1

Electromagnetics Behind Shielding

Shielding an electromagnetic field is a complex and sometimes formidable task. The reasons are many, since the effectiveness of any strategy or technique aimed at the reduction of the electromagnetic field levels in a prescribed region depends largely upon the source characteristics, the shield geometry, and the involved materials. Moreover, as it often happens when common terms are adopted in a technical context, different definitions of shielding exist. In electromagnetics the *shielding effectiveness* (SE) is a concise parameter generally applied to quantify shielding performance. However, a variety of standards are adopted for the measurement or the assessment of the performance of a given shielding structure. Unfortunately, they all call for very specific conditions in the measurement setup. The results therefore are often useless if the source or system configurations differ even slightly. Last among the difficulties that arise in the solution of actual shielding problems are the difficulties inherent in both the solution of the boundary value problem and the description of the electromagnetic problem in mathematical form.

1.1 Definitions

To establish a common ground, we will begin with some useful definitions. An electromagnetic shield can be defined as [1]:

> [A] housing, screen, or other object, usually conducting, that substantially reduces the effect of electric or magnetic fields on one side thereof, upon devices or circuits on the other side.

This definition is restrictive because it implicitly assumes the presence of a "victim." The definition is also based on the misconception that the source and observation points are in opposite positions with respect to the shield, and it

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includes the word "substantially" whose meaning is obscure and introduces an unacceptable level of arbitrariness.

Another definition of electromagnetic shielding even more restrictive is [2]:

> [A] means of preventing two circuits from electromagnetic coupling by placing at least one of the circuits in a grounded enclosure of magnetic conductive material.

The most appropriate definition entails a broad view of the phenomenon:

> [A]ny means used for the reduction of the electromagnetic field in a prescribed region.

Notice that no reference to shape, material, and grounding of the shield is necessary to define its purpose.

In general, electromagnetic shielding represents a way toward the improvement of the electromagnetic-compatibility (EMC) (defined as the capability of electronic equipment or systems to be operated in the intended electromagnetic environment at design levels of efficiency) performance of single devices, apparatus, or systems. Biological systems are included, for which it is correct to talk about health rather than EMC. Electromagnetic shielding is also used to prevent sensitive information from being intercepted, that is, to guarantee communication security.

Electromagnetic shielding is not implemented only for such purposes. Some sort of electromagnetic shielding is almost always used in electrical and electronic systems to reduce their electromagnetic emissions and to increase their electromagnetic immunity against external fields. In cases where the available methodologies for reducing the source levels of electromagnetic emission or strengthening the victim immunity are not available or are not sufficient to ensure the correct operation of devices or systems, a reduction of the coupling between the source and the victim (either present or only potentially present) is often the preferred choice.

The immunity of the victims is generally obtained by means of filters that are analogous to electromagnetic shielding with respect to conducted emissions and immunity. The main advantage of filters is that they are "local" devices. Thus, where the number of sensitive components to be protected is limited, the cost of filtering may be much lower than that of shielding. The main disadvantage of using a filter is that it is able to arrest only interferences whose characteristics (e.g., level or mode of transmission) are different from those of the device, so the correct operation in the presence of some types of interference is not guaranteed. Another serious disadvantage of the filter is its inadequacy or its low efficiency for the prevention of data detection.

In general, designing a filter is much simpler than designing a shield. The filter designer has only to consider the waveform of the interference (in terms of voltage or current) and the values of the input and output impedance [3], whereas the shield designer must include a large amount of input information and constraints, as it will be discussed throughout the book.

Any shielding analysis begins by an accurate examination of the shield geometry [4–7]. Although the identification of the coupling paths between the main space regions is often trivial, sometimes it deserves more care, especially in complex configurations. The complexity of a shield is associated with its shape, apertures, the components identified as the most susceptible, the source characteristics, and so forth. Subdividing its configuration into several subsystems (each simpler than the original one and interacting with the others in a definite way) is always a useful approach to identify critical problems and find ways to fix and improve the overall performance [5]. This approach is based on the assumption that each subsystem can be analyzed, and hence its behavior can be characterized, independently of the others components/subsystems. For instance, in the frequency domain and for a linear subsystem, for each coupling path and for each susceptible element, it is possible to investigate the transfer function $T(\omega)$ relating the external source input $S(\omega)$ and the victim output $V(\omega)$ characteristics as $V(\omega) = U(\omega) + T(\omega)\,S(\omega)$, where $U(\omega)$ represents the subsystem output in the absence of external-source excitation. In the presence of multilevel barriers, the transfer function $T(\omega)$ may ensue from the product of the transfer functions associated with each barrier level.

The foregoing approach can be generalized for a better understanding of the shielding problem in complex configurations. However, it is often sufficient to consider only the most critical subsystems and components, on one hand, and the most important coupling paths, on the other hand, in order to solve the principal shielding problems and thus improve the overall performance [8]. The general approach is obviously suitable in a design context. A complete analysis of the relations between shielding and grounding is left to the specific literature [4, 9–11].

1.2 Notation, Symbology, and Acronyms

The abbreviations and symbols used throughout the book are briefly summarized here in order to make clear the standard we have chosen to adopt. Of course, we will warn the reader anytime an exception occurs.

Scalar quantities are shown in italic type (e.g., V and t), while vectors are shown in boldface (e.g., $\mathbf{e}$ and $\mathbf{H}$); dyadics are shown in boldface with an underbar (e.g., $\underline{\boldsymbol{\varepsilon}}$ and $\underline{\mathbf{G}}$). A physical quantity that depends on time and space variables is indicated with a lowercase letter (e.g., $\mathbf{e}(\mathbf{r}, t)$ for the electric field). The Fourier transform with respect to the time variable is indicated with the corresponding uppercase letter (e.g., $\mathbf{E}(\mathbf{r}, \omega)$ while the Fourier transform with respect to the spatial variables is indicated by a tilde (e.g., $\tilde{\mathbf{e}}(\mathbf{k}, t)$); when the Fourier transform with respect to both time and spatial variables is considered, the two symbologies are combined (e.g., $\tilde{\mathbf{E}}(\mathbf{k}, \omega)$).

The sets of spatial variables in rectangular, cylindrical, and spherical coordinates are denoted by (x, y, z), (ρ, ϕ, z), and (r, ϕ, θ), respectively. The boldface Latin letter $\mathbf{u}$ is used to indicate a unit vector and a subscript is used to indicate its direction: for instance, $\left(\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z\right)$, $\left(\mathbf{u}_\rho, \mathbf{u}_\phi, \mathbf{u}_z\right)$, and $\left(\mathbf{u}_r, \mathbf{u}_\phi, \mathbf{u}_\theta\right)$ denote the unit vectors in the rectangular, cylindrical, and spherical coordinate system, respectively.

We will use the "del" notation ∇ with the suitable product type to indicate gradient ($\nabla[\cdot]$), curl ($\nabla \times [\cdot]$) and divergence operators ($\nabla \cdot [\cdot]$); the Laplacian operator is indicated as $\nabla^2[\cdot]$. The imaginary unit is denoted with $\mathrm{j} = \sqrt{-1}$ and the asterisk * as a superscript of a complex quantity denotes its complex conjugate. The real and imaginary parts of a complex quantity are indicated by $\mathrm{Re}[\cdot]$ and $\mathrm{Im}[\cdot]$, respectively, while the principal argument is indicated by the function $\mathrm{Arg}[\cdot]$. The base-10 logarithm and the natural logarithm are indicated by means of the $\log(\cdot)$ and $\ln(\cdot)$ functions, respectively.

Finally, throughout the book, the international system of units SI is adopted, electromagnetic is abbreviated as EM, and shielding effectiveness as SE.

1.3 Macroscopic Electromagnetism and Maxwell's Equations

A complete description of the macroscopic electromagnetism is provided by Maxwell's equations whose validity is taken as a postulate. Maxwell's equations can be used either in a differential (local) form or in an integral (global) form, and there has been a long debate over which is the best representation (e.g., David Hilbert preferred the integral form whereas Arnold Sommerfeld found more suitable the differential form, from which special relativity follows more naturally [12]). When stationary media are considered, the main difference between the two representations consists in how they account for discontinuities of materials and/or sources. Basically, if one adopts the differential form, some boundary conditions at surface discontinuities must be postulated; on the other hand, if the integral forms are chosen, one must postulate their validity across such discontinuities [13, 14].

Maxwell's equations can be expressed in scalar, vector, or tensor form, and different vector fields can be considered as fundamental. A full description of all these details can be found, e.g., in [12]. In this book we assume the following differential form of the Maxwell equations:

$$\begin{aligned}
\nabla \times \mathbf{e}(\mathbf{r}, t) &= -\frac{\partial}{\partial t} \mathbf{b}(\mathbf{r}, t), \\
\nabla \times \mathbf{h}(\mathbf{r}, t) &= \mathbf{j}(\mathbf{r}, t) + \frac{\partial}{\partial t} \mathbf{d}(\mathbf{r}, t), \\
\nabla \cdot \mathbf{d}(\mathbf{r}, t) &= \rho_{\mathrm{e}}(\mathbf{r}, t), \\
\nabla \cdot \mathbf{b}(\mathbf{r}, t) &= 0 .
\end{aligned} \tag{1.1}$$

From these equations the continuity equation can be derived as

$$\nabla \cdot \mathbf{j}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \rho_{\mathrm{e}}(\mathbf{r}, t). \tag{1.2}$$

In this framework the EM field—described by vectors **e** (electric field, unit of measure V/m), **h** (magnetic field, unit of measure A/m), **d** (electric displacement, unit of measure C/m^2), and **b** (magnetic induction, unit of measure Wb/m^2 or T)—arises from sources **j** (electric current density, unit of measure A/m^2) and ρ_e (electric charge density, unit of measure C/m^3). Further, except for static fields, if a time can be found before which all the fields and sources are identically zero, the divergence equations in (1.1) are a consequence of the curl equations [12], so under this assumption the curl equations can be taken as independent.

It can be useful to make the Maxwell equations symmetric by introducing fictitious magnetic current and charge densities **m** and ρ_m (units of measure V/m^2 and Wb/m^3, respectively), which satisfy a continuity equation similar to (1.2) so that (1.1) can be rewritten as

$$\begin{aligned}
\nabla \times \mathbf{e}(\mathbf{r}, t) &= -\frac{\partial}{\partial t} \mathbf{b}(\mathbf{r}, t) - \mathbf{m}(\mathbf{r}, t), \\
\nabla \times \mathbf{h}(\mathbf{r}, t) &= \mathbf{j}(\mathbf{r}, t) + \frac{\partial}{\partial t} \mathbf{d}(\mathbf{r}, t), \\
\nabla \cdot \mathbf{d}(\mathbf{r}, t) &= \rho_{\mathrm{e}}(\mathbf{r}, t), \\
\nabla \cdot \mathbf{b}(\mathbf{r}, t) &= \rho_{\mathrm{m}}(\mathbf{r}, t).
\end{aligned} \tag{1.3}$$

As it will be shown later, the equivalence principle indeed requires the introduction of such fictitious quantities.

It is also useful to identify in Maxwell's equations some "impressed" source terms, which are independent of the unknown fields and are instead due to other external sources (magnetic sources can be only of this type). Such "impressed" sources are considered as known terms in Maxwell's differential equations and indicated by the subscript "i." In this connection, (1.3) can be expressed as

$$\nabla \times \mathbf{e}(\mathbf{r}, t) = -\frac{\partial}{\partial t} \mathbf{b}(\mathbf{r}, t) - \mathbf{m}_{\mathrm{i}}(\mathbf{r}, t) \ , \tag{1.4a}$$

$$\nabla \times \mathbf{h}(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) + \frac{\partial}{\partial t} \mathbf{d}(\mathbf{r}, t) + \mathbf{j}_{\mathrm{i}}(\mathbf{r}, t) \ , \tag{1.4b}$$

$$\nabla \cdot \mathbf{d}(\mathbf{r}, t) = \rho_{\mathrm{e}}(\mathbf{r}, t) + \rho_{\mathrm{ei}}(\mathbf{r}, t) \ , \tag{1.4c}$$

$$\nabla \cdot \mathbf{b}(\mathbf{r}, t) = \rho_{\mathrm{mi}}(\mathbf{r}, t) \ . \tag{1.4d}$$

The impressed-source concept is well known in circuit theory. For example, independent voltage sources are voltage excitations that are independent of possible loads.

Although both the sources and the fields cannot have true spatial discontinuities, from a modeling point of view, it is useful to consider additionally sources in one or two dimensions: surface- and line-source densities can be introduced in terms of the Dirac delta distribution δ, as (singular) idealizations of actual continuous volume densities [12, 15].

Finally, in the frequency domain, Maxwell's curl equations are expressed as

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r},\omega) &= -\mathrm{j}\omega \mathbf{B}(\mathbf{r},\omega) - \mathbf{M}_\mathrm{i}(\mathbf{r},\omega) \ , \\ \nabla \times \mathbf{H}(\mathbf{r},\omega) &= \mathbf{J}(\mathbf{r},\omega) + \mathrm{j}\omega \mathbf{D}(\mathbf{r},\omega) + \mathbf{J}_\mathrm{i}(\mathbf{r},\omega) \ , \end{aligned} \tag{1.5}$$

where the uppercase quantities indicate either the Fourier transform or the phasors associated with the corresponding time-domain fields. Note that in this text the following definition of temporal Fourier transform will be adopted:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\ \mathrm{e}^{-\mathrm{j}\omega t}\ \mathrm{d}t, \tag{1.6}$$

with the corresponding inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\ \mathrm{e}^{+\mathrm{j}\omega t}\ \mathrm{d}\omega \ , \tag{1.7}$$

whereas in the phasor domain a time-harmonic dependence $\exp\left(\mathrm{j}\omega t\right)$ is assumed:

$$f(t) = F_0 \cos(\omega t + \alpha) = \mathrm{Re}\left[F\,\mathrm{e}^{+\mathrm{j}\omega t}\right] \ , \tag{1.8}$$

where $F_0 \geq 0$ and $F = F_0\,\mathrm{e}^{\mathrm{j}\alpha}$ is the phasor associated with $f(t)$. The same definitions also apply for vector functions.

1.4 Constitutive Relations

By direct inspection of Maxwell's curl equations in (1.1), it is immediately clear that they represent six scalar equations with 15 unknown quantities. With fewer equations than unknowns no unique solution can be identified (the problem is said to be indefinite). The additional equations required to make the problem definite are those that describe the relations among the field quantities **e**, **h**, **d**, **b**, and **j**, enforced by the medium filling the region where the EM phenomena occur. Such relations are called *constitutive relations*, and they depend on the properties of the medium supporting the EM field.

In non-moving media, with the exclusion of magnetoelectric and chiral materials, the **d** field depends only on the **e** field, **b** depends only on **h**, and **j** depends only on **e**. These dependences are expressed as constitutive relations, with the **e** and **h** fields regarded as causes and the **d**, **b**, and **j** fields as effects.

If a linear combination of causes (with given coefficients) produces a linear combination of effects (with the same coefficients), the medium is said to be *linear* (otherwise *nonlinear*). In general, the constitutive relations are described by a set of constitutive parameters and a set of constitutive operators that relate the above-mentioned fields inside a region of space. The constitutive parameters can be constants of proportionality between the fields (the medium is thus said *isotropic*), or they can be components in a tensor relationship (the medium is said anisotropic). If the constitutive parameters are constant within a certain region of space, the medium is said *homogeneous* in that region (otherwise, the medium is inhomogeneous). If the constitutive parameters are constant with time, the medium is said *stationary* (otherwise, the medium is nonstationary).

If the constitutive operators are expressed in terms of time integrals, the medium is said to be *temporally dispersive*. If these operators involve space integrals, the medium is said to be *spatially dispersive*. Finally, we note that the constitutive parameters may depend on other nonelectromagnetic properties of the material and external conditions (temperature, pressure, etc.).

The simplest medium is *vacuum*. In vacuum the following constitutive relations hold:

$$\begin{aligned} \mathbf{d}(\mathbf{r},t) &= \varepsilon_0 \mathbf{e}(\mathbf{r},t), \\ \mathbf{b}(\mathbf{r},t) &= \mu_0 \mathbf{h}(\mathbf{r},t), \\ \mathbf{j}(\mathbf{r},t) &= \mathbf{0}. \end{aligned} \tag{1.9}$$

The quantities $\mu_0 \simeq 4\pi \cdot 10^{-7}$ H/m[1] and $\varepsilon_0 \simeq 10^{-9}/(36\pi) \simeq 8.854 \cdot 10^{-12}$ F/m are the free-space magnetic permeability and dielectric permittivity, respectively. Their values are related to the speed of light in free space c_0 through $c_0 = 1/\sqrt{\mu_0 \varepsilon_0}$, whose exact value is $c_0 = 2.99792458 \cdot 10^8$ m/s; the above two values for ε_0 correspond to approximating $c_0 \simeq 3 \cdot 10^8$ m/s and $c_0 \simeq 2.998 \cdot 10^8$ m/s, respectively.

For a linear, homogeneous, isotropic, and nondispersive material, the constitutive relations can be expressed as

$$\begin{aligned} \mathbf{d}(\mathbf{r},t) &= \varepsilon \mathbf{e}(\mathbf{r},t), \\ \mathbf{b}(\mathbf{r},t) &= \mu \mathbf{h}(\mathbf{r},t), \\ \mathbf{j}(\mathbf{r},t) &= \sigma \mathbf{e}(\mathbf{r},t), \end{aligned} \tag{1.10}$$

where μ and ε are the magnetic permeability and dielectric permittivity of the medium, respectively. These quantities can be related to the corresponding

1 Until May 20th 2020 this was an exact value. Since that date, a redefinition of the SI base units assigned to the elementary charge (e.g., the electron charge) an exact value: $q_e = 1.602176634 \cdot 10^{-19}$ C, thereby making the reported value of the vacuum permeability μ_0 an approximate quantity.

free-space quantities through the dimensionless relative permeability μ_r and relative permittivity ε_r, such that $\mu = \mu_r\mu_0$ and $\varepsilon = \varepsilon_r\varepsilon_0$. The dimensionless quantities $\chi_m = \mu_r - 1$ and $\chi_e = \varepsilon_r - 1$ (known as magnetic and electric susceptibilities, respectively) are also used. The third equation of (1.10) expresses the Ohm law in local form, and σ is the conductivity of the medium (unit of measure S/m).

For such a simple medium, thanks to (1.10), Maxwell's equations (1.4) can be rewritten as

$$\nabla \times \mathbf{e}(\mathbf{r}, t) = -\mu \frac{\partial}{\partial t} \mathbf{h}(\mathbf{r}, t) - \mathbf{m}_i(\mathbf{r}, t) \ , \tag{1.11a}$$

$$\nabla \times \mathbf{h}(\mathbf{r}, t) = \sigma \mathbf{e}(\mathbf{r}, t) + \varepsilon \frac{\partial}{\partial t} \mathbf{e}(\mathbf{r}, t) + \mathbf{j}_i(\mathbf{r}, t) \ , \tag{1.11b}$$

$$\nabla \cdot \mathbf{e}(\mathbf{r}, t) = \frac{\rho_e(\mathbf{r}, t) + \rho_{ei}(\mathbf{r}, t)}{\varepsilon} \ , \tag{1.11c}$$

$$\nabla \cdot \mathbf{h}(\mathbf{r}, t) = \frac{\rho_{mi}(\mathbf{r}, t)}{\mu} \ . \tag{1.11d}$$

If the medium is inhomogeneous, μ, ε, or σ are quantities that depend on the vector position $\mathbf{r}$. If the medium is anisotropic (but still linear and nondispersive) the constitutive relations can be written as

$$\begin{aligned} \mathbf{d}(\mathbf{r}, t) &= \underline{\boldsymbol{\varepsilon}} \cdot \mathbf{e}(\mathbf{r}, t) \ , \\ \mathbf{b}(\mathbf{r}, t) &= \underline{\boldsymbol{\mu}} \cdot \mathbf{h}(\mathbf{r}, t) \ , \\ \mathbf{j}(\mathbf{r}, t) &= \underline{\boldsymbol{\sigma}} \cdot \mathbf{e}(\mathbf{r}, t) \ , \end{aligned} \tag{1.12}$$

where $\underline{\boldsymbol{\varepsilon}}$, $\underline{\boldsymbol{\mu}}$, and $\underline{\boldsymbol{\sigma}}$ are called the permittivity tensor, the permeability tensor, and the conductivity tensor, respectively (they are space-dependent quantities for inhomogeneous media).

For linear, inhomogeneous, anisotropic, stationary, and temporally dispersive materials, the constitutive relation between $\mathbf{d}$ and $\mathbf{e}$ is expressed by a superposition integral as

$$\mathbf{d}(\mathbf{r}, t) = \int_{-\infty}^{t} \underline{\boldsymbol{\varepsilon}}\left(\mathbf{r}, t - t'\right) \cdot \mathbf{e}\left(\mathbf{r}, t'\right) \, \mathrm{d}t' \ . \tag{1.13}$$

The constitutive relations for other field quantities have similar expressions. Causality is implied by the upper limit t in the integrals (this means that the effect cannot depend on future values of the cause). If the medium is nonstationary, $\underline{\boldsymbol{\varepsilon}}\left(\mathbf{r}, t, t'\right)$ has to be used instead of $\underline{\boldsymbol{\varepsilon}}\left(\mathbf{r}, t - t'\right)$. The important concept expressed by (1.13) is that the behavior of $\mathbf{d}$ at the time t depends not only on the value of $\mathbf{e}$ at the same time t but also on its values at all past times, thus giving rise to a time-lag between cause and effect. The upper limit of the integral in (1.13) can be extended to $+\infty$ by assuming that $\underline{\boldsymbol{\varepsilon}}\left(\mathbf{r}, t, t'\right) = \underline{\mathbf{0}}$ whenever $t' > t$, thus obtaining a *convolution* integral. Hence, in the frequency domain the constitutive relation (1.13) is expressed as

$$\mathbf{D}(\mathbf{r},\omega) = \underline{\varepsilon}(\mathbf{r},\omega)\cdot\mathbf{E}(\mathbf{r},\omega)\ , \tag{1.14}$$

where, with a little abuse of notation, $\underline{\varepsilon}(\mathbf{r},\omega)$ indicates the Fourier transform of the corresponding quantity in the time domain. The important point to note here is that, in the frequency domain, temporal dispersion is associated with complex values of the constitutive parameters; causality establishes a relationship between their real and imaginary parts (known as the Kramers–Kronig relation) [12] for which neither part can be constant with frequency.

Finally, if the medium is also spatially dispersive (and nonstationary), the constitutive relation takes the form

$$\mathbf{d}(\mathbf{r},t) = \int_V \left[\int_{-\infty}^{t} \underline{\varepsilon}\left(\mathbf{r},\mathbf{r}',t,t'\right)\cdot\mathbf{e}\left(\mathbf{r},\mathbf{r}',t'\right)\,\mathrm{d}t\right]\mathrm{d}V'\ , \tag{1.15}$$

where V indicates the whole three-dimensional space; as before, similar expressions hold for the constitutive relations of other field quantities as well. The integral over the volume V in (1.15) expresses the physical phenomenon for which the effect at the point $\mathbf{r}$ depends on the value of the cause in all the neighboring points $\mathbf{r}'$. An important point is that if the medium is spatially dispersive but homogeneous, the constitutive relations involve a convolution integral in the space domain. Therefore the constitutive relations in a linear, homogeneous, and stationary medium for the Fourier transforms of the fields with respect to both time and space can be written as

$$\tilde{\mathbf{D}}(k,\omega) = \underline{\tilde{\varepsilon}}(\mathbf{k},\omega)\cdot\tilde{\mathbf{E}}(\mathbf{k},\omega)\ . \tag{1.16}$$

Very often, in the frequency domain, the contributions in Maxwell's equations (1.5) from the conductivity current and the electric displacement are combined in a unique term by introducing an equivalent complex permittivity. For simplicity, we consider isotropic materials for which complex permittivity is a scalar quantity defined as

$$\varepsilon_\mathrm{c} = \varepsilon - \mathrm{j}\sigma/\omega\ . \tag{1.17}$$

Thus we can rewrite (1.5) in a dual form as

$$\begin{aligned}\nabla\times\mathbf{E}(\mathbf{r},\omega) &= -\mathrm{j}\omega\mu(\mathbf{r},\omega)\,\mathbf{H}(\mathbf{r},\omega) - \mathbf{M}_\mathrm{i}(\mathbf{r},\omega)\ ,\\ \nabla\times\mathbf{H}(\mathbf{r},\omega) &= \mathrm{j}\omega\varepsilon_\mathrm{c}(\mathbf{r},\omega)\,\mathbf{E}(\mathbf{r},\omega) + \mathbf{J}_\mathrm{i}(\mathbf{r},\omega)\ .\end{aligned} \tag{1.18}$$

Alternatively to the constitutive parameters ε_c and μ, the medium can also be described by the (possibly complex) *medium wavenumber*

$$k = \omega\sqrt{\mu\varepsilon_\mathrm{c}} \tag{1.19}$$

(where the principal branch of the square root is chosen so that the imaginary part of k is nonpositive) and the (possibly complex) *intrinsic impedance*

$$\eta = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon - j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \tag{1.20}$$

(where the principal branch of the square root is chosen so that the real part of η is nonnegative) [16]. In particular, the free-space wavenumber and the free-space impedance are

$$k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{\omega}{c_0} \tag{1.21}$$

and

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377\ \Omega\ , \tag{1.22}$$

respectively. Accordingly, the free-space wavelength is defined as

$$\lambda_0 = \frac{2\pi}{k_0}. \tag{1.23}$$

Finally, it is important to note that for the study of electromagnetism in matter, the EM field can be represented by four vectors other than **e**, **h**, **d**, and **b** (provided that the new vectors are a linear mapping of these vectors). In particular, the common alternative is to use vectors **e**, **b**, **p**, and **m** (not to be confused with the magnetic current density), where the new vectors **p** and **m** are called polarization and magnetization vectors, respectively, and Maxwell's equations are consequently written as

$$\begin{aligned}
&\nabla \times \mathbf{e}(\mathbf{r},t) = -\frac{\partial}{\partial t}\,\mathbf{b}(\mathbf{r},t)\ ,\\
&\nabla \times \left[\frac{\mathbf{b}(\mathbf{r},t)}{\mu_0} - \mathbf{m}(\mathbf{r},t)\right] = \mathbf{j}(\mathbf{r},t) + \frac{\partial}{\partial t}\left[\varepsilon_0\mathbf{e}(\mathbf{r},t) + \mathbf{p}(\mathbf{r},t)\right]\ ,\\
&\nabla \cdot \left[\varepsilon_0\mathbf{e}(\mathbf{r},t) + \mathbf{p}(\mathbf{r},t)\right] = \rho_e(\mathbf{r},t)\ ,\\
&\nabla \cdot \mathbf{b}(\mathbf{r},t) = 0\ .
\end{aligned} \tag{1.24}$$

From (1.1) and (1.24), it follows that

$$\begin{aligned}
&\mathbf{p}(\mathbf{r},t) = \mathbf{d}(\mathbf{r},t) - \varepsilon_0\mathbf{e}(\mathbf{r},t)\ ,\\
&\mathbf{m}(\mathbf{r},t) = \frac{\mathbf{b}(\mathbf{r},t)}{\mu_0} - \mathbf{h}(\mathbf{r},t)\ ,
\end{aligned} \tag{1.25}$$

or, in the frequency domain,

$$\begin{aligned}
&\mathbf{P}(\mathbf{r},\omega) = \mathbf{D}(\mathbf{r},\omega) - \varepsilon_0\mathbf{E}(\mathbf{r},\omega)\ ,\\
&\mathbf{M}(\mathbf{r},\omega) = \frac{\mathbf{B}(\mathbf{r},\omega)}{\mu_0} - \mathbf{H}(\mathbf{r},\omega)\ .
\end{aligned} \tag{1.26}$$

Next we introduce the equivalent polarization current density $\mathbf{j}_\mathrm{P} = \partial \mathbf{p}/\, \partial t$, the equivalent magnetization current density $\mathbf{j}_\mathrm{M} = \nabla \times \mathbf{m}$, and the equivalent polarization charge density $\rho_\mathrm{P} = -\nabla \cdot \mathbf{p}$ so that the Maxwell equations take the form

$$\begin{aligned} &\nabla \times \mathbf{e}(\mathbf{r}, t) = -\frac{\partial}{\partial t}\, \mathbf{b}(\mathbf{r}, t) \ , \\ &\nabla \times \frac{\mathbf{b}(\mathbf{r}, t)}{\mu_0} = \mathbf{j}(\mathbf{r}, t) + \mathbf{j}_\mathrm{P}(\mathbf{r}, t) + \mathbf{j}_\mathrm{M}(\mathbf{r}, t) + \varepsilon_0 \frac{\partial \mathbf{e}(\mathbf{r}, t)}{\partial t} \ , \\ &\varepsilon_0 \nabla \cdot \mathbf{e}(\mathbf{r}, t) = \rho_\mathrm{e}(\mathbf{r}, t) + \rho_\mathrm{P}(\mathbf{r}, t) \ , \\ &\nabla \cdot \mathbf{b}(\mathbf{r}, t) = 0 \ . \end{aligned} \tag{1.27}$$

formally equivalent to Maxwell's equations in vacuum.

1.5 Discontinuities and Singularities

As mentioned in Section 1.4, in the absence of discontinuities, Maxwell's equations in differential form are valid everywhere in space; nevertheless, for modeling purposes, discontinuities of material parameters or singular sources are often considered. In such cases other field relationships must be postulated (alternatively, they can be derived from Maxwell's equations in the integral form if such integral forms are postulated to be valid also across the discontinuities).

Let us consider the presence of singular sources in the form of electric and magnetic surface densities: electric $\mathbf{j}_S$ (A/m), $\rho_{\mathrm{e}S}$ (C/m^2) and magnetic $\mathbf{m}_S$ (V/m), $\rho_{\mathrm{m}S}$ (Wb/m^2), distributed over a surface S, which separates two regions (regions 1 and 2, respectively), or discontinuities in the material parameters across the surface S; the EM field in each region is indicated by the subscript 1 or 2. Let $\mathbf{u}_n$ be the unit vector normal to the surface S directed from region 2 to region 1. In such conditions the following jump conditions hold:

$$\begin{aligned} &\mathbf{u}_n \times (\mathbf{h}_1 - \mathbf{h}_2) = \mathbf{j}_S \ , \\ &\mathbf{u}_n \times (\mathbf{e}_1 - \mathbf{e}_2) = -\mathbf{m}_S \ , \\ &\mathbf{u}_n \cdot (\mathbf{d}_1 - \mathbf{d}_2) = \rho_{\mathrm{e}S} \ , \\ &\mathbf{u}_n \cdot (\mathbf{b}_1 - \mathbf{b}_2) = \rho_{\mathrm{m}S} \ , \end{aligned} \tag{1.28}$$

and

$$\begin{aligned} &\mathbf{u}_n \cdot (\mathbf{j}_1 - \mathbf{j}_2) = -\nabla_S \cdot \mathbf{j}_S - \frac{\partial \rho_{\mathrm{e}S}}{\partial t} \ , \\ &\mathbf{u}_n \cdot (\mathbf{m}_1 - \mathbf{m}_2) = -\nabla_S \cdot \mathbf{m}_S - \frac{\partial \rho_{\mathrm{m}S}}{\partial t} \ , \end{aligned} \tag{1.29}$$

where $\nabla_S [\cdot] = \nabla [\cdot] - \mathbf{u}_n\, \partial [\cdot] \,/\, \partial n$. It is clear that when $\mathbf{j}_S$ and $\mathbf{m}_S$ are zero, the tangential components of both electric and magnetic fields are continuous across the surface S. In particular, if discontinuities in the material parameters are present, the electric surface current density $\mathbf{j}_S$ may be different from zero at the boundary of a perfect electric conductor (PEC, within which $\mathbf{e}_2 = 0$), and the magnetic surface current density $\mathbf{m}_S$ may be different from zero at the boundary of a perfect magnetic conductor (PMC) (within which $\mathbf{h}_2 = 0$). Then the jump conditions at the interface between the conventional medium and the PEC are written as

$$\begin{aligned} &\mathbf{u}_n \times \mathbf{h} = \mathbf{j}_S \ , \\ &\mathbf{u}_n \times \mathbf{e} = 0 \ , \\ &\mathbf{u}_n \cdot \mathbf{d} = \rho_{eS} \ , \\ &\mathbf{u}_n \cdot \mathbf{b} = 0 \ , \\ &\mathbf{u}_n \cdot \mathbf{j} = -\nabla_S \cdot \mathbf{j}_S - \frac{\partial \rho_{eS}}{\partial t} \ , \\ &\mathbf{u}_n \cdot \mathbf{m} = 0 \ . \end{aligned} \tag{1.30}$$

Likewise, at the interface between a conventional medium and a PMC, the results are

$$\begin{aligned} &\mathbf{u}_n \times \mathbf{h} = 0 \ , \\ &\mathbf{u}_n \times \mathbf{e} = -\mathbf{m}_S \ , \\ &\mathbf{u}_n \cdot \mathbf{d} = 0 \ , \\ &\mathbf{u}_n \cdot \mathbf{b} = \rho_{mS} \ , \\ &\mathbf{u}_n \cdot \mathbf{j} = 0 \ , \\ &\mathbf{u}_n \cdot \mathbf{m} = -\nabla_S \cdot \mathbf{m}_S - \frac{\partial \rho_{mS}}{\partial t} \ . \end{aligned} \tag{1.31}$$

In these jump conditions the $\mathbf{u}_n$ unit vector points outside the conductors.

Finally, some other singular behaviors of fields and currents worthy of mention occur in correspondence to the edge of a dielectric or conducting wedge and to the tip of a dielectric or conducting cone. The solution of the EM problem in such cases can be made unique by enforcing the physical constraint that the energy stored in proximity of the edge or tip is finite. The order of singularity generally depends on the boundary conditions that hold on the surface boundaries; further details can be found in [15] and [17].

1.6 Initial Conditions, Boundary Conditions, and Causality

As was noted earlier, Maxwell's equations together with the constitutive relations represent a set of partial differential equations. However, it is well known that in order to obtain a solution for this set of equations, both initial and boundary conditions must be specified. The initial conditions are represented by the constraints that the EM field must satisfy at a given time, while boundary conditions are, in general, constraints that the EM field must satisfy over certain surfaces of the three-dimensional space, usually surfaces that separate regions of space filled with different materials. In these cases the boundary conditions coincide with the jump conditions illustrated in Section 1.5. Other important examples of boundary conditions that can easily be formulated in the frequency domain are the *impedance boundary condition* and *radiation condition at infinity*. The impedance boundary condition (also known as the Leontovich condition) relates the component $\mathbf{E}_\mathrm{t}$ of the electric field tangential to a surface S with the magnetic field as

$$\mathbf{E}_\mathrm{t} = Z_S \left(\mathbf{u}_n \times \mathbf{H}\right) \ , \tag{1.32}$$

where Z_S (surface impedance) is a complex scalar quantity. The radiation condition at infinity (also known as the Sommerfeld radiation condition in scalar radiation problems or the Silver–Müller radiation condition in vector radiation problems) postulates that in free space, in the absence of sources at infinity, there results

$$\lim_{r\to+\infty} r\left[\mathbf{E} - \eta_0 \left(\mathbf{H} \times \mathbf{u}_r\right)\right] = \mathbf{0}. \tag{1.33}$$

As concerns transient fields, physical grounds require that all the fields satisfy the law of causality. Two different causality conditions can be defined, i.e., a weak and a strong causality condition. The *weak causality* condition states that all fields have to be zero for $t \leq t_0$ if the sources are zero for $t \leq t_0$. In other words, an output does not exist until an input is applied. If such a weak causality condition is assumed for electromagnetic fields, Maxwell's equations imply a *strong causality* condition, i.e., that the fields are zero beyond a distance $c\left(t - t_0\right)$ from a point source that is zero for $t \leq t_0$, where

$$c = \frac{1}{\sqrt{\mu\varepsilon}} \tag{1.34}$$

is the speed of light in a medium with constitutive parameters μ and ε.

1.7 Poynting's Theorem and Energy Considerations

A fundamental consequence of Maxwell's equations is *Poynting's theorem*, which provides a power balance for the electromagnetic field. In the time domain, given a region V bounded by a surface S with unit normal vector $\mathbf{u}_n$ oriented towards the exterior of V, its statement is

$$\oint_S \mathbf{u}_n \cdot \boldsymbol{\pi}\, \mathrm{d}S + \int_V p_{\mathrm{d}}\, \mathrm{d}V + \int_V \left(p_H + p_E\right)\, \mathrm{d}V = \int_V \left(p_{\mathrm{i}} + p_{\mathrm{mi}}\right)\, \mathrm{d}V\,, \tag{1.35}$$

where

$$\begin{aligned} \boldsymbol{\pi} &= \mathbf{e} \times \mathbf{h}\,, \\ p_{\mathrm{d}} &= \mathbf{e} \cdot \mathbf{j}\,, \\ p_H &= \mathbf{h} \cdot \frac{\partial}{\partial t}\mathbf{b}\,, \\ p_E &= \mathbf{e} \cdot \frac{\partial}{\partial t}\mathbf{d}\,, \\ p_{\mathrm{i}} &= -\mathbf{e} \cdot \mathbf{j}_{\mathrm{i}}\,, \\ p_{\mathrm{mi}} &= -\mathbf{h} \cdot \mathbf{j}_{\mathrm{mi}}\,. \end{aligned} \tag{1.36}$$

The right-hand side of (1.35) expresses the power furnished by the impressed currents, i.e., by the sources of the electromagnetic field. The three addends at left-hand side are the destinations of such power: the surface integral is the power that leaves the volume V by crossing its boundary S; the volume integral with p_{d} represents the power transferred from the field to the charges in V (and eventually dissipated into heat via Joule effect inside, e.g., a metal); finally, the volume integral with p_H and p_E represents the power exchanged with the electromagnetic field inside V in the form of stored magnetic and electric energy.

A complex version of the Poynting theorem is also available, valid for time-harmonic fields and sources, which can be expressed using the phasor notation in the frequency domain[2].

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2 It is understood that this is an idealization, since true monochromatic fields cannot exist. However, the simplicity of the formalism and the fact that a monochromatic wave is an elemental component of the complete frequency-domain spectrum of an arbitrary time-varying field make the assumption of monochromatic fields an invaluable tool for the investigation of the EM-field theory. Nevertheless, great care must be given to the use of such an assumption because it can lead to nonphysical consequences: a classical example consists in determining the energy stored in a lossless cavity. An infinite value is actually obtained, since the cavity stores energy starting from a remote instant $t = -\infty$. The problem can be overcome by considering time-averaged quantities, but some other problems can arise when the filling material is dispersive.

Given a region V bounded by a surface S, from the time-harmonic Maxwell equations the following identity holds

$$\oint_S \mathbf{u}_n \cdot \mathbf{\Pi}\, \mathrm{d}S + \int_V P_\mathrm{d}\, \mathrm{d}V + \int_V \left(P_H + P_E\right)\, \mathrm{d}V = \int_V \left(P_\mathrm{i} + P_\mathrm{mi}\right)\, \mathrm{d}V\,, \tag{1.37}$$

where

$$\begin{aligned} \mathbf{\Pi} &= \frac{1}{2}\, \mathbf{E} \times \mathbf{H}^*\,, \\ P_\mathrm{d} &= \frac{1}{2}\, \mathbf{E} \cdot \mathbf{J}^*\,, \\ P_H &= \frac{1}{2}\, \mathrm{j}\omega \mathbf{B} \cdot \mathbf{H}^*\,, \\ P_E &= -\frac{1}{2}\, \mathrm{j}\omega \mathbf{E} \cdot \mathbf{D}^*\,, \\ P_\mathrm{i} &= -\frac{1}{2}\, \mathbf{E} \cdot \mathbf{J}_\mathrm{i}^*\,, \\ P_\mathrm{mi} &= -\frac{1}{2}\, \mathbf{J}_\mathrm{mi} \cdot \mathbf{H}^*\,. \end{aligned} \tag{1.38}$$

The real part of each term in (1.37) is equal to the average value (over one period $T = 2\pi/\omega$ in time) of the corresponding term in the time-domain version (1.35). Hence, the real part of the complex Poynting theorem expresses an average balance of power for the given volume V.

In particular, for simple isotropic materials (with complex constitutive parameters μ and ε_c), the Poynting theorem can also be expressed in a local form as

$$\begin{aligned} &\nabla \cdot\ \mathrm{Re}\,[\mathbf{\Pi}] - \omega\, \mathrm{Im}\left[\varepsilon_\mathrm{c}\right] \frac{|E|^2}{2} - \omega\, \mathrm{Im}\,[\mu]\, \frac{|H|^2}{2} = -\mathrm{Re}\left[\frac{\mathbf{J}_\mathrm{i}^* \cdot \mathbf{E}}{2} + \frac{\mathbf{M}_\mathrm{i} \cdot \mathbf{H}^*}{2}\right], \\ &\nabla \cdot\ \mathrm{Im}\,[\mathbf{\Pi}] + 2\omega \left(\mathrm{Re}\,[\mu]\, \frac{|H|^2}{4} - \mathrm{Re}\left[\varepsilon_\mathrm{c}\right] \frac{|E|^2}{4}\right) = -\mathrm{Im}\left[\frac{\mathbf{J}_\mathrm{i}^* \cdot \mathbf{E}}{2} + \frac{\mathbf{M}_\mathrm{i} \cdot \mathbf{H}^*}{2}\right]. \end{aligned} \tag{1.39}$$

In the first equation of (1.39) the terms involving the imaginary parts of μ and ε_c correspond to time-averaged power densities dissipated through a conduction current or for different mechanisms (magnetic and dielectric hysteresis); moreover for media that cannot transfer energy (mechanical or chemical) into the field (i.e., *passive media*) such imaginary parts must be nonpositive. In general, we will refer to *lossless* isotropic media as those materials having the imaginary parts of μ and ε_c identically zero. It can be shown that lossless anisotropic media are characterized by complex tensor permeability and permittivity which are both Hermitian.

For non-dispersive media the term into the brackets in the second equation of (1.39) represents the difference between the time-averaged magnetic and electric energy densities. The right-hand side is called *reactive power density*. As the second equation of (1.39) shows, such reactive power density (divided by 2ω) represents a sort of energy exchange between the external and the internal region.

1.8 Fundamental Theorems

1.8.1 Uniqueness Theorem

As in any other problem of mathematical physics, the uniqueness property is a fundamental condition for a problem to be well-posed. First of all, a uniqueness theorem establishes the mandatory information that one needs to obtain the solution of the problem. Second, it is of critical importance to know that the solution that one can obtain through different techniques is also unique. Third, the uniqueness theorem is a fundamental tool for the development of other important theorems, such as the equivalence theorem and the reciprocity theorem.

The uniqueness theorem can be formulated in the time domain as follows: Given a stationary region Ω in space, filled with a linear, stationary, non-dispersive medium, hosting prescribed impressed currents $\mathbf{j}_\mathrm{i}$, $\mathbf{j}_\mathrm{mi}$, there exists a unique EM field $\mathbf{e}$, $\mathbf{h}$ that satisfies both the Maxwell equations and the constitutive relations in Ω for $t \geq t_0$, provided that the tangential component of $\mathbf{e}$ over the boundary, or the tangential component of $\mathbf{h}$ over the boundary, or the former over part of the boundary and the latter over the remaining part of the boundary, are specified for $t \geq t_0$ (*boundary conditions*), and the electromagnetic field is assigned everywhere in Ω at $t = t_0$ (*initial conditions*).

A time-harmonic version of the theorem is also available, in which the assumptions are essentially the same except that no initial conditions are given. In the case of regions of infinite extent the uniqueness theorem still holds, provided that the boundary conditions for the tangential components of the field are replaced by the radiation condition at infinity. It is important to stress, however, that in the time-harmonic case the proof of the theorem is strictly valid only for lossy media (and, in turn, this restriction is a consequence of the ideal time-harmonic assumption). However, the lossless case can be obtained in the limit of vanishing losses [18].

1.8.2 Reciprocity Theorem

Another important theorem of electromagnetism is the reciprocity theorem, which follows directly from Maxwell's equations in the time-harmonic regime.

In fact, given a set of sources $\{\mathbf{J}_{i1}, \mathbf{M}_{i1}\}$ that produce the fields $\{\mathbf{E}_1, \mathbf{H}_1, \mathbf{D}_1, \mathbf{B}_1\}$ and a second set of sources $\{\mathbf{J}_{i2}, \mathbf{M}_{i2}\}$ that produce the fields $\{\mathbf{E}_2, \mathbf{H}_2, \mathbf{D}_2, \mathbf{B}_2\}$, from (1.9) the following identity can be obtained

$$\nabla \cdot \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) = \mathrm{j}\omega \left(\mathbf{H}_1 \cdot \mathbf{B}_2 - \mathbf{H}_2 \cdot \mathbf{B}_1 - \mathbf{E}_1 \cdot \mathbf{D}_2 + \mathbf{E}_2 \cdot \mathbf{D}_1\right) + \left(\mathbf{H}_1 \cdot \mathbf{M}_{i2} - \mathbf{H}_2 \cdot \mathbf{M}_{i1} - \mathbf{E}_1 \cdot \mathbf{J}_{i2} + \mathbf{E}_2 \cdot \mathbf{J}_{i1}\right) . \tag{1.40}$$

The media for which the first term in the right-hand side of (1.40) is zero are called *reciprocal*. It can be shown that this is the case for isotropic media and also for anisotropic media provided that both the tensor permittivity and permeability are symmetric; examples of nonreciprocal media are lossless gyrotropic materials (for which the tensor constitutive parameters are Hermitian but not symmetric). Therefore, from (1.40), for reciprocal media there results

$$\nabla \cdot \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) = \left(\mathbf{H}_1 \cdot \mathbf{M}_{i2} - \mathbf{H}_2 \cdot \mathbf{M}_{i1} - \mathbf{E}_1 \cdot \mathbf{J}_{i2} + \mathbf{E}_2 \cdot \mathbf{J}_{i1}\right) . \tag{1.41}$$

By integrating (1.41) over a finite volume V bounded by a closed surface S, we obtain the *Lorentz reciprocity theorem*, that is,

$$\oint_S \left(\mathbf{E}_1 \times \mathbf{H}_2 - \mathbf{E}_2 \times \mathbf{H}_1\right) \cdot \mathbf{u}_n \, \mathrm{d}S = \int_V \left(\mathbf{H}_1 \cdot \mathbf{M}_{i2} - \mathbf{H}_2 \cdot \mathbf{M}_{i1} - \mathbf{E}_1 \cdot \mathbf{J}_{i2} + \mathbf{E}_2 \cdot \mathbf{J}_{i1}\right) \, \mathrm{d}V . \tag{1.42}$$

A system for which the integral at the left-hand side of (1.42) vanishes is said to be *reciprocal*. It can be shown that this is the case if the region V is source free or if an impedance boundary condition holds over the surface S. In such cases, from (1.42), the *reaction theorem* can be obtained, which is expressed by

$$\int_V \left(\mathbf{H}_1 \cdot \mathbf{M}_{i2} - \mathbf{E}_1 \cdot \mathbf{J}_{i2}\right) \, \mathrm{d}V = \int_V \left(\mathbf{H}_2 \cdot \mathbf{M}_{i1} - \mathbf{E}_2 \cdot \mathbf{J}_{i1}\right) \, \mathrm{d}V . \tag{1.43}$$

These results can be extended to infinite regions if the impedance boundary condition is replaced by the radiation condition at infinity.

The usefulness of the reciprocity theorem can be understood by considering the EM problem of an elemental electric dipole $\mathbf{J}_{i1} = \mathbf{u}_1 \delta \left(\mathbf{r} - \mathbf{r}_1\right)$ placed in free space and a second elemental electric dipole $\mathbf{J}_{i2} = \mathbf{u}_2 \delta \left(\mathbf{r} - \mathbf{r}_2\right)$ placed inside a metallic enclosure having an aperture in one of its walls ($\mathbf{u}_1$ and $\mathbf{u}_2$ are the unit vectors along two arbitrary directions). According to the discussion above, the system is reciprocal and from (1.43) we obtain

$$\mathbf{u}_2 \cdot \mathbf{E}_1 \left(\mathbf{r}_2\right) = \mathbf{u}_1 \cdot \mathbf{E}_2 \left(\mathbf{r}_1\right) . \tag{1.44}$$

Equation (1.44) expresses the fact that the component along $\mathbf{u}_2$ of the electric field radiated by the dipole (placed in free space) at the point $\mathbf{r}_2$ inside the cavity is equal to the component along $\mathbf{u}_1$ of the electric field radiated by the dipole (placed inside the enclosure) at the point $\mathbf{r}_1$ in free space.

The reciprocity theorem can also be used to show that the EM field produced by an electric surface current density distributed over a PEC surface is identically zero (and, dually, that produced by a magnetic surface current density distributed over a PMC surface). Other applications of the reciprocity theorem regard the mode excitation in waveguides and cavities and receiving and transmitting properties of antennas [18].

It is worth noting that, while in time-harmonic reciprocity the interaction is only spatial, in time-domain reciprocity the interaction is spatio-temporal. In particular, two types of temporal interactions can be used, i.e., a *convolution-type* and a *correlation-type reciprocity*. Historically, in [19] the first time-domain reciprocity theorem (of correlation type) was derived for electromagnetic fields, while in [20] a convolution-type reciprocity theorem was developed. These theorems apply to homogeneous, isotropic and lossless [19] or lossy [20] media. In [21] the convolution-type reciprocity theorem was extended to dispersive and possibly nonhomogeneous or anisotropic media. In [22] both convolution-type and correlation-type reciprocity theorems have been presented for homogeneous, isotropic and lossless media. Next, time-domain reciprocity theorems of both types have been developed for a large class of media by A. de Hoop [23, 24]. Applications of time-domain reciprocity range from the study of coupling between antennas to the study of both direct and inverse scattering problems, to integral-equation formulations of electromagnetic problems, and to the development of numerical methods [24–27].

1.8.3 Equivalence Principle

The equivalence principle is a consequence of Maxwell's equations and the uniqueness theorem. Basically, it allows the original EM problem to be replaced with an equivalent problem whose solution coincides with that of the original problem in a finite region of space. To be effective, the equivalent problem should be easier to solve than the original one. The principle is usually formulated in the time-harmonic regime, in different forms.

The first form of the equivalence principle (also known as the *Love equivalence principle*) establishes that the EM field $\{\mathbf{E}, \mathbf{H}\}$ outside a region V bounded by a surface S enclosing the sources $\{\mathbf{J}_\mathrm{i}, \mathbf{M}_\mathrm{i}\}$ is equal to that produced by the equivalent sources $\{\mathbf{J}_S, \mathbf{M}_S\}$ distributed over the surface S and given by

$$\begin{aligned} \mathbf{J}_S &= \mathbf{u}_n \times \mathbf{H}_S \, , \\ \mathbf{M}_S &= -\mathbf{u}_n \times \mathbf{E}_S \, , \end{aligned} \tag{1.45}$$

where $\{\mathbf{E}_S, \mathbf{H}_S\}$ is the EM field $\{\mathbf{E}, \mathbf{H}\}$ in correspondence of the surface S and $\mathbf{u}_n$ is the unit vector normal to S pointing outside the region V. It can be shown that the field produced by the equivalent sources inside V is identically zero. The equivalent sources are considered as known terms in the formulation of the problem. However, they depend on the field $\{\mathbf{E}, \mathbf{H}\}$, which is unknown. In practice, there are many problems for which approximate expressions can be found for the equivalent currents, and in any case they are extensively used to formulate exact boundary integral equations for the considered problem.

A second form of the equivalence principle is known as the *Schelkunoff equivalence principle*. It is based on the fact that, according to the Love equivalence principle, the field produced by the equivalent sources inside V is identically zero. It differs from the Love equivalence principle since the medium filling the region V is replaced with a PEC (the boundary conditions on S are not changed). In this way, the equivalent magnetic current $\mathbf{M}_S$ is the only radiating source. Dually, the region V could be replaced with a PMC; in this case, $\mathbf{J}_S$ would be the only radiating source. However, it must be pointed out that the two situations considered by Love and Schelkunoff are different: in the latter, the presence of a PEC (or PMC) body must be explicitly taken into account.

1.8.4 Duality

Duality is a fundamental internal symmetry of Maxwell's equations. With reference to a linear, time-invariant, isotropic, and local medium, it consists in the invariance of Maxwell's equations (1.18) upon the substitutions $\mathbf{E} \to \mathbf{H}$, $\mathbf{H} \to -\mathbf{E}$, $\mathbf{J}_\mathrm{i} \to \mathbf{M}_\mathrm{i}$, $\mathbf{M}_\mathrm{i} \to -\mathbf{J}_\mathrm{i}$, $\varepsilon_\mathrm{c} \to \mu$, $\mu \to \varepsilon_\mathrm{c}$.

Given a solution $\mathbf{E}$, $\mathbf{H}$ of the Maxwell equations, this property allows for obtaining another (dual) solution $\mathbf{E}^\mathrm{d}$, $\mathbf{H}^\mathrm{d}$ given by

$$\begin{aligned} \mathbf{E}^\mathrm{d} &= \eta \mathbf{H} \ , \\ \mathbf{H}^\mathrm{d} &= -\frac{1}{\eta}\mathbf{E} \end{aligned} \tag{1.46}$$

produced by the dual sources

$$\begin{aligned} \mathbf{J}_\mathrm{i}^\mathrm{d} &= \frac{1}{\eta}\mathbf{M}_\mathrm{i} \ , \\ \mathbf{M}_\mathrm{i}^\mathrm{d} &= -\eta \mathbf{J}_\mathrm{i} \ . \end{aligned} \tag{1.47}$$

If the original solution $\mathbf{E}$, $\mathbf{H}$ is considered inside a spatial region V and satisfies certain boundary conditions on the boundary S_V, then the dual solution $\mathbf{E}^\mathrm{d}$, $\mathbf{H}^\mathrm{d}$ satisfies on S_V different boundary conditions. For instance, assume that the boundary conditions for $\mathbf{E}$, $\mathbf{H}$ are those valid at the boundary of a PEC plane, i.e., $\mathbf{E}_\tau = \mathbf{0}$ (where the subscript τ indicates the tangential component of the vector field) and $\mathbf{H}_\tau = \mathbf{J}_S \times \mathbf{u}_n$. Then $\mathbf{E}^\mathrm{d}$ and $\mathbf{H}^\mathrm{d}$ satisfy the dual boundary conditions

$\mathbf{H}_\tau = \mathbf{0}$, $\mathbf{E}_\tau = -\mathbf{M}_S \times \mathbf{u}_n$ (which are those characterizing the boundary of a PMC plane).

1.8.5 Symmetry

Let the plane $z = 0$ be a symmetry plane for the media configuration within the region V under analysis. The electromagnetic field can then have two kinds of symmetries, even and odd, respectively. With reference to the electric and magnetic-induction fields $\{\mathbf{E}, \mathbf{B}\}$, an *even* field is characterized by the properties

$$\begin{aligned} E_x^{\text{even}}(x, y, -z) &= E_x^{\text{even}}(x, y, z) \ , \\ E_y^{\text{even}}(x, y, -z) &= E_y^{\text{even}}(x, y, z) \ , \\ E_z^{\text{even}}(x, y, -z) &= -E_z^{\text{even}}(x, y, z) \ , \end{aligned} \tag{1.48}$$

$$\begin{aligned} B_x^{\text{even}}(x, y, -z) &= -B_x^{\text{even}}(x, y, z) \ , \\ B_y^{\text{even}}(x, y, -z) &= -B_y^{\text{even}}(x, y, z) \ , \\ B_z^{\text{even}}(x, y, -z) &= B_z^{\text{even}}(x, y, z) \ , \end{aligned} \tag{1.49}$$

whereas an *odd* field is characterized by the properties

$$\begin{aligned} E_x^{\text{odd}}(x, y, -z) &= -E_x^{\text{odd}}(x, y, z) \ , \\ E_y^{\text{odd}}(x, y, -z) &= -E_y^{\text{odd}}(x, y, z) \ , \\ E_z^{\text{odd}}(x, y, -z) &= E_z^{\text{odd}}(x, y, z) \ , \end{aligned} \tag{1.50}$$

$$\begin{aligned} B_x^{\text{odd}}(x, y, -z) &= B_x^{\text{odd}}(x, y, z) \ , \\ B_y^{\text{odd}}(x, y, -z) &= B_y^{\text{odd}}(x, y, z) \ , \\ B_z^{\text{odd}}(x, y, -z) &= -B_z^{\text{odd}}(x, y, z) \ . \end{aligned} \tag{1.51}$$

In short, for an even electromagnetic field $\{\mathbf{E}^{\text{even}}, \mathbf{B}^{\text{even}}\}$, the electric-field components parallel to the symmetry plane are even functions of the coordinate orthogonal to the symmetry plane, whereas the electric-field component orthogonal to the plane is an odd function of the same coordinate; the opposite is true for the magnetic induction. For an odd electromagnetic field $\{\mathbf{E}^{\text{odd}}, \mathbf{B}^{\text{odd}}\}$, the properties of two fields are exchanged. Note that the different behavior of the electric field and magnetic induction upon reflection originates from their fundamentally different nature: the electric field is a true (or polar) vector, whereas the magnetic induction is a pseudo- (or axial) vector, its true nature being that of an antisymmetric, second-order tensor (see [28] for more details).

Similar definitions can be given for even and odd impressed currents, the behavior of the electric current being analogous to that of the electric field and that of the magnetic current to that of the magnetic induction. It is then readily seen, by virtue of the uniqueness theorem, that even/odd currents generate even/odd fields. Note that the most general electromagnetic field or impressed current in V need not be even nor odd; however, they can always be represented as the sum of even and odd fields/currents.

1.8.6 Image Principle

For an odd field, the tangential component of the electric field and the normal component of the magnetic induction vanish on the symmetry plane, which then behaves as a PEC plane. By filling one of the two half spaces $z > 0$ or $z < 0$ with a PEC medium, the uniqueness theorem then guarantees that the field in the other half space remains unchanged. This is the basis of the so-called *image principle*, by which the field produced by an arbitrary system of impressed sources in a half space bounded by an infinite PEC plane is the same field that would be produced in free space by the same sources and by their odd-symmetric counterparts (their *images*) placed in the complementary half space.

By duality, the field produced by an arbitrary system of impressed sources in a half space bounded by an infinite PMC plane is the same field that would be produced in free space by the same sources and by their even-symmetric counterparts (their *images*) placed in the complementary half space.

1.8.7 Babinet's Principle

Let now $\mathbf{E}^{\mathrm{i}}$, $\mathbf{H}^{\mathrm{i}}$ be the field produced in a uniform medium with parameters k and η by given sources $\mathbf{J}_{\mathrm{i}}$, $\mathbf{M}_{\mathrm{i}}$, located in $z < 0$ (see Figure 1.1*a*). Let S be the plane $z = 0$, where an infinitesimally thin metal screen is assumed to be located; PEC boundary conditions hold on the metalized surface S_{m}, whereas the screen apertures constitute the complementary surface S_{a} (see Figure 1.1*b*). The *Babinet principle* [18, 29] relates the field $\mathbf{E}^{\mathrm{e}}$, $\mathbf{H}^{\mathrm{e}}$ transmitted through such a metal screen to the field $\mathbf{E}^{\mathrm{m}}$, $\mathbf{H}^{\mathrm{m}}$ transmitted through the complementary dual screen, in which PMC boundary conditions hold on S_{a} and apertures are in S_{m}, produced by the same sources (see Figure 1.1*c*). In alternative versions, the principle relates the field $\mathbf{E}^{\mathrm{e}}$, $\mathbf{H}^{\mathrm{e}}$ to the field $\mathbf{E}^{\mathrm{d}}$, $\mathbf{H}^{\mathrm{d}}$ transmitted through the complementary metal screen, in which PEC boundary conditions hold on S_{a} and apertures are in S_{m}, produced by the dual sources $\mathbf{J}_{\mathrm{i}}^{\mathrm{d}} = (1/\eta)\mathbf{M}_{\mathrm{i}}$, $\mathbf{M}_{\mathrm{i}}^{\mathrm{d}} = -\eta\mathbf{J}_{\mathrm{i}}^{\mathrm{d}}$ (see Figure 1.1*d*).

Let us write any of the field in Figures 1.1*b*–*d* as the sum of an incident field (indicated with the superscript i) and a scattered field (indicated with the superscript s). The scattered field in Figure 1.1*b* is due to the induced electric currents on the PEC S_{m}, while in Figure 1.1*c* it is due to the induced magnetic currents on the PMC S_{a}. The incident field in Figures 1.1*b,c* is $\mathbf{E}^{\mathrm{i}}$, $\mathbf{H}^{\mathrm{i}}$, while in Figure 1.1*d* it is the dual field $\mathbf{E}^{\mathrm{di}} = \eta\mathbf{H}^{\mathrm{i}}$, $\mathbf{H}^{\mathrm{di}} = -(1/\eta)\,\mathbf{E}^{\mathrm{i}}$. With reference to Figure 1.1*b* we can

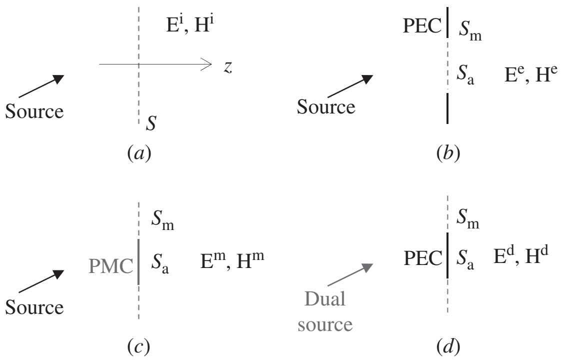

Figure 1.1 Illustrating the Babinet principle: *(a)* definition of the incident field; *(b)* the reference configuration: electromagnetic transmission through an aperture in an infinitely extended, infinitesimally thin PEC screen; *(c)* scattering from the dual screen; *(d)* scattering of the dual field from the complementary PEC screen.

write

$$\begin{aligned} \mathbf{E}_\tau^{\mathrm{e}} &= \mathbf{0} \quad \text{on } S_{\mathrm{m}} \ , \\ \mathbf{H}_\tau^{\mathrm{e}} &= \mathbf{H}_\tau^{\mathrm{i}} \quad \text{on } S_{\mathrm{a}} \ , \end{aligned} \tag{1.52}$$

(where the subscript τ indicates the tangential component of the field) since, by symmetry, $\mathbf{H}_\tau^{\mathrm{es}} = \mathbf{0}$ on S_{a}. With reference to Figure 1.1*c* we can write

$$\begin{aligned} \mathbf{E}_\tau^{\mathrm{m}} &= \mathbf{E}_\tau^{\mathrm{i}} \quad \text{on } S_{\mathrm{m}} \ , \\ \mathbf{H}_\tau^{\mathrm{m}} &= \mathbf{0} \quad \text{on } S_{\mathrm{a}} \ , \end{aligned} \tag{1.53}$$

since, by symmetry, $\mathbf{E}_\tau^{\mathrm{ms}} = \mathbf{0}$ on S_{m}. By summing (1.52) and (1.53) we have

$$\begin{aligned} \mathbf{E}_\tau^{\mathrm{e}} + \mathbf{E}_\tau^{\mathrm{m}} &= \mathbf{E}_\tau^{\mathrm{i}} \quad \text{on } S_{\mathrm{m}} \ , \\ \mathbf{H}_\tau^{\mathrm{e}} + \mathbf{H}_\tau^{\mathrm{m}} &= \mathbf{H}_\tau^{\mathrm{i}} \quad \text{on } S_{\mathrm{a}} \ . \end{aligned} \tag{1.54}$$

By the uniqueness theorem, we then deduce that in $z > 0$ it results (*Babinet's principle*)

$$\begin{aligned} \mathbf{E}^{\mathrm{e}} + \mathbf{E}^{\mathrm{m}} &= \mathbf{E}^{\mathrm{i}} \ , \\ \mathbf{H}^{\mathrm{e}} + \mathbf{H}^{\mathrm{m}} &= \mathbf{H}^{\mathrm{i}} \ . \end{aligned} \tag{1.55}$$

On the other hand, with reference to Figure 1.1*d* we can write, by duality

$$\begin{aligned} \mathbf{E}^{\mathrm{d}} &= \eta \mathbf{H}^{\mathrm{m}} \ , \\ \mathbf{H}^{\mathrm{d}} &= -\frac{1}{\eta} \mathbf{E}^{\mathrm{m}} \ . \end{aligned} \tag{1.56}$$

Hence in $z > 0$, by inserting (1.55) in (1.56), it results (*Babinet's principle, first alternative version*)

$$\begin{aligned} &\mathbf{E}^{\mathrm{e}} - \eta \mathbf{H}^{\mathrm{d}} = \mathbf{E}^{\mathrm{i}} \,, \\ &\mathbf{H}^{\mathrm{e}} + \frac{1}{\eta}\mathbf{E}^{\mathrm{d}} = \mathbf{H}^{\mathrm{i}} \,. \end{aligned} \tag{1.57}$$

Again with reference to Figure 1.1*d*, from (1.56) and (1.53) we have

$$\begin{aligned} &\mathbf{E}_\tau^{\mathrm{d}} = \eta \mathbf{H}_\tau^{\mathrm{m}} = \mathbf{0} \quad \text{on } S_{\mathrm{a}} \,, \\ &\mathbf{H}_\tau^{\mathrm{d}} = -\frac{1}{\eta}\mathbf{E}_\tau^{\mathrm{m}} = -\frac{1}{\eta}\mathbf{E}_\tau^{\mathrm{i}} \quad \text{on } S_{\mathrm{m}} \,, \end{aligned} \tag{1.58}$$

that are equivalent to

$$\begin{aligned} &\mathbf{E}_\tau^{\mathrm{ds}} = -\eta \mathbf{E}_\tau^{\mathrm{di}} = -\eta \mathbf{H}_\tau^{\mathrm{i}} \quad \text{on } S_{\mathrm{a}} \,, \\ &\mathbf{H}_\tau^{\mathrm{ds}} = -\mathbf{H}_\tau^{\mathrm{di}} - \frac{1}{\eta}\mathbf{E}_\tau^{\mathrm{i}} = \frac{1}{\eta}\mathbf{E}_\tau^{\mathrm{i}} - \frac{1}{\eta}\mathbf{E}_\tau^{\mathrm{i}} = \mathbf{0} \quad \text{on } S_{\mathrm{m}} \,. \end{aligned} \tag{1.59}$$

These can be written as

$$\begin{aligned} &-\frac{1}{\eta}\mathbf{E}_\tau^{\mathrm{ds}} = \mathbf{H}_\tau^{\mathrm{i}} \quad \text{on } S_{\mathrm{a}} \,, \\ &\eta \mathbf{H}_\tau^{\mathrm{ds}} = \mathbf{0} \quad \text{on } S_{\mathrm{m}} \,. \end{aligned} \tag{1.60}$$

By comparison with (1.52) we deduce, by the uniqueness theorem, that in $z > 0$ it results (*Babinet's principle, second alternative version*)

$$\begin{aligned} &\mathbf{E}^{\mathrm{e}} = \eta \mathbf{H}^{\mathrm{ds}} \,, \\ &\mathbf{H}^{\mathrm{e}} = -\frac{1}{\eta}\mathbf{E}^{\mathrm{ds}} \,. \end{aligned} \tag{1.61}$$

1.9 Wave Equations, Helmholtz's Equations, Potentials, and Green's Functions

In linear, homogeneous, and isotropic media, we can take the curl of Maxwell's equations in (1.18) and obtain the electric field and the magnetic field *wave equations* as

$$\begin{aligned} &\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2 \mathbf{E}(\mathbf{r}) = -\mathrm{j}\omega\mu \mathbf{J}_{\mathrm{i}}(\mathbf{r}) - \nabla \times \mathbf{M}_{\mathrm{i}}(\mathbf{r}) \,, \\ &\nabla \times \nabla \times \mathbf{H}(\mathbf{r}) - k^2 \mathbf{H}(\mathbf{r}) = -\mathrm{j}\omega\varepsilon_{\mathrm{c}} \mathbf{M}_{\mathrm{i}}(\mathbf{r}) + \nabla \times \mathbf{J}_{\mathrm{i}}(\mathbf{r}) \,, \end{aligned} \tag{1.62}$$

where the dependence on frequency of the fields is assumed and suppressed. From the vector identity $\nabla \times \nabla \times [\cdot] = \nabla\nabla [\cdot] - \nabla^2 [\cdot]$ applied to (1.62), the Maxwell divergence equations, and the equation of continuity, the *vector Helmholtz equations* for the electric and magnetic fields can be derived as

$$\begin{aligned}\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) &= \mathrm{j}\omega\mu\mathbf{J}_\mathrm{i}(\mathbf{r}) - \frac{\nabla\nabla\cdot\mathbf{J}_\mathrm{i}(\mathbf{r})}{\mathrm{j}\omega\varepsilon_\mathrm{c}} + \nabla\times\mathbf{M}_\mathrm{i}(\mathbf{r}) \ , \\ \nabla^2\mathbf{H}(\mathbf{r}) + k^2\mathbf{H}(\mathbf{r}) &= \mathrm{j}\omega\varepsilon_\mathrm{c}\mathbf{M}_\mathrm{i}(\mathbf{r}) - \frac{\nabla\nabla\cdot\mathbf{M}_\mathrm{i}(\mathbf{r})}{\mathrm{j}\omega\mu} - \nabla\times\mathbf{J}_\mathrm{i}(\mathbf{r}) \ .\end{aligned} \tag{1.63}$$

Both the vector wave equations and the vector Helmholtz equations are inhomogeneous differential equations whose forcing terms can be quite complicated functions. Therefore auxiliary quantities (known as *potentials*) are usually introduced to simplify the analysis. Different choices are possible, although the most common are the magnetic and electric (vector and scalar) potentials $\{\mathbf{A}, \mathbf{F}, V, W\}$ in the Lorenz gauge, which are defined as solutions of the following equations:

$$\begin{aligned}\nabla^2\mathbf{A}(\mathbf{r}) + k^2\mathbf{A}(\mathbf{r}) &= -\mu\mathbf{J}_\mathrm{i}(\mathbf{r}) \ , \\ \nabla^2\mathbf{F}(\mathbf{r}) + k^2\mathbf{F}(\mathbf{r}) &= -\varepsilon_\mathrm{c}\mathbf{M}_\mathrm{i}(\mathbf{r}) \ , \\ \nabla^2 V(\mathbf{r}) + k^2 V(\mathbf{r}) &= -\frac{\rho_\mathrm{e}(\mathbf{r})}{\varepsilon_\mathrm{c}} \ , \\ \nabla^2 W(\mathbf{r}) + k^2 W(\mathbf{r}) &= -\frac{\rho_\mathrm{m}(\mathbf{r})}{\mu} \ .\end{aligned} \tag{1.64}$$

The Lorenz gauge implies that $\nabla\cdot\mathbf{A} = -\mathrm{j}\omega\mu\varepsilon_\mathrm{c}V$ and $\nabla\cdot\mathbf{F} = -\mathrm{j}\omega\mu\varepsilon_\mathrm{c}W$. The electric and magnetic fields are expressed in terms of the potentials as

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= -\mathrm{j}\omega\mathbf{A}(\mathbf{r}) - \nabla V(\mathbf{r}) - \frac{1}{\varepsilon_\mathrm{c}}\nabla\times\mathbf{F}(\mathbf{r}) \ , \\ \mathbf{H}(\mathbf{r}) &= -\mathrm{j}\omega\mathbf{F}(\mathbf{r}) - \nabla W(\mathbf{r}) + \frac{1}{\mu}\nabla\times\mathbf{A}(\mathbf{r}) \ .\end{aligned} \tag{1.65}$$

All the equations in (1.64) are again inhomogeneous (vector or scalar) Helmholtz's equations, but now with a simple forcing term. They can be solved by means of the Green function method. Basically the scalar Helmholtz equation can be written as an operator equation of the kind

$$L\left[f(\mathbf{r})\right] = f_0(\mathbf{r}) \ , \tag{1.66}$$

where $L[\cdot] = \nabla^2[\cdot] + k^2[\cdot]$, f is the unknown function, and f_0 is the forcing term (furthermore appropriate boundary conditions must be specified, which define the domain of the operator L). The scalar Green function $G(\mathbf{r}, \mathbf{r}')$ is thus defined as the solution of the equation

$$L\left[G(\mathbf{r}, \mathbf{r}')\right] = -\delta(\mathbf{r} - \mathbf{r}') \ , \tag{1.67}$$

subjected to the same boundary conditions. This way it can be shown that the function $f(\mathbf{r})$ can be expressed through a superposition integral in terms of the

Green function and the forcing term as

$$f(\mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}') f_0(\mathbf{r}') \, \mathrm{d}V' \; . \tag{1.68}$$

In particular, for a scalar Helmholtz equation in free space (subjected to the radiation condition at infinity) it results

$$G(\mathbf{r}, \mathbf{r}') = \frac{\mathrm{e}^{-\mathrm{j}k_0 |\mathbf{r}-\mathbf{r}'|}}{4\pi |\mathbf{r}-\mathbf{r}'|} \; . \tag{1.69}$$

Alternatively, the Hertzian electric and magnetic potentials, $\mathbf{\Pi}_\mathrm{e}$ and $\mathbf{\Pi}_\mathrm{h}$, can be introduced. When $\mathbf{M}_\mathrm{i} = \mathbf{0}$, the electric Hertzian potential $\mathbf{\Pi}_\mathrm{e}$ is defined as

$$\mathbf{\Pi}_\mathrm{e}(\mathbf{r}) = \frac{\mathbf{A}(\mathbf{r})}{\mathrm{j}\omega\mu\varepsilon_\mathrm{c}} \tag{1.70}$$

and obeys the Helmholtz equation

$$\nabla^2 \mathbf{\Pi}_\mathrm{e}(\mathbf{r}) + k^2 \mathbf{\Pi}_\mathrm{e}(\mathbf{r}) = -\frac{\mathbf{J}_\mathrm{i}(\mathbf{r})}{\mathrm{j}\omega\varepsilon_\mathrm{c}} \; . \tag{1.71}$$

In (1.71), $\mathbf{J}_\mathrm{i}$ can represent either an impressed electric current source or an impressed polarization current source $\mathbf{J}_\mathrm{i} = \mathrm{j}\omega\mathbf{P}$. For $\mathbf{J}_\mathrm{i} = \mathbf{0}$, the magnetic Hertzian potential $\mathbf{\Pi}_\mathrm{h}$ is defined as

$$\mathbf{\Pi}_\mathrm{h}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{\mathrm{j}\omega\mu\varepsilon_\mathrm{c}} \; , \tag{1.72}$$

which obeys the Helmholtz equation

$$\nabla^2 \mathbf{\Pi}_\mathrm{h}(\mathbf{r}) + k^2 \mathbf{\Pi}_\mathrm{h}(\mathbf{r}) = -\frac{\mathbf{M}_\mathrm{i}(\mathbf{r})}{\mathrm{j}\omega\mu} \; , \tag{1.73}$$

where $\mathbf{M}_\mathrm{i}$ can represent either an impressed magnetic current source or an impressed magnetization current source $\mathbf{M}_\mathrm{i} = \mathrm{j}\omega\mathbf{M}_\mathrm{m}$.

In free space the vector Helmholtz equations in (1.64) can easily be separated in three scalar Helmholtz's equations, each characterized by the same scalar Green function (1.69). The electric and magnetic fields can be expressed by using the scalar Green function for the potentials in terms of electric and magnetic sources:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathrm{j}\omega\mu \int_V \underline{\mathbf{G}}_\mathrm{e}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_\mathrm{i}(\mathbf{r}') \, \mathrm{d}V' + \int_V \underline{\mathbf{G}}_\mathrm{m}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}_\mathrm{i}(\mathbf{r}') \, \mathrm{d}V' \; , \\ \mathbf{H}(\mathbf{r}) &= \int_V \underline{\mathbf{G}}_\mathrm{m}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_\mathrm{i}(\mathbf{r}') \, \mathrm{d}V' - \mathrm{j}\omega\varepsilon_\mathrm{c} \int_V \underline{\mathbf{G}}_\mathrm{e}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{M}_\mathrm{i}(\mathbf{r}') \, \mathrm{d}V' \; . \end{aligned} \tag{1.74}$$

The free-space *electric dyadic Green function* $\underline{\mathbf{G}}_\mathrm{e}$ is

$$\underline{\mathbf{G}}_\mathrm{e}(\mathbf{r}, \mathbf{r}') = \left(\underline{\mathbf{I}} + \frac{\nabla\nabla}{k_0^2} \right) G(\mathbf{r}, \mathbf{r}') \tag{1.75}$$

and the free-space *magnetic dyadic Green function* $\underline{\mathbf{G}}_{\mathrm{m}}$ is

$$\underline{\mathbf{G}}_{\mathrm{m}}\left(\mathbf{r},\mathbf{r}'\right)=\nabla G\left(\mathbf{r},\mathbf{r}'\right)\times\underline{\mathbf{I}}\ , \tag{1.76}$$

where $\underline{\mathbf{I}}$ is the identity 3×3 tensor.

A similar approach can be adopted in the time domain, assuming, for simplicity, a homogeneous, isotropic, and nondispersive medium, with a real wavenumber $k=\omega/c$.

Taking the curl of both sides of (1.11a) and (1.11b), we have

$$\nabla\times\nabla\times\mathbf{e}\left(\mathbf{r},t\right)=-\mu\ \frac{\partial}{\partial t}\ \nabla\times\mathbf{h}\left(\mathbf{r},t\right)-\nabla\times\mathbf{m}_{\mathrm{i}}\left(\mathbf{r},t\right)\ , \tag{1.77a}$$

$$\nabla\times\nabla\times\mathbf{h}\left(\mathbf{r},t\right)=\sigma\nabla\times\mathbf{e}\left(\mathbf{r},t\right)+\varepsilon\ \frac{\partial}{\partial t}\ \nabla\times\mathbf{e}\left(\mathbf{r},t\right)+\nabla\times\mathbf{j}_{\mathrm{i}}\left(\mathbf{r},t\right)\ . \tag{1.77b}$$

Substituting (1.11b) in (1.77a) and using the vector identity $\nabla\times\nabla\times[\cdot]=\nabla\left(\nabla\cdot[\cdot]\right)-\nabla^2\left[\cdot\right]$, (1.77a) can be rewritten as

$$\nabla\left[\nabla\cdot\mathbf{e}\left(\mathbf{r},t\right)\right]-\nabla^2\mathbf{e}\left(\mathbf{r},t\right)=-\mu\ \frac{\partial}{\partial t}\ \nabla\times\mathbf{h}\left(\mathbf{r},t\right)-\nabla\times\mathbf{m}_{\mathrm{i}}\left(\mathbf{r},t\right)\ . \tag{1.78}$$

By using (1.11c) and rearranging the terms, we thus obtain

$$\nabla^2\mathbf{e}\left(\mathbf{r},t\right)-\frac{1}{c^2}\ \frac{\partial^2\mathbf{e}}{\partial t^2}-\mu\sigma\ \frac{\partial\mathbf{e}}{\partial t}=\nabla\times\mathbf{m}_{\mathrm{i}}\left(\mathbf{r},t\right)+\mu\ \frac{\partial\mathbf{j}_{\mathrm{i}}}{\partial t}+\frac{\nabla q_{\mathrm{e}}}{\varepsilon}\ . \tag{1.79}$$

Equation (1.79) is the vector wave equation for **e**.

Similarly, an uncoupled second-order partial differential equation for **h** is obtained as

$$\nabla^2\mathbf{h}\left(\mathbf{r},t\right)-\frac{1}{c^2}\ \frac{\partial^2\mathbf{h}}{\partial t^2}-\mu\sigma\ \frac{\partial\mathbf{h}}{\partial t}=-\nabla\times\mathbf{j}_{\mathrm{i}}\left(\mathbf{r},t\right)+\sigma\ \frac{\partial\mathbf{m}_{\mathrm{i}}}{\partial t}+\frac{\nabla q_{\mathrm{m}}}{\mu}\ , \tag{1.80}$$

which is the vector wave equation for **h**.

For lossless media ($\sigma=0$) the wave equations (1.79) and (1.80) become

$$\nabla^2\mathbf{e}\left(\mathbf{r},t\right)-\frac{1}{c^2}\ \frac{\partial^2\mathbf{e}}{\partial t^2}=\nabla\times\mathbf{m}_{\mathrm{i}}\left(\mathbf{r},t\right)+\mu\ \frac{\partial\mathbf{j}_{\mathrm{i}}}{\partial t}+\frac{\nabla q_{\mathrm{e}}}{\varepsilon}\ , \tag{1.81a}$$

$$\nabla^2\mathbf{h}\left(\mathbf{r},t\right)-\frac{1}{c^2}\ \frac{\partial^2\mathbf{h}}{\partial t^2}=-\nabla\times\mathbf{j}_{\mathrm{i}}\left(\mathbf{r},t\right)+\frac{\nabla q_{\mathrm{m}}}{\mu}\ . \tag{1.81b}$$

Time-dependent Lorenz potentials $\{\mathbf{a},\mathbf{f},v,w\}$ and Hertzian potentials $\left\{\boldsymbol{\pi}_{\mathrm{e}},\boldsymbol{\pi}_{\mathrm{h}}\right\}$ can also be introduced and they are shown to obey the *time-domain wave* *equations*:

$$\begin{aligned}
&\nabla^2 \mathbf{a}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{a}(\mathbf{r},\mathbf{t}) = -\mu \mathbf{j}_{\mathrm{i}}(\mathbf{r},t) \ , \\
&\nabla^2 \mathbf{f}(\mathbf{r}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{f}(\mathbf{r},t) = -\varepsilon \mathbf{m}_{\mathrm{i}}(\mathbf{r},t) \ , \\
&\nabla^2 v(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} v(\mathbf{r},t) = -\frac{\rho_{\mathrm{e}}(\mathbf{r},t)}{\varepsilon} \ , \\
&\nabla^2 w(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} w(\mathbf{r},t) = -\frac{\rho_{\mathrm{m}}(\mathbf{r},t)}{\mu}
\end{aligned} \tag{1.82}$$

and

$$\begin{aligned}
&\nabla^2 \boldsymbol{\pi}_{\mathrm{e}}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{\pi}_{\mathrm{e}}(\mathbf{r},\mathbf{t}) = -\frac{1}{\varepsilon}\mathbf{p}_{\mathrm{i}}(\mathbf{r},t) \ , \\
&\nabla^2 \boldsymbol{\pi}_{\mathrm{h}}(\mathbf{r}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{\pi}_{\mathrm{h}}(\mathbf{r},t) = -\mathbf{m}_{\mathrm{mi}}(\mathbf{r},t) \ ,
\end{aligned} \tag{1.83}$$

where the impressed polarization $\mathbf{p}_{\mathrm{i}}$ and magnetization $\mathbf{m}_{\mathrm{mi}}$ currents are such that

$$\begin{aligned}
&\mathbf{j}_{\mathrm{i}}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{p}_{\mathrm{i}}(\mathbf{r},t) \ , \\
&\mathbf{m}_{\mathrm{i}}(\mathbf{r},t) = \frac{\partial}{\partial t}\mathbf{m}_{\mathrm{mi}}(\mathbf{r},t) \ .
\end{aligned} \tag{1.84}$$

The Lorenz gauge now implies that $\nabla \cdot \mathbf{a} = -\mu\varepsilon \frac{\partial v}{\partial t}$ and $\nabla \cdot \mathbf{f} = -\mu\varepsilon \frac{\partial w}{\partial t}$. The time-domain electric and magnetic fields are expressed in terms of the Lorenz potentials as

$$\mathbf{e}(\mathbf{r},t) = -\frac{\partial \mathbf{a}}{\partial t} - \nabla v(\mathbf{r},t) - \frac{1}{\varepsilon}\nabla \times \mathbf{f}(\mathbf{r},t) \ , \tag{1.85a}$$

$$\mathbf{h}(\mathbf{r},t) = \frac{1}{\mu}\nabla \times \mathbf{a}(\mathbf{r},t) - \frac{\partial \mathbf{f}}{\partial t} - \nabla w(\mathbf{r},t) \ . \tag{1.85b}$$

On the other hand the expressions of the time-domain electric and magnetic fields in terms of the Hertzian potentials are

$$\mathbf{e}(\mathbf{r},t) = \nabla \times \nabla \times \boldsymbol{\pi}_{\mathrm{e}}(\mathbf{r},t) - \mu\nabla \times \frac{\partial \boldsymbol{\pi}_{\mathrm{h}}}{\partial t} - \frac{\mathbf{p}_{\mathrm{i}}(\mathbf{r},t)}{\varepsilon} \ , \tag{1.86a}$$

$$\mathbf{h}(\mathbf{r},t) = \varepsilon\nabla \times \frac{\partial \boldsymbol{\pi}_{\mathrm{e}}}{\partial t} + \nabla \times \nabla \times \boldsymbol{\pi}_{\mathrm{h}}(\mathbf{r},t) - \mathbf{m}_{\mathrm{mi}}(\mathbf{r},t) \ . \tag{1.86b}$$

The Green function method can be used also in the time domain to solve the inhomogeneous equations (1.82)–(1.83). In free space, the relevant Green function can be found by inverse-Fourier transforming the time-harmonic Green function (1.69), thus obtaining:

$$\mathcal{G}\left(\mathbf{r},t,\mathbf{r}',t'\right) = \frac{\delta\left(t - t' - \frac{|\,\mathbf{r}-\mathbf{r}'\,|}{c_0}\right)}{4\pi\,|\,\mathbf{r}-\mathbf{r}'\,|} \ . \tag{1.87}$$

Note that now the Green function has a distributional character, containing the Dirac delta.

The potentials can then be obtained through superposition integrals. For instance, for the time-domain magnetic potential $\mathbf{a}$ in free space we have

$$\begin{aligned}\mathbf{a}(\mathbf{r},t) &= \mu_0 \int_V \int_{-\infty}^{+\infty} \mathcal{G}\left(\mathbf{r},t,\mathbf{r}',t'\right) \mathbf{j}_\mathrm{i}\left(\mathbf{r}',t'\right)\ \mathrm{d}V'\ \mathrm{d}t' \\ &= \mu_0 \int_V \frac{1}{4\pi \mid \mathbf{r}-\mathbf{r}' \mid} \mathbf{j}_\mathrm{i}\left(\mathbf{r}',t_\mathrm{ret}\right)\ \mathrm{d}V'\ ,\end{aligned} \tag{1.88}$$

where $t_\mathrm{ret} = t - \mid \mathbf{r}-\mathbf{r}' \mid / c_0$ is the *retarded time*. Similar expressions hold for all the time-domain potentials, which are thus known as *retarded potentials*. The electric and magnetic fields can finally be calculated using (1.85) or (1.86) with $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$. The involved calculations are algebraically cumbersome and the resulting expressions, which can be cast in various alternative forms, are quite elaborate and thus not reported here; for details, see, e.g., [28]. The expressions for impulsive sources will be reported later in Chapter 6.

1.10 Basic Shielding Mechanisms

EM shielding may be pursued by any of the following main strategies, or by a combination of them:

- Interposition of a "barrier" between the source and the area (volume) where the EM field has to be reduced.
- Introduction of a mean capable of diverging the EM field from the area of interest.
- Introduction of an additional source whose effect is the reduction of the EM field levels in the prescribed area with respect to a situation involving the original source or source system.

The choice of the strategy is made according to the characteristics of the source (electromagnetic or physical) and to the characteristics of the area to be protected. Of course, several other factors such as costs or insensitivity to source variations must be accounted for as well.

The interposition of a barrier is particularly effective in reducing the EM field levels when the shield material is highly conducting or when it is characterized by constitutive parameters such that the level of attenuation of the field propagating through the shield is high. A simple situation, which will be studied in detail in Chapter 4, may help to clarify this point. A uniform plane wave propagating in a medium with permeability μ, permittivity ε, and conductivity σ, has a propagation constant expressed by $\gamma = \sqrt{\mathrm{j}\omega\mu\left(\sigma + \mathrm{j}\omega\varepsilon\right)} = \alpha + \mathrm{j}\beta$, where α is the attenuation constant and β the phase constant. Any combination of the values μ, ε, and σ giving rise to a high value of the attenuation constant α is suitable for shielding purposes.

An EM field can be diverted by means of an alternative path, not necessarily enclosing the area to be shielded. Such a path may offer better propagation characteristics to the electric field (by means of highly conducting materials), the electric induction (by means of high-permittivity materials), the magnetic induction (high-permeability materials).

Generally speaking, at relatively low frequencies, e.g., below tens of MHz the dominant coupling mechanism is related to pass-through cables and connectors. Above such an approximate threshold the propagation of EM waves through apertures and shield discontinuities becomes more and more important.

Discontinuities treatment is a major issue in shielding theory and practice. Chapters 10 and 11 provide a sound approach and practical solutions, respectively, to this very special key point around which the whole shielding problem revolves.

1.11 Source Inside or Outside the Shielding Structure and Reciprocity

In general the techniques for introducing a shield that excludes EM interference from a certain region are identical to those used for confining an EM field in the neighborhood of the source. This is an immediate consequence of the reciprocity theorem as formulated in (1.43) or (1.44). The simplest shield consists in an infinite planar screen that divides two regions of space, region 1 and region 2. When a source is placed in $\mathbf{r}_1$ (in region 1), it produces a certain field at $\mathbf{r}_2$ (in region 2). If the assumptions of the reciprocity theorem are fulfilled, it is easy to see that such a field is the same as that produced at $\mathbf{r}_1$ by the *same* source placed at $\mathbf{r}_2$. There is no difficulty in generalizing these considerations to the more involved case of a source in the presence of an enclosure. In this case region 1 is the interior of the enclosure while region 2 is the external region: thus the field radiated at $\mathbf{r}_2$ by a source placed in $\mathbf{r}_1$ is the same as that radiated at $\mathbf{r}_1$ by the same source placed at $\mathbf{r}_2$. This also means, for example, that the shielding performance of a shielding structure can be calculated (or measured) by placing either the source outside the structure and determining the field inside it or the source inside the structure and determining the field outside it.

Although the previous considerations are quite simple, the basic assumptions must be clear. First of all, the reciprocity theorem (or some of its modifications that account also for nonreciprocal media [17]) must hold: this implies, for example, that the above described conclusions for linear media are not valid in the presence of nonlinear media. The two considered situations (with source inside and outside the shielding structure) must be identical. Finally, from a practical point of view, since a sensor is always used to measure the field at a point, its interactions with the rest of the system (and, in particular, with the field source) must be negligible (at least, within the accuracy limits of measurements).

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