# CHAPTER ONE
# Starting Out



In spite of its formidable name, calculus is not a particularly difficult subject. The fundamental concepts of calculus are straightforward. Your appreciation of their value will grow as you develop the skills to use them.

After working through *Quick Calculus* you will be able to handle many problems and be prepared to acquire more elaborate techniques if you need them. The important word here is *working,* though we hope that you find that the work is enjoyable.

*Quick Calculus* comprises four chapters that consist of sections and subsections. We refer to the subsections as *frames*. Each chapter concludes with a summary. Following these chapters there are two appendixes on supplementary material and a collection of review problems with solutions.

The frames are numbered sequentially throughout the text. Working *Quick Calculus* involves studying the frames and doing the problems. You can check your answers immediately: they will be located at the bottom of one of the following pages or, if the solutions are longer, in a separate frame. Also a summary of frame problems answers start on page 273.

Your path through *Quick Calculus* will depend on your answers. The reward for a correct answer is to go on to new material. If you have difficulty, the solution will usually be explained and you may be directed to another problem.

Go on to frame **1**.

## 1.1 A Few Preliminaries

**1**

Chapter 1 will review topics that are foundational for the discussions to come. These are:

- the definition of a mathematical function;
- graphs of functions;

*(continued)*

- the properties of the most widely used functions: linear and quadratic functions, trigonometric functions, exponentials, and logarithms.

A note about calculators: a few problems in *Quick Calculus* need a simple calculator. The calculator in a typical smartphone is more than adequate. If you do not happen to have access to a calculator, simply skip the problem: you can master the text without it.

Go on to frame **2**.

**2**

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Here is what's ahead: this first chapter is a review, which will be useful later on; Chapter 2 is on differential calculus; and Chapter 3 introduces integral calculus. Chapter 4 presents some more advanced topics. At the end of each chapter there is a summary to help you review the material in that chapter. There are two appendixes—one gives proofs of a number of relations used in the book, and the other describes some supplementary topics. In addition, there is a list of extra problems with answers in the Review Problems on page 277, and a section of tables you may find useful.

First we review the definition of a function. If you are already familiar with this and with the idea of dependent and independent variables, skip to frame **14**. (In fact, in this chapter there is ample opportunity for skipping if you already know the material. On the other hand, some of the material may be new to you, and a little time spent on review can be a good thing.)

A word of caution about the next few frames. Because we start with some definitions, the first section must be somewhat more formal than most other parts of the book.

Go on to frame **3**.

## 1.2 Functions

**3**

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The definition of a function makes use of the idea of a *set*. If you know what a set is, go to **4**. If not, read on.

A *set* is a collection of objects—not necessarily material objects—described in such a way that we have no doubt as to whether a particular object does or does not belong to it. A set may be described by listing its elements. Example: 23, 7, 5, 10 is a set of numbers. Another example: Reykjavik, Ottawa, and Rome is a set of capitals.

We can also describe a set by a rule, such as all the even positive integers (this set contains an infinite number of objects).

A particularly useful set is the set of all real numbers. This includes all numbers such as 5, −4, 0, ½, $\pi$, −3.482, $\sqrt{2}$. The set of real numbers does *not* include quantities involving the square root of negative numbers. Such quantities are called *complex numbers*; in this book we will be concerned only with real numbers.

The mathematical use of the word "set" is similar to the use of the same word in ordinary conversation, as "a set of chess pieces."

Go to **4**.

**4**

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In the blank below, list the elements of the set that consists of all the odd integers between −10 and +10.

Elements: ____________________

Go to **5** for the correct answer.

**5**

---

Here are the elements of the set of all odd integers between −10 and +10:

$$-9, -7, -3, -5, -1, \ 1, \ 3, \ 5, \ 7, \ 9.$$

Go to **6**.

**6**

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Now we are ready to talk about functions. Here is the definition.

A *function* is a rule that assigns to each element in a set $A$ one and only one element in a set $B$.

The rule can be specified by a mathematical formula such as $y = x^2$, or by tables of associated numbers, for instance, the temperature at each hour of the day. If $x$ is one of the elements in set $A$, then the element in set $B$ that the function $f$ associates with $x$ is denoted by the symbol $f(x)$. This symbol $f(x)$ is the value of $f$ evaluated at the element $x$. It is usually read as "$f$ of $x$."

The set $A$ is called the *domain* of the function. The set of all possible values of $f(x)$ as $x$ varies over the domain is called the *range* of the function. The range of $f$ need not be all of $B$.

*(continued)*

In general, $A$ and $B$ need not be restricted to sets of real numbers. However, as mentioned in frame **3**, in this book we will be concerned only with real numbers.

Go to **7**.

**7**

---

For example, for the function $f(x) = x^2$, with the domain being all real numbers, the range is ______________________________.

Go to **8**.

**8**

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The range is *all nonnegative real numbers*. For an explanation, go to **9**.

Otherwise, skip to **10**.

**9**

---

Recall that the product of two negative numbers is positive. Thus for any real value of $x$ positive or negative, $x^2$ is positive. When $x$ is 0, $x^2$ is also 0. Therefore, the range of $f(x) = x^2$ is all nonnegative numbers.

Go to **10**.

**10**

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Our chief interest will be in rules for evaluating functions defined by formulas. If the domain is not specified, it will be understood that the domain is the set of all real numbers for which the formula produces a real number, and for which it makes sense. For instance,

(a) $f(x) = \sqrt{x}$ Range = __________________.

(b) $f(x) = \dfrac{1}{x}$ Range = __________________.

For the answers go to **11**.

**11**

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The function $\sqrt{x}$ is real for $x$ nonnegative, so the answer to (a) is all nonnegative real numbers. The function $1/x$ is defined for all values of $x$ except zero, so the range in (b) is all real numbers except zero.

Go to **12**.

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**12**

When a function is defined by a formula such as $f(x) = ax^3 + b$, $x$ is called the *independent variable* and $f(x)$ is called the *dependent variable*. One advantage of this notation is that the value of the dependent variable, say for $x = 3$, can be indicated by $f(3)$.

Often, a single letter is used to represent the dependent variable, as in

$$y = f(x).$$

Here $x$ is the independent variable, and $y$ is the dependent variable.

Go to **13**.

**13**

In mathematics the symbol $x$ frequently represents an independent variable, $f$ often represents the function, and $y = f(x)$ usually denotes the dependent variable. However, any other symbols may be used for the function, the independent variable, and the dependent variable. For example, we might have $z = H(r)$, which is read as "$z$ equals $H$ of $r$." Here $r$ is the independent variable, $z$ is the dependent variable, and $H$ is the function.

Now that we know what a function means, let's describe a function with a graph instead of a formula.

Go to **14**.

## 1.3 Graphs

**14**

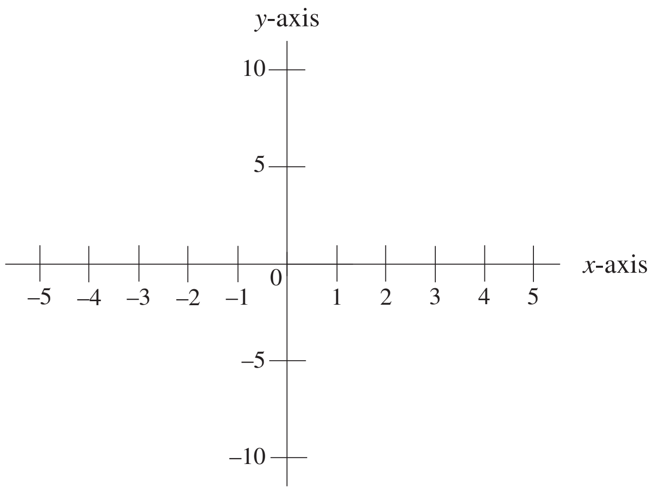
If you know how to plot graphs of functions, skip to frame **19**.

Otherwise, go to **15**.

**15**

We start by constructing coordinate axes. In the most common cases we construct a pair of mutually perpendicular intersecting lines, one horizontal, the other vertical. The horizontal line is often called the *x-axis* and the vertical line the *y-axis*. The point of intersection is the origin, and the axes together are called the *coordinate axes*.

*(continued)*




Next we select a convenient unit of length and, starting from the origin, mark off a number scale on the $x$-axis, positive to the right and negative to the left. In the same way, we mark off a scale along the $y$-axis with positive numbers going upward and negative downward. The scale of the $y$-axis does not need to be the same as that for the $x$-axis (as in the drawing). In fact, $y$ and $x$ can have different units, such as distance and time.

Go to **16**.

**16**

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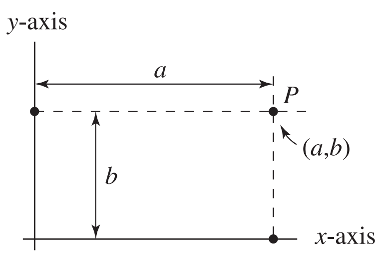
We can represent one specific pair of values associated by the function in the following way: let $a$ represent some particular value for the independent variable $x$, and let $b$ indicate the corresponding value of $y = f(x)$. Thus, $b = f(a)$.




We now draw a line parallel to the $y$-axis at distance $a$ from the $y$-axis and another line parallel to the $x$-axis at distance $b$ from that axis. The point $P$ at which these two lines intersect is designated by the pair of values $(a, b)$ for $x$ and $y$, respectively.

The number $a$ is called the $x$-coordinate of $P$, and the number $b$ is called the $y$-coordinate of $P$. (Sometimes the x-coordinate is called the *abscissa*, and the $y$-coordinate is called the *ordinate*.) In the designation of a typical point by the notation $(a, b)$ we will always designate the $x$-coordinate first and the $y$-coordinate second.

As a review of this terminology, encircle the correct answers below. For the point (5, −3):

$x$-coordinate: [−5 | − 3 | 3 | 5]

$y$-coordinate: [−5 | − 3 | 3 | 5]

Go to **17**.

**17**

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The most direct way to plot the graph of a function $y = f(x)$ is to make a table of reasonably spaced values of $x$ and of the corresponding values of $y = f(x)$. Then each pair of values $(x, y)$ can be represented by a point as in the previous frame. A graph of the function is obtained by connecting the points with a smooth curve. Of course, the points on the curve may be only approximate. If we want an accurate plot, we just have to be very careful and use many points. (On the other hand, crude plots are pretty good for many purposes.)

Go to **18**.

**18**

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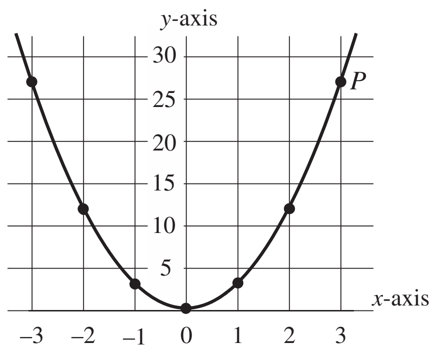
As an example, here is a plot of the function $y = 3x^2$. A table of values of $x$ and $y$ is shown, and these points are indicated on the graph.

| $x$ | $y$ |
|---|---|
| −3 | 27 |
| −2 | 12 |
| −1 | 3 |
| 0 | 0 |
| 1 | 3 |
| 2 | 12 |
| 3 | 27 |




To test yourself, encircle the pair of coordinates that corresponds to the point $P$ indicated in the figure.

[(3, 27) | (27, 3) | none of these]

If incorrect, study frame **16** once again and then go to **19**. If correct,

Go to **19**.

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**19**

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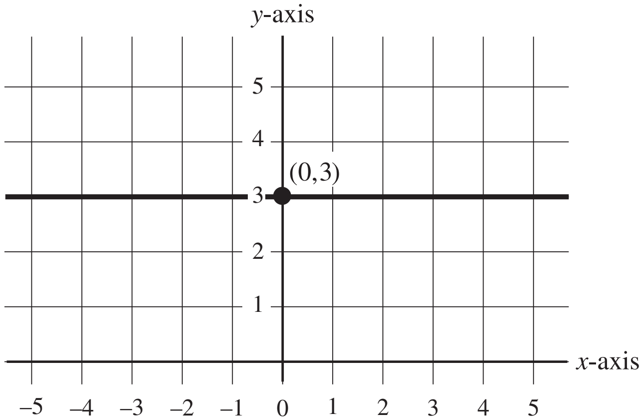
Here is a rather special function. It is called a *constant function* and assigns a single fixed number $c$ to every value of the independent variable, $x$. Hence, $f(x) = c$.




This is a peculiar function because the value of the dependent variable is the same for all values of the independent variable. Nevertheless, the relation $f(x) = c$ assigns exactly one value of $f(x)$ to each value of $x$ as required in the definition of a function. All the values of $f(x)$ happen to be the same.

Try to convince yourself that the graph of the constant function $y = f(x) = 3$ is a straight line parallel to the $x$-axis passing through the point (0,3) as shown in the figure.

Go to **20**.

**20**

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Another special function is the *absolute value function*. The absolute value of $x$ is indicated by the symbol $|x|$. The absolute value of a number $x$ determines the size, or magnitude, of the number without regard to its sign. For example,

$$|-3| = |3| = 3$$

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Answers: Frame 16: 5, −3
Frame 18: (3, 27)

Now we will define $|x|$ in a general way. But first we need to recall the inequality symbols:

$a > b$ means $a$ is greater than $b$.

$a \geq b$ means $a$ is greater than or equal to $b$.

$a < b$ means $a$ is less than $b$.

$a \leq b$ means $a$ is less than or equal to $b$.

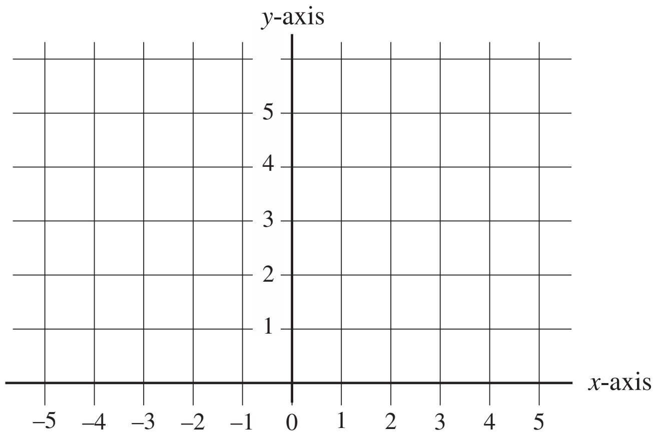
With this notation we can define the absolute value function, $|x|$, by the following two rules:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Go to **21**.

**21**

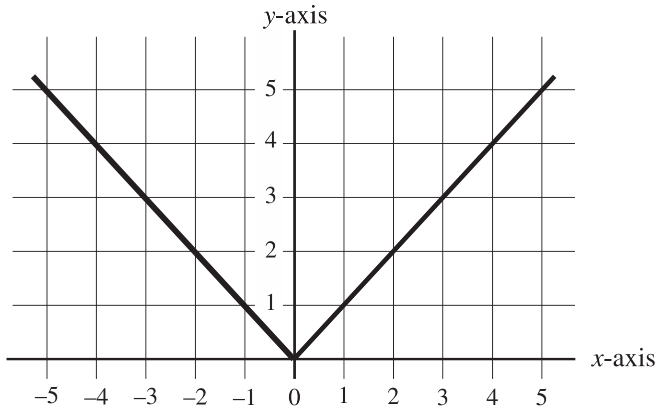
A good way to show the behavior of a function is to plot its graph. Therefore, as an exercise, plot a graph of the function $y = |x|$ in the accompanying figure.




To check your answer, go to **22**.

**22**

The graph for | $x$ | is




This can be seen by preparing a table of $x$ and $y$ values as follows:

| $x$ | $y = \|x\|$ |
|---|---|
| −4 | +4 |
| −2 | +2 |
| 0 | 0 |
| +2 | +2 |
| +4 | +4 |

These points may be plotted as in frames **16** and **18** and the lines drawn with the results in the above figure.

The graph and $x$, $y$ coordinates described here are known as a *Cartesian coordinate system*. There are other coordinate systems better suited to other geometries, such as cylindrical or spherical coordinate systems, but Cartesian coordinates are the best known.

With this introduction on functions and graphs, we are now going to familiarize ourselves with some important elementary functions.

These functions are the linear, quadratic, trigonometric, exponential, and logarithmic functions.

Go to **23**.

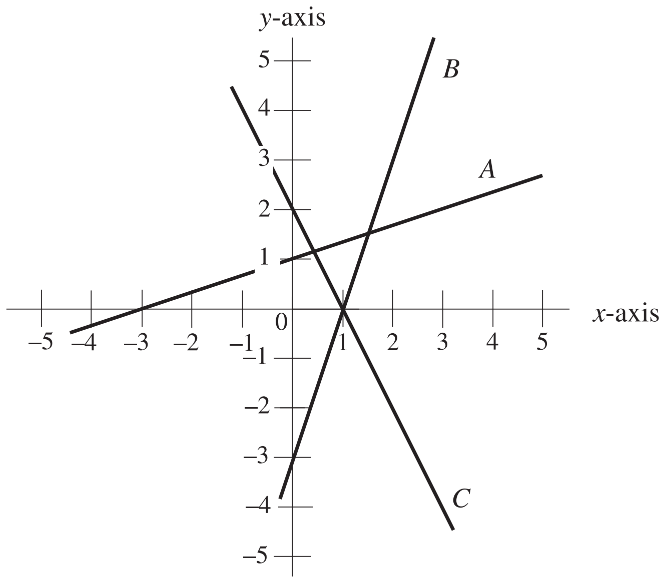
# 1.4 Linear and Quadratic Functions

**23**

A function defined by an equation in the form $y = mx + b$, where $m$ and $b$ are constants, is called a *linear function* because its graph is a straight line. This is a simple and useful function, and you need to become familiar with it.

Here is an example: Encircle the letter that identifies the graph (as labeled in the figure) of

$$y = 3x - 3. \qquad [A \mid B \mid C]$$



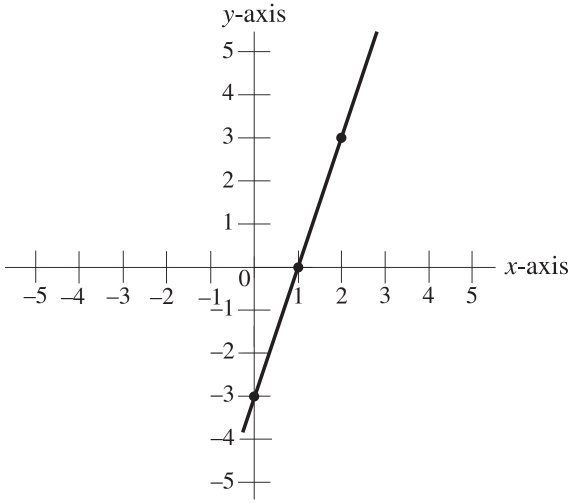

If you missed this or if you do not feel entirely sure of the answer, go to **24**.

Otherwise, go to **25**.

**24**

The table below gives a few values of $x$ and $y$ for the function $y = 3x - 3$.

| $x$ | $y$ |
|---|---|
| –2 | –9 |
| –1 | –6 |
| 0 | –3 |
| 1 | 0 |
| 2 | 3 |



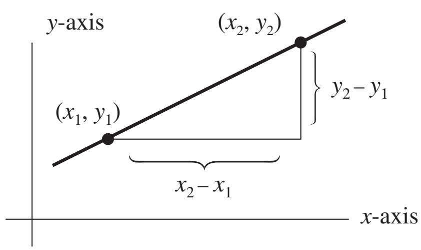

A few of these points are shown on the graph, and a straight line has been drawn through them. This is line $B$ of the figure in frame **23**.

Go to **25**.

**25**

Here is the graph of a typical linear function. Let us take any two different points on the line, $(x_2, y_2)$ and $(x_1, y_1)$. We define the slope of the line in the following way:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

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Answer: Frame 23: $B$

The idea of slope will be important in our later work, so let's spend a little time learning more about it.

Go to **26**.

**26**

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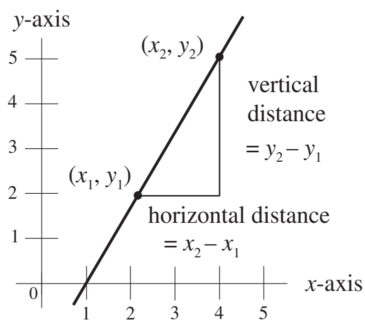
If the $x$ and $y$ scales are the same, as in the figure, then the slope is the ratio of the vertical distance $y_2 - y_1$ to the horizontal distance $x_2 - x_1$ as we go from the point $(x_1, y_1)$ on the line to $(x_2, y_2)$. If the line is vertical, the slope is infinite (or, more strictly, undefined). Test for yourself that the slope is the same for any pair of two separate points on the line.




Go to **27**.

**27**

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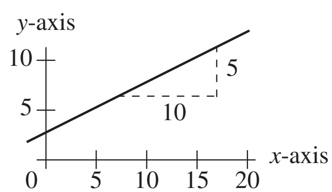
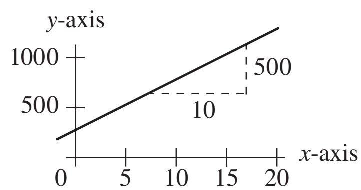
If the vertical and horizontal scales are not the same, the slope is still defined by

$$\text{slope} = \frac{\text{vertical distance}}{\text{horizontal distance}},$$

but now the distance is measured using the appropriate scale. For instance, the two figures below may look similar, but the slopes are quite different. In the first figure the $x$ and $y$ scales are identical, and the slope is ½. In the second figure the $y$ scale has been changed by a factor of 100, and the slope is 50.

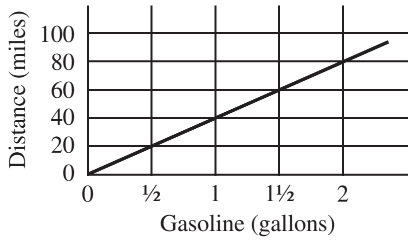



*(continued)*

Because the slope is the ratio of two lengths, the slope is a pure number if the lengths are pure numbers. However, if the variables have different dimensions, the slope will also have a dimension.

Below is a plot of the distance traveled by a car vs. the amount of gasoline consumed.




Here the slope has the units of miles per gallon (mpg). What is the slope of the line shown?

Slope = [20 | 40 | 60 | 80] mpg

If right, go to **29**.
Otherwise, go to **28**.

**28**

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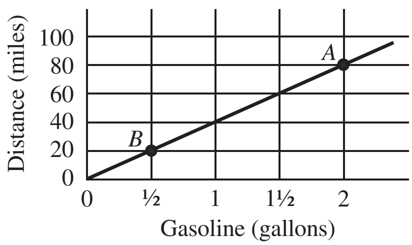
To evaluate the slope, let us find the coordinates of any two points on the line.




For instance, *A* has the coordinates (2 gallons, 80 miles) and *B* has the coordinates (½ gallon, 20 miles). Therefore, the slope is

$$\frac{(80 - 20)\text{ miles}}{(2 - \frac{1}{2})\text{ gallon}} = 40\,\frac{\text{miles}}{\text{gallon}} = 40\text{ mpg.}$$

We would have obtained the same value for the slope no matter which two points we used, because two points determine a straight line.

Go to **29**.

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**29**

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If the line is described by an equation of the form $y = mx + b$, then the slope is given by

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substituting in the above expression for $y$, we have

$$\text{slope} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{mx_2 - mx_1}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$

What is the slope of $y = 7x - 5$?

$\left[5/7 \mid 7/5 \mid -5 \mid -7 \mid 5 \mid 7\right]$

If right, go to **31**.
Otherwise, go to **30**.

**30**

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The equation $y = 7x - 5$ can be written in the form $y = mx + b$ if $m = 7$ and $b = -5$. Because slope $= m$, the line given has a slope of 7.

Go to **31**.

**31**

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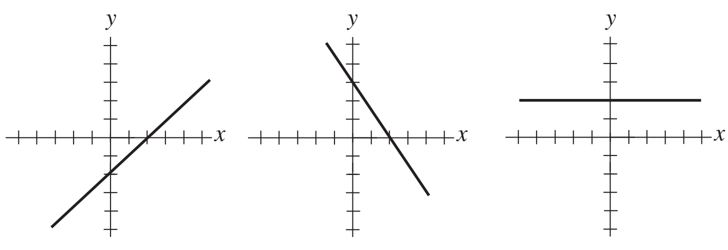
The slope of a line can be positive (greater than 0), negative (less than 0), or 0. An example of each is shown graphically below.




Positive slope Figure 1 | Negative slope Figure 2 | Zero slope Figure 3

Note how a line with positive slope rises in going from left to right, a line with negative slope falls in going from left to right, and a line of zero slope is horizontal. (It was pointed out in frame **26** that the slope of a vertical line is not defined.)

*(continued)*

Indicate whether the slope of the graph of each of the following equations is positive, negative, or zero by encircling your choice.

| | Equation | Slope |
|---|---|---|
| 1. | $y = 2x - 5$ | { + \| − \| 0 } |
| 2. | $y = -3x$ | { + \| − \| 0 } |
| 3. | $p = q - 2$ | { + \| − \| 0 } |
| 4. | $y = 4$ | { + \| − \| 0 } |

The answer is in the next frame.

**32**

Here are the answers to the questions in frame **31**.

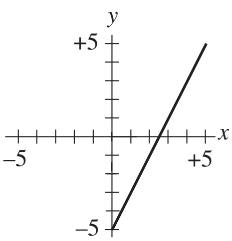
In frame **29** we saw that for a linear equation in the form $y = mx + b$ the slope is $m$.

**1.** $y = 2x - 5$. Here $m = 2$ and the slope is 2. Clearly this is a positive number. See Figure 1 below.

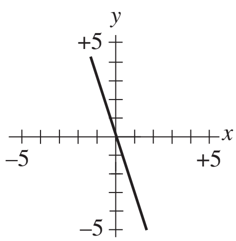
**2.** $y = -3x$. Here $m = -3$. The slope is $-3$, which is negative. See Figure 2 below.

**3.** $p = q - 2$. In this equation the variables are $p$ and $q$, rather than $y$ and $x$. Written in the form $p = mq + b$, it is evident that $m = 1$, which is positive. See Figure 3 below.

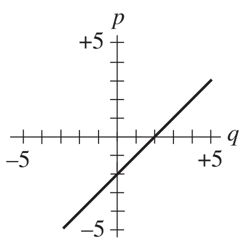
**4.** $y = 4$. This is an example of a constant function. Here $m = 0$, $b = 4$, and the slope is 0. See Figure 4 below.



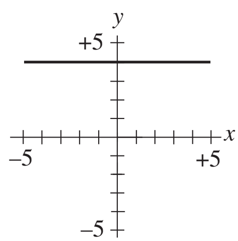

Positive slope
$y = 2x - 5$
Figure 1




Negative slope
$y = -3x$
Figure 2




Positive slope
$p = q - 2$
Figure 3




Zero slope
$y = 4$
Figure 4

Go to **33**.

Answers: Frame 27: 40 mpg
Frame 29: 7

**33**

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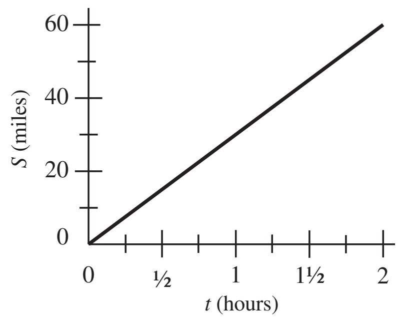
Here is a linear equation in which the slope has a familiar meaning. The graph below shows the position $S$ of a car on a straight road at different times. The position $S = 0$ means the car is at the starting point.




Try to guess the correct word to fill in the blank below:
The slope of the line has the same value as the car's ____________________.

Go to **34**.

**34**

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The slope of the line has the same value as the car's *speed* (or, for this one-dimensional motion *velocity*).

The slope is the ratio of the distance traveled to the time required. But, by definition, the speed is also the distance traveled divided by the time. Thus the value of the slope of the line is equal to the speed.

Go to **35**.

**35**

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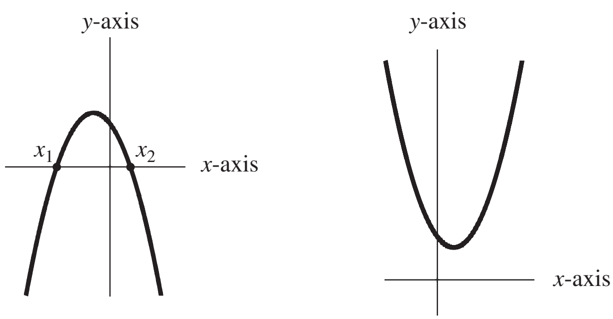
Now let's look at another type of equation. An equation in the form $y = ax^2 + bx + c$, where $a$, $b$, and $c$ are constants ($a \neq 0$), is called a *quadratic function* and its graph is called a *parabola*. Two typical parabolas are shown in the figure.




Go to **36**.

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**36**

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*Roots of an Equation:*

The values of $x$ for which $f(x) = 0$ are called the *roots* of the equation. The values at $y = 0$, shown by $x_1$ and $x_2$ in the figure on the left in frame **35**, correspond to values of $x$ which satisfy $ax^2 + bx + c = 0$ and are thus the roots of the equation. Not all quadratic equations have real roots. For example, the curve on the right represents an equation with no real value of $x$ when $y = 0$.

Although you will not need to find the roots of any quadratic equation later in this book, you may want to know the formula anyway. If you would like to see a discussion of this, go to frame **37**.

Otherwise, skip to frame **39**.

**37**

---

The equation $ax^2 + bx + c = 0$ has two roots. These are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The subscripts 1 and 2 serve merely to identify the roots. The two roots can be summarized by a single equation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We will not prove these results, though they can be checked by substituting the values for $x$ in the original equation.

Here is a practice problem on finding roots: Which answer correctly gives the roots of $3x - 2x^2 = 1$?

(a) $\frac{1}{4}\left(3 + \sqrt{17}\right)$; $\frac{1}{4}\left(3 - \sqrt{17}\right)$

(b) $-1$; $-\frac{1}{2}$

(c) $\frac{1}{4}$; $-\frac{1}{4}$

(d) $1$; $\frac{1}{2}$

---

Answers: Frame 31: $+, -, +, 0$

Encircle the letter of the correct answer.

$$[a \mid b \mid c \mid d]$$

If you got the right answer, go to **39**.
The answer is in the following frame.

**38**

---

Here is the solution to the problem in frame **37**.
The equation $3x - 2x^2 = 1$ can be written in the standard form

$$2x^2 - 3x + 1 = 0.$$

Here $a = 2$, $b = -3$, $c = 1$.

$$x = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right] = \frac{1}{4}\left[-(-3) \pm \sqrt{(-3)^2 - (4)(2)(1)}\right]$$

$$= \frac{1}{4}(3 \pm 1).$$

$$x_1 = \frac{1}{4}(3 + 1) = 1.$$

$$x_2 = \frac{1}{4}(3 - 1) = \frac{1}{2}.$$

Go to **39**.

**39**

---

This ends our brief discussion of linear and quadratic functions. Perhaps you would like some more practice on these topics before continuing. If so, try working Review Problems 1–5 on page 277. At the end of this chapter there is a concise summary of the material we have had so far, which you may find useful.

Whenever you are ready, go to **40**.

## 1.5 Angles and Their Measurements

**40**

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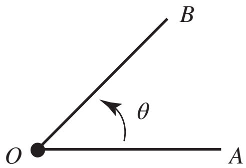
*Elementary features of rotations and angles:*

If you are already familiar with rotations, angle, and degrees and radians, you can jump to frame **50**.

*(continued)*




The concept of the angle is the bedrock of trigonometry. Although the general idea of an angle is probably familiar, it is important to agree on the conventions and units for describing angles. For two straight-line segments *OA* and *OB* that intersect at a point *O*, the angle between them is a measure of how far the line segment *OA* must be rotated about the point *O* to coincide with the line segment *OB*.

If the two segments initially coincide, for instance, half a revolution of either segment will leave them pointing in opposite directions and a full revolution will bring them back to their original positions.

The Greek letter $\theta$ (theta) symbolizes the *rotation angle*. The sense of rotation is shown by the curved arrow. We will follow the convention that if the segment *OA* is rotated in the counterclockwise direction to coincide with the segment *OB*, then the rotation is positive. Conversely, a rotation in the clockwise direction is negative. The direction can be indicated by a small arrowhead on the curve between the two segments. If the sense of rotation is unimportant, the arrowhead is usually omitted.

*Measuring the size of rotations:*

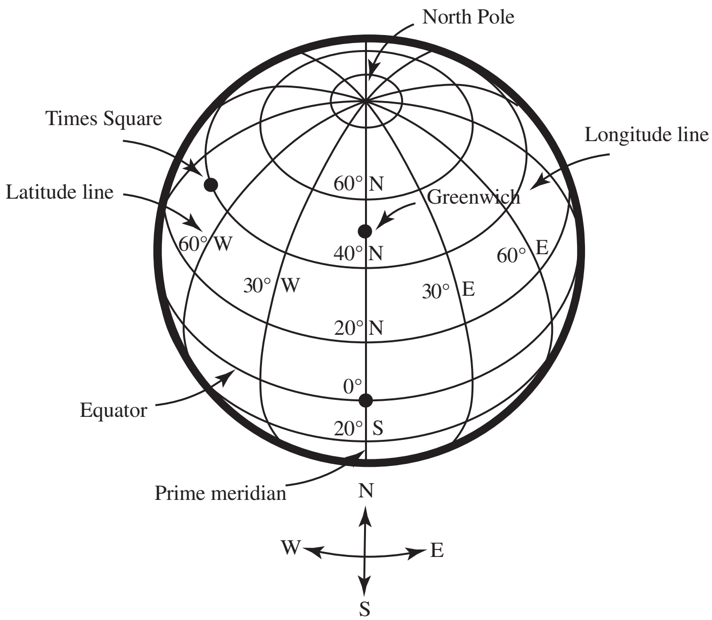
There are two conventions used for measuring the size of rotation. The first divides one revolution into 360 parts called *degrees* (not to be confused with degrees of temperature). The number 360 has the attraction of being large—providing good angular resolution—and possessing many divisors. The symbol for a degree of rotation is °; hence, a quarter turn is 90°. The degree can be subdivided into 60 *minutes* (60′), and the minute subdivided into 60 *seconds* (60″). Until recent years degrees, minutes, and seconds (*DMS coordinates*) were commonly used in map-making and navigation. With advent of GPS and laser metrology, the convention for subdividing the degree has been changed: instead of minutes and seconds, the convention is to express the fraction of a degree by a decimal number, typically with four digits.

*Location on Earth: latitude and longitude:*

Two numbers are needed to describe positions on a surface. Because the Earth is spherical, Cartesian coordinates are not useful for specifying a position. Instead, a system based on coordinates known as *latitude* and *longitude* is employed. One of the coordinates is based on the idea of a *meridian*. This is an imaginary half circle on the Earth that connects the Earth's

Answers: Frame 37: d

South and North Poles. The *prime meridian* is the half circle passing through Earth's South and North Poles and Greenwich, England. Longitude is the angle between the prime meridian and the meridian through the point of interest. Points east of the prime meridian require a clockwise rotation around the polar axis and by convention are positive; those to the west are negative. The sign of longitude reverses when passing the meridian 180 degrees east or west of the prime meridian.



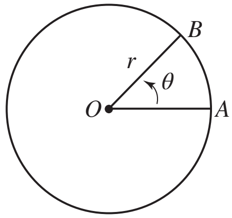

Instead of using positive and negative values of longitude, the convention is to assign east (E) for positive values and west (W) for negative values.

Latitude lines are parallel to the equator. Latitudes north of the equator use the symbol N and latitudes south of the equator use the symbol S. For example, the latitude and longitude of Times Square in New York City are 40.7580°N and 73.9855°W. In old-fashioned DMS units these are 77° 2′ 7.008″W and 38° 53′ 22.1424″N.

Go to **41**.

**41**

As we have seen, the full circle contains 360°, and so it follows that a semicircle contains 180°.



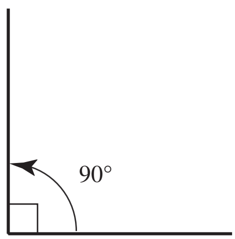

Which of the following angles is equal to the angle $\theta$ shown in the figure?

[25° | 45° | 90° | 180°]

If right, go to **43**.
Otherwise, go to **42**.

**42**

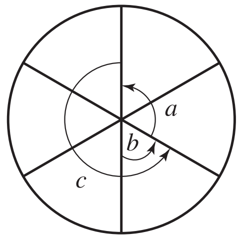
To find the angle $\theta$, let's first look at a related example.




The angle shown is a *right angle*, constructed from two perpendicular lines. (The symbol between sides indicates a right angle.) Because there are four right angles in a full revolution, it is apparent that the angle equals

$$\frac{360^\circ}{4} = 90^\circ.$$

The angle $\theta$ shown in frame **41** is just half as big as the right angle; thus it is 45°.
Here is a circle divided into equal segments by three straight lines.



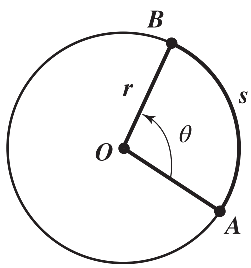

Which of the angles in the figure equals 240°?

[*a* | *b* | *c*]

Go to **43**.

**43**

The second convention for measuring the size of rotations is the *radian*. The symbol for the radian is *rad*.




To find the value of an angle in radians, we draw a circle of radius *r*, about the vertex, *O*, of the angle so that it intersects the sides of the angle at two points, shown in the figure as *A* and *B*. The length of the arc between *A* and *B* is designated by *s*. Then,

$$\theta \text{ (in radians)} = \frac{s}{r} = \frac{\text{length of arc}}{\text{radius}}.$$

Radians are used widely in scientific applications. For example, to calculate numerical values of trigonometric functions in this chapter naturally calls for radians. If no unit is given for the numerical value of an angle, the angle is in radians.

*(continued)*

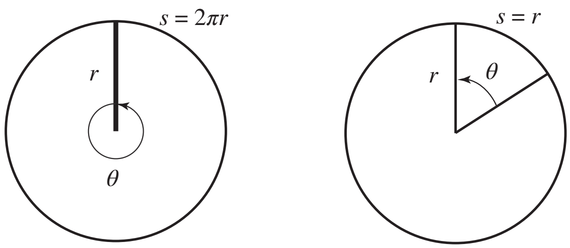
To see whether you have caught on, answer this question: There are 360 degrees in a circle; how many radians are there?

$$[1 \mid 2 \mid \pi \mid 2\pi \mid 360/\pi]$$

If right, go to **45**.
Otherwise, go to **44**.

**44**

---




The circumference of a circle is $\pi d$ or $2\pi r$, where $d$ is the diameter and $r$ is the radius. The length of an arc going completely around a circle is the circumference, $2\pi r$, so the angle enclosed is $2\pi r/r = 2\pi$ radians, as shown in the figure on the left. In the figure on the right the angle $\theta$ subtends an arc $s = r$.

Encircle the answer, which gives $\theta$.

$$\left[1 \text{ rad} \mid {}^1\!/_4 \text{ rad} \mid {}^1\!/_2 \text{ rad} \mid \pi \text{ rad} \mid \text{none of these}\right]$$

Go to **45**.

**45**

---

Because $2\pi$ rad = 360°, the rule for converting angles from degrees to radians is

$$1 \text{ rad} = \frac{360°}{2\pi}.$$

---

Answers: Frame 41: 45°
Frame 42: c

Conversely,

$$1^\circ = \frac{2\pi \text{ rad}}{360}.$$

Try the following problems.

$$60^\circ = [2\pi/3 \mid \pi/3 \mid \pi/4 \mid \pi/6] \text{ rad}$$
$$\pi/4 = \left[22\tfrac{1}{2}^\circ \mid 45^\circ \mid 60^\circ \mid 90^\circ\right]$$

Which angle is closest to 1 rad? (Remember that $\pi = 3.14\ldots$)

$$[30^\circ \mid 45^\circ \mid 60^\circ \mid 90^\circ]$$

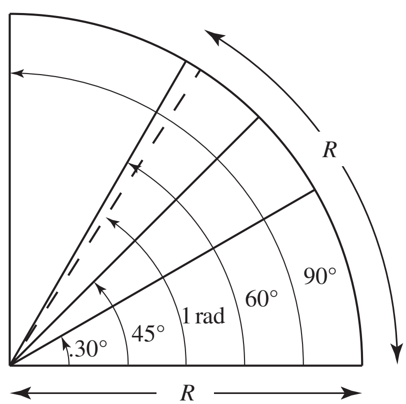
If correct, go to **47**.
If you made any mistake, go to **46**.

**46**

Here are the solutions to the problems in frame **45**. From the formulas in frame **45**, one obtains

$$60^\circ = 60 \times \frac{2\pi \text{ rad}}{360} = \frac{2\pi \text{ rad}}{6} = \frac{\pi}{3}\text{rad}.$$
$$\frac{\pi}{4}\text{rad} = \frac{\pi}{4} \times \frac{360^\circ}{2\pi} = \frac{360^\circ}{8} = 45^\circ.$$

Because $2\pi$ is just a little greater than 6, 1 rad is slightly less than $360^\circ/6 = 60^\circ$. (A closer approximation to the radian is $57.3^\circ$.) The figure below shows all the angles in this question.




Go to **47**.

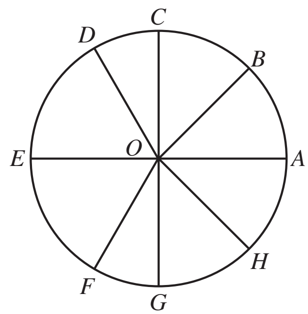
**47**

---

In the circle shown, $CG$ is perpendicular to $AE$ and

arc $AB$ = arc $BC$ = arc $AH$,

arc $AD$ = arc $DF$ = arc $FA$.




(Arc $AB$ means the length of the arc along the circle between $A$ and $B$, going the shortest way.)

We will designate angles by three letters. For example, $\angle AOB$ (read as "angle AOB") designates the angle between OA and OB.

Try the following:

$$\angle AOD = \{60^\circ \mid 90^\circ \mid 120^\circ \mid 150^\circ \mid 180^\circ\}$$

$$\angle FOH = \{15^\circ \mid 30^\circ \mid 45^\circ \mid 60^\circ \mid 75^\circ \mid 90 \text{ degrees}\}$$

$$\angle HOB = \{1/4 \mid 1 \mid \pi/2 \mid \pi/4 \mid \pi/8\}$$

If you did all these correctly, go to **49**.
If you made any mistakes, go to **48**.

**48**

---

Because *arc* $AD$ = *arc* $DF$ = *arc* $FA$, and the sum of their angles is 360°, $\angle AOD = 360^\circ/3 = 120^\circ$.

$$\angle FOA = 120^\circ, \quad \angle GOA = 90^\circ, \quad \angle GOH = 45^\circ.$$

---

Answers: Frame 43: $2\pi$
Frame 44: 1 rad
Frame 45: $\pi/3$, 45°, 60°

Thus

$$\angle FOH = \angle FOG + \angle GOH = 30^\circ + 45^\circ = 75^\circ.$$
$$\angle HOB = \angle HOA + \angle AOB = 45^\circ + 45^\circ = 90^\circ.$$

Now try the following:

$$90^\circ = [2\pi \mid \pi/6 \mid \pi/2 \mid \pi/8 \mid 1/4]$$
$$3\pi = [240^\circ \mid 360^\circ \mid 540^\circ \mid 720^\circ]$$
$$\pi/6 = [15^\circ \mid 30^\circ \mid 45^\circ \mid 60^\circ \mid 90^\circ \mid 120^\circ]$$

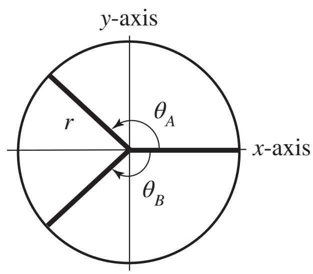
Go to **49**.

**49**

---

Rotations can be clockwise or counterclockwise. By choosing a convention for the sign of an angle, we can indicate which direction is meant. As previously explained, an angle formed by rotating in a counterclockwise direction is positive; an angle formed by moving in a clockwise direction is negative.

Here is a circle of radius *r* drawn with *x*- and *y*-axes, as shown:




We will usually choose the positive *x*-axis as the initial side and, in general, we will measure angles from the initial to the final or terminal side, denoted by the curved arrow. For example, the angle $\theta_A$ measured in the counterclockwise direction is positive and $\theta_B$ is negative, as shown in the figure. If there is no curved arrow associated with the angle, then we shall assume that the angle is positive.

Go to **50**.

---

# 1.6 Trigonometry

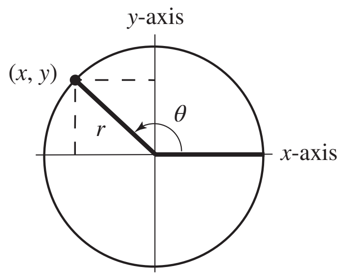
**50**

If you are not familiar with trigonometric functions, proceed with this frame. Otherwise, check yourself with frame **51**, or go right to frame **52**.

Our next task is to introduce the trigonometric functions. These functions relate the various sides and angles of triangles.

Do you know the general definitions of the trigonometric functions of angle $\theta$? If you do, test yourself with the quiz below. If you don't, go right on to frame **51**.

The trigonometric functions of $\theta$ can be expressed in terms of the coordinates $x$ and $y$ and the radius of the circle, $r = \sqrt{x^2 + y^2}$.




These are shown in the figure. Try to fill in the blanks (the answers are in frame **51**):

$\sin\theta =$ _____ $\csc\theta =$ _____

$\cos\theta =$ _____ $\sec\theta =$ _____

$\tan\theta =$ _____ $\cot\theta =$ _____

Go to frame **51** to check your answers.

Answers: Frame 47: 120°, 75°, $\pi/2$
Frame 48: $\pi/2$, 540°, 30°

**51**

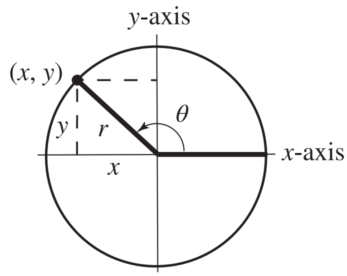
Here are the definitions of the trigonometric functions:

$$\text{sine}: \quad \sin\theta = \frac{y}{r}, \quad \text{cotangent}: \quad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y},$$

$$\text{cosine}: \quad \cos\theta = \frac{x}{r}, \quad \text{secant}: \quad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x},$$

$$\text{tangent}: \quad \tan\theta = \frac{y}{x}, \quad \text{cosecant}: \quad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}.$$

Notice that the definitions in the right-hand equations are the reciprocal of those on the left.




For the angle shown in the figure, $x$ is negative and $y$ is positive ($r = \sqrt{x^2 + y^2}$ and is always positive) so that $\cos\theta$, $\tan\theta$, $\cot\theta$, and $\sec\theta$ are negative, while $\sin\theta$ and $\csc\theta$ are positive.

Go to **52**.

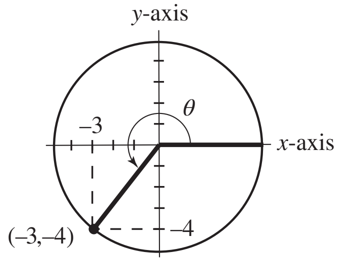
**52**

Below is a circle with a radius of 5. The point shown is (−3, −4). On the basis of the definition in the last frame, you should be able to answer the following:

$\sin\theta = [\,3/5 \mid 5/3 \mid 3/4 \mid -4/5 \mid -3/5 \mid 4/3\,]$

$\cos\theta = [\,3/5 \mid 5/3 \mid 3/4 \mid -4/5 \mid -3/5 \mid 4/3\,]$

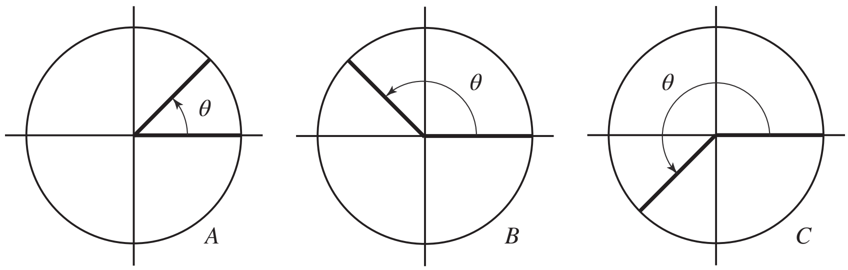
$\tan\theta = [\,3/5 \mid 5/3 \mid 3/4 \mid -4/5 \mid -3/5 \mid 4/3\,]$




If all right, go to **55**.
Otherwise, go to **53**.

**53**

Perhaps you had difficulty because you did not realize that $x$ and $y$ have different signs in different quadrants (quarters of the circle) while $r$, a radius, is always positive. Try this problem.




Indicate whether the function specified is positive or negative, for each of the figures, by checking the correct box.

| | Figure *A* + | Figure *A* − | Figure *B* + | Figure *B* − | Figure *C* + | Figure *C* − |
|---|---|---|---|---|---|---|
| $\sin\theta$ | | | | | | |
| $\cos\theta$ | | | | | | |
| $\tan\theta$ | | | | | | |

See frame **54** for the correct answers.

**54**

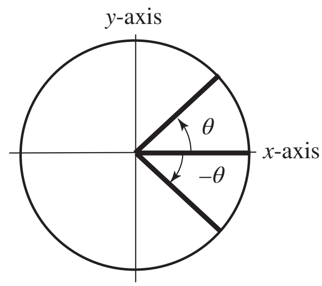
Here are the answers to the questions in frame **53**.

| | Figure *A* + | Figure *A* − | Figure *B* + | Figure *B* − | Figure *C* + | Figure *C* − |
|---|---|---|---|---|---|---|
| $\sin\theta$ | ✓ | | ✓ | | | ✓ |
| $\cos\theta$ | ✓ | | | ✓ | | ✓ |
| $\tan\theta$ | ✓ | | | ✓ | ✓ | |

Go to **55**.

**55**

In the figure both $\theta$ and $-\theta$ are shown. The trigonometric functions for these two angles are simply related.




Can you do these problems? Encircle the correct sign.

$$\sin(-\theta) = [+ \mid -]\sin\theta$$

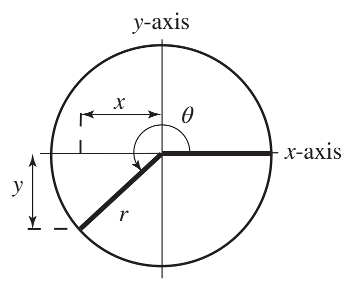
$$\cos(-\theta) = [+ \mid -]\cos\theta$$

$$\tan(-\theta) = [+ \mid -]\tan\theta$$

Go to **56**.

**56**

There are many relationships among the trigonometric functions.




*(continued)*

For instance, using $r^2 = x^2 + y^2$, we have

$$\sin^2\theta = \frac{y^2}{r^2} = \frac{r^2 - x^2}{r^2} = 1 - \left(\frac{x}{r}\right)^2 = 1 - \cos^2\theta.$$

Try these:

**1.** $\sin^2\theta + \cos^2 = \{\sec^2\theta \mid 1 \mid \tan^2\theta \mid \cot^2\theta\}$

**2.** $1 + \tan^2\theta = \{1 \mid \tan^2\theta \mid \cot^2\theta \mid \sec^2\theta\}$

**3.** $\sin^2\theta - \cos^2\theta = \{1 - 2\cos^2\theta \mid 1 - 2\sin^2\theta \mid \cot^2\theta \mid 1\}$

If any mistakes, go to **57**.
Otherwise, go to **58**.

**57**

---

Here are the solutions to the problems in frame **56**.

**1.** $\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1.$

This is an important identity, which is worth remembering.
The other solutions are

**2.** $1 + \tan^2\theta = 1 + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta} = \sec^2\theta.$

**3.** $\sin^2\theta - \cos^2 = (1 - \cos^2\theta) - \cos^2\theta = 1 - 2\cos^2\theta.$

Go to **58**.

**58**

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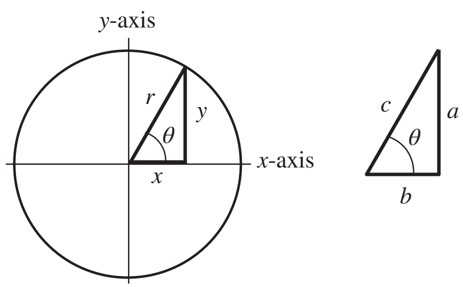


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Answers: Frame 52: $-4/5, -3/5, 4/3$
Frame 55: $-, +, -$

The trigonometric functions are particularly useful when applied to right triangles (triangles with one 90°, or right angle). In this case $\theta$ is always acute (less than 90°, or $\pi/2$). You can then write the trigonometric functions in terms of the sides $a$ and $b$ of the right triangle shown, and its hypotenuse $c$. Fill in the blanks.

$\sin\theta =$ ________ $\csc\theta =$ ________

$\cos\theta =$ ________ $\sec\theta =$ ________

$\tan\theta =$ ________ $\cot\theta =$ ________

Check your answer in **59**.

**59**

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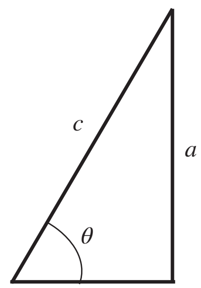


The answers are:

$$\sin\theta = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}, \csc\theta = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{opposite side}},$$

$$\cos\theta = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}, \sec\theta = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{adjacent side}},$$

$$\tan\theta = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}, \cot\theta = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}.$$

These results follow from the definitions in frame **51**, providing we let $a$, $b$, and $c$ correspond to $y$, $x$, and $r$, respectively. (Remember that here $\theta$ is less than 90°.) If you are not familiar with the terms *opposite side*, *adjacent side*, and *hypotenuse*, they should be evident from the figure.

Go to **60**.

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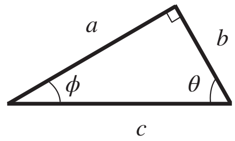
**60**




The following problems refer to the figure shown. ($\phi$ is the Greek letter "phi.")

$$\sin\theta = [b/c \mid a/c \mid c/a \mid c/b \mid b/a \mid a/b]$$

$$\tan\phi = [b/c \mid a/c \mid c/a \mid c/b \mid b/a \mid a/b]$$

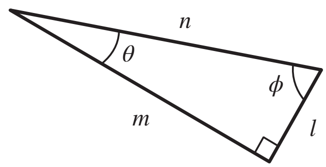
If all right, go to **62.**
Otherwise, go to **61.**

**61**

You may have become confused because the triangle was drawn in a new position. Review the definitions in **51**, and then do the following problems:

$$\cos\ \theta = [l/n \mid n/l \mid m/n \mid m/l \mid n/m \mid l/m]$$

$$\cot\ \phi = [l/n \mid n/l \mid m/n \mid m/l \mid n/m \mid l/m]$$

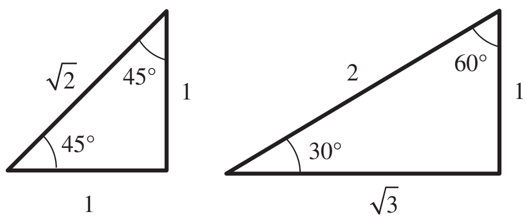



If you missed either of these, you will have to put in more work learning and memorizing the definitions.

Meanwhile go to **62.**

**62**

It is helpful to be familiar with the trigonometric functions of 30°,45°, and 60°. The triangles for these angles are particularly simple.

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Answer: Frame 56: 1, $\sec^2\theta$, $1 - 2\cos^2\theta$

Try these problems:

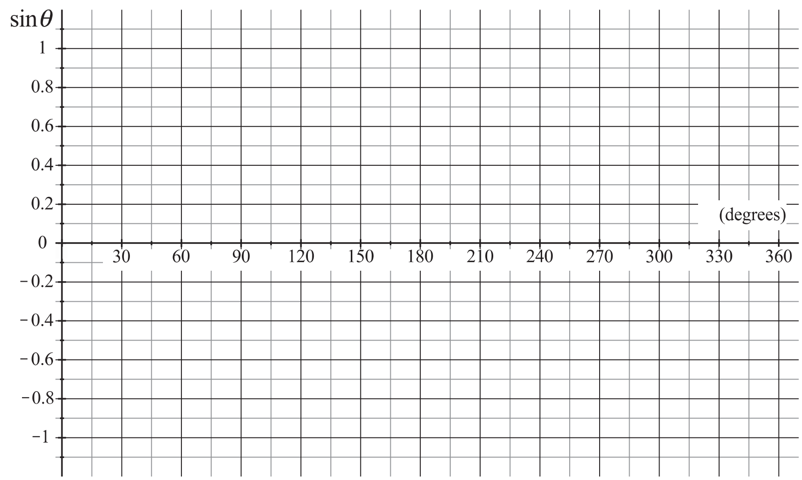
$$\cos 45^\circ = [1/2 \mid 1/\sqrt{2} \mid 2\sqrt{2} \mid 2]$$
$$\sin 30^\circ = [3 \mid \sqrt{3/2} \mid 2/3 \mid 1/2]$$
$$\sin 45^\circ = [1/2 \mid 1/\sqrt{2} \mid \sqrt{2/2} \mid 2]$$
$$\tan 30^\circ = [1 \mid \sqrt{3} \mid 1/\sqrt{3} \mid 2]$$

Make sure you understand these problems. Then go to **63.**

**63**

---

Many calculators provide values of trigonometric functions. With such a calculator, it is simple to plot enough points to make a good graph of the function. If you have a calculator, plot $\sin\theta$ for values between 0° and 360° on the coordinate axes below, and then compare your result with frame **64**. If you do not have a suitable calculator, go directly to **64** and check that $\sin\theta$ has the correct values for the angles you know.

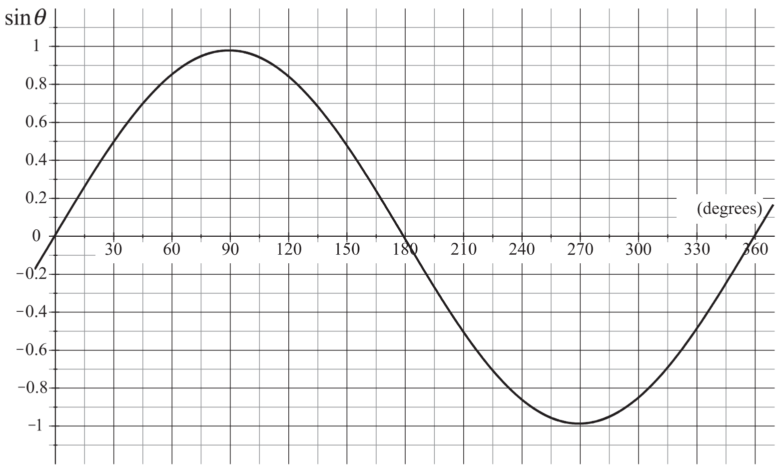



Go to **64**.

---

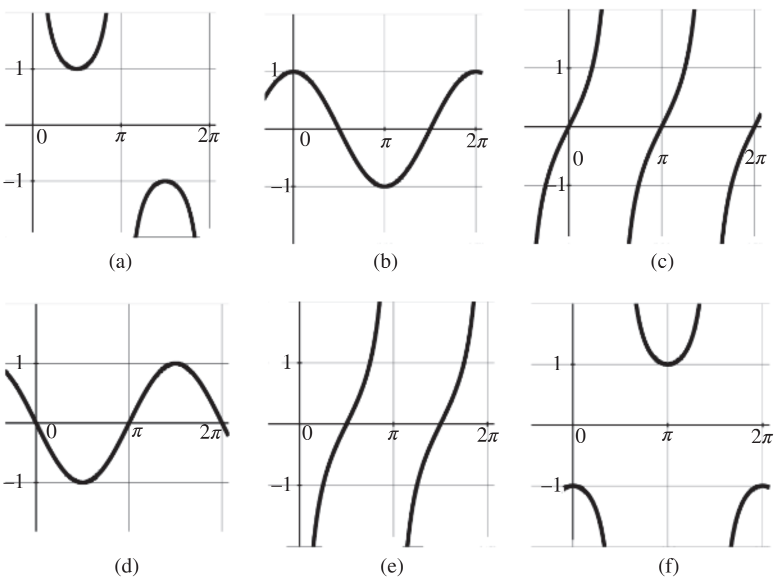
**64**

Here is the graph of the sine function.




Go to **65**.

**65**

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Answers: Frame 60: $a/c$, $b/a$
Frame 61: $m/n$, $l/m$
Frame 62: $1/\sqrt{2}$, $1/2$, $1/\sqrt{2}$, $1/\sqrt{3}$

Try to decide which graph represents each function.

$\cos\theta$: [$a$ | $b$ | $c$ | $d$ | $e$ | $f$ | none of these]

$\tan\theta$: [$a$ | $b$ | $c$ | $d$ | $e$ | $f$ | none of these]

$\sin(-\theta)$: [$a$ | $b$ | $c$ | $d$ | $e$ | $f$ | none of these]

$\tan(-\theta)$: [$a$ | $b$ | $c$ | $d$ | $e$ | $f$ | none of these]

If you got these all right, go to **67**.
Otherwise go to **66**.

**66**

---

Knowing the values of the trigonometric functions at a few important points will help you identify them. Try these ($\infty$ is the symbol for infinity, here meaning that the function is undefined):

$\sin 0^\circ$ = [0 | 1 | $-1$ | $-\infty$ | $+\infty$]

$\cos 0^\circ$ = [0 | 1 | $-1$ | $-\infty$ | $+\infty$]

$\cos 30^\circ$ = [1 | 1/2 | $\sqrt{3}$ | $\sqrt{3}/2$]

$\tan 45^\circ$ = [0 | 1 | $-1$ | $-\infty$ | $+\infty$]

$\cos 60^\circ$ = [1 | 1/2 | $\sqrt{3}$ | $\sqrt{3}/2$]

$\sin 90^\circ$ = [0 | 1 | $-1$ | $-\infty$ | $+\infty$]

$\cos 90^\circ$ = [0 | 1 | $-1$ | $-\infty$ | $+\infty$]

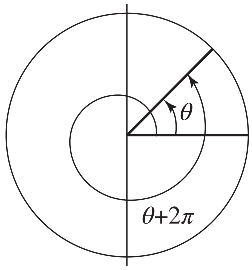
Go to **67**.

**67**

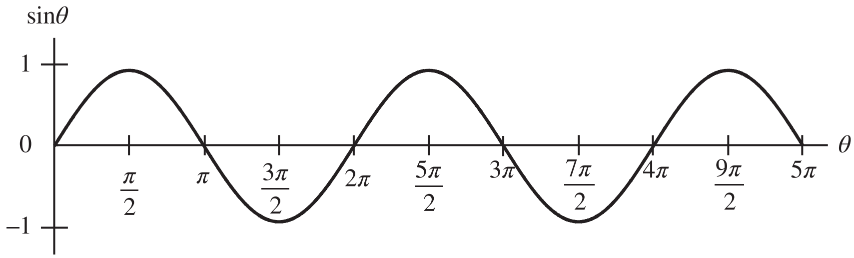
---

Because the angle $\theta + 2\pi n$, where $n$ is any integer, is equivalent to $\theta$ as far as the trigonometric functions are concerned (i.e. for any trig function $f$, $f(\theta + 2\pi n) = f(\theta)$), we can add $2\pi n$ to any angle without changing the value of the trigonometric functions. Thus, the sine and cosine (as well as its reciprocals, csc and sec) functions repeat their values whenever $\theta$ increases by $2\pi n$ where $n$ is an integer; we say that these functions are *periodic* in $\theta$ with a *fundamental period* of $2\pi$, or $360^\circ$. (The fundamental period of the tangent and the cotangent is $\pi$.)

*(continued)*



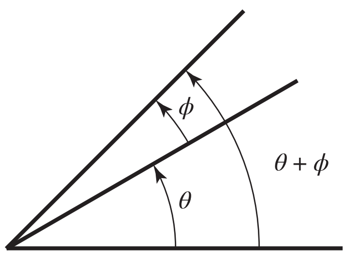

Using this property, you can extend the graph of $\sin\theta$ in frame **64** to the following. (For variety, the angle here is in radians.)




Go to **68**.

**68**

It is helpful to know the sine and cosine of the sum and the difference of two angles.

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Answers: Frame 65: *b*, *c*, *d*, none of these;
Frame 66: $\sin 0° = 0$, $\cos 0° = 1$, $\cos 30° = \sqrt{3}/2$, $\tan 45° = 1$,
$\cos 60° = 1/2$, $\sin 90° = 1$, $\cos 90° = 0$.

Do you happen to remember the formulas from previous studies of trigonometry? If not, go to **69**. If you do, try the quiz below.

$$\sin(\theta + \phi) = \underline{\hspace{4cm}}.$$
$$\cos(\theta + \phi) = \underline{\hspace{4cm}}.$$

Go to **69** to see the correct answer.

**69**

---

Here are the formulas. They are derived in Appendix A1.

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

These formulas hold for both positive and negative values of the angles. (Note that $\tan(\theta + \phi)$ and $\cot(\theta + \phi)$ can be obtained from these formulas and the relation $\tan\theta = \sin\theta/\cos\theta$.

By using what you have already learned, circle the correct sign in each of the following:

(a) $\sin(\theta - \phi) = \{+ \mid -\} \sin\theta\cos\phi \ \{+ \mid -\} \cos\theta\sin\phi$

(b) $\cos(\theta - \phi) = \{+ \mid -\} \cos\theta\cos\phi \ \{+ \mid -\} \sin\theta\sin\phi$

If right, go to **71**.
If wrong, go to **70**.

**70**

---

If you made a mistake in problem **69,** recall from frame **55** that

$$\sin(-\phi) = -\sin\phi,$$
$$\cos(-\phi) = +\cos\phi.$$

Then

$$\begin{aligned}\sin(\theta - \phi) &= \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)\\ &= \sin\theta\cos\phi - \cos\theta\sin\phi,\\ \cos(\theta - \phi) &= \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)\\ &= \cos\theta\cos\phi + \sin\theta\sin\phi.\end{aligned}$$

Go to **71**.

---

**71**

By using the expressions for $\sin(\theta + \phi)$ and $\cos(\theta + \phi)$, one can obtain the formulas for $\sin(2\theta)$ and $\cos(2\theta)$. Simply let $\theta = \phi$. Fill in the blanks.

$$\sin 2\theta = \_\_\_\_\_\_\_\_\_\_\_\_.$$
$$\cos 2\theta = \_\_\_\_\_\_\_\_\_\_\_\_.$$

See **72** for the correct answers.

**72**

Here are the answers:

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta\cos\theta, \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= 1 - 2\sin^2\theta \\ &= 2\cos^2\theta - 1. \end{aligned}$$

(Note, by convention, $(\sin\theta)^2$ is usually written $\sin^2\theta$, and $(\cos\theta)^2$is usually written $\cos^2\theta$.)

Go to **73**

**73**

It is often useful to use the *inverse trigonometric* function. This is the value of the angle for which the trigonometric function has a specified value. The inverse sine of $x$ is denoted by $\sin^{-1}x$. (Warning: This notation is standard, but it can be confusing. $\sin^{-1}x$ always represents the inverse sine of $x$, not $1/\sin x$. The latter would be written $(\sin x)^{-1}$. An older notation for $\sin^{-1}x$ is arcsin $x$.)

For example, because the sine of 30°is 1/2, $\sin^{-1}(1/2) = 30°$. Note, however, that the sine of 150°is also 1/2. Furthermore, the trigonometric functions are periodic: there is an endless sequence of angles (all differing by 360°) having the same value for the sine, cosine, etc.

---

Answer: Frame 69 (a): +, −; (b): +, +

Because the definition of function (frame **6**) specifies the assignment of one and only one value of $f(x)$ for each value of $x$, the domain of the inverse trigonometric function must be suitably restricted.

The inverse functions are defined by

$$y = \sin^{-1}x \quad \text{Domain}: -1 \leq x \leq +1 \quad \text{Range}: -\frac{\pi}{2} \leq y \leq +\frac{\pi}{2}$$

$$y = \cos^{-1}x \quad \text{Domain}: -1 \leq x \leq +1 \quad \text{Range}: 0 \leq y \leq \pi$$

$$y = \tan^{-1}x \quad \text{Domain}: -\infty < x < +\infty \quad \text{Range}: -\frac{\pi}{2} < y < +\frac{\pi}{2}$$

Go to **74**.

**74**

---

Try these problems:

(a) $\sin^{-1}(1/\sqrt{2}) = [\pi/6 \mid \pi/4 \mid \pi/3 \mid \pi/2]$

(b) $\tan^{-1}(1) = [\pi/6 \mid \pi/4 \mid \pi/3 \mid \pi]$

(c) $\cos^{-1}(1/2) = [\pi/6 \mid \pi/4 \mid \pi/3 \mid \pi]$

If you have a calculator with inverse trigonometric functions, try the following:

(d) $\sin^{-1}(0.8) = [46.9 \mid 28.2 \mid 53.1 \mid 67.2]$ degrees

(e) $\tan^{-1}(12) = [0.82 \mid 1.49 \mid 1.62 \mid 1.83]$ radians

(f) $\cos^{-1}(0.05) = [4.3 \mid 12.6 \mid 77.2 \mid 87.1]$ degrees

Check your answers, and then go on to the next section, which is the last one in our reviews.

Go to **75**.

---

## 1.7 Exponentials and Logarithms

**75**

Are you already familiar with exponentials? If not, go to **76**. If you are, try this short quiz.

$$a^5 = [5^a \mid 5 \log a \mid a \log 5 \mid \text{none of these}]$$
$$a^{b+c} = [a^b a^c \mid a^b + a^c \mid ca^b \mid (b+c)\log a]$$
$$a^f/a^g = [(f-g)\log a \mid a^{f/g} \mid a^{f-g} \mid \text{none of these}]$$
$$a^0 = [0 \mid 1 \mid a \mid \text{none of these}];\ a \neq 0$$
$$(a^b)^c = [a^b a^c \mid a^{b+c} \mid a^{bc} \mid \text{none of these}]$$

If any mistakes, go to **76**.
Otherwise, go to **77**.

**76**

By definition $a^m$, where $m$ is a positive integer, is the product of $m$ factors of $a$. Hence,

$$2^3 = (2)(2)(2) = 8 \text{ and } 10^2 = (10)(10) = 100.$$

Furthermore, by definition $a^{-m} = 1/a^m$. It is easy to see, then, that

$$a^m a^n = a^{m+n},$$
$$\frac{a^m}{a^n} = a^{m-n},$$
$$a^0 = \frac{a^m}{a^m} = 1 \qquad (\text{a} \neq 0,\ m \text{ can be any integer})$$
$$(a^m)^n = a^{mn},$$
$$(ab)^m = a^m b^m.$$

Note that $a^{m+n}$ is evaluated as $a^{(m+n)}$; the expression in the exponential is always evaluated before any other operation is carried out.

If you have not yet tried the quiz in frame **75**, try it now. Otherwise,

Go to **77**.

Answer: Frame 74: (a) $\pi/4$, (b) $\pi/4$, (c) $\pi/3$, (d) 53.1°, (e) 1.49, (f) 87.1°

**77**

Here are a few problems:

$$3^2 = [6 \mid 8 \mid 9 \mid \text{none of these}]$$

$$1^3 = \left[1 \mid 3 \mid \frac{1}{3} \mid \text{none of these}\right]$$

$$2^{-3} = \left[-6 \mid \frac{1}{8} \mid -9 \mid \text{none of these}\right]$$

$$\frac{4^3}{4^5} = [4^8 \mid 4^{-8} \mid 16^{-1} \mid \text{none of these}]$$

If you did these all correctly, go to **79**.
If you made any mistakes, go to **78**.

**78**

Below are the solutions to problem **77**. Refer back to the rules in **76** if you have trouble understanding the solution.

$$3^2 = (3)(3) = 9,$$

$$1^3 = (1)(1)(1) = 1 \qquad (1^m = 1 \text{ for any } m),$$

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8},$$

$$\frac{4^3}{4^5} = 4^{3-5} = 4^{-2} = \frac{1}{16} = 16^{-1}.$$

Now try these:

$$(3^{-3})^3 = [1 \mid 3^{-9} \mid 3^{-27} \mid \text{none of these}]$$

$$\frac{5^2}{3^2} = \left[\left(\frac{5}{3}\right)^2 \middle| \left(\frac{5}{3}\right)^{-1} \middle| 5^{-6} \middle| \text{none of these}\right]$$

$$4^3 = [12 \mid 16 \mid 2^6 \mid \text{none of these}]$$

Check your answers and try to track down any mistakes.

Then go to **79**.

**79**

Here are a few more problems.

$$10^0 = [0 \mid 1 \mid 10]$$

$$10^{-1} = [-1 \mid 1 \mid 0.1]$$

$$0.00003 = \left[\frac{1}{3} \times 10^{-3} \mid 10^{-3} \mid 3 \times 10^{-5}\right]$$

$$0.4 \times 10^{-4} = [4 \times 10^{-5} \mid 4 \times 10^{-3} \mid 2.5 \times 10^{-5}]$$

$$\frac{3 \times 10^{-7}}{6 \times 10^{-3}} = \left[\frac{1}{2} \times 10^{10} \,\middle|\, 5 \times 10^4 \,\middle|\, 0.5 \times 10^{-4}\right]$$

If these were all correct, go to **81**.
If you made any mistakes, go to **80**.

**80**

Here are the solutions to the problems in **79**:

$$10^0 = \frac{10}{10} = 1$$

$$10^{-1} = \frac{1}{10} = 0.1,$$

$$0.00003 = 0.00001 \times 3 = 3 \times 10^{-5},$$

$$0.4 \times 10^{-4} = (4 \times 10^{-1}) \times 10^{-4} = 4 \times 10^{-5},$$

$$\frac{3 \times 10^{-7}}{6 \times 10^{-3}} = \frac{3}{6} \times \frac{10^{-7}}{10^{-3}} = \frac{1}{2} \times 10^{-7+3} = 0.5 \times 10^{-4}.$$

Go to **81**.

**81**

Let's introduce the idea of *fractional exponents*. If $b^n = a$, then $b$ is called the *nth root* of $a$ and is written $b = a^{1/n}$. Hence $16^{1/4}$ = (fourth root of 16) = 2. That is, $2^4 = 16$.

---

Answers: Frame 75: $a^5$ = none of these, $a^{b+c} = a^b a^c$, $a^f / a^g = a^{f-g}$, $a^0 = 1$, $(a^b)^c = a^{bc}$

Frame 77: 9, 1, $1/8$, $16^{-1}$

Frame 78: $3^{-9}$, $(5/3)^2$, $2^6$

If $y = a^{m/n}$, where $m$ and $n$ are integers, then $y = (a^{1/n})^m$. For instance

$$8^{2/3} = (8^{1/3})^2 = 2^2 = 4.$$

Try these:

$$27^{-2/3} = \left[{}^1\!/_{18} \mid {}^1\!/_{18} \mid {}^1\!/_9 \mid -18 \mid \text{none of these}\right]$$
$$16^{3/4} = [12 \mid 8 \mid 6 \mid 64]$$

If right, go to **84**.
If wrong, go to **82**.

**82**

---

The answers are:

$$27^{-2/3} = (27^{1/3})^{-2} = 3^{-2} = {}^1\!/_9,$$
$$16^{3/4} = (16^{1/4})^3 = 2^3 = 8.$$

Do these problems:

$$25^{3/2} = [125 \mid 5 \mid 15 \mid \text{none of these}]$$
$$(0.00001)^{-3/5} = [0.001 \mid 1000 \mid 10^{-15} \mid 10^{-25}]$$

If your answers were correct, go to **84**.
Otherwise, go to **83**.

**83**

---

Here are the solutions to the problems in **82**.

$$25^{3/2} = (25^{1/2})^3 = 5^3 = 125,$$
$$(0.00001)^{-3/5} = (10^{-5})^{-3/5} = 10^{15/5} = 10^3 = 1000.$$

Here are a few more problems. Encircle the correct answers.

$$\left({}^{27}\!/_{64} \times 10^{-6}\right)^{1/3} = \left[{}^3\!/_{400} \mid {}^3\!/_{16} \times 10^{-2} \mid {}^9\!/_{64} \times 10^{-4}\right],$$
$$(49 \times 10^{-4})^{1/4} = \left[\sqrt{7}/10 \,\middle|\, (10 \times 7)^{-2} \,\middle|\, \sqrt{7}/1000\right].$$

Go to **84** after checking your answers.

---

**84**

---

Although our original definition of $a^m$ applied only to integral values of $m$, we have also defined $(a^m)^{1/n} = a^{m/n}$, where both $m$ and $n$ are integers. Thus we have a meaning for $a^p$, where $p$ is either an integer or a fraction (ratio of integers).

As yet we do not know how to evaluate $a^p$ if $p$ is an irrational number, such as $\pi$ or $\sqrt{2}$. However, we can approximate an irrational number as closely as we desire by a fraction. For instance, $\pi$ is approximately 31,416/10,000. This is in the form $m/n$, where $m$ and $n$ are integers, and we know how to evaluate it. Therefore, $y = a^x$, where $x$ is any real number, is a meaningful expression in the sense that we can evaluate it as accurately as we please. (A more rigorous treatment of irrational exponents can be based on the properties of suitably defined logarithms.)

Try the following problem.

$$\frac{a^\pi a^x}{a^3} = \left[a^{\pi x/3} \;\middle|\; a^{\pi+x-3} \;\middle|\; a^{3\pi x} \;\middle|\; a^{(\pi+x)/3}\right]$$

If right, go to **86**.
If wrong, go to **85**.

**85**

---

The rules given in frame **76** apply here as if all exponents were integers. Hence

$$\frac{a^\pi a^x}{a^3} = a^{\pi+x-3}.$$

Here is another problem:

$$(\pi^2)(2^\pi) = \left[1 \;\middle|\; (2\pi)^{2\pi} \;\middle|\; 2\pi^{2+\pi} \;\middle|\; \text{none of these}\right]$$

If right, go to **87**.
If wrong, go to **86**.

---

Answers: Frame 79: 1, 0.1, $3 \times 10^{-5}$, $4 \times 10^{-5}$, $0.5 \times 10^{-4}$

Frame 81: 1/9, 8

Frame 82: 125, 1000

Frame 83: 3/400, $\sqrt{7}/10$

**86**

---

The quantity $(\pi^2)(2^\pi)$ is the product of two different numbers raised to two different exponents. None of our rules apply to this and, in fact, there is no way to simplify this expression.

Now go to **87**.

**87**

---

If you do not clearly remember logarithms, go to **88**.

If you do, try the following test. Let $x$ be any positive number, and let $\log x$ represent the log of $x$ to the *base* 10. Then:

$$10^{\log x} = \underline{\qquad\qquad}.$$

Go to **88** for the correct answer.

**88**

---

The answer to **87** is $x$; in fact we will take the logarithm of $x$ to the *base* 10 to be defined by

$$\boxed{10^{\log x} = x.}$$

That is, the logarithm of a number $x$ is the power to which 10 must be raised to produce the number $x$ itself. This definition only applies for $x > 0$. Here are two examples:

$$100 = 10^2, \quad \text{therefore} \quad \log 100 = 2;$$
$$0.001 = 10^{-3}, \quad \text{therefore} \quad \log 0.001 = -3.$$

Now try these problems:

$$\log 1{,}000{,}000 = [1{,}000{,}000 \mid 6 \mid 60 \mid 600]$$

$$\log 1 = [0 \mid 1 \mid 10 \mid 100]$$

If right, go to **90**.
If wrong, go to **89**.

**89**

---

Here are the answers:

$$\log 1{,}000{,}000 = \log(10^6) = 6 \quad (\text{check, } 10^6 = 1{,}000{,}000),$$

$$\log 1 = \log(10^0) = 0 \quad (\text{check, } 10^0 = 1).$$

*(continued)*

Try the following:

$$\log(10^4/10^{-3}) = [10^7 \mid 1 \mid 10 \mid 7 \mid 70]$$
$$\log(10^n) = [10n \mid n \mid 10^n \mid 10/n]$$
$$\log(10^{-n}) = [-10n \mid -n \mid -10^n \mid -10/n]$$

If you had trouble with these, carefully review the material in this section.

Then go to **90**.

**90**

---

Here are three important relations for manipulating logarithms, $a$ and $b$ are any positive numbers:

$$\log(ab) = \log a + \log b,$$
$$\log(a/b) = \log a - \log b,$$
$$\log(a^n) = n \log a.$$

If you are familiar with these rules, go to **92**. If you want to see how they are derived,

Go to **91**.

**91**

---

We can derive the required rules as follows. From the definition of $\log x$, $a = 10^{\log a}$ and $b = 10^{\log b}$. Consequently, from the properties of exponentials,

$$ab = (10^{\log a})(10^{\log b}) = 10^{\log a + \log b}.$$

Taking the log of both sides, and again using $\log 10^x = x$ gives

$$\log\,(ab) = \log 10^{\log a + \log b} = \log a + \log b.$$

Similarly,

$$a/b = 10^{\log a} 10^{-\log b} = 10^{\log a - \log b}.$$
$$\log(a/b) = \log a - \log b$$

---

Answers: Frame 84: $a^{\pi + x - 3}$
Frame 85: None of these
Frame 88: 6, 0

Likewise,

$$a^n = (10^{\log a})^n = 10^{n \log a},$$

so that

$$\log(a^n) = n \log a.$$

Go to **92**.

**92**

---

Try these problems:

$$\text{If } \log n = -3, \quad n = \left[{}^{1}\!/_{3} \mid {}^{1}\!/_{300} \mid {}^{1}\!/_{1000}\right]$$

$$10^{\log 100} = [10^{10} \mid 20 \mid 100 \mid \text{none of these}]$$

$$\frac{\log 1000}{\log 100} = \left[{}^{3}\!/_{2} \mid 1 \mid -1 \mid 10\right]$$

If right, go to **94**.
If wrong, go to **93**.

**93**

---

The answers are:

$$10^{\log n} = n, \text{so if } \log n = -3, n = 10^{-3} = {}^{1}\!/_{1000}.$$

For the same reason,

$$10^{\log 100} = 100.$$

$$\frac{\log 1000}{\log 100} = \frac{\log 10^3}{\log 10^2} = \frac{3}{2}.$$

Try these problems:

$${}^{1}\!/_{2} \log 16 = [2 \mid 4 \mid 8 \mid \log 2 \mid \log 4]$$

$$\log(\log 10) = [10 \mid 1 \mid 0 \mid -1 \mid -10]$$

Go to **94**.

**94**

---

In this section we have discussed only logarithms to the base 10. However, any positive number except 1 can be used as a base. Bases other than 10 are usually indicated by a subscript.

*(continued)*

For instance, the logarithm of 8 to the base 2 is written $\log_2 8$, and is equal to 3 because $2^3 = 8$. If our base is denoted by $r$, then the defining equation for $\log_r x$ is

$$\boxed{r^{\log_r x} = x.}$$

All the relations explained in frame **91** are true for logarithms to any base (provided, of course, that the same base is used for all the logarithms in each equation).

There is a special base number

$$e = 2.71828\ldots,$$

called *Euler's number*, that is used to define *natural logarithms* that are usually designated by the symbol $\ln x = \log_e x$. Euler's number is an irrational number, and the three dots, known as an *ellipsis*, indicate the indefinite continuation of that number. The defining equation for natural logarithms is then

$$e^{\ln x} = x.$$

From the defining equation, set $x = e$, then $e^{\ln e} = e$, thus

$$\ln e = 1.$$

The significance of this special property will be described in Chapter 2.

Go to **95**.

**95**

---

From the definition of logarithm in the last frame we can obtain the rule for changing logarithms from one base to another, for instance from base 10 to the base $e$. (Many calculators give both $\log x$, i.e. $\log_{10} x$, and $\ln x$.) Take $\log_{10}$ of both sides of the defining equation $e^{\ln x} = x$,

$$\log(e^{\ln x}) = \log x.$$

Because $\log x^n = n \log x$ (frame **91**), this gives $\ln x \log e = \log x$ or

$$\ln x = \frac{\log x}{\log e}.$$

---

Answers: Frame 89: 7, $n$, $-n$
Frame 92: $1/1000$, 100, $3/2$
Frame 93: log 4, 0

The numerical value of $\log e$ is $1/(2.303\ \ldots)$ so

$$\ln x = (2.303)\log x.$$

If you have a calculator which evaluates both $\ln x$ and $\log x$, check this relation for a few values of $x$.

The $\ln x$ satisfies the same properties as $\log x$ as listed in frame **90**,

$$\ln(ab) = \ln a + \ln b,$$
$$\ln(a/b) = \ln a - \ln b,$$
$$\ln(a^n) = n\ln a.$$

Go to **96**.

**96**

Before concluding Chapter 1 it is worth commenting on how to find the values of the functions in this chapter. In former times one had to consult bulky books of tables. Today the values are essentially instantly generated on simple and inexpensive calculators. The technique for doing this will be explained in Chapter 4 in the section on Taylor's formula. This technique requires differential calculus, which is introduced in the next chapter.

On page 277, following the appendices and the solutions to the problems, there is a collection of review problems with answers, an index to the symbols, and an index to the text.

Before going on, here is a summary of Chapter 1 to help you review what you have learned. Take a look if you feel that this would be helpful.

As soon as you are ready, go to Chapter 2.

# Summary of Chapter 1

## 1.2 Functions (frames 3–13)

A *set* is a collection of objects—not necessarily material objects—described in such a way that we have no doubt as to whether a particular object does or does not belong to the set. A set may be described by listing its elements or by a rule.

A *function* is a rule that assigns to each element in a set $A$ one and only one element in a set $B$. The rule can be specified by a mathematical formula such as $y = x^2$, or by tables of associated numbers. If $x$ is one of the elements of set $A$, then the element in set $B$ that the function $f$ associates with $x$ is denoted by the symbol $f(x)$, which is usually read as "$f$ of $x$."

The set $A$ is called the *domain* of the function. The set of all possible values of $f(x)$ as $x$ varies over the domain is called the *range* of the function. The range of $f$ need not be all of $B$.

When a function is defined by a formula such as $f(x) = ax^3 + b$, then $x$ is often called the *independent variable* and $f(x)$ is called the *dependent variable*. Often, however, a single letter is used to represent the single variable as in $y = f(x)$.

Here $x$ is the independent variable and $y$ is the dependent variable. In mathematics the symbol $x$ frequently represents an independent variable, $f$ often represents the function, and $y = f(x)$ usually denotes the dependent variable. However, any other symbols may be used for the function, the independent variable, and the dependent variable, for example, $x = H(r)$.

## 1.3 Graphs (frames 14–22)

A convenient way to represent a function is to plot a graph as described in frames **15**–**18**. The mutually perpendicular coordinate axes intersect at the origin. The axis that runs horizontally is called the horizontal axis, or $x$-axis. The axis that runs vertically is called the vertical axis, or $y$-axis. Sometimes the value of the $x$-coordinate of a point is called the *abscissa*, and the value of the $y$-coordinate is called the *ordinate*. In the designation of a typical point by the notation $(a, b)$, we will always designate the $x$-coordinate first and the $y$-coordinate second.

The constant function assigns a single fixed number $c$ to each value of the independent variable $x$. The absolute value function $|x|$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

## 1.4 Linear and Quadratic Functions (frames 23–39)

An equation of the form $y = mx + b$ where $m$ and $b$ are constants is called *linear* because its graph is a straight line. The slope of a linear function is defined by

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}.$$

From the definition it is easy to see (frame **29**) that the slope of the above linear equation is $m$.

An equation of the form $y = ax^2 + bx + c$, where $a$, $b$, and $c$, are constants (and $a \neq 0$), is called a *quadratic equation*. Its graph is called a *parabola*. The values of $x$ at $y = 0$ satisfy $ax^2 + bx + c = 0$ and are called the *roots* of the equation. Not all quadratic equations have real roots. The equation $ax^2 + bx + c = 0$ has two roots given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

## 1.5–1.6 Angles and Their Measurements; Trigonometry (frames 40–74)

Angles are measured in either *degrees* or *radians*. A circle is divided into 360 equal *degrees*. The number of *radians* in an angle is equal to the length of the subtending arc divided by the length of the radius (frame **42**). The relation between degrees and radians is

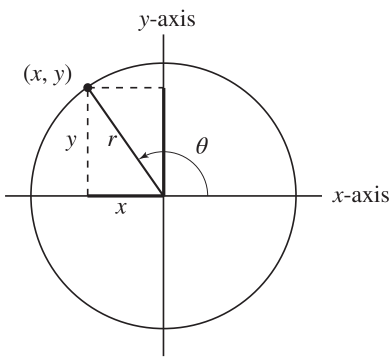
$$1 \text{ rad} = \frac{360^\circ}{2\pi}.$$

Rotations can be clockwise or counterclockwise. An angle formed by rotating in a counterclockwise direction is taken to be positive.

The trigonometric functions are defined in conjunction with the figure.

The definitions are

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r},$$
$$\tan\theta = \frac{y}{x}, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y},$$
$$\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}, \quad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}.$$




Although $r = \sqrt{x^2 + y^2}$ is always positive, $x$ and $y$ can be either positive or negative and the above quantities may be positive or negative depending on the value of $\theta$. From the Pythagorean theorem it is easy to see (frame **56**) that

$$\sin^2\theta + \cos^2\theta = 1.$$

The sines and cosines for the sum of two angles are given by:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi,$$
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

The inverse trigonometric function designates the angle for which the trigonometric function has the specified value. Thus the inverse trigonometric function to $x = \sin\theta$ is $\theta = \sin^{-1}x$ and similar definitions apply to $\cos^{-1}x$, $\tan^{-1}x$, etc. [Warning: This notation is standard, but it can be confusing: $\sin^{-1}x \neq (\sin x)^{-1}$. An older notation for $\sin^{-1}x$ is arc $\sin x$.]

## 1.7 Exponentials and Logarithms (frames 75–95)

If $a$ is multiplied by itself as $aaa \cdots$ with $m$ factors, the product is written as $a^m$. Furthermore, by definition, $a^{-m} = 1/a^m$. From this it follows that

$$a^m a^n = a^{m+n},$$
$$\frac{a^m}{a^n} = a^{m-n},$$
$$a^0 = \frac{a^m}{a^m} = 1,$$
$$(a^m)^n = a^{mn},$$
$$(ab)^m = a^m b^m.$$

If $b^n = a$, $b$ is called the *nth root* of $a$ and is written as $b = a^{1/n}$. If $m$ and $n$ are integers,

$$a^{m/n} = (a^{1/n})^m.$$

The meaning of exponents can be extended to irrational numbers (frame **84**) and the above relations also apply with irrational exponents, so $(a^x)^b = a^{xb}$, etc.

The definition of $\log x$ (the logarithm of $x$ to the base 10) is

$$x = 10^{\log x}.$$

The following important relations can easily be seen to apply to logarithms (frame **91**):

$$\log(ab) = \log a + \log b,$$
$$\log(a/b) = \log a - \log b,$$
$$\log(a^n) = n \log a.$$

The logarithm of $x$ to another base $r$ is written as $\log_r x$ and is defined by

$$x = r^{\log_r x}.$$

The above three relations for logarithms of $a$ and $b$ are correct for logarithms to any base provided the same base is used for all the logarithms in each equation.

A particular important base is $r = e = 2.71828\ldots$ as defined in frame **109**. Logarithms to the base $e$ are so important in calculus that they are given a different name; they are called *natural logarithms* and written as ln. With this notation the natural logarithm of $x$ is defined by

$$e^{\ln x} = x.$$

If we take the logarithm to base 10 of both sides of the equation,

$$\log e^{\ln x} = \log x,$$
$$\ln x \log e = \log x,$$
$$\ln x = \frac{\log x}{\log e}.$$

Because the numerical value of $1/\log e = 2.303\ldots$,

$$\ln x = (2.303) \log x.$$

The special value of $e$ and the importance that $\ln e = 1$ will be discussed in Chapter 2.

Continue to Chapter **2**.