CHAPTER 1

Equations and Inequalities

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Golf courses usually charge both greens fees (cost of playing the course) and cart fees (cost of renting a golf cart). Two friends who enjoy playing golf decide to investigate becoming members at a golf course.

The course they enjoy playing the most charges \$40 for greens fees and \$15 for cart rental (per person), so it currently costs each of them \$55 every time they play. The membership offered at that course costs \$160 per month with no greens fees, but there is still the per person cart rental fee.

How many times a month would they have to play golf in order for the membership option to be the better deal?* This is just one example of how the real world can be modeled with equations and inequalities.

EQUATIONS AND INEQUALITIES

*See Section 1.5, Exercises 115 and 116.

LEARNING OBJECTIVES

- Solve linear equations.
- Solve application problems involving linear equations.
- Solve quadratic equations.
- Solve rational, polynomial, and radical equations.
- Solve linear inequalities.
- Solve polynomial and rational inequalities.
- Solve absolute value equations and inequalities.

In This Chapter

You will solve linear and quadratic equations. You will then solve more complicated equations (polynomial, rational, radical, and absolute value) by first transforming them into linear or quadratic equations. Then you will solve linear, quadratic, polynomial, rational, and absolute value inequalities. Throughout this chapter you will solve applications of equations and inequalities.

1.1 Linear Equations

SKILLS OBJECTIVES

- Solve linear equations in one variable.
- Solve rational equations that are reducible to linear equations.

CONCEPTUAL OBJECTIVES

- Understand the definition of a linear equation in one variable.
- Eliminate values that result in a denominator being equal to zero.

1.1.1 Solving Linear Equations in One Variable

1.1.1 Skill Solve linear equations in one variable.

1.1.1 Conceptual Understand the definition of a linear equation in one variable.

An **algebraic expression** (see Chapter 0) consists of one or more terms that are combined through basic operations such as addition, subtraction, multiplication, or division; for example:

$$3x + 2 \qquad 5 - 2y \qquad x + y$$

An **equation** is a statement that says two expressions are equal. For example, the following are all equations in one variable, x:

$$x + 7 = 11 \qquad x^2 = 9 \qquad 7 - 3x = 2 - 3x \qquad 4x + 7 = x + 2 + 3x + 5$$

To **solve** an equation means to find all the values of x that make the equation true. These values are called **solutions**, or **roots**, of the equation. The first of these statements shown above, $x + 7 = 11$, is true when $x = 4$ and false for any other values of x. We say that $x = 4$ is the solution to the equation. Sometimes an equation can have more than one solution, as in $x^2 = 9$. In this case, there are actually two values of x that make this equation true, $x = -3$ and $x = 3$.

We say the **solution set** of this equation is $\{-3, 3\}$. In the third equation, $7 - 3x = 2 - 3x$, no values of x make the statement true. Therefore, we say this equation has **no solution**. And the fourth equation, $4x + 7 = x + 2 + 3x + 5$, is true for any values of x. An equation that is true for any value of the variable x is called an **identity**. In this case, we say the solution set is the **set of all real numbers**.

Two equations that have the same solution set are called **equivalent equations**. For example,

$$3x + 7 = 13 \qquad 3x = 6 \qquad x = 2$$

are all equivalent equations because each of them has the solution set $\{2\}$. Note that $x^2 = 4$ is not equivalent to these three equations because it has the solution set $\{-2, 2\}$.

When solving equations, it helps to find a simpler equivalent equation in which the variable is isolated (alone). The following table summarizes the procedures for generating equivalent equations.

Generating Equivalent Equations

Original Equation	Description	Equivalent Equation
$3(x - 6) = 6x - x$	• Eliminate parentheses. • Combine like terms on one or both sides of an equation.	$3x - 18 = 5x$
$7x + 8 = 29$	Add (or subtract) the same quantity to (from) *both* sides of an equation. $7x + 8 - \mathbf{8} = 29 - \mathbf{8}$	$7x = 21$
$5x = 15$	Multiply (or divide) both sides of an equation by the same nonzero quantity: $\frac{5x}{5} = \frac{15}{5}$.	$x = 3$
$-7 = x$	Interchange the two sides of the equation.	$x = -7$

You probably already know how to solve simple linear equations. Solving a linear equation in one variable is done by finding an equivalent equation. In generating an equivalent equation, remember that whatever operation is performed on one side of an equation must also be performed on the other side of the equation.

EXAMPLE 1 | Solving a Linear Equation

Solve the equation $3x + 4 = 16$.

Solution

Subtract 4 from both sides of the equation.

$$\begin{aligned} 3x + 4 &= 16 \\ -4 \quad &\ \ -4 \\ \hline 3x \quad &= 12 \end{aligned}$$

Divide both sides by 3.

$$\frac{3x}{3} = \frac{12}{3}$$

The solution is $x = 4$.

$$\boxed{x = 4}$$

The solution set is $\{4\}$.

Your Turn Solve the equation $2x + 3 = 9$.

Answer

The solution is $x = 3$. The solution set is $\{3\}$.

Example 1 illustrates solving linear equations in one variable. What is a linear equation in one variable?

Linear Equation

A **linear equation in one variable**, x, can be written in the form

$$ax + b = 0$$

where a and b are real numbers and $a \neq 0$.

What makes this equation linear is that x is raised to the first power. We can also classify a linear equation as a **first-degree** equation.

Equation	Degree	General Name
$x - 7 = 0$	First	Linear
$x^2 - 6x - 9 = 0$	Second	Quadratic
$x^3 + 3x^2 - 8 = 0$	Third	Cubic

Video **EXAMPLE 2** | **Solving a Linear Equation**

Solve the equation $5x - (7x - 4) - 2 = 5 - (3x + 2)$.

Solution

Eliminate the parentheses. $5x - (7x - 4) - 2 = 5 - (3x + 2)$

Don't forget to distribute the negative sign through *both* terms inside the parentheses. $5x - 7x + 4 - 2 = 5 - 3x - 2$

Combine x terms on the left, constants on the right. Add $3x$ to both sides.

$$-2x + 2 = 3 - 3x$$
$$+3x \qquad +3x$$
$$x + 2 = 3$$

Subtract 2 from both sides.

$$-2 \quad -2$$
$$x = 1$$

Check to verify that $x = 1$ is a solution to the original equation.

$$5 \cdot 1 - (7 \cdot 1 - 4) - 2 = 5 - (3 \cdot 1 + 2)$$
$$5 - (7 - 4) - 2 = 5 - (3 + 2)$$
$$5 - (3) - 2 = 5 - (5)$$
$$0 = 0$$

Since the solution $x = 1$ makes the equation true, the solution set is $\{1\}$.

Your Turn Solve the equation $4(x - 1) - 2 = x - 3(x - 2)$.

Answer

The solution is $x = 2$. The solution set is $\{2\}$.

STUDY TIP

Prime Factors

$$2 = 2$$
$$6 = 2 \cdot 3$$
$$5 = \qquad \cdot 5$$
$$\text{LCD} = 2 \cdot 3 \cdot 5 = 30$$

To solve a linear equation involving fractions, find the least common denominator (LCD) of all terms and multiply both sides of the equation by the LCD. We will first review how to find the LCD.

To add the fractions $\frac{1}{2} + \frac{1}{6} + \frac{2}{5}$, we must first find a common denominator. Some people are taught to find the lowest number that 2, 6, and 5 all divide evenly into. Others prefer a more systematic approach in terms of prime factors.

Concept Check

Which one of these is a linear equation in one variable?

(A) $3x + 2 = 11$ (B) $x^2 = 9$ (C) $y = x + 1$

Answer: (A) $3x + 2 = 11$ (B) is nonlinear in one variable (C) is linear in two variables

EXAMPLE 3 | Solving a Linear Equation Involving Fractions

Solve the equation $\frac{1}{2}p - 5 = \frac{3}{4}p$.

Solution

Write the equation. $\frac{1}{2}p - 5 = \frac{3}{4}p$

Multiply each term in the equation by the LCD, 4. $(4)\frac{1}{2}p - (4)5 = (4)\frac{3}{4}p$

The result is a linear equation with no fractions. $2p - 20 = 3p$

Subtract $2p$ from both sides.

$$\begin{array}{r} 2p - 20 = 3p \\ \underline{-2p \qquad -2p} \\ -20 = p \\ \boxed{p = -20} \end{array}$$

Since $p = -20$ satisfies the original equation, the solution set is $\{-20\}$.

Your Turn Solve the equation $\frac{1}{4}m = \frac{1}{12}m - 3$.

Answer

The solution is $m = -18$. The solution set is $\{-18\}$.

Solving a Linear Equation in One Variable

Step	Description	Example
1	Simplify the algebraic expressions on both sides of the equation.	$-3(x - 2) + 5 = 7(x - 4) - 1$ $-3x + 6 + 5 = 7x - 28 - 1$ $-3x + 11 = 7x - 29$
2	Gather all variable terms on one side of the equation and all constant terms on the other side.	$-3x + 11 = 7x - 29$ $\underline{+3x \qquad +3x}$ $11 = 10x - 29$ $\underline{+29 \qquad +29}$ $40 = 10x$
3	Isolate the variable.	$10x = 40$ $\boxed{x = 4}$

1.1.2 Solving Rational Equations That Are Reducible to Linear Equations

1.1.2 Skill Solve rational equations that are reducible to linear equations.

1.1.2 Conceptual Eliminate values that result in a denominator being equal to zero.

A **rational equation** is an equation that contains one or more rational expressions (Chapter 0). Some rational equations can be transformed into linear equations that you can then solve, but as you will see momentarily, you must be certain that the solution to the linear equation also satisfies the original rational equation.

Video EXAMPLE 4 | Solving a Rational Equation That Can Be Reduced to a Linear Equation

Solve the equation $\frac{2}{3x} + \frac{1}{2} = \frac{4}{x} + \frac{4}{3}$.

Solution

State the excluded values (those that make any denominator equal 0). $\quad \frac{2}{3x} + \frac{1}{2} = \frac{4}{x} + \frac{4}{3} \quad x \neq 0$

Eliminate fractions by multiplying *each term* by the LCD, $6x$. $\quad 6x\left(\frac{2}{3x}\right) + 6x\left(\frac{1}{2}\right) = 6x\left(\frac{4}{x}\right) + 6x\left(\frac{4}{3}\right)$

Simplify both sides. $\quad 4 + 3x = 24 + 8x$

Subtract 4. $\quad -4 \qquad -4$

$3x = 20 + 8x$

Subtract $8x$. $\quad -8x \qquad -8x$

$-5x = 20$

Divide by -5. $\quad \boxed{x = -4}$

Since $x = -4$ satisfies the original equation, the solution set is $\{-4\}$.

Your Turn Solve the equation $\frac{3}{y} + 2 = \frac{7}{2y}$.

Answer

The solution is $y = \frac{1}{4}$. The solution set is $\left\{\frac{1}{4}\right\}$.

STUDY TIP

Since dividing by 0 is not defined, we exclude values of the variable that correspond to a denominator equaling 0.

Extraneous solutions are solutions that satisfy a transformed equation but do not satisfy the original equation. It is important to first state any values of the variable that must be eliminated based on the original rational equation. Once the rational equation is transformed to a linear equation and solved, remove any excluded values of the variable.

EXAMPLE 5 | Solving Rational Equations That Can Be Reduced to Linear Equations

Solve the equation $\frac{3x}{x-1} + 2 = \frac{3}{x-1}$.

Solution

State the excluded values (those that make any denominator equal 0). $\quad \frac{3x}{x-1} + 2 = \frac{3}{x-1} \quad \boxed{x \neq 1}$

Eliminate the fractions by multiplying each term by the LCD, $x - 1$. $\quad \frac{3x}{x-1} \cdot (x-1) + 2 \cdot (x-1) = \frac{3}{x-1} \cdot (x-1)$

Simplify. $\quad \frac{3x}{\cancel{x-1}} \cdot \cancel{(x-1)} + 2 \cdot (x-1) = \frac{3}{\cancel{x-1}} \cdot \cancel{(x-1)}$

$3x + 2(x-1) = 3$

Distribute the 2. $3x + 2x - 2 = 3$

Combine x terms on the left. $5x - 2 = 3$

Add 2 to both sides. $5x = 5$

Divide both sides by 5. $x = 1$

It may seem that $x = 1$ is the solution. However, the original equation had the restriction $x \neq 1$. Therefore, $x = 1$ is an extraneous solution and must be eliminated as a possible solution.

Thus, the equation $\frac{3x}{x-1} + 2 = \frac{3}{x-1}$ has no solution.

Your Turn Solve the equation $\frac{2x}{x-2} - 3 = \frac{4}{x-2}$.

Answer

no solution

STUDY TIP

When a variable is in the denominator of a fraction, the LCD will contain the variable. This sometimes results in an extraneous solution.

We have reviewed finding the least common denominator (LCD) for real numbers. Now let us consider finding the LCD for rational equations that have different denominators. We multiply the denominators in order to get a common denominator.

Rational expression: $\frac{1}{x} + \frac{2}{x-1}$ LCD: $x(x-1)$

In order to find a *least* common denominator, it is useful to first factor the denominators to identify common multiples.

Rational equation: $\frac{1}{3x-3} + \frac{1}{2x-2} = \frac{1}{x^2-x}$

Factor the denominators: $\frac{1}{3(x-1)} + \frac{1}{2(x-1)} = \frac{1}{x(x-1)}$

LCD: $6x(x-1)$

Video EXAMPLE 6 | Solving Rational Equations

Solve the equation $\frac{1}{3x+18} - \frac{1}{2x+12} = \frac{1}{x^2+6x}$.

Solution

Factor the denominators. $\frac{1}{3(x+6)} - \frac{1}{2(x+6)} = \frac{1}{x(x+6)}$

State the excluded values. $x \neq 0, -6$

Multiply the equation by the LCD, $6x(x + 6)$.

$$6x(x+6) \cdot \frac{1}{3(x+6)} - 6x(x+6) \cdot \frac{1}{2(x+6)} = 6x(x+6) \cdot \frac{1}{x(x+6)}$$

Divide out the common factors.

$$6x\cancel{(x+6)} \cdot \frac{1}{3\cancel{(x+6)}} - 6x\cancel{(x+6)} \cdot \frac{1}{2\cancel{(x+6)}} = 6x\cancel{(x+6)} \cdot \frac{1}{x\cancel{(x+6)}}$$

Simplify. $2x - 3x = 6$

Solve the linear equation. $x = -6$

Since one of the excluded values is $x \neq -6$, we say that $x = -6$ is an extraneous solution. Therefore, this rational equation has no solution.

Your Turn Solve the equation $\frac{2}{x} + \frac{1}{x+1} = -\frac{1}{x(x+1)}$.

Answer

no solution

EXAMPLE 7 | Solving Rational Equations

Solve the equation $\frac{2}{(x-3)} = \frac{-3}{(2-x)}$.

Solution

What values make *either* denominator equal to zero? The values $x = 2$ and $x = 3$ must be excluded from possible solutions to the equation.

$$\frac{2}{(x-3)} = \frac{-3}{(2-x)} \qquad \boxed{x \neq 2, x \neq 3}$$

Multiply the equation by the LCD, $(x-3)(2-x)$.

$$\frac{2}{(x-3)}(x-3)(2-x) = \frac{-3}{(2-x)}(x-3)(2-x)$$

Divide out the common factors.

$$2(2-x) = -3(x-3)$$

Eliminate the parentheses.

$$4 - 2x = -3x + 9$$

Collect x terms on the left, constants on the right.

$$\boxed{x = 5}$$

Since $x = 5$ satisfies the original equation, the solution set is $\{5\}$.

Your Turn Solve the equation $\frac{-4}{x+8} = \frac{3}{x-6}$.

Answer

The solution is $x = 0$. The solution set is $\{0\}$.

Concept Check

Which values of x must be eliminated as potential solutions of the rational equation $\frac{1}{x-a} = \frac{1}{x+b}$?

Answer: $x = a$ and $x = -b$

EXAMPLE 8 | Automotive Service

A car dealership charges for parts and an hourly rate for labor. If parts cost \$273, labor is \$53 per hour, and the total bill is \$458.50, how many hours did the dealership spend working on your car?

Solution

Let x equal the number of hours the dealership worked on your car.

Write the cost equation.

$$\underbrace{53x}_{\text{labor}} + \underbrace{273}_{\text{parts}} = \underbrace{458.50}_{\text{total cost}}$$

Subtract 273 from both sides of the equation.

$$53x = 185.50$$

Divide both sides of the equation by 53.

$$\boxed{x = 3.5}$$

The dealership charged for 3.5 hours of labor.

EXAMPLE 9 | Grades

Dante currently has the following three test scores: 82, 79, and 90. If the score on the final exam is worth two test scores and his goal is to earn an 85 for his class average, what score on the final exam does Dante need to achieve his course goal?

Solution

Let x equal final exam grade.

Write the equation that determines the course grade.

$$\frac{\overbrace{82 + 79 + 90 +}^{\text{scores 1, 2, and 3}} \overbrace{2x}^{\text{final is worth two test scores}}}{\underbrace{5}_{\text{total of five test scores}}} = \underbrace{85}_{\text{average}}$$

Simplify the numerator. $\frac{251 + 2x}{5} = 85$

Multiply the equation by 5 (or cross multiply). $251 + 2x = 425$

Solve the linear equation. $\boxed{x = 87}$

Dante needs to score at least an 87 on the final exam.

Section 1.1 Summary

Linear equations, $ax + b = 0$, are solved by:

1. Simplifying the algebraic expressions on both sides of the equation.
2. Gathering all variable terms on one side of the equation and all constant terms on the other side.
3. Isolating the variable.

Rational equations are solved by:

1. Determining any excluded values (denominator equals 0).
2. Multiplying the equation by the LCD.
3. Solving the resulting equation.
4. Eliminating any extraneous solutions.

Section 1.1 Exercises

Skills

In Exercises 1–36, solve for the indicated variable.

1. $5x = 35$
2. $4t = 32$
3. $-3 + n = 12$
4. $4 = -5 + y$
5. $24 = -3x$
6. $-50 = -5t$
7. $\frac{1}{5}n = 3$
8. $6 = \frac{1}{3}p$
9. $3x - 5 = 7$
10. $4p + 5 = 9$
11. $9m - 7 = 11$
12. $2x + 4 = 5$
13. $5t + 11 = 18$
14. $7x + 4 = 21 + 24x$
15. $3x - 5 = 25 + 6x$
16. $5x + 10 = 25 + 2x$
17. $20n - 30 = 20 - 5n$
18. $14c + 15 = 43 + 7c$
19. $4(x - 3) = 2(x + 6)$
20. $5(2y - 1) = 2(4y - 3)$
21. $-3(4t - 5) = 5(6 - 2t)$
22. $2(3n + 4) = -(n + 2)$
23. $2(x - 1) + 3 = x - 3(x + 1)$
24. $4(y + 6) - 8 = 2y - 4(y + 2)$
25. $5p + 6(p + 7) = 3(p + 2)$
26. $3(z + 5) - 5 = 4z + 7(z - 2)$
27. $7x - (2x + 3) = x - 2$
28. $3x - (4x + 2) = x - 5$
29. $2 - (4x + 1) = 3 - (2x - 1)$
30. $5 - (2x - 3) = 7 - (3x + 5)$
31. $2a - 9(a + 6) = 6(a + 3) - 4a$
32. $25 - [2 + 5y - 3(y + 2)] = -3(2y - 5) - [5(y - 1) - 3y + 3]$
33. $32 - [4 + 6x - 5(x + 4)] = 4(3x + 4) - [6(3x - 4) + 7 - 4x]$
34. $12 - [3 + 4m - 6(3m - 2)] = -7(2m - 8) - 3[(m - 2) + 3m - 5]$
35. $20 - 4[c - 3 - 6(2c + 3)] = 5(3c - 2) - [2(7c - 8) - 4c + 7]$
36. $46 - [7 - 8y + 9(6y - 2)] = -7(4y - 7) - 2[6(2y - 3) - 4 + 6y]$

Exercises 37–48 involve fractions. Clear the fractions by first multiplying by the least common denominator, and then solve the resulting linear equation.

37. $\frac{1}{5}m = \frac{1}{60}m + 1$

38. $\frac{1}{12}z = \frac{1}{24}z + 3$

39. $\frac{x}{7} = \frac{2x}{63} + 4$

40. $\frac{a}{11} = \frac{a}{22} + 9$

41. $\frac{1}{3}p = 3 - \frac{1}{24}p$

42. $\frac{3x}{5} - x = \frac{x}{10} - \frac{5}{2}$

43. $\frac{5y}{3} - 2y = \frac{2y}{84} + \frac{5}{7}$

44. $2m - \frac{5m}{8} = \frac{3m}{72} + \frac{4}{3}$

45. $p + \frac{p}{4} = \frac{5}{2}$

46. $\frac{c}{4} - 2c = \frac{5}{4} - \frac{c}{2}$

47. $\frac{x-3}{3} - \frac{x-4}{2} = 1 - \frac{x-6}{6}$

48. $1 - \frac{x-5}{3} = \frac{x+2}{5} - \frac{6x-1}{15}$

In Exercises 49–70, specify any values that must be excluded from the solution set and then solve the equation.

49. $\frac{4}{y} - 5 = \frac{5}{2y}$

50. $\frac{4}{x} + 10 = \frac{2}{3x}$

51. $7 - \frac{1}{6x} = \frac{10}{3x}$

52. $\frac{7}{6t} = 2 + \frac{5}{3t}$

53. $\frac{2}{a} - 4 = \frac{4}{3a}$

54. $\frac{4}{x} - 2 = \frac{5}{2x}$

55. $\frac{x}{x-2} + 5 = \frac{2}{x-2}$

56. $\frac{n}{n-5} + 2 = \frac{n}{n-5}$

57. $\frac{2p}{p-1} = 3 + \frac{2}{p-1}$

58. $\frac{4t}{t+2} = 3 - \frac{8}{t+2}$

59. $\frac{3x}{x+2} - 4 = \frac{2}{x+2}$

60. $\frac{5y}{2y-1} - 3 = \frac{12}{2y-1}$

61. $\frac{1}{n} + \frac{1}{n+1} = \frac{-1}{n(n+1)}$

62. $\frac{1}{x} + \frac{1}{x-1} = \frac{1}{x(x-1)}$

63. $\frac{3}{a} - \frac{2}{a+3} = \frac{9}{a(a+3)}$

64. $\frac{1}{c-2} + \frac{1}{c} = \frac{2}{c(c-2)}$

65. $\frac{n-5}{6n-6} = \frac{1}{9} - \frac{n-3}{4n-4}$

66. $\frac{5}{m} + \frac{3}{m-2} = \frac{6}{m(m-2)}$

67. $\frac{2}{5x+1} = \frac{1}{2x-1}$

68. $\frac{3}{4n-1} = \frac{2}{2n-5}$

69. $\frac{t-1}{1-t} = \frac{3}{2}$

70. $\frac{2-x}{x-2} = \frac{3}{4}$

Applications

71. **Temperature.** To calculate temperature in degrees Fahrenheit, we use the formula $F = \frac{9}{5}C + 32$, where F is degrees Fahrenheit and C is degrees Celsius. Find the formula to convert from Fahrenheit to Celsius.

72. **Geometry.** The perimeter P of a rectangle is related to the length L and width W of the rectangle through the equation $P = 2L + 2W$. Determine the width in terms of the perimeter and length.

73. **Costs: Cell Phones.** Michael is studying abroad in Europe. Before he gets a local number, he uses his old cell phone with an international travel pass. This pass consists of a flat rate of \$3 per day plus a charge of 15 cents per text sent. How many texts did he send if his bill for three days was \$20.25?

74. **Costs: Rental Car.** Becky rented a car on her Ft. Lauderdale vacation. The car was \$25 a day plus 10 cents per mile. She kept the car for 5 days and her bill was \$185. How many miles did she drive the car?

75. **Sales: Income.** For a summer job, Dwayne works as an independent contractor inside of Costco selling solar power systems. He will be paid \$72 per day plus \$50 per system he sells. If he works for 25 days and makes \$2550, how many solar power systems did he sell?

76. **Business.** Jovan is selling essential oils. She submits one order for four people which includes \$20 bottles as well as a single \$15.95 flat rate for shipping for the order. If the order total was \$575.95, and each person ordered the same number of bottles, how many bottles did each person order?

77. **Business.** The operating costs for a local business per year are a fixed amount of \$15,000 and \$2,500 per day.

a. Find C that represents operating costs for the company, which depend on the number of days open, x.

b. If the business accrues \$5,515,000 in annual operating costs, how many days did the business operate during the year?

78. **Business.** Negotiated contracts for a technical support provider produce monthly revenue of \$5000 and \$0.75 per minute per phone call.

a. Find R that represents the revenue for the technical support provider, which depends on the number of minutes of phone calls x.

b. In one month the provider received \$98,750 in revenue. How many minutes of technical support were provided?

In Exercises 79 and 80 refer to the following:

Medications are often packaged in liquid form (known as a suspension) so that a precise dose of a drug is delivered within a volume of inert liquid—for example, 250 milligrams amoxicillin in 5 milliliters of a liquid suspension. If a patient is prescribed a dose of a drug, medical personnel must compute the volume of the liquid with a known concentration to administer. The formula

$$a = \frac{d}{c}$$

defines the relationship between the dose of the drug prescribed d, the concentration of the liquid suspension c, and the amount of the liquid administered a.

79. **Medicine.** A physician has ordered a 600-milligram dose of amoxicillin. The pharmacy has a suspension of amoxicillin with a concentration of 125 milligrams per 5 milliliters. How much liquid suspension must be administered to the patient?

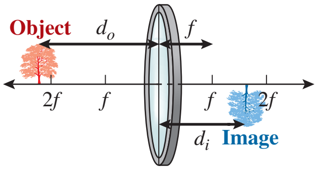
80. **Medicine.** A physician has ordered a 600-milligram dose of carbamazepine. The pharmacy has a suspension of carbamazepine with a concentration of 100 milligrams per 5 milliliters. How much liquid suspension must be administered to the patient?

81. Speed of Light. The frequency f of an optical signal in hertz (Hz) is related to the wavelength λ in meters (m) of a laser through the equation $f = \frac{c}{l}$, where c is the speed of light in a vacuum and is typically taken to be $c = 3.0 \times 10^8$ meters per second (m/s). What values must be eliminated from the wavelengths?

82. Optics. For an object placed near a lens, an image forms on the other side of the lens at a distinct position determined by the distance from the lens to the object. The position of the image is found using the thin lens equation:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$$

where d_o is the distance from the object to the lens, d_i is the distance from the lens to the image, and f is the *focal length* of the lens. Solve for the object distance d_o in terms of the focal length and image distance.



Catch the Mistake

In Exercises 83–86, explain the mistake that is made.

83. Solve the equation $4x + 3 = 6x - 7$.

Solution

Subtract $4x$ and add 7 to the equation. $3 = 6x$

Divide by 3. $x = 2$

This is incorrect. What mistake was made?

84. Solve the equation $3(x + 1) + 2 = x - 3(x - 1)$.

Solution

$$3x + 3 + 2 = x - 3x - 3$$
$$3x + 5 = -2x - 3$$
$$5x = -8$$
$$x = -\frac{8}{5}$$

This is incorrect. What mistake was made?

85. Solve the equation $\frac{4}{p} - 3 = \frac{2}{5p}$.

Solution

$$(p - 3)2 = 4(5p)$$

Cross multiply. $2p - 6 = 20p$

$$-6 = 18p$$
$$p = -\frac{6}{18}$$
$$p = -\frac{1}{3}$$

This is incorrect. What mistake was made?

86. Solve the equation $\frac{1}{x} + \frac{1}{x - 1} = \frac{1}{x(x - 1)}$.

Solution

Multiply by the LCD, $x(x - 1)$.

$$\frac{x(x-1)}{x} + \frac{x(x-1)}{x-1} = \frac{x(x-1)}{x(x-1)}$$

Simplify. $(x - 1) + x = 1$

$$x - 1 + x = 1$$
$$2x = 2$$
$$x = 1$$

This is incorrect. What mistake was made?

Conceptual

In Exercises 87–90, determine whether each of the statements is true or false.

87. The solution to the equation $x = \frac{1}{1/x}$ is the set of all real numbers.

88. The solution to the equation

$$\frac{1}{(x-1)(x+2)} = \frac{1}{x^2 + x - 2}$$

is the set of all real numbers.

89. $x = -1$ is a solution to the equation $\frac{x^2 - 1}{x - 1} = x + 1$.

90. $x = 1$ is a solution to the equation $\frac{x^2 - 1}{x - 1} = x + 1$.

91. Solve for x, given that a, b, and c are real numbers and $a \neq 0$:

$$ax + b = c$$

92. Solve for x, given that a, b, and c are real numbers and $c \neq 0$:

$$\frac{a}{x} - \frac{b}{x} = c$$

Challenge

93. Solve the equation for x: $\frac{b + c}{x + a} = \frac{b - c}{x - a}$. Are there any restrictions given that $a \neq 0$, $x \neq 0$?

94. Solve the equation for y: $\frac{1}{y - a} + \frac{1}{y + a} = \frac{2}{y - 1}$. Does y have any restrictions?

95. Solve for x: $\dfrac{1-\dfrac{1}{x}}{1+\dfrac{1}{x}}=1$.

96. Solve for t: $\dfrac{\dfrac{t+1}{t}}{\dfrac{1}{t-1}}=1$.

97. Solve the equation for x in terms of y:

$$y=\frac{a}{\dfrac{1+b}{x+c}}$$

98. Find the number a for which $y=2$ is a solution of the equation $y-a=y+5-3ay$.

Technology

In Exercises 99–106, graph the function represented by each side of the equation in the same viewing rectangle and solve for *x*.

99. $3(x+2)-5x=3x-4$

100. $-5(x-1)-7=10-9x$

101. $2x+6=4x-2x+8-2$

102. $10-20x=10x-30x+20-10$

103. $\dfrac{x(x-1)}{x^2}=1$

104. $\dfrac{2x(x+3)}{x^2}=2$

105. $0.035x+0.029(8706-x)=285.03$

106. $\dfrac{1}{0.75x}-\dfrac{0.45}{x}=\dfrac{1}{9}$

1.2 Applications Involving Linear Equations

SKILLS OBJECTIVES

- Solve application problems involving common formulas.
- Solve geometry problems using linear equations.
- Solve simple interest problems.
- Solve mixture problems.
- Solve distance–rate–time problems.

CONCEPTUAL OBJECTIVES

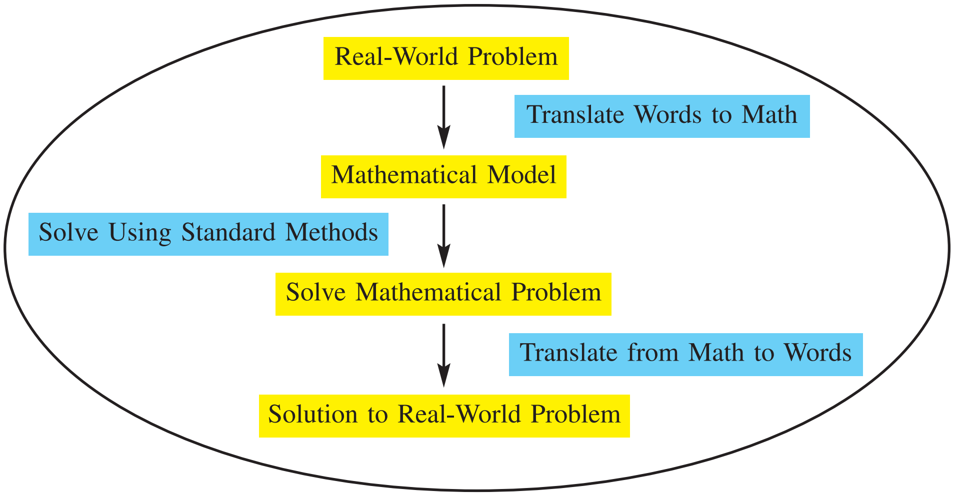
- Understand the mathematical modeling process.
- Estimate a practical solution (guess) prior to solving a problem and then check solution at the end.
- Use intuition to confirm answers in multiple investment problems.
- Use intuition to confirm answers to mixture problems.
- Estimate distance–rate–time problem solutions prior to solving and then confirm with a check.

1.2.1 Solving Application Problems Using Mathematical Models

1.2.1 Skill Solve application problems involving common formulas.

1.2.1 Conceptual Understand the mathematical modeling process.

In this section, we will use algebra to solve problems that occur in our day-to-day lives. You typically will read the problem in words, develop a mathematical model (equation) for the problem, solve the equation, and write the answer in words.



You will have to come up with a unique formula to solve each kind of word problem, but there is a universal *procedure* for approaching all word problems.

Procedure for Solving Word Problems

STEP 1 **Identify the question.** Read the problem *one* time and note what you are asked to find.

STEP 2 **Make notes.** Read until you can note something (an amount, a picture, anything). Continue reading and making notes until you have read the problem a second* time.

STEP 3 **Assign a variable** to whatever is being asked for (if there are two choices, then let it be the smaller of the two).

STEP 4 **Set up an equation.**

STEP 5 **Solve the equation.**

STEP 6 **Check the solution.** Run the solution past the "common sense department" using estimation.

*Step 2 often requires multiple readings of the problem.

EXAMPLE 1 | How Long Was the Trip?

During a camping trip in North Bay, Ontario, a couple went one-third of the way by boat, 10 miles by foot, and one-sixth of the way by horse. How long was the trip?

Solution

STEP 1 **Identify the question.**

How many miles was the trip?

STEP 2 **Make notes.**

Read	Write
. . . one-third of the way by boat	BOAT: $\frac{1}{3}$ of the trip
. . . 10 miles by foot	FOOT: 10 miles
. . . one-sixth of the way by horse	HORSE: $\frac{1}{6}$ of the trip

STEP 3 **Assign a variable.**

Distance of total trip in miles $= x$

STEP 4 **Set up an equation.**

The total distance of the trip is the sum of all the distances by boat, foot, and horse.

Distance by boat + Distance by foot + Distance by horse = Total distance of trip

Distance by boat = $\frac{1}{3}x$

Distance by foot = 10 miles

Distance by horse = $\frac{1}{6}x$

$$\underbrace{\frac{1}{3}x}_{\text{boat}} + \underbrace{10}_{\text{foot}} + \underbrace{\frac{1}{6}x}_{\text{horse}} = \underbrace{x}_{\text{total}}$$

STEP 5 **Solve the equation.** $\frac{1}{3}x + 10 + \frac{1}{6}x = x$

Multiply by the LCD, 6. $2x + 60 + x = 6x$

Collect x terms on the right. $60 = 3x$

Divide by 3. $20 = x$

The trip was 20 miles. $x = 20$

STEP 6 **Check the solution.**

Estimate: The boating distance, $\frac{1}{3}$ of 20 miles, is approximately 7 miles; the riding distance on horse, $\frac{1}{6}$ of 20 miles, is approximately 3 miles. Adding these two distances to the 10 miles by foot gives a trip distance of 20 miles.

Your Turn A family arrives at the Walt Disney World parking lot. To get from their car in the parking lot to the gate at the Magic Kingdom they walk $\frac{1}{4}$ mile, take a tram for $\frac{1}{3}$ of their total distance, and take a monorail for $\frac{1}{2}$ of their total distance. How far is it from their car to the gate of Magic Kingdom?

Answer

The distance from their car to the gate is 1.5 miles.

Concept Check

TRUE OR FALSE In Example 1 we could have stopped at $x = 20$.

Answer: False. A problem asked in words "How long was the trip?" needs an answer in words such as "The trip was 20 miles."

EXAMPLE 2 | Find the Numbers

Find three consecutive even integers so that the sum of the three numbers is 2 more than twice the third.

Solution

STEP 1 **Identify the question.**

What are the three consecutive even integers?

STEP 2 **Make notes.**

Examples of three consecutive even integers are 14, 16, 18 or −8, −6, −4 or 2, 4, 6.

STEP 3 **Assign a variable.**

Let n represent the first even integer. The next consecutive even integer is $n + 2$ and the next consecutive even integer after that is $n + 4$.

n = 1st integer

$n + 2$ = 2nd consecutive even integer

$n + 4$ = 3rd consecutive even integer

STEP 4 Set up an equation.

Read	Write
. . . sum of the three numbers	$n + (n+2) + (n+4)$
. . . is	$=$
. . . two more than	$+2$
. . . twice the third	$2(n+4)$

$$\underbrace{n + (n+2) + (n+4)}_{\text{sum of the three numbers}} \underbrace{=}_{\text{is}} \underbrace{2 +}_{\text{2 more than}} \underbrace{2(n+4)}_{\text{twice the third}}$$

STEP 5 Solve the equation. $n + (n+2) + (n+4) = 2 + 2(n+4)$

Eliminate the parentheses. $n + n + 2 + n + 4 = 2 + 2n + 8$

Simplify both sides. $3n + 6 = 2n + 10$

Collect n terms on the left and constants on the right. $n = 4$

The three consecutive even integers are 4, 6, and 8.

STEP 6 Check the solution.

Substitute the solution into the problem to see whether it makes sense. The sum of the three integers $(4 + 6 + 8)$ is 18. Twice the third is 16. Since 2 more than twice the third is 18, the solution checks.

Your Turn Find three consecutive odd integers so that the sum of the three integers is 5 less than 4 times the first.

Answer The three consecutive odd integers are 11, 13, and 15.

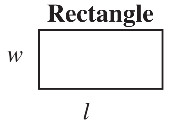
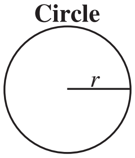
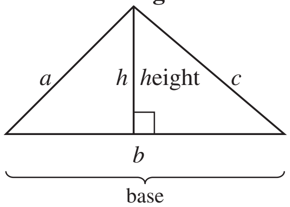
1.2.2 Geometry Problems

1.2.2 Skill Solve geometry problems using linear equations.

1.2.2 Conceptual Estimate a practical solution (guess) prior to solving a problem and then check the solution at the end.

Some problems require geometric formulas in order to be solved. The following geometric formulas may be useful.

Geometric Formulas

Rectangle 	**Perimeter** $P = 2l + 2w$	**Area** $A = l \cdot w$
Circle 	**Circumference** $C = 2\pi r$	**Area** $A = \pi r^2$
Triangle 	**Perimeter** $P = a + b + c$	**Area** $A = \frac{1}{2}bh$

Video EXAMPLE 3 | Geometry

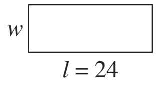
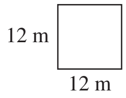
A rectangle 24 meters long has the same area as a square with 12-meter sides. What are the dimensions of the rectangle?

Solution

STEP 1 Identify the question.

What are the dimensions (length and width) of the rectangle?

STEP 2 Make notes.

Read	Write/Draw
A rectangle 24 meters long	 area of rectangle $= l \cdot w = 24w$
. . . a square with 12-meter sides	 area of square $= 12 \cdot 12 = 144$

STEP 3 Assign a variable.

Let $w =$ width of the rectangle.

STEP 4 Set up an equation.

The area of the rectangle is equal to the area of the square. — rectangle area = square area

Substitute in known quantities. — $24w = 144$

STEP 5 Solve the equation.

Divide by 24. — $w = \frac{144}{24} = 6$

The rectangle is 24 meters long and 6 meters wide.

STEP 6 Check the solution.

A 24 meter by 6 meter rectangle has an area of 144 square meters.

Your Turn A rectangle 3 inches wide has the same area as a square with 9-inch sides. What are the dimensions of the rectangle?

Answer

The rectangle is 27 inches long and 3 inches wide.

Concept Check

If we were to *guess* an answer to Example 3, what would be our guess?

(A) 24 m by 5 m (B) 24 m by 20 m (C) 24 m by 1 m

Answer: The answer is (A). Option (B) cannot be correct because that would have area *much* larger than a 12-m square, and (C) cannot be correct because that would have area much smaller than a 12-m square.

1.2.3 Interest Problems

1.2.3 Skill Solve simple interest problems.

1.2.3 Conceptual Use intuition to confirm answers in multiple investment problems.

In our personal or business financial planning, a particular concern we have is interest. **Interest** is money paid for the use of money; it is the cost of borrowing money. The total amount borrowed is called the **principal**. The principal can be the price of our new car; we pay the bank interest for loaning us money. The principal can also be the amount we keep in a CD or money market account; the bank uses this money and pays us interest. Typically interest rate, expressed as a percentage, is the amount charged for the use of the principal for a given time, usually in years.

Simple interest is interest that is paid only on principal during a period of time. Later we will discuss *compound interest*, which is interest paid on both principal and the interest accrued over a period of time.

Simple Interest

If a principal of P dollars is borrowed for a period of t years at an annual interest rate r (expressed in decimal form), the interest I charged is

$$I = Prt$$

This is the formula for **simple interest**.

EXAMPLE 4 | Simple Interest

Through a summer job Morgan is able to save \$2500. If she puts that money into a 6-month certificate of deposit (CD) that pays a simple interest rate of 3% a year, how much money will she have in her CD at the end of the 6 months?

Solution

STEP 1 Identify the question.

How much money does Morgan have after 6 months?

STEP 2 Make notes.

The principal is \$2500.

The annual interest rate is 3%, which in decimal form is 0.03.

The time the money spends accruing interest is 6 months, or $\frac{1}{2}$ of a year.

STEP 3 Assign a variable.

Label the known quantities. $P = 2500$, $r = 0.03$, and $t = 0.5$

STEP 4 Set up an equation.

Write the simple interest formula. $I = Prt$

STEP 5 Solve the equation. $I = Prt$

$I = (2500)(0.03)(0.5) = 37.5$

The interest paid on the CD is \$37.50. Adding this to the principal gives a total of

$$\$2500 + \$37.50 = \boxed{\$2537.50}$$

STEP 6 Check the solution.

This answer agrees with our intuition. Had we made a mistake, say, of moving one decimal place to the right, then the interest would have been \$375, which is much larger than we would expect on a principal of only \$2500.

Video EXAMPLE 5 | Multiple Investments

Theresa earns a full athletic scholarship for college, and her parents have given her the \$20,000 they had saved to pay for her college tuition. She decides to invest that money with an overall goal of earning 11% interest. She wants to put some of the money in a low-risk investment that has been earning 8% a year and the rest of the money in a medium-risk investment that typically earns 12% a year. How much money should she put in each investment to reach her goal?

Solution

STEP 1 Identify the question.

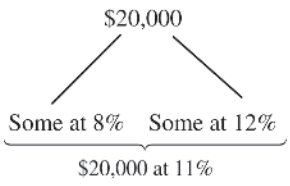
How much money is invested in each (the 8% and the 12%) account?

STEP 2 Make notes.

Read

Theresa has \$20,000 to invest. If part is invested at 8% and the rest at 12%, how much should be invested at each rate to yield 11% on the total amount invested?

Write/Draw



STEP 3 Assign a variable.

If we let x represent the amount Theresa puts into the 8% investment, how much of the \$20,000 is left for her to put in the 12% investment?

Amount in the 8% investment: x

Amount in the 12% investment: $20{,}000 - x$

STEP 4 Set up an equation.

Simple interest formula: $I = Prt$

Investment	Principal	Rate	Time (yr)	Interest
8% Account	x	0.08	1	$0.08x$
12% Account	$20{,}000 - x$	0.12	1	$0.12(20{,}000 - x)$
Total	20,000	0.11	1	$0.11(20{,}000)$

Adding the interest earned in the 8% investment to the interest earned in the 12% investment should earn an average of 11% on the total investment.

$$0.08x + 0.12(20{,}000 - x) = 0.11(20{,}000)$$

STEP 5 Solve the equation.

Eliminate the parentheses. $\quad 0.08x + 2400 - 0.12x = 2200$

Collect x terms on the left, constants on the right. $\quad -0.04x = -200$

Divide by -0.04. $\quad x = 5000$

Calculate the amount at 12%. $\quad 20{,}000 - 5000 = 15{,}000$

Theresa should invest \$5000 at 8% and \$15,000 at 12% to reach her goal.

STEP 6 Check the solution.

If money is invested at 8% and 12% with a goal of averaging 11%, our intuition tells us that more should be invested at 12% than 8%, which is what we found. The exact check is as follows:

$$0.08(5000) + 0.12(15{,}000) = 0.11(20{,}000)$$
$$400 + 1800 = 2200$$
$$2200 = 2200$$

Your Turn You win \$24,000 and you decide to invest the money in two different investments: one paying 18% and the other paying 12%. A year later you have \$27,480 total. How much did you originally invest in each account?

Answer

\$10,000 is invested at 18% and \$14,000 is invested at 12%.

Concept Check

If Teresa is going to invest in two accounts, one at 8% and one at 12% with a result of an average of 11% earnings, does your intuition think she should have (A) more at 8% than at 12% or (B) more at 12% than at 8%?

Answer: (B) because 11% is closer to 12% than 8%

1.2.4 Mixture Problems

1.2.4 **Skill** Solve mixture problems.

1.2.4 **Conceptual** Use intuition to confirm answers to mixture problems.

Mixtures are something we come across every day. Different candies that sell for different prices may make up a movie snack. New blends of coffees are developed by coffee connoisseurs. Chemists mix different concentrations of acids in their labs. Whenever two or more distinct ingredients are combined, the result is a **mixture**.

Our choice at a gas station is typically 87, 89, and 93 octane. The octane number is the number that represents the percentage of iso-octane in fuel; 89 octane is significantly overpriced. Therefore, if your car requires 89 octane, it would be more cost effective to mix 87 and 93 octane.

Video EXAMPLE 6 | Mixture Problem

The manual for your new car suggests using gasoline that is 89 octane. In order to save money, you decide to use some 87 octane and some 93 octane in combination with the 89 octane currently in your tank in order to have an approximate 89 octane mixture. Assuming you have 1 gallon of 89 octane remaining in your tank (your tank capacity is 16 gallons), how many gallons of 87 and 93 octane should be used to fill up your tank to achieve a mixture of 89 octane?

Solution

STEP 1 Identify the question.

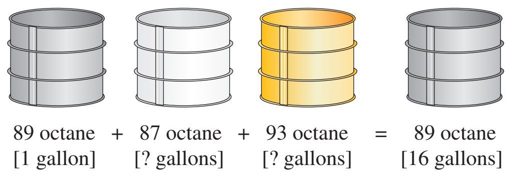
How many gallons of 87 octane and how many gallons of 93 octane should be used?

STEP 2 Make notes.

Read

Assuming you have one gallon of 89 octane remaining in your tank (your tank capacity is 16 gallons), how many gallons of 87 and 93 octane should you add?

Write/Draw



STEP 3 Assign a variable.

x = gallons of 87 octane gasoline added at the pump

$15 - x$ = gallons of 93 octane gasoline added at the pump

1 = gallons of 89 octane gasoline already in the tank

STEP 4 Set up an equation.

$$0.89(1) + 0.87x + 0.93(15 - x) = 0.89(16)$$

STEP 5 Solve the equation. $\qquad 0.89(1) + 0.87x + 0.93(15 - x) = 0.89(16)$

Eliminate the parentheses. $\qquad 0.89 + 0.87x + 13.95 - 0.93x = 14.24$

Collect x terms on the left side. $\qquad -0.06x + 14.84 = 14.24$

Subtract 14.84 from both sides of the equation. $-0.06x = -0.6$

Divide both sides by -0.06. $x = 10$

Calculate the amount of 93 octane. $15 - 10 = 5$

Add 10 gallons of 87 octane and 5 gallons of 93 octane.

STEP 6 Check the solution.

Estimate: Our intuition tells us that if the desired mixture is 89 octane, then we should add approximately one part 93 octane and two parts 87 octane. The solution we found, 10 gallons of 87 octane and 5 gallons of 93 octane, agrees with this.

Your Turn For a certain experiment, a student requires 100 milliliters of a solution that is 11% HCl (hydrochloric acid). The storeroom has only solutions that are 5% HCl and 15% HCl. How many milliliters of each available solution should be mixed to get 100 milliliters of 11% HCl?

Answer

40 milliliters of 5% HCl and 60 milliliters of 15% HCl

Concept Check

If a chemistry student has HCl concentrations of 5% and 15% and the desired solution is 11% HCl, do we expect (A) more 15% than 5% or (B) more 5% than 15%?

Answer: (A) because 11% is closer to 15% than 5%.

1.2.5 Distance–Rate–Time Problems

1.2.5 Skill Solve distance–rate–time problems.

1.2.5 Conceptual Estimate distance–rate–time problem solutions prior to solving and then confirm with a check.

The next example deals with distance, rate, and time. On a road trip, you see a sign that says your destination is 90 miles away, and your speedometer reads 60 miles per hour. Dividing 90 miles by 60 miles per hour tells you that your arrival will be in 1.5 hours. Here is how you know.

If the rate, or speed, is assumed to be constant, then the equation that relates distance (d), rate (r), and time (t) is given by $d = r \cdot t$. In the above driving example,

$$d = 90 \text{ miles} \qquad r = 60\,\frac{\text{miles}}{\text{hour}}$$

Substituting these into $d = r \cdot t$, we arrive at
$$90 \text{ miles} = \left[60\,\frac{\text{miles}}{\text{hour}}\right] \cdot t$$

Solving for t, we get
$$t = \frac{90 \text{ miles}}{60\,\frac{\text{miles}}{\text{hour}}} = 1.5 \text{ hours}$$

Video **EXAMPLE 7 | Distance–Rate–Time**

It takes 8 hours to fly from Orlando to London and 9.5 hours to return. If an airplane averages 550 miles per hour in still air, what is the average rate of the wind blowing in the direction from Orlando to London? Assume the speed of the wind (jet stream) is constant and the same for both legs of the trip. Round your answer to the nearest miles per hour.

Solution

STEP 1 Identify the question.

At what rate in mph is the wind blowing?

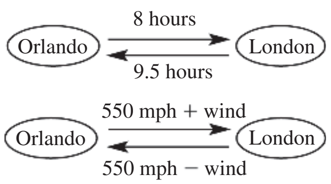
STEP 2 Make notes.

Read

It takes 8 hours to fly from Orlando to London and 9.5 hours to return.

If the airplane averages 550 miles per hour in still air . . .

Write/Draw



STEP 3 Assign a variable. w = wind speed

STEP 4 Set up an equation.

The formula relating distance, rate, and time is $d = r \cdot t$. The distance d of each flight is the same. On the Orlando to London flight the time is 8 hours due to an increased speed from a tailwind. On the London to Orlando flight the time is 9.5 hours, and the speed is decreased due to the headwind. Let w represent the wind speed.

Orlando to London: $d = (550 + w)8$

London to Orlando: $d = (550 - w)9.5$

These distances are the same, so set them equal to each other:

$$(550 + w)8 = (550 - w)9.5$$

STEP 5 Solve the equation.

Eliminate the parentheses. $4400 + 8w = 5225 - 9.5w$

Collect w terms on the left, constants on the right. $17.5w = 825$

Divide by 17.5. $w = 47.1429 \approx 47$

The wind is blowing approximately **47 miles per hour** in the direction from Orlando to London.

STEP 6 Check the solution.

Estimate: Going from Orlando to London, the tailwind is approximately 50 miles per hour, which added to the plane's 550 miles per hour speed yields a ground speed of 600 miles per hour. The Orlando to London route took 8 hours. The distance of that flight is (600 mph) (8 hr), which is 4800 miles. The return trip experienced a headwind of approximately 50 miles per hour, so subtracting the 50 from 550 gives an average speed of 500 miles per hour. That route took 9.5 hours, so the distance of the London to Orlando flight was (500 mph)(9.5 hr), which is 4750 miles. Note that the estimates of 4800 and 4750 miles are close.

Your Turn A Cessna 150 averages 150 miles per hour in still air. With a tailwind it is able to make a trip in $2\frac{1}{3}$ hours. Because of the headwind, it is only able to make the return trip in $3\frac{1}{2}$ hours. What is the average wind speed?

Answer

The wind is blowing 30 mph.

Concept Check

If Connie cleans her house alone it takes 2 hours, and if Alvaro helps her it takes 1 hour and 15 minutes. Which of the following is true?

(A) Alvaro cleans faster than Connie.
(B) Alvaro and Connie clean at the same rate.
(C) Alvaro cleans slower than Connie.

Answer: (C) is correct. (B) cannot be true or it would take them exactly 1 hour if they were working together. (A) cannot be true or the combined time would be less than 1 hour.

Video

EXAMPLE 8 | Work

Connie can clean her house in 2 hours. If Alvaro helps her, they can clean the house in 1 hour and 15 minutes together. How long would it take Alvaro to clean the house by himself?

Solution

STEP 1 Identify the question.

How long would it take Alvaro to clean the house by himself?

STEP 2 Make notes.

Connie can clean her house in 2 hours, so Connie can clean $\frac{1}{2}$ of the house per hour. Together Connie and Alvaro can clean the house in 1 hour and 15 minutes, or $\frac{5}{4}$ of an hour. Therefore, together they can clean $\frac{1}{5/4} = \frac{4}{5}$ of the house per hour.

STEP 3 Assign a variable.

Let $x =$ number of hours it takes Alvaro to clean the house by himself. So Alvaro can clean $\frac{1}{x}$ of the house per hour.

STEP 4 Set up an equation.

	AMOUNT OF TIME TO DO ONE JOB	AMOUNT OF JOB DONE PER UNIT OF TIME
Connie	2	$\frac{1}{2}$
Alvaro	x	$\frac{1}{x}$
Together	$\frac{5}{4}$	$\frac{4}{5}$

$$\overbrace{\frac{1}{2}}^{\text{Amount of house Connie can clean per hour}} + \overbrace{\frac{1}{x}}^{\text{Amount of house Alvaro can clean per hour}} = \overbrace{\frac{4}{5}}^{\text{Amount of house they can clean per hour if they work together}}$$

STEP 5 Solve the equation.

Multiply $\frac{1}{2} + \frac{1}{x} = \frac{4}{5}$ by the LCD, $10x$. $\quad 5x + 10 = 8x$

Solve for x. $\quad x = \frac{10}{3} = 3\frac{1}{3}$

It takes Alvaro 3 hours and 20 minutes to clean the house by himself.

STEP 6 Check the solution.

Connie cleans the house in 2 hours. If Alvaro could clean it in 2 hours, then together it would take them 1 hour. Since together it takes them 1 hour and 15 minutes, we expect that it takes Alvaro more than 2 hours by himself.

Section 1.2 Summary

In the real world, many kinds of application problems can be solved through modeling with linear equations. The following six-step procedure will help you develop the model. Some problems require development of a mathematical model, while others rely on common formulas.

1. Identify the quantity you are to determine.
2. Make notes on any clues that will help you set up an equation.
3. Assign a variable.
4. Set up the equation.
5. Solve the equation.
6. Check the solution against your intuition.

Section 1.2 Exercises

Applications

1. Discount Price. Donna redeems a 10% off coupon at her local nursery. After buying azaleas, bougainvillea, and bags of potting soil, her checkout price before tax is \$217.95. How much would she have paid without the coupon?

2. Discount Price. The original price of a pair of binoculars is \$74. The sale price is \$51.80. How much was the markdown?

3. Cost: Fair Share. Jeff, Tom, and Chelsea order a large pizza. They decide to split the cost according to how much they will eat. Tom pays \$5.16, Chelsea eats $\frac{1}{8}$ of the pizza, and Jeff eats $\frac{1}{2}$ of the pizza. How much did the pizza cost?

4. Event Planning. A couple decide to analyze their monthly spending habits. The monthly bills are 50% of their take-home pay, and they invest 20% of their take-home pay. They spend \$560 on groceries, and 23% goes to miscellaneous. How much is their take-home pay per month?

5. Discounts. A builder of tract homes reduced the price of a model by 15%. If the new price is \$125,000, what was its original price? How much can be saved by purchasing the model?

6. Markups. A college bookstore marks up the price it pays the publisher for a book by 25%. If the selling price of a book is \$79, how much did the bookstore pay for the book?

7. Puzzle. Angela is on her way from home in Jersey City into New York City for dinner. She walks 1 mile to the train station, takes the train $\frac{3}{4}$ of the way, and takes a taxi $\frac{1}{6}$ of the way to the restaurant. How far does Angela live from the restaurant?

8. Puzzle. An employee at Kennedy Space Center (KSC) lives in Daytona Beach and works in the vehicle assembly building (VAB). She carpools to work with a colleague. She drives 7 miles from her house to the park-and-ride. Then she rides with her colleague from the park-and-ride in Daytona Beach to the KSC headquarters building, and then takes the KSC shuttle from the headquarters building to the VAB. The drive from the park-and-ride to the headquarters building is $\frac{5}{6}$ of her total trip, and the shuttle ride is $\frac{1}{20}$ of her total trip. How many miles does she travel from her house to the VAB on days when her colleague drives?

9. Puzzle. A typical college student spends $\frac{1}{3}$ of her waking time in class, $\frac{1}{5}$ of her waking time eating, $\frac{1}{10}$ of her waking time working out, 3 hours studying, and $2\frac{1}{2}$ hours doing other things. How many hours of sleep does the typical college student get?

10. Diet. A particular 1550-calories-per-day diet suggests eating breakfast, lunch, dinner, and two snacks. Dinner is twice the calories of breakfast. Lunch is 100 calories more than breakfast. The two snacks are 100 and 150 calories. How many calories are each meal?

11. Budget. A company has a total of \$20,000 allocated for monthly costs. Fixed costs are \$15,000 per month and variable costs are \$18.50 per unit. How many units can be manufactured in a month?

12. Budget. A woman decides to start a small business making monogrammed cocktail napkins. She can set aside \$1870 for monthly costs. Fixed costs are \$1329.50 per month and variable costs are \$3.70 per set of napkins. How many sets of napkins can she afford to make per month?

13. Numbers. Find a number such that 10 less than $\frac{2}{3}$ the number is $\frac{1}{4}$ the number.

14. Numbers. Find a positive number such that 10 times the number is 16 more than twice the number.

15. Numbers. Find two consecutive even integers such that 4 times the smaller number is 2 more than 3 times the larger number.

16. Numbers. Find three consecutive integers such that the sum of the three is equal to 2 times the sum of the first two integers.

17. Geometry. Find the perimeter of a triangle if one side is 11 inches, another side is $\frac{1}{5}$ the perimeter, and the third side is $\frac{1}{4}$ the perimeter.

18. Geometry. Find the dimensions of a rectangle whose length is a foot longer than twice its width and whose perimeter is 20 feet.

19. Geometry. An NFL playing field is a rectangle. The length of the field (excluding the end zones) is 40 more yards than twice the width. The perimeter of the playing field is 260 yards. What are the dimensions of the field in yards?

20. Geometry. The length of a rectangle is 2 more than 3 times the width, and the perimeter is 28 inches. What are the dimensions of the rectangle?

21. Geometry. Consider two circles, a smaller one and a larger one. If the larger one has a radius that is 3 feet larger than that of the smaller circle and the ratio of the circumferences is 2:1, what are the radii of the two circles?

22. Geometry. The perimeter of a semicircle is doubled when the radius is increased by 1. Find the radius of the semicircle.

23. Home Improvement. A man wants to remove a tall pine tree from his yard. Before he goes to Home Depot, he needs to know how tall an extension ladder he needs to purchase. He measures the shadow of the tree to be 225 feet long. At the same time he measures the shadow of a 4-foot stick to be 3 feet. Approximately how tall is the pine tree?

24. Home Improvement. The same man in Exercise 23 realizes he also wants to remove a dead oak tree. Later in the day he measures the shadow of the oak tree to be 880 feet long, and the 4-foot stick now has a shadow of 10 feet. Approximately how tall is the oak tree?

25. **Biology: Alligators.** It is common to see alligators in ponds, lakes, and rivers in Florida. The ratio of head size (back of the head to the end of the snout) to the full body length of an alligator is typically constant. If a $3\frac{1}{2}$-foot alligator has a head length of 6 inches, how long would you expect an alligator to be whose head length is 9 inches?

26. **Biology: Snakes.** In the African rainforest there is a snake called a Gaboon viper. The fang size of this snake is proportional to the length of the snake. A 3-foot snake typically has 2-inch fangs. If a herpetologist finds Gaboon viper fangs that are 2.6 inches long, how long a snake would she expect to find?

27. **Investing.** Ashley has $120,000 to invest and decides to put some in a CD that earns 4% interest per year and the rest in a low-risk stock that earns 7%. How much did she invest in each to earn $7800 interest in the first year?

28. **Investing.** You inherit $13,000 and you decide to invest the money in two different investments: one paying 10% and the other paying 14%. A year later your investments are worth $14,580. How much did you originally invest in each account?

29. **Investing.** Wendy was awarded a volleyball scholarship to the University of Michigan, so on graduation her parents gave her the $14,000 they had saved for her college tuition. She opted to invest some money in a privately held company that pays 10% per year and evenly split the remaining money between a money market account yielding 2% and a high-risk stock that yielded 40%. At the end of the first year she had $16,610 total. How much did she invest in each of the three?

30. **Interest.** A high school student was able to save $5000 by working a part-time job every summer. He invested half the money in a money market account and half the money in a stock that paid three times as much interest as the money market account. After a year he earned $150 in interest. What were the interest rates of the money market account and the stock?

31. **Budget: Home Improvement.** When landscaping their yard, a couple budgeted $4200. The irrigation system costs $2400 and the sod costs $1500. The rest they will spend on trees and shrubs. Trees each cost $32 and shrubs each cost $4. They plant a total of 33 trees and shrubs. How many of each did they plant in their yard?

32. **Budget: Shopping.** At the deli Jennifer bought spicy turkey and provolone cheese. The turkey costs $6.32 per pound and the cheese costs $4.27 per pound. In total, she bought 3.2 pounds and the price was $17.56. How many pounds of each did she buy?

33. **Chemistry.** For a certain experiment, a student requires 100 milliliters of a solution that is 8% HCl (hydrochloric acid). The storeroom has only solutions that are 5% HCl and 15% HCl. How many milliliters of each available solution should be mixed to get 100 milliliters of 8% HCl?

34. **Chemistry.** How many gallons of pure alcohol must be mixed with 5 gallons of a solution that is 20% alcohol to make a solution that is 50% alcohol?

35. **Automobiles.** A mechanic has tested the amount of antifreeze in your radiator. He says it is only 40% antifreeze and the remainder is water. How many gallons must be drained from your 5 gallon radiator and replaced with pure antifreeze to make the mixture in your radiator 80% antifreeze?

36. **Costs: Overhead.** A professor is awarded two research grants, each having different overhead rates. The research project conducted on campus has a rate of 42.5% overhead, and the project conducted in the field, off campus, has a rate of 26% overhead. If she was awarded $1,170,000 total for the two projects with an average overhead rate of 39%, how much was the research project on campus and how much was the research project off campus?

37. **Theater.** On the way to the movies a family picks up a custom-made bag of candies. The parents like caramels ($1.50/lb) and the children like gummy bears ($2.00/lb). They bought a 1.25-pound bag of combined candies that cost $2.50. How much of each candy did they buy?

38. **Coffee.** Joy is an instructional assistant in one of the college labs. She is on a very tight budget. She loves Jamaican Blue Mountain coffee, but it costs $12 a pound. She decides to blend this with regular coffee beans that cost $4.20 a pound. If she spends $14.25 on 2 pounds of coffee, how many pounds of each did she purchase?

39. **Communications.** The speed of light is approximately 3.0×10^8 meters per second (670,616,629 mph). The distance from Earth to Mars varies because of the orbits of the planets around the Sun. On average, Mars is 100 million miles from Earth. If we use laser communication systems, what will be the delay between Houston and NASA astronauts on Mars?

40. **Speed of Sound.** The speed of sound is approximately 760 miles per hour in air. If a gun is fired $\frac{1}{2}$ mile away, how long will it take the sound to reach you?

41. **Business.** Because of holiday travel during the month of November, the average price of gasoline rose 4.7% in the United States. If the average price of gasoline at the end of November was $3.21 per gallon, what was the price of gasoline at the beginning of November?

42. **Business.** During the Christmas shopping season, the average price of a flat screen television fell by 40%. A shopper purchased a 42-inch flat screen television for $299 in late November. How much would the shopper have paid, to the nearest dollar, for the same television if it was purchased in September?

43. **Medicine.** A patient requires an IV of 0.9% saline solution, also known as normal saline solution. How much distilled water, to the nearest milliliter, must be added to 100 milliliters of a 3% saline solution to produce normal saline?

44. **Medicine.** A patient requires an IV of D5W, a 5% solution of dextrose (sugar) in water. To the nearest milliliter, how much D20W, a 20% solution of dextrose in water, must be added to 100 milliliters of distilled water to produce a D5W solution?

45. **Boating.** A motorboat can maintain a constant speed of 16 miles per hour relative to the water. The boat makes a trip upstream to a marina in 20 minutes. The return trip takes 15 minutes. What is the speed of the current?

46. **Aviation.** A Cessna 175 can average 130 miles per hour. If a trip takes 2 hours one way and the return takes 1 hour and 15 minutes, find the wind speed, assuming it is constant.

47. Exercise. A jogger and a walker cover the same distance. The jogger finishes in 40 minutes. The walker takes an hour. How fast is each exerciser moving if the jogger runs 2 miles per hour faster than the walker?

48. Travel. A high school student in Seattle, Washington, attended the University of Central Florida. On the way to UCF he took a southern route. After graduation he returned to Seattle via a northern trip. On both trips he had the same average speed. If the southern trek took 45 hours and the northern trek took 50 hours, and the northern trek was 300 miles longer, how long was each trip?

49. Distance–Rate–Time. College roommates leave for their first class in the same building. One walks at 2 miles per hour and the other rides his bike at a slow 6 miles per hour pace. How long will it take each to get to class if the walker takes 12 minutes longer to get to class and they travel on the same path?

50. Distance–Rate–Time. A long-distance delivery service sends out a truck with a package at 7 A.M. At 7:30 A.M., the manager realizes there was another package going to the same location. He sends out a car to catch the truck. If the truck travels at an average speed of 50 miles per hour and the car travels at 70 miles per hour, how long will it take the car to catch the truck?

51. Work. Christopher can paint the interior of his house in 15 hours. If he hires Cynthia to help him, they can do the same job together in 9 hours. If he lets Cynthia work alone, how long will it take her to paint the interior of his house?

52. Work. Jay and Morgan work in the summer for a landscaper. It takes Jay 3 hours to complete the company's largest yard alone. If Morgan helps him, it takes only 1 hour. How much time would it take Morgan alone?

53. Work. Tracey and Robin deliver soft drinks to local convenience stores. Tracey can complete the deliveries in 4 hours alone. Robin can do it in 6 hours alone. If they decide to work together on a Saturday, how long will it take?

54. Work. Joshua can deliver his newspapers in 30 minutes. It takes Amber 20 minutes to do the same route. How long would it take them to deliver the newspapers if they worked together?

55. Music. A major chord in music is composed of notes whose frequencies are in the ratio 4:5:6. If the first note of a chord has a frequency of 264 hertz (middle C on the piano), find the frequencies of the other two notes. *Hint:* Set up two proportions using 4:5 and 4:6.

56. Music. A minor chord in music is composed of notes whose frequencies are in the ratio 10:12:15. If the first note of a minor chord is A, with a frequency of 220 hertz, what are the frequencies of the other two notes?

57. Grades. Danielle's test scores are 86, 80, 84, and 90. The final exam will count as $\frac{2}{3}$ of the final grade. What score does Danielle need on the final in order to earn a B, which requires an average score of 80? What score does she need to earn an A, which requires an average of 90?

58. Grades. Sam's final exam will count as two tests. His test scores are 80, 83, 71, 61, and 95. What score does Sam need on the final in order to have an average score of 80?

59. Sports. In Super Bowl XXXVII, the Tampa Bay Buccaneers scored a total of 48 points. All of their points came from field goals and touchdowns. Field goals are worth 3 points and each touchdown was worth 7 points (Martin Gramatica was successful in every extra point attempt). They scored a total of 8 times. How many field goals and touchdowns were scored?

60. Sports. A tight end can run the 100-yard dash in 12 seconds. A defensive back can do it in 10 seconds. The tight end catches a pass at his own 20-yard line with the defensive back at the 15-yard line. If no other players are nearby, at what yard line will the defensive back catch up to the tight end?

61. Recreation. How do two children of different weights balance on a seesaw? The heavier child sits closer to the center and the lighter child sits farther away. When the product of the weight of the child and the distance from the center is equal on both sides, the seesaw should be horizontal to the ground. Suppose Max weighs 42 pounds and Maria weighs 60 pounds. If Max sits 5 feet from the center, how far should Maria sit from the center in order to balance the seesaw horizontal to the ground?

62. Recreation. Refer to Exercise 61. Suppose Martin, who weighs 33 pounds, sits on the side of the seesaw with Max. If their average distance to the center is 4 feet, how far should Maria sit from the center in order to balance the seesaw horizontal to the ground?

63. Recreation. If a seesaw has an adjustable bench, then the board can slide along the fulcrum. Maria and Max in Exercise 61 decide to sit on the very edge of the board on each side. Where should the fulcrum be placed along the board in order to balance the seesaw horizontally to the ground? Give the answer in terms of the distance from each child's end.

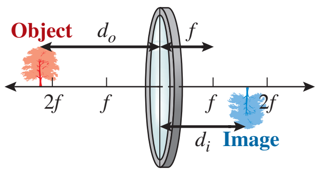
64. Recreation. Add Martin (Exercise 62) to Max's side of the seesaw and recalculate Exercise 63.

In Exercises 65–68, refer to this lens law. (See Exercise 82 in Section 1.1.)

The position of the image is found using the thin lens equation:

$$\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i},$$

where d_o is the distance from the object to the lens, d_i is the distance from the lens to the image, and f is the focal length of the lens.



65. **Optics.** If the focal length of a lens is 3 centimeters and the image distance is 5 centimeters from the lens, what is the distance from the object to the lens?

66. **Optics.** If the focal length of the lens is 8 centimeters and the image distance is 2 centimeters from the lens, what is the distance from the object to the lens?

67. **Optics.** The focal length of a lens is 2 centimeters. If the image distance from the lens is half the distance from the object to the lens, find the object distance.

68. **Optics.** The focal length of a lens is 8 centimeters. If the image distance from the lens is half the distance from the object to the lens, find the object distance.

Conceptual

In Exercises 69–76, solve each formula for the specified variable.

69. $P = 2l + 2w$ for w

70. $P = 2l + 2w$ for l

71. $A = \frac{1}{2}bh$ for h

72. $C = 2\pi r$ for r

73. $A = lw$ for w

74. $d = rt$ for t

75. $V = lwh$ for h

76. $V = \pi r^2 h$ for h

Challenge

77. Tricia and Janine are roommates and leave Houston on Interstate 10 at the same time to visit their families for a long weekend. Tricia travels west and Janine travels east. If Tricia's average speed is 12 miles per hour faster than Janine's, find the speed of each if they are 320 miles apart in 2 hours and 30 minutes.

78. Rick and Mike are roommates and leave Gainesville on Interstate 75 at the same time to visit their girlfriends for a long weekend. Rick travels north and Mike travels south. If Mike's average speed is 8 miles per hour faster than Rick's, find the speed of each if they are 210 miles apart in 1 hour and 30 minutes.

Technology

79. Suppose you bought a house for \$132,500 and sold it 3 years later for \$168,190. Plot these points using a graphing utility. Assuming a linear relationship, how much could you have sold the house for had you waited 2 additional years?

80. Suppose you bought a house for \$132,500 and sold it 3 years later for \$168,190. Plot these points using a graphing utility. Assuming a linear relationship, how much could you have sold the house for had you sold it 1 year after buying it?

81. A golf club membership has two options. Option A is a \$300 monthly fee plus a \$15 cart fee every time you play. Option B has a \$150 monthly fee and a \$42 fee every time you play. Write a mathematical model for monthly costs for each plan and graph both in the same viewing rectangle using a graphing utility. Explain when Option A is the better deal and when Option B is the better deal.

82. A phone provider offers two calling plans. Plan A has a \$30 monthly charge and a \$0.10 per minute charge on every call. Plan B has a \$50 monthly charge and a \$0.03 per minute charge on every call. Explain when Plan A is the better deal and when Plan B is the better deal.

1.3 Quadratic Equations

SKILLS OBJECTIVES

- Solve quadratic equations by factoring.
- Use the square root method to solve quadratic equations.
- Solve quadratic equations by completing the square.
- Use the Quadratic Formula to solve quadratic equations.
- Solve application problems using quadratic equations.

CONCEPTUAL OBJECTIVES

- Understand the zero product property in factoring.
- Understand that the square root method can only be used when there is no linear term in the quadratic equation.
- Understand that completing the square transforms a standard quadratic equation into a perfect square.
- Derive the Quadratic Formula.
- Understand why eliminated nonphysical answers should be eliminated.

1.3.1 Factoring

1.3.1 **Skill** Solve quadratic equations by factoring.

1.3.1 **Conceptual** Understand the zero product property in factoring.

In a linear equation, the variable is raised only to the first power in any term where it occurs. In a *quadratic equation*, the variable is raised to the second power in at least one term. Examples of *quadratic equations*, also called second-degree equations, are:

$$x^2 + 3 = 7 \qquad 5x^2 + 4x - 7 = 0 \qquad x^2 - 3 = 0$$

Quadratic Equation

A **quadratic equation** in x is an equation that can be written in the **standard form**

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers and $a \neq 0$.

There are several methods for solving quadratic equations: *factoring*, the *square root method*, *completing the square*, and the *Quadratic Formula.*

Factoring Method

The **factoring method** applies the **zero product property**:

Words	Math
If a product is zero, then at least one of its factors has to be zero.	If $B \cdot C = 0$, then $B = 0$ or $C = 0$ or both.

Consider $(x - 3)(x + 2) = 0$. The zero product property says that $x - 3 = 0$ or $x + 2 = 0$, which leads to $x = -2$ or $x = 3$. The solution set is $\{-2, 3\}$.

When a quadratic equation is written in the standard form $ax^2 + bx + c = 0$, it may be possible to factor the left side of the equation as a product of two first-degree polynomials. We use the zero product property and set each linear factor equal to zero. We solve the resulting two linear equations to obtain the solutions of the quadratic equation.

Video

EXAMPLE 1 | Solving a Quadratic Equation by Factoring

Solve the equation $x^2 - 6x - 16 = 0$.

Solution

The quadratic equation is already in standard form. $x^2 - 6x - 16 = 0$

Factor the left side into a product of two linear factors. $(x-8)(x+2) = 0$

If a product equals zero, one of its factors has to be equal to zero. $x - 8 = 0$ or $x + 2 = 0$

Solve both linear equations. $\boxed{x = 8 \text{ or } x = -2}$

The solution set is $\boxed{\{-2, 8\}}$.

Your Turn Solve the quadratic equation $x^2 + x - 20 = 0$ by factoring.

Answer

The solution is $x = -5, 4$. The solution set is $\{-5, 4\}$.

Caution

Don't forget to put the quadratic equation in standard form first.

EXAMPLE 2 | Solving a Quadratic Equation by Factoring

Solve the equation $x^2 - 6x + 5 = -4$.

Common Mistake

A common mistake is to forget to put the equation in standard form first and then use the zero product property incorrectly.

Correct

Write the original equation.

$$x^2 - 6x + 5 = -4$$

Write the equation in standard form by adding 4 to both sides.

$$x^2 - 6x + 9 = 0$$

Factor the left side.

$$(x-3)(x-3) = 0$$

Use the zero product property and set each factor equal to zero.

$$x - 3 = 0 \quad \text{or} \quad x - 3 = 0$$

Solve each linear equation.

$$\boxed{x = 3}$$

Incorrect

Factor the left side.

$$(x-5)(x-1) = -4$$

The **error** occurs here.

$$x - 5 = -4 \quad \text{or} \quad x - 1 = -4$$

Don't forget to put the quadratic equation in standard form first.

Note: The equation has one solution, or root, which is 3. The solution set is $\{3\}$. Since the linear factors were the same, or repeated, we say that 3 is a **double root**, or **repeated root**.

Your Turn Solve the quadratic equation $9p^2 = 24p - 16$ by factoring.

Answer

The solution is $p = \frac{4}{3}$, which is a double root. The solution set is $\left\{\frac{4}{3}\right\}$.

Concept Check

If something times something is equal to zero, then _____ has to be zero:

(A) One of those somethings;
(B) Both of those somethings

Answer: (A) One of those somethings

Video **EXAMPLE 3** | **Solving a Quadratic Equation by Factoring**

Solve the equation $2x^2 = 3x$.

Common Mistake

The common mistake here is dividing both sides by x, which is not allowed because x might be zero.

Correct

Write the equation in standard form by subtracting $3x$.

$$2x^2 - 3x = 0$$

Factor the left side.

$$x(2x - 3) = 0$$

Use the zero product property and set each factor equal to zero.

$$x = 0 \quad \text{or} \quad 2x - 3 = 0$$

Solve each linear equation.

$$\boxed{x = 0 \quad \text{or} \quad x = \frac{3}{2}}$$

The solution set is $\boxed{\left\{0, \frac{3}{2}\right\}}$.

Incorrect

Write the original equation.

$$2x^2 = 3x$$

The **error** occurs here when both sides are divided by x.

$$2x = 3$$

Caution

Do not divide by a variable (because the value of that variable may be zero). Bring all terms to one side first and then factor.

In Example 3, the root $x = 0$ is lost when the original quadratic equation is divided by x. Remember to put the equation in standard form first and then factor.

1.3.2 Square Root Method

1.3.2 Skill Use the square root method to solve quadratic equations.

1.3.2 Conceptual Understand that the square root method can only be used when there is no linear term in the quadratic equation.

The square root of 16, $\sqrt{16}$, is 4, *not* ± 4. In the review (Chapter 0) the **principal square root** was discussed. The solutions to $x^2 = 16$, however, are $x = -4$ and $x = 4$. Let us now investigate quadratic equations that do not have a first-degree term. They have the form

$$ax^2 + c = 0 \qquad a \neq 0$$

The method we use to solve such equations uses the square root property.

Square Root Property

Words	Math
If an expression squared is equal to a constant, then that expression is equal to the positive or negative square root of the constant.	If $x^2 = P$, then $x = \pm\sqrt{P}$.

Note: The variable squared must be isolated first (coefficient equal to 1).

Concept Check

Which of the following can be solved using the square root method?

(A) $x^2 = x$

(B) $x^2 = 9$

Answer: (B) $x^2 = 9$ can be solved by the square root method because it is in the form of $x^2 =$ constant. (A) cannot be solved using the square root method.

EXAMPLE 4 | Using the Square Root Property

Solve the equation $3x^2 - 27 = 0$.

Solution

Add 27 to both sides.	$3x^2 = 27$
Divide both sides by 3.	$x^2 = 9$
Apply the square root property.	$x = \pm\sqrt{9} = \pm 3$

The solution set is $\boxed{\{-3, 3\}}$.

If we alter Example 4 by changing subtraction to addition, we see in Example 5 that we get imaginary roots (as opposed to real roots), which we discussed in Chapter 0.

EXAMPLE 5 | Using the Square Root Property

Solve the equation $3x^2 + 27 = 0$.

Solution

Subtract 27 from both sides.	$3x^2 = -27$
Divide by 3.	$x^2 = -9$
Apply the square root property.	$x = \pm\sqrt{-9}$
Simplify.	$x = \pm i\sqrt{9} = \pm 3i$

The solution set is $\boxed{\{-3i, 3i\}}$.

Your Turn Solve the equations $y^2 - 147 = 0$ and $v^2 + 64 = 0$.

Answer

The solution is $y = \pm 7\sqrt{3}$. The solution set is $\{-7\sqrt{3}, 7\sqrt{3}\}$. The solution is $v = \pm 8i$. The solution set is $\{-8i, 8i\}$.

EXAMPLE 6 | Using the Square Root Property

Solve the equation $(x - 2)^2 = 16$.

Solution

Approach 1:

If an expression squared is 16, then the expression equals $\pm\sqrt{16}$. $\qquad (x - 2) = \pm\sqrt{16}$

Separate into two equations.	$x-2=\sqrt{16}$ or $x-2=-\sqrt{16}$
	$x-2=4$ $\quad$ $x-2=-4$
	$x=6$ $\quad$ $x=-2$

The solution set is $\boxed{\{-2, 6\}}$.

Approach 2:

It is acceptable notation to keep the equations together.	$(x-2)=\pm\sqrt{16}$
	$x-2=\pm 4$
	$x=2\pm 4$
	$\boxed{x=-2, 6}$

1.3.3 Completing the Square

1.3.3 Skill Solve quadratic equations by completing the square.

1.3.3 Conceptual Understand that completing the square transforms a standard quadratic equation into a perfect square.

Factoring and the square root method are two efficient, quick procedures for solving many quadratic equations. However, some equations, such as $x^2-10x-3=0$, cannot be solved directly by these methods. A more general procedure to solve this kind of equation is called **completing the square**. The idea behind completing the square is to transform any standard quadratic equation $ax^2+bx+c=0$ into the form $(x+A)^2=B$, where A and B are constants and the left side, $(x+A)^2$, has the form of a **perfect square**. This last equation can then be solved by the square root method. How do we transform the first equation into the second equation?

Note that the above-mentioned example, $x^2-10x-3=0$, cannot be factored into expressions in which all numbers are integers (or even rational numbers). We can, however, transform this quadratic equation into a form that contains a perfect square.

Words	Math
Write the original equation.	$x^2-10x-3=0$
Add 3 to both sides.	$x^2-10x=3$
Add 25 to both sides.*	$x^2-10x+\mathbf{25}=3+\mathbf{25}$
The left side can be written as a perfect square.	$(x-5)^2=28$
Apply the square root method.	$x-5=\pm\sqrt{28}$
Add 5 to both sides.	$x=5\pm 2\sqrt{7}$

*Why did we add 25 to both sides? Recall that $(x-c)^2=x^2-2xc+c^2$. In this case $c=5$ in order for $-2xc=-10x$. Therefore, the desired perfect square $(x-5)^2$ results in $x^2-10x+25$. Applying this product, we see that $+25$ is needed. A systematic approach is to take the coefficient of the first-degree term $x^2-10x-3=0$, which is -10. Take half of (-10), which is (-5), and then square it $(-5)^2=25$.

Solving a Quadratic Equation by Completing the Square

Words	Math
Express the quadratic equation in the following form.	$x^2+bx=c$
Divide b by 2 and square the result, then add the square to both sides.	$x^2+bx+\left(\frac{b}{2}\right)^2=c+\left(\frac{b}{2}\right)^2$
Write the left side of the equation as a perfect square.	$\left(x+\frac{b}{2}\right)^2=c+\left(\frac{b}{2}\right)^2$
Solve using the square root method.	

Concept Check

The quadratic equation $x^2 - 4x + 3 = 0$ can be transformed to which perfect square?
(A) $(x+2)^2 = 1$ (B) $(x-2)^2 = 1$

Answer: (B)

EXAMPLE 7 | Completing the Square

Solve the quadratic equation $x^2 + 8x - 3 = 0$ by completing the square.

Solution

Add 3 to both sides. $x^2 + 8x = 3$

Add $\left(\frac{1}{2} \cdot 8\right)^2 = 4^2$ to both sides. $x^2 + 8x + \mathbf{4^2} = 3 + \mathbf{4^2}$

Write the left side as a perfect square and simplify the right side. $(x+4)^2 = 19$

Apply the square root method to solve. $x + 4 = \pm\sqrt{19}$

Subtract 4 from both sides. $\boxed{x = -4 \pm \sqrt{19}}$

The solution set is $\boxed{\{-4-\sqrt{19}, -4+\sqrt{19}\}}$.

In Example 7, the leading coefficient (the coefficient of the x^2 term) is 1. When the leading coefficient is not 1, start by first dividing the equation by that leading coefficient.

Video EXAMPLE 8 | Completing the Square When the Leading Coefficient Is Not Equal to 1

STUDY TIP

When the leading coefficient is not 1, start by first dividing the equation by that leading coefficient.

Solve the equation $3x^2 - 12x + 13 = 0$ by completing the square.

Solution

Divide by the leading coefficient, 3. $x^2 - 4x + \frac{13}{3} = 0$

Collect variables to one side of the equation and constants to the other side. $x^2 - 4x = -\frac{13}{3}$

Add $\left(-\frac{4}{2}\right)^2 = 4$ to both sides. $x^2 - 4x + \mathbf{4} = -\frac{13}{3} + \mathbf{4}$

Write the left side of the equation as a perfect square and simplify the right side. $(x-2)^2 = -\frac{1}{3}$

Solve using the square root method. $x - 2 = \pm\sqrt{-\frac{1}{3}}$

Simplify. $x = 2 \pm i\sqrt{\frac{1}{3}}$

Rationalize the denominator (Chapter 0). $x = 2 \pm \frac{i}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

Simplify. $\boxed{x = 2 - \frac{i\sqrt{3}}{3}, x = 2 + \frac{i\sqrt{3}}{3}}$

The solution set is $\boxed{\left\{2 - \frac{i\sqrt{3}}{3}, 2 + \frac{i\sqrt{3}}{3}\right\}}$.

Your Turn Solve the equation $2x^2 - 4x + 3 = 0$ by completing the square.

Answer

The solution is $x = 1 \pm \frac{i\sqrt{2}}{2}$. The solution set is $\left\{1 - \frac{i\sqrt{2}}{2}, 1 + \frac{i\sqrt{2}}{2}\right\}$.

1.3.4 Quadratic Formula

1.3.4 Skill Use the Quadratic Formula to solve quadratic equations.

1.3.4 Conceptual Derive the Quadratic Formula.

Let us now consider the most general quadratic equation:

$$ax^2 + bx + c = 0 \quad a \neq 0$$

We can solve this equation by completing the square.

Words	Math
Divide the equation by the leading coefficient a.	$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
Subtract $\frac{c}{a}$ from both sides.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$
Square half of $\frac{b}{a}$ and add the result $\left(\frac{b}{2a}\right)^2$ to both sides.	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$
Write the left side of the equation as a perfect square and the right side as a single fraction.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Solve using the square root method.	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$
Subtract $\frac{b}{2a}$ from both sides and simplify the radical.	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
Write as a single fraction.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

We have derived the **Quadratic Formula**.

Concept Check

TRUE OR FALSE Completing the square is still necessary in certain cases when the Quadratic Formula cannot be used to solve a particular Quadratic Equation.

Answer: False. The Quadratic Formula was derived by solving the general Quadratic Equation by completing the square and can be used for solving any quadratic equation.

Quadratic Formula

If $ax^2 + bx + c = 0$, $a \neq 0$, then the solution is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The quadratic equation must be in standard form ($ax^2 + bx + c = 0$) in order to identify the parameters:

a—coefficient of x^2 $\quad$ b—coefficient of x $\quad$ c—constant

STUDY TIP

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Read as "negative b plus or minus the square root of the quantity b squared minus $4ac$ all over $2a$."

We read this formula as *negative b plus or minus the square root of the quantity b squared minus 4ac all over 2a.* It is important to note that negative b could be positive (if b is negative). For this reason, an alternate form is "opposite b. . ." The Quadratic Formula should be memorized and used when simpler methods (factoring and the square root method) cannot be used. The Quadratic Formula works for *any* quadratic equation.

STUDY TIP

The Quadratic Formula works for *any* quadratic equation.

EXAMPLE 9 | Using the Quadratic Formula and Finding Two Distinct Real Roots

STUDY TIP

Using parentheses as placeholders helps avoid $\pm$ errors.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$$

Use the Quadratic Formula to solve the quadratic equation $x^2 - 4x - 1 = 0$.

Solution

For this problem $a = 1$, $b = -4$, and $c = -1$.

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Use parentheses to avoid losing a minus sign. $x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$

Substitute values for a, b, and c into the parentheses. $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$

Simplify. $x = \frac{4 \pm \sqrt{16 + 4}}{2} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = \frac{4}{2} \pm \frac{2\sqrt{5}}{2} = \boxed{2 \pm \sqrt{5}}$

The solution set $\boxed{\{2 - \sqrt{5}, 2 + \sqrt{5}\}}$ contains two distinct real numbers.

Your Turn Use the Quadratic Formula to solve the quadratic equation $x^2 + 6x - 2 = 0$.

Answer

The solution is $x = -3 \pm \sqrt{11}$. The solution set is $\{-3 - \sqrt{11}, -3 + \sqrt{11}\}$.

Video

EXAMPLE 10 | Using the Quadratic Formula and Finding Two Complex Roots

Use the Quadratic Formula to solve the quadratic equation $x^2 + 8 = 4x$.

Solution

Write this equation in standard form $x^2 - 4x + 8 = 0$ in order to identify $a = 1$, $b = -4$, and $c = 8$.

Write the Quadratic Formula. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Use parentheses to avoid overlooking a minus sign. $x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$

Substitute the values for a, b, and c into the parentheses. $x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(8)}}{2(1)}$

Simplify. $x = \frac{4 \pm \sqrt{16 - 32}}{2} = \frac{4 \pm \sqrt{-16}}{2} = \frac{4 \pm 4i}{2} = \frac{4}{2} \pm \frac{4i}{2} = \boxed{2 \pm 2i}$

The solution set $\boxed{\{2 - 2i, 2 + 2i\}}$ contains two complex numbers. Note that they are complex conjugates of each other.

Your Turn Use the Quadratic Formula to solve the quadratic equation $x^2 + 2 = 2x$.

Answer

The solution set is $\{1 - i, 1 + i\}$.

EXAMPLE 11 | Using the Quadratic Formula and Finding One Repeated Real Root

Use the Quadratic Formula to solve the quadratic equation $4x^2 - 4x + 1 = 0$.

Solution

Identify a, b, and c.	$a = 4, b = -4, c = 1$
Write the Quadratic Formula.	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Use parentheses to avoid losing a minus sign.	$x = \dfrac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$
Substitute values $a = 4$, $b = -4$, and $c = 1$.	$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(1)}}{2(4)}$
Simplify.	$x = \dfrac{4 \pm \sqrt{16 - 16}}{8} = \dfrac{4 \pm 0}{8} = \dfrac{1}{2}$

The solution set is a repeated real root $\left\{\dfrac{1}{2}\right\}$.

Note: This quadratic equation also could have been solved by factoring: $(2x - 1)^2 = 0$.

Your Turn Use the Quadratic Formula to solve the quadratic equation $9x^2 - 6x + 1 = 0$.

Answer

$\left\{\dfrac{1}{3}\right\}$

Types of Solutions

The term inside the radical, $b^2 - 4ac$, is called the **discriminant**. The discriminant gives important information about the corresponding solutions or roots of $ax^2 + bx + c = 0$, where a, b, and c are real numbers.

$b^2 - 4ac$	Solutions (Roots)
Positive	Two distinct real roots
0	One real root (a double or repeated root)
Negative	Two complex roots (complex conjugates)

In Example 9, the discriminant is positive and the solution has two distinct real roots. In Example 10, the discriminant is negative and the solution has two complex (conjugate) roots. In Example 11, the discriminant is zero and the solution has one repeated real root.

1.3.5 Applications Involving Quadratic Equations

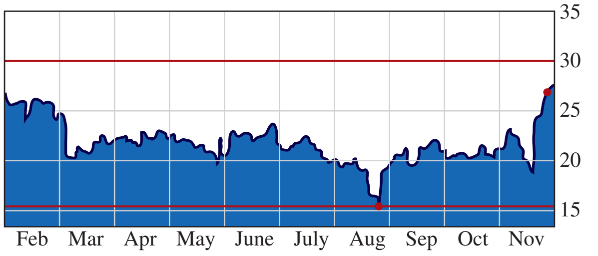
1.3.5 Skill Solve application problems using quadratic equations.

1.3.5 Conceptual Understand why nonphysical answers should be eliminated.

In Section 1.2, we developed a procedure for solving word problems involving linear equations. The procedure is the same for applications involving quadratic equations. The only difference is that the mathematical equations will be quadratic, as opposed to linear.

EXAMPLE 12 | Stock Value

From March 1 to May 1 the price of Abercrombie & Fitch's (ANF) stock was approximately given by $P = -3t^2 + 6t + 20$, where P is the price of stock in dollars, t is in months, and $t = 0$ corresponds to March 1. When was the value of the stock worth $22?



Solution

STEP 1 Identify the question.

When is the price of the stock equal to $22?

STEP 2 Make notes.

Stock price: $P = -3t^2 + 6t + 20$

$P = 22$

STEP 3 Set up an equation. $-3t^2 + 6t + 20 = 22$

STEP 4 Solve the equation.

Subtract 22 from both sides. $-3t^2 + 6t - 2 = 0$

Solve for t using the Quadratic Formula. $t = \dfrac{-(6) \pm \sqrt{6^2 - 4(-3)(-2)}}{2(-3)}$

Simplify. $t = \dfrac{-6 \pm \sqrt{6^2 - 4(-3)(-2)}}{2(-3)} = \dfrac{-6 \pm \sqrt{12}}{-6} \approx 1.57, 0.42$

Rounding these two numbers, we find that $t \approx 1.5$ and $t \approx 0.5$. Since $t = 1$ corresponds to March 1, the value of $t = 0.5$ corresponds to March 15, and the value $t = 1.5$ corresponds to April 15.

STEP 5 Check the solution.

Look at the figure. The horizontal axis represents the month, and the vertical axis represents the stock price. Estimating when the stock price is approximately $22, we find March 15 and April 15.

EXAMPLE 13 | Pythagorean Theorem

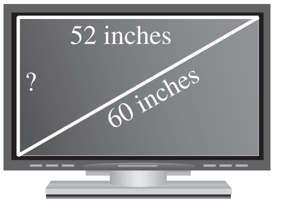
Hitachi makes a 60-inch HDTV that has a 60-inch diagonal. If the width of the screen is approximately 52 inches, what is the approximate height of the screen?

Solution

STEP 1 Identify the question.

What is the approximate height of the HDTV screen?

STEP 2 Make notes.



STEP 3 Set up an equation.

Recall the Pythagorean theorem. $a^2 + b^2 = c^2$

Substitute in the known values. $h^2 + 52^2 = 60^2$

STEP 4 Solve the equation.

Simplify the constants. $h^2 + 2704 = 3600$

Subtract 2704 from both sides. $h^2 = 896$

Solve using the square root method. $h = \pm\sqrt{896} \approx \pm 30$

Distance is positive, so the negative value is eliminated.

The height is approximately 30 inches.

STEP 5 Check the solution.

$$30^2 + 52^2 \stackrel{?}{=} 60^2$$
$$900 + 2704 \stackrel{?}{=} 3600$$
$$3604 \approx 3600$$

STUDY TIP

Dimensions such as length and width are distances, which are defined as positive quantities. Although the mathematics may yield both positive and negative values, the negative values are excluded.

Concept Check

Which of the following would we NOT eliminate negative values for physical answers:
(A) distance
(B) height
(C) width
(D) temperature

Answer: (D) temperature. All others are defined as positive.

Section 1.3 Summary

The four methods for solving quadratic equations

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

are factoring, the square root method, completing the square, and the Quadratic Formula. Factoring and the square root method are the quickest and easiest but cannot always be used. The Quadratic Formula and completing the square work for all quadratic equations can yield three types of solutions: two distinct real roots, one real root (repeated), or two complex roots (conjugates of each other).

Quadratic Formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Section 1.3 Exercises

Skills

In Exercises 1–24, solve by factoring.

1. $x^2 - 5x + 6 = 0$ **2.** $v^2 + 7v + 6 = 0$ **3.** $p^2 - 8p + 15 = 0$ **4.** $u^2 - 2u - 24 = 0$

5. $x^2 = 12 - x$ **6.** $11x = 2x^2 + 12$ **7.** $16x^2 + 8x = -1$ **8.** $3x^2 + 10x - 8 = 0$

9. $9y^2 + 1 = 6y$ **10.** $4x = 4x^2 + 1$ **11.** $8y^2 = 16y$ **12.** $3A^2 = -12A$

13. $-4x^2 = 12x$ **14.** $7p^2 = 28p$ **15.** $9p^2 = 12p - 4$ **16.** $4u^2 = 20u - 25$

17. $x^2 - 9 = 0$ **18.** $16v^2 - 25 = 0$ **19.** $x(x + 4) = 12$ **20.** $3t^2 - 48 = 0$

21. $2p^2 - 50 = 0$ **22.** $5y^2 - 45 = 0$ **23.** $3x^2 = 12$ **24.** $7v^2 = 28$

In Exercises 25–42, solve using the square root method.

25. $x^2 - 16 = 0$ **26.** $t^2 - 49 = 0$ **27.** $p^2 - 8 = 0$ **28.** $y^2 - 72 = 0$

29. $x^2 + 9 = 0$ **30.** $v^2 + 16 = 0$ **31.** $5y^2 - 20 = 0$ **32.** $3x^2 - 147 = 0$

33. $(x - 3)^2 = 36$ **34.** $(x - 1)^2 = 25$ **35.** $(3x + 5)^2 = 16$ **36.** $(4y - 1)^2 = 49$

37. $(2x + 3)^2 = -4$ **38.** $(4x - 1)^2 = -16$ **39.** $(5x - 2)^2 = 27$ **40.** $(3x + 8)^2 = 12$

41. $(1 - x)^2 = 9$ **42.** $(1 - x)^2 = -9$

In Exercises 43–56, what number should be added to complete the square of each expression?

43. $x^2 + 6x$ **44.** $x^2 - 8x$ **45.** $x^2 - 12x$ **46.** $x^2 + 20x$ **47.** $x^2 + 9x$ **48.** $x^2 - 3x$ **49.** $x^2 - 15x$

50. $x^2 + 11x$ **51.** $x^2 - \frac{1}{2}x$ **52.** $x^2 - \frac{1}{3}x$ **53.** $x^2 + \frac{2}{5}x$ **54.** $x^2 + \frac{4}{5}x$ **55.** $x^2 - 2.4x$ **56.** $x^2 + 1.6x$

In Exercises 57–72, solve by completing the square.

57. $x^2 + 2x = 3$ **58.** $y^2 + 8y - 2 = 0$ **59.** $t^2 - 6t = -5$ **60.** $x^2 + 10x = -21$

61. $y^2 - 4y + 3 = 0$ **62.** $x^2 - 7x + 12 = 0$ **63.** $x^2 - 5x = 9$ **64.** $x^2 + 7x = -11$

65. $x^2 + 3x = 4$ **66.** $x^2 - 9x + 7 = 0$ **67.** $2p^2 + 8p = -3$ **68.** $2x^2 - 4x + 3 = 0$

69. $2x^2 - 7x + 3 = 0$ **70.** $3x^2 - 5x - 10 = 0$ **71.** $\frac{x^2}{2} - 2x = \frac{1}{4}$ **72.** $\frac{t^2}{3} + \frac{2t}{3} + \frac{5}{6} = 0$

In Exercises 73–84, solve using the Quadratic Formula.

73. $t^2 + 3t - 1 = 0$ **74.** $t^2 + 2t = 1$ **75.** $s^2 + s + 1 = 0$ **76.** $2s^2 + 5s = -2$

77. $3x^2 - 3x - 4 = 0$ **78.** $4x^2 - 2x = 7$ **79.** $x^2 - 2x + 17 = 0$ **80.** $4m^2 + 7m + 8 = 0$

81. $5x^2 + 7x = 3$ **82.** $3x^2 + 5x = -11$ **83.** $\frac{1}{4}x^2 + \frac{2}{3}x - \frac{1}{2} = 0$ **84.** $\frac{1}{4}x^2 - \frac{2}{3}x - \frac{1}{3} = 0$

In Exercises 85–90, determine whether the discriminant is positive, negative, or zero, and indicate the number and type of root to expect. Do not solve.

85. $x^2 - 22x + 121 = 0$ **86.** $x^2 - 28x + 196 = 0$ **87.** $2y^2 - 30y + 68 = 0$

88. $-3y^2 + 27y + 66 = 0$ **89.** $9x^2 - 7x + 8 = 0$ **90.** $-3x^2 + 5x - 7 = 0$

In Exercises 91–110, solve using any method.

91. $v^2 - 8v = 20$ **92.** $v^2 - 8v = -20$ **93.** $t^2 + 5t - 6 = 0$ **94.** $t^2 + 5t + 6 = 0$

95. $(x + 3)^2 = 16$ **96.** $(x + 3)^2 = -16$ **97.** $(p - 2)^2 = 4p$ **98.** $(u + 5)^2 = 16u$

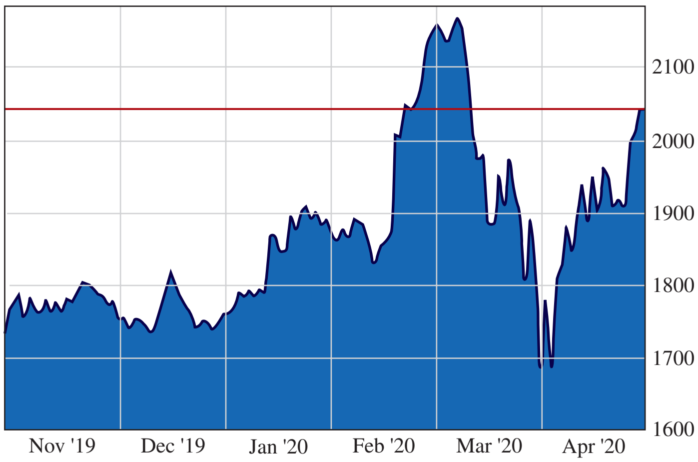
99. $8w^2 + 2w + 21 = 0$ **100.** $8w^2 + 2w - 21 = 0$ **101.** $3p^2 - 9p + 1 = 0$ **102.** $3p^2 - 9p - 1 = 0$

103. $\frac{2}{3}t^2 + \frac{4}{3}t = \frac{1}{5}$ **104.** $\frac{1}{2}x^2 + \frac{2}{3}x = \frac{2}{5}$ **105.** $x + \frac{12}{x} = 7$ **106.** $x - \frac{10}{x} = -3$

107. $\frac{4(x-2)}{x-3} + \frac{3}{x} = \frac{-3}{x(x-3)}$ **108.** $\frac{5}{y+4} = 4 + \frac{3}{y-2}$ **109.** $x^2 - 0.1x = 0.12$ **110.** $y^2 - 0.5y = -0.06$

Applications

111. Stock Value. From November 2019 to April 2020, Amazon stock was approximately worth $P = -200t^2 + 1000t + 900$, where P is the price of the stock in dollars, t is months, and $t = 0$ corresponds to November 2019. During what month(s) is the stock equal to \$1700?



Source: Based on Market activity, Common Stock, Amazon.com, Nasdaq, Inc. https://www.nasdaq.com/market-activity/stocks/amzn

112. Stock Value. From November 2019 to April 2020, Apple stock was approximately worth $P = -25t^2 + 175t + 15$, where P is the price of the stock in dollars, t is months, and $t = 0$ corresponds to November 2019. During what month(s) is the stock equal to \$265?

Source: Based on Market activity, Common Stock, Apple Inc, Nasdaq, Inc. https://www.nasdaq.com/market-activity/stocks/aapl

In Exercises 113 and 114 refer to the following:

Research indicates that monthly profit for Widgets R Us is modeled by the function

$$P = -100 + (0.2q - 3)q$$

where P is profit measured in millions of dollars and q is the quantity of widgets produced measured in thousands.

113. Business. Find the break-even point for a month to the nearest unit.

114. Business. Find the production level that produces a monthly profit of \$40 million.

In Exercises 115 and 116 refer to the following:

In response to economic conditions, a local business explores the effect of a price increase on weekly profit. The function

$$P = -5(x + 3)(x - 24)$$

models the effect that a price increase of x dollars on a bottle of wine will have on the profit P measured in dollars.

115. Business/Economics. What is the smallest price increase that will produce a weekly profit of \$460?

116. Business/Economics. What is the smallest price increase that will produce a weekly profit of \$630?

In Exercises 117 and 118 refer to the following:

An epidemiological study of the spread of COVID-19 in a small city finds that the total number P of people who contracted the coronavirus t days into an outbreak is modeled by the function

$$P = -t^2 + 13t + 130 \qquad 1 \le t \le 6$$

117. Health/Medicine. After approximately how many days will 160 people have contracted the coronavirus?

118. Health/Medicine. After approximately how many days will 172 people have contracted the coronavirus?

119. Environment: Reduce Your Margins, Save a Tree. Let's define the *usable area* of an 8.5-inch by 11-inch piece of paper as the rectangular space between the margins of that piece of paper. Assume the default margins are set up to be 1.25 inches wide (top and bottom) and 1 inch wide (left and right). Answer the following questions using this information.

- **a.** Determine the amount of usable space, in square inches, on one side of an 8.5-inch by 11-inch piece of paper with the default margins of 1.25-inch and 1-inch.
- **b.** If the default margins are reduced by x inches. Create and simplify the quadratic expression that represents the new usable area, in square inches, of one side of an 8.5-inch by 11-inch piece of paper if the default margins are each reduced by x inches.
- **c.** Subtract the usable space in part (a) from the expression in part (b). Explain what this difference represents.
- **d.** If 10 pages are printed using the new margins and as a result saved one whole sheet of paper, then by how much were the margins reduced? Round to the nearest tenth of an inch.

120. Environment: Reduce Your Margins, Save a Tree. Repeat Exercise 119 assuming the default margins are 1 inch all the way around (left, right, top, and bottom). If 15 pages are printed using the new margins and as a result saved one whole sheet of paper, then by how much were the margins reduced? Round to the nearest tenth of an inch.

121. Television. A standard 32-inch television has a 32-inch diagonal and a 25-inch width. What is the height of the 32-inch television?

122. Television. A 42-inch LCD television has a 42-inch diagonal and a 20-inch height. What is the width of the 42-inch LCD television?

123. Numbers. Find two consecutive numbers such that their sum is 35 and their product is 306.

124. Numbers. Find two consecutive odd integers such that their sum is 24 and their product is 143.

125. Geometry. The area of a rectangle is 135 square feet. The width is 6 feet less than the length. Find the dimensions of the rectangle.

126. Geometry. A rectangle has an area of 31.5 square meters. If the length is 2 more than twice the width, find the dimensions of the rectangle.

127. Geometry. A triangle has a height that is 2 more than 3 times the base and an area of 60 square units. Find the base and height.

128. Geometry. A square's side is increased by 3 yards, which corresponds to an increase in the area by 69 square yards. How many yards is the side of the initial square?

129. Falling Objects. If a person drops a water balloon off the rooftop of a 100-foot building, the height of the water balloon is given by the equation $h = -16t^2 + 100$, where t is in seconds. When will the water balloon hit the ground?

130. Falling Objects. If the person in Exercise 129 throws the water balloon downward with a speed of 5 feet per second, the height of the water balloon is given by the equation $h = -16t^2 - 5t + 100$, where t is in seconds. When will the water balloon hit the ground?

131. Gardening. A square garden has an area of 900 square feet. If a sprinkler (with a circular pattern) is placed in the center of the garden, what is the minimum radius of spray the sprinkler would need in order to water all of the garden?

132. Sports. A baseball diamond is a square. The distance from base to base is 90 feet. What is the distance from home plate to second base?

133. Volume. A flat square piece of cardboard is used to construct an open box. Cutting a 1-foot by 1-foot square off of each corner and folding up the edges will yield an open box (assuming these edges are taped together). If the desired volume of the box is 9 cubic feet, what are the dimensions of the original square piece of cardboard?

134. Volume. A rectangular piece of cardboard whose length is twice its width is used to construct an open box. Cutting a 1-foot by 1-foot square off of each corner and folding up the edges will yield an open box. If the desired volume is 12 cubic feet, what are the dimensions of the original rectangular piece of cardboard?

135. Gardening. A landscaper has planted a rectangular garden that measures 8 feet by 5 feet. He has ordered 1 cubic yard (27 cubic feet) of stones for a border along the outside of the garden. If the border needs to be 4 inches deep and he wants to use all of the stones, how wide should the border be?

136. Gardening. A gardener has planted a semicircular rose garden with a radius of 6 feet, and 2 cubic yards of mulch (1 cubic yard = 27 cubic feet) are being delivered. Assuming she uses all of the mulch, how deep will the layer of mulch be?

137. Work. Lindsay and Kimmie, working together, can balance the financials for the Kappa Kappa Gamma sorority in 6 days. Lindsay by herself can complete the job in 5 days less than Kimmie. How long will it take Lindsay to complete the job by herself?

138. Work. When Jack cleans the house, it takes him 4 hours. When Ryan cleans the house, it takes him 6 hours. How long would it take both of them if they worked together?

Catch the Mistake

In Exercises 139–142, explain the mistake that is made.

139.
$$\begin{aligned} t^2 - 5t - 6 &= 0 \\ (t-3)(t-2) &= 0 \\ t &= 2, 3 \end{aligned}$$

140.
$$\begin{aligned} (2y-3)^2 &= 25 \\ 2y - 3 &= 5 \\ 2y &= 8 \\ y &= \frac{5}{4} \\ y &= 4 \end{aligned}$$

141.
$$\begin{aligned} 16a^2 + 9 &= 0 \\ 16a^2 &= -9 \\ a^2 &= -\frac{9}{16} \\ a &= \pm\sqrt{\frac{9}{16}} \\ a &= \pm\frac{3}{4} \end{aligned}$$

142.
$$\begin{aligned} 2x^2 - 4x &= 3 \\ 2(x^2 - 2x) &= 3 \\ 2(x^2 - 2x + 1) &= 3 + 1 \\ 2(x-1)^2 &= 4 \\ (x-1)^2 &= 2 \\ x - 1 &= \pm\sqrt{2} \\ x &= 1 \pm \sqrt{2} \end{aligned}$$

Conceptual

In Exercises 143–146, determine whether the following statements are true or false.

143. The equation $(3x + 1)^2 = 16$ has the same solution set as the equation $3x + 1 = 4$.

144. The quadratic equation $ax^2 + bx + c = 0$ can be solved by the square root method only if $b = 0$.

145. All quadratic equations can be solved exactly.

146. The Quadratic Formula can be used to solve any quadratic equation.

147. Write a quadratic equation in general form that has $x = a$ as a repeated real root.

148. Write a quadratic equation in general form that has $x = bi$ as a root.

149. Write a quadratic equation in general form that has the solution set $\{2, 5\}$.

150. Write a quadratic equation in general form that has the solution set $\{-3, 0\}$.

In Exercises 151–158, solve for the indicated variable in terms of other variables.

151. Solve $s = \frac{1}{2}gt^2$ for t.

152. Solve $A = P(1 + r)^2$ for r.

153. Solve $a^2 + b^2 = c^2$ for c.

154. Solve $P = EI - RI^2$ for I.

155. Solve the equation by factoring: $x^4 - 4x^2 = 0$.

156. Solve the equation by factoring: $3x - 6x^2 = 0$.

157. Solve the equation using factoring by grouping: $x^3 + x^2 - 4x - 4 = 0$.

158. Solve the equation using factoring by grouping: $x^3 + 2x^2 - x - 2 = 0$.

Challenge

159. Show that the sum of the roots of a quadratic equation is equal to $-\frac{b}{a}$.

160. Show that the product of the roots of a quadratic equation is equal to $\frac{c}{a}$.

161. Write a quadratic equation in general form whose solution set is $\{3 + \sqrt{5}, 3 - \sqrt{5}\}$.

162. Write a quadratic equation in general form whose solution set is $\{2 - i, 2 + i\}$.

163. Aviation. An airplane takes 1 hour longer to go a distance of 600 miles flying against a headwind than on the return trip with a tailwind. If the speed of the wind is 50 miles per hour, find the speed of the plane in still air.

164. Boating. A speedboat takes 1 hour longer to go 24 miles up a river than to return. If the boat cruises at 10 miles per hour in still water, what is the rate of the current?

165. Find a quadratic equation whose two distinct real roots are the negatives of the two distinct real roots of the equation $ax^2 + bx + c = 0$.

166. Find a quadratic equation whose two distinct real roots are the reciprocals of the two distinct real roots of the equation $ax^2 + bx + c = 0$.

167. A small jet and a 757 leave Atlanta at 1 P.M. The small jet is traveling due west. The 757 is traveling due south. The speed of the 757 is 100 miles per hour faster than the small jet. At 3 P.M. the planes are 1000 miles apart. Find the average speed of each plane.

168. Two boats leave Key West at noon. The smaller boat is traveling due west. The larger boat is traveling due south. The speed of the larger boat is 10 miles per hour faster than the speed of the smaller boat. At 3 P.M. the boats are 150 miles apart. Find the average speed of each boat.

Technology

169. Solve the equation $x^2 - x = 2$ by first writing it in standard form and then factoring. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 - x$ and $y_2 = 2$). At what x-values do these two graphs intersect? Do those points agree with the solution set you found?

170. Solve the equation $x^2 - 2x = -2$ by first writing it in standard form and then using the quadratic formula. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 - 2x$ and $y_2 = -2$). Do these graphs intersect? Does this agree with the solution set you found?

171. a. Solve the equation $x^2 - 2x = b$, $b = 8$ by first writing it in standard form. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 - 2x$ and $y_2 = b$). At what x values do these two graphs intersect? Do those points agree with the solution set you found?

b. Repeat part (a) for $b = -3, -1, 0$, and 5.

172. a. Solve the equation $x^2 + 2x = b$, $b = 8$ by first writing it in standard form. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 + 2x$ and $y_2 = b$). At what x values do these two graphs intersect? Do those points agree with the solution set you found?

b. Repeat part (a) for $b = -3, -1, 0$, and 5.

1.4 Other Types of Equations

SKILLS OBJECTIVES

- Solve radical equations.
- Solve equations that are quadratic in form using *u*-substitutions.
- Solve equations that are factorable.

CONCEPTUAL OBJECTIVES

- Check for extraneous solutions.
- Recognize the *u*-substitution required to transform the equation into a simpler quadratic equation.
- Recognize when a polynomial equation or an equation with rational exponents can be factored either by grouping or by first factoring out a greatest common factor.

1.4.1 Radical Equations

1.4.1 Skill Solve radical equations.

1.4.1 Conceptual Check for extraneous solutions.

Radical equations are equations in which the variable is inside a radical (that is, under a square root, cube root, or higher root). Examples of radical equations follow.

$$\sqrt{x-3} = 2 \qquad \sqrt{2x+3} = x \qquad \sqrt{x+2} + \sqrt{7x+2} = 6$$

Until now your experience has been with linear and quadratic equations. Often you can transform a radical equation into a simple linear or quadratic equation. Sometimes the transformation process yields **extraneous solutions**, or apparent solutions that may solve the transformed problem but are not solutions of the original radical equation. Therefore, it is very important to check your answers.

EXAMPLE 1 | Solving an Equation Involving a Radical

Solve the equation $\sqrt{x-3} = 2$.

Solution

Square both sides of the equation. $(\sqrt{x-3})^2 = 2^2$

Simplify. $x - 3 = 4$

Solve the resulting linear equation. $\boxed{x = 7}$

The solution set is $\{7\}$.

Check: $\sqrt{7-3} = \sqrt{4} = 2$

Your Turn Solve the equation $\sqrt{3p+4} = 5$.

Answer $p = 7$ or $\{7\}$

STUDY TIP

Extraneous solutions are common when we deal with radical equations, so remember to check your answers.

When both sides of an equation are squared, extraneous solutions can arise. For example, take the equation

$$x = 2$$

If we square both sides of this equation, then the resulting equation, $x^2 = 4$, has two solutions: $x = -2$ and $x = 2$. Notice that the value $x = -2$ is not in the solution set of the original equation $x = 2$. Therefore, we say that $x = -2$ is an extraneous solution.

In solving a radical equation, we square both sides of the equation and then solve the resulting equation. The solutions to the resulting equation can sometimes be extraneous in that they do not satisfy the *original* radical equation.

EXAMPLE 2 | Solving an Equation Involving a Radical

Solve the equation $\sqrt{2x+3} = x$.

Solution

Square both sides of the equation. $(\sqrt{2x+3})^2 = x^2$

Simplify. $2x + 3 = x^2$

Write the quadratic equation in standard form. $x^2 - 2x - 3 = 0$

Factor. $(x-3)(x+1) = 0$

Use the zero product property. $x = 3 \text{ or } x = -1$

Check these values to see whether they *both* make the original equation statement true.

$$x = 3: \sqrt{2(3)+3} = 3 \Rightarrow \sqrt{6+3} = 3 \Rightarrow \sqrt{9} = 3 \Rightarrow 3 = 3 \quad \checkmark$$

$$x = -1: \sqrt{2(-1)+3} = -1 \Rightarrow \sqrt{-2+3} = -1 \Rightarrow \sqrt{1} = -1 \Rightarrow 1 \neq -1 \quad \text{X}$$

The solution is $\boxed{x = 3}$. The solution set is $\{3\}$.

Your Turn Solve the equation $\sqrt{12+t} = t$.

Answer $t = 4$ or $\{4\}$

Your Turn Solve the equation $\sqrt{2x+6} = x + 3$.

Answer $x = -1$ and $x = -3$ or $\{-3, -1\}$

Concept Check

Check the extraneous solution $x = -1$ and explain why it does not satisfy the original equation.

Answer: $\sqrt{2(-1)+3} \stackrel{?}{=} -1$

$\sqrt{1} \stackrel{?}{=} -1$

$x = -1$ is extraneous.

What happened in Example 2? When we transformed the radical equation into a quadratic equation, we created an **extraneous solution**, $x = -1$, a solution that appears to solve the original equation but does not. When solving radical equations, answers must be checked to avoid including extraneous solutions in the solution set.

EXAMPLE 3 | Solving an Equation That Involves a Radical

Solve the equation $4x - 2\sqrt{x+3} = -10$.

Solution

Subtract $4x$ from both sides. $-2\sqrt{x+3} = -10 - 4x$

Divide both sides by -2. $\sqrt{x+3} = 2x + 5$

Square both sides. $x + 3 = \underbrace{(2x+5)(2x+5)}_{(2x+5)^2}$

Eliminate the parentheses. $x + 3 = 4x^2 + 20x + 25$

Rewrite the quadratic equation in standard form. $4x^2 + 19x + 22 = 0$

Factor. $(4x + 11)(x + 2) = 0$

Solve. $x = -\frac{11}{4}$ and $x = -2$

The apparent solutions are $-\frac{11}{4}$ and -2. Note that $-\frac{11}{4}$ does not satisfy the original equation; therefore it is extraneous. The solution is $\boxed{x = -2}$. The solution set is $\{-2\}$.

Your Turn Solve the equation $2x - 4\sqrt{x+2} = -6$.

Answer

$x = -1$ or $\{-1\}$

Common Mistake

Correct

Square the expression.

$$(3 + \sqrt{x+2})^2$$

Write the square as a product of two factors.

$$(3 + \sqrt{x+2})(3 + \sqrt{x+2})$$

Use the FOIL method.

$$9 + 6\sqrt{x+2} + (x+2)$$

Incorrect

Square the expression.

$$(3 + \sqrt{x+2})^2$$

The **error** occurs here when only individual terms are squared.

$$\neq 9 + (x+2)$$

In Examples 1 through 3 each equation only contained one radical each. The next example contains two radicals. Our technique will be to isolate one radical on one side of the equation with the other radical on the other side of the equation.

Video **EXAMPLE 4** | **Solving an Equation with More Than One Radical**

Solve the equation $\sqrt{x+2} + \sqrt{7x+2} = 6$.

Solution

Subtract $\sqrt{x+2}$ from both sides.	$\sqrt{7x+2} = 6 - \sqrt{x+2}$
Square both sides.	$(\sqrt{7x+2})^2 = (6 - \sqrt{x+2})^2$
Simplify.	$7x + 2 = (6 - \sqrt{x+2})(6 - \sqrt{x+2})$
Multiply the expressions on the right side of the equation.	$7x + 2 = 36 - 12\sqrt{x+2} + (x+2)$
Isolate the term with the radical on the left side.	$12\sqrt{x+2} = 36 + x + 2 - 7x - 2$
Combine like terms on the right side.	$12\sqrt{x+2} = 36 - 6x$
Divide by 6.	$2\sqrt{x+2} = 6 - x$
Square both sides.	$4(x+2) = (6-x)^2$
Simplify.	$4x + 8 = 36 - 12x + x^2$
Rewrite the quadratic equation in standard form.	$x^2 - 16x + 28 = 0$
Factor.	$(x-14)(x-2) = 0$
Solve.	$x = 14$ and $x = 2$

The apparent solutions are 2 and 14. Note that $x = 14$ does not satisfy the original equation; therefore, it is extraneous. The solution is $\boxed{x = 2}$. The solution set is $\{2\}$.

STUDY TIP

Remember to check both solutions.

Your Turn Solve the equation $\sqrt{x-4} = 5 - \sqrt{x+1}$.

Answer

$x = 8$ or $\{8\}$

Procedure for Solving Radical Equations

STEP 1 Isolate the term with a radical on one side.

STEP 2 Raise both (*entire*) sides of the equation to the power that will eliminate this radical, and simplify the equation.

STEP 3 If a radical remains, repeat Steps 1 and 2.

STEP 4 Solve the resulting linear or quadratic equation.

STEP 5 Check the solutions and eliminate any extraneous solutions.

Note: If there is more than one radical in the equation, it does not matter which radical is isolated first.

1.4.2 Equations Quadratic in Form: *u*-Substitution

1.4.2 **Skill** Solve equations that are quadratic in form using *u*-substitutions.

1.4.2 **Conceptual** Recognize the *u*-substitution required to transform the equation into a simpler quadratic equation.

Equations that are higher order or that have fractional powers often can be transformed into a quadratic equation by introducing a *u*-substitution. When this is the case, we say that equations are **quadratic in form**. In the following table, the two original equations are quadratic in form because they can be transformed into a quadratic equation given the correct substitution.

Original Equation	Substitution	New Equation
$x^4 - 3x^2 - 4 = 0$	$u = x^2$	$u^2 - 3u - 4 = 0$
$t^{2/3} + 2t^{1/3} + 1 = 0$	$u = t^{1/3}$	$u^2 + 2u + 1 = 0$
$\frac{2}{y} - \frac{1}{\sqrt{y}} + 1 = 0$	$u = y^{-1/2}$	$2u^2 - u + 1 = 0$

For example, the equation $x^4 - 3x^2 - 4 = 0$ is a fourth-degree equation in x. How did we know that $u = x^2$ would transform the original equation into a quadratic equation? If we rewrite the original equation as $(x^2)^2 - 3(x^2) - 4 = 0$, the expression in parentheses is the u-substitution.

Let us introduce the substitution $u = x^2$. Note that squaring both sides implies $u^2 = x^4$. We then replace x^2 in the original equation with u, and x^4 in the original equation with u^2, which leads to a quadratic equation in u: $u^2 - 3u - 4 = 0$.

Words	**Math**
Solve for x.	$x^4 - 3x^2 - 4 = 0$
Introduce u-substitution.	$u = x^2$ [Note that $u^2 = x^4$.]
Write the quadratic equation in u.	$u^2 - 3u - 4 = 0$
Factor.	$(u - 4)(u + 1) = 0$
Solve for u.	$u = 4$ or $u = -1$
Transform back to x, $u = x^2$.	$x^2 = 4$ or $x^2 = -1$
Solve for x.	$x = \pm 2$ or $x = \pm i$

The solution set is $\{\pm 2, \pm i\}$.

It is important to correctly determine the appropriate substitution in order to arrive at an equation quadratic in form. For example, $t^{2/3} + 2t^{1/3} + 1 = 0$ is an original equation given in the table. If we rewrite this equation as $(t^{1/3})^2 + 2(t^{1/3}) + 1 = 0$, then it becomes apparent that the correct substitution is $u = t^{1/3}$, which transforms the equation in t into a quadratic equation in u: $u^2 + 2u + 1 = 0$.

Procedure for Solving Equations Quadratic in Form

STEP 1 Identify the substitution.
STEP 2 Transform the equation into a quadratic equation.
STEP 3 Solve the quadratic equation.
STEP 4 Apply the substitution to rewrite the solution in terms of the original variable.
STEP 5 Solve the resulting equation.
STEP 6 Check the solutions in the original equation.

EXAMPLE 5 | Solving an Equation Quadratic in Form with Negative Exponents

Find the solutions to the equation $x^{-2} - x^{-1} - 12 = 0$.

Solution

Rewrite the original equation.	$(x^{-1})^2 - (x^{-1}) - 12 = 0$
Determine the u-substitution.	$u = x^{-1}$ [Note that $u^2 = x^{-2}$.]
The original equation in x corresponds to a quadratic equation in u.	$u^2 - u - 12 = 0$
Factor.	$(u - 4)(u + 3) = 0$
Solve for u.	$u = 4$ or $u = -3$

The most common mistake is forgetting to transform back to *x*.

Transform back to *x*. Let $u = x^{-1}$. $x^{-1} = 4$ or $x^{-1} = -3$

Write x^{-1} as $\frac{1}{x}$. $\frac{1}{x} = 4$ or $\frac{1}{x} = -3$

Solve for *x*. $\boxed{x = \frac{1}{4}}$ or $\boxed{x = -\frac{1}{3}}$

The solution set is $\left\{-\frac{1}{3}, \frac{1}{4}\right\}$.

Your Turn Find the solutions to the equation $x^{-2} - x^{-1} - 6 = 0$.

Answer

The solution is $x = -\frac{1}{2}$ or $x = \frac{1}{3}$. The solution set is $\left\{-\frac{1}{2}, \frac{1}{3}\right\}$.

Video

EXAMPLE 6 | Solving an Equation Quadratic in Form with Fractional Exponents

Find the solutions to the equation $x^{2/3} - 3x^{1/3} - 10 = 0$.

Solution

Rewrite the original equation. $(x^{1/3})^2 - 3x^{1/3} - 10 = 0$

Identify the substitution as $u = x^{1/3}$. $u^2 - 3u - 10 = 0$

Factor. $(u - 5)(u + 2) = 0$

Solve for *u*. $u = 5$ or $u = -2$

Let $u = x^{1/3}$ again. $x^{1/3} = 5$ $x^{1/3} = -2$

Cube both sides of the equations. $(x^{1/3})^3 = (5)^3$ $(x^{1/3})^3 = (-2)^3$

Simplify. $\boxed{x = 125}$ $\boxed{x = -8}$

The solution set is $\boxed{\{-8, 125\}}$, which a check will confirm.

STUDY TIP

Remember to transform back to the original variable.

Your Turn Find the solution to the equation $2t - 5t^{1/2} - 3 = 0$.

Answer $t = 9$ or $\{9\}$.

Concept Check

What do we let *u* be equal to in order to transform the equation

$$2t - 5t^{1/2} - 3 = 0$$

into a quadratic equation?

Answer: $u = t^{1/2}$

1.4.3 Factorable Equations

1.4.3 Skill Solve equations that are factorable.

1.4.3 Conceptual Recognize when a polynomial equation or an equation with rational exponents can be factored either by grouping or by first factoring out a greatest common factor.

Some equations (both polynomial and with rational exponents) that are factorable can be solved using the zero product property.

Video **EXAMPLE 7** | **Solving an Equation with Rational Exponents by Factoring**

Solve the equation $x^{7/3} - 3x^{4/3} - 4x^{1/3} = 0$.

Solution

Factor the left side of the equation. $x^{1/3}(x^2 - 3x - 4) = 0$

Factor the quadratic expression. $x^{1/3}(x-4)(x+1) = 0$

Apply the zero product property. $x^{1/3} = 0$ or $x - 4 = 0$ or $x + 1 = 0$

Solve for x. $\boxed{x = 0}$ or $\boxed{x = 4}$ or $\boxed{x = -1}$

The solution set is $\{-1, 0, 4\}$.

Video **EXAMPLE 8** | **Solving a Polynomial Equation Using Factoring by Grouping**

Solve the equation $x^3 + 2x^2 - x - 2 = 0$.

Solution

Factor by grouping (Chapter 0). $(x^3 - x) + (2x^2 - 2) = 0$

Identify the common factors. $x(x^2 - 1) + 2(x^2 - 1) = 0$

Factor. $(x+2)(x^2 - 1) = 0$

Factor the quadratic expression. $(x+2)(x-1)(x+1) = 0$

Apply the zero product property. $x + 2 = 0$ or $x - 1 = 0$ or $x + 1 = 0$

Solve for x. $\boxed{x = -2}$ or $\boxed{x = 1}$ or $\boxed{x = -1}$

The solution set is $\{-2, -1, 1\}$.

Your Turn Solve the equation $x^3 + x^2 - 4x - 4 = 0$.

Answer

$x = -1$ or $x = \pm 2$ or $\{-2, -1, 2\}$

Concept Check

How do we initially group the cubic equation in $x^3 + x^2 - 4x - 4 = 0$?

Answer: $(x^3 - 4x) + (x^2 - 4) = 0$

Section 1.4 Summary

Radical equations, equations quadratic in form, and factorable equations can often be solved by transforming them into simpler linear or quadratic equations.

- **Radical Equations:** Isolate the term containing a radical and raise it to the appropriate power that will eliminate the radical. If there is more than one radical, it does not matter which radical is isolated first. Raising radical equations to powers may cause extraneous solutions, so check each solution.
- **Equations Quadratic in Form:** Identify the *u*-substitution that transforms the equation into a quadratic equation. Solve the quadratic equation and then remember to transform back to the original variable.
- **Factorable Equations:** Look for a factor common to all terms or factor by grouping.

Section 1.4 Exercises

Skills

In Exercises 1–40, solve the radical equation for the given variable.

1. $\sqrt{t-5}=2$
2. $\sqrt{2t-7}=3$
3. $(4p-7)^{1/2}=5$
4. $11=(21-p)^{1/2}$
5. $\sqrt{u+1}=-4$
6. $-\sqrt{3-2u}=9$
7. $\sqrt[3]{5x+2}=3$
8. $\sqrt[3]{1-x}=-2$
9. $(4y+1)^{1/3}=-1$
10. $(5x-1)^{1/3}=4$
11. $\sqrt{12+x}=x$
12. $x=\sqrt{56-x}$
13. $y=5\sqrt{y}$
14. $\sqrt{y}=\dfrac{y}{4}$
15. $s=3\sqrt{s-2}$
16. $-2s=\sqrt{3-s}$
17. $\sqrt{2x+6}=x+3$
18. $\sqrt{8-2x}=2x-2$
19. $\sqrt{1-3x}=x+1$
20. $\sqrt{2-x}=x-2$
21. $3x-6\sqrt{x-1}=3$
22. $5x-10\sqrt{x+2}=-10$
23. $3x-6\sqrt{x+2}=3$
24. $2x-4\sqrt{x+1}=4$
25. $3\sqrt{x+4}-2x=9$
26. $2\sqrt{x+1}-3x=-5$
27. $\sqrt{x^2-4}=x-1$
28. $\sqrt{25-x^2}=x+1$
29. $\sqrt{x^2-2x-5}=x+1$
30. $\sqrt{2x^2-8x+1}=x-3$
31. $\sqrt{3x+1}-\sqrt{6x-5}=1$
32. $\sqrt{2-x}+\sqrt{6-5x}=6$
33. $\sqrt{x+12}+\sqrt{8-x}=6$
34. $\sqrt{5-x}+\sqrt{3x+1}=4$
35. $\sqrt{2x-1}-\sqrt{x-1}=1$
36. $\sqrt{8-x}=2+\sqrt{2x+3}$
37. $\sqrt{3x-5}=7-\sqrt{x+2}$
38. $\sqrt{x+5}=1+\sqrt{x-2}$
39. $\sqrt{2+\sqrt{x}}=\sqrt{x}$
40. $\sqrt{2-\sqrt{x}}=\sqrt{x}$

In Exercises 41–70, solve the equations by introducing a substitution that transforms these equations to quadratic form.

41. $x^{2/3}+2x^{1/3}=0$
42. $x^{1/2}-2x^{1/4}=0$
43. $x^4-3x^2+2=0$
44. $x^4-8x^2+16=0$
45. $2x^4+7x^2+6=0$
46. $x^8-17x^4+16=0$
47. $(2x+1)^2+5(2x+1)+4=0$
48. $(x-3)^2+6(x-3)+8=0$
49. $4(t-1)^2-9(t-1)=-2$
50. $2(1-y)^2+5(1-y)-12=0$
51. $x^{-8}-17x^{-4}+16=0$
52. $2u^{-2}+5u^{-1}-12=0$
53. $3y^{-2}+y^{-1}-4=0$
54. $5a^{-2}+11a^{-1}+2=0$
55. $z^{2/5}-2z^{1/5}+1=0$
56. $2x^{1/2}+x^{1/4}-1=0$
57. $(x+3)^{5/3}=32$
58. $(x+2)^{4/3}=16$
59. $(x+1)^{2/3}=4$
60. $(x-7)^{4/3}=81$
61. $6t^{-2/3}-t^{-1/3}-1=0$
62. $t^{-2/3}-t^{-1/3}-6=0$
63. $3=\dfrac{1}{(x+1)^2}+\dfrac{2}{(x+1)}$
64. $\dfrac{1}{(x+1)^2}+\dfrac{4}{(x+1)}+4=0$
65. $\left(\dfrac{1}{2x-1}\right)^2+\left(\dfrac{1}{2x-1}\right)-12=0$
66. $\dfrac{5}{(2x+1)^2}-\dfrac{3}{(2x+1)}=2$
67. $u^{4/3}-5u^{2/3}=-4$
68. $u^{4/3}+5u^{2/3}=-4$
69. $t=\sqrt[4]{t^2+6}$
70. $u=\sqrt[4]{-2u^2-1}$

In Exercises 71–86, solve by factoring.

71. $x^3-x^2-12x=0$
72. $2y^3-11y^2+12y=0$
73. $4p^3-9p=0$
74. $25x^3=4x$
75. $u^5-16u=0$
76. $t^5-81t=0$
77. $x^3-5x^2-9x+45=0$
78. $2p^3-3p^2-8p+12=0$
79. $y(y-5)^3-14(y-5)^2=0$
80. $v(v+3)^3-40(v+3)^2=0$
81. $x^{9/4}-2x^{5/4}-3x^{1/4}=0$
82. $u^{7/3}+u^{4/3}-20u^{1/3}=0$
83. $t^{5/3}-25t^{-1/3}=0$
84. $4x^{9/5}-9x^{-1/5}=0$
85. $y^{3/2}-5y^{1/2}+6y^{-1/2}=0$
86. $4p^{5/3}-5p^{2/3}-6p^{-1/3}=0$

Applications

In Exercises 87 and 88 refer to the following:

An analysis of sales indicates that demand for a product during a calendar year is modeled by

$$d=3\sqrt{t+1}-0.75t$$

where d is demand in millions of units and t is the month of the year where $t=0$ represents January.

87. **Economics.** During which month(s) is demand 3 million units?
88. **Economics.** During which month(s) is demand 4 million units?

In Exercises 89 and 90 refer to the following:

Body Surface Area (BSA) is used in physiology and medicine for many clinical purposes. BSA can be modeled by the function

$$BSA=\sqrt{\frac{wh}{3600}}$$

where w is weight in kilograms and h is height in centimeters.

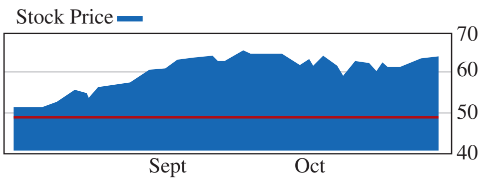
89. Health. The BSA of a 72-kilogram female is 1.8. Find the height of the female to the nearest centimeter.

90. Health. The BSA of a 177-centimeter-tall male is 2.1. Find the weight of the male to the nearest kilogram.

91. Insurance: Health. Cost for health insurance with a private policy is given by $C = \sqrt{10 + a}$, where C is the cost per day and a is the insured's age in years. Health insurance for a 6-year-old, $a = 6$, is \$4 a day (or \$1460 per year). At what age would someone be paying \$9 a day (or \$3285 per year)?

92. Insurance: Life. Cost for life insurance is given by $C = \sqrt{5a + 1}$, where C is the cost per day and a is the insured's age in years. Life insurance for a newborn, $a = 0$, is \$1 a day (or \$365 per year). At what age would someone be paying \$20 a day (or \$7300 per year)?

93. Stock Value. The stock price of a certain pharmaceutical company from August to November can be approximately modeled by the equation $P = 5\sqrt{t^2 + 1} + 50$, where P is the price of the stock in dollars and t is the month with $t = 0$ corresponding to August. Assuming this trend continues, when would the stock be worth \$85?



94. Grades. The average combined math and verbal SAT score of incoming freshman at a university is given by the equation $S = 1200 + 10\sqrt{2t}$, where t is in years and $t = 0$ corresponds to 1990. What year will the incoming class have an average SAT score of 1295?

95. Speed of Sound. A man buys a house with an old well but does not know how deep the well is. To get an estimate he decides to drop a rock in the opening of the well and time how long it takes until he hears the splash. The total elapsed time T given by $T = t_1 + t_2$, is the sum of the time it takes for the rock to reach the water, t_1, and the time it takes for the sound of the splash to travel to the top of the well, t_2. The time (seconds) that it takes for the rock to reach the water is given by $t_1 = \frac{\sqrt{d}}{4}$, where d is the depth of the well in feet. Since the speed of sound is 1100 ft/s, the time (seconds) it takes for the sound to reach the top of the well is $t_2 = \frac{d}{1100}$. If the splash is heard after 3 seconds, how deep is the well?

96. Speed of Sound. If the owner of the house in Exercise 95 forgot to account for the speed of sound, what would he have calculated the depth of the well to be?

97. Physics: Pendulum. The period (T) of a pendulum is related to the length (L) of the pendulum and acceleration due to gravity (g) by the formula $T = 2\pi\sqrt{\frac{L}{g}}$. If gravity is 9.8 m/s^2 and the period is 1 second, find the approximate length of the pendulum. Round to the nearest centimeter. *Note:* 100 cm = 1 m.

98. Physics: Pendulum. The period (T) of a pendulum is related to the length (L) of the pendulum and acceleration due to gravity (g) by the formula $T = 2\pi\sqrt{\frac{L}{g}}$. If gravity is 32 ft/s^2 and the period is 1 second, find the approximate length of the pendulum. Round to the nearest inch. *Note:* 12 in. = 1 ft.

In Exercises 99 and 100, refer to the following:

Einstein's special theory of relativity states that time is relative: Time speeds up or slows down, depending on how fast one object is moving with respect to another. For example, a space probe traveling at a velocity v near the speed of light c will have "clocked" a time t hours, but for a stationary observer on Earth that corresponds to a time t_0. The formula governing this relativity is given by

$$t = t_0\sqrt{1 - \frac{v^2}{c^2}}$$

99. Physics: Special Theory of Relativity. If the time elapsed on a space probe mission is 18 years but the time elapsed on Earth during that mission is 30 years, how fast is the space probe traveling? Give your answer relative to the speed of light.

100. Physics: Special Theory of Relativity. If the time elapsed on a space probe mission is 5 years but the time elapsed on Earth during that mission is 30 years, how fast is the space probe traveling? Give your answer relative to the speed of light.

Catch the Mistake

In Exercises 101–104, explain the mistake that is made.

101. Solve the equation $\sqrt{3t + 1} = -4$.

Solution

$$3t + 1 = 16$$
$$3t = 15$$
$$t = 5$$

This is incorrect. What mistake was made?

102. Solve the equation $x = \sqrt{x + 2}$.

Solution

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = -1, x = 2$$

This is incorrect. What mistake was made?

103. Solve the equation $x^{2/3} - x^{1/3} - 20 = 0$.

Solution
$$u = x^{1/3}$$
$$u^2 - u - 20 = 0$$
$$(u-5)(u+4) = 0$$
$$x = 5, x = -4$$

This is incorrect. What mistake was made?

104. Solve the equation $x^4 - 2x^2 = 3$.

Solution
$$x^4 - 2x^2 - 3 = 0$$
$$u = x^2$$
$$u^2 - 2u - 3 = 0$$
$$(u-3)(u+1) = 0$$
$$u = -1, u = 3$$
$$u = x^2$$
$$x^2 = -1, x^2 = 3$$
$$x = \pm 1, x = \pm 3$$

This is incorrect. What mistake was made?

Conceptual

In Exercises 105–108, determine whether each statement is true or false.

105. The equation $(2x-1)^6 + 4(2x-1)^3 + 3 = 0$ is quadratic in form.

106. The equation $t^{25} + 2t^5 + 1 = 0$ is quadratic in form.

107. If two solutions are found and one does not check, then the other does not check.

108. Squaring both sides of $\sqrt{x+2} + \sqrt{x} = \sqrt{x+5}$ leads to $x + 2 + x = x + 5$.

Challenge

109. Solve $\sqrt{x^2} = x$.

110. Solve $\sqrt{x^2} = -x$.

111. Solve the equation $3x^2 + 2x = \sqrt{3x^2 + 2x}$ *without squaring* both sides.

112. Solve the equation $3x^{7/12} - x^{5/6} - 2x^{1/3} = 0$.

113. Solve the equation $\sqrt{x+6} + \sqrt{11+x} = 5\sqrt{3+x}$.

114. Solve the equation $\sqrt[4]{2x\sqrt[3]{x\sqrt{x}}} = 2$.

Technology

115. Solve the equation $\sqrt{x-3} = 4 - \sqrt{x+2}$. Plot both sides of the equation in the same viewing screen, $y_1 = \sqrt{x-3}$ and $y_2 = 4 - \sqrt{x+2}$, and zoom in on the x-coordinate of the point of intersection. Does the graph agree with your solution?

116. Solve the equation $2\sqrt{x+1} = 1 + \sqrt{3-x}$. Plot both sides of the equation in the same viewing screen, $y_1 = 2\sqrt{x+1}$ and $y_2 = 1 + \sqrt{3-x}$, and zoom in on the x-coordinate of the points of intersection. Does the graph agree with your solution?

117. Solve the equation $-4 = \sqrt{x+3}$. Plot both sides of the equation in the same viewing screen, $y_1 = -4$ and $y_2 = \sqrt{x+3}$. Does the graph agree or disagree with your solution?

118. Solve the equation $x^{1/4} = -4x^{1/2} + 21$. Plot both sides of the equation in the same viewing screen, $y_1 = x^{1/4}$ and $y_2 = -4x^{1/2} + 21$. Does the point(s) of intersection agree with your solution?

119. Solve the equation $x^{1/2} = -4x^{1/4} + 21$. Plot both sides of the equation in the same viewing screen, $y_1 = x^{1/2}$ and $y_2 = -4x^{1/4} + 21$. Does the point(s) of intersection agree with your solution?

120. Solve the equation $x^{-1} = 3x^{-2} - 10$. Plot both sides of the equation in the same viewing screen, $y_1 = x^{-1}$ and $y_2 = 3x^{-2} - 10$. Does the point(s) of intersection agree with your solution?

121. Solve the equation $x^{-2} = 3x^{-1} - 10$. Plot both sides of the equation in the same viewing screen, $y_1 = x^{-2}$ and $y_2 = 3x^{-1} - 10$. Does the point(s) of intersection agree with your solution?

1.5 Linear Inequalities

SKILLS OBJECTIVES

- Use interval notation.
- Solve linear inequalities in one variable.

CONCEPTUAL OBJECTIVES

- Apply intersection and union concepts.
- Understand that a linear inequality in one variable has an interval solution.

1.5.1 Graphing Inequalities and Interval Notation

1.5.1 Skill Use interval notation.

1.5.1 Conceptual Apply intersection and union concepts.

An example of a linear equation is $3x - 2 = 7$, whereas $3x - 2 \leq 7$ is an example of a **linear inequality**. One difference between a linear equation and a linear inequality is that the equation has at most only one solution, or value of x, that makes the statement true, whereas the inequality can have a range or continuum of numbers that make the statement true. For example, the inequality $x \leq 4$ denotes all real numbers x that are less than or equal to 4. Four inequality symbols are used.

Symbol	In Words
$<$	Less than
$>$	Greater than
$\leq$	Less than or equal to
$\geq$	Greater than or equal to

We call $<$ and $>$ **strict inequalities**. For any two real numbers a and b, one of three things must be true:

$$a < b \qquad \text{or} \qquad a = b \qquad \text{or} \qquad a > b$$

This property is called the **trichotomy property** of real numbers.

If x is less than 5 ($x < 5$) and x is greater than or equal to -2 ($x \geq -2$), then we can represent this as a **double** (or **combined**) **inequality**, $-2 \leq x < 5$, which means that x is greater than or equal to -2 and less than 5.

We will express solutions to inequalities in four ways: an inequality, a solution set, an interval, and a graph. The following are ways of expressing all real numbers greater than or equal to a and less than b.

Inequality Notation	Solution Set	Interval Notation	Graph/ Number Line
$a \leq x < b$	$\{x \mid a \leq x < b\}$	$[a, b)$	 or 

In this example, a is referred to as the **left endpoint** and b is referred to as the **right endpoint**. If an inequality is a strict inequality ($<$ or $>$), then the graph and interval notation use *parentheses*. If it includes an endpoint ($\geq$ or $\leq$), then the graph and interval notation use *brackets*. Number lines are drawn with either closed/open circles or brackets/parentheses. In this text, the brackets/parentheses notation will be used. Intervals are classified as follows:

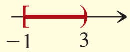
Open (,) Closed [,] Half open (,] or [,)

Let x Be a Real Number. x Is . . .	Inequality	Set Notation	Interval	Graph
greater than a and less than b	$a < x < b$	$\{x \mid a < x < b\}$	(a, b)	
greater than or equal to a and less than b	$a \leq x < b$	$\{x \mid a \leq x < b\}$	$[a, b)$	
greater than a and less than or equal to b	$a < x \leq b$	$\{x \mid a < x \leq b\}$	$(a, b]$	
greater than or equal to a and less than or equal to b	$a \leq x \leq b$	$\{x \mid a \leq x \leq b\}$	$[a, b]$	
less than a	$x < a$	$\{x \mid x < a\}$	$(-\infty, a)$	
less than or equal to a	$x \leq a$	$\{x \mid x \leq a\}$	$(-\infty, a]$	

(*Continued*)

Let *x* Be a Real Number. *x* Is . . .	Inequality	Set Notation	Interval	Graph
greater than *b*	$x > b$	$\{x \mid x > b\}$	(b, ∞)	
greater than or equal to *b*	$x \geq b$	$\{x \mid x \geq b\}$	$[b, \infty)$	
all real numbers	$\mathbb{R}$	$\mathbb{R}$	$(-\infty, \infty)$	

1. *Infinity* (∞) is not a number. It is a symbol that means continuing indefinitely to the right on the number line. Similarly, *negative infinity* $(-\infty)$ means continuing indefinitely to the left on the number line. Since both are unbounded, we use a parenthesis, never a bracket.
2. In interval notation, the lower number is always written to the left. Write the inequality in interval notation: $-1 \leq x < 3$.



Correct $[-1, 3)$ **Incorrect** $(3, -1]$

EXAMPLE 1 | Expressing Inequalities Using Interval Notation and a Graph

Express the following as an inequality, an interval, and a graph.

a. x is greater than -3.

b. x is less than or equal to 5.

c. x is greater than or equal to -1 and less than 4.

d. x is greater than or equal to 0 and less than or equal to 4.

Solution

Inequality	Interval	Graph
a. $x > -3$	$(-3, \infty)$	
b. $x \leq 5$	$(-\infty, 5]$	
c. $-1 \leq x < 4$	$[-1, 4)$	
d. $0 \leq x \leq 4$	$[0, 4]$	

Since the solutions to inequalities are sets of real numbers, it is useful to discuss two operations on sets called **intersection** and **union**.

Union and Intersection

The **union** of sets A and B, denoted $A \cup B$, is the set formed by combining all the elements in A with all the elements in B.

$$A \cup B = \{x \mid x \text{ is in } A \textbf{ or } B \textbf{ or } \text{both}\}$$

The **intersection** of sets A and B, denoted $A \cap B$, is the set formed by the elements that are in both A and B.

$$A \cap B = \{x \mid x \text{ is in } A \textbf{ and } B\}$$

The notation "$x \mid x$ is in" is read "all x such that x is in." The vertical line represents "such that."

As an example of intersection and union, consider the following sets of people:

$A = \{\text{Austin, Brittany, Jonathan}\}$ $\qquad B = \{\text{Anthony, Brittany, Elise}\}$

Intersection: $A \cap B = \{\text{Brittany}\}$

Union: $A \cup B = \{\text{Anthony, Austin, Brittany, Elise, Jonathan}\}$

Concept Check

If A is the set of all of the students who are enrolled in a math class and B is the set of all students who are enrolled in a history class, then which set is larger?
(A) the intersection of A and B or
(B) the union of A and B?
Assume that the two classes do not contain exactly the same students.

Answer: (B) the set of students who are enrolled in either a math class or a history class. (A) is the smaller set because it is all of the students who are enrolled in BOTH math and history.

EXAMPLE 2 | Determining Unions and Intersections: Intervals and Graphs

If $A = [-3, 2]$ and $B = (1, 7)$, determine $A \cup B$ and $A \cap B$. Write these sets in interval notation, and graph.

Solution

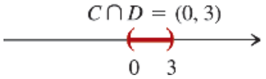
Set	**Interval notation**	**Graph**
A	$[-3, 2]$	
B	$(1, 7)$	
$A \cup B$	$[-3, 7)$	
$A \cap B$	$(1, 2]$	

Your Turn If $C = [-3, 3)$ and $D = (0, 5]$, find $C \cup D$ and $C \cap D$. Express the intersection and union in interval notation, and graph.

Answer

$C \cup D = [-3, 5]$

$C \cap D = (0, 3)$



1.5.2 Solving Linear Inequalities

1.5.2 Skill Solve linear inequalities in one variable.

1.5.2 Conceptual Understand that a linear inequality in one variable has an interval solution.

As mentioned at the beginning of this section, if we were to solve the equation $3x - 2 = 7$, we would add 2 to both sides, divide by 3, and find that $x = 3$ is the solution, the *only* value that makes the equation true. If we were to solve the linear inequality $3x - 2 \leq 7$, we would follow the same procedure: add 2 to both sides, divide by 3, and find that $x \leq 3$, which is an *interval* or *range* of numbers that make the inequality true.

In solving linear inequalities, we follow the same procedures that we used in solving linear equations with one general exception: *if you multiply or divide an inequality by a negative number, then you must change the direction of the inequality sign*. For example, if $-2x < -10$, then the solution set includes real numbers such as $x = 6$ and $x = 7$. Note that real numbers such as $x = -6$ and $x = -7$ are not included in the solution set. Therefore, when this inequality is divided by -2, the inequality sign must also be reversed: $x > 5$. If $a < b$, then $ac < bc$ if $c > 0$ and $ac > bc$ if $c < 0$.

The most common mistake that occurs when solving an inequality is forgetting to change the direction of, or reverse, the inequality symbol when the inequality is multiplied or divided by a negative number.

STUDY TIP

If you multiply or divide an inequality by a negative number, remember to change the direction of the inequality sign.

Inequality Properties

Procedures That Do Not Change the Inequality Sign

1. Simplifying by eliminating parentheses and collecting like terms. $3(x - 6) < 6x - x$; $3x - 18 < 5x$
2. Adding or subtracting the same quantity on both sides. $7x + 8 \geq 29$; $7x \geq 21$
3. Multiplying or dividing by the same *positive* real number. $5x \leq 15$; $x \leq 3$

Procedures That Change (Reverse) the Inequality Sign

1. Interchanging the two sides of the inequality. $x \leq 4$ is equivalent to $4 \geq x$.
2. Multiplying or dividing by the same *negative* real number. $-5x \leq 15$ is equivalent to $x \geq -3$.

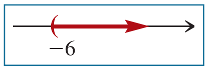
Video EXAMPLE 3 | Solving a Linear Inequality

Solve and graph the inequality $5 - 3x < 23$.

Solution

Write the original inequality.	$5 - 3x < 23$
Subtract 5 from both sides.	$-3x < 18$
Divide both sides by -3 and reverse the inequality sign.	$\frac{-3x}{-3} > \frac{18}{-3}$
Simplify.	$x > -6$

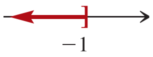
Solution set: $\{x \mid x > -6\}$ Interval notation: $(-6, \infty)$ Graph:



Your Turn Solve the inequality $5 \le 3 - 2x$. Express the solution in set and interval notation, and graph.

Answer

Solution set: $\{x \mid x \le -1\}$. Interval notation: $(-\infty, -1]$ Graph:



EXAMPLE 4 | Solving Linear Inequalities with Fractions

Solve the inequality $\frac{5x}{3} \le \frac{4 + 3x}{2}$.

Common Mistake

A common mistake is using cross multiplication to solve inequalities. Cross multiplication should not be used because the expression by which you are multiplying might be negative for some values of x, and that would require the direction of the inequality sign to be reversed.

Correct

Eliminate the fractions by multiplying by the LCD, 6.

$$6\left(\frac{5x}{3}\right) \le 6\left(\frac{4 + 3x}{2}\right)$$

Simplify.

$$10x \le 3(4 + 3x)$$

Eliminate the parentheses.

$$10x \le 12 + 9x$$

Subtract $9x$ from both sides.

$$\boxed{x \le 12}$$

Incorrect

Cross multiply. $3(4 + 3x) \le 2(5x)$

The error is in cross multiplying.

Although it is not possible to "check" inequalities since the solutions are often intervals, it is possible to confirm that some points that lie in your solution do satisfy the inequality. It is important to remember that cross multiplication cannot be used in solving inequalities.

Caution

Cross multiplication should not be used in solving inequalities.

Video EXAMPLE 5 | Solving a Double Linear Inequality

Solve the inequality $-2 < 3x + 4 \le 16$.

Solution

This double inequality can be written as two inequalities. $-2 < 3x + 4 \le 16$

Both inequalities must be satisfied.	$-2 < 3x + 4$	and	$3x + 4 \le 16$
Subtract 4 from both sides of each inequality.	$-6 < 3x$	and	$3x \le 12$
Divide each inequality by 3.	$-2 < x$	and	$x \le 4$

Combining these two inequalities gives us $-2 < x \le 4$ in inequality notation; in interval notation we have $(-2, \infty) \cap (-\infty, 4]$ or $(-2, 4]$.

Notice that the steps we took in solving these inequalities individually were identical. This leads us to a **shortcut method** in which we solve them together:

Write the combined inequality.	$-2 < 3x + 4 \le 16$
Subtract 4 from each part.	$-6 < 3x \le 12$
Divide each part by 3.	$-2 < x \le 4$

Interval notation: $\boxed{(-2, 4]}$

For the remainder of this section we will use the shortcut method for solving inequalities.

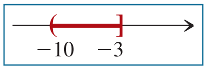
EXAMPLE 6 | Solving a Double Linear Inequality

Solve the inequality $1 \le \frac{-2-3x}{7} < 4$. Express the solution set in interval notation, and graph.

Solution

Write the original double inequality.	$1 \le \frac{-2-3x}{7} < 4$
Multiply each part by 7.	$7 \le -2 - 3x < 28$
Add 2 to each part.	$9 \le -3x < 30$
Divide each part by -3 and reverse the inequality signs.	$-3 \ge x > -10$
Write in standard form.	$-10 < x \le -3$

Interval notation: $(-10, -3]$ Graph:



EXAMPLE 7 | Solving a Double Linear Inequality

Solve the inequality $x - 1 \le 4x - 4 \le x + 8$. Express the solution in interval notation.

Solution

Subtract x from all three parts.	$-1 \le 3x - 4 \le 8$
Add 4 to all three parts.	$3 \le 3x \le 12$
Divide all three parts by 3.	$1 \le x \le 4$
Express the solution in interval notation.	$[1, 4]$

Your Turn Solve the inequality $2x + 1 < 4x + 2 < 2x + 5$. Express the solution in interval notation.

Answer

$\left(-\frac{1}{2}, \frac{3}{2}\right)$

Applications Involving Linear Inequalities

EXAMPLE 8 | Temperature Ranges

New York City on average has a yearly temperature range of 23 degrees Fahrenheit to 86 degrees Fahrenheit. What is the range in degrees Celsius given that the conversion relation is $F = 32 + \frac{9}{5}C$?

Solution

The temperature ranges from 23°F to 86°F.	$23 \le F \le 86$
Replace F using the Celsius conversion.	$23 \le 32 + \frac{9}{5}C \le 86$
Subtract 32 from all three parts.	$-9 \le \frac{9}{5}C \le 54$
Multiply all three parts by $\frac{5}{9}$.	$-5 \le C \le 30$

New York City has an average yearly temperature range of −5°C to 30°C.

Video **EXAMPLE 9 | Comparative Shopping**

Two car rental companies have advertised weekly specials on full-size cars. Hertz is advertising an $80 rental fee plus an additional $0.10 per mile. Thrifty is advertising $60 and $0.20 per mile. How many miles must you drive for the rental car from Hertz to be the better deal?

Solution

Let $x =$ number of miles driven during the week.

Write the cost for the Hertz rental.	$80 + 0.1x$
Write the cost for the Thrifty rental.	$60 + 0.2x$
Write the inequality if Hertz is less than Thrifty.	$80 + 0.1x < 60 + 0.2x$
Subtract $0.1x$ from both sides.	$80 < 60 + 0.1x$
Subtract 60 from both sides.	$20 < 0.1x$
Divide both sides by 0.1.	$200 < x$

You must drive more than 200 miles for Hertz to be the better deal.

Section 1.5 Summary

The solutions of linear inequalities are solution sets that can be expressed four ways:

1. Inequality notation $a < x \le b$
2. Set notation $\{x \mid a < x \le b\}$
3. Interval notation $(a, b]$
4. Graph (number line)

Linear inequalities are solved using the same procedures as linear equations with one exception: **When you multiply or divide by a negative number, you must reverse the inequality sign**.

Note: Cross multiplication cannot be used with inequalities.

Section 1.5 Exercises

Skills

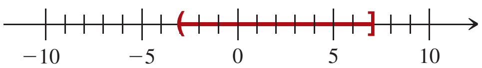
In Exercises 1–16, rewrite in interval notation and graph.

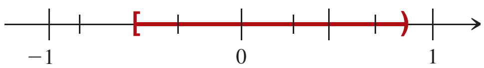
1. $x \ge 3$
2. $x < -2$
3. $x \le -5$
4. $x > -7$
5. $-2 \le x < 3$
6. $-4 \le x \le -1$
7. $-3 < x \le 5$
8. $0 < x < 6$
9. $0 \le x \le 0$
10. $-7 \le x \le -7$
11. $x \le 6$ and $x \ge 4$
12. $x > -3$ and $x \le 2$
13. $x \le -6$ and $x \ge -8$
14. $x < 8$ and $x < 2$
15. $x > 4$ and $x \le -2$
16. $x \ge -5$ and $x < -6$

In Exercises 17–24, rewrite in set notation.

17. $[0, 2)$
18. $(0, 3]$
19. $(-7, -2)$
20. $[-3, 2]$
21. $(-\infty, 6]$
22. $(5, \infty)$
23. $(-\infty, \infty)$
24. $[4, 4]$

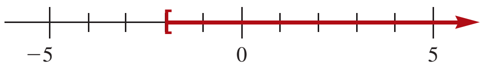
In Exercises 25–32, write in inequality and interval notation.

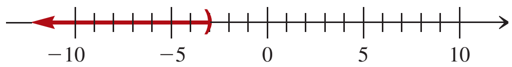
25. 

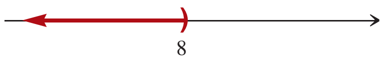
26. 

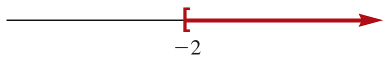
27. 

28. 

29. 

30. 

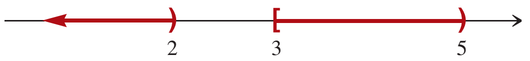
31. 

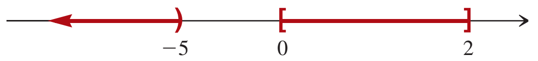
32. 

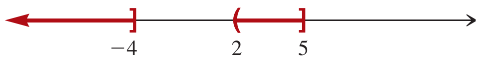
In Exercises 33–50, graph the indicated set and write as a single interval, if possible.

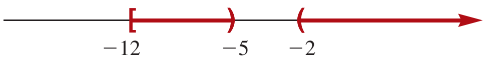
33. $(-5, 2] \cup (-1, 3)$
34. $(2, 7) \cup [-5, 3)$
35. $[-6, 4) \cup [-2, 5)$
36. $[-3, 1) \cup [-6, 0)$
37. $(-\infty, 1] \cap [-1, \infty)$
38. $(-\infty, -5) \cap (-\infty, 7]$
39. $(-\infty, 4) \cap [1, \infty)$
40. $(-3, \infty) \cap [-5, \infty)$
41. $[-5, 2) \cap [-1, 3]$
42. $[-4, 5) \cap [-2, 7)$
43. $(-\infty, 4) \cup (4, \infty)$
44. $(-\infty, -3] \cup [-3, \infty)$
45. $(-\infty, -3] \cup [3, \infty)$
46. $(-2, 2) \cap [-3, 1]$
47. $(-\infty, \infty) \cap (-3, 2]$
48. $(-\infty, \infty) \cup (-4, 7)$
49. $(-6, -2) \cap [1, 4)$
50. $(-\infty, -2) \cap (-1, \infty)$

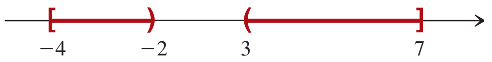
In Exercises 51–58, write in interval notation.

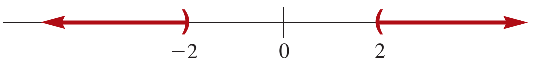
51. 

52. 

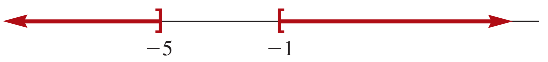
53. 

54. 

55. 

56. 

57. 

58. 

In Exercises 59–94, solve and express the solution in interval notation.

59. $x - 3 < 7$
60. $x + 4 > 9$
61. $3x - 2 \le 4$
62. $3x + 7 \ge -8$
63. $-5p \ge 10$
64. $-4u < 12$
65. $3 - 2x \le 7$
66. $4 - 3x > -17$
67. $-3x + 7 \le 22$
68. $5 + 2x > -15$
69. $-6 + 8x < 26$
70. $-4x + 3 \le -21$
71. $-1.8x + 2.5 > 3.4$
72. $2.7x - 1.3 < 6.8$
73. $3(t + 1) > 2t$
74. $2(y + 5) \le 3(y - 4)$
75. $7 - 2(1 - x) > 5 + 3(x - 2)$
76. $4 - 3(2 + x) < 5$
77. $\frac{x + 2}{3} - 2 \ge \frac{x}{2}$
78. $\frac{y - 3}{5} - 2 \le \frac{y}{4}$
79. $\frac{t - 5}{3} \le -4$
80. $\frac{2p + 1}{5} > -3$
81. $\frac{2}{3}y - \frac{1}{2}(5 - y) < \frac{5y}{3} - (2 + y)$
82. $\frac{s}{2} - \frac{(s - 3)}{3} > \frac{s}{4} - \frac{1}{12}$
83. $-2 < x + 3 < 5$
84. $1 < x + 6 < 12$
85. $-8 \le 4 + 2x < 8$
86. $0 < 2 + x \le 5$
87. $-3 < 1 - x \le 9$
88. $3 \le -2 - 5x \le 13$
89. $0 < 2 - \frac{1}{3}y < 4$
90. $3 < \frac{1}{2}A - 3 < 7$
91. $\frac{1}{2} \le \frac{1 + y}{3} \le \frac{3}{4}$
92. $-1 < \frac{2 - z}{4} \le \frac{1}{5}$
93. $-0.7 \le 0.4x + 1.1 \le 1.3$
94. $7.1 > 4.7 - 1.2x > 1.1$

Applications

95. Weight. A healthy weight range for a woman is given by the following formula:

- 110 pounds for the first 5 feet (tall)
- 2–6 pounds per inch for every inch above 5 feet

Write an inequality representing a healthy weight, w, for a 5 foot 9 inch woman.

96. Weight. NASA has more stringent weight allowances for its astronauts. Write an inequality representing allowable weight for a female 5 foot 9 inch mission specialist given 105 pounds for the first 5 feet, and 1–5 pounds per inch for every additional inch.

97. Profit. A seamstress decides to open a dress shop. Her fixed costs are \$4000 per month, and it costs her \$20 to make each dress. If the price of each dress is \$100, how many dresses does she have to sell per month to make a profit?

98. Profit. Labrador retrievers that compete in field trials typically cost \$2000 at birth. Professional trainers charge \$400 to \$1000 per month to train the dogs. If the dog is a champion by age 2, it sells for \$30,000. What is the range of profit for a champion at age 2?

In Exercises 99 and 100 refer to the following:

The annual revenue for a small company is modeled by

$$R = 5000 + 1.75x$$

where x is hundreds of units sold and R is revenue in thousands of dollars.

99. Business. Find the number of units (to the nearest 100) that must be sold to generate at least \$10 million in revenue.

100. Business. Find the number of units (to the nearest 100) that must be sold to generate at least \$7.5 million in revenue.

In Exercises 101 and 102 refer to the following:

The Target or Training Heart Rate (THR) is a range of heart rate (measured in beats per minute) that enables a person's heart and lungs to benefit the most from an aerobic workout. THR can be modeled by the formula

$$THR = (HR_{\text{max}} - HR_{\text{rest}}) \times I + HR_{\text{rest}}$$

where HR_{max} is the maximum heart rate that is deemed safe for the individual, HR_{rest} is the resting heart rate, and I is the intensity of the workout that is reported as a percentage.

101. Health. A female with a resting heart rate of 65 beats per minute has a maximum safe heart rate of 170 beats per minute. If her target heart rate is between 100 and 140 beats per minute, what percent intensities of workout can she consider?

102. Health. A male with a resting heart rate of 75 beats per minute has a maximum safe heart rate of 175 beats per minute. If his target heart rate is between 110 and 150 beats per minute, what percent intensities of workout can he consider?

103. Cost: Television. A cable company charges \$30 per month for a basic package. On-demand movies can be rented for an additional \$3.99 per movie. If a customer's bill ranged from a low of \$53.94 to a high of \$73.89 over a 6-month period, what were the most movies rented in a single month? What were the least?

104. Cost: Food. Club FroYo is a frozen yogurt shop that lets customers fill a container with yogurt and toppings themselves. For a large size cup, it costs \$6 for the first 8 ounces and then an additional \$0.30/oz for any amount over 8 ounces. Jimmy goes to Club FroYo once a week during his 6-week summer school and his totals ranged from a low of \$7.20 to a high of \$9.00. What is the least number of ounces that he ordered? What is the most?

105. Business. Javier is paid by his company to create YouTube videos. He is paid \$400 per video that he creates. Additionally, he is paid \$0.18 per view. He needs to make at least \$535 per video in order to pay his monthly bills. How many views does he need to get per month?

106. Business. A local convenience store has a minimum \$5 purchase required when using a credit card. Janelle wants to buy a soda for \$1.50 and some hot dogs that cost \$1.25 each. How many hot dogs does she need to buy to use her credit card?

107. Grades. In your general biology class, your first three test scores are 67, 77, and 84. What is the lowest score you can get on the fourth test to earn at least a B for the course? Assume that each test is of equal weight and the minimum score required to earn a B is an 80.

108. Grades. In your Economics I class there are four tests and a final exam, all of which count equally. Your four test grades are 96, 87, 79, and 89. What grade on your final exam is needed to earn between 80 and 90 for the course?

109. Markups. Typical markup on new cars is 15–30%. If the sticker price is \$27,999, write an inequality that gives the range of the invoice price (what the dealer paid the manufacturer for the car).

110. Markups. Repeat Exercise 109 with a sticker price of \$42,599.

111. Lasers. A circular laser beam with a radius r_T is transmitted from one tower to another tower. If the received beam radius r_R fluctuates 10% from the transmitted beam radius due to atmospheric turbulence, write an inequality representing the received beam radius.

112. Electronics: Communications. Communication systems are often evaluated based on their signal-to-noise ratio (SNR), which is the ratio of the average power of received signal, S, to average power of noise, N, in the system. If the SNR is required to be at least 2 at all times, write an inequality representing the received signal power if the noise can fluctuate 10%.

113. Real Estate. The Aguileras are listing their house with a real estate agent. They are trying to determine a listing price, L, for the house. Their realtor advises them that most buyers traditionally offer a buying price, B, that is 85–95% of the listing price. Write an inequality that relates the buying price to the listing price.

114. Humidity. The National Oceanic and Atmospheric Administration (NOAA) has stations on buoys in the oceans to measure atmosphere and ocean characteristics such as temperature, humidity, and wind. The humidity sensors have an error of 5%. Write an inequality relating the measured humidity h_m, and the true humidity h_t.

115. Recreation: Golf. Two friends enjoy playing golf. Their favorite course charges \$40 for greens fees (to play the course) and a \$15 cart rental (per person), so it currently costs each of them \$55 every time they play. The membership offered at that course is \$160 per month. The membership allows them to play as much as they want (no greens fees), but does still charge a cart rental fee of \$10 every time they play. What is the least number of times they should play a month in order for the membership to be the better deal?

116. Recreation: Golf. The same friends in Exercise 115 have a second favorite course. That golf course charges \$30 for greens fees (to play the course) and a \$10 cart rental (per person), so it currently costs each of them \$40 every time they play. The membership offered at that course is \$125 per month. The membership allows them to play as much as they want (no greens fees), but does still charge a cart rental fee of \$10. What is the least number of times they should play a month in order for the membership to be the better deal?

The following table is the 2020 Federal Tax Rate Schedule for people filing as single:

Tax Bracket #	If Taxable Income Is:	The Tax Is:
I	\$0 to \$9700	10% of amount over \$0
II	\$9701 to \$39,475	\$970 plus 12% of the amount over \$9,700
III	\$39,476 to \$84,200	\$4543 plus 22% of the amount over \$39,475
IV	\$84,201 to \$160,725	\$14,382.50 plus 24% of the amount over \$84,200
V	\$160,726 to \$204,100	\$32,748.50 plus 32% of the amount over \$160,725
VI	\$204,101 to \$510,300	\$46,628.50 plus 35% of the amount over \$204,100
VII	\$510,301 or more	\$153,798.50 plus 37% of the amount over \$510,300

117. Federal Income Tax. What is the range of federal income taxes a person in tax bracket III will pay the IRS?

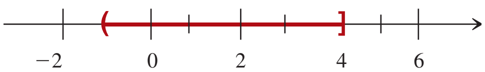
118. Federal Income Tax. What is the range of federal income taxes a person in tax bracket IV will pay the IRS?

Catch the Mistake

In Exercises 119–122, explain the mistake that is made.

119. Rewrite in interval notation.

$-1 \le x < 4$

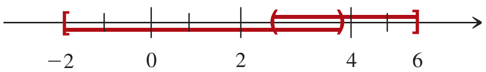


$(-1, 4]$

This is incorrect. What mistake was made?

120. Graph the indicated set and write as a single interval if possible.

$[-2, 4) \cap (3, 6]$



$[-2, 6]$

This is incorrect. What mistake was made?

121. Solve the inequality $2 - 3p \le -4$ and express the solution in interval notation.

Solution

$$2 - 3p \le -4$$
$$-3p \le -6$$
$$p \le 2$$
$$(-\infty, 2]$$

This is incorrect. What mistake was made?

122. Solve the inequality $3 - 2x \le 7$ and express the solution in interval notation.

Solution

$$3 - 2x \le 7$$
$$-2x \le 4$$
$$x \ge -2$$
$$(-\infty, -2]$$

This is incorrect. What mistake was made?

Conceptual

In Exercises 123 and 124, determine whether each statement is true or false.

123. If $x < a$, then $a > x$.

124. If $-x \ge a$, then $x \ge -a$.

In Exercises 125–128, select any of the statements *a* – *d* that could be true.

a. $m > 0$ and $n > 0$ **b.** $m < 0$ and $n < 0$

c. $m > 0$ and $n < 0$ **d.** $m < 0$ and $n > 0$

125. $mn > 0$

126. $mn < 0$

127. $\frac{m}{n} > 0$

128. $\frac{m}{n} < 0$

In Exercises 129 and 130, select any of the statements *a* – *c* that could be true.

a. $n = 0$ **b.** $n > 0$ **c.** $n < 0$

129. $m + n < m - n$

130. $m + n \ge m - n$

Challenge

131. Solve the inequality $x \le -x$ mentally (without doing any algebraic manipulation).

132. Solve the inequality $x > -x$ mentally (without doing any algebraic manipulation).

133. Solve the inequality $ax + b < ax - c$, where $0 < b < c$.

134. Solve the inequality $-ax + b < -ax + c$, where $0 < b < c$.

Technology

135. **a.** Solve the inequality $2.7x + 3.1 < 9.4x - 2.5$.

b. Graph each side of the inequality in the same viewing screen. Find the range of x-values when the graph of the left side lies *below* the graph of the right side.

c. Do (a) and (b) agree?

136. **a.** Solve the inequality $-0.5x + 2.7 > 4.1x - 3.6$.

b. Graph each side of the inequality in the same viewing screen. Find the range of x-values when the graph of the left side lies *above* the graph of the right side.

c. Do (a) and (b) agree?

137. **a.** Solve the inequality $x - 3 < 2x - 1 < x + 4$.

b. Graph all three expressions of the inequality in the same viewing screen. Find the range of x-values when the graph of the middle expression lies above the graph of the left side and below the graph of the right side.

c. Do (a) and (b) agree?

138. **a.** Solve the inequality $x - 2 < 3x + 4 \leq 2x + 6$.

b. Graph all three expressions of the inequality in the same viewing screen. Find the range of x-values when the graph of the middle expression lies above the graph of the left side and on top of and below the graph of the right side.

c. Do (a) and (b) agree?

139. **a.** Solve the inequality $x + 3 < x + 5$.

b. Graph each side of the inequality in the same viewing screen. Find the range of x-values when the graph of the left side lies *below* the graph of the right side.

c. Do (a) and (b) agree?

140. **a.** Solve the inequality $\frac{1}{2}x - 3 > -\frac{2}{3}x + 1$.

b. Graph each side of the inequality in the same viewing screen. Find the range of x-values when the graph of the left side lies above the graph of the right side.

c. Do (a) and (b) agree?

1.6 Polynomial and Rational Inequalities

SKILLS OBJECTIVES

- Solve polynomial inequalities.
- Solve rational inequalities.

CONCEPTUAL OBJECTIVES

- Understand zeros and test intervals.
- Realize that a rational inequality has an implied domain restriction on the variable.

1.6.1 Polynomial Inequalities

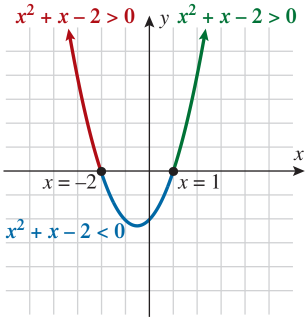
1.6.1 **Skill** Solve polynomial inequalities.

1.6.1 **Conceptual** Understand zeros and test intervals.

In this section we will focus primarily on quadratic inequalities, but the procedures outlined are also valid for higher degree polynomial inequalities. An example of a quadratic inequality is $x^2 + x - 2 < 0$. This statement is true when the value of the polynomial on the left side is negative. For any value of x, a polynomial has a positive, negative, or zero value. A polynomial must pass through zero before its value changes from positive to negative or from negative to positive. **Zeros** of a polynomial are the values of x that make the polynomial equal to zero. These zeros divide the real number line into **test intervals** where the value of the polynomial is either positive or negative. For example, if we set the given polynomial equal to zero and solve:

$$\begin{aligned} x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2 \quad \text{or} \quad x &= 1 \end{aligned}$$

we find that $x = -2$ and $x = 1$ are the zeros. These zeros divide the real number line into three test intervals: $(-\infty, -2)$, $(-2, 1)$, and $(1, \infty)$.



Since the polynomial is equal to zero at $x = -2$ and $x = 1$, the value of the polynomial in each of these three intervals is either positive or negative. We select one real number that lies in each of the three intervals and test to see whether the value of the polynomial at each point is either positive or negative. In this example, we select the real numbers: $x = -3$, $x = 0$, and $x = 2$. At this point, there are two ways we can determine whether the value of the polynomial is positive or negative on the interval. One approach is to substitute each of the test points into the polynomial $x^2 + x - 2$.

$x = -3$	$(-3)^2 + (-3) - 2 = 9 - 3 - 2 = 4$	Positive
$x = 0$	$(0)^2 + (0) - 2 = 0 - 0 - 2 = -2$	Negative
$x = 2$	$(2)^2 + (2) - 2 = 4 + 2 - 2 = 4$	Positive

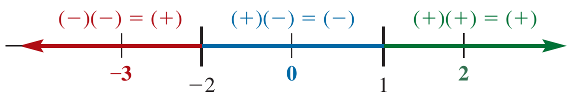
The second approach is to simply determine the sign of the result as opposed to actually calculating the exact number. This alternate approach is often used when the expressions or test points get more complicated to evaluate. The polynomial is written as the product $(x + 2)(x - 1)$; therefore, we simply look for the sign in each set of parentheses.

$$(x + 2)(x - 1)$$

$$x = -3: \quad (-3 + 2)(-3 - 1) = (-1)(-4) \rightarrow (-)(-) = (+)$$

$$x = 0: \quad (0 + 2)(0 - 1) = (2)(-1) \rightarrow (+)(-) = (-)$$

$$x = 2: \quad (2 + 2)(2 - 1) = (4)(1) \rightarrow (+)(+) = (+)$$



In this second approach we find the same result: $(-\infty, -2)$ and $(1, \infty)$ correspond to a positive value of the polynomial, and $(-2, 1)$ corresponds to a negative value of the polynomial.

In this example, the statement $x^2 + x - 2 < 0$ is true when the value of the polynomial (in factored form), $(x + 2)(x - 1)$, is negative. In the interval $(-2, 1)$, the value of the polynomial is negative. Thus, the solution to the inequality $x^2 + x - 2 < 0$ is $(-2, 1)$. To check the solution, select any number in the interval and substitute it into the original inequality to make sure it makes the statement true. The value $x = -1$ lies in the interval $(-2, 1)$. Upon substituting into the original inequality, we find that $x = -1$ satisfies the inequality $(-1)^2 + (-1) - 2 = -2 < 0$.

STUDY TIP

If the original polynomial is < 0, then the interval(s) that yield(s) *negative* products should be selected. If the original polynomial is > 0, then the interval(s) that yield(s) *positive* products should be selected.

Procedure for Solving Polynomial Inequalities

STEP 1 Write the inequality in *standard form*.

STEP 2 Identify zeros.

STEP 3 Draw the number line with zeros labeled.

STEP 4 Determine the sign of the polynomial in each interval.

STEP 5 Identify which interval(s) make the inequality true.

STEP 6 Write the solution in interval notation.

Note: Be careful in Step 5. If the original polynomial is < 0, then the interval(s) that correspond(s) to the value of the polynomial being negative should be selected. If the original polynomial is > 0, then the interval(s) that correspond(s) to the value of the polynomial being positive should be selected.

Video EXAMPLE 1 | Solving a Quadratic Inequality

Solve the inequality $x^2 - x > 12$.

Solution

STEP 1 Write the inequality in standard form. $\quad x^2 - x - 12 > 0$

Factor the left side. $\quad (x + 3)(x - 4) > 0$

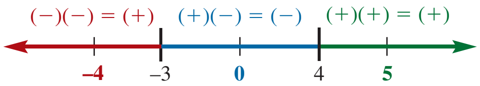
STEP 2 Identify the zeros.

$(x + 3)(x - 4) = 0$

$x = -3 \quad \text{or} \quad x = 4$

STEP 3 Draw the number line with the zeros labeled.

STEP 4 Determine the sign of $(x + 3)(x - 4)$ in each interval.



STEP 5 Intervals in which the value of the polynomial is *positive* make this inequality true.

$(-\infty, -3)$ or $(4, \infty)$

STEP 6 Write the solution in interval notation.

$(-\infty, -3) \cup (4, \infty)$

Your Turn Solve the inequality $x^2 - 5x \le 6$ and express the solution in interval notation.

Answer $[-1, 6]$

The inequality in Example 1, $x^2 - x > 12$, is a strict inequality, so we use parentheses when we express the solution in interval notation $(-\infty, -3) \cup (4, \infty)$. It is important to note that if we change the inequality sign from $>$ to $\ge$, then the zeros $x = -3$ and $x = 4$ also make the inequality true. Therefore, the solution to $x^2 - x \ge 12$ is $(-\infty, -3] \cup [4, \infty)$.

EXAMPLE 2 | Solving a Quadratic Inequality

Solve the inequality $x^2 \le 4$.

Common Mistake

Do not take the square root of both sides. You must write the inequality in standard form and factor.

Correct

STEP 1 Write the inequality in standard form.

$$x^2 - 4 \le 0$$

Factor.

$$(x - 2)(x + 2) \le 0$$

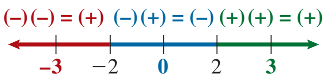
STEP 2 Identify the zeros.

$$(x - 2)(x + 2) = 0$$

$$x = 2 \quad \text{and} \quad x = -2$$

STEP 3 Draw the number line with the zeros labeled.

STEP 4 Determine the sign of $(x - 2)(x + 2)$, in each interval.



STEP 5 Intervals in which the value of the polynomial is *negative* make the inequality true.

$$(-2, 2)$$

The endpoints, $x = -2$ and $x = 2$, satisfy the inequality, so they are included in the solution.

Incorrect

Error:

Take the square root of both sides.

$$x \le \pm 2$$

Caution

The square root method cannot be used for quadratic inequalities.

STEP 6 Write the solution in interval notation.

$$[-2, 2]$$

When solving quadratic inequalities, you must first write the inequality in standard form and then factor to identify zeros.

Not all inequalities have a solution. For example, $x^2 < 0$ has no real solution. Any real number squared is always nonnegative, so there are no real values that when squared will yield a negative number. The zero is $x = 0$, which divides the real number line into two intervals: $(-\infty, 0)$ and $(0, \infty)$. Both of these intervals, however, correspond to the value of x^2 being positive, so there are no intervals that satisfy the inequality. We say that this inequality has no real solution.

STUDY TIP

When solving quadratic inequalities, you must first write the inequality in standard form and then factor to identify zeros.

EXAMPLE 3 | Solving a Quadratic Inequality

Solve the inequality $x^2 + 2x \geq -3$.

Solution

STEP 1 Write the inequality in standard form. $x^2 + 2x + 3 \geq 0$

STEP 2 Identify the zeros. $x^2 + 2x + 3 = 0$

Apply the quadratic formula. $x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)}$

Simplify. $x = \dfrac{-2 \pm \sqrt{-8}}{2} = \dfrac{-2 \pm 2i\sqrt{2}}{2} = -1 \pm i\sqrt{2}$

Since there are no real zeros, the quadratic expression $x^2 + 2x + 3$ never equals zero; hence its value is either always positive or always negative. If we select any value for x, say, $x = 0$, we find that $(0)^2 + 2(0) + 3 \geq 0$. Therefore, the quadratic expression is always positive, and so the solution is the set of all real numbers, $(-\infty, \infty)$.

Caution

Do not divide inequalities by a variable.

EXAMPLE 4 | Solving a Quadratic Inequality

Solve the inequality $x^2 > -5x$.

Common Mistake

A common mistake is to divide by x. Never divide by a variable because the value of the variable might be zero. Always start by writing the inequality in standard form and then factor to determine the zeros.

Correct

STEP 1 Write the inequality in standard form.

$$x^2 - 5x > 0$$

Factor.

$$x(x + 5) > 0$$

STEP 2 Identify the zeros.

$$x = 0, x = -5$$

Incorrect

Write the original inequality.

$$x^2 > -5x$$

Error:

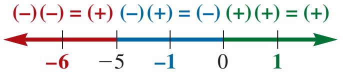
Divide both sides by x.

$$x > -5$$

Dividing by x is the mistake. If x is negative, the inequality sign must be reversed. What if x is zero?

STEP 3 Draw the number line with the zeros labeled.

STEP 4 Determine the sign of $x(x + 5)$ in each interval.



STEP 5 Intervals in which the value of the polynomial is *positive* satisfy the inequality.

$(-\infty, -5)$ and $(0, -\infty)$

STEP 6 Express the solution in interval notation.

$\boxed{(-\infty, -5) \cup (0, \infty)}$

EXAMPLE 5 | Solving a Quadratic Inequality

Solve the inequality $x^2 + 2x < 1$.

Solution

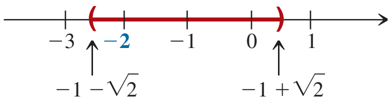
Write the inequality in standard form. $\quad x^2 + 2x - 1 < 0$

Identify the zeros. $\quad x^2 + 2x - 1 = 0$

Apply the quadratic formula. $\quad x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$

Simplify. $\quad x = \dfrac{-2 \pm \sqrt{8}}{2} = \dfrac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Draw the number line with the intervals labeled.
Note: $-1 - \sqrt{2} \approx -2.41$
$-1 + \sqrt{2} \approx 0.41$



Test each interval.

$(-\infty, -1 - \sqrt{2})$ $\quad x = -3:$ $\quad (-3)^2 + 2(-3) - 1 = 2 > 0$

$(-1 - \sqrt{2}, -1 + \sqrt{2})$ $\quad x = 0:$ $\quad (0)^2 + 2(0) - 1 = -1 < 0$

$(-1 + \sqrt{2}, \infty)$ $\quad x = 1:$ $\quad (1)^2 + 2(1) - 1 = 2 > 0$

Intervals in which the value of the polynomial is *negative* make this inequality true. $\quad \boxed{(-1 - \sqrt{2}, -1 + \sqrt{2})}$

Your Turn Solve the inequality $x^2 - 2x \geq 1$.

Answer

$(-\infty, 1 - \sqrt{2}] \cup [1 + \sqrt{2}, \infty)$

Video EXAMPLE 6 | Solving a Polynomial Inequality

Solve the inequality $x^3 - 3x^2 \geq 10x$.

Solution

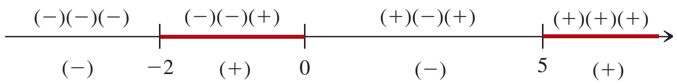
Write the inequality in standard form. $x^3 - 3x^2 - 10x \geq 0$

Factor. $x(x-5)(x+2) \geq 0$

Identify the zeros. $x = 0, x = 5, x = -2$

Draw the number line with the zeros (intervals) labeled.

Test each interval.



Intervals in which the value of the polynomial is *positive* make this inequality true. $\boxed{[-2, 0] \cup [5, \infty)}$

Your Turn Solve the inequality $x^3 - x^2 - 6x < 0$.

Answer $(-\infty, -2) \cup (0, 3)$

Concept Check

Solve: $(x - a)(x + b) < 0$, where $a > 0$ and $b > 0$.

Answer: $(-b, a)$

1.6.2 Rational Inequalities

1.6.2 Skills Solve rational inequalities.

1.6.2 Conceptual Realize that a rational inequality has an implied domain restriction on the variable.

Rational expressions have numerators and denominators. Recalling the properties of negative real numbers (Chapter 0), we see that the following possible combinations correspond to either positive or negative rational expressions.

$$\frac{(+)}{(+)} = (+) \qquad \frac{(-)}{(+)} = (-) \qquad \frac{(-)}{(-)} = (+) \qquad \frac{(+)}{(-)} = (-)$$

A rational expression can change signs if either the numerator or denominator changes signs. In order to go from positive to negative or vice versa, you must pass through zero. Therefore, to solve rational inequalities such as $\frac{x-3}{x^2-4} \geq 0$ we use a similar procedure to the one used for solving polynomial inequalities, with one exception. You must eliminate values for x that make the denominator equal to zero. In this example, we must eliminate $x = -2$ and $x = 2$ because these values make the denominator equal to zero. Rational inequalities have implied domains. In this example, $x \neq \pm 2$ is a domain restriction and these values ($x = -2$ and $x = 2$) must be eliminated from a possible solution.

STUDY TIP

Values that make the denominator equal to zero are always excluded.

We will proceed with a similar procedure involving zeros and test intervals that was outlined for polynomial inequalities. However, in rational inequalities once expressions are combined into a single fraction, any values that make *either* the numerator *or* the denominator equal to zero divide the number line into intervals.

EXAMPLE 7 | Solving a Rational Inequality

Solve the inequality $\dfrac{x-3}{x^2-4} \geq 0$.

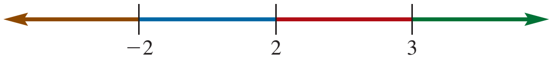
Solution

Factor the denominator. $\dfrac{(x-3)}{(x-2)(x+2)} \geq 0$

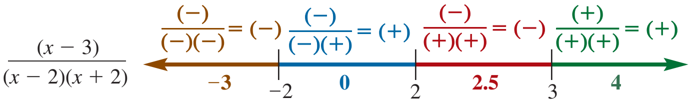
State the domain restrictions on the variable. $x \neq 2, x \neq -2$

Identify the zeros of numerator and denominator. $x = -2, x = 2, x = 3$

Draw the number line and divide into intervals.



Test the intervals.



Intervals in which the value of the rational expression is *positive* satisfy this inequality. **(−2, 2)** and **(3, ∞)**

Since this inequality is greater than or equal to, we include $x = 3$ in our solution because it satisfies the inequality. However, $x = -2$ and $x = 2$ are not included in the solution because they make the denominator equal to zero.

The solution is $\boxed{(-2, 2) \cup [3, \infty)}$.

Your Turn Solve the inequality $\dfrac{x+2}{x-1} \leq 0$.

Answer $[-2, 1)$

EXAMPLE 8 | Solving a Rational Inequality

Solve the inequality $\dfrac{x-5}{x^2+9} < 0$.

Solution

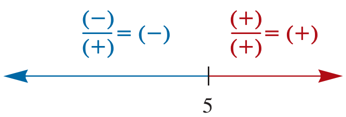
Identify the zero(s) of the numerator. $x = 5$

Note that the denominator is never equal to zero when x is any real number.

Draw the number line and divide into intervals.

Test the intervals.

The denominator is always positive.



Intervals in which the value of the rational expression is *negative* satisfy the inequality. $\boxed{(-\infty, 5)}$

Notice that $x = 5$ is not included in the solution because of the strict inequality.

Your Turn Solve the inequality $\dfrac{x+4}{x^2+25} \geq 0$.

Answer $[-4, \infty)$

Concept Check

Which of the following has an implied domain restriction on the variable:

(A) $\dfrac{1}{x^2+9}$ or (B) $\dfrac{1}{x^2-9}$

Answer: (B) $x \neq \pm 3$. (A) does not have a domain restriction because $x^2 + 9$ is never equal to zero when x is a real number.

Caution

Rational inequalities should not be solved using cross multiplication.

Video

EXAMPLE 9 | Solving a Rational Inequality

Solve the inequality $\dfrac{x}{x+2} \leq 3$.

Common Mistake

Do not cross multiply. The LCD or expression by which you are multiplying might be negative for some values of x, and that would require the direction of the inequality sign to be reversed.

Correct

Subtract 3 from both sides.

$$\frac{x}{x+2} - 3 \leq 0$$

Write as a single rational expression.

$$\frac{x-3(x+2)}{x+2} \leq 0$$

Eliminate the parentheses.

$$\frac{x-3x-6}{x+2} \leq 0$$

Simplify the numerator.

$$\frac{-2x-6}{x+2} \leq 0$$

Factor the numerator.

$$\frac{-2(x+3)}{x+2} \leq 0$$

Identify the zeros of the numerator and the denominator.

$$x = -3 \text{ and } x = -2$$

Draw the number line and test the intervals.

$$\frac{-2(x+3)}{x+2} \leq 0$$

$$\frac{(-)(-)}{(-)} = (-) \qquad \frac{(-)(+)}{(-)} = (+) \qquad \frac{(-)(+)}{(+)} = (-)$$

Intervals in which the value of the rational expression is *negative* satisfy the inequality, **$(-\infty, -3]$** and **$(-2, \infty)$**. Note that $x = -2$ is not included in the solution because it makes the denominator zero, and $x = -3$ is included because it satisfies the inequality.

The solution is:

$$\boxed{(-\infty, -3] \cup (-2, \infty)}$$

Incorrect

Error:

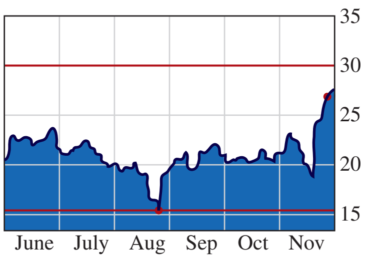
Do not cross multiply.

$$x \leq 3(x+2)$$

Applications

EXAMPLE 10 | Stock Prices

From June 2015 to November 2015, the price of Abercrombie & Fitch's (ANF) stock was approximately given by $P = 0.5t^2 - 2t + 17$, where P is the price of the stock in dollars, t is months, $t = 0$ corresponds to June 2015, and $t = 5$ corresponds to November 2015. During which months was the value of the stock worth no more than \$21?



Solution

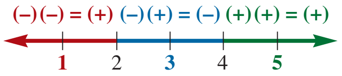
Set the price less than or equal to 21: $0.5t^2 - 3t + 25 \le 21, \quad 0 \le t \le 5$

Write in standard form: $0.5t^2 - 3t + 4 \le 0$

Multiply by 2: $t^2 - 6t + 8 \le 0$

Factor. $(t - 4)(t - 2) \le 0$

Identify zeros: $t = 2$ and $t = 4$



Test the intervals: $(t - 4)(t - 2) \le 0$

Negative interval satisfies the inequality: $[2, 4]$

> The Abercrombie & Fitch price was no more than \$21 during August 2015, September 2015, and October 2015.

Section 1.6 Summary

The following procedure can be used for solving polynomial and rational inequalities.

1. Write in standard form—zero on one side.
2. Determine the zeros; if it is a rational function, note the domain restrictions.
   Polynomial Inequality
   – Factor if possible.
   – Otherwise, use the quadratic formula.
   Rational Inequality
   – Write as a single fraction.
   – Determine values that make the numerator or denominator equal to zero. Always exclude values that make the denominator $= 0$.
3. Draw the number line labeling the intervals.
4. Test the intervals to determine whether they are positive or negative.
5. Select the intervals according to the sign of the inequality.
6. Write the solution in interval notation.

Section 1.6 Exercises

Skills

In Exercises 1–28, solve the polynomial inequality and express the solution set in interval notation.

1. $x^2 - 3x - 10 \ge 0$
2. $x^2 + 2x - 3 < 0$
3. $u^2 - 5u - 6 \le 0$
4. $u^2 - 6u - 40 > 0$
5. $p^2 + 4p < -3$
6. $p^2 - 2p \ge 15$
7. $2t^2 - 3 \le t$
8. $3t^2 \ge -5t + 2$

9. $5v - 1 > 6v^2$
10. $12t^2 < 37t + 10$
11. $2s^2 - 5s \geq 3$
12. $8s + 12 \leq -s^2$
13. $y^2 + 2y \geq 4$
14. $y^2 + 3y \leq 1$
15. $x^2 - 4x < 6$
16. $x^2 - 2x > 5$
17. $u^2 \geq 3u$
18. $u^2 \leq -4u$
19. $-2x \leq -x^2$
20. $-3x \leq x^2$
21. $x^2 > 9$
22. $x^2 \geq 16$
23. $t^2 < 81$
24. $t^2 \leq 49$
25. $z^2 > -16$
26. $z^2 \geq -2$
27. $y^2 < -4$
28. $y^2 \leq -25$

In Exercises 29–58, solve the rational inequality and graph the solution on the real number line.

29. $-\frac{3}{x} \leq 0$
30. $\frac{3}{x} \leq 0$
31. $\frac{y}{y+3} > 0$
32. $\frac{y}{2-y} \leq 0$
33. $\frac{t+3}{t-4} \geq 0$
34. $\frac{2t-5}{t-6} < 0$
35. $\frac{s+1}{4-s^2} \geq 0$
36. $\frac{s+5}{4-s^2} \leq 0$
37. $\frac{x-3}{x^2-25} \geq 0$
38. $\frac{1-x}{x^2-9} \leq 0$
39. $\frac{2u^2+u}{3} < 1$
40. $\frac{u^2-3u}{3} \geq 6$
41. $\frac{3t^2}{t+2} \geq 5t$
42. $\frac{-2t-t^2}{4-t} \geq t$
43. $\frac{3p-2p^2}{4-p^2} < \frac{3+p}{2-p}$
44. $-\frac{7p}{p^2-100} \leq \frac{p+2}{p+10}$
45. $\frac{x^2}{5+x^2} < 0$
46. $\frac{x^2}{5+x^2} \leq 0$
47. $\frac{x^2+10}{x^2+16} > 0$
48. $-\frac{x^2+2}{x^2+4} < 0$
49. $\frac{v^2-9}{v-3} \geq 0$
50. $\frac{v^2-1}{v+1} \leq 0$
51. $\frac{2}{t-3} + \frac{1}{t+3} \geq 0$
52. $\frac{1}{t-2} + \frac{1}{t+2} \leq 0$
53. $\frac{3}{x+4} - \frac{1}{x-2} \leq 0$
54. $\frac{2}{x-5} - \frac{1}{x-1} \geq 0$
55. $\frac{1}{p+4} + \frac{1}{p-4} > \frac{p^2-48}{p^2-16}$
56. $\frac{1}{p-3} - \frac{1}{p+3} \leq 2$
57. $\frac{1}{p-2} - \frac{1}{p+2} \geq \frac{3}{p^2-4}$
58. $\frac{2}{2p-3} - \frac{1}{p+1} \leq \frac{1}{2p^2-p-3}$

Applications

59. **Profit.** A web-based embroidery company makes monogrammed napkins. The profit associated with producing x orders of napkins is governed by the equation

$$P = -x^2 + 130x - 3000$$

Determine the range of orders the company should accept in order to make a profit.

60. **Profit.** Repeat Exercise 59 using $P = x^2 - 130x + 3600$.

61. **Car Value.** The term "upside down" on car payments refers to owing more than a car is worth. Assume you buy a new car and finance 100% over 5 years. The difference between the value of the car and what is owed on the car is governed by the expression $\frac{t}{t-3}$ where t is age (in years) of the car. Determine the time period when the car is worth more than you owe $\left(\frac{t}{t-3} > 0\right)$. When do you owe more than it's worth $\left(\frac{t}{t-3} < 0\right)$?

62. **Car Value.** Repeat Exercise 61 using the expression $-\frac{2-t}{4-t}$.

63. **Bullet Speed.** A .22-caliber gun fires a bullet at a speed of 1200 feet per second. If a .22-caliber gun is fired straight upward into the sky, the height of the bullet in feet is given by the equation $h = -16t^2 + 1200t$, where t is the time in seconds with $t = 0$ corresponding to the instant the gun is fired. How long is the bullet in the air?

64. **Bullet Speed.** A .38-caliber gun fires a bullet at a speed of 600 feet per second. If a .38-caliber gun is fired straight upward into the sky, the height of the bullet in feet is given by the equation $h = -16t^2 + 600t$. How many seconds is the bullet in the air?

65. **Geometry.** A rectangular area is fenced in with 100 feet of fence. If the minimum area enclosed is to be 600 square feet, what is the range of feet allowed for the length of the rectangle?

66. **Stock Value.** From November 2014 to November 2015, Amazon.com stock was approximately worth $P = 6.25t^2 - 25t + 325$, where P is the price of the stock in dollars, t is months, and $t = 0$ corresponds to November 2014. During what months was the stock worth no more than $525?

www.nasdaq.com/symbol/amzn/stock-chart

In Exercises 67 and 68 refer to the following:

In response to economic conditions, a local business explores the effect of a price increase on weekly profit. The function

$$P = -5(x+3)(x-24)$$

models the effect that a price increase of x dollars on a bottle of wine will have on the profit P measured in dollars.

67. Economics. What price increase will lead to a weekly profit of less than $460?

68. Economics. What price increases will lead to a weekly profit of more than $550?

69. Real Estate. A woman is selling a piece of land that she advertises as 400 acres (±7 acres) for $1.36 million. If you pay that price, what is the range of dollars per acre you have paid? Round to the nearest dollar.

70. Real Estate. A woman is selling a piece of land that she advertises as 1000 acres (±10 acres) for $1 million. If you pay that price, what is the range of dollars per acre you have paid? Round to the nearest dollar.

Catch the Mistake

In Exercises 71–74, explain the mistake that is made.

71. Solve the inequality $3x < x^2$.

Solution

Divide by x.	$3 < x$
Write the solution in interval notation.	$(3, \infty)$

This is incorrect. What mistake was made?

72. Solve the inequality $u^2 < 25$.

Solution

Take the square root of both sides.	$u < -5$
Write the solution in interval notation.	$(-\infty, -5)$

This is incorrect. What mistake was made?

73. Solve the inequality $\dfrac{x^2-4}{x+2} > 0$.

Solution

Factor the numerator and denominator.	$\dfrac{(x-2)(x+2)}{(x+2)} < 0$
Cancel the $(x+2)$ common factor.	$x - 2 > 0$
Solve.	$x > 2$

This is incorrect. What mistake was made?

74. Solve the inequality $\dfrac{x+4}{x} < -\dfrac{1}{3}$.

Solution

Cross multiply.	$3(x+4) < -1(x)$
Eliminate the parentheses.	$3x + 12 < -x$
Combine like terms.	$4x < -12$
Divide both sides by 4.	$x < -3$

This is incorrect. What mistake was made?

Conceptual

In Exercises 75 and 76, determine whether each statement is true or false. Assume that *a* is a positive real number.

75. If $x < a^2$, then the solution is $(-\infty, a)$.

76. If $x \geq a^2$, then the solution is $[a, \infty)$.

77. Assume the quadratic inequality $ax^2 + bx + c < 0$ is true. If $b^2 - 4ac < 0$, then describe the solution.

78. Assume the quadratic inequality $ax^2 + bx + c > 0$ is true. If $b^2 - 4ac < 0$, then describe the solution.

Challenge

In Exercises 79–82, solve for *x* given that *a* and *b* are both positive real numbers.

79. $-x^2 \leq a^2$

80. $\dfrac{x^2-b^2}{x+b} < 0$

81. $\dfrac{x^2+a^2}{x^2+b^2} \geq 0$

82. $\dfrac{a}{x^2} < -b$

Technology

In Exercises 83–90, plot the left side and the right side of each inequality in the same screen and use the zoom feature to determine the range of values for which the inequality is true.

83. $1.4x^2 - 7.2x + 5.3 > -8.6x + 3.7$

84. $17x^2 + 50x - 19 < 9x^2 + 2$

85. $11x^2 < 8x + 16$

86. $0.1x + 7.3 > 0.3x^2 - 4.1$

87. $x < x^2 - 3x < 6 - 2x$

88. $x^2 + 3x - 5 \geq -x^2 + 2x + 10$

89. $\dfrac{2p}{5-p} > 1$

90. $\dfrac{3p}{4-p} < 1$

1.7 Absolute Value Equations and Inequalities

SKILLS OBJECTIVES

- Solve absolute value equations.
- Solve absolute value inequalities.

CONCEPTUAL OBJECTIVES

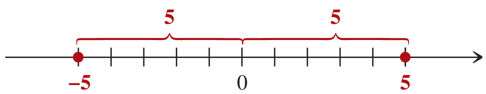
- Understand absolute value in terms of distance on the number line.
- Apply intersection and union concepts to expressing solutions to linear inequalities in one variable.

1.7.1 Equations Involving Absolute Value

1.7.1 Skill Solve absolute value equations.

1.7.1 Conceptual Understand absolute value in terms of distance on the number line.

The **absolute value** of a real number can be interpreted algebraically and graphically. Algebraically, the absolute value of 5 is 5, or in mathematical notation, $|5| = 5$; and the absolute value of -5 is 5 or $|-5| = 5$. Graphically, the absolute value of a real number is the distance on the real number line between the real number and the origin; thus, the distance from 0 to either -5 or 5 is 5.



Absolute Value

The **absolute value** of a real number a, denoted by the symbol $|a|$, is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

The absolute value of a real number is never negative. When $a = -5$, this definition says $|-5| = -(-5) = 5$.

Properties of Absolute Value

For all real numbers a and b,

1. $|a| \geq 0$ **2.** $|-a| = |a|$ **3.** $|ab| = |a||b|$ **4.** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $b \neq 0$

Absolute value can be used to define the distance between two points on the real number line.

Distance Between Two Points on the Real Number Line

If a and b are real numbers, the **distance between a and b** is the absolute value of their difference given by $|a - b|$ or $|b - a|$.

EXAMPLE 1 | Finding the Distance Between Two Points on the Number Line

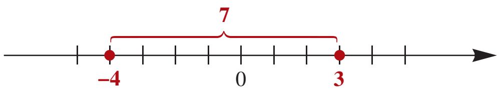
Find the distance between -4 and 3 on the real number line.

Solution

The distance between -4 and 3 is given by the absolute value of the difference. $|-4-3| = |-7| = 7$

Note that if we reverse the numbers the result is the same. $|3-(-4)| = |7| = 7$

We check this by counting the units between -4 and 3 on the number line.



When absolute value is involved in algebraic equations, we interpret the definition of absolute value as follows.

Absolute Value Equation

If $|x| = a$, then $x = -a$ or $x = a$, where $a \geq 0$.

In words, "if the absolute value of a number is a, then that number equals $-a$ or a." For example, the equation $|x| = 7$ is true if $x = -7$ or $x = 7$. We say the equation $|x| = 7$ has the solution set $\{-7, 7\}$. *Note:* $|x| = -3$ does not have a solution because there is no value of x such that its absolute value is -3.

Video EXAMPLE 2 | Solving an Absolute Value Equation

Solve the equation $|x-3| = 8$ algebraically and graphically.

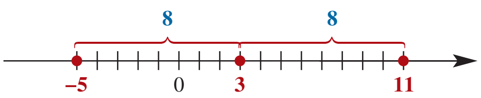
Solution

Using the absolute value equation definition, we see that if the absolute value of an expression is 8, then that expression is either -8 or 8. Rewrite as two equations:

$$x - 3 = -8 \quad \text{or} \quad x - 3 = 8$$
$$x = -5 \qquad\qquad x = 11$$

The solution set is $\boxed{\{-5, 11\}}$.

Graph: The absolute value equation $|x-3| = 8$ is interpreted as "what numbers are eight units away from 3 on the number line?" We find that eight units to the right of 3 is 11 and eight units to the left of 3 is -5.



Your Turn Solve the equation $|x+5| = 7$.

Answer

$x = 2$ or $x = -12$. The solution set is $\{-12, 2\}$.

EXAMPLE 3 | Solving an Absolute Value Equation

STUDY TIP

Rewrite an absolute value equation as two equations.

Solve the equation $|1 - 3x| = 7$.

Solution

If the absolute value of an expression is 7, then that expression is −7 or 7.

$$1 - 3x = -7 \quad \text{or} \quad 1 - 3x = 7$$
$$-3x = -8 \qquad\qquad -3x = 6$$
$$x = \frac{8}{3} \qquad\qquad x = -2$$

The solution set is $\boxed{\left\{-2, \frac{8}{3}\right\}}$.

Your Turn Solve the equation $|1 + 2x| = 5$.

Answer

$x = -3$ or $x = 2$. The solution set is $\{-3, 2\}$.

Video EXAMPLE 4 | Solving an Absolute Value Equation

Solve the equation $2 - 3|x - 1| = -4|x - 1| + 7$.

Solution

Isolate the absolute value expressions to one side.

Add $4|x - 1|$ to both sides. $\quad 2 + |x - 1| = 7$

Subtract 2 from both sides. $\quad |x - 1| = 5$

If the absolute value of an expression is equal to 5, then the expression is equal to either −5 or 5.

$$x - 1 = -5 \quad \text{or} \quad x - 1 = 5$$
$$x = -4 \qquad\qquad x = 6$$

The solution set is $\boxed{\{-4, 6\}}$.

Your Turn Solve the equation $3 - 2|x - 4| = -3|x - 4| + 11$.

Answer

$x = -4$ or $x = 12$. The solution set is $\{-4, 12\}$.

EXAMPLE 5 | Finding That an Absolute Value Equation Has No Solution

Solve the equation $|1 - 3x| = -7$.

Solution

The absolute value of an expression is never negative. Therefore, no values of x make this equation true.

$\boxed{\text{No solution}}$

EXAMPLE 6 | Solving a Quadratic Absolute Value Equation

Solve the equation $|5 - x^2| = 1$.

Solution

If the absolute value of an expression is 1, that expression is either −1 or 1, which leads to two equations.

$$5 - x^2 = -1 \quad \text{or} \quad 5 - x^2 = 1$$

$$\begin{aligned} -x^2 &= -6 & \qquad -x^2 &= -4 \\ x^2 &= 6 & x^2 &= 4 \\ x &= \pm\sqrt{6} & x &= \pm\sqrt{4} = \pm 2 \end{aligned}$$

The solution set is $\boxed{\{\pm 2, \pm\sqrt{6}\}}$.

Your Turn Solve the equation $|7 - x^2| = 2$.

Answer

$x = \pm\sqrt{5}$ or $x = \pm 3$. The solution set is $\{\pm\sqrt{5}, \pm 3\}$.

Concept Check

$|x - a| = b$ is interpreted on the number line as (A) b units from a or (B) a units from b?

Answer: (A) b units from a

1.7.2 Inequalities Involving Absolute Value

1.7.2 Skills Solve absolute value inequalities.

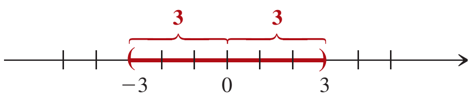
1.7.2 Conceptual Apply intersection and union concepts to expressing solutions to linear inequalities in one variable.

To solve the inequality $|x| < 3$, look for all real numbers that make this statement true. Some numbers that make it true are $-2, -\frac{3}{2}, -1, 0, \frac{1}{5}, 1$, and 2. Some numbers that make it false are $-7, -5, -3.5, -3, 3$, and 4. If we interpret this inequality as distance, we ask *what numbers are less than three units from the origin?* We can represent the solution in the following ways.

Inequality notation: $-3 < x < 3$

Interval notation: $(-3, 3)$

Graph:

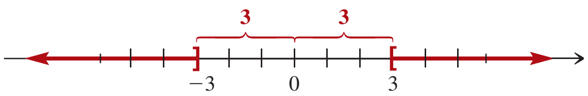


Similarly, to solve the inequality $|x| \geq 3$, look for all real numbers that make the statement true. If we interpret this inequality as a distance, we ask *what numbers are at least three units from the origin?* We can represent the solution in the following three ways.

Inequality notation: $x \leq -3$ or $x \geq 3$

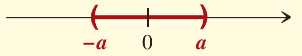
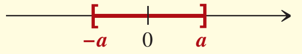
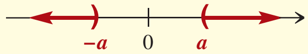
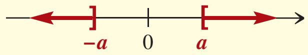
Interval notation: $(-\infty, -3] \cup [3, \infty)$

Graph:



This discussion leads us to the following equivalence relations.

Properties of Absolute Value Inequalities

1. $\lvert x\rvert < a$	is equivalent to	$-a < x < a$	
2. $\lvert x\rvert \le a$	is equivalent to	$-a \le x \le a$	
3. $\lvert x\rvert > a$	is equivalent to	$x < -a$ or $x > a$	
4. $\lvert x\rvert \ge a$	is equivalent to	$x \le -a$ or $x \ge a$	

Note: $a > 0$.

It is important to realize that for these four properties, the variable x can be any algebraic expression.

EXAMPLE 7 | Solving an Inequality Involving an Absolute Value

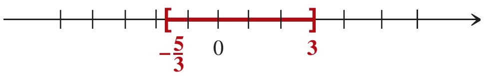
Solve the inequality $\lvert 3x - 2\rvert \le 7$.

Solution

We apply property (2) and squeeze the absolute value expression between −7 and 7.	$-7 \le 3x - 2 \le 7$
Add 2 to all three parts.	$-5 \le 3x \le 9$
Divide all three parts by 3.	$-\frac{5}{3} \le x \le 3$

The solution in interval notation is $\boxed{\left[-\frac{5}{3}, 3\right]}$.

Graph:



Your Turn Solve the inequality $\lvert 2x + 1\rvert < 11$.

Answer Inequality notation: $-6 < x < 5$. Interval notation: $(-6, 5)$.

STUDY TIP

Less than **inequalities can be written as a single statement.**

Greater than **inequalities must be written as two statements.**

It is often helpful to note that for absolute value inequalities,

- *less than* inequalities can be written as a single statement (see Example 7).
- *greater than* inequalities must be written as two statements (see Example 8).

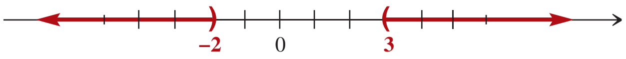
Video EXAMPLE 8 | Solving an Inequality Involving an Absolute Value

Solve the inequality $\lvert 1 - 2x\rvert > 5$.

Solution

Apply property (3).	$1 - 2x < -5$	or	$1 - 2x > 5$
Subtract 1 from all expressions.	$-2x < -6$		$-2x > 4$
Divide by −2 and reverse the inequality sign.	$x > 3$		$x < -2$
Express the solution in interval notation.	$\boxed{(-\infty, -2) \cup (3, \infty)}$		

Graph:



Your Turn Solve the inequality $|5 - 2x| \geq 1$.

Answer

Inequality notation: $x \leq 2$ or $x \geq 3$. Interval notation: $(-\infty, 2] \cup [3, \infty)$.

Concept Check

If the solution to a linear inequality in one variable is all the values less than a OR greater than b (where a and b are both positive), then the appropriate notation is (A) intersection or (B) union.

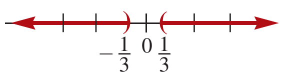
Answer: (B) Union $(-\infty, a) \cup (b, \infty)$

Notice that if we change the problem in Example 8 to $|1 - 2x| > -5$, the answer is all real numbers because the absolute value of any expression is greater than or equal to zero. Similarly, $|1 - 2x| < -5$ would have no solution because the absolute value of an expression can never be negative.

Video **EXAMPLE 9** | **Solving an Inequality Involving an Absolute Value**

Solve the inequality $2 - |3x| < 1$.

Solution

Subtract 2 from both sides.	$-\|3x\| < -1$
Multiply by (-1) and reverse the inequality sign.	$\|3x\| > 1$
Apply property (3).	$3x < -1$ or $3x > 1$
Divide both inequalities by 3.	$x < -\frac{1}{3}$ $x > \frac{1}{3}$
Express in interval notation.	$\boxed{\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)}$
Graph.	

Section 1.7 Summary

Absolute value equations and absolute value inequalities are solved by writing the equations or inequalities in terms of two equations or inequalities. *Note:* $A > 0$.

Equations

$|x| = A$ is equivalent to $x = -A$ or $x = A$

Inequalities

$|x| < A$ is equivalent to $-A < x < A$

$|x| > A$ is equivalent to $x < -A$ or $x > A$

Section 1.7 Exercises

Skills

In Exercises 1–46, solve the equation.

1. $|x| = 3$
2. $|x| = 2$
3. $|x| = -4$
4. $|x| = -2$
5. $|t + 3| = 2$
6. $|t - 3| = 2$
7. $|p - 7| = 3$
8. $|p + 7| = 3$
9. $|4 - y| = 1$
10. $|2 - y| = 11$
11. $|4 - y| = 2$
12. $|7 - y| = 5$
13. $|3x| = 9$
14. $|5x| = 50$
15. $|-7x| = 21$
16. $|-12y| = 144$
17. $|2x + 7| = 9$
18. $|2x - 5| = 7$
19. $|3t - 9| = 3$
20. $|4t + 2| = 2$
21. $|7 - 2x| = 9$
22. $|6 - 3y| = 12$
23. $|1 - 3y| = 1$
24. $|5 - x| = 2$
25. $|3 - 2x| = -9$
26. $|7 + 4x| = 5$
27. $|4.7 - 2.1x| = 3.3$
28. $|5.2x + 3.7| = 2.4$
29. $\left|\frac{2}{3}x - \frac{4}{7}\right| = \frac{5}{3}$
30. $\left|\frac{1}{2}x + \frac{3}{4}\right| = \frac{1}{16}$
31. $|x - 5| + 4 = 12$
32. $|x + 3| - 9 = 2$
33. $3|x - 2| + 1 = 19$
34. $2|1 - x| - 4 = 2$
35. $5 = 7 - |2 - x|$
36. $-1 = 3 - |x - 3|$
37. $2|p + 3| - 15 = 5$
38. $8 - 3|p - 4| = 2$
39. $5|y - 2| - 10 = 4|y - 2| - 3$
40. $3 - |y + 9| = 11 - 3|y + 9|$
41. $|4 - x^2| = 1$
42. $|7 - x^2| = 3$
43. $|x^2 + 1| = 5$
44. $|4x^2 - 9| = 0$
45. $|2x^2 + 5| = 45$
46. $|x^2 - 1| = 5$

In Exercises 47–80, solve the inequality and express the solution in interval notation.

47. $|x| < 7$
48. $|y| < 9$
49. $|y| \geq 5$
50. $|x| \geq 2$
51. $|x + 3| < 7$
52. $|x + 2| \leq 4$
53. $|x - 4| > 2$
54. $|x - 1| < 3$
55. $|4 - x| \leq 1$
56. $|1 - y| < 3$
57. $|2x| > -3$
58. $|2x| < -3$
59. $|x - 7| > -2$
60. $|x + 8| \leq -7$
61. $|2t + 3| < 5$
62. $|3t - 5| > 1$
63. $|7 - 2y| \geq 3$
64. $|6 - 5y| \leq 1$
65. $|4 - 3x| \geq 0$
66. $|4 - 3x| \geq 1$
67. $2|4x| - 9 \geq 3$
68. $5|x - 1| + 2 \leq 7$
69. $2|x + 1| - 3 \leq 7$
70. $3|x - 1| - 5 > 4$
71. $3 - 2|x + 4| < 5$
72. $7 - 3|x + 2| \geq -14$
73. $9 - |2x| < 3$
74. $4 - |x + 1| > 1$
75. $|1 - 2x| < \frac{1}{2}$
76. $\left|\frac{2 - 3x}{5}\right| \geq \frac{2}{5}$
77. $|2.6x + 5.4| < 1.8$
78. $|3.7 - 5.5x| > 4.3$
79. $|x^2 - 1| \leq 8$
80. $|x^2 + 4| \geq 29$

In Exercises 81–86, write an inequality that fits the description.

81. Any real numbers less than seven units from 2.
82. Any real numbers more than three units from −2.
83. Any real numbers at least $\frac{1}{2}$ unit from $\frac{3}{2}$.
84. Any real number no more than $\frac{5}{3}$ units from $\frac{11}{3}$.
85. Any real numbers no more than two units from a.
86. Any real number at least a units from −3.

Applications

87. **Temperature.** If the average temperature in Hawaii is 83°F(±15°), write an absolute value inequality representing the temperature in Hawaii.

88. **Temperature.** If the average temperature of a human is 97.8°F(±1.2°), write an absolute value inequality describing normal human body temperature.

89. **Sports.** Two women tee off the green of a par-3 hole on a golf course. They are playing "closest to the pin." If the first woman tees off and lands exactly 4 feet from the hole, write an inequality that describes where the second woman must land in order to win the hole. What equation would suggest a tie? Let d = the distance from where the second woman lands to the tee.

90. **Electronics.** A band-pass filter in electronics allows certain frequencies within a range (or band) to pass through to the receiver and eliminates all other frequencies. Write an absolute value inequality that allows any frequency f within 15 Hertz of the carrier frequency f_c to pass.

In Exercises 91 and 92 refer to the following:

A company is reviewing revenue for the prior sales year. The model for projected revenue and the model for actual revenue are

$$R_{\text{projected}} = 200 + 5x$$
$$R_{\text{actual}} = 210 + 4.8x$$

where x represents the number of units sold and R represents the revenue in thousands of dollars. Since the two revenue models are not identical, an error in projected revenue occurred. This error is represented by

$$E = |R_{\text{projected}} - R_{\text{actual}}|$$

91. Business. For what number of units sold was the error in projected revenue less than \$5000?

92. Business. For what number of units sold was the error in projected revenue less than \$3000?

Catch the Mistake

In Exercises 93–96, explain the mistake that is made.

93. Solve the absolute value equation $|x - 3| = 7$.

Solution

Eliminate the absolute value symbols. $x - 3 = 7$

Add 3 to both sides. $x = 10$

Check. $|10 - 3| = 7$

This is incorrect. What mistake was made?

94. Solve the inequality $|x - 3| < 7$.

Solution

Eliminate the absolute value symbols. $x - 3 < -7$ or $x - 3 > 7$

Add 3 to both sides. $x < -4$ $\quad x > 10$

The solution is $(-\infty, -4) \cup (10, \infty)$.

This is incorrect. What mistake was made?

95. Solve the inequality $|5 - 2x| \le 1$.

Solution

Eliminate the absolute value symbols. $-1 \le 5 - 2x \le 1$

Subtract 5. $-6 \le -2x \le -4$

Divide by -2. $3 \le x \le 2$

Write the solution in interval notation. $(-\infty, 2] \cup [3, \infty)$

This is incorrect. What mistake was made?

96. Solve the equation $|5 - 2x| = -1$.

Solution

$5 - 2x = -1$ or $5 - 2x = 1$

$-2x = -6$ $\quad -2x = -4$

$x = 3$ $\quad x = 2$

The solution is $\{2, 3\}$.

This is incorrect. What mistake was made?

Conceptual

In Exercises 97–100, determine whether each statement is true or false.

97. $-|m| \le m \le |m|$

98. $|n^2| = n^2$

99. $|m + n| = |m| + |n|$ is true only when m and n are both nonnegative.

100. For what values of x does the absolute value equation $|x - 7| = x - 7$ hold?

In Exercises 101–106, assuming *a* and *b* are real positive numbers, solve the equation or inequality and express the solution in interval notation.

101. $|x - a| < b$

102. $|a - x| > b$

103. $|x| \ge -a$

104. $|x| \le -b$

105. $|x - a| = b$

106. $|x - a| = -b$

Challenge

107. For what values of x does the absolute value equation $|x + 1| = 4 + |x - 2|$ hold?

108. Solve the inequality $|3x^2 - 7x + 2| > 8$.

Technology

109. Graph $y_1 = |x - 7|$ and $y_2 = x - 7$ in the same screen. Do the x-values where these two graphs coincide agree with your result in Exercise 100?

110. Graph $y_1 = |x + 1|$ and $y_2 = |x - 2| + 4$ in the same screen. Do the x-values where these two graphs coincide agree with your result in Exercise 107?

111. Graph $y_1 = |3x^2 - 7x + 2|$ and $y_2 = 8$ in the same screen. Do the x-values where y_1 lies above y_2 agree with your result in Exercise 108?

112. Solve the inequality $|2.7x^2 - 7.9x + 5| \le |5.3x^2 - 9.2|$ by graphing both sides of the inequality and identify which x-values make this statement true.

113. Solve the inequality $\left|\dfrac{x}{x+1}\right| < 1$ by graphing both sides of the inequality, and identify which x-values make this statement true.

114. Solve the inequality $\left|\dfrac{x}{x+1}\right| < 2$ by graphing both sides of the inequality, and identify which x-values make this statement true.

Chapter 1 Review

Section	Concept	Key Ideas/Formulas
1.1	**Linear equations**	$ax + b = 0$
	Solving linear equations in one variable	Isolate variables on one side and constants on the other side.
	Solving rational equations that are reducible to linear equations	Any values that make the denominator equal to 0 must be eliminated as possible solutions.
1.2	**Applications involving linear equations**	
	Solving application problems using mathematical models	Five-step procedure: **STEP 1** Identify the question. **STEP 2** Make notes. **STEP 3** Set up an equation. **STEP 4** Solve the equation. **STEP 5** Check the solution.
	Geometry problems	Formulas for rectangles, triangles, and circles
	Interest problems	Simple interest: $I = Prt$
	Mixture problems	Whenever two *distinct* quantities are mixed, the result is a mixture.
	Distance–rate–time problems	$d = r \cdot t$
1.3	**Quadratic equations**	$ax^2 + bx + c = 0 \qquad a \neq 0$
	Factoring	If $(x - h)(x - k) = 0$, then $x = h$ or $x = k$.
	Square root method	If $x^2 = P$, then $x = \pm\sqrt{P}$.
	Completing the square	Find half of b; square that quantity; add the result to both sides.
	Quadratic Formula	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
1.4	**Other types of equations**	
	Radical equations	Check solutions to avoid extraneous solutions.
	Equations quadratic in form: u-substitution	Use a u-substitution to write the equation in quadratic form.
	Factorable equations	Extract common factor or factor by grouping.
1.5	**Linear inequalities**	Solutions are a range of real numbers.
	Graphing inequalities and interval notation	• $a < x < b$ is equivalent to (a, b). • $x \leq a$ is equivalent to $(-\infty, a]$. • $x > a$ is equivalent to (a, ∞).
	Solving linear inequalities	If an inequality is multiplied or divided by a *negative* number, the inequality sign must be reversed.
1.6	**Polynomial and rational inequalities**	
	Polynomial inequalities	Zeros are values that make the polynomial equal to 0.
	Rational inequalities	The number line is divided into intervals. The endpoints of these intervals are values that make either the numerator or denominator equal to 0. Always exclude values that make the denominator = 0.
1.7	**Absolute value equations and inequalities**	$\lvert b - a\rvert$ is the distance between points a and b on the number line.
	Equations involving absolute value	If $\lvert x\rvert = a$, then $x = -a$ or $x = a$.
	Inequalities involving absolute value	• $\lvert x\rvert \leq a$ is equivalent to $-a \leq x \leq a$. • $\lvert x\rvert > a$ is equivalent to $x < -a$ or $x > a$.

Chapter 1 Review Exercises

1.1 Linear Equations

Solve for the variable.

1. $7x - 4 = 12$
2. $13d + 12 = 7d + 6$
3. $20p + 14 = 6 - 5p$
4. $4(x - 7) - 4 = 4$
5. $3(x + 7) - 2 = 4(x - 2)$
6. $7c + 3(c - 5) = 2(c + 3) - 14$
7. $14 - [-3(y - 4) + 9] = [4(2y + 3) - 6] + 4$
8. $[6 - 4x + 2(x - 7)] - 52 = 3(2x - 4) + 6[3(2x - 3) + 6]$
9. $\frac{12}{b} - 3 = \frac{6}{b} + 4$
10. $\frac{g}{3} + g = \frac{7}{9}$
11. $\frac{13x}{7} - x = \frac{x}{4} - \frac{3}{14}$
12. $5b + \frac{b}{6} = \frac{b}{3} - \frac{29}{6}$

Specify any values that must be excluded from the solution set and then solve.

13. $\frac{1}{x} - 4 = \frac{3}{x} - 5$
14. $\frac{4}{x+1} - \frac{8}{x-1} = 3$
15. $\frac{2}{t+4} - \frac{7}{t} = \frac{6}{t(t+4)}$
16. $\frac{3}{2x-7} = \frac{-2}{3x+1}$
17. $\frac{3}{2x} - \frac{6}{x} = 9$
18. $\frac{3 - (5/m)}{2 + (5/m)} = 1$
19. $7x - (2 - 4x) = 3[-6 + (4 - 2x + 7)] + 12$
20. $\frac{x}{5} - \frac{x-3}{15} = -6$

Solve for the specified variable.

21. Solve for x in terms of y:

$$3x - 2[(y + 4)3 - 7] = y - 2x + 6(x - 3)$$

22. If $y = \frac{x+3}{1+2x}$, find $\frac{y+2}{1-2y}$ in terms of x.

1.2 Applications Involving Linear Equations

23. **Transportation.** Maria is on her way from her home near Orlando to the Sun Dome in Tampa for a rock concert. She drives 16 miles to the Orlando park-n-ride, takes a bus $\frac{3}{4}$ of the way to a bus station in Tampa, and then takes a cab $\frac{1}{12}$ of the way to the Sun Dome. How far does Maria live from the Sun Dome?

24. **Diet.** A particular 2000 calorie per day diet suggests eating breakfast, lunch, dinner, and four snacks. Each snack is $\frac{1}{4}$ the calories of lunch. Lunch has 100 calories less than dinner. Dinner has 1.5 times as many calories as breakfast. How many calories are in each meal and snack?

25. **Numbers.** Find a number such that 12 more than $\frac{1}{4}$ the number is $\frac{1}{3}$ the number.

26. **Numbers.** Find four consecutive odd integers such that the sum of the four numbers is equal to three more than three times the fourth integer.

27. **Geometry.** The length of a rectangle is one more than two times the width, and the perimeter is 20 inches. What are the dimensions of the rectangle?

28. **Geometry.** Find the perimeter of a triangle if one side is 10 inches, another side is $\frac{1}{3}$ of the perimeter, and the third side is $\frac{1}{6}$ of the perimeter.

29. **Investments.** You win \$25,000 and you decide to invest the money in two different investments: one paying 20% and the other paying 8%. A year later you have \$27,600 total. How much did you originally invest in each account?

30. **Investments.** A college student on summer vacation was able to make \$5000 by working a full-time job every summer. He invested half the money in a mutual fund and half the money in a stock that yielded four times as much interest as the mutual fund. After a year he earned \$250 in interest. What were the interest rates of the mutual fund and the stock?

31. **Chemistry.** For an experiment, a student requires 150 milliliters of a solution that is 8% NaCl (sodium chloride). The storeroom has only solutions that are 10% NaCl and 5% NaCl. How many milliliters of each available solution should be mixed to get 150 milliliters of 8% NaCl?

32. **Chemistry.** A mixture containing 8% salt is to be mixed with 4 ounces of a mixture that is 20% salt, in order to obtain a solution that is 12% salt. How much of the first solution must be used?

33. **Grades.** Going into the College Algebra final exam, which will count as two tests, Danny has test scores of 95, 82, 90, and 77. If his final exam is higher than his lowest test score, then it will count for the final exam and replace the lowest test score. What score does Danny need on the final in order to have an average score of at least 90?

34. **Car Value.** A car salesperson reduced the price of a model car by 20%. If the new price is \$25,000, what was its original price? How much can be saved by purchasing the model?

1.3 Quadratic Equations

Solve by factoring.

35. $b^2 = 4b + 21$
36. $x(x - 3) = 54$
37. $x^2 = 8x$
38. $6y^2 - 7y - 5 = 0$

Solve by the square root method.

39. $q^2 - 169 = 0$
40. $c^2 + 36 = 0$
41. $(2x - 4)^2 = -64$
42. $(d + 7)^2 - 4 = 0$

Solve by completing the square.

43. $x^2 - 4x - 12 = 0$
44. $2x^2 - 5x - 7 = 0$
45. $\frac{x^2}{2} = 4 + \frac{x}{2}$
46. $8m = m^2 + 15$

Solve by the Quadratic Formula.

47. $3t^2 - 4t = 7$
48. $4x^2 + 5x + 7 = 0$
49. $8f^2 - \frac{1}{3}f = \frac{7}{6}$
50. $x^2 = -6x + 6$

Solve by any method.

51. $5q^2 - 3q - 3 = 0$
52. $(x - 7)^2 = -12$
53. $2x^2 - 3x - 5 = 0$
54. $(g - 2)(g + 5) = -7$
55. $7x^2 = -19x + 6$
56. $7 = 2b^2 + 1$

Solve for the indicated variable.

57. $S = \pi r^2 h$ for r

58. $V = \frac{\pi r^3 h}{3}$ for r

59. $h = vt - 16t^2$ for v

60. $A = 2\pi r^2 + 2\pi rh$ for h

61. Geometry. Find the base and height of a triangle with an area of 2 square feet if its base is 3 feet longer than its height.

62. Falling Objects. A man is standing on top of a building 500 feet tall. If he drops a penny off the roof, the height of the penny is given by $h = -16t^2 + 500$, where t is in seconds. Determine how many seconds it takes until the penny hits the ground.

1.4 Other Types of Equations

Solve the radical equation for the given variable.

63. $\sqrt[3]{2x-4} = 2$

64. $\sqrt{x-2} = -4$

65. $(2x-7)^{1/5} = 3$

66. $x = \sqrt{7x-10}$

67. $x - 4 = \sqrt{x^2 + 5x + 6}$

68. $\sqrt{2x-7} = \sqrt{x+3}$

69. $\sqrt{x+3} = 2 - \sqrt{3x+2}$

70. $4 + \sqrt{x-3} = \sqrt{x-5}$

71. $x - 2 = \sqrt{49 - x^2}$

72. $\sqrt{2x-5} - \sqrt{x+2} = 3$

73. $-x = \sqrt{3-x}$

74. $\sqrt{15 + 2\sqrt{x-4}} + \sqrt{x} = 5$

Solve the equation by introducing a substitution that transforms the equation to quadratic form.

75. $-28 = (3x-2)^2 - 11(3x-2)$

76. $x^4 - 6x^2 + 9 = 0$

77. $\left(\frac{x}{1-x}\right)^2 = 15 - 2\left(\frac{x}{1-x}\right)$

78. $3(x-4)^4 - 11(x-4)^2 - 20 = 0$

79. $y^{-2} - 5y^{-1} + 4 = 0$

80. $p^{-2} + 4p^{-1} = 12$

81. $3x^{1/3} + 2x^{2/3} = 5$

82. $2x^{2/3} - 3x^{1/3} - 5 = 0$

83. $x^{-2/3} + 3x^{-1/3} + 2 = 0$

84. $y^{-1/2} - 2y^{-1/4} + 1 = 0$

85. $x^4 + 5x^2 = 36$

86. $3 - 4x^{-1/2} + x^{-1} = 0$

Solve the equation by factoring.

87. $x^3 + 4x^2 - 32x = 0$

88. $9t^3 - 25t = 0$

89. $p^3 - 3p^2 - 4p + 12 = 0$

90. $4x^3 - 9x^2 + 4x - 9 = 0$

91. $p(2p-5)^2 - 3(2p-5) = 0$

92. $2(t^2-9)^3 - 20(t^2-9)^2 = 0$

93. $y - 81y^{-1} = 0$

94. $9x^{3/2} - 37x^{1/2} + 4x^{-1/2} = 0$

1.5 Linear Inequalities

Rewrite using interval notation.

95. $x \le -4$

96. $-1 < x \le 7$

97. $2 \le x \le 6$

98. $x > -1$

Rewrite using inequality notation.

99. $(-6, \infty)$

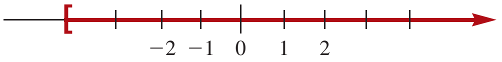
100. $(-\infty, 0]$

101. $[-3, 7]$

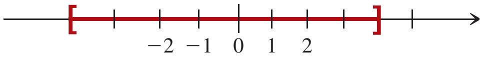
102. $(-5, 2]$

Express each interval using inequality and interval notation.

103.



104.



Graph the indicated set and write as a single interval, if possible.

105. $(4, 6] \cup [5, \infty)$

106. $(-\infty, -3) \cup [-7, 2]$

107. $(3, 12] \cap [8, \infty)$

108. $(-\infty, -2) \cap [-2, 9)$

Solve and graph.

109. $2x < 5 - x$

110. $6x + 4 \le 2$

111. $4(x-1) > 2x - 7$

112. $\frac{x+3}{3} \ge 6$

113. $6 < 2 + x \le 11$

114. $-6 \le 1 - 4(x+2) \le 16$

115. $\frac{2}{3} \le \frac{1+x}{6} \le \frac{3}{4}$

116. $\frac{x}{3} + \frac{x+4}{9} > \frac{x}{6} - \frac{1}{3}$

Applications

117. Grades. In your algebra class your first four exam grades are 72, 65, 69, and 70. What is the lowest score you can get on the fifth exam to earn a *C* for the course? Assume that each exam is equal in weight and a *C* is any score greater than or equal to 70.

118. Profit. A tailor decided to open a men's custom suit business. His fixed costs are $8500 per month, and it costs him $50 for the materials to make each suit. If the price he charges per suit is $300, how many suits does he have to tailor per month to make a profit?

1.6 Polynomial and Rational Inequalities

Solve the polynomial inequality and express the solution set using interval notation.

119. $x^2 \le 36$

120. $6x^2 - 7x < 20$

121. $4x \le x^2$

122. $-x^2 \ge 9x + 14$

123. $-x^2 \le -7x$

124. $x^2 < -4$

125. $4x^2 - 12 > 13x$

126. $3x \le x^2 + 2$

Solve the rational inequality and express the solution set using interval notation.

127. $\frac{x}{x-3} < 0$

128. $\frac{x-1}{x-4} > 0$

129. $\frac{x^2 - 3x}{3} \ge 18$

130. $\frac{x^2 - 49}{x-7} \ge 0$

131. $\frac{3}{x-2} - \frac{1}{x-4} \le 0$

132. $\frac{4}{x-1} \le \frac{2}{x+3}$

133. $\frac{x^2+9}{x-3} \ge 0$

134. $x < \frac{5x+6}{x}$

1.7 Absolute Value Equations and Inequalities

Solve the equation.

135. $|x-3| = -4$

136. $|2+x| = 5$

137. $|3x-4| = 1.1$

138. $|x^2 - 6| = 3$

Solve the inequality and express the solution using interval notation.

139. $|x| < 4$

140. $|x - 3| < 6$

141. $|x + 4| > 7$

142. $|-7 + y| \le 4$

143. $|2x| > 6$

144. $\left|\frac{4 + 2x}{3}\right| \ge \frac{1}{7}$

145. $|2 + 5x| \ge 0$

146. $|1 - 2x| \le 4$

Applications

147. Temperature. If the average temperature in Phoenix is 85°F (±10°), write an inequality representing the average temperature T in Phoenix.

148. Blood Alcohol Level. If a person registers a 0.08 blood alcohol level, he will be issued a DUI ticket in the state of Florida. If the test is accurate within 0.007, write a linear inequality representing an actual blood alcohol level that will not be issued a ticket.

Technology Exercises

Section 1.1

Graph the function represented by each side of the question in the same viewing rectangle, and solve for *x*.

149. $0.031x + 0.017(4000 - x) = 103.14$

150. $\frac{1}{0.16x} - \frac{0.2}{x} = \frac{1}{4}$

Section 1.3

151. **a.** Solve the equation $x^2 + 4x = b$, $b = 5$ by first writing in standard form and then factoring. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 + 4x$ and $y_2 = b$). At what x-values do these two graphs intersect? Do those points agree with the solution set you found?

b. Repeat part (a) for $b = -5, 0, 7$, and 12.

152. **a.** Solve the equation $x^2 - 4x = b$, $b = 5$ by first writing in standard form and then factoring. Now plot both sides of the equation in the same viewing screen ($y_1 = x^2 - 4x$ and $y_2 = b$). At what x-values do these two graphs intersect? Do those points agree with the solution set you found?

b. Repeat part (a) for $b = -5, 0, 7$, and 12.

Section 1.4

153. Solve the equation $2x^{1/4} = -x^{1/2} + 6$. Round your answer to two decimal places. Plot both sides of the equation in the same viewing screen, $y_1 = 2x^{1/4}$ and $y_2 = -x^{1/2} + 6$. Does the point(s) of intersection agree with your solution?

154. Solve the equation $2x^{-1/2} = x^{-1/4} + 6$. Plot both sides of the equation in the same viewing screen, $y_1 = 2x^{-1/2}$ and $y_2 = x^{-1/4} + 6$. Does the point(s) of intersection agree with your solution?

Section 1.5

155. **a.** Solve the inequality $-0.61x + 7.62 > 0.24x - 5.47$. Express the solution set using interval notation.

b. Graph each side of the inequality in the same viewing screen. Find the range of x-values when the graph of the left side lies above the graph of the right side.

c. Do parts (a) and (b) agree?

156. **a.** Solve the inequality $-\frac{1}{2}x + 7 < \frac{3}{4}x - 5$. Express the solution set using interval notation.

b. Graph each side of the inequality in the same viewing screen. Find the range of x-values when the graph of the left side lies below the graph of the right side.

c. Do parts (a) and (b) agree?

Section 1.6

Plot the left side and the right side of each inequality in the same screen, and use the zoom feature to determine the range of values for which the inequality is true.

157. $0.2x^2 - 2 > 0.05x + 3.25$

158. $12x^2 - 7x - 10 < 2x^2 + 2x - 1$

159. $\frac{3p}{7 - 2p} > 1$

160. $\frac{7p}{15 - 2p} < 1$

Section 1.7

161. Solve the inequality $|1.6x^2 - 4.5| < 3.2$ by graphing both sides of the inequality, and identify which x-values make this statement true. Express the solution using interval notation and round to two decimal places.

162. Solve the inequality $|0.8x^2 - 5.4x| > 4.5$ by graphing both sides of the inequality, and identify which x-values make this statement true. Express the solution using interval notation and round to two decimal places.

Chapter 1 Practice Test

Solve the equation.

1. $4p - 7 = 6p - 1$
2. $-2(z - 1) + 3 = -3z + 3(z - 1)$
3. $3t = t^2 - 28$
4. $8x^2 - 13x = 6$
5. $6x^2 - 13x = 8$
6. $\frac{3}{x-1} = \frac{5}{x+2}$
7. $\frac{5}{y-3} + 1 = \frac{30}{y^2 - 9}$
8. $x^4 - 5x^2 - 36 = 0$
9. $\sqrt{2x+1} + x = 7$
10. $2x^{2/3} + 3x^{1/3} - 2 = 0$
11. $\sqrt{3y-2} = 3 - \sqrt{3y+1}$
12. $x(3x - 5)^3 - 2(3x - 5)^2 = 0$
13. $x^{7/3} - 8x^{4/3} + 12x^{1/3} = 0$

Solve for the specified variable.

14. $F = \frac{9}{5}C + 32$ for C
15. $P = 2L + 2W$ for L

Solve the inequality and express the solution in interval notation.

16. $7 - 5x > -18$
17. $3x + 19 \geq 5(x - 3)$
18. $-1 \leq 3x + 5 < 26$
19. $\frac{2}{5} < \frac{x+8}{4} \leq \frac{1}{2}$
20. $3x \geq 2x^2$
21. $3p^2 \geq p + 4$
22. $|5 - 2x| > 1$
23. $\frac{x-3}{2x+1} \leq 0$
24. $\frac{x+4}{x^2-9} \geq 0$
25. **Puzzle.** A piling supporting a bridge sits so that $\frac{1}{4}$ of the piling is in the sand, 150 feet is in the water, and $\frac{3}{5}$ of the piling is in the air. What is the total height of the piling?
26. **Real Estate.** As a real estate agent you earn 7% of the sale price. The owners of a house you have listed at $150,000 will entertain offers within 10% of the list price. Write an inequality that models the commission you could make on this sale.
27. **Costs: Food.** Club FroYo is a frozen yogurt shop that lets customers fill a container with yogurt and toppings themselves. For a medium size cup, it costs $4 for the first 6 ounces and an additional $0.35/oz for any amount over 6 ounces. Jimmy goes to Club FroYo once a week during his 6-week summer school and his totals ranged from a low of $5.75 to a high of $7.15 Write an inequality representing the range of frozen yogurt amounts in ounces he bought during summer school.
28. **Social Media.** Instagram posts and stories are classified in ratios of width to height. Stories have a 9:16 ratio, and main feed posts are made in a format with a 4:5 ratio. If you own a 6-inch Samsung Galaxy 10 phone (2.8 inch wide × 6 inch screen height) and you check your Instagram on it, what are the dimensions of the story you view? What are the dimensions of the main feed posts?
29. Solve the equation $\frac{1}{0.75x} - \frac{0.45}{x} = \frac{1}{9}$. Graph the function represented by each side in the same viewing rectangle and solve for x.
30. Solve the inequality $0.3 + |2.4x^2 - 1.5| \leq 6.3$ by graphing both sides of the inequality, and identify which x-values make this statement true. Express the solution using interval notation.

Chapter 1 Cumulative Test

Simplify.

1. $5 \cdot (7 - 3 \cdot 4 + 2)$

Simplify and express in terms of positive exponents.

2. $(4x^{-3}b^4)^{-3}$

3. $\dfrac{(x^2y^{-2})^3}{(x^2y)^{-3}}$

Perform the operations and simplify.

4. $(-x^4 + 2x^3) + (x^3 - 5x - 6) - (5x^4 + 4x^3 - 6x + 8)$

5. $x^2(x - 5)(x - 3)$

Factor completely.

6. $3x^3 - 3x^2 - 60x$

7. $2a^3 + 2000$

Perform the operations and simplify.

8. $\dfrac{3 - x}{x^2 - 1} \div \dfrac{5x - 15}{x + 1}$

9. $\dfrac{6x}{x - 2} - \dfrac{5x}{x + 2}$

Solve for *x*.

10. $x^3 - x^2 - 30x = 0$

11. $\frac{2}{7}x = \frac{1}{8}x + 9$

12. Perform the operation and express in standard form: $\dfrac{45}{6 - 3i}$.

Solve for *x*.

13. $\dfrac{6x}{5} - \dfrac{8x}{3} = 4 - \dfrac{7x}{15}$

14. $\dfrac{x - 6}{6 - x} = \dfrac{3}{2}$

15. Tim can paint the interior of a condo in 9 hours. If Chelsea is hired to help him, they can do a similar condo in 5 hours. Working alone, how long will it take Chelsea to paint a similar condo?

16. Solve using the square root method: $y^2 + 36 = 0$.

17. Solve by completing the square: $x^2 + 12x + 40 = 0$.

18. Solve using the Quadratic Formula: $x^2 + x + 9 = 0$.

19. Solve and check: $\sqrt{4 - x} = x - 4$.

20. Solve using substitution: $3x^{-2} + 8x^{-1} + 4 = 0$.

Solve and express the solution in interval notation.

21. $0 < 4 - x \leq 7$

22. $4x^2 < 9x - 11$

23. $\dfrac{x + 2}{9 - x^2} \geq 0$

24. $\left|\dfrac{4 - 5x}{7}\right| \geq \dfrac{3}{14}$

25. Solve for x: $\left|\frac{1}{5}x + \frac{2}{3}\right| = \frac{7}{15}$.

26. Solve the equation $x^6 + \frac{37}{8}x^3 = 27$. Plot both sides of the equation in the same viewing screen, $y_1 = x^6 + \frac{37}{8}x^3$ and $y_2 = 27$. Does the point(s) of intersection agree with your solution?

27. Solve the inequality $\left|\dfrac{3x}{x - 2}\right| < 1$ by graphing both sides of the inequality, and identify which x-values make this statement true.