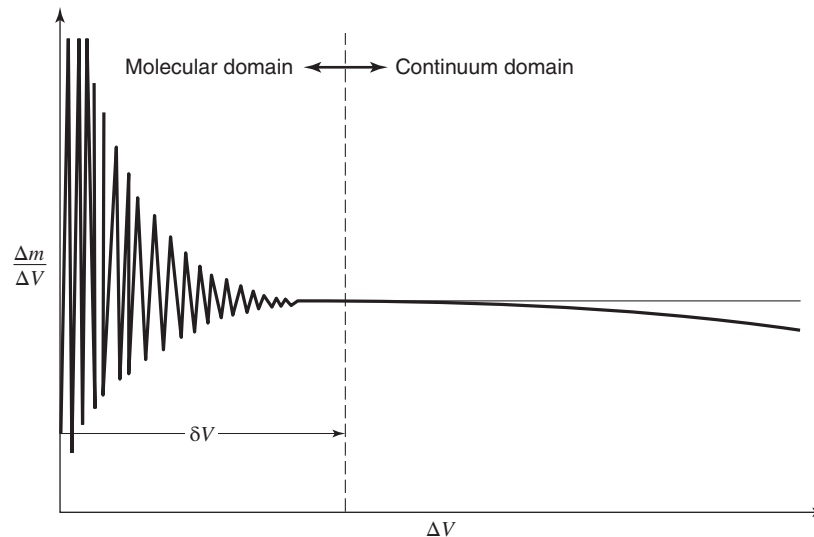


Figure 1.1 Density at a point.

The concept of the density at a mathematical point—that is, at $\Delta V = 0$, is seen to be fictitious; however, taking $\rho = \lim_{\Delta V \rightarrow \delta V} (\Delta m / \Delta V)$ is extremely useful, as it allows us to describe fluid flow in terms of continuous functions. The density, in general, may vary from point to point in a fluid and may also vary with respect to time, as in a punctured automobile tire. A table of densities of common fluids is given in Table 1.1.

Fluid Properties and Flow Properties Some fluids, particularly liquids, have densities that remain almost constant over wide ranges of pressure and temperature. Fluids that exhibit this quality are usually treated as being incompressible. The effects of compressibility, however, are more a property of the situation than of the fluid itself. For example, the flow of air at low velocities is described by the same equations that describe the flow of water. From a static viewpoint, air is a compressible fluid and water incompressible. Instead of being classified according to the fluid, compressibility effects are considered a property of the flow. A distinction, often subtle, is made between the properties of the fluid and the properties of the flow, and the student is hereby alerted to the importance of this concept.

Table 1.1 Densities of various fluids (at 20 C) unless otherwise noted

Fluid	ρ (g/cm ³)
Acetone	0.792
Ethanol	0.791
Benzene	0.899
Gasoline	0.670
Glycerin	1.260
Mercury	13.6
Sea water	1.025
Water	1.00
Soap solution	0.900
Blood	1.060 (37 C)

Source: *Handbook of Chemistry and Physics*, 62nd ed., Chemical Rubber Publishing Co., Cleveland OH, 1980.

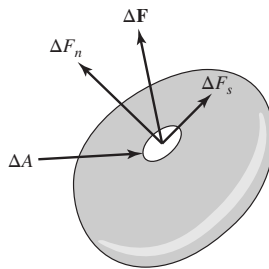


Figure 1.2 Force on an element of fluid.

Stress at a Point Consider the force $\Delta \mathbf{F}$ acting on an element ΔA of the body shown in Figure 1.2. The force $\Delta \mathbf{F}$ is resolved into components normal and parallel to the surface of the element. The force per unit area or stress at a point is defined as the limit of $\Delta \mathbf{F}/\Delta A$ as $\Delta A \rightarrow \delta A$ where δA is the smallest area for which statistical averages are meaningful:

$$\lim_{\Delta A \rightarrow \delta A} \frac{\Delta F_n}{\Delta A} = \sigma_{ii} \quad \lim_{\Delta A \rightarrow \delta A} \frac{\Delta F_s}{\Delta A} = \tau_{ij}$$

Here σ_{ii} is the normal stress and τ_{ij} the shear stress. In this text, the double-subscript stress notation as used in solid mechanics will be employed. The student will recall that normal stress is positive in tension. The limiting process for the normal stress is illustrated in Figure 1.3.

Forces acting on a fluid are divided into two general groups: body forces and surface forces. Body forces are those that act without physical contact—for example, gravity and electrostatic forces. On the contrary, pressure and frictional forces require physical contact for transmission. As a surface is required for the action of these forces, they are called surface forces. Stress is therefore a surface force per unit area.¹

Pressure at a point in a Static Fluid For a static fluid, the normal stress at a point may be determined from the application of Newton's laws to a fluid element as the fluid element

¹ Mathematically, stress is classed as a tensor of second order, as it requires magnitude, direction, and orientation with respect to a plane for its determination.

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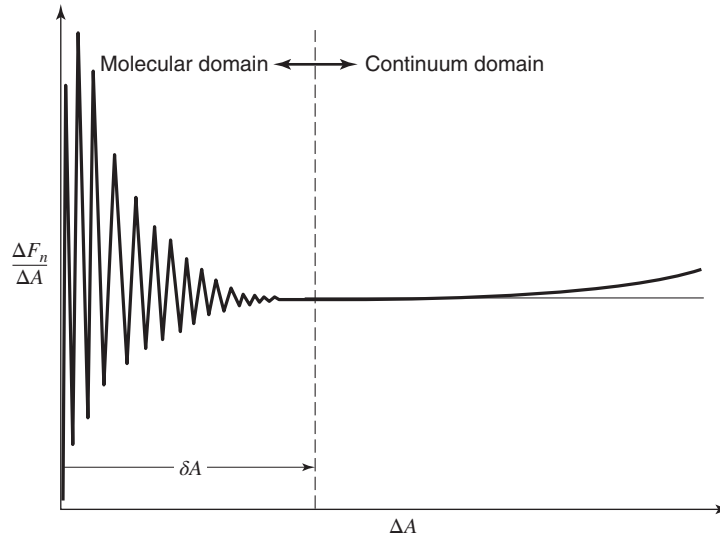


Figure 1.3 Normal stress at a point.

approaches zero size. It may be recalled that there can be no shearing stress in a static fluid. Thus, the only surface forces present will be those due to normal stresses. Consider the element shown in Figure 1.4. This element, while at rest, is acted upon by gravity and normal stresses. The weight of the fluid element is $\rho g (\Delta x \Delta y \Delta z / 2)$.

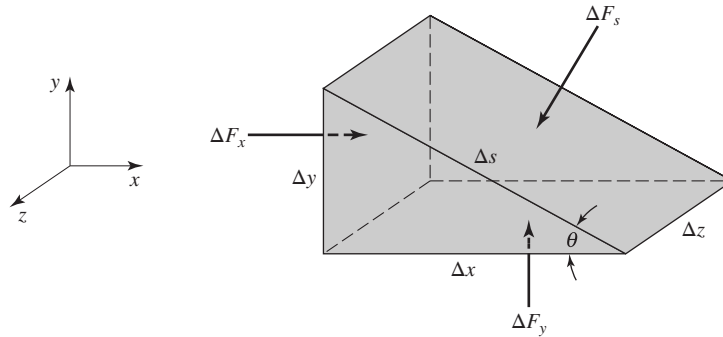


Figure 1.4 Element in a static fluid.

For a body at rest, $\Sigma \mathbf{F} = 0$. In the x direction,

$$\Delta F_x - \Delta F_s \sin \theta = 0$$

Since $\sin \theta = \Delta y / \Delta s$, the above equation becomes

$$\Delta F_x - \Delta F_s \frac{\Delta y}{\Delta s} = 0$$

Dividing through by $\Delta y \Delta z$ and taking the limit as the volume of the element approaches zero, we obtain

$$\lim_{\Delta V \rightarrow 0} \left[\frac{\Delta F_x}{\Delta y \Delta z} - \frac{\Delta F_s}{\Delta s \Delta z} \right] = 0$$

Recalling that normal stress is positive in tension, we obtain, by evaluating the above equation,

$$\sigma_{xx} = \sigma_{ss} \quad (1-1)$$

In the y direction, applying $\Sigma \mathbf{F} = 0$ yields

$$\Delta F_y - \Delta F_s \cos \theta - \rho g \frac{\Delta x \Delta y \Delta z}{2} = 0$$

Since $\cos \theta = \Delta x / \Delta s$, one has

$$\Delta F_y - \Delta F_s \frac{\Delta x}{\Delta s} - \rho g \frac{\Delta x \Delta y \Delta z}{2} = 0$$

Dividing through by $\Delta x \Delta z$ and taking the limit as before, we obtain

$$\lim_{\Delta V \rightarrow 0} \left[\frac{\Delta F_y}{\Delta x \Delta z} - \frac{\Delta F_s}{\Delta s \Delta z} - \frac{\rho g \Delta y}{2} \right] = 0$$

which becomes

$$-\sigma_{yy} + \sigma_{ss} - \frac{\rho g}{2}(0) = 0$$

or

$$\sigma_{yy} = \sigma_{ss} \quad (1-2)$$

It may be noted that the angle θ does not appear in equation (1-1) or (1-2); thus, the normal stress at a point in a static fluid is independent of direction, and is therefore a scalar quantity.

As the element is at rest, the only surface forces acting are those due to the normal stress. If we were to measure the force per unit area acting on a submerged element, we would observe that it acts inward or places the element in compression. The quantity measured is, of course, pressure, which, in light of the preceding development, must be the negative of the normal stress. This important simplification—the reduction of stress, a tensor, to pressure, a scalar—may also be shown for the case of zero shear stress in a flowing fluid. When shearing stresses are present, the normal stress components at a point may not be equal; however, the pressure is still equal to the average normal stress—that is,

$$P = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

with very few exceptions, one being flow in shock waves.

Now that certain properties at a point have been discussed, let us investigate the manner in which fluid properties vary from point to point.

▶ 1.3

POINT-TO-POINT VARIATION OF PROPERTIES IN A FLUID

In the continuum approach to momentum transfer, use will be made of pressure, temperature, density, velocity, and stress fields. In previous studies, the concept of a gravitational field has been introduced. Gravity, of course, is a vector, and thus a gravitational field is a vector field. In this book, vectors will be written in boldfaced type. Weather maps illustrating the pressure variation over this country are published in newspapers and on

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websites. As pressure is a scalar quantity, such maps are an illustration of a scalar field. Scalars in this book will be set in regular type.

In Figure 1.5, the lines drawn are the loci of points of equal pressure. The pressure varies continuously throughout the region, and one may observe the pressure levels and infer the manner in which the pressure varies by examining such a map.

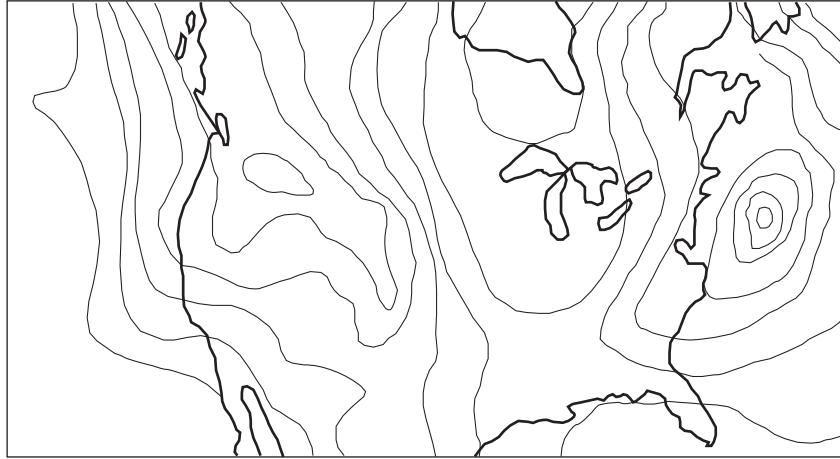


Figure 1.5 Weather map—an example of a scalar field.

Of specific interest in momentum transfer is the description of the point-to-point variation in the pressure. Denoting the directions east and north in Figure 1.5 by x and y , respectively, we may represent the pressure throughout the region by the general function $P(x,y)$.

The change in P , written as dP , between two points in the region separated by the distances dx and dy is given by the total differential

$$dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy \quad (1-3)$$

In equation (1-3), the partial derivatives represent the manner in which P changes along the x and y axes, respectively.

Along an arbitrary path s in the xy plane the total derivative is

$$\frac{dP}{ds} = \frac{\partial P}{\partial x} \frac{dx}{ds} + \frac{\partial P}{\partial y} \frac{dy}{ds} \quad (1-4)$$

In equation (1-4), the term dP/ds is the directional derivative, and its functional relation describes the rate of change of P in the s direction.

A small portion of the pressure field depicted in Figure 1.5 is shown in Figure 1.6. The arbitrary path s is shown, and it is easily seen that the terms dx/ds and dy/ds are the cosine and sine of the path angle, α , with respect to the x axis. The directional derivative, therefore, may be written as

$$\frac{dP}{ds} = \frac{\partial P}{\partial x} \cos \alpha + \frac{\partial P}{\partial y} \sin \alpha \quad (1-5)$$

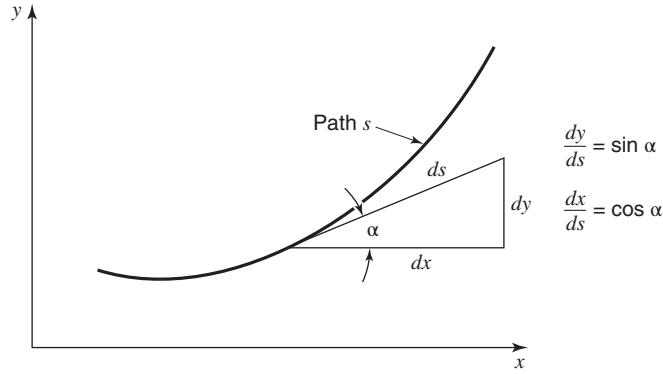


Figure 1.6 Path s in the xy plane.

There are an infinite number of paths to choose from in the xy plane; however, two particular paths are of special interest: the path for which dP/ds is zero and that for which dP/ds is maximum.

The path for which the directional derivative is zero is quite simple to find. Setting dP/ds equal to zero, we have

$$\frac{\sin \alpha}{\cos \alpha} \Big|_{dP/ds=0} = \tan \alpha \Big|_{dP/ds=0} = -\frac{\partial P/\partial x}{\partial P/\partial y}$$

or, since $\tan \alpha = dy/dx$, we have

$$\frac{dy}{dx} \Big|_{dP/ds=0} = -\frac{\partial P/\partial x}{\partial P/\partial y} \quad (1-6)$$

Along the path whose slope is defined by equation (1-6), we have $dP=0$, and thus P is constant. Paths along which a scalar is constant are called *isolines*.

In order to find the direction for which dP/ds is a maximum, we must have the derivative $(d/d\alpha)(dP/ds)$ equal to zero, or

$$\frac{d}{d\alpha} \frac{dP}{ds} = -\sin \alpha \frac{\partial P}{\partial x} + \cos \alpha \frac{\partial P}{\partial y} = 0$$

or

$$\tan \alpha \Big|_{dP/ds \text{ is max}} = \frac{\partial P/\partial y}{\partial P/\partial x} \quad (1-7)$$

Comparing equations (1-6) and (1-7), we see that the two directions defined by these equations are perpendicular. The magnitude of the directional derivative when the directional derivative is maximum is

$$\frac{dP}{ds} \Big|_{\max} = \frac{\partial P}{\partial x} \cos \alpha + \frac{\partial P}{\partial y} \sin \alpha$$

where $\cos \alpha$ and $\sin \alpha$ are evaluated along the path given by equation (1-7). As the cosine is related to the tangent by

$$\cos \alpha = \frac{1}{\sqrt{1 + \tan^2 \alpha}}$$

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we have

$$\cos \alpha \Big|_{dP/ds \text{ is max}} = \frac{\partial P / \partial x}{\sqrt{(\partial P / \partial x)^2 + (\partial P / \partial y)^2}}$$

Evaluating $\sin \alpha$ in a similar manner gives

$$\frac{dP}{ds} \Big|_{\text{max}} = \frac{(\partial P / \partial x)^2 + (\partial P / \partial y)^2}{\sqrt{(\partial P / \partial x)^2 + (\partial P / \partial y)^2}} = \sqrt{\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2} \quad (1-8)$$

Equations (1-7) and (1-8) suggest that the maximum directional derivative is a vector of the form

$$\frac{\partial P}{\partial x} \mathbf{e}_x + \frac{\partial P}{\partial y} \mathbf{e}_y$$

where \mathbf{e}_x and \mathbf{e}_y are unit vectors in the x and y directions, respectively.

The directional derivative along the path of maximum value is frequently encountered in the analysis of transfer processes and is given a special name, the *gradient*. Thus, the gradient of P , $\text{grad } P$, is

$$\text{grad } P = \frac{\partial P}{\partial x} \mathbf{e}_x + \frac{\partial P}{\partial y} \mathbf{e}_y$$

where $P = P(x, y)$. This concept can be extended to cases in which $P = P(x, y, z)$. For this more general case,

$$\text{grad } P = \frac{\partial P}{\partial x} \mathbf{e}_x + \frac{\partial P}{\partial y} \mathbf{e}_y + \frac{\partial P}{\partial z} \mathbf{e}_z \quad (1-9)$$

Equation (1-9) may be written in more compact form by use of the operation ∇ (pronounced “del”), giving

$$\nabla P = \frac{\partial P}{\partial x} \mathbf{e}_x + \frac{\partial P}{\partial y} \mathbf{e}_y + \frac{\partial P}{\partial z} \mathbf{e}_z$$

where

$$\nabla = \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y + \frac{\partial}{\partial z} \mathbf{e}_z \quad (1-10)$$

Equation (1-10) is the defining relationship for the ∇ operator in Cartesian coordinates. This symbol indicates that differentiation is to be performed in a prescribed manner. In other coordinate systems, such as cylindrical and spherical coordinates, the gradient takes on a different form.² However, the geometric meaning of the gradient remains the same; it is a vector having the direction and magnitude of the maximum rate of change of the dependent variable with respect to distance.

► 1.4

UNITS

In addition to the International Standard (SI) system of units, there are two different English systems of units commonly used in engineering. These systems have their roots in Newton’s second law of motion: force is equal to the time rate of change of momentum. In defining

²The forms of the gradient operator in rectangular, cylindrical, and spherical coordinate systems are listed in Appendix B.

each term of this law, a direct relationship has been established between the four basic physical quantities used in mechanics: force, mass, length, and time. Through the arbitrary choice of fundamental dimensions, some confusion has occurred in the use of the English systems of units. Using the SI system of units has greatly reduced these difficulties.

The relationship between force and mass may be expressed by the following statement of Newton's second law of motion:

$$\mathbf{F} = \frac{m\mathbf{a}}{g_c}$$

where g_c is a conversion factor that is included to make the equation dimensionally consistent.

In the SI system, mass, length, and time are taken as basic units. The basic units are mass in kilograms (kg), length in meters (m), and time in seconds (s). The corresponding unit of force is the newton (N). One newton is the force required to accelerate a mass of one kilogram at a rate of one meter per second per second (1 m/s^2). The conversion factor, g_c , is then equal to one kilogram meter per newton per second per second ($1 \text{ kg} \cdot \text{m/N} \cdot \text{s}^2$).

In engineering practice, force, length, and time have been frequently chosen as defining fundamental units. With this system, force is expressed in pounds force (lb_f), length in feet, and time in seconds. The corresponding unit of mass will be that which will be accelerated at the rate of $1 \text{ ft}/(\text{s})^2$ by 1 lb_f .

This unit of mass having the dimensions of $(\text{lb}_f)(\text{s})^2/(\text{ft})$ is called the *slug*. The conversion factor, g_c , is then a multiplying factor to convert slugs into $(\text{lb}_f)(\text{s})^2/(\text{ft})$, and its value is $1 \text{ (slug)(ft)/(lb}_f)(\text{s})^2$.

A third system encountered in engineering practice involves all four fundamental units. The unit of force is 1 lb_f , the unit of mass is 1 lb_m ; length and time are given in units of feet and seconds, respectively. When 1 lb_m at sea level is allowed to fall under the influence of gravity, its acceleration will be $32.174 \text{ (ft)/(s)}^2$. The force exerted by gravity on 1 lb_m at sea level is defined as 1 lb_f . Therefore, the conversion factor, g_c , for this system is $32.174 \text{ (lb}_m)(\text{ft)/(lb}_f)(\text{s})^2$.³

A summary of the values of g_c is given in Table 1.2 for these three English systems of engineering units, along with the units of length, time, force, and mass.

Table 1.2

System	Length	Time	Force	Mass	g_c
1	meter	second	newton	kilogram	$1 \frac{\text{kg} \cdot \text{m}}{\text{N} \cdot \text{s}^2}$
2	foot	second	lb_f	slug	$\frac{1 \text{ (slug)(ft)}}{(\text{lb}_f)(\text{s})^2}$
3	foot	second	lb_f	lb_m	$\frac{32.174 \text{ (lb}_m)(\text{ft})}{(\text{lb}_f)(\text{s})^2}$

As all three systems are in current use in the technical literature, the student should be able to use formulas given in any particular situation. Careful checking for dimensional consistency will be required in *all* calculations. The conversion factor, g_c , will correctly relate the units corresponding to a system. There will be no attempt by the authors to

³ In subsequent calculations in this book, g_c will be rounded off to a value of $32.2 \text{ lb}_m\text{ft}/\text{lb}_f\text{s}^2$.

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incorporate the conversion factor in any equations; instead, it will be the reader's responsibility to use units that are consistent with every term in the equation.

► 1.5

COMPRESSIBILITY

A fluid is considered *compressible* or *incompressible* depending on whether its density is variable or constant. Liquids are generally considered to be incompressible, whereas gases are certainly compressible.

The *bulk modulus of elasticity*, often referred to as simply the *bulk modulus*, is a fluid property that characterizes compressibility. It is defined according to

$$\beta \equiv \frac{dP}{dV/V} \quad (1-11a)$$

or as

$$\beta \equiv -\frac{dP}{d\rho/\rho} \quad (1-11b)$$

and has the dimensions N/m^2 .

Disturbances introduced at some location in a fluid continuum will be propagated at a finite velocity. The velocity is designated the *acoustic velocity*—that is, the speed of sound in the fluid is symbolized C .

It can be shown that the acoustic velocity is related to changes in pressure and density according to

$$C = \left(\frac{dP}{d\rho}\right)^{1/2} \quad (1-12)$$

Introducing equation (1-11b) into this relationship yields

$$C = \left(-\frac{\beta}{\rho}\right)^{1/2} \quad (1-13)$$

For a gas, undergoing an isentropic process where $PV^k = C$, a constant, we have

$$C = \left(\frac{kP}{\rho}\right)^{1/2}, \quad k = C_p/C_v \quad (1-14)$$

or

$$C = (kRT)^{1/2} \quad (1-15)$$

The question arises concerning when a gas, which is compressible, may be treated in a flow situation as incompressible—that is, when density variations are negligibly small. A common criterion for such a consideration involves the *Mach number*. The Mach number, a dimensionless parameter, is defined as the ratio of the fluid velocity, v , to the speed of sound, C , in the fluid:

$$M = \frac{v}{C} \quad (1-16)$$

A general rule of thumb is that when $M < 0.2$, the flow may be treated as incompressible with negligible error.

Example 1

A jet aircraft is flying at an altitude of 15,500 m, where the air temperature is 239 K. Determine whether compressibility effects are significant at airspeeds of (a) 220 km/h and (b) 650 km/h.

The test for compressibility effects requires calculating the Mach number, M , which, in turn, requires that the acoustic velocity at each airspeed be evaluated.

For air, $k = 1.4$, $R = 0.287 \text{ kJ/kg} \cdot \text{K}$, and

$$\begin{aligned} C &= (kRT)^{1/2} \\ &= [1.4(0.287 \text{ kJ/kg} \cdot \text{K})(239 \text{ K})(1000 \text{ N} \cdot \text{m/kJ})(\text{kg} \cdot \text{m/N} \cdot \text{s}^2)]^{1/2} \\ &= 310 \text{ m/s} \end{aligned}$$

(a) At $v = 220 \text{ km/hr}$ (61.1 m/s),

$$M = \frac{v}{C} = \frac{61.1 \text{ m/s}}{310 \text{ m/s}} = 0.197$$

The flow may be treated as incompressible.

(b) At $v = 650 \text{ km/hr}$ (180.5 m/s),

$$M = \frac{v}{C} = \frac{180.5 \text{ m/s}}{310 \text{ m/s}} = 0.582$$

Compressible effects must be accounted for.

▶ 1.6

SURFACE TENSION

The situation where a small amount of unconfined liquid forms a spherical drop is familiar to most of us. The phenomenon is the consequence of the attraction that exists between liquid molecules. Within a drop a molecule of liquid is completely surrounded by many others. Particles near the surface, on the contrary, will experience an imbalance of net force because of the nonuniformity in the numbers of adjacent particles. The extreme condition is the density discontinuity at the surface. Particles at the surface experience a relatively strong inwardly directed attractive force.

Given this behavior, it is evident that some work must be done when a liquid particle moves toward the surface. As more fluid is added, the drop will expand, creating additional surface. The work associated with creating this new surface is the *surface tension*, symbolized, σ . Quantitatively, σ is the work per unit area, $\text{N} \cdot \text{m/m}^2$ or force per unit length of interface in N/m .

A surface is, in reality, an interface between two phases. Thus, both phases will have the property of surface tension. The most common materials involving phase interfaces are water and air, but many others are also possible. For a given interfacial composition, the surface tension property is a function of both pressure and temperature, but a much stronger function of temperature. Table 1.3 lists values of σ for several fluids in air at 1 atm and 20 C. For water in air, the surface tension is expressed as a function of temperature according to

$$\sigma = 0.123(1 - 0.00139 T)\text{N/m} \quad (1-17)$$

where T is in Kelvin.

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Table 1.3 Surface tension of some fluids in air at 1 atm and 20 C

Fluid	σ (N/m)
Ammonia	0.021
Ethyl alcohol	0.028
Gasoline	0.022
Glycerin	0.063
Kerosene	0.028
Mercury	0.440
Soap solution	0.025
SAE 30 oil	0.035

Source: *Handbook of Chemistry and Physics*, 62nd ed., Chemical Rubber Publishing Co., Cleveland, OH, 1980.

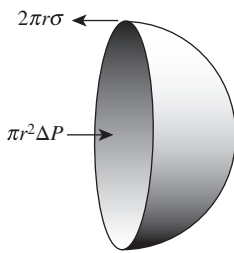


Figure 1.7 A free body diagram of a hemispherical liquid droplet.

In Figure 1.7, we show a free body diagram of a hemispherical drop of liquid with the pressure and surface tension forces in balance. The condition examined is typically used for this analysis as a sphere represents the minimum surface area for a prescribed volume. The pressure difference, ΔP , between the inside and outside of the hemisphere produces a net pressure force that is balanced by the surface tension force. This force balance can be expressed as

$$\pi r^2 \Delta P = 2\pi r \sigma$$

and the pressure difference is given by

$$\Delta P = \frac{2\sigma}{r} \quad (1-18)$$

The Young–Laplace equation is a general condition for the equilibrium of normal stresses across a static interface separating a pair of immiscible fluids,

$$\Delta P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

where R_1 and R_2 are the radii of curvature at the surface of the body. This equation states that the surface tension causes an increased pressure on the inside of a surface whose magnitude depends on the radii of curvature of the surface. In most systems, $R_1 = R_2$.

For the case of a soap bubble, which has a very thin wall, there are two interfaces, and the pressure difference will be

$$\Delta P = \frac{4\sigma}{r} \quad (1-19)$$

Equations (1-18) and (1-19) indicate that the pressure difference is inversely proportional to r . The limit of this relationship is the case of a fully wetted surface where $r \cong \infty$, and the pressure difference due to surface tension is zero.

A consequence of the pressure difference resulting from surface tension is the phenomenon of *capillary action*. This effect is related to how well a liquid *wets* a solid boundary. The indicator for wetting or nonwetting is the *contact angle*, θ , defined as illustrated in Figure 1.8. With θ measured through the liquid, a nonwetting case, as shown in the figure, is associated with $\theta > 90^\circ$. For a wetting case $\theta < 90^\circ$. For mercury in contact

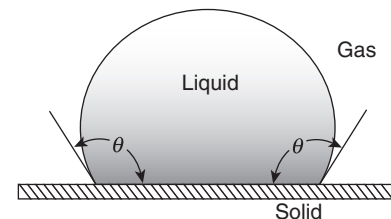


Figure 1.8 Contact angle for a non-wetting gas–liquid–solid interface.

with a clean glass tube, $\theta \cong 130^\circ$. Water in contact with a clean glass surface will completely wet the surface, and, for this case, $\theta \cong 0$.

Illustrated in Figure 1.9 is the case of a small glass tube inserted into a pool of (a) water and (b) mercury. Note that water will rise in the tube and that in mercury the level in the tube is depressed.

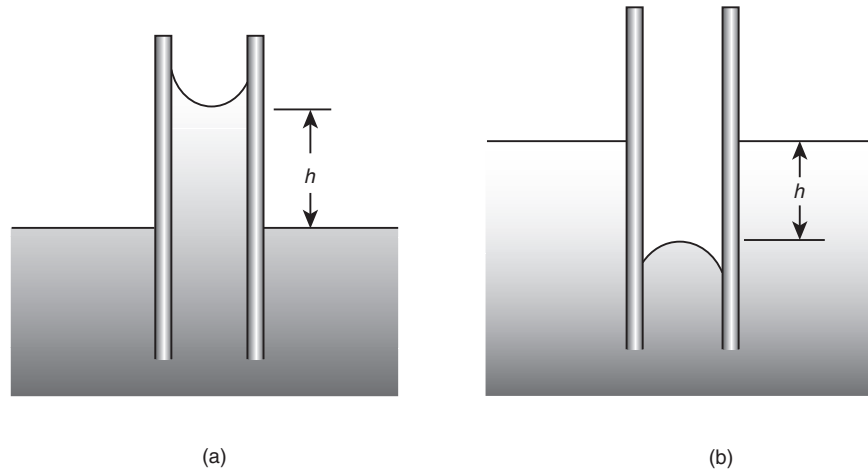


Figure 1.9 Capillary effects with a tube inserted in (a) water and (b) mercury.

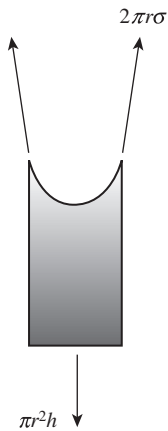


Figure 1.10 Free body diagram of a wetting liquid in a tube.

For the water case, the liquid rises a distance h above the level in the pool. This is the result of attraction between the liquid molecules and the tube wall being greater than the attraction between water molecules at the liquid surface. For the mercury case, the intermolecular forces at the liquid surface are greater than the attractive forces between liquid mercury and the glass surface. The mercury is depressed a distance h below the level of the pool.

A free body diagram of the wetting liquid is shown in Figure 1.10. The upward force, due to surface tension,

$$2\pi r \sigma \cos \theta$$

will be equal to the downward force due to the weight of liquid having volume $V = \pi r^2 h$.

Equating these forces, we obtain

$$2\pi r \sigma \cos \theta = \rho g \pi r^2 h$$

and the value of h becomes

$$h = \frac{2\sigma \cos \theta}{\rho g r} \tag{1-20}$$

Example 2

Determine the distance h that mercury will be depressed with a 4-mm-diameter glass tube inserted into a pool of mercury at 20 C (Figure 1.11).

Equation (1-20) applies, so we have

$$h = \frac{2\sigma \cos \theta}{\rho g r}$$

Recall that, for mercury and glass, $\theta = 130^\circ$.

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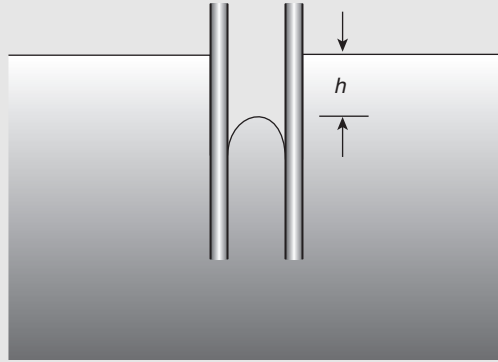


Figure 1.11 Capillary depression of mercury in a glass tube.

For mercury at 20 C, $\rho = 13,580 \text{ kg/m}^3$, and for mercury in air $\sigma = 0.44 \text{ N/m}$ (Table 1.3), giving

$$\begin{aligned}
 h &= \frac{2(0.44 \text{ N/m})(\cos 130^\circ)}{(13,580 \text{ kg/m}^3)(9.81 \text{ m/s}^2)(2 \times 10^{-3} \text{ m})} \\
 &= 2.12 \times 10^{-3} \text{ m} \quad (2.12 \text{ mm})
 \end{aligned}$$