

Chapter 1

FOUNDATION FOR CALCULUS: FUNCTIONS AND LIMITS

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1.1 FUNCTIONS AND CHANGE

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. In 2015, Boston, Massachusetts, had the highest annual snowfall, 110.6 inches, since recording started in 1872. Table 1.1 shows one 14-day period in which the city broke another record with a total of 64.4 inches.¹

Table 1.1 Daily snowfall in inches for Boston, January 27 to February 9, 2015

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Snowfall	22.1	0.2	0	0.7	1.3	0	16.2	0	0	0.8	0	0.9	7.4	14.8

You may not have thought of something so unpredictable as daily snowfall as being a function, but it *is* a function of day, because each day gives rise to one snowfall total. There is no formula for the daily snowfall (otherwise we would not need a weather bureau), but nevertheless the daily snowfall in Boston does satisfy the definition of a function: Each day, t , has a unique snowfall, S , associated with it.

We define a function as follows:

A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the snowfall example, the domain is the set of days $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ and the range is the set of daily snowfalls $\{0, 0.2, 0.7, 0.8, 0.9, 1.3, 7.4, 14.8, 16.2, 22.1\}$. We call the function f and write $S = f(t)$. Notice that a function may have identical outputs for different inputs (Days 8 and 9, for example).

Some quantities, such as a day or date, are *discrete*, meaning they take only certain isolated values (days must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \leq t \leq b$ is called a *closed interval* and written $[a, b]$.

The set of numbers t such that $a < t < b$ is called an *open interval* and written (a, b) .

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily snowfall in Boston can be represented by the graph in Figure 1.1, as well as by Table 1.1.

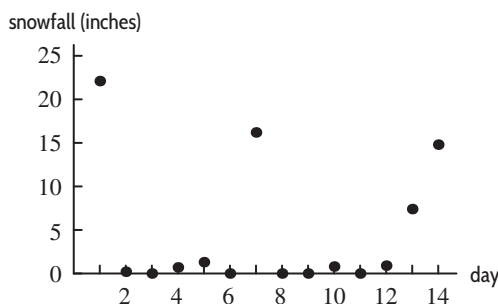


Figure 1.1: Boston snowfall, starting January 27, 2015

As another example of a function, consider the snowy tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine

¹w2.weather.gov/climate/xmacis.php?wfo=box, accessed June 2015.

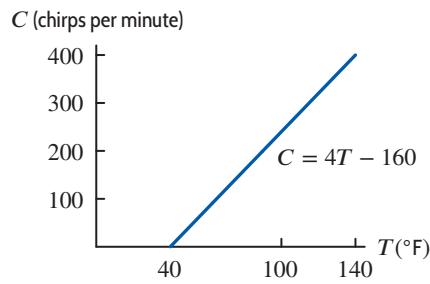


Figure 1.2: Cricket chirp rate versus temperature

the chirp rate. Even more surprisingly, the chirp rate, C , in chirps per minute, increases steadily with the temperature, T , in degrees Fahrenheit, and can be computed by the formula

$$C = 4T - 160$$

to a fair level of accuracy. We write $C = f(T)$ to express the fact that we think of C as a function of T and that we have named this function f . The graph of this function is in Figure 1.2.

Notice that the graph of $C = f(T)$ in Figure 1.2 is a solid line. This is because $C = f(T)$ is a *continuous function*. Roughly speaking, a continuous function is one whose graph has no breaks, jumps, or holes. This means that the independent variable must be continuous. (We give a more precise definition of continuity of a function in Section 1.7.)

Examples of Domain and Range

If the domain of a function is not specified, we usually take it to be the largest possible set of real numbers. For example, we usually think of the domain of the function $f(x) = x^2$ as all real numbers. However, the domain of the function $g(x) = 1/x$ is all real numbers except zero, since we cannot divide by zero.

Sometimes we restrict the domain to be smaller than the largest possible set of real numbers. For example, if the function $f(x) = x^2$ is used to represent the area of a square of side x , we restrict the domain to nonnegative values of x .

Example 1

The function $C = f(T)$ gives chirp rate as a function of temperature. We restrict this function to temperatures for which the predicted chirp rate is positive, and up to the highest temperature ever recorded at a weather station, 134°F .² What is the domain of this function f ?

Solution

If we consider the equation

$$C = 4T - 160$$

simply as a mathematical relationship between two variables C and T , any T value is possible. However, if we think of it as a relationship between cricket chirps and temperature, then C cannot be less than 0. Since $C = 0$ leads to $0 = 4T - 160$, and so $T = 40^\circ\text{F}$, we see that T cannot be less than 40°F . (See Figure 1.2.) In addition, we are told that the function is not defined for temperatures above 134° . Thus, for the function $C = f(T)$ we have

$$\begin{aligned} \text{Domain} &= \text{All } T \text{ values between } 40^\circ\text{F and } 134^\circ\text{F} \\ &= \text{All } T \text{ values with } 40 \leq T \leq 134 \\ &= [40, 134]. \end{aligned}$$

Example 2

Find the range of the function f , given the domain from Example 1. In other words, find all possible values of the chirp rate, C , in the equation $C = f(T)$.

Solution

Again, if we consider $C = 4T - 160$ simply as a mathematical relationship, its range is all real C values. However, when thinking of the meaning of $C = f(T)$ for crickets, we see that the function predicts cricket chirps per minute between 0 (at $T = 40^\circ\text{F}$) and 376 (at $T = 134^\circ\text{F}$). Hence,

$$\begin{aligned} \text{Range} &= \text{All } C \text{ values from 0 to 376} \\ &= \text{All } C \text{ values with } 0 \leq C \leq 376 \\ &= [0, 376]. \end{aligned}$$

²www.guinnessworldrecords.com, accessed January 2017.

In using the temperature to predict the chirp rate, we thought of the temperature as the *independent variable* and the chirp rate as the *dependent variable*. However, we could do this backward, and calculate the temperature from the chirp rate. From this point of view, the temperature is dependent on the chirp rate. Thus, which variable is dependent and which is independent may depend on your viewpoint.

Linear Functions

The chirp-rate function, $C = f(T)$, is an example of a *linear function*. A function is linear if its slope, or rate of change, is the same at every point. The rate of change of a function that is not linear may vary from point to point.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 20 cm every four years. Table 1.2 shows that the height started at 330 cm in 1900, and increased by the equivalent of 5 cm a year. So the height was a linear function of time from 1900 to 1912. If y is the winning height in centimeters and t is the number of years since 1900, then y is predicted approximately by

$$y = f(t) = 330 + 5t.$$

Since $y = f(t)$ increases with t , we say that f is an *increasing function*. The coefficient 5 tells us the rate, in centimeters per year, at which the height increases.

Table 1.2 Men's Olympic pole vault winning height (approximate)

Year	1900	1904	1908	1912
Height (centimeters)	330	350	370	390

This rate of increase is the *slope* of the line in Figure 1.3. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{370 - 350}{8 - 4} = \frac{20}{4} = 5 \text{ centimeters/year.}$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

What about the constant 330? This represents the initial height in 1900, when $t = 0$. Geometrically, 330 is the *intercept* on the vertical axis.

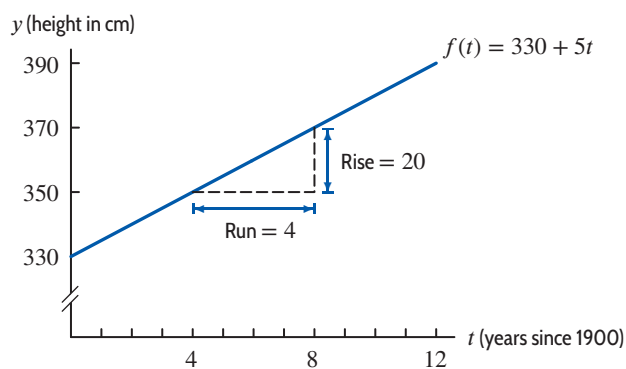


Figure 1.3: Olympic pole vault records

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it does not give a good prediction. The formula $y = 330 + 5t$ predicts that the height in the 2016 Olympics would be 910 centimeters, which is considerably higher than the actual value of 603 centimeters. There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.2 is discrete, because it is given only at specific points (every four years). However, we have treated the variable t as though it were continuous, because the function $y = 330 + 5t$ makes

sense for all values of t . The graph in Figure 1.3 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

As the pole vault heights have increased over the years, the time to run the mile has decreased. If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.39t.$$

The 260 tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.39 , tells us that the world record decreased by about 0.39 seconds per year. We say that g is a *decreasing function*.

Difference Quotients and Delta Notation

We use the symbol Δ (the Greek letter capital delta) to mean “change in,” so Δx means change in x and Δy means change in y .

The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2 , using the formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.4.) Since $m = \Delta y/\Delta x$, the units of m are y -units over x -units.

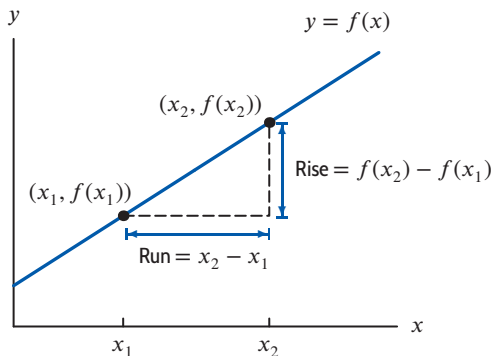


Figure 1.4: Difference quotient = $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Families of Linear Functions

A **linear function** has the form

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x .
- b is the **vertical intercept**, or value of y when x is zero.

Notice that if the slope, m , is zero, we have $y = b$, a horizontal line.

To recognize that a table of x and y values comes from a linear function, $y = b + mx$, look for differences in y -values that are constant for equally spaced x -values.

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, give a *family of functions*. All the functions in a family share certain properties—in this case, all the

graphs are straight lines. The constants m and b are called *parameters*; their meaning is shown in Figures 1.5 and 1.6. Notice that the greater the magnitude of m , the steeper the line.

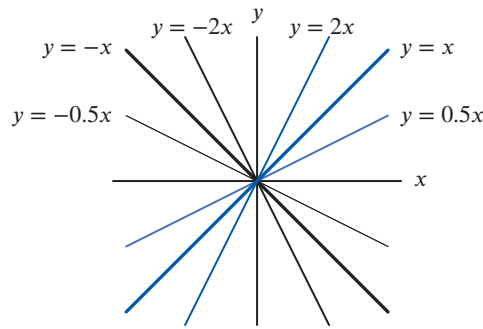


Figure 1.5: The family $y = mx$ (with $b = 0$)

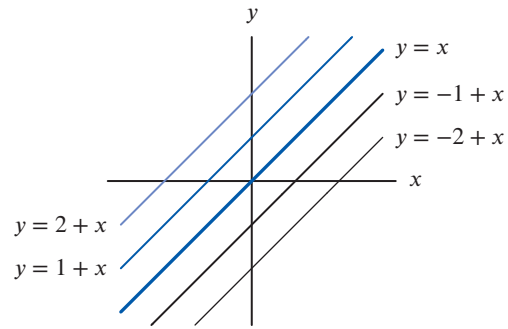


Figure 1.6: The family $y = b + x$ (with $m = 1$)

Increasing versus Decreasing Functions

The terms increasing and decreasing can be applied to other functions, not just linear ones. See Figure 1.7. In general,

A function f is **increasing** if the values of $f(x)$ increase as x increases.

A function f is **decreasing** if the values of $f(x)$ decrease as x increases.

The graph of an *increasing* function *climbs* as we move from left to right.

The graph of a *decreasing* function *falls* as we move from left to right.

A function $f(x)$ is **monotonic** if it increases for all x or decreases for all x .



Figure 1.7: Increasing and decreasing functions

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, the area, A , of a circle is proportional to the square of the radius, r , because

$$A = f(r) = \pi r^2.$$

We say y is (directly) **proportional** to x if there is a nonzero constant k such that

$$y = kx.$$

This k is called the constant of proportionality.

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v , at which you make a 50-mile trip is inversely proportional to the time, t , taken, because v is proportional to $1/t$:

$$v = 50 \left(\frac{1}{t} \right) = \frac{50}{t}.$$

Summary for Section 1.1

- **Definition of function:** a rule which takes numbers as inputs and assigns to each input exactly one output number.
- The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.
- **Function notation:** $y = f(t)$, where t is the **independent variable** and y is the **dependent variable**.
- A **linear function** has values of y that change at a constant rate with the values of x .
- **Formula** for linear functions:

$$y = \underbrace{b}_{\text{Initial value}} + \underbrace{m}_{\text{Slope}} \cdot x$$

- The graph of a linear function is a **line**.
 - b is the **vertical intercept**, or **y-intercept**, and gives the value of y for $x = 0$.
 - m is the **slope** of the line, and gives the rate of change of y with respect to x :
- $$m = \frac{\Delta y}{\Delta x}.$$
- Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, represent a **family of functions**.
 - A function f is **increasing** if the values of $f(x)$ increase as x increases.
 - A function f is **decreasing** if the values of $f(x)$ decrease as x increases.
 - A function f is **monotonic** if it increases for all x or decreases for all x .
 - **Proportionality:** We say y is (directly) **proportional** to x if there is a nonzero constant k such that $y = kx$ and k is called the constant of proportionality.

Exercises and Problems for Section 1.1

EXERCISES

1. The population of a city, P , in millions, is a function of t , the number of years since 2020, so $P = f(t)$. Explain the meaning of the statement $f(5) = 8$ in terms of the population of this city.
2. The pollutant PCB (polychlorinated biphenyl) can affect the thickness of pelican eggshells. Thinking of the thickness, T , of the eggshells, in mm, as a function of the concentration, P , of PCBs in ppm (parts per million), we have $T = f(P)$. Explain the meaning of $f(200)$ in terms of thickness of pelican eggs and concentration of PCBs.
3. Describe what Figure 1.8 tells you about an assembly line whose productivity is represented as a function of the number of workers on the line.

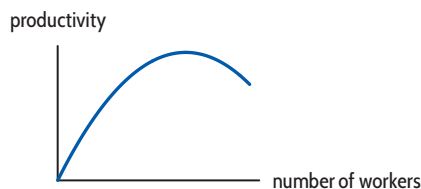


Figure 1.8

4. Match the graphs in Figure 1.9 with the following equations. (Note that the x and y scales may be unequal.)

- | | |
|-----------------|-------------------|
| (a) $y = x - 5$ | (b) $-3x + 4 = y$ |
| (c) $5 = y$ | (d) $y = -4x - 5$ |
| (e) $y = x + 6$ | (f) $y = x/2$ |

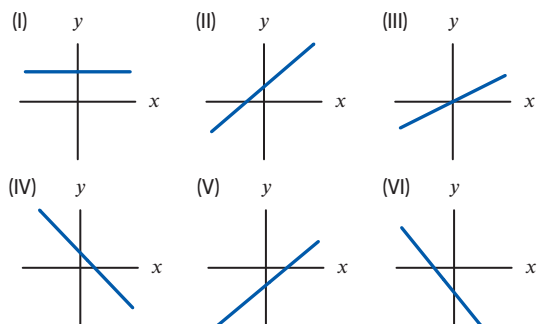


Figure 1.9

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5. Match the graphs in Figure 1.10 with the following equations. (Note that the x and y scales may be unequal.)

- (a) $y = -2.72x$ (b) $y = 0.01 + 0.001x$
 (c) $y = 27.9 - 0.1x$ (d) $y = 0.1x - 27.9$
 (e) $y = -5.7 - 200x$ (f) $y = x/3.14$

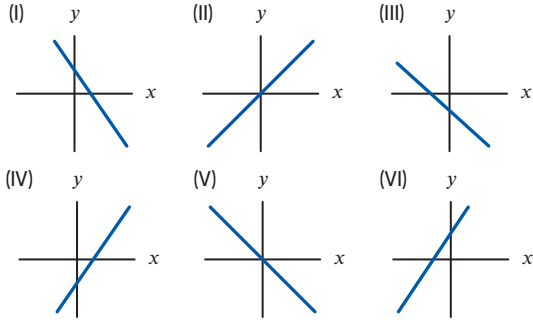


Figure 1.10

■ For Exercises 6–9, find an equation for the line that passes through the given points.

6. (0, 0) and (1, 1) 7. (0, 2) and (2, 3)
 8. (-2, 1) and (2, 3) 9. (-1, 0) and (2, 6)

■ For Exercises 10–13, determine the slope and the y -intercept of the line whose equation is given.

10. $2y + 5x - 8 = 0$ 11. $7y + 12x - 2 = 0$
 12. $-4y + 2x + 8 = 0$ 13. $12x = 6y + 4$

14. Estimate the slope and the equation of the line in Figure 1.11.

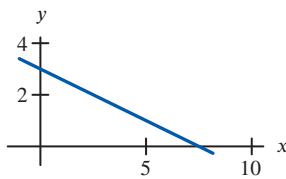


Figure 1.11

15. Find an equation for the line with slope m through the point (a, c) .
 16. Find a linear function that generates the values in Table 1.3.

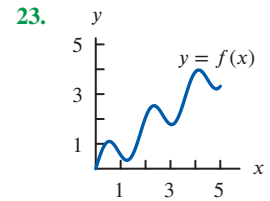
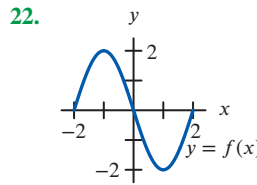
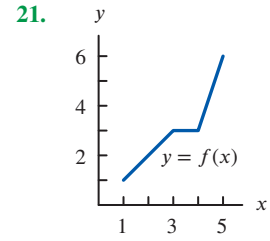
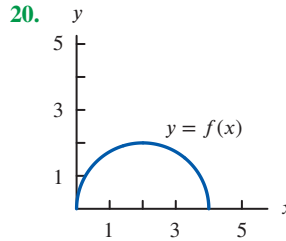
Table 1.3

x	5.2	5.3	5.4	5.5	5.6
y	27.8	29.2	30.6	32.0	33.4

■ For Exercises 17–19, use the facts that parallel lines have equal slopes and that the slopes of perpendicular lines are negative reciprocals of one another.

17. Find an equation for the line through the point (2, 1) which is perpendicular to the line $y = 5x - 3$.
 18. Find equations for the lines through the point (1, 5) that are parallel to and perpendicular to the line with equation $y + 4x = 7$.
 19. Find equations for the lines through the point (a, b) that are parallel and perpendicular to the line $y = mx + c$, assuming $m \neq 0$.

■ For Exercises 20–23, give the approximate domain and range of each function. Assume the entire graph is shown.



■ Find the domain and range in Exercises 24–25.

24. $y = x^2 + 2$ 25. $y = \frac{1}{x^2 + 2}$

26. If $f(t) = \sqrt{t^2 - 16}$, find all values of t for which $f(t)$ is a real number. Solve $f(t) = 3$.

■ In Exercises 27–31, write a formula representing the function.

27. The volume of a sphere is proportional to the cube of its radius, r .
 28. The average velocity, v , for a trip over a fixed distance, d , is inversely proportional to the time of travel, t .
 29. The strength, S , of a beam is proportional to the square of its thickness, h .
 30. The energy, E , expended by a swimming dolphin is proportional to the cube of the speed, v , of the dolphin.
 31. The number of animal species, N , of a certain body length, l , is inversely proportional to the square of l .

PROBLEMS

32. In December 2010, the snowfall in Minneapolis was unusually high,³ leading to the collapse of the roof of the Metrodome. Figure 1.12 gives the snowfall, S , in Minneapolis for December 6–15, 2010.
- How do you know that the snowfall data represents a function of date?
 - Estimate the snowfall on December 12.
 - On which day was the snowfall more than 10 inches?
 - During which consecutive two-day interval was the increase in snowfall largest?

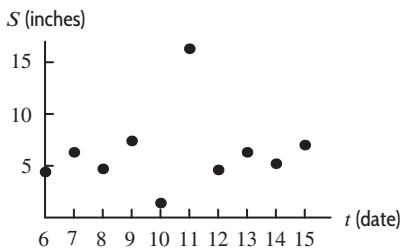


Figure 1.12

33. The value of a car, $V = f(a)$, in thousands of dollars, is a function of the age of the car, a , in years.
- Interpret the statement $f(5) = 6$.
 - Sketch a possible graph of V against a . Is f an increasing or decreasing function? Explain.
 - Explain the significance of the horizontal and vertical intercepts in terms of the value of the car.
34. Which graph in Figure 1.13 best matches each of the following stories?⁴ Write a story for the remaining graph.
- I had just left home when I realized I had forgotten my books, so I went back to pick them up.
 - Things went fine until I had a flat tire.
 - I started out calmly but sped up when I realized I was going to be late.

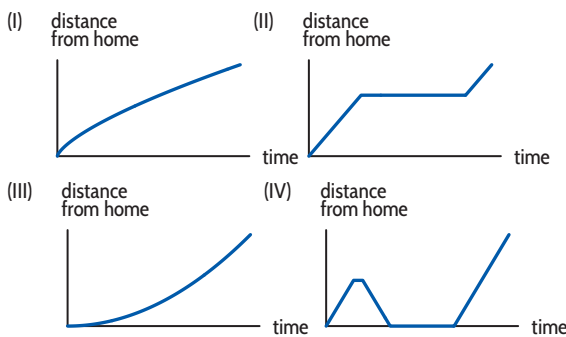


Figure 1.13

- In Problems 35–38, the function $S = f(t)$ gives the average annual sea level, S , in meters above a fixed reference level, in Aberdeen, Scotland,⁵ as a function of t , the number of years before 2020. Write a mathematical expression that represents the given statement.

- In 2018 the average annual sea level in Aberdeen was 7.088 meters.
- The average annual sea level in Aberdeen in 2020.
- The average annual sea level in Aberdeen was the same in 1949 and 2000.
- The average annual sea level in Aberdeen decreased by 11 millimeters from 2017 to 2018.

- Problems 39–42 ask you to plot graphs based on the following story: “As I drove down the highway this morning, at first traffic was fast and uncongested, then it crept nearly bumper-to-bumper until we passed an accident, after which traffic flow went back to normal until I exited.”

- Driving speed against time on the highway
 - Distance driven against time on the highway
 - Distance from my exit vs time on the highway
 - Distance between cars vs distance driven on the highway
43. An object is put outside on a cold day at time $t = 0$ minutes. Its temperature, $H = f(t)$, in $^{\circ}\text{C}$, is graphed in Figure 1.14.
- What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.
 - Explain what the vertical intercept, a , and the horizontal intercept, b , represent in terms of temperature of the object and time outside.

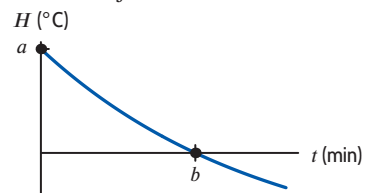


Figure 1.14

44. A rock is dropped from a window and falls to the ground below. The height, s (in meters), of the rock above ground is a function of the time, t (in seconds), since the rock was dropped, so $s = f(t)$.
- Sketch a possible graph of s as a function of t .
 - Explain what the statement $f(7) = 12$ tells us about the rock’s fall.
 - The graph drawn as the answer for part (a) should have a horizontal and vertical intercept. Interpret each intercept in terms of the rock’s fall.

³<http://www.crh.noaa.gov/mpx/Climate/DisplayRecords.php>

⁴Adapted from Jan Terwel, “Real Math in Cooperative Groups in Secondary Education”, *Cooperative Learning in Mathematics*, ed. Neal Davidson, p. 234 (Reading: Addison Wesley, 1990).

⁵www.psm1.org, accessed August 12, 2019.

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45. You drive at a constant speed from Chicago to Detroit, a distance of 275 miles. About 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.
46. US imports of crude oil and petroleum were increasing between 1992 and 2007.⁶ There were many ups and downs, but the general trend is shown by the line in Figure 1.15.

- (a) Find the slope of the line. Include its units of measurement.
- (b) Write an equation for the line. Define your variables, including their units.
- (c) Assuming the trend continues, when does the linear model predict imports will reach 18 million barrels per day? Do you think this is a reliable prediction? Give reasons.

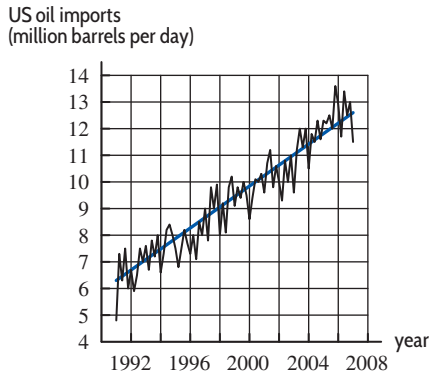


Figure 1.15

- Problems 47–49 use Figure 1.16 showing how the quantity, Q , of grass (kg/hectare) in different parts of Namibia depended on the average annual rainfall, r , (mm), in two different years.⁷

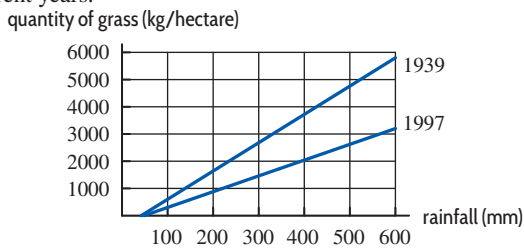


Figure 1.16

47. (a) For 1939, find the slope of the line, including units.
 (b) Interpret the slope in this context.
 (c) Find the equation of the line.
48. (a) For 1997, find the slope of the line, including units.
 (b) Interpret the slope in this context.
 (c) Find the equation of the line.
49. Which of the two functions in Figure 1.16 has the larger difference quotient $\Delta Q/\Delta r$? What does this tell us about grass in Namibia?

⁶www.theoilrum.com/node/2767, accessed August 16th, 2019.

⁷David Ward and Ben T. Ngairorue, “Are Namibia’s Grasslands Desertifying?”, *Journal of Range Management* 53, 2000, 138–144.

⁸David W. Inouye, Billy Barr, Kenneth B. Armitage, and Brian D. Inouye, “Climate Change is Affecting Altitudinal Migrants and Hibernating Species”, *PNAS* 97, 2000, 1630–1633.

⁹*Ibid.*

50. For t in years since 1950, the quantity of carbon dioxide in the atmosphere, in ppm (parts per million), is predicted to be

$$g(t) = 300 + 1.4t.$$

- (a) Find $g(10)$, $g(20)$ and $g(100)$.
 (b) Suppose another model predicts the quantity of carbon dioxide to be

$$h(t) = 300 + 1.5t.$$

Find $h(10)$, $h(20)$ and $h(100)$.

- (c) Do the functions $g(t)$ and $h(t)$ differ in their intercept or in their slope? Does this lead to a larger difference in values of $g(t)$ and $h(t)$ for small t or for large t ? (Use your answers to parts (a) and (b) to decide.)

51. Marmots are large squirrels that hibernate in the winter and come out in the spring. Figure 1.17 shows the date (days after Jan 1) that they are first sighted each year in Colorado as a function of the average minimum daily temperature for that year.⁸

- (a) Find the slope of the line, including units.
 (b) What does the sign of the slope tell you about marmots?
 (c) Use the slope to determine how much difference 6°C warming makes to the date of first appearance of a marmot.
 (d) Find the equation of the line.

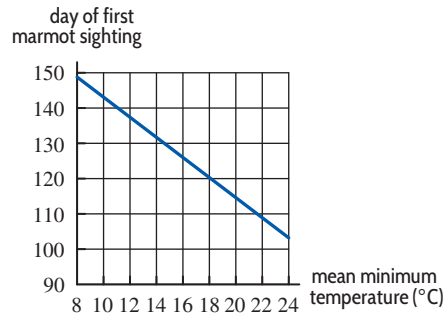


Figure 1.17

52. In Colorado spring has arrived when the bluebell first flowers. Figure 1.18 shows the date (days after Jan 1) that the first flower is sighted in one location as a function of the first date (days after Jan 1) of bare (snow-free) ground.⁹

- (a) If the first date of bare ground is 140, how many days later is the first bluebell flower sighted?
 (b) Find the slope of the line, including units.
 (c) What does the sign of the slope tell you about bluebells?
 (d) Find the equation of the line.

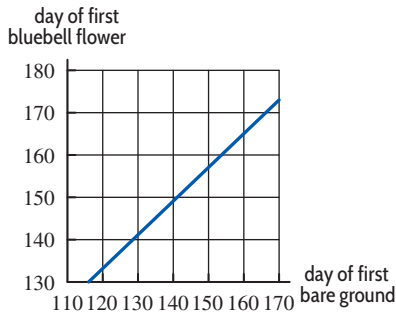


Figure 1.18

53. On March 5, 2015, Capracotta, Italy, received 256 cm (100.787 inches) of snow in 18 hours.¹⁰
- Assuming the snow fell at a constant rate and there were already 100 cm of snow on the ground, find a formula for $f(t)$, in cm, for the depth of snow as a function of t hours since the snowfall began on March 5.
 - What are the domain and range of f ?
54. In a Washington town, the charge for commercial waste collection is \$694.55 for 5 tons and \$1098.32 for 8 tons of waste.
- Find a linear formula for the cost, C , of waste collection as a function of the weight, w , in tons.
 - What is the slope of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.
 - What is the vertical intercept of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.
55. For tax purposes, you may have to report the value of your assets, such as cars or refrigerators. The value you report drops with time. “Straight-line depreciation” assumes that the value is a linear function of time. If a \$950 refrigerator depreciates completely in seven years, find a formula for its value as a function of time.
56. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount monthly plus a charge for each cubic foot of water used. A household using 1000 cubic feet was billed \$40, while one using 1600 cubic feet was billed \$55.
- What is the charge per cubic foot?
 - Write an equation for the total cost of a resident’s water as a function of cubic feet of water used.
 - How many cubic feet of water used would lead to a bill of \$100?
57. A company rents cars at \$40 a day and 15 cents a mile. Its competitor’s cars are \$50 a day and 10 cents a mile.
- For each company, give a formula for the cost of renting a car for a day as a function of the distance traveled.
 - On the same axes, graph both functions.
 - How should you decide which company is cheaper?
58. A controversial 1992 Danish study¹¹ reported that men’s average sperm count decreased from 113 million per milliliter in 1940 to 66 million per milliliter in 1990.
- Express the average sperm count, S , as a linear function of the number of years, t , since 1940.
 - A man’s fertility is affected if his sperm count drops below about 20 million per milliliter. If the linear model found in part (a) is accurate, in what year will the average male sperm count fall below this level?
59. Let $f(t)$ be the number of US billionaires in year t .
- Express the following statements¹² in terms of f .
 - In 2001 there were 272 US billionaires.
 - In 2019 there were 607 US billionaires.
 - Find the average yearly increase in the number of US billionaires from 2001 to 2019. Express this using f .
 - Assuming the yearly increase remains constant, find a formula predicting the number of US billionaires in year t .
60. The cost of planting seed is usually a function of the number of acres sown. The cost of the equipment is a *fixed cost* because it must be paid regardless of the number of acres planted. The costs of supplies and labor vary with the number of acres planted and are called *variable costs*. Suppose the fixed costs are \$10,000 and the variable costs are \$200 per acre. Let C be the total cost, measured in thousands of dollars, and let x be the number of acres planted.
- Find a formula for C as a function of x .
 - Graph C against x .
 - Which feature of the graph represents the fixed costs? Which represents the variable costs?
61. An airplane uses a fixed amount of fuel for takeoff, a (different) fixed amount for landing, and a third fixed amount per mile when it is in the air. How does the total quantity of fuel required depend on the length of the trip? Write a formula for the function involved. Explain the meaning of the constants in your formula.

¹⁰iceagenow.info, accessed April 2015.

¹¹“Investigating the Next Silent Spring,” *US News and World Report*, pp. 50–52 (March 11, 1996).

¹²www.forbes.com/billionaires, accessed August 15, 2019.

12 Chapter 1 FOUNDATION FOR CALCULUS: FUNCTIONS AND LIMITS

62. For the line $y = f(x)$ in Figure 1.19, evaluate

- (a) $f(423) - f(422)$ (b) $f(517) - f(513)$

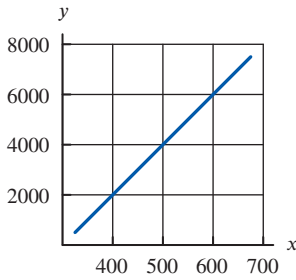


Figure 1.19

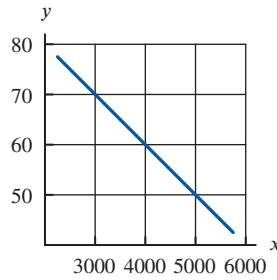


Figure 1.20

63. For the line $y = g(x)$ in Figure 1.20, evaluate

- (a) $g(4210) - g(4209)$ (b) $g(3760) - g(3740)$

64. Let $V(t)$ be the average speed, in miles per hour, for a journey of 100 miles taking t hours. See Figure 1.21.

- (a) Find $V(1)$ and $V(2)$. Give units.
 (b) Do you think it is true for all t that

$$V(t + 1) = V(t) + 1?$$

If so, give an argument. If not, give a counterexample.

- (c) Answer the same question as part (b), but with

$$V(t + 1) = V(t) + V(1).$$

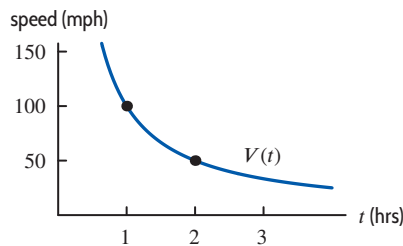


Figure 1.21

65. The percentage of people who have heard some news t days after it was announced is $N(t)$. See Figure 1.22.

- (a) Find $N(1)$ and $N(2)$.
 (b) Do you think it is true for all t that

$$N(t + 1) = N(t) + 1?$$

If so, give an argument. If not, give a counterexample.

- (c) Answer the same question as part (b), but with

$$N(t + 1) = N(t) + N(1).$$

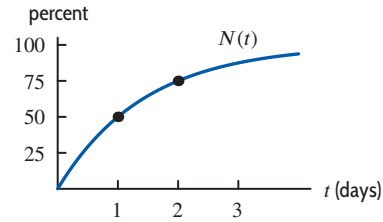


Figure 1.22

66. An alternative to petroleum-based diesel fuel, biodiesel, is derived from renewable resources such as food crops, algae, and animal oils. The table shows the recent annual percent growth in US biodiesel exports.¹³

- (a) Find the largest time interval over which the percentage growth in the US exports of biodiesel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.
 (b) Find the largest time interval over which the actual US exports of biodiesel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.

Year	2012	2013	2014	2015	2016	2017	2018
% growth over previous yr	69.9	53.0	-57.8	5.93	0.33	6.20	10.10

67. Hydroelectric power is electric power generated by the force of moving water. Figure 1.23 shows¹⁴ the annual percent growth in hydroelectric power consumption by the US industrial sector between 2008 and 2018.

- (a) Find the largest time interval over which the percentage growth in the US consumption of hydroelectric power was an increasing function of time. Interpret what increasing means, practically speaking, in this case.
 (b) Find the largest time interval over which the actual US consumption of hydroelectric power was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.

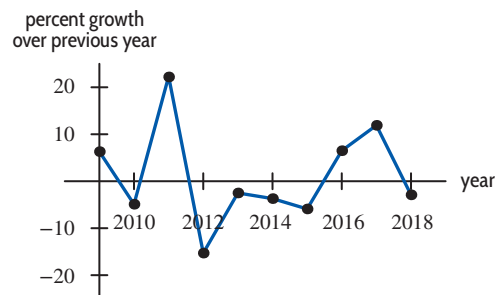


Figure 1.23

¹³www.eia.doe.gov, accessed August 19, 2019.

¹⁴Yearly values have been joined with line segments to highlight trends in the data; however, values in between years should not be inferred from the segments. From www.eia.doe.gov, accessed August 19, 2019.

68. Solar panels are arrays of photovoltaic cells that convert solar radiation into electricity. The table shows the annual percent change in the US price per watt of a solar panel.¹⁵

- (a) Find the largest time interval over which the percentage growth in the US price per watt of a solar panel was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.
- (b) Find the largest time interval over which the actual price per watt of a solar panel was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.

Year	2005	2006	2007	2008	2009	2010
% growth over previous yr	6.7	9.7	-3.7	3.6	-20.1	-29.7

69. Table 1.4 shows the average annual sea level, S , in meters, in Aberdeen, Scotland,¹⁶ as a function of time, t , measured in years before 2014.

Table 1.4

t	0	25	50	75	100	125
S	7.071	7.083	6.990	6.964	6.985	6.900

- (a) What was the average sea level in Aberdeen in 2014?
- (b) In what year was the average sea level 7.083 meters? 6.985 meters?
- (c) Table 1.5 gives the average sea level, S , in Aberdeen as a function of the year, x . Complete the missing values.

Table 1.5

x	1889	?	1939	1964	1989	2014
S	?	6.985	?	6.990	?	?

70. The table gives the required standard weight, w , in kilograms, of American soldiers, aged between 21 and 27, for height, h , in centimeters.¹⁷

- (a) How do you know that the data in this table could represent a linear function?
- (b) Find weight, w , as a linear function of height, h . What is the slope of the line? What are the units for the slope?
- (c) Find height, h , as a linear function of weight, w . What is the slope of the line? What are the units for the slope?

h (cm)	172	176	180	184	188	192	196
w (kg)	79.7	82.4	85.1	87.8	90.5	93.2	95.9

71. Table 1.6 shows the pressure P , in torr,¹⁸ at a depth of h meters below the surface of a lake.

- (a) Explain why P could be a linear function of h .

- (b) Find pressure, P , as a linear function of height h and give practical interpretations of the slope and vertical intercept.
- (c) Use part (b) to approximate the depth at which the pressure is twice that at the surface of the lake.

Table 1.6

h (m)	6	8	10	12	14	16
P (torr)	1201	1348	1495	1642	1789	1936

72. A \$25,000 vehicle depreciates \$2000 a year as it ages. Repair costs are \$1500 per year.

- (a) Write formulas for each of the two linear functions at time t , value, $V(t)$, and repair costs to date, $C(t)$. Graph them.
- (b) One strategy is to replace a vehicle when the total cost of repairs is equal to the current value. Find this time.
- (c) Another strategy is to replace the vehicle when the value of the vehicle is some percent of the original value. Find the time when the value is 6%.

73. A bakery owner knows that customers buy a total of q cakes when the price, p , is no more than $p = d(q) = 20 - q/20$ dollars. She is willing to make and supply as many as q cakes at a price of $p = s(q) = 11 + q/40$ dollars each. (The graphs of the functions $d(q)$ and $s(q)$ are called a *demand curve* and a *supply curve*, respectively.) The graphs of $d(q)$ and $s(q)$ are in Figure 1.24.

- (a) Why, in terms of the context, is the slope of $d(q)$ negative and the slope of $s(q)$ positive?
- (b) Is each of the ordered pairs (q, p) a solution to the inequality $p \leq 20 - q/20$? Interpret your answers in terms of the context.

(60, 18) (120, 12)

- (c) Graph in the qp -plane the solution set of the system of inequalities $p \leq 20 - q/20$, $p \geq 11 + q/40$. What does this solution set represent in terms of the context?
- (d) What is the rightmost point of the solution set you graphed in part (c)? Interpret your answer in terms of the context.

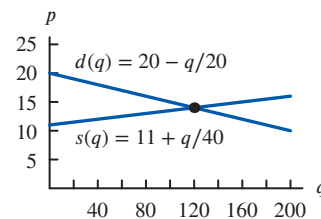


Figure 1.24

¹⁵We use the official price per peak watt, which uses the maximum number of watts a solar panel can produce under ideal conditions. From www.eia.doe.gov, accessed March 29, 2015.

¹⁶www.psmf.org, accessed August 19, 2019.

¹⁷Adapted from usmilitary.about.com, accessed March 29, 2015.

¹⁸A torr is a unit of pressure.

74. (a) Consider the functions graphed in Figure 1.25(a). Find the coordinates of C .
 (b) Consider the functions in Figure 1.25(b). Find the coordinates of C in terms of b .

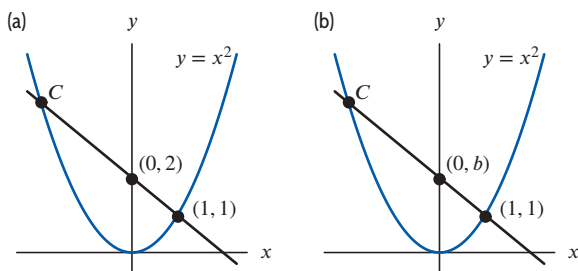


Figure 1.25

75. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and

falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen.

What do the experimental data in Table 1.7 tell you about Galileo's hypothesis? What alternative hypothesis is suggested by the two sets of data in Table 1.7 and Table 1.8?

Table 1.7

Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

Table 1.8

Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

Strengthen Your Understanding

- In Problems 76–80, explain what is wrong with the statement.
 76. For constants m and b , the slope of the linear function $y = b + mx$ is $m = \Delta x / \Delta y$.
 77. The lines $x = 3$ and $y = 3$ are both linear functions of x .
 78. The line $y - 3 = 0$ has slope 1 in the xy -plane.
 79. Values of y on the graph of $y = 0.5x - 3$ increase more slowly than values of y on the graph of $y = 0.5 - 3x$.
 80. The equation $y = 2x + 1$ indicates that y is directly proportional to x with a constant of proportionality 2.
- In Problems 81–82, give an example of:
 81. A linear function with a positive slope and a negative x -intercept.
 82. A formula representing the statement “ q is inversely proportional to the cube root of p and has a positive constant of proportionality.”
- In Problems 83–88, is the statement true or false? Give an explanation for your answer.
 83. For any two points in the plane, there is a linear function whose graph passes through them.
 84. If $y = f(x)$ is a linear function, then increasing x by 1 unit changes the corresponding y by m units, where m is the slope.
 85. The linear functions $y = -x + 1$ and $x = -y + 1$ have the same graph.
 86. The linear functions $y = 2 - 2x$ and $x = 2 - 2y$ have the same graph.
 87. If y is a linear function of x , then the ratio y/x is constant for all points on the graph at which $x \neq 0$.
 88. If $y = f(x)$ is a linear function, then increasing x by 2 units adds $m + 2$ units to the corresponding y , where m is the slope.
- 89. Which of the following functions has its domain identical with its range?

(a) $f(x) = x^2$	(b) $g(x) = \sqrt{x}$
(c) $h(x) = x^3$	(d) $i(x) = x $

1.2 EXPONENTIAL FUNCTIONS

Population Growth

The population of Burkina Faso, a sub-Saharan African country,¹⁹ is given in Table 1.9. To see how the population is growing, we look at the increase in population in the third column. If the population had been growing linearly, all the numbers in the third column would be the same.

¹⁹www.worldometers.inf, accessed August 19, 2019.

Table 1.9 Population of Burkina Faso (estimated), 2009–2018

Year	Population (millions)	Change in population (millions)
2009	15.141	
2010	15.605	0.464
2011	16.081	0.476
2012	16.572	0.490
2013	17.073	0.502
2014	17.586	0.513
2015	18.111	0.525
2016	18.646	0.535
2017	19.193	0.547
2018	19.751	0.559

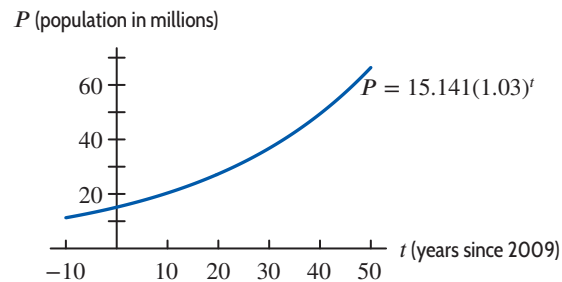


Figure 1.26: Population of Burkina Faso (estimated): Exponential growth

Suppose we divide each year's population by the previous year's population. For example,

$$\frac{\text{Population in 2010}}{\text{Population in 2009}} = \frac{15.605 \text{ million}}{15.141 \text{ million}} = 1.03$$

$$\frac{\text{Population in 2011}}{\text{Population in 2010}} = \frac{16.081 \text{ million}}{15.605 \text{ million}} = 1.03.$$

The fact that both calculations give 1.03 shows the population grew by about 3% between 2009 and 2010 *and* between 2010 and 2011. Similar calculations for other years show that the population grew by a factor of about 1.03, or 3%, every year. Whenever we have a constant growth factor (here 1.03), we have exponential growth. The population t years after 2009 is given by the *exponential* function

$$P = 15.141(1.03)^t.$$

If we assume that the formula holds for 50 years, the population graph has the shape shown in Figure 1.26. Since the population is growing faster and faster as time goes on, the graph is bending upward; we say it is *concave up*. Even exponential functions which climb slowly at first, such as this one, eventually climb extremely quickly.

To recognize that a table of t and P values comes from an exponential function, look for ratios of P values that are constant for equally spaced t values.

Concavity

We have used the term concave up²⁰ to describe the graph in Figure 1.26. In words:

The graph of a function is **concave up** if it bends upward as we move left to right; it is **concave down** if it bends downward. (See Figure 1.27 for four possible shapes.) A line is neither concave up nor concave down.



Figure 1.27: Concavity of a graph

²⁰In Chapter 2 we consider concavity in more depth.

Elimination of a Drug from the Body

Now we look at a quantity which is decreasing exponentially instead of increasing. When a patient is given medication, the drug enters the bloodstream. As the drug passes through the liver and kidneys, it is metabolized and eliminated at a rate that depends on the particular drug. For the antibiotic ampicillin, approximately 40% of the drug is eliminated every hour. A typical dose of ampicillin is 250 mg. Suppose $Q = f(t)$, where Q is the quantity of ampicillin, in mg, in the bloodstream at time t hours since the drug was given. At $t = 0$, we have $Q = 250$. Since every hour the amount remaining is 60% of the previous amount, we have

$$\begin{aligned}f(0) &= 250 \\f(1) &= 250(0.6) \\f(2) &= (250(0.6))(0.6) = 250(0.6)^2,\end{aligned}$$

and after t hours,

$$Q = f(t) = 250(0.6)^t.$$

This is an *exponential decay function*. Some values of the function are in Table 1.10; its graph is in Figure 1.28.

Notice the way in which the function in Figure 1.28 is decreasing. Each hour a smaller quantity of the drug is removed than in the previous hour. This is because as time passes, there is less of the drug in the body to be removed. Compare this to the exponential growth in Figure 1.26, where each step upward is larger than the previous one. Notice, however, that both graphs are concave up.

Table 1.10 Drug elimination

t (hours)	Q (mg)
0	250
1	150
2	90
3	54
4	32.4
5	19.4

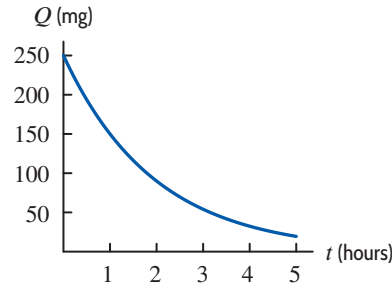


Figure 1.28: Drug elimination: Exponential decay

The General Exponential Function

We say P is an **exponential function** of t with base a if

$$P = P_0 a^t,$$

where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1.

If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.

Provided $a > 0$, the largest possible domain for the exponential function is all real numbers. The reason we do not want $a \leq 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 1^t = P_0$ is then a constant function.

The value of a is closely related to the percent growth (or decay) rate. For example, if $a = 1.03$, then P is growing at 3%; if $a = 0.94$, then P is decaying at 6%, so the growth rate is $r = a - 1$.

Example 1 Suppose that $Q = f(t)$ is an exponential function of t . If $f(20) = 88.2$ and $f(23) = 91.4$:

(a) Find the base. (b) Find the growth rate. (c) Evaluate $f(25)$.

Solution (a) Let

$$Q = Q_0 a^t.$$

Substituting $t = 20$, $Q = 88.2$ and $t = 23$, $Q = 91.4$ gives two equations for Q_0 and a :

$$88.2 = Q_0 a^{20} \quad \text{and} \quad 91.4 = Q_0 a^{23}.$$

Dividing the two equations enables us to eliminate Q_0 :

$$\frac{91.4}{88.2} = \frac{Q_0 a^{23}}{Q_0 a^{20}} = a^3.$$

Solving for the base, a , gives

$$a = \left(\frac{91.4}{88.2} \right)^{1/3} = 1.012.$$

(b) Since $a = 1.012$, the growth rate is $1.012 - 1 = 0.012 = 1.2\%$.

(c) We want to evaluate $f(25) = Q_0 a^{25} = Q_0 (1.012)^{25}$. First we find Q_0 from the equation

$$88.2 = Q_0 (1.012)^{20}.$$

Solving gives $Q_0 = 69.5$. Thus,

$$f(25) = 69.5(1.012)^{25} = 93.6.$$

Half-Life and Doubling Time

Radioactive substances, such as uranium, decay exponentially. A certain percentage of the mass disintegrates in a given unit of time; the time it takes for half the mass to decay is called the *half-life* of the substance.

A well-known radioactive substance is carbon-14, which is used to date organic objects. When a piece of wood or bone was part of a living organism, it accumulated small amounts of radioactive carbon-14. Once the organism dies, it no longer picks up carbon-14. Using the half-life of carbon-14 (about 5730 years), we can estimate the age of the object. We use the following definitions:

The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.

The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.

The Family of Exponential Functions

The formula $P = P_0 a^t$ gives a family of exponential functions with positive parameters P_0 (the initial quantity) and a (the base, or growth/decay factor). The base tells us whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.29 and 1.30.) All members of the family $P = P_0 a^t$ are concave up.

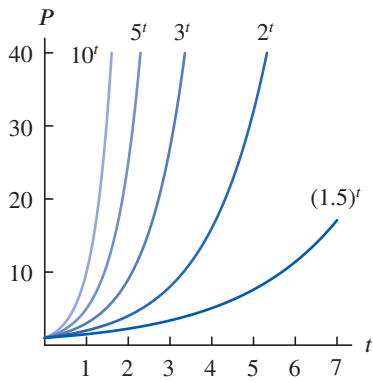


Figure 1.29: Exponential growth: $P = a^t$, for $a > 1$

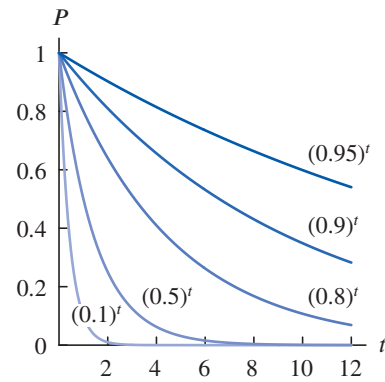


Figure 1.30: Exponential decay: $P = a^t$, for $0 < a < 1$

Example 2 Figure 1.31 is the graph of three exponential functions. What can you say about the values of the six constants a, b, c, d, p, q ?

Solution All the constants are positive. Since a, c, p represent y -intercepts, we see that $a = c$ because these graphs intersect on the y -axis. In addition, $a = c < p$, since $y = p \cdot q^x$ crosses the y -axis above the other two.

Since $y = a \cdot b^x$ is decreasing, we have $0 < b < 1$. The other functions are increasing, so $1 < d$ and $1 < q$.

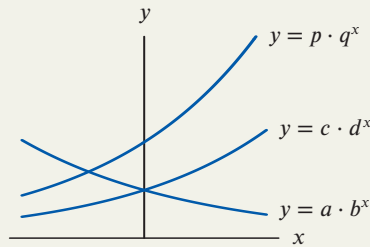


Figure 1.31: Three exponential functions

Exponential Functions with Base e

The most frequently used base for an exponential function is the famous number $e = 2.71828 \dots$. This base is used so often that you will find an e^x button on most scientific calculators. At first glance, this is all somewhat mysterious. Why is it convenient to use the base 2.71828 ...? The full answer to that question must wait until Chapter 3, where we show that many calculus formulas come out neatly when e is used as the base. We often use the following result:

Any **exponential growth** function can be written, for some $a > 1$ and $k > 0$, in the form

$$P = P_0 a^t \quad \text{or} \quad P = P_0 e^{kt}$$

and any **exponential decay** function can be written, for some $0 < a < 1$ and $-k < 0$, as

$$Q = Q_0 a^t \quad \text{or} \quad Q = Q_0 e^{-kt},$$

where P_0 and Q_0 are the initial quantities.

We say that P and Q are growing or decaying at a *continuous*²¹ rate of k . (For example, $k = 0.02$ corresponds to a continuous rate of 2%.)

²¹The reason that k is called the continuous rate is explored in detail in Chapter 11.

Example 3 Convert the functions $P = e^{0.5t}$ and $Q = 5e^{-0.2t}$ into the form $y = y_0a^t$. Use the results to explain the shape of the graphs in Figures 1.32 and 1.33.

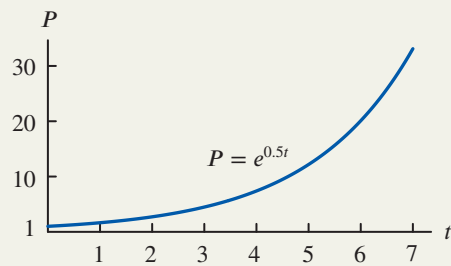


Figure 1.32: An exponential growth function

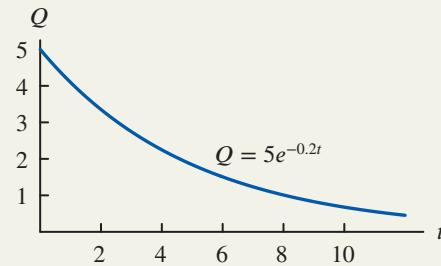


Figure 1.33: An exponential decay function

Solution We have

$$P = e^{0.5t} = (e^{0.5})^t = (1.65)^t.$$

Thus, P is an exponential growth function with $P_0 = 1$ and $a = 1.65$. The function is increasing and its graph is concave up, similar to those in Figure 1.29. Also,

$$Q = 5e^{-0.2t} = 5(e^{-0.2})^t = 5(0.819)^t,$$

so Q is an exponential decay function with $Q_0 = 5$ and $a = 0.819$. The function is decreasing and its graph is concave up, similar to those in Figure 1.30.

Example 4 The quantity, Q , of a drug in a patient's body at time t is represented for positive constants S and k by the function $Q = S(1 - e^{-kt})$. For $t \geq 0$, describe how Q changes with time. What does S represent?

Solution The graph of Q is shown in Figure 1.34. Initially none of the drug is present, but the quantity increases with time. Since the graph is concave down, the quantity increases at a decreasing rate. This is realistic because as the quantity of the drug in the body increases, so does the rate at which the body excretes the drug. Thus, we expect the quantity to level off. Figure 1.34 shows that S is the saturation level. The line $Q = S$ is called a *horizontal asymptote*.

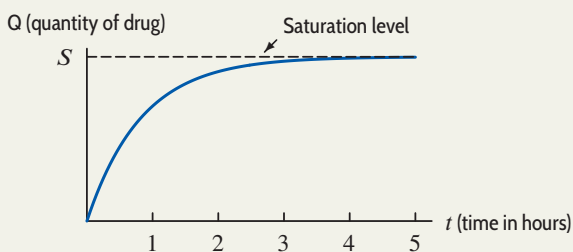


Figure 1.34: Buildup of the quantity of a drug in body

Summary for Section 1.2

- We say that P is an **exponential function** of t with base a if

$$P = P_0a^t.$$

- P_0 is the initial quantity (when $t = 0$).
- a is the factor by which P changes when t increases by 1.
- If $a > 1$, we have **exponential growth**.
- If $0 < a < 1$, we have **exponential decay**.
- To recognize that a table of t and P values comes from an exponential function, look for ratios of P values that are constant for equally-spaced t values.

- The graph of a function is **concave up** if it bends upward as we move left to right; it is **concave down** if it bends downward.
- The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.
- The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.
- The formula $P = P_0a^t$ gives a **family of exponential functions** with parameters P_0 (the initial quantity) and a (the base). In practice, the most commonly used base is the number $e = 2.71828$.
- Any **exponential growth** function can be written, for some $a > 1$ and $k > 0$, in the form

$$P = P_0a^t \quad \text{or} \quad P = P_0e^{kt}$$

and any **exponential decay** function can be written, for some $0 < a < 1$ and $-k < 0$, as

$$Q = Q_0a^t \quad \text{or} \quad Q = Q_0e^{-kt},$$

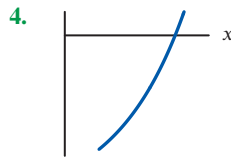
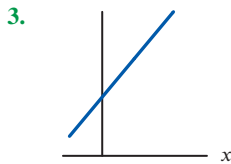
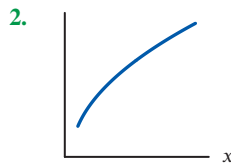
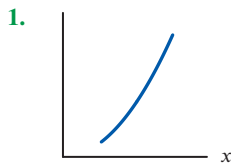
where P_0 and Q_0 are the initial quantities.

- We say that P and Q are growing or decaying at a **continuous rate** of k .

Exercises and Problems for Section 1.2

EXERCISES

■ In Exercises 1–4, decide whether the graph is concave up, concave down, or neither.



■ The functions in Exercises 5–8 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

5. $P = 5(1.07)^t$

6. $P = 7.7(0.92)^t$

7. $P = 3.2e^{0.03t}$

8. $P = 15e^{-0.06t}$

■ Write the functions in Exercises 9–12 in the form $P = P_0a^t$. Which represent exponential growth and which represent exponential decay?

9. $P = 15e^{0.25t}$

10. $P = 2e^{-0.5t}$

11. $P = P_0e^{0.2t}$

12. $P = 7e^{-\pi t}$

■ In Exercises 13–14, let $f(t) = Q_0a^t = Q_0(1 + r)^t$.

(a) Find the base, a .

(b) Find the percentage growth rate, r .

13. $f(5) = 75.94$ and $f(7) = 170.86$

14. $f(0.02) = 25.02$ and $f(0.05) = 25.06$

15. A town has a population of 1000 people at time $t = 0$. In each of the following cases, write a formula for the population, P , of the town as a function of year t .

(a) The population increases by 50 people a year.

(b) The population increases by 5% a year.

16. An air-freshener starts with 30 grams and evaporates over time. In each of the following cases, write a formula for the quantity, Q grams, of air-freshener remaining t days after the start and sketch a graph of the function. The decrease is:

(a) 2 grams a day

(b) 12% a day

17. For which pairs of consecutive points in Figure 1.35 is the function graphed:

(a) Increasing and concave up?

(b) Increasing and concave down?

(c) Decreasing and concave up?

(d) Decreasing and concave down?

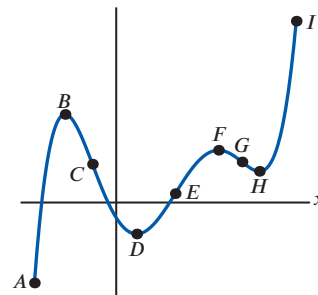


Figure 1.35

18. The table gives the average temperature in Wallingford, Connecticut, for the first 10 days in March.
- (a) Over which intervals was the average temperature increasing? Decreasing?
 - (b) Find a pair of consecutive intervals over which the average temperature was increasing at a decreasing rate. Find another pair of consecutive intervals

over which the average temperature was increasing at an increasing rate.

Day	1	2	3	4	5	6	7	8	9	10
°F	42°	42°	34°	25°	22°	34°	38°	40°	49°	49°

PROBLEMS

19. (a) Which (if any) of the functions in the following table could be linear? Find formulas for those functions.
- (b) Which (if any) of these functions could be exponential? Find formulas for those functions.

x	$f(x)$	$g(x)$	$h(x)$
-2	12	16	37
-1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

23. Match the functions $h(s)$, $f(s)$, and $g(s)$, whose values are in Table 1.12, with the formulas

$$y = a(1.1)^s, \quad y = b(1.05)^s, \quad y = c(1.03)^s,$$

assuming a , b , and c are constants. Note that the function values have been rounded to two decimal places.

Table 1.12

s	$h(s)$	s	$f(s)$	s	$g(s)$
2	1.06	1	2.20	3	3.47
3	1.09	2	2.42	4	3.65
4	1.13	3	2.66	5	3.83
5	1.16	4	2.93	6	4.02
6	1.19	5	3.22	7	4.22

■ In Problems 20–21, find all the tables that have the given characteristic.

- (A)

x	0	40	80	160
y	2.2	2.2	2.2	2.2
- (B)

x	-8	-4	0	8
y	51	62	73	95
- (C)

x	-4	-3	4	6
y	18	0	4.5	-2.25
- (D)

x	3	4	5	6
y	18	9	4.5	2.25

24. Each of the functions g , h , k in Table 1.13 is increasing, but each increases in a different way. Which of the graphs in Figure 1.36 best fits each function?

Table 1.13

t	$g(t)$	$h(t)$	$k(t)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

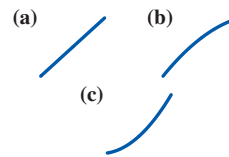


Figure 1.36

20. y could be a linear function of x .
21. y could be an exponential function of x .
22. Table 1.11 shows some values of a linear function f and an exponential function g . Find exact values (not decimal approximations) for each of the missing entries.

Table 1.11

x	0	1	2	3	4
$f(x)$	10	?	20	?	?
$g(x)$	10	?	20	?	?

25. Each of the functions in Table 1.14 decreases, but each decreases in a different way. Which of the graphs in Figure 1.37 best fits each function?

Table 1.14

x	$f(x)$	$g(x)$	$h(x)$
1	100	22.0	9.3
2	90	21.4	9.1
3	81	20.8	8.8
4	73	20.2	8.4
5	66	19.6	7.9
6	60	19.0	7.3

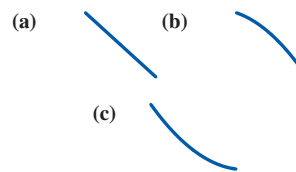


Figure 1.37

26. Figure 1.38 shows $Q = 50(1.2)^t$, $Q = 50(0.6)^t$, $Q = 50(0.8)^t$, and $Q = 50(1.4)^t$. Match each formula to a graph.

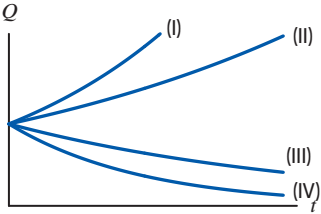
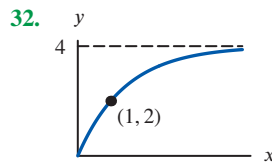
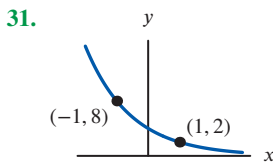
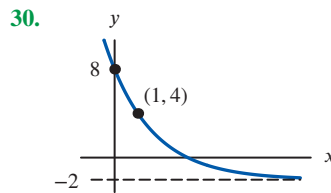
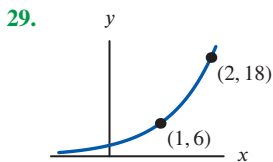
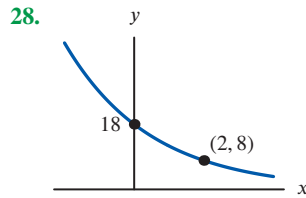
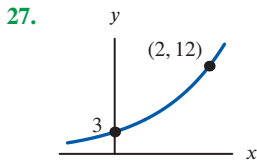


Figure 1.38

■ In Problems 27–32, give a possible formula for the function.



33. The table gives the number of North American houses (millions) with analog cable TV.²²

- (a) Plot the number of houses, H , in millions, with cable TV versus year, Y .
 (b) Could H be a linear function of Y ? Why or why not?
 (c) Could H be an exponential function of Y ? Why or why not?

Year	2010	2011	2012	2013	2014	2015
Houses	18.3	13	7.8	3.9	1	0.5

34. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time goes on.

- (a) Graph the total number of people who have tried such a product against time.
 (b) What do you know about the concavity of the graph?

35. Sketch reasonable graphs for the following. Pay particular attention to the concavity of the graphs.

- (a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.
 (b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.
36. (a) A population, P , grows at a continuous rate of 2% a year and starts at 1 million. Write P in the form $P = P_0 e^{kt}$, with P_0, k constants.
 (b) Plot the population in part (a) against time.

37. A 2008 study of 300 oil fields producing a total of 84 million barrels per day reported that daily production was decaying at a continuous rate of 9.1% per year.²³ Find the estimated production in these fields in 2025 if the decay continues at the same rate.

38. In 2017, the world's population reached 7.41 billion²⁴ and was increasing at a rate of 1.1% per year. Assume that this growth rate remains constant. (In fact, the growth rate has been decreasing since 2008.)

- (a) Write a formula for the world population (in billions) as a function of the number of years since 2017.
 (b) Estimate the population of the world in the year 2025.
 (c) Sketch world population as a function of years since 2017. Use the graph to estimate the doubling time of the population of the world.

39. The population of Nevada was 3.0 million in 2018 and 2.7 million in 2010. Assuming an exponential model:

- (a) Write the population of Nevada in the form $N = N_0 a^t$, where N is the population of Nevada in millions, N_0 and a are constants, and t is time in years since 2010.
 (b) What was the estimated population of Nevada in 2000?

40. Aircraft require longer takeoff distances, called takeoff rolls, at high-altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0° C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

Elevation (ft)	Sea level	1000	2000	3000	4000
Takeoff roll (ft)	670	734	805	882	967

²²<http://www.statista.com>, accessed May 2015.

²³International Energy Agency, *World Energy Outlook*, 2008.

²⁴www.indexmundi.com, accessed August 20, 2019.

41. One of the main contaminants of a nuclear accident, such as that at Chernobyl, is strontium-90, which decays exponentially at a continuous rate of approximately 2.47% per year. After the Chernobyl disaster, it was suggested that it would be about 100 years before the region would again be safe for human habitation. What percent of the original strontium-90 would still remain then?

42. The decrease in the number of colonies of honey bees, essential to pollinate crops providing one quarter of US food consumption, worries policy makers. US beekeepers say that over the three winter months a 5.9% decline in the number of colonies per month is economically sustainable, but higher rates are not.²⁵

- (a) Assuming a constant percent colony loss, which function I–III could describe a winter monthly colony loss trend that is economically sustainable? Assume y is the number of US bee colonies, t is time in months, and a is a positive constant.

I. $y = a(1.059)^t$ II. $y = a(0.962)^t$
 III. $y = a(0.935)^t$

- (b) What is the annual bee colony trend described by each of the functions in part (a)?

43. Let $f(t) = 3e^{kt}$ and $g(t) = ae^{2kt}$ for positive constants k and a .

- (a) Is there a value of a such that $g(t) = 2f(t)$ for all t ? If so, what is it?
 (b) Is there a value of a such that $g(t) = (f(t))^2$? If so, what is it?

44. A certain region has a population of 10,000,000 and an annual growth rate of 2%. Estimate the doubling time by guessing and checking.

45. According to the EPA, sales of electronic devices in the US doubled between 1997 and 2009, when 438 million electronic devices sold.²⁶

- (a) Find an exponential function, $S(t)$, to model sales in millions since 1997.
 (b) What was the annual percentage growth rate between 1997 and 2009?

46. (a) Estimate graphically the doubling time of the exponentially growing population shown in Figure 1.39. Check that the doubling time is independent of where you start on the graph.

- (b) Show algebraically that if $P = P_0a^t$ doubles between time t and time $t + d$, then d is the same number for any t .

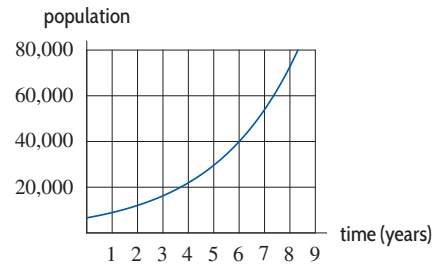


Figure 1.39

47. A deposit of P_0 into a bank account has a doubling time of 50 years. No other deposits or withdrawals are made.

- (a) How much money is in the bank account after 50 years? 100 years? 150 years? (Your answer will involve P_0 .)
 (b) How many times does the amount of money double in t years? Use this to write a formula for P , the amount of money in the account after t years.

48. A 325 mg aspirin has a half-life of H hours in a patient's body.

- (a) How long does it take for the quantity of aspirin in the patient's body to be reduced to 162.5 mg? To 81.25 mg? To 40.625 mg? (Note that $162.5 = 325/2$, etc. Your answers will involve H .)
 (b) How many times does the quantity of aspirin, A mg, in the body halve in t hours? Use this to give a formula for A after t hours.

49. (a) The half-life of radium-226 is 1620 years. If the initial quantity of radium is Q_0 , explain why the quantity, Q , of radium left after t years, is given by

$$Q = Q_0(0.999572)^t.$$

- (b) What percentage of the original amount of radium is left after 500 years?

50. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 2010? [Hint: Write the function in the form $Q = Q_0(1/2)^{t/29}$.]

51. The amount of a 10-gram sample of cesium-137 remaining after t years is given by $f(t) = 10(1/2)^{t/30}$ grams. Find the half-life and the continuous rate of decay of cesium-137.

52. Food bank usage in Britain grew rapidly from 2015 to 2019. The number of emergency three-day food parcels given by the Trussell Trust, in millions, is estimated to

²⁵“A Key to America's Crops Is Disappearing at a Staggering Rate,” www.businessinsider.com, accessed January 2016.

²⁶<http://www.epa.gov/osw/conservation/materials/recycling/docs/summarybaselinereport2011.pdf>, accessed March 2015.

²⁷Estimates for the Trussell Trust: www.trusselltrust.org/stats, accessed August 2019.

be $N(t) = 1.109e^{0.12t}$, where t is the number of years since 2015.²⁷

- (a) What does the 1.109 represent in this context? Give units.
- (b) What is the continuous growth rate of parcels per year?
- (c) What is the annual percent growth rate of parcels per year?
- (d) Using only your answer for part (c), decide if the doubling time is more or less than 1 year.

53. Figure 1.40 shows the concentration of a drug in the body, $f(t) = a - be^{kt}$, over time, t :

- (a) What does the constant a represent in this context?
- (b) What is the sign of the constant a ?
- (c) What is the relation between the constants a and b ?
- (d) What is the sign of the constant k ?

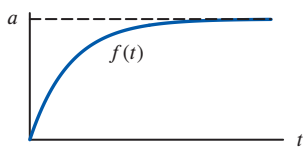


Figure 1.40

54. The remaining charge on a capacitor t seconds after discharge through a circuit is $q = f(t)$ microcoulombs (μC). Table 1.15 gives approximate values of f .

- (a) Explain why f appears to be roughly exponential and find an approximate formula for f .
- (b) Approximate when the charge remaining on the capacitor is half its initial value.
- (c) What is the approximate continuous decay rate of the charge on the capacitor?

Table 1.15

t (sec)	0	1	2	3	4
$f(t)$ (μC)	60	46.7	36.4	28.3	22.1

55. Atmospheric pressure decreases exponentially as elevation increases, measured from 760 torr²⁸ at ground level to 341 torr at an elevation of 5.6 kilometers.

- (a) Find a formula for the atmospheric pressure, p , at an elevation of h km.
- (b) By what percentage does atmospheric pressure decrease for each 1 km increase in elevation?
- (c) At what elevation is atmospheric pressure 100 torr?

²⁸A torr is a unit of pressure.

²⁹www.eia.doe.gov, accessed August 20, 2019.

³⁰From www.eia.doe.gov, accessed September 16, 2019.

³¹1 quadrillion BTU= 10^{15} BTU.

■ Problems 56–57 concern biodiesel, a fuel derived from renewable resources such as food crops, algae, and animal oils. Table 1.16 shows the percent change over the previous year in US biodiesel consumption.²⁹

Table 1.16

Year	2012	2013	2014	2015	2016	2017	2018
% change over previous year	1.47	58.95	−0.84	5.43	39.56	−4.80	−4.48

56. (a) According to the US Department of Energy, the US consumed 899 million gallons of biodiesel in 2012. Approximately how much biodiesel (in millions of gallons) did the US consume in 2013? In 2014?

(b) Graph the points showing the annual US consumption of biodiesel, in millions of gallons of biodiesel, for the years 2012 to 2018. Label the scales on the horizontal and vertical axes.

57. (a) True or false: The annual US consumption of biodiesel grew exponentially from 2012 to 2018. Justify your answer without doing any calculations.

(b) According to this data, in which year did US consumption have the greatest percentage

- (a) Growth?
- (b) Decline?

58. Hydroelectric power is electric power generated by the force of moving water. Table 1.17 shows the percent change over the previous year in hydroelectric power generated in the US.³⁰

(a) According to the US Department of Energy, the US generated about 2.686 quadrillion³¹ BTUs of hydroelectric power in 2013. Approximately how much hydroelectric power (in quadrillion BTUs) did the US consume in 2014? In 2015?

(b) Graph the points showing the annual US consumption of hydroelectric power, in quadrillion BTUs, for the years 2013 to 2018. Label the scales on the horizontal and vertical axes.

(c) According to this data, when did the largest yearly increase, in quadrillion BTUs, in the US consumption of hydroelectric power occur? What was this increase?

Table 1.17

Year	2013	2014	2015	2016	2017	2018
% change over previous year	−2.87	−3.42	−3.97	7.52	12.14	−2.84

- Problems 59–60 concern wind power, which has been used for centuries to propel ships and mill grain. Modern wind power is obtained from windmills that convert wind energy into electricity. Figure 1.41 shows the annual percent growth in US wind power consumption³² between 2013 and 2018. For example, the 20% growth rate in 2013 indicates there was 20% more wind power consumed in 2013 than in 2012.

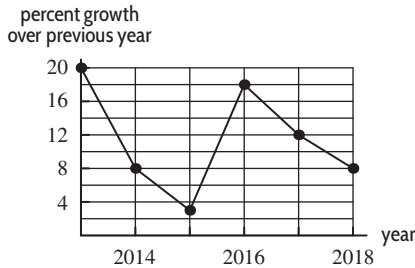


Figure 1.41

59. (a) According to the US Department of Energy, the US consumption of wind power was 1601 trillion BTUs in 2013. How much wind power did the US consume in 2014? In 2015?
- (b) Graph the points showing the annual US consumption of wind power, in trillion BTUs, for the years 2013 to 2018. Label the scales on the horizontal and vertical axes.
- (c) Based on this data, in what year did the largest yearly increase, in trillion BTUs, in the US con-

sumption of wind power occur? What was this increase?

60. (a) According to Figure 1.41, in which years did the US consumption of wind power energy increase by at least 10% over the previous year? Decrease by at least 10%?
- (b) True or false: The US consumption of wind power energy grew by more than 40% from the beginning of 2015 to the end of 2018.
61. (a) The exponential functions in Figure 1.42 have b , d , q positive. Which of the constants a , c , and p must be positive?
- (b) Which of the constants a , b , c , d , p , and q must be between 0 and 1?
- (c) Which two of the constants a , b , c , d , p , and q must be equal?
- (d) What information about the constants a and b does the point $(1, 1)$ provide?

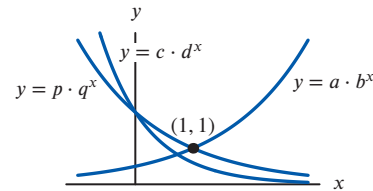


Figure 1.42

Strengthen Your Understanding

- In Problems 62–65, explain what is wrong with the statement.
62. A quantity that doubles daily has an exponential growth rate of 200% per day.
63. The function $y = e^{-0.25x}$ is decreasing and its graph is concave down.
64. The function $y = 2x$ is increasing, and its graph is concave up.
65. The points $(0, 1)$, $(1, e)$, $(2, 2e)$, and $(3, 3e)$ are all on the graph of $y = e^x$.
- In Problems 66–70, give an example of:
66. A decreasing exponential function with a vertical intercept of π .
67. A formula representing the statement “ q decreases at a constant percent rate, and $q = 2.2$ when $t = 0$.”
68. A function that is increasing at a constant percent rate and that has the same vertical intercept as $f(x) = 0.3x + 2$.
69. A function with a horizontal asymptote at $y = -5$ and range $y > -5$.
70. An exponential function that grows slower than $y = e^x$ for $x > 0$.
- Are the statements in Problems 71–78 true or false? Give an explanation for your answer.
71. The function $f(x) = e^{2x}/(2e^{5x})$ is exponential.
72. The function $y = 2 + 3e^{-t}$ has a y -intercept of $y = 3$.
73. The function $y = 5 - 3e^{-4t}$ has a horizontal asymptote of $y = 5$.
74. If $y = f(x)$ is an exponential function and if increasing x by 1 increases y by a factor of 5, then increasing x by 2 increases y by a factor of 10.
75. If $y = Ab^x$ and increasing x by 1 increases y by a factor of 3, then increasing x by 2 increases y by a factor of 9.
76. An exponential function can be decreasing.
77. If a and b are positive constants, $b \neq 1$, then $y = a + ab^x$ has a horizontal asymptote.
78. The function $y = 20/(1 + 2e^{-kt})$, with $k > 0$, has a horizontal asymptote at $y = 20$.

³²Yearly values have been joined with line segments to highlight trends in the data. Actual values in between years should not be inferred from the segments. From www.eia.doe.gov, accessed September 18, 2019.

1.3 NEW FUNCTIONS FROM OLD

Shifts and Stretches

The graph of a constant multiple of a given function is easy to visualize: each y -value is stretched or shrunk by that multiple. For example, consider the function $f(x)$ and its multiples $y = 3f(x)$ and $y = -2f(x)$. Their graphs are shown in Figure 1.43. The factor 3 in the function $y = 3f(x)$ stretches each $f(x)$ value by multiplying it by 3; the factor -2 in the function $y = -2f(x)$ stretches $f(x)$ by multiplying by 2 and reflects it across the x -axis. You can think of the multiples of a given function as a family of functions.

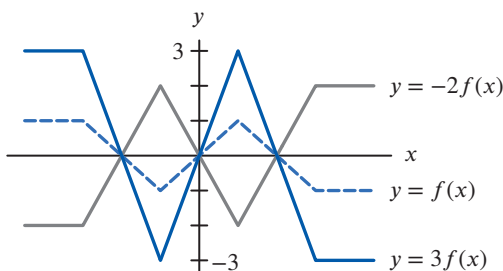


Figure 1.43: Multiples of the function $f(x)$

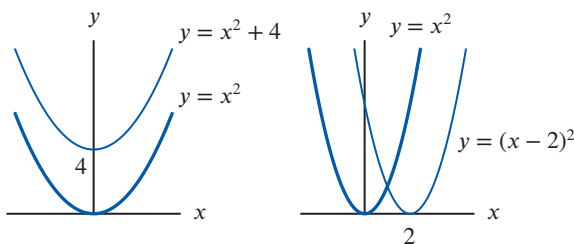


Figure 1.44: Graphs of $y = x^2$ with $y = x^2 + 4$ and $y = (x - 2)^2$

It is also easy to create families of functions by shifting graphs. For example, $y - 4 = x^2$ is the same as $y = x^2 + 4$, which is the graph of $y = x^2$ shifted up by 4. Similarly, $y = (x - 2)^2$ is the graph of $y = x^2$ shifted right by 2. (See Figure 1.44.)

- Multiplying a function by a constant, c , stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$). A negative sign (if $c < 0$) reflects the graph across the x -axis, in addition to shrinking or stretching.
- Replacing y by $(y - k)$ moves a graph up by k (down if k is negative).
- Replacing x by $(x - h)$ moves a graph to the right by h (to the left if h is negative).

Composite Functions

If oil is spilled from a tanker, the area of the oil slick grows with time. Suppose that the oil slick is always a perfect circle. Then the area, A , of the oil slick is a function of its radius, r :

$$A = f(r) = \pi r^2.$$

The radius is also a function of time, because the radius increases as more oil spills. Thus, the area, being a function of the radius, is also a function of time. If, for example, the radius is given by

$$r = g(t) = 1 + t,$$

then the area is given as a function of time by substitution:

$$A = \pi r^2 = \pi(1 + t)^2.$$

We are thinking of A as a *composite function* or a “function of a function,” which is written

$$A = \underbrace{f(g(t))}_{\text{Composite function; } f \text{ is outside function, } g \text{ is inside function}} = \pi(g(t))^2 = \pi(1 + t)^2.$$

Composite function;
 f is outside function,
 g is inside function

To calculate A using the formula $\pi(1 + t)^2$, the first step is to find $1 + t$, and the second step is to square and multiply by π . The first step corresponds to the inside function $g(t) = 1 + t$, and the second step corresponds to the outside function $f(r) = \pi r^2$.

Example 1 If $f(x) = x^2$ and $g(x) = x - 2$, find each of the following:

- (a) $f(g(3))$ (b) $g(f(3))$ (c) $f(g(x))$ (d) $g(f(x))$

Solution

- (a) Since $g(3) = 1$, we have $f(g(3)) = f(1) = 1$.
 (b) Since $f(3) = 9$, we have $g(f(3)) = g(9) = 7$. Notice that $f(g(3)) \neq g(f(3))$.
 (c) $f(g(x)) = f(x - 2) = (x - 2)^2$.
 (d) $g(f(x)) = g(x^2) = x^2 - 2$. Again, notice that $f(g(x)) \neq g(f(x))$.
 Notice that the horizontal shift in Figure 1.44 can be thought of as a composition $f(g(x)) = (x - 2)^2$.

Example 2 Express each of the following functions as a composition:

- (a) $h(t) = (1 + t^3)^{27}$ (b) $k(y) = e^{-y^2}$ (c) $l(y) = -(e^y)^2$

Solution

In each case think about how you would calculate a value of the function. The first stage of the calculation gives you the inside function, and the second stage gives you the outside function.

- (a) For $(1 + t^3)^{27}$, the first stage is cubing and adding 1, so an inside function is $g(t) = 1 + t^3$. The second stage is taking the 27th power, so an outside function is $f(y) = y^{27}$. Then

$$f(g(t)) = f(1 + t^3) = (1 + t^3)^{27}.$$

In fact, there are lots of different answers: $g(t) = t^3$ and $f(y) = (1 + y)^{27}$ is another possibility.

- (b) To calculate e^{-y^2} we square y , take its negative, and then take e to that power. So if $g(y) = -y^2$ and $f(z) = e^z$, then we have

$$f(g(y)) = e^{-y^2}.$$

- (c) To calculate $-(e^y)^2$, we find e^y , square it, and take the negative. Using the same definitions of f and g as in part (b), the composition is

$$g(f(y)) = -(e^y)^2.$$

Since parts (b) and (c) give different answers, we see the order in which functions are composed is important.

Odd and Even Functions: Symmetry

There is a certain symmetry apparent in the graphs of $f(x) = x^2$ and $g(x) = x^3$ in Figure 1.45. For each point (x, x^2) on the graph of f , the point $(-x, x^2)$ is also on the graph; for each point (x, x^3) on the graph of g , the point $(-x, -x^3)$ is also on the graph. The graph of $f(x) = x^2$ is symmetric about the y -axis, whereas the graph of $g(x) = x^3$ is symmetric about the origin. The graph of any polynomial involving only even powers of x has symmetry about the y -axis, while polynomials with only odd powers of x are symmetric about the origin. Consequently, any functions with these symmetry properties are called *even* and *odd*, respectively.

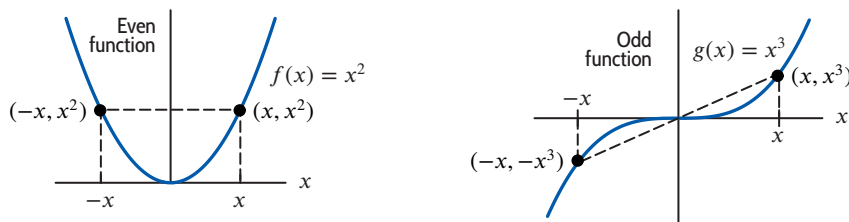


Figure 1.45: Symmetry of even and odd functions

For any function f ,

f is an **even** function if $f(-x) = f(x)$ for all x .

f is an **odd** function if $f(-x) = -f(x)$ for all x .

For example, $g(x) = e^{x^2}$ is even and $h(x) = x^{1/3}$ is odd. However, many functions do not have any symmetry and are neither even nor odd.

Inverse Functions

On August 26, 2005, the runner Kenenisa Bekele³³ of Ethiopia set a still-standing world record for the 10,000-meter race. His times, in seconds, at 2000-meter intervals are recorded in Table 1.18, where $t = f(d)$ is the number of seconds Bekele took to complete the first d meters of the race. For example, Bekele ran the first 4000 meters in 629.98 seconds, so $f(4000) = 629.98$. The function f was useful to athletes planning to compete with Bekele.

Let us now change our point of view and ask for distances rather than times. If we ask how far Bekele ran during the first 629.98 seconds of his race, the answer is clearly 4000 meters. Going backward in this way from numbers of seconds to numbers of meters gives f^{-1} , the *inverse function*³⁴ of f . We write $f^{-1}(629.98) = 4000$. Thus, $f^{-1}(t)$ is the number of meters that Bekele ran during the first t seconds of his race. See Table 1.19, which contains values of f^{-1} .

The independent variable for f is the dependent variable for f^{-1} , and vice versa. The domains and ranges of f and f^{-1} are also interchanged. The domain of f is all distances d such that $0 \leq d \leq 10000$, which is the range of f^{-1} . The range of f is all times t such that $0 \leq t \leq 1577.53$, which is the domain of f^{-1} .

Table 1.18 Bekele's running time

d (meters)	$t = f(d)$ (seconds)
0	0.00
2000	315.63
4000	629.98
6000	944.66
8000	1264.63
10000	1577.53

Table 1.19 Distance run by Bekele

t (seconds)	$d = f^{-1}(t)$ (meters)
0.00	0
315.63	2000
629.98	4000
944.66	6000
1264.63	8000
1577.53	10000

Which Functions Have Inverses?

If a function has an inverse, we say it is *invertible*. Let's look at a function which is not invertible. Consider the flight of the Mercury spacecraft *Freedom 7*, which carried Alan Shepard, Jr. into space in May 1961. Shepard was the first American to journey into space. After launch, his spacecraft rose to an altitude of 116 miles, and then came down into the sea. The function $f(t)$ giving the altitude in miles t minutes after lift-off does not have an inverse. To see why it does not, try to decide on a value for $f^{-1}(100)$, which should be the time when the altitude of the spacecraft was 100 miles. However, there are two such times, one when the spacecraft was ascending and one when it was descending. (See Figure 1.46.)

The reason the altitude function does not have an inverse is that the altitude has the same value for two different times. The reason the Bekele time function did have an inverse is that each running time, t , corresponds to a unique distance, d .

³³<https://www.worldathletics.org/records/by-category/world-records>, accessed January 10, 2020.

³⁴The notation f^{-1} represents the inverse function, which is not the same as the reciprocal, $1/f$.

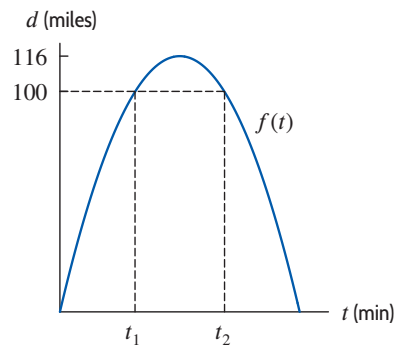


Figure 1.46: Two times, t_1 and t_2 , at which altitude of spacecraft is 100 miles

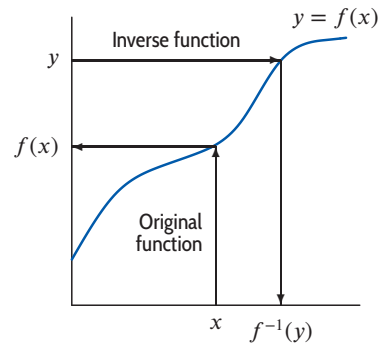


Figure 1.47: A function which has an inverse

Figure 1.47 suggests when an inverse exists. The original function, f , takes us from an x -value to a y -value, as shown in Figure 1.47. Since having an inverse means there is a function going from a y -value to an x -value, the crucial question is whether we can get back. In other words, does each y -value correspond to a unique x -value? If so, there's an inverse; if not, there is not. This principle may be stated geometrically, as follows:

A function has an inverse if (and only if) its graph intersects any horizontal line at most once.

For example, the function $f(x) = x^2$ does not have an inverse because many horizontal lines intersect the parabola twice.

Definition of an Inverse Function

If the function f is invertible, its inverse is defined as follows:

$$f^{-1}(y) = x \quad \text{means} \quad y = f(x).$$

Formulas for Inverse Functions

If a function is defined by a formula, it is sometimes possible to find a formula for the inverse function. In Section 1.1, we looked at the snowy tree cricket, whose chirp rate, C , in chirps per minute, is approximated at the temperature, T , in degrees Fahrenheit, by the formula

$$C = f(T) = 4T - 160.$$

So far we have used this formula to predict the chirp rate from the temperature. But it is also possible to use this formula backward to calculate the temperature from the chirp rate.

Example 3 Find the formula for the function giving temperature in terms of the number of cricket chirps per minute; that is, find the inverse function f^{-1} such that

$$T = f^{-1}(C).$$

Solution Since C is an increasing function, f is invertible. We know $C = 4T - 160$. We solve for T , giving

$$T = \frac{C}{4} + 40,$$

so

$$f^{-1}(C) = \frac{C}{4} + 40.$$

Graphs of Inverse Functions

The function $f(x) = x^3$ is increasing everywhere and so has an inverse. To find the inverse, we solve

$$y = x^3$$

for x , giving

$$x = y^{1/3}.$$

The inverse function is

$$f^{-1}(y) = y^{1/3}$$

or, if we want to call the independent variable x ,

$$f^{-1}(x) = x^{1/3}.$$

The graphs of $y = x^3$ and $y = x^{1/3}$ are shown in Figure 1.48. Notice that these graphs are the reflections of one another across the line $y = x$. For example, $(8, 2)$ is on the graph of $y = x^{1/3}$ because $2 = 8^{1/3}$, and $(2, 8)$ is on the graph of $y = x^3$ because $8 = 2^3$. The points $(8, 2)$ and $(2, 8)$ are reflections of one another across the line $y = x$.

In general, we have the following result.

If the x - and y -axes have the same scales, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

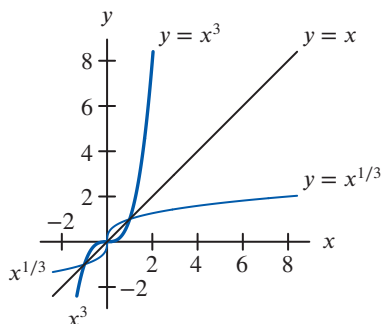


Figure 1.48: Graphs of inverse functions, $y = x^3$ and $y = x^{1/3}$, are reflections across the line $y = x$

Summary for Section 1.3

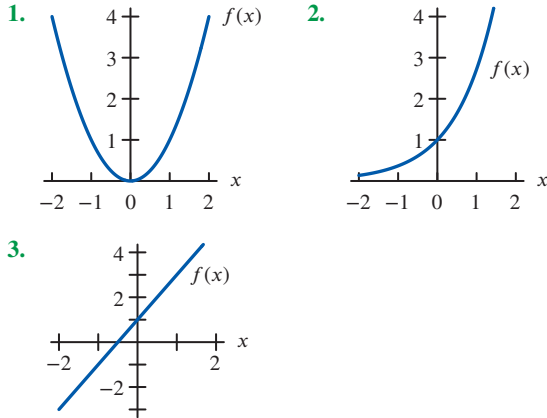
- **Stretches of graphs:** Multiplying a function by a constant, c ,
 - stretches the graph vertically (if $c > 1$) or
 - shrinks the graph vertically (if $0 < c < 1$).
 - A negative sign (if $c < 0$) reflects the graph across the x -axis, in addition to shrinking or stretching.
- **Shifted graphs:**
 - Replacing y by $(y - k)$ moves a graph up by k (down if k is negative).
 - Replacing x by $(x - h)$ moves a graph to the right by h (to the left if h is negative).
- **Composite functions:** The function $f(g(t))$ is a “function of a function,” or a **composite function**, in which there is an **inside function** and an **outside function**.
- **Symmetry:** For any function f ,
 - f is an **even** function if $f(-x) = f(x)$ for all x .
 - f is an **odd** function if $f(-x) = -f(x)$ for all x .
- **Inverse functions:** A function has an inverse if (and only if) its graph intersects any horizontal line at most once.
 - If the function f is invertible, its inverse is defined as follows: $f^{-1}(y) = x$ means $y = f(x)$.
 - If the x - and y -axes have the same scales, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

Exercises and Problems for Section 1.3

EXERCISES

■ For the functions f in Exercises 1–3, graph:

- (a) $f(x + 2)$ (b) $f(x - 1)$ (c) $f(x) - 4$
 (d) $f(x + 1) + 3$ (e) $3f(x)$ (f) $-f(x) + 1$



■ In Exercises 4–7, use Figure 1.49 to graph the functions.

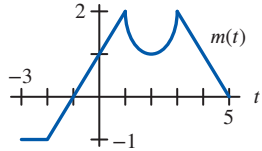


Figure 1.49

4. $n(t) = m(t) + 2$ 5. $p(t) = m(t - 1)$
 6. $k(t) = m(t + 1.5)$
 7. $w(t) = m(t - 0.5) - 2.5$

8. Use Figure 1.50 to graph each of the following. Label any intercepts or asymptotes that can be determined.

- (a) $y = f(x) + 3$ (b) $y = 2f(x)$
 (c) $y = f(x + 4)$ (d) $y = 4 - f(x)$

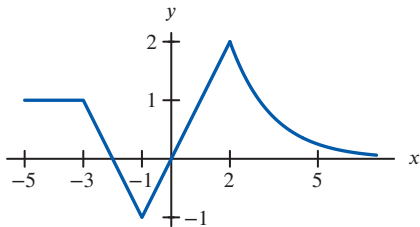


Figure 1.50

■ For the functions f and g in Exercises 9–12, find

- (a) $f(g(1))$ (b) $g(f(1))$ (c) $f(g(x))$
 (d) $g(f(x))$ (e) $f(t)g(t)$

9. $f(x) = x^2, g(x) = x + 1$
 10. $f(x) = \sqrt{x + 4}, g(x) = x^2$
 11. $f(x) = e^x, g(x) = x^2$
 12. $f(x) = 1/x, g(x) = 3x + 4$

■ In Exercises 13–16, find $f(g(x)) - g(f(x))$.

13. $f(x) = x^2, g(x) = \sqrt{x}, x > 0$.
 14. $f(x) = 3x, g(x) = 2x$.
 15. $f(x) = 3x + 1, g(x) = 2x$.
 16. $f(x) = x^2, g(x) = e^x$.

17. If $f(x) = x^2 + 1$, find and simplify:

- (a) $f(t + 1)$ (b) $f(t^2 + 1)$ (c) $f(2)$
 (d) $2f(t)$ (e) $(f(t))^2 + 1$

18. For $g(x) = x^2 + 2x + 3$, find and simplify:

- (a) $g(2 + h)$ (b) $g(2)$
 (c) $g(2 + h) - g(2)$

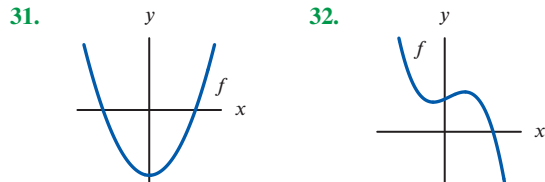
■ Simplify the quantities in Exercises 19–22 using $m(z) = z^2$.

19. $m(z + 1) - m(z)$ 20. $m(z + h) - m(z)$
 21. $m(z) - m(z - h)$ 22. $m(z + h) - m(z - h)$

■ Are the functions in Exercises 23–30 even, odd, or neither?

23. $f(x) = x^6 + x^3 + 1$ 24. $f(x) = x^3 + x^2 + x$
 25. $f(x) = x^4 - x^2 + 3$ 26. $f(x) = x^3 + 1$
 27. $f(x) = 2x$ 28. $f(x) = e^{x^2 - 1}$
 29. $f(x) = x(x^2 - 1)$ 30. $f(x) = e^x - x$

■ For Exercises 31–32, decide if the function $y = f(x)$ is invertible.



■ For Exercises 33–35, use a graph of the function to decide whether or not it is invertible.

33. $f(x) = x^2 + 3x + 2$ 34. $f(x) = x^3 - 5x + 10$
 35. $f(x) = x^3 + 5x + 10$

36. Let p be the price of an item and q be the number of items sold at that price, where $q = f(p)$. What do the following quantities mean in terms of prices and quantities sold?

- (a) $f(25)$ (b) $f^{-1}(30)$

37. Let $C = f(A)$ be the cost, in dollars, of building a store of area A square feet. In terms of cost and square feet, what do the following quantities represent?

- (a) $f(10,000)$ (b) $f^{-1}(20,000)$

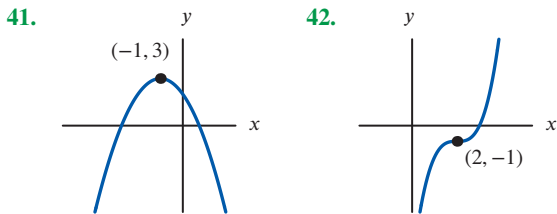
38. Let $f(x)$ be the temperature ($^{\circ}\text{F}$) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(75)$ in practical terms?
39. (a) Write an equation for a graph obtained by vertically stretching the graph of $y = x^2$ by a factor of 2, followed by a vertical upward shift of 1 unit.

- Sketch it.
- (b) What is the equation if the order of the transformations (stretching and shifting) in part (a) is interchanged?
- (c) Are the two graphs the same? Explain the effect of reversing the order of transformations.

PROBLEMS

40. How does the graph of $Q = S(1 - e^{-kt})$ in Example 4 on page 19 relate to the graph of the exponential decay function, $y = Se^{-kt}$?

■ In Problems 41–42 find possible formulas for the graphs using shifts of x^2 or x^3 .



■ In Problems 43–46, use Figure 1.51 to estimate the function value or explain why it cannot be done.

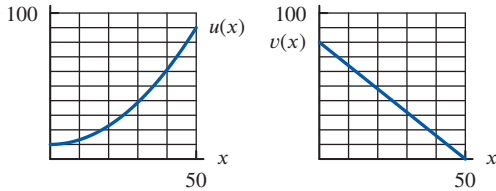


Figure 1.51

43. $u(v(10))$ 44. $u(v(40))$
 45. $v(u(10))$ 46. $v(u(40))$

■ For Problems 47–52, use the graphs in Figure 1.52.

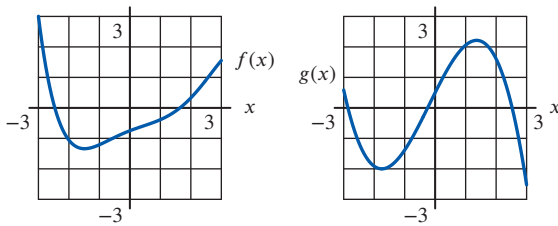


Figure 1.52

47. Estimate $f(g(1))$. 48. Estimate $g(f(2))$.
 49. Estimate $f(f(1))$. 50. Graph $f(g(x))$.
 51. Graph $g(f(x))$. 52. Graph $f(f(x))$.

■ For Problems 53–56, determine functions f and g such that $h(x) = f(g(x))$. (Note: There is more than one correct answer. Do not choose $f(x) = x$ or $g(x) = x$.)

53. $h(x) = (x + 1)^3$ 54. $h(x) = x^3 + 1$
 55. $h(x) = \sqrt{x^2 + 4}$ 56. $h(x) = e^{2x}$

57. A tree of height y meters has, on average, B branches, where $B = y - 1$. Each branch has, on average, n leaves, where $n = 2B^2 - B$. Find the average number of leaves on a tree as a function of height.
58. A spherical balloon is growing with radius $r = 3t + 1$, in centimeters, for time t in seconds. Find the volume of the balloon at 3 seconds.
59. Complete the following table with values for the functions f , g , and h , given that:

- (a) f is an even function.
 (b) g is an odd function.
 (c) h is the composition $h(x) = g(f(x))$.

x	$f(x)$	$g(x)$	$h(x)$
-3	0	0	
-2	2	2	
-1	2	2	
0	0	0	
1			
2			
3			

60. Write a table of values for f^{-1} , where f is as given below. The domain of f is the integers from 1 to 7. State the domain of f^{-1} .

x	1	2	3	4	5	6	7
$f(x)$	3	-7	19	4	178	2	1

61. (a) Use Figure 1.53 to estimate $f^{-1}(2)$.
 (b) Sketch a graph of f^{-1} on the same axes.

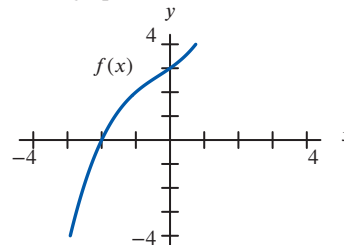


Figure 1.53

- For Problems 62–65, decide if the function f is invertible.
62. $f(d)$ is the total number of gallons of fuel an airplane has used by the end of d minutes of a particular flight.
63. $f(t)$ is the number of customers in Saks Fifth Avenue at t minutes past noon on December 18, 2014.
64. $f(n)$ is the number of students in your calculus class whose birthday is on the n^{th} day of the year.
65. $f(w)$ is the cost of mailing a letter weighing w grams.

■ In Problems 66–70, interpret the expression in terms of carbon footprint, a measure of the environmental impact in kilograms of greenhouse gas (GHG) emissions. Assume that a bottle of drinking water that travels d km from its production source has a carbon footprint $f(d)$ kg of GHGs.³⁵

- 66. $f(150)$
- 67. $f(8700) = 0.25$
- 68. $f^{-1}(1.1)$
- 69. $f(150) + f(0)$
- 70. $\frac{f(1500) - f(150)}{1500 - 150}$

■ In Problems 71–74 the functions $r = f(t)$ and $V = g(r)$ give the radius and the volume of a commercial hot air balloon being inflated for testing. The variable t is in minutes, r is in feet, and V is in cubic feet. The inflation begins at $t = 0$. In each case, give a mathematical expression that represents the given statement.

- 71. The volume of the balloon t minutes after inflation began.
- 72. The volume of the balloon if its radius were twice as big.
- 73. The time that has elapsed when the radius of the balloon is 30 feet.
- 74. The time that has elapsed when the volume of the balloon is 10,000 cubic feet.
- 75. The cost of producing q articles is given by the function $C = f(q) = 100 + 2q$.
 - (a) Find a formula for the inverse function.
 - (b) Explain in practical terms what the inverse function tells you.

- 76. Figure 1.54 shows $f(t)$, the number (in millions) of motor vehicles registered³⁶ in the world in the year t .
 - (a) Is f invertible? Explain.
 - (b) What is the meaning of $f^{-1}(400)$ in practical terms? Evaluate $f^{-1}(400)$.
 - (c) Sketch the graph of f^{-1} .

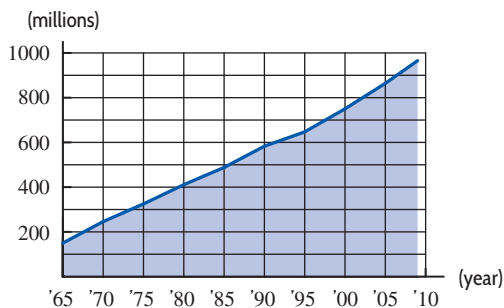


Figure 1.54

- 77. Figure 1.55 is a graph of the function $f(t)$. Here $f(t)$ is the depth in meters below the Atlantic Ocean floor where t million-year-old rock can be found.³⁷

- (a) Evaluate $f(15)$, and say what it means in practical terms.
- (b) Is f invertible? Explain.
- (c) Evaluate $f^{-1}(120)$, and say what it means in practical terms.
- (d) Sketch a graph of f^{-1} .

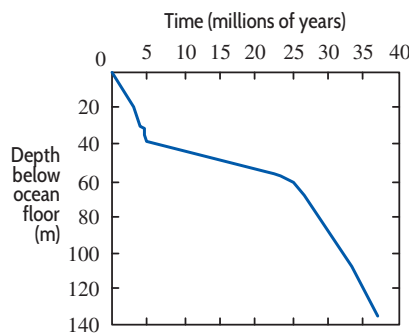


Figure 1.55

- 78. Figure 1.56 shows graphs of four useful functions: the step, the sign, the ramp, and the absolute value. We have

$$\text{step}(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Match the remaining three graphs with the following formulas in terms of the step function.

- (a) $x \text{ step}(x)$
- (b) $\text{step}(x) - \text{step}(-x)$
- (c) $x \text{ step}(x) - x \text{ step}(-x)$

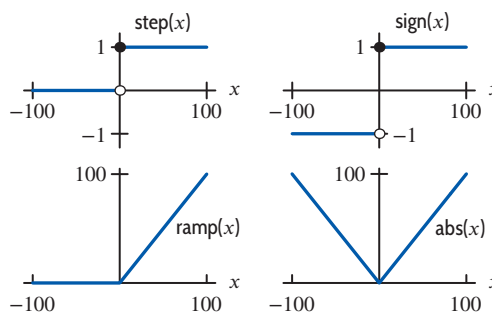


Figure 1.56

³⁵For the data in Problems 67–70, see www.triplepundit.com, accessed March 1, 2015.

³⁶www.earth-policy.org, accessed June 5, 2011. In 2000, about 30% of the registered vehicles were in the US.

³⁷Data of Dr. Murlene Clark based on core samples drilled by the research ship *Glomar Challenger*, taken from *Initial Reports of the Deep Sea Drilling Project*.

Strengthen Your Understanding

- In Problems 79–83, explain what is wrong with the statement.
79. The graph of $f(x) = -(x + 1)^3$ is the graph of $g(x) = -x^3$ shifted right by 1 unit.
80. The functions $f(x) = 3^x$ and $g(x) = x^3$ are inverses of each other.
81. $f(x) = 3x + 5$ and $g(x) = -3x - 5$ are inverse functions of each other.
82. The function $f(x) = e^x$ is its own inverse.
83. The inverse of $f(x) = x$ is $f^{-1}(x) = 1/x$.
- In Problems 84–87, give an example of:
84. An invertible function whose graph contains the point $(0, 3)$.
85. An even function whose graph does not contain the point $(0, 0)$.
86. An increasing function $f(x)$ whose values are greater than those of its inverse function $f^{-1}(x)$ for $x > 0$.
87. Two functions $f(x)$ and $g(x)$ such that moving the graph of f to the left 2 units gives the graph of g and moving the graph of f up 3 also gives the graph of g .
- Are the statements in Problems 88–97 true or false? Give an explanation for your answer.
88. The composition of the exponential functions $f(x) = 2^x$ and $g(x) = 3^x$ is exponential.
89. The graph of $f(x) = 100(10^x)$ is a horizontal shift of the graph of $g(x) = 10^x$.
90. If f is an increasing function, then f^{-1} is an increasing function.
91. If a function is even, then it does not have an inverse.
92. If a function is odd, then it does not have an inverse.
93. The function $f(x) = e^{-x^2}$ is decreasing for all x .
94. If $g(x)$ is an even function then $f(g(x))$ is even for every function $f(x)$.
95. If $f(x)$ is an even function then $f(g(x))$ is even for every function $g(x)$.
96. There is a function which is both even and odd.
97. The composition of two linear functions is linear.
- In Problems 98–101, suppose f is an increasing function and g is a decreasing function. Give an example of functions f and g for which the statement is true, or say why such an example is impossible.
98. $f(x) + g(x)$ is decreasing for all x .
99. $f(x) - g(x)$ is decreasing for all x .
100. $f(x)g(x)$ is decreasing for all x .
101. $f(g(x))$ is increasing for all x .

1.4 LOGARITHMIC FUNCTIONS

In Section 1.2, we approximated the population of Burkina Faso (in millions) by the function

$$P = f(t) = 15.141(1.03)^t,$$

where t is the number of years since 2009. Now suppose that instead of calculating the population at time t , we ask when the population will reach 20 million. We want to find the value of t for which

$$20 = f(t) = 15.141(1.03)^t.$$

We use logarithms to solve for a variable in an exponent.

Logarithms to Base 10 and to Base e

We define the *logarithm* function, $\log_{10} x$, to be the inverse of the exponential function, 10^x , as follows:

The **logarithm** to base 10 of x , written $\log_{10} x$, is the power of 10 we need to get x . In other words,

$$\log_{10} x = c \quad \text{means} \quad 10^c = x.$$

We often write $\log x$ in place of $\log_{10} x$.

The other frequently used base is e . The logarithm to base e is called the *natural logarithm* of x , written $\ln x$ and defined to be the inverse function of e^x , as follows:

The **natural logarithm** of x , written $\ln x$, is the power of e needed to get x . In other words,

$$\ln x = c \quad \text{means} \quad e^c = x.$$

Values of $\log x$ are in Table 1.20. Because no power of 10 gives 0, $\log 0$ is undefined. The graph of $y = \log x$ is shown in Figure 1.57. The domain of $y = \log x$ is positive real numbers; the range is all real numbers. In contrast, the inverse function $y = 10^x$ has domain all real numbers and range all positive real numbers. The graph of $y = \log x$ has a vertical asymptote at $x = 0$, whereas $y = 10^x$ has a horizontal asymptote at $y = 0$.

One big difference between $y = 10^x$ and $y = \log x$ is that the exponential function grows extremely quickly whereas the log function grows extremely slowly. However, $\log x$ does go to infinity, albeit slowly, as x increases. Since $y = \log x$ and $y = 10^x$ are inverse functions, the graphs of the two functions are reflections of one another across the line $y = x$, provided the scales along the x - and y -axes are equal.

Table 1.20 Values for $\log x$ and 10^x

x	$\log x$	x	10^x
0	undefined	0	1
1	0	1	10
2	0.3	2	100
3	0.5	3	10^3
4	0.6	4	10^4
\vdots	\vdots	\vdots	\vdots
10	1	10	10^{10}

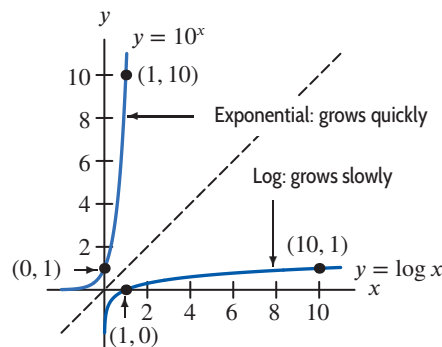


Figure 1.57: Graphs of $\log x$ and 10^x

The graph of $y = \ln x$ in Figure 1.58 has roughly the same shape as the graph of $y = \log x$. The x -intercept is $x = 1$, since $\ln 1 = 0$. The graph of $y = \ln x$ also climbs very slowly as x increases. Both graphs, $y = \log x$ and $y = \ln x$, have *vertical asymptotes* at $x = 0$.

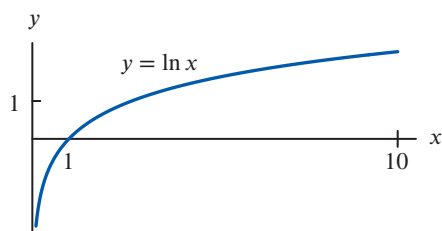


Figure 1.58: Graph of the natural logarithm

The following properties of logarithms may be deduced from the properties of exponents:

Properties of Logarithms

Note that $\log x$ and $\ln x$ are not defined when x is negative or 0.

- | | |
|---|--|
| 1. $\log(AB) = \log A + \log B$ | 1. $\ln(AB) = \ln A + \ln B$ |
| 2. $\log\left(\frac{A}{B}\right) = \log A - \log B$ | 2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$ |
| 3. $\log(A^p) = p \log A$ | 3. $\ln(A^p) = p \ln A$ |
| 4. $\log(10^x) = x$ | 4. $\ln e^x = x$ |
| 5. $10^{\log x} = x$ | 5. $e^{\ln x} = x$ |

In addition, $\log 1 = 0$ because $10^0 = 1$, and $\ln 1 = 0$ because $e^0 = 1$.

Solving Equations Using Logarithms

Logs are frequently useful when we have to solve for unknown exponents, as in the next examples.

Example 1 Find t such that $2^t = 7$.

Solution First, notice that we expect t to be between 2 and 3 (because $2^2 = 4$ and $2^3 = 8$). To calculate t , we take logs to base 10 of both sides. (Natural logs could also be used.)

$$\log(2^t) = \log 7.$$

Then use the third property of logs, which says $\log(2^t) = t \log 2$, and get:

$$t \log 2 = \log 7.$$

Using a calculator to find the logs gives

$$t = \frac{\log 7}{\log 2} \approx 2.81.$$

Example 2 Find when the population of Burkina Faso reaches 20 million by solving $20 = 15.141(1.03)^t$.

Solution Dividing both sides of the equation by 15.141, we get

$$\frac{20}{15.141} = (1.03)^t.$$

Now take logs of both sides:

$$\log\left(\frac{20}{15.141}\right) = \log(1.03^t).$$

Using the fact that $\log(A^t) = t \log A$, we get

$$\log\left(\frac{20}{15.141}\right) = t \log(1.03).$$

Solving this equation using a calculator to find the logs, we get

$$t = \frac{\log(20/15.141)}{\log(1.03)} = 9.42 \text{ years,}$$

which is between $t = 9$ and $t = 10$. This value of t corresponds to the year 2018.

Example 3 Traffic pollution is harmful to school-age children. The concentration of carbon monoxide, CO, in the air near a busy road is a function of distance from the road. The concentration decays exponentially at a continuous rate of 3.3% per meter.³⁸ At what distance from the road is the concentration of CO half what it is on the road?

Solution If C_0 is the concentration of CO on the road, then the concentration x meters from the road is

$$C = C_0 e^{-0.033x}.$$

³⁸Rickwood, P. and Knight, D. (2009). "The Health Impacts of Local Traffic Pollution on Primary School Age Children", *State of Australian Cities 2009 Conference Proceedings*, www.be.unsw.edu.au, accessed March 24, 2016.

We want to find the value of x making $C = C_0/2$, that is,

$$C_0 e^{-0.033x} = \frac{C_0}{2}.$$

Dividing by C_0 and then taking natural logs yields

$$\ln(e^{-0.033x}) = -0.033x = \ln\left(\frac{1}{2}\right) = -0.6931,$$

so

$$x = 21 \text{ meters.}$$

At 21 meters from the road the concentration of CO in the air is half the concentration on the road.

In Example 3 the decay rate was given. However, in many situations where we expect to find exponential growth or decay, the rate is not given. To find it, we must know the quantity at two different times and then solve for the growth or decay rate, as in the next example.

Example 4 The population of Mexico was 100.3 million in 2000 and 124.6 million in 2017.³⁹ Assuming it increases exponentially, find a formula for the population of Mexico as a function of time.

Solution If we measure the population, P , in millions and time, t , in years since 2000, we can say

$$P = P_0 e^{kt} = 100.3e^{kt},$$

where $P_0 = 100.3$ is the initial value of P . We find k by using the fact that $P = 124.6$ when $t = 17$, so

$$124.6 = 100.3e^{k \cdot 17}.$$

To find k , we divide both sides by 100.3, giving

$$\frac{124.6}{100.3} = 1.24227 = e^{17k}.$$

Now take natural logs of both sides:

$$\ln(1.24227) = \ln(e^{17k}).$$

Using a calculator and the fact that $\ln(e^{17k}) = 17k$, this becomes

$$0.2169 = 17k.$$

So

$$k = 0.013,$$

and therefore

$$P = 100.3e^{0.013t}.$$

Since $k = 0.013 = 1.3\%$, the population of Mexico was growing at a continuous rate of 1.3% per year.

In Example 4 we chose to use e for the base of the exponential function representing Mexico's population, making clear that the continuous growth rate was 1.3%. If we had wanted to emphasize the annual growth rate, we could have expressed the exponential function in the form $P = P_0 a^t$.

³⁹www.indexmundi.com, accessed September 10, 2019.

Example 5 Give a formula for the inverse of the following function (that is, solve for t in terms of P):

$$P = f(t) = 15.141(1.03)^t.$$

Solution We want a formula expressing t as a function of P . Take logs:

$$\log P = \log(15.141(1.03)^t).$$

Since $\log(AB) = \log A + \log B$, we have

$$\log P = \log 15.141 + \log((1.03)^t).$$

Now use $\log(A^t) = t \log A$:

$$\log P = \log 15.141 + t \log 1.03.$$

Solve for t in two steps, using a calculator at the final stage:

$$t \log 1.03 = \log P - \log 15.141$$

$$t = \frac{\log P}{\log 1.03} - \frac{\log 15.141}{\log 1.03} = 77.8985 \log P - 91.9322.$$

Thus,

$$f^{-1}(P) = 77.8985 \log P - 91.9322.$$

Note that

$$f^{-1}(20) = 77.8985(\log 20) - 91.9322 = 9.42,$$

which agrees with the result of Example 2.

Summary for Section 1.4

- The **logarithm** to base 10 of x , written $\log_{10} x$, is the power of 10 we need to get x . In other words,

$$\log_{10} x = c \quad \text{means} \quad 10^c = x.$$

We often write $\log x$ in place of $\log_{10} x$.

- The **natural logarithm** of x , written $\ln x$, is the power of e needed to get x . In other words,

$$\ln x = c \quad \text{means} \quad e^c = x.$$

- **Properties of logarithms:**

- | | |
|--|---|
| • $\log(AB) = \log A + \log B$ | • $\ln(AB) = \ln A + \ln B$ |
| • $\log\left(\frac{A}{B}\right) = \log A - \log B$ | • $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$ |
| • $\log(A^p) = p \log A$ | • $\ln(A^p) = p \ln A$ |
| • $\log(10^x) = x$ | • $\ln e^x = x$ |
| • $10^{\log x} = x$ | • $e^{\ln x} = x$ |

Exercises and Problems for Section 1.4

EXERCISES

■ In Exercises 1–6, simplify the expression completely.

1. $e^{\ln(1/2)}$

2. $10^{\log(AB)}$

3. $5e^{\ln(A^2)}$

4. $\ln(e^{2AB})$

5. $\ln(1/e) + \ln(AB)$

6. $2 \ln(e^A) + 3 \ln B^e$

■ For Exercises 7–18, solve for x using logs.

7. $3^x = 11$

8. $17^x = 2$

9. $20 = 50(1.04)^x$

10. $4 \cdot 3^x = 7 \cdot 5^x$

11. $7 = 5e^{0.2x}$

12. $2^x = e^{x+1}$

13. $50 = 600e^{-0.4x}$ 14. $2e^{3x} = 4e^{5x}$
 15. $7^{x+2} = e^{17x}$ 16. $10^{x+3} = 5e^{7-x}$
 17. $2x - 1 = e^{\ln x^2}$ 18. $4e^{2x-3} - 5 = e$

■ For Exercises 19–24, solve for t . Assume a and b are positive, a and $b \neq 1$, and k is nonzero.

19. $a = b^t$ 20. $P = P_0 a^t$
 21. $Q = Q_0 a^{nt}$ 22. $P_0 a^t = Q_0 b^t$
 23. $a = be^t$ 24. $P = P_0 e^{kt}$

PROBLEMS

32. The exponential function $y(x) = Ce^{ax}$ satisfies the conditions $y(0) = 2$ and $y(1) = 1$. Find the constants C and a . What is $y(2)$?

■ For Problems 33–34, find k such that $p = p_0 e^{kt}$ has the given doubling time.

33. 10 34. 0.4

35. Write $f(t) = \ln t + 2.3$ as the natural log of one quantity (that is, as $\ln(\underline{\hspace{2cm}})$).

36. A culture of bacteria originally numbers 500. After 2 hours there are 1500 bacteria in the culture. Assuming exponential growth, how many are there after 6 hours?

37. One hundred kilograms of a radioactive substance decay to 40 kg in 10 years. How much remains after 20 years?

38. The population of the US was 282.2 million in 2000 and 327.2 million in 2018.⁴⁰ Assuming exponential growth,

- (a) In what year is the population expected to go over 350 million?
 (b) What population is predicted for the 2020 census?

39. The population of a region is growing exponentially. There were 40,000,000 people in 2005 ($t = 0$) and 48,000,000 in 2015. Find an expression for the population at any time t , in years. What population would you predict for the year 2020? What is the doubling time?

40. Oil consumption in China grew exponentially⁴¹ from 11.986 million barrels per day in 2015 to 13.525 million barrels per day in 2018. Assuming exponential growth continues at the same rate, what will oil consumption be in 2025?

41. The concentration of the car exhaust fume nitrous oxide, NO_2 , in the air near a busy road is a function of distance from the road. The concentration decays exponentially at a continuous rate of 2.54% per meter.⁴² At what distance from the road is the concentration of NO_2 half what it is on the road?

⁴⁰data.worldbank.org, accessed September 10, 2019.

⁴¹Based on www.ceicdata.com, accessed September, 2019.

⁴²P. Rickwood, and D. Knight, “The Health Impacts of Local Traffic Pollution on Primary School Age Children”, *State of Australian Cities 2009 Conference Proceedings*, www.be.unsw.edu.au, accessed September 12, 2019.

⁴³R. D. Brook, B. Franklin, W. Cascio, Y. Hong, G. Howard, M. Lipsett, R. Luepker, M. Mittleman, J. Samet, and S. C. Smith, “Air Pollution and Cardiovascular Disease.” *Circulation*, 2004;109:2655–2671.

⁴⁴www.eia.gov, accessed September 18, 2019.

■ In Exercises 25–28, put the functions in the form $P = P_0 e^{kt}$.

25. $P = 15(1.5)^t$ 26. $P = 10(1.7)^t$
 27. $P = 174(0.9)^t$ 28. $P = 4(0.55)^t$

■ Find the inverse function in Exercises 29–31.

29. $p(t) = (1.04)^t$ 30. $f(t) = 50e^{0.1t}$
 31. $f(t) = 1 + \ln t$

42. For children and adults with diseases such as asthma, the number of respiratory deaths per year increases by 0.33% when pollution particles increase by a microgram per cubic meter of air.⁴³

- (a) Write a formula for the number of respiratory deaths per year as a function of quantity of pollution in the air. (Let Q_0 be the number of deaths per year with no pollution.)
 (b) What quantity of air pollution results in twice as many respiratory deaths per year as there would be without pollution?

43. The number of alternative fuel vehicles⁴⁴ running on E85, a fuel that is up to 85% plant-derived ethanol, increased exponentially in the US between 2012 and 2017.

- (a) By what percent did the number of E85-powered vehicles grow from 2012 to 2017?
 (b) Assuming exponential growth of the form Ae^{rt} , where t is years since 2012, find r and A , and fill in the missing table values.
 (c) Use this exponential model to estimate the number of E85-powered vehicles in the US in 2011.

Year	2012	2013	2014	2015	2016	2017
E85 vehicles	302,341	?	?	?	?	393,553

44. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

- (a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.
 (b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.
 (c) Use logarithms to find the half-life of caffeine.

45. Persistent organic pollutants (POPs) are a serious environmental hazard. Figure 1.59 shows their natural decay over time in human fat.⁴⁵
- How long does it take for the concentration to decrease from 100 units to 50 units?
 - How long does it take for the concentration to decrease from 50 units to 25 units?
 - Explain why your answers to parts (a) and (b) suggest that the decay may be exponential.
 - Find an exponential function that models concentration, C , as a function of t , the number of years since 1970.

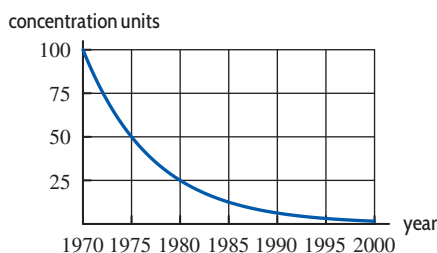


Figure 1.59

46. At time t hours after taking the cough suppressant hydrocodone bitartrate, the amount, A , in mg, remaining in the body is given by $A = 10(0.82)^t$.
- What was the initial amount taken?
 - What percent of the drug leaves the body each hour?
 - How much of the drug is left in the body 6 hours after the dose is administered?
 - How long is it until only 1 mg of the drug remains in the body?
47. Why does snowfall in the northeastern US seem to be increasing, in spite of global warming? The explanation lies in the fact that warm air can hold more moisture than cold air; a 1°F increase in temperature enables air to hold roughly 4% more moisture.⁴⁶ Since the oceans are warming, the air above them contains more moisture and dumps more snow or rain when over land.
- Find a model expressing the quantity, Q , of moisture that can be held in air as a function of temperature, T , in $^\circ\text{F}$. Let Q_0 be the quantity at 32°F .
 - Compare the quantities of moisture that air can hold at 14°F and 50°F by computing their proportions.
 - What temperature increase doubles the amount of water that can be held?

48. Different isotopes (versions) of the same element can have very different half-lives. With t in years, the decay of plutonium-240 is described by the formula

$$Q = Q_0 e^{-0.00011t},$$

whereas the decay of plutonium-242 is described by

$$Q = Q_0 e^{-0.000018t}.$$

Find the half-lives of plutonium-240 and plutonium-242.

49. The size of an exponentially growing bacteria colony doubles in 5 hours. How long will it take for the number of bacteria to triple?
50. Air pressure, P , decreases exponentially with height, h , above sea level. If P_0 is the air pressure at sea level and h is in meters, then

$$P = P_0 e^{-0.00012h}.$$

- At the top of Denali, height 6194 meters (about 20,320 feet), what is the air pressure, as a percent of the pressure at sea level?
 - The maximum cruising altitude of an ordinary commercial jet is around 12,000 meters (about 39,000 feet). At that height, what is the air pressure, as a percent of the sea level value?
51. With time, t , in years since the start of 1980, textbook prices have increased at 6.7% per year while inflation has been 3.3% per year.⁴⁷ Assume both rates are continuous growth rates.
- Find a formula for $B(t)$, the price of a textbook in year t if it cost $\$B_0$ in 1980.
 - Find a formula for $P(t)$, the price of an item in year t if it cost $\$P_0$ in 1980 and its price rose according to inflation.
 - A textbook cost $\$50$ in 1980. When is its price predicted to be double the price that would have resulted from inflation alone?
52. In November 2010, a “tiger summit” was held in St. Petersburg, Russia.⁴⁸ In 1900, there were 100,000 wild tigers worldwide; in 2010 the number was 3200.
- Assuming the tiger population has decreased exponentially, find a formula for $f(t)$, the number of wild tigers t years since 1900.
 - Between 2000 and 2010, the number of wild tigers decreased by 40%. Is this percentage larger or smaller than the decrease in the tiger population predicted by your answer to part (a)?

⁴⁵K.C. Jones, P. de Voegt, “Persistent Organic Pollutants (POPs): State of the Science,” *Environmental Pollution* 100, 1999, pp. 209–221.

⁴⁶<http://theconversation.com/does-global-warming-mean-more-or-less-snow-36936>, accessed September 2015.

⁴⁷Data from “Textbooks Healed for Ash Heap of History,” <http://educationtechnews.com>, Vol 5, 2010.

⁴⁸“Tigers Would be Extinct in Russia if Unprotected,” Yahoo! News, Nov. 21, 2010.

53. In 2017, the populations of China and India were approximately 1.379 and 1.282 billion people,⁴⁹ respectively. However, the annual population growth rate of China was 0.41% while the population of India was growing by 1.17% each year. If these growth rates remain constant, when will the population of India exceed that of China?
54. The revenue of Apple® went from \$54.5 billion in 2013 to \$74.6 billion⁵⁰ in 2015. Find an exponential function to model the revenue as a function of years since 2013. What is the continuous percent growth rate, per year, of revenue?
55. The world population was 6.9 billion at the end of 2010 and is predicted to reach 9 billion by the end of 2050.⁵¹
- Assuming the population is growing exponentially, what is the continuous growth rate per year?
 - The United Nations celebrated the “Day of 6 Billion” on October 12, 1999, and “Day of 7 Billion” on October 31, 2011. Using the growth rate in part (a), when is the “Day of 8 Billion” predicted to be?
56. In the early 1920s, Germany had tremendously high inflation, called hyperinflation. Photographs of the time show people going to the store with wheelbarrows full of money. If a loaf of bread cost 1/4 marks in 1919 and 2,400,000 marks in 1922, what was the average yearly inflation rate between 1919 and 1922?
57. In 2010, there were about 246 million vehicles (cars and trucks) and about 308.7 million people in the US.⁵² The number of vehicles grew 15.5% over the previous decade, while the population has been growing at 9.7% per decade. If the growth rates remain constant, when will there be, on average, one vehicle per person?
58. Tiny marine organisms reproduce at different rates. Phytoplankton doubles in population twice a day, but foraminifera doubles every five days. If the two populations are initially the same size and grow exponentially, how long does it take for
- The phytoplankton population to be double the foraminifera population.
 - The phytoplankton population to be 1000 times the foraminifera population.
59. A picture supposedly painted by Vermeer (1632–1675) contains 99.5% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.
60. Cyanide is used in solution to isolate gold in a mine.⁵³ This may result in contaminated groundwater near the mine, which needs to be filtered. Table 1.21 gives the

concentration, $c(t)$ (in parts per million), of cyanide in the groundwater, where t is in years since 2017.

- Find an exponential model for $c(t)$.
- Use the model in part (a) to find the number of years it takes for the cyanide concentration to fall to 10 ppm.
- If the filtering process removing the cyanide is sped up so that the new model is $D(t) = c(2t)$, find $D(t)$.
- If the cyanide removal was started three years earlier, but run at the speed of part (a), find a new model, $E(t)$.

Table 1.21

t (years)	0	1	2
$c(t)$ (ppm)	25.0	21.8	19.01

61. In 2015, Nepal had two massive earthquakes, the first measuring 7.8 on the Richter scale and the second measuring 7.3. The Richter scale compares the strength, W , of the seismic waves of an earthquake with the strength, W_0 , of the waves of a standard earthquake giving a Richter rating of $R = \log(W/W_0)$. By what factor were the seismic waves in Nepal’s first earthquake stronger than the seismic waves in
- A standard earthquake?
 - The second Nepal earthquake?
62. Find the equation of the line l in Figure 1.60.

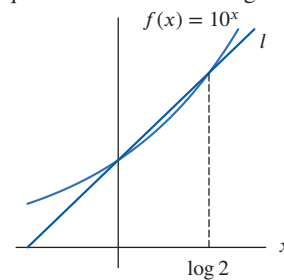


Figure 1.60

63. Figure 1.61 shows $f(x) = \log_e x$ and $g(x) = \log_{10} x$.
- What is the x -intercept of the graphs?
 - Which graph is which? (Answer both for x greater and for x smaller than the x -intercept.)

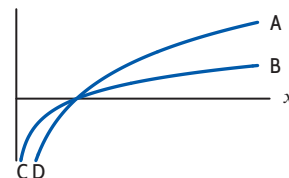


Figure 1.61

⁴⁹www.indexmundi.com, accessed September 12, 2019.

⁵⁰techcrunch.com, accessed April 2, 2015.

⁵¹“Reviewing the Bidding on the Climate Files”, in About Dot Earth, *New York Times*, Nov. 19, 2010.

⁵²http://www.autoblog.com/2010/01/04/report-number-of-cars-in-the-u-s-dropped-by-four-million-in-20/ and http://2010.census.gov/news/releases/operations/cb10-cn93.html, accessed February 2012.

⁵³www.waterboards.ca.gov, accessed September 16, 2019.

64. Without a calculator or computer, match the functions e^x , $\ln x$, x^2 , and $x^{1/2}$ to their graphs in Figure 1.62.

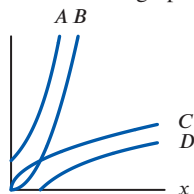


Figure 1.62

65. A quantity, Q , grows exponentially, $Q = Q_0 e^{kt}$. If you double k , does the doubling time double? If not, what change takes place?
66. Is there a difference between $\ln(\ln(x))$ and $\ln^2(x)$? (Note: $\ln^2(x)$ is another way of writing $(\ln x)^2$.)
67. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on
- (a) The y -intercept? (b) The x -intercept?

68. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on the vertical asymptote?

69. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on

- (a) The y -intercept? (b) The x -intercept?

70. If $f(x) = a \ln(x + 2)$, what is the effect of increasing a on the vertical asymptote?

71. If $g(x) = \ln(ax + 2)$, where $a \neq 0$, what is the effect of increasing a on the vertical asymptote?

72. Show that the growth rate k of the exponential function $f(t) = P_0 e^{kt}$, with $P_0 > 0$, can be computed from two values of f by a difference quotient of the form:

$$k = \frac{\ln f(t_2) - \ln f(t_1)}{t_2 - t_1}.$$

Strengthen Your Understanding

- In Problems 73–78, explain what is wrong with the statement.

73. The function $-\log|x|$ is odd.

74. For all $x > 0$, the value of $\ln(100x)$ is 100 times larger than $\ln x$.

75. $\ln x > \log x$.

76. $\ln(A + B) = (\ln A)(\ln B)$.

77. $\ln(A + B) = \ln A + \ln B$.

78. $\frac{1}{\ln x} = e^x$

- In Problems 79–81, give an example of:

79. A function that grows slower than $y = \log x$ for $x > 1$.

80. A function $f(x)$ such that $\ln(f(x))$ is only defined for $x < 0$.

81. A function with a vertical asymptote at $x = 3$ and defined only for $x > 3$.

- Are the statements in Problems 82–85 true or false? Give an explanation for your answer.

82. The graph of $f(x) = \ln x$ is concave down.

83. The graph of $g(x) = \log(x - 1)$ crosses the x -axis at $x = 1$.

84. The inverse function of $y = \log x$ is $y = 1/\log x$.

85. If a and b are positive constants, then $y = \ln(ax + b)$ has no vertical asymptote.

1.5 TRIGONOMETRIC FUNCTIONS

Trigonometry originated as part of the study of triangles. The name *tri-gon-o-metry* means the measurement of three-cornered figures, and the first definitions of the trigonometric functions were in terms of triangles. However, the trigonometric functions can also be defined using the unit circle, a definition that makes them periodic, or repeating. Many naturally occurring processes are also periodic. The water level in a tidal basin, the blood pressure in a heart, an alternating current, and the position of the air molecules transmitting a musical note all fluctuate regularly. Such phenomena can be represented by trigonometric functions.

Radians

There are two commonly used ways to represent the input of the trigonometric functions: radians and degrees. The formulas of calculus, as you will see, are neater in radians than in degrees.

An angle of 1 **radian** is defined to be the angle at the center of a unit circle which cuts off an arc of length 1, measured counterclockwise. (See Figure 1.63(a).) A unit circle has radius 1.

An angle of 2 radians cuts off an arc of length 2 on a unit circle. A negative angle, such as $-1/2$ radians, cuts off an arc of length $1/2$, but measured clockwise. (See Figure 1.63(b).)

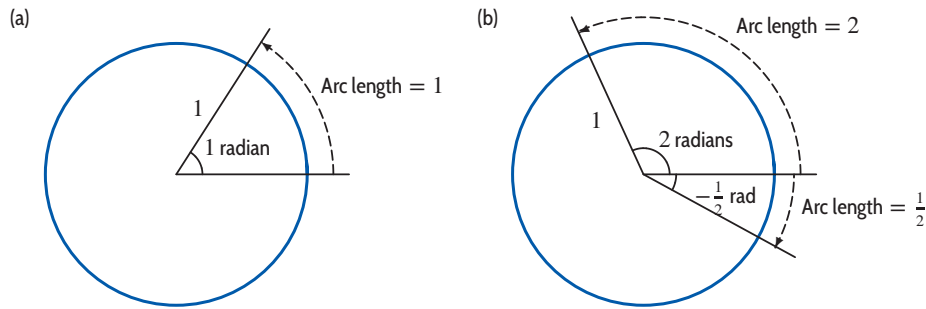


Figure 1.63: Radians defined using unit circle

It is useful to think of angles as rotations, since then we can make sense of angles larger than 360° ; for example, an angle of 720° represents two complete rotations counterclockwise. Since one full rotation of 360° cuts off an arc of length 2π , the circumference of the unit circle, it follows that

$$360^\circ = 2\pi \text{ radians, so } 180^\circ = \pi \text{ radians.}$$

In other words, $1 \text{ radian} = 180^\circ/\pi$, so one radian is about 60° . The word radians is often dropped, so if an angle or rotation is referred to without units, it is understood to be in radians.

Radians are useful for computing the length of an arc in any circle. If the circle has radius r and the arc cuts off an angle θ , as in Figure 1.64, then we have the following relation:

$$\text{Arc length} = s = r\theta.$$

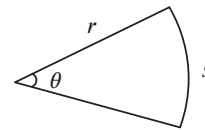


Figure 1.64: Arc length of a sector of a circle

The Sine and Cosine Functions

The two basic trigonometric functions—the sine and cosine—are defined using a unit circle. In Figure 1.65, an angle of t radians is measured counterclockwise around the circle from the point $(1, 0)$. If P has coordinates (x, y) , we define

$$\cos t = x \quad \text{and} \quad \sin t = y.$$

We assume that the angles are *always* in radians unless specified otherwise.

Since the equation of the unit circle is $x^2 + y^2 = 1$, writing $\cos^2 t$ for $(\cos t)^2$, we have the identity

$$\cos^2 t + \sin^2 t = 1.$$

As t increases and P moves around the circle, the values of $\sin t$ and $\cos t$ oscillate between 1 and -1 , and eventually repeat as P moves through points where it has been before. If t is negative, the angle is measured clockwise around the circle.

Amplitude, Period, and Phase

The graphs of sine and cosine are shown in Figure 1.66. Notice that sine is an odd function, and cosine is even. The maximum and minimum values of sine and cosine are $+1$ and -1 , because those are the maximum and minimum values of y and x on the unit circle. After the point P has moved around the complete circle once, the values of $\cos t$ and $\sin t$ start to repeat; we say the functions are *periodic*.

For any periodic function of time, the

- **Amplitude** is half the distance between the maximum and minimum values (if it exists).
- **Period** is the smallest time needed for the function to execute one complete cycle.

The amplitude of $\cos t$ and $\sin t$ is 1, and the period is 2π . Why 2π ? Because that's the value of t when the point P has gone exactly once around the circle. (Remember that $360^\circ = 2\pi$ radians.)

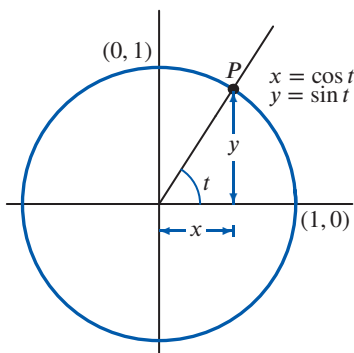


Figure 1.65: The definitions of $\sin t$ and $\cos t$

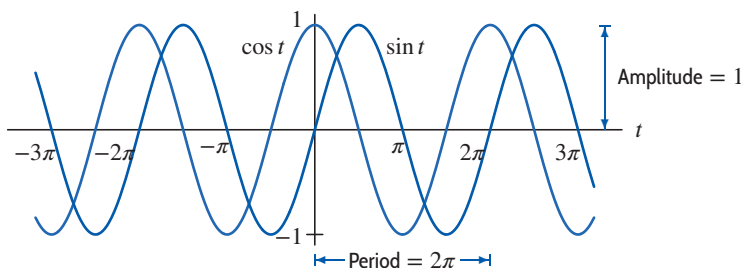


Figure 1.66: Graphs of $\cos t$ and $\sin t$

In Figure 1.66, we see that the sine and cosine graphs are exactly the same shape, only shifted horizontally. Since the cosine graph is the sine graph shifted $\pi/2$ to the left,

$$\cos t = \sin(t + \pi/2).$$

Equivalently, the sine graph is the cosine graph shifted $\pi/2$ to the right, so

$$\sin t = \cos(t - \pi/2).$$

We say that the *phase difference* or *phase shift* between $\sin t$ and $\cos t$ is $\pi/2$.

Functions whose graphs are the shape of a sine or cosine curve are called *sinusoidal* functions.

To describe arbitrary amplitudes and periods of sinusoidal functions, we use functions of the form

$$f(t) = A \sin(Bt) \quad \text{and} \quad g(t) = A \cos(Bt),$$

where $|A|$ is the amplitude and $2\pi/|B|$ is the period.

The graph of a sinusoidal function is shifted horizontally by a distance $|h|$ when t is replaced by $t - h$ or $t + h$.

Functions of the form $f(t) = A \sin(Bt) + C$ and $g(t) = A \cos(Bt) + C$ have graphs which are shifted vertically by C and oscillate about this value.

Example 1

Find and show on a graph the amplitude and period of the functions

(a) $y = 5 \sin(2t)$

(b) $y = -5 \sin\left(\frac{t}{2}\right)$

(c) $y = 1 + 2 \sin t$

Solution

(a) From Figure 1.67, you can see that the amplitude of $y = 5 \sin(2t)$ is 5 because the factor of 5 stretches the oscillations up to 5 and down to -5 . The period of $y = \sin(2t)$ is π , because when t changes from 0 to π , the quantity $2t$ changes from 0 to 2π , so the sine function goes through one complete oscillation.

(b) Figure 1.68 shows that the amplitude of $y = -5 \sin(t/2)$ is again 5, because the negative sign

reflects the oscillations in the t -axis, but does not change how far up or down they go. The period of $y = -5 \sin(t/2)$ is 4π because when t changes from 0 to 4π , the quantity $t/2$ changes from 0 to 2π , so the sine function goes through one complete oscillation.

- (c) The 1 shifts the graph $y = 2 \sin t$ up by 1. Since $y = 2 \sin t$ has an amplitude of 2 and a period of 2π , the graph of $y = 1 + 2 \sin t$ goes up to 3 and down to -1 , and has a period of 2π . (See Figure 1.69.) Thus, $y = 1 + 2 \sin t$ also has amplitude 2.

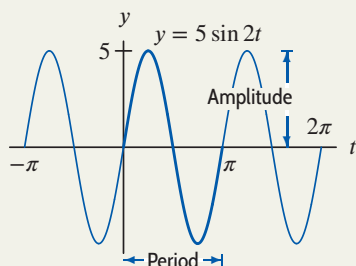


Figure 1.67: Amplitude = 5, period = π

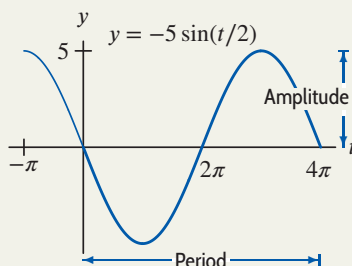


Figure 1.68: Amplitude = 5, period = 4π

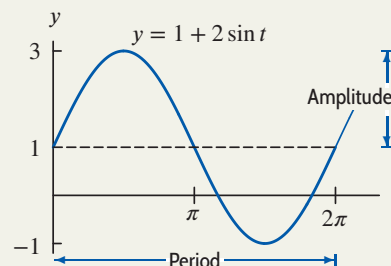
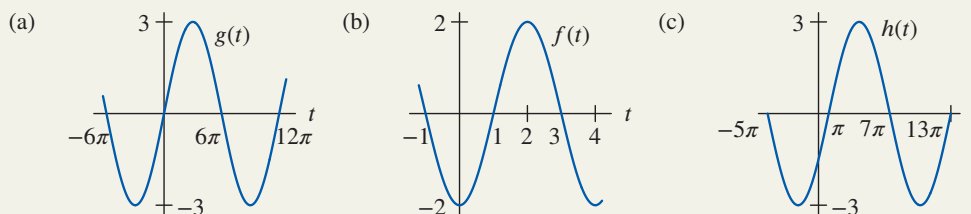


Figure 1.69: Amplitude = 2, period = 2π

Example 2 Find possible formulas for the following sinusoidal functions.



Solution

- (a) This function looks like a sine function with amplitude 3, so $g(t) = 3 \sin(Bt)$. Since the function executes one full oscillation between $t = 0$ and $t = 12\pi$, when t changes by 12π , the quantity Bt changes by 2π . This means $B \cdot 12\pi = 2\pi$, so $B = 1/6$. Therefore, $g(t) = 3 \sin(t/6)$ has the graph shown.
- (b) This function looks like an upside-down cosine function with amplitude 2, so $f(t) = -2 \cos(Bt)$. The function completes one oscillation between $t = 0$ and $t = 4$. Thus, when t changes by 4, the quantity Bt changes by 2π , so $B \cdot 4 = 2\pi$, or $B = \pi/2$. Therefore, $f(t) = -2 \cos(\pi t/2)$ has the graph shown.
- (c) This function looks like the function $g(t)$ in part (a), but shifted a distance of π to the right. Since $g(t) = 3 \sin(t/6)$, we replace t by $(t - \pi)$ to obtain $h(t) = 3 \sin[(t - \pi)/6]$.

Example 3 On October 14, 2019, high tide in Boston was at midnight. The water level at high tide was 9.8 feet; later, at low tide, it was 0.6 feet. Assuming the next high tide is at 12 noon and that the height of the water is given by a sine or cosine curve, find a formula for the water level in Boston as a function of time.

Solution Let y be the water level in feet, and let t be the time measured in hours from midnight. The oscillations have amplitude 4.6 feet $(= (9.8 - 0.6)/2)$ and period 12, so $12B = 2\pi$ and $B = \pi/6$. Since the water is highest at midnight, when $t = 0$, the oscillations are best represented by a cosine function. (See Figure 1.70.) We can say

$$\text{Height above average} = 4.6 \cos\left(\frac{\pi}{6} t\right).$$

Since the average water level was 5.2 feet ($= (9.8 + 0.6)/2$), we shift the cosine up by adding 5.2:

$$y = 5.2 + 4.6 \cos\left(\frac{\pi}{6}t\right).$$

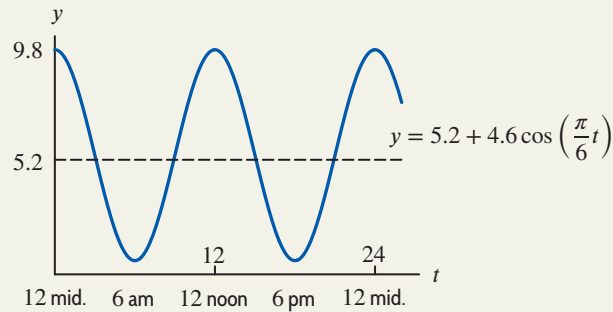


Figure 1.70: Function approximating the tide in Boston on October 14, 2019

Example 4 Of course, there's something wrong with the assumption in Example 3 that the next high tide is at noon. If so, the high tide would always be at noon or midnight, instead of progressing slowly through the day, as in fact it does. The interval between successive high tides actually averages about 12 hours 24 minutes. Using this, give a more accurate formula for the height of the water as a function of time.

Solution The period is 12 hours 24 minutes $= 12.4$ hours, so $B = 2\pi/12.4$, giving

$$y = 5.2 + 4.6 \cos\left(\frac{2\pi}{12.4}t\right) = 5.2 + 4.6 \cos(0.507t).$$

Example 5 Use the information from Example 4 to write a formula for the water level in Boston on a day when the high tide is at 2 pm.

Solution When the high tide is at midnight,

$$y = 5.2 + 4.6 \cos(0.507t).$$

Since 2 pm is 14 hours after midnight, we replace t by $(t - 14)$. Therefore, on a day when the high tide is at 2 pm,

$$y = 5.2 + 4.6 \cos(0.507(t - 14)).$$

The Tangent Function

If t is any number with $\cos t \neq 0$, we define the tangent function as follows:

$$\tan t = \frac{\sin t}{\cos t}.$$

Figure 1.65 on page 44 shows the geometrical meaning of the tangent function: $\tan t$ is the slope of the line through the origin $(0, 0)$ and the point $P = (\cos t, \sin t)$ on the unit circle.

The tangent function is undefined wherever $\cos t = 0$, namely, at $t = \pm\pi/2, \pm3\pi/2, \dots$, and it has a vertical asymptote at each of these points. The function $\tan t$ is positive where $\sin t$ and $\cos t$ have the same sign. The graph of the tangent is shown in Figure 1.71.

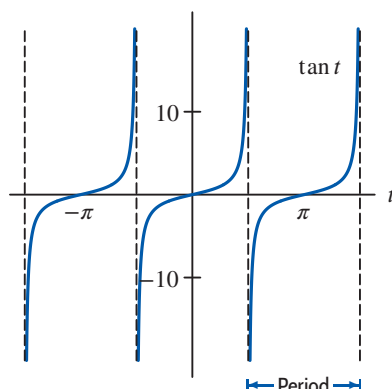


Figure 1.71: The tangent function

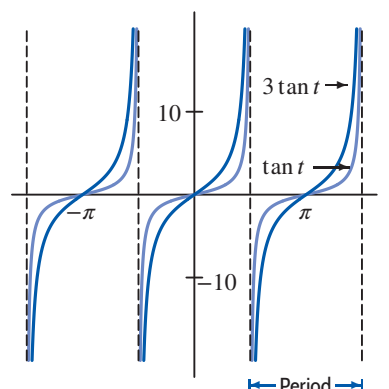


Figure 1.72: Multiple of tangent

The tangent function has period π , because it repeats every π units. Does it make sense to talk about the amplitude of the tangent function? Not if we're thinking of the amplitude as a measure of the size of the oscillation, because the tangent becomes infinitely large near each vertical asymptote. We can still multiply the tangent by a constant, but that constant no longer represents an amplitude. (See Figure 1.72.)

The Inverse Trigonometric Functions

On occasion, you may need to find a number with a given sine. For example, you might want to find x such that

$$\sin x = 0$$

or such that

$$\sin x = 0.3.$$

The first of these equations has solutions $x = 0, \pm\pi, \pm2\pi, \dots$. The second equation also has infinitely many solutions. Using a calculator and a graph, we get

$$x \approx 0.305, 2.84, 0.305 \pm 2\pi, 2.84 \pm 2\pi, \dots$$

For each equation, we pick out the solution between $-\pi/2$ and $\pi/2$ as the preferred solution. For example, the preferred solution to $\sin x = 0$ is $x = 0$, and the preferred solution to $\sin x = 0.3$ is $x = 0.305$. We define the inverse sine, written “arcsin” or “ \sin^{-1} ,” as the function which gives the preferred solution.

For $-1 \leq y \leq 1$,

means $\arcsin y = x$
 $\sin x = y$ with $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Thus the arcsine is the inverse function to the piece of the sine function having domain $[-\pi/2, \pi/2]$. (See Table 1.22 and Figure 1.73.) On a calculator, the arcsine function⁵⁴ is usually denoted by \sin^{-1} .

⁵⁴Note that $\sin^{-1} x = \arcsin x$ is not the same as $(\sin x)^{-1} = 1/\sin x$.

Table 1.22 Values of $\sin x$ and $\sin^{-1} x$

x	$\sin x$	x	$\sin^{-1} x$
$-\frac{\pi}{2}$	-1.000	-1.000	$-\frac{\pi}{2}$
-1.0	-0.841	-0.841	-1.0
-0.5	-0.479	-0.479	-0.5
0.0	0.000	0.000	0.0
0.5	0.479	0.479	0.5
1.0	0.841	0.841	1.0
$\frac{\pi}{2}$	1.000	1.000	$\frac{\pi}{2}$

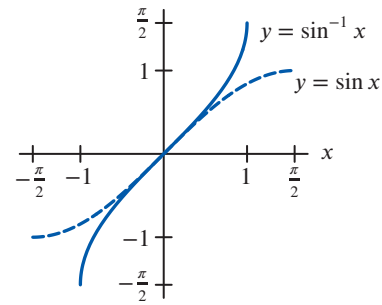


Figure 1.73: The arcsine function

The inverse tangent, written “arctan” or “ \tan^{-1} ,” is the inverse function for the piece of the tangent function having the domain $-\pi/2 < x < \pi/2$. On a calculator, the inverse tangent is usually denoted by \tan^{-1} . The graph of the arctangent is shown in Figure 1.75.

For any y ,

$\arctan y = x$

means $\tan x = y$ with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The inverse cosine function, written “arccos” or “ \cos^{-1} ,” is discussed in Problem 75. The range of the arccosine function is $0 \leq x \leq \pi$.

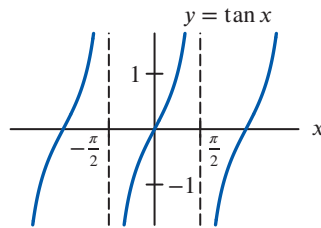


Figure 1.74: The tangent function

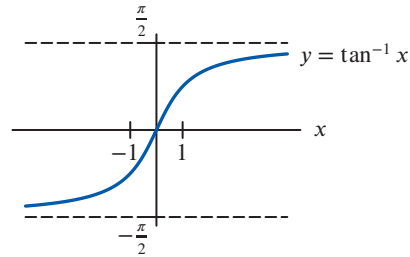


Figure 1.75: The arctangent function

Summary for Section 1.5

- An angle of **1 radian** is defined to be the angle at the center of a unit circle which cuts off an arc of length 1, measured counterclockwise. A unit circle has radius 1.
- If the circle has radius r and the arc cuts off an angle θ , then **Arc length** $= s = r\theta$.
- **The sine and cosine functions:** Both are defined using a unit circle.
 - If P with coordinates (x, y) corresponds to the point on the unit circle with an angle of t radians measured counterclockwise around the circle from the point $(1, 0)$, then

$$\cos t = x \quad \text{and} \quad \sin t = y.$$

- $\cos^2 t + \sin^2 t = 1$
- **Amplitude and period:** Functions whose values repeat at regular intervals are called **periodic**. For any periodic function of time, the
 - **Amplitude** is half the distance between the maximum and minimum values (if it exists).
 - **Period** is the smallest time needed for the function to execute one complete cycle.
- Functions whose graphs are the shape of a sine or cosine curve are called **sinusoidal** functions. To describe arbitrary amplitudes and periods of sinusoidal functions, we use functions of the form $f(t) = A \sin(Bt)$ and $g(t) = A \cos(Bt)$, where $|A|$ is the amplitude and $2\pi/|B|$ is the period.

- The graph of a sinusoidal function is shifted horizontally by a distance $|h|$ when t is replaced by $t - h$ or $t + h$.
- Functions of the form $f(t) = A \sin(Bt) + C$ and $g(t) = A \cos(Bt) + C$ have graphs which are shifted vertically by C and oscillate about this value.
- **The tangent function:** $\tan t = \frac{\sin t}{\cos t}$
- **The inverse trigonometric functions:**
 - For $-1 \leq y \leq 1$, $\arcsin y = x$ means $\sin x = y$ with $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
 - For any y , $\arctan y = x$ means $\tan x = y$ with $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Exercises and Problems for Section 1.5

EXERCISES

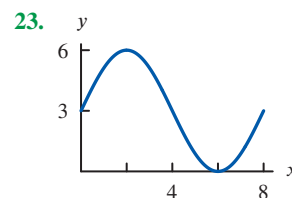
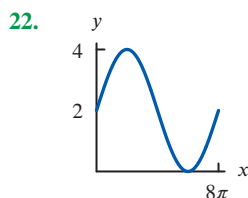
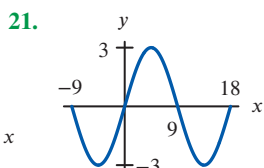
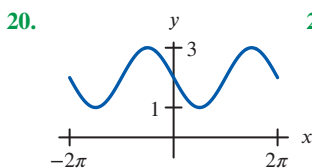
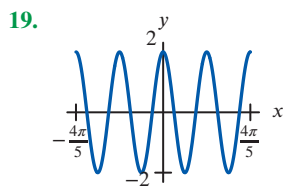
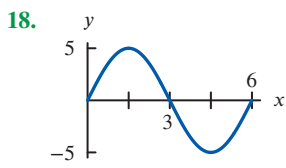
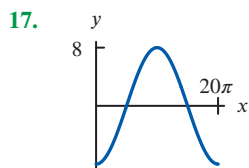
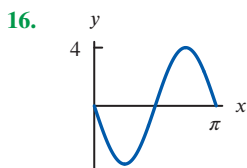
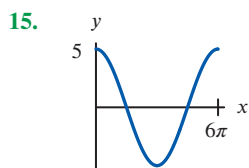
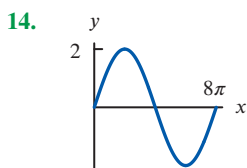
■ For Exercises 1–9, draw the angle using a ray through the origin, and determine whether the sine, cosine, and tangent of that angle are positive, negative, zero, or undefined.

- | | | |
|----------------------|--------------------|---------------------|
| 1. $\frac{3\pi}{2}$ | 2. 2π | 3. $\frac{\pi}{4}$ |
| 4. 3π | 5. $\frac{\pi}{6}$ | 6. $\frac{4\pi}{3}$ |
| 7. $\frac{-4\pi}{3}$ | 8. 4 | 9. -1 |

■ Find the period and amplitude in Exercises 10–13.

- | | |
|--------------------------------|-------------------------------|
| 10. $y = 7 \sin(3t)$ | 11. $z = 3 \cos(u/4) + 5$ |
| 12. $w = 8 - 4 \sin(2x + \pi)$ | 13. $r = 0.1 \sin(\pi t) + 2$ |

■ For Exercises 14–23, find a possible formula for each graph.



■ In Exercises 24–29, calculate the quantity without using the trigonometric functions on a calculator. You are given that $\sin(\pi/12) = 0.259$ and $\cos(\pi/5) = 0.809$. You may want to draw a picture showing the angles involved.

- | | | |
|----------------------------|--------------------------|---------------------------|
| 24. $\cos(-\frac{\pi}{5})$ | 25. $\sin \frac{\pi}{5}$ | 26. $\cos \frac{\pi}{12}$ |
| 27. $\sin(-\pi/12)$ | 28. $\tan \pi/12$ | 29. $\tan \pi/5$ |

■ In Exercises 30–34, find a solution to the equation if possible. Give the answer in exact form and in decimal form.

- | | |
|----------------------|------------------------------|
| 30. $2 = 5 \sin(3x)$ | 31. $1 = 8 \cos(2x + 1) - 3$ |
| 32. $8 = 4 \tan(5x)$ | 33. $1 = 8 \tan(2x + 1) - 3$ |
| 34. $8 = 4 \sin(5x)$ | |

35. What is the period of the earth's revolution around the sun?
36. What is the approximate period of the moon's revolution around the earth?
37. When a car's engine makes less than about 200 revolutions per minute, it stalls. What is the period of the rotation of the engine when it is about to stall?
38. A compact disc spins at a rate of 200 to 500 revolutions per minute. What are the equivalent rates measured in radians per second?
39. Find the angle, in degrees, that a wheelchair ramp makes with the ground if the ramp rises 1 foot over a horizontal distance of
- | |
|---|
| (a) 12 ft, the normal requirement ⁵⁵ |
| (b) 8 ft, the steepest ramp legally permitted |
| (c) 20 ft, the recommendation if snow can be expected on the ramp |

⁵⁵www.modular-wheelchair-ramps.com, accessed September 16, 2019.

PROBLEMS

40. (a) Use a graphing calculator or computer to estimate the period of $2 \sin \theta + 3 \cos(2\theta)$.
 (b) Explain your answer, given that the period of $\sin \theta$ is 2π and the period of $\cos(2\theta)$ is π .
41. Without a calculator or computer, match the formulas with the graphs in Figure 1.76.
- (a) $y = 2 \cos(t - \pi/2)$ (b) $y = 2 \cos t$
 (c) $y = 2 \cos(t + \pi/2)$

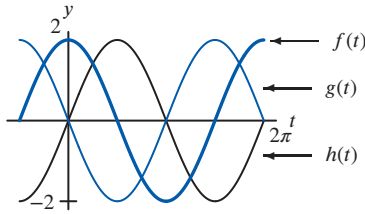


Figure 1.76

42. Figure 1.77 shows four periodic functions of the family $f(t) = A \cos(B(t-h))$, all with the same amplitude and period but with different horizontal shifts.
- (a) Find the value of A .
 (b) Find the value of B .
 (c) Which graph corresponds to each of the following values of h : 0, 20, 50, 60?

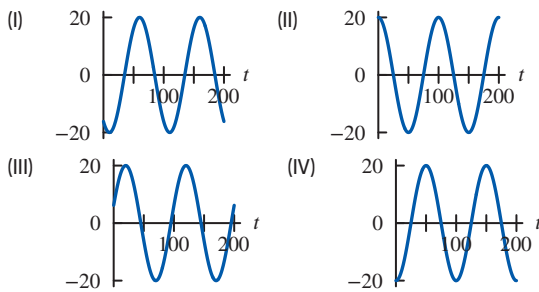


Figure 1.77

■ In Problems 43–49, graph the given function on the axes in Figure 1.78.

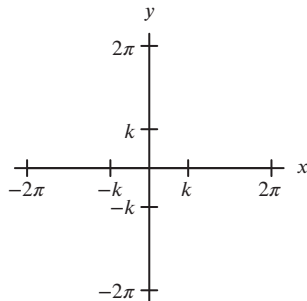


Figure 1.78

43. $y = k \sin x$ 44. $y = -k \cos x$

45. $y = k(\cos x) + k$ 46. $y = k(\sin x) - k$
 47. $y = k \sin(x - k)$ 48. $y = k \cos(x + k)$
 49. $y = k \sin(2\pi x/k)$

50. Coober Pedy is a town in Australia that mines most of the world's opals (precious gemstones). Because of the scorching heat, much of the population lives in underground “dugouts.” The Coober Pedy monthly high temperatures are shown in Figure 1.79, with $H(n)$ the temperature in month n , where $n = 1$ is January. We fit the data with a function,

$$H(n) = A \cos(Bn) + C.$$

- (a) What does the value of C represent and what is its value?
 (b) What does the value of A represent and what is its value?
 (c) What is the value of B ?
 (d) Graph the data and the function.

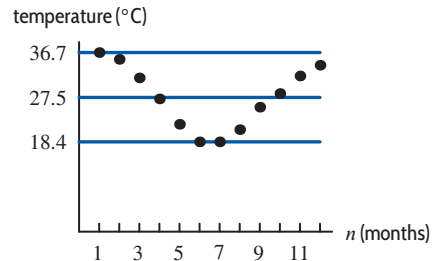


Figure 1.79

■ For Problems 51–54, use Figure 1.80 to estimate the given value for $f(t) = A \cos(B(t-h)) + C$, which approximates the plotted average monthly water temperature of the Mississippi River in Louisiana.⁵⁶

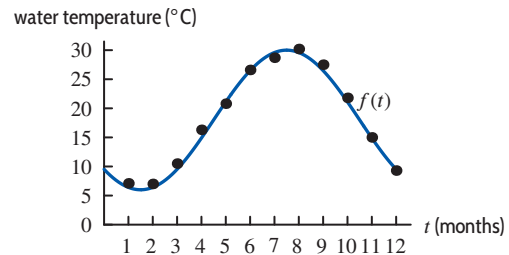


Figure 1.80

51. C 52. A 53. B 54. h

⁵⁶waterdata.usgs.gov/nwis, accessed September 16, 2019.

■ For Problems 55–58, use Figure 1.80 to estimate the given value for $f(t) = A \sin(B(t - h)) + C$, which approximates the plotted average monthly water flow of the Mississippi River.⁵⁷

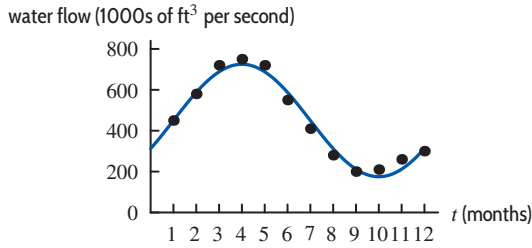


Figure 1.81

55. C 56. A 57. B 58. h

59. The visitors' guide to St. Petersburg, Florida, contains the chart shown in Figure 1.82 to advertise their good weather. Fit a trigonometric function approximately to the data, where H is temperature in degrees Fahrenheit, and the independent variable is time in months. In order to do this, you will need to estimate the amplitude and period of the data, and when the maximum occurs. (There are many possible answers to this problem, depending on how you read the graph.)

H (°F)	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
100°												
90°												
80°												
70°												
60°												
50°												

Figure 1.82: “St. Petersburg...where we’re famous for our wonderful weather and year-round sunshine.” (Reprinted with permission)

60. What is the difference between $\sin x^2$, $\sin^2 x$, and $\sin(\sin x)$? Express each of the three as a composition. (Note: $\sin^2 x$ is another way of writing $(\sin x)^2$.)
61. On the graph of $y = \sin x$, points P and Q are at consecutive lowest and highest points. Find the slope of the line through P and Q .
62. A population of animals oscillates sinusoidally between a low of 700 on January 1 and a high of 900 on July 1.
- (a) Graph the population against time.
 (b) Find a formula for the population as a function of time, t , in months since the start of the year.
63. The desert temperature, H , oscillates daily between 40°F at 5 am and 80°F at 5 pm. Write a possible formula for H in terms of t , measured in hours from 5 am.

64. The depth of water in a tank oscillates sinusoidally once every 6 hours. If the smallest depth is 5.5 feet and the largest depth is 8.5 feet, find a possible formula for the depth in terms of time in hours.
65. The voltage, V , of an electrical outlet in a home as a function of time, t (in seconds), is $V = V_0 \cos(120\pi t)$.
- (a) What is the period of the oscillation?
 (b) What does V_0 represent?
 (c) Sketch the graph of V against t . Label the axes.
66. The power output, P , of a solar panel varies with the position of the sun. Let $P = 10 \sin \theta$ watts, where θ is the angle between the sun's rays and the panel, $0 \leq \theta \leq \pi$. On a typical summer day in Ann Arbor, Michigan, the sun rises at 6 am and sets at 8 pm and the angle is $\theta = \pi t/14$, where t is time in hours since 6 am and $0 \leq t \leq 14$.
- (a) Write a formula for a function, $f(t)$, giving the power output of the solar panel (in watts) t hours after 6 am on a typical summer day in Ann Arbor.
 (b) Graph the function $f(t)$ in part (a) for $0 \leq t \leq 14$.
 (c) At what time is the power output greatest? What is the power output at this time?
 (d) On a typical winter day in Ann Arbor, the sun rises at 8 am and sets at 5 pm. Write a formula for a function, $g(t)$, giving the power output of the solar panel (in watts) t hours after 8 am on a typical winter day.
67. A baseball hit at an angle of θ to the horizontal with initial velocity v_0 has horizontal range, R , given by

$$R = \frac{v_0^2}{g} \sin(2\theta).$$

Here g is the acceleration due to gravity. Sketch R as a function of θ for $0 \leq \theta \leq \pi/2$. What angle gives the maximum range? What is the maximum range?

68. (a) Match the functions $\omega = f(t)$, $\omega = g(t)$, $\omega = h(t)$, $\omega = k(t)$, whose values are in the table, with the functions with formulas:
- (i) $\omega = 1.5 + \sin t$ (ii) $\omega = 0.5 + \sin t$
 (iii) $\omega = -0.5 + \sin t$ (iv) $\omega = -1.5 + \sin t$
- (b) Based on the table, what is the relationship between the values of $g(t)$ and $k(t)$? Explain this relationship using the formulas you chose for g and k .
- (c) Using the formulas you chose for g and h , explain why all the values of g are positive, whereas all the values of h are negative.

t	$f(t)$	t	$g(t)$	t	$h(t)$	t	$k(t)$
6.0	-0.78	3.0	1.64	5.0	-2.46	3.0	0.64
6.5	-0.28	3.5	1.15	5.1	-2.43	3.5	0.15
7.0	0.16	4.0	0.74	5.2	-2.38	4.0	-0.26
7.5	0.44	4.5	0.52	5.3	-2.33	4.5	-0.48
8.0	0.49	5.0	0.54	5.4	-2.27	5.0	-0.46

⁵⁷www.americaswetlandresources.com, accessed September 16, 2019.

69. For a boat to float in a tidal bay, the water must be at least 2.5 meters deep. The depth of water around the boat, $d(t)$, in meters, where t is measured in hours since midnight, is

$$d(t) = 5 + 4.6 \sin(0.5t).$$

- (a) What is the period of the tides in hours?
 (b) If the boat leaves the bay at midday, what is the latest time it can return before the water becomes too shallow?
70. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, y meters, is given as a function of time, t , in hours since midnight by

$$y = D + A \cos(B(t - C)).$$

- (a) What is the physical meaning of D ?
 (b) What is the value of A ?
 (c) What is the value of B ? Assume the time between successive high tides is 12.4 hours.
 (d) What is the physical meaning of C ?
71. Match graphs A - D in Figure 1.83 with the functions below. Assume a, b, c and d are positive constants.

$$\begin{array}{ll} f(t) = \sin t + b & h(t) = \sin t + e^{ct} + d \\ g(t) = \sin t + at + b & r(t) = \sin t - e^{ct} + b \end{array}$$

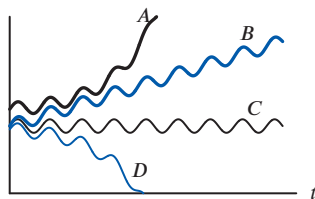


Figure 1.83

72. In Figure 1.84, the blue curve shows monthly mean carbon dioxide (CO_2) concentration, in parts per million (ppm) at Mauna Loa Observatory, Hawaii, as a function of t , in months, since December 2013. The black curve shows the monthly mean concentration adjusted for seasonal CO_2 variation.⁵⁸
- (a) Approximately how much did the monthly mean CO_2 increase between December 2013 and December 2018?
 (b) Find the average monthly rate of increase of the monthly mean CO_2 between December 2013 and December 2018. Use this information to find a linear function that approximates the black curve.
 (c) The seasonal CO_2 variation between December 2013 and December 2018 can be approximated by a sinusoidal function. What is the approximate period of the function? What is its amplitude? Give a formula for the function.

- (d) The blue curve may be approximated by a function of the form $h(t) = f(t) + g(t)$, where $f(t)$ is sinusoidal and $g(t)$ is linear. Using your work in parts (b) and (c), find a possible formula for $h(t)$. Graph $h(t)$ using the scale in Figure 1.84.

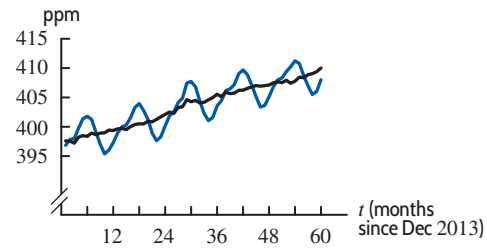


Figure 1.84

73. Find the area of the trapezoidal cross-section of the irrigation canal shown in Figure 1.85.

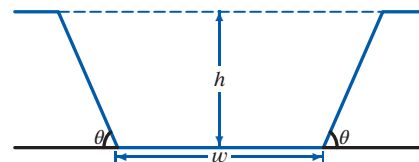


Figure 1.85

74. Graph $y = \sin x$, $y = 0.4$, and $y = -0.4$.
- (a) From the graph, estimate to one decimal place all the solutions of $\sin x = 0.4$ with $-\pi \leq x \leq \pi$.
 (b) Use a calculator to find $\arcsin(0.4)$. What is the relation between $\arcsin(0.4)$ and each of the solutions you found in part (a)?
 (c) Estimate all the solutions to $\sin x = -0.4$ with $-\pi \leq x \leq \pi$ (again, to one decimal place).
 (d) What is the relation between $\arcsin(0.4)$ and each of the solutions you found in part (c)?
75. This problem introduces the arccosine function, or inverse cosine, denoted by $\boxed{\cos^{-1}}$ on most calculators.
- (a) Using a calculator set in radians, make a table of values, to two decimal places, of $g(x) = \arccos x$, for $x = -1, -0.8, -0.6, \dots, 0, \dots, 0.6, 0.8, 1$.
 (b) Sketch the graph of $g(x) = \arccos x$.
 (c) Why is the domain of the arccosine the same as the domain of the arcsine?
 (d) What is the range of the arccosine?
 (e) Why is the range of the arccosine *not* the same as the range of the arcsine?

⁵⁸www.esrl.noaa.gov/gmd/ccgg/trends, accessed March, 2019. Monthly means joined by segments to highlight trends.

Strengthen Your Understanding

- In Problems 76–79, explain what is wrong with the statement.
 - 76. The functions $f(x) = 3 \cos x$ and $g(x) = \cos 3x$ have the same period.
 - 77. For the function $f(x) = \sin(Bx)$ with $B > 0$, increasing the value of B increases the period.
 - 78. The function $y = \sin x \cos x$ is periodic with period 2π .
 - 79. For positive A, B, C , the maximum value of the function $y = A \sin(Bx) + C$ is $y = A$.
- In Problems 80–81, give an example of:
 - 80. A sine function with period 23.
 - 81. A cosine function which oscillates between values of 1200 and 2000.
- Are the statements in Problems 82–98 true or false? Give an explanation for your answer.
 - 82. The family of functions $y = a \sin x$, with a a positive constant, all have the same period.
 - 83. The family of functions $y = \sin ax$, a a positive constant, all have the same period.
 - 84. The function $f(\theta) = \cos \theta - \sin \theta$ is increasing on $0 \leq \theta \leq \pi/2$.
 - 85. The function $f(t) = \sin(0.05\pi t)$ has period 0.05.
 - 86. If t is in seconds, $g(t) = \cos(200\pi t)$ executes 100 cycles in one second.
 - 87. The function $f(\theta) = \tan(\theta - \pi/2)$ is not defined at $\theta = \pi/2, 3\pi/2, 5\pi/2, \dots$
 - 88. $\sin |x| = \sin x$ for $-2\pi < x < 2\pi$
 - 89. $\sin |x| = |\sin x|$ for $-2\pi < x < 2\pi$
 - 90. $\cos |x| = |\cos x|$ for $-2\pi < x < 2\pi$
 - 91. $\cos |x| = \cos x$ for $-2\pi < x < 2\pi$
 - 92. The function $f(x) = \sin(x^2)$ is periodic, with period 2π .
 - 93. The function $g(\theta) = e^{\sin \theta}$ is periodic.
 - 94. If $f(x)$ is a periodic function with period k , then $f(g(x))$ is periodic with period k for every function $g(x)$.
 - 95. If $g(x)$ is a periodic function, then $f(g(x))$ is periodic for every function $f(x)$.
 - 96. The function $f(x) = |\sin x|$ is even.
 - 97. $\sin^{-1}(\sin t) = t$ for all t .
 - 98. The function $f(t) = \sin^{-1}(\sin t)$ is periodic with period 2π .

1.6 POWERS, POLYNOMIALS, AND RATIONAL FUNCTIONS

Power Functions

A *power function* is a function in which the dependent variable is proportional to a power of the independent variable:

A **power function** has the form

$$f(x) = kx^p, \quad \text{where } k \text{ and } p \text{ are constant.}$$

For example, the volume, V , of a sphere of radius r is given by

$$V = g(r) = \frac{4}{3}\pi r^3.$$

As another example, the gravitational force, F , on a unit mass at a distance r from the center of the earth is given by Newton's Law of Gravitation, which says that, for some positive constant k ,

$$F = \frac{k}{r^2} \quad \text{or} \quad F = kr^{-2}.$$

We consider the graphs of the power functions x^n , with n a positive integer. Figures 1.86 and 1.87 show that the graphs fall into two groups: odd and even powers. For n greater than 1, the odd powers

have a “seat” at the origin and are increasing everywhere else. The even powers are first decreasing and then increasing. For large x , the higher the power of x , the faster the function climbs.

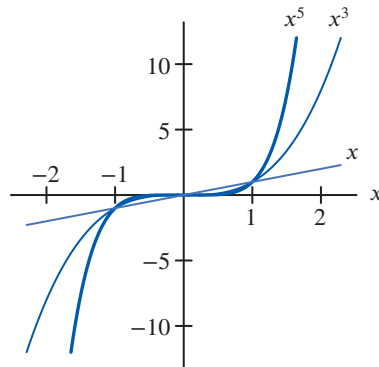


Figure 1.86: Odd powers of x : “Seat”-shaped for $n > 1$

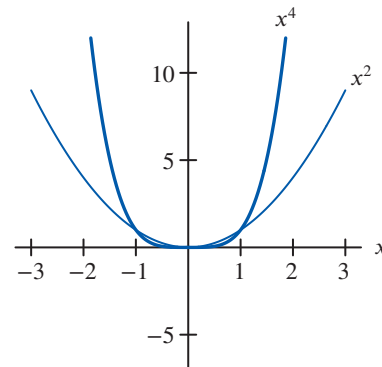


Figure 1.87: Even powers of x : U-shaped

Exponentials and Power Functions: Which Dominate?

In everyday language, the word exponential is often used to imply very fast growth. But do exponential functions always grow faster than power functions? To determine what happens “in the long run,” we often want to know which functions *dominate* as x gets arbitrarily large.

Let’s consider $y = 2^x$ and $y = x^3$. The close-up view in Figure 1.88(a) shows that between $x = 2$ and $x = 4$, the graph of $y = 2^x$ lies below the graph of $y = x^3$. The far-away view in Figure 1.88(b) shows that the exponential function $y = 2^x$ eventually overtakes $y = x^3$. Figure 1.88(c), which gives a very far-away view, shows that, for large x , the value of x^3 is insignificant compared to 2^x . Indeed, 2^x is growing so much faster than x^3 that the graph of 2^x appears almost vertical in comparison to the more leisurely climb of x^3 .

We say that Figure 1.88(a) gives a *local* view of the functions’ behavior, whereas Figure 1.88(c) gives a *global* view.

In fact, *every* exponential growth function eventually dominates *every* power function. Although an exponential function may be below a power function for some values of x , if we look at large enough x -values, a^x (with $a > 1$) will eventually dominate x^n , no matter what n is.

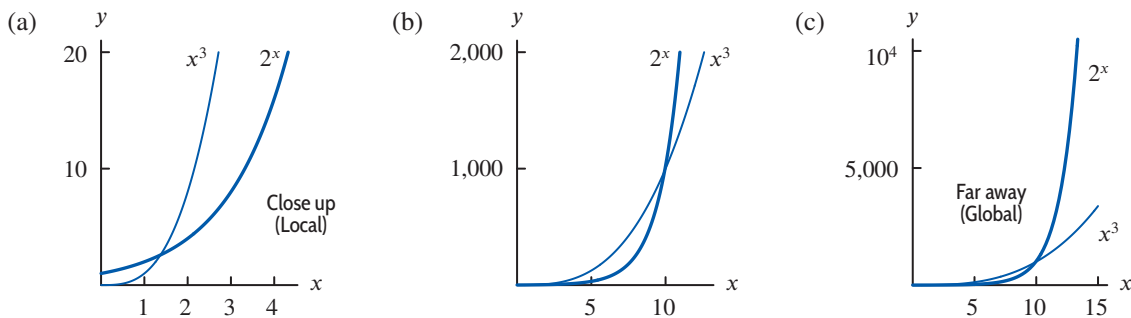


Figure 1.88: Comparison of $y = 2^x$ and $y = x^3$: Notice that $y = 2^x$ eventually dominates $y = x^3$

Polynomials

Polynomials are the sums of power functions with nonnegative integer exponents:

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

Here n is a nonnegative integer called the *degree* of the polynomial, and $a_n, a_{n-1}, \dots, a_1, a_0$ are constants, with leading coefficient $a_n \neq 0$. An example of a polynomial of degree $n = 3$ is

$$y = p(x) = 2x^3 - x^2 - 5x - 7.$$

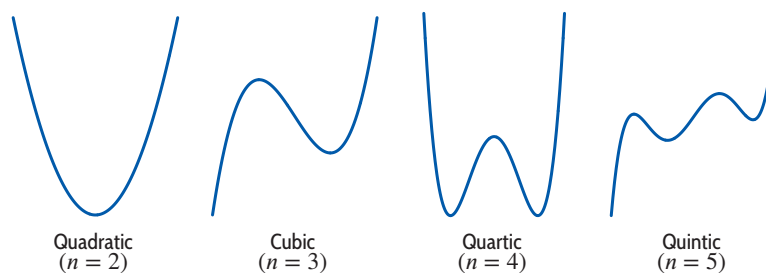


Figure 1.89: Graphs of typical polynomials of degree n

In this case $a_3 = 2$, $a_2 = -1$, $a_1 = -5$, and $a_0 = -7$. The shape of the graph of a polynomial depends on its degree; typical graphs are shown in Figure 1.89. These graphs correspond to a positive coefficient for x^n ; a negative leading coefficient turns the graph upside down. Notice that the quadratic “turns around” once, the cubic “turns around” twice, and the quartic (fourth degree) “turns around” three times. An n^{th} -degree polynomial “turns around” at most $n - 1$ times (where n is a positive integer), but there may be fewer turns.

Example 1

Find possible formulas for the polynomials whose graphs are in Figure 1.90.

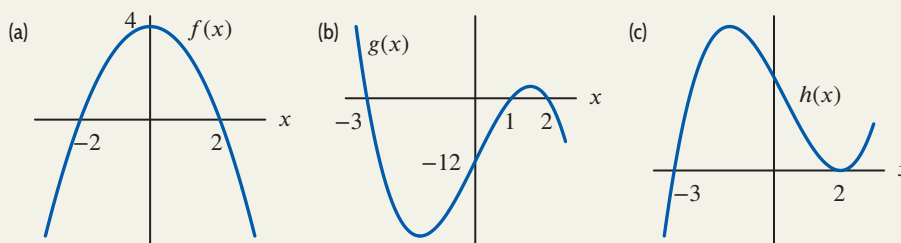


Figure 1.90: Graphs of polynomials

Solution

- (a) This graph appears to be a parabola, turned upside down, and moved up by 4, so

$$f(x) = -x^2 + 4.$$

The negative sign turns the parabola upside down and the $+4$ moves it up by 4. Notice that this formula does give the correct x -intercepts since $0 = -x^2 + 4$ has solutions $x = \pm 2$. These values of x are called *zeros* of f .

We can also solve this problem by looking at the x -intercepts first, which tell us that $f(x)$ has factors of $(x + 2)$ and $(x - 2)$. So

$$f(x) = k(x + 2)(x - 2).$$

To find k , use the fact that the graph has a y -intercept of 4, so $f(0) = 4$, giving

$$4 = k(0 + 2)(0 - 2),$$

or $k = -1$. Therefore, $f(x) = -(x + 2)(x - 2)$, which multiplies out to $-x^2 + 4$.

Note that $f(x) = 4 - x^4/4$ also has the same basic shape, but is flatter near $x = 0$. There are many possible answers to these questions.

- (b) This looks like a cubic with factors $(x + 3)$, $(x - 1)$, and $(x - 2)$, one for each intercept:

$$g(x) = k(x + 3)(x - 1)(x - 2).$$

Since the y -intercept is -12 , we have

$$-12 = k(0 + 3)(0 - 1)(0 - 2).$$

So $k = -2$, and we get the cubic polynomial

$$g(x) = -2(x + 3)(x - 1)(x - 2).$$

- (c) This also looks like a cubic with zeros at $x = 2$ and $x = -3$. Notice that at $x = 2$ the graph of $h(x)$ touches the x -axis but does not cross it, whereas at $x = -3$ the graph crosses the x -axis. We say that $x = 2$ is a *double zero*, but that $x = -3$ is a single zero.

To find a formula for $h(x)$, imagine the graph of $h(x)$ to be slightly lower down, so that the graph has one x -intercept near $x = -3$ and two near $x = 2$, say at $x = 1.9$ and $x = 2.1$. Then a formula would be

$$h(x) \approx k(x + 3)(x - 1.9)(x - 2.1).$$

Now move the graph back to its original position. The zeros at $x = 1.9$ and $x = 2.1$ move toward $x = 2$, giving

$$h(x) = k(x + 3)(x - 2)(x - 2) = k(x + 3)(x - 2)^2.$$

The double zero leads to a repeated factor, $(x - 2)^2$. Notice that when $x > 2$, the factor $(x - 2)^2$ is positive, and when $x < 2$, the factor $(x - 2)^2$ is still positive. This reflects the fact that $h(x)$ does not change sign near $x = 2$. Compare this with the behavior near the single zero at $x = -3$, where h does change sign.

We cannot find k , as no coordinates are given for points off of the x -axis. Any positive value of k stretches the graph vertically but does not change the zeros, so any positive k works.

Example 2

Using a calculator or computer, graph $y = x^4$ and $y = x^4 - 15x^2 - 15x$ for $-4 \leq x \leq 4$ and for $-20 \leq x \leq 20$. Set the y range to $-100 \leq y \leq 100$ for the first domain, and to $-100 \leq y \leq 200,000$ for the second. What do you observe?

Solution

From the graphs in Figure 1.91 we see that close up ($-4 \leq x \leq 4$) the graphs look different; from far away, however, they are almost indistinguishable. The reason is that the leading terms (those with the highest power of x) are the same, namely x^4 , and for large values of x , the leading term dominates the other terms.

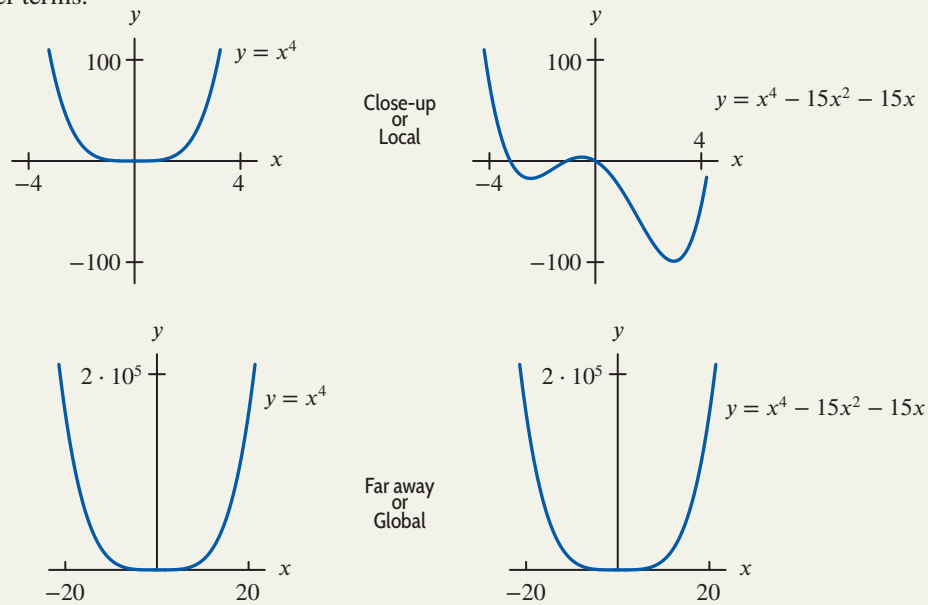


Figure 1.91: Local and global views of $y = x^4$ and $y = x^4 - 15x^2 - 15x$

Rational Functions

Rational functions are ratios of polynomials, p and q :

$$f(x) = \frac{p(x)}{q(x)}.$$

Example 3 Look at a graph and explain the behavior of $y = \frac{1}{x^2 + 4}$.

Solution The function is even, so the graph is symmetric about the y -axis. As x gets larger, the denominator gets larger, making the value of the function closer to 0. Thus the graph gets arbitrarily close to the x -axis as x increases without bound. See Figure 1.92.

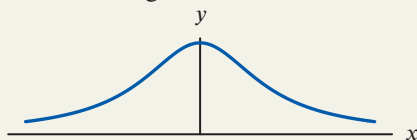


Figure 1.92: Graph of $y = \frac{1}{x^2 + 4}$

In the previous example, we say that $y = 0$ (i.e. the x -axis) is a *horizontal asymptote*. Writing “ \rightarrow ” to mean “tends to,” we have $y \rightarrow 0$ as $x \rightarrow \infty$ and $y \rightarrow 0$ as $x \rightarrow -\infty$.

If the graph of $y = f(x)$ approaches a horizontal line $y = L$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$, then the line $y = L$ is called a **horizontal asymptote**.⁵⁹ This occurs when

$$f(x) \rightarrow L \text{ as } x \rightarrow \infty \quad \text{or} \quad f(x) \rightarrow L \text{ as } x \rightarrow -\infty.$$

If the graph of $y = f(x)$ approaches the vertical line $x = K$ as $x \rightarrow K$ from one side or the other, that is, if

$$y \rightarrow \infty \text{ or } y \rightarrow -\infty \text{ when } x \rightarrow K,$$

then the line $x = K$ is called a **vertical asymptote**.

The graphs of rational functions may have vertical asymptotes where the denominator is zero. For example, the function in Example 3 has no vertical asymptotes as the denominator is never zero. The function in Example 4 has two vertical asymptotes corresponding to the two zeros in the denominator.

Rational functions have horizontal asymptotes if $f(x)$ approaches a finite number as $x \rightarrow \infty$ or $x \rightarrow -\infty$. We call the behavior of a function as $x \rightarrow \pm\infty$ its *end behavior*.

Example 4 Look at a graph and explain the behavior of $y = \frac{3x^2 - 12}{x^2 - 1}$, including end behavior.

Solution Factoring gives

$$y = \frac{3x^2 - 12}{x^2 - 1} = \frac{3(x + 2)(x - 2)}{(x + 1)(x - 1)}$$

so $x = \pm 1$ are vertical asymptotes. If $y = 0$, then $3(x + 2)(x - 2) = 0$ or $x = \pm 2$; these are the x -intercepts. Note that zeros of the denominator give rise to the vertical asymptotes, whereas zeros of the numerator give rise to x -intercepts. Substituting $x = 0$ gives $y = 12$; this is the y -intercept. The function is even, so the graph is symmetric about the y -axis.

To see what happens as $x \rightarrow \pm\infty$, look at the y -values in Table 1.23. Clearly y is getting closer to 3 as x gets large positively or negatively. Alternatively, realize that as $x \rightarrow \pm\infty$, only the highest powers of x matter. For large x , the 12 and the 1 are insignificant compared to x^2 , so

$$y = \frac{3x^2 - 12}{x^2 - 1} \approx \frac{3x^2}{x^2} = 3 \quad \text{for large } x.$$

⁵⁹We are assuming that $f(x)$ gets arbitrarily close to L as $x \rightarrow \infty$.

So $y \rightarrow 3$ as $x \rightarrow \pm\infty$, and therefore the horizontal asymptote is $y = 3$. See Figure 1.93. Since, for $x > 1$, the value of $(3x^2 - 12)/(x^2 - 1)$ is less than 3, the graph lies *below* its asymptote. (Why doesn't the graph lie below $y = 3$ when $-1 < x < 1$?)

Table 1.23 Values of

$y = \frac{3x^2 - 12}{x^2 - 1}$	
x	$y = \frac{3x^2 - 12}{x^2 - 1}$
± 10	2.909091
± 100	2.999100
± 1000	2.999991

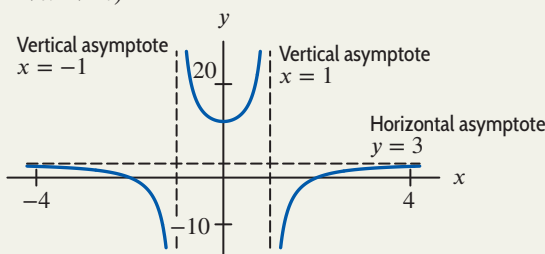


Figure 1.93: Graph of the function $y = \frac{3x^2 - 12}{x^2 - 1}$

Summary for Section 1.6

- **Power functions:** A power function $f(x) = kx^p$, where k and p are constant.
- **Polynomials:** Sums of power functions with nonnegative integer exponents are called *polynomials*. With a_n, a_{n-1}, \dots, a_0 constants, polynomials are functions of the form

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0.$$

- n is a nonnegative integer called the **degree** of the polynomial.
- a_n is a nonzero number called the **leading coefficient**.
- $a_n x^n$ is the **leading term**.
- **Rational functions** are ratios of polynomials, p and q : $f(x) = \frac{p(x)}{q(x)}$
- **Asymptotes:** If the graph of $y = f(x)$
 - approaches a horizontal line $y = L$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$, then the line $y = L$ is called a **horizontal asymptote**.
 - approaches the vertical line $x = K$ as $x \rightarrow K$ from one side or the other, that is, if

$$y \rightarrow \infty \text{ or } y \rightarrow -\infty \text{ when } x \rightarrow K,$$

then the line $x = K$ is called a **vertical asymptote**.

Exercises and Problems for Section 1.6 Online Resource: Additional Problems for Section 1.6

EXERCISES

- For Exercises 1–2, what happens to the value of the function as $x \rightarrow \infty$ and as $x \rightarrow -\infty$?
 - $y = 0.25x^3 + 3$
 - $y = 2 \cdot 10^{4x}$
- In Exercises 3–10, determine the end behavior of each function as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$.
 - $f(x) = -10x^4$
 - $f(x) = 3x^5$
 - $f(x) = 5x^4 - 25x^3 - 62x^2 + 5x + 300$
 - $f(x) = 1000 - 38x + 50x^2 - 5x^3$
 - $f(x) = \frac{3x^2 + 5x + 6}{x^2 - 4}$
 - $f(x) = \frac{10 + 5x^2 - 3x^3}{2x^3 - 4x + 12}$
 - $f(x) = 3x^{-4}$
 - $f(x) = e^x$
- In Exercises 11–16, which function dominates as $x \rightarrow \infty$?
 - $1000x^4$ or $0.2x^5$
 - $10e^{0.1x}$ or $5000x^2$
 - $100x^5$ or 1.05^x
 - $2x^4$ or $10x^3 + 25x^2 + 50x + 100$
 - $20x^4 + 100x^2 + 5x$ or $25 - 40x^2 + x^3 + 3x^5$
 - \sqrt{x} or $\ln x$
- 17. Each of the graphs in Figure 1.94 is of a polynomial. The windows are large enough to show end behavior.
 - What is the minimum possible degree of the polynomial?

(b) Is the leading coefficient of the polynomial positive or negative?

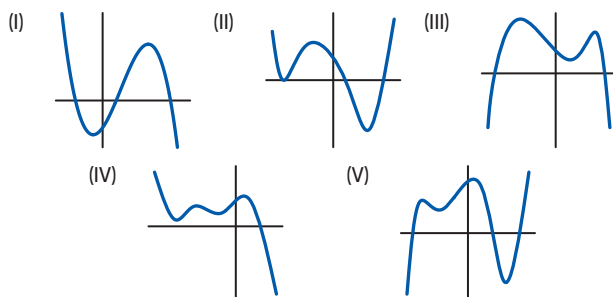
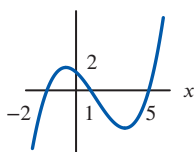


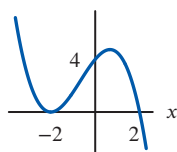
Figure 1.94

Find cubic polynomials for the graphs in Exercises 18–19.

18.

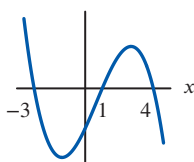


19.

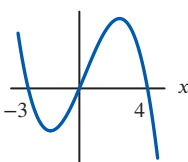


Find possible formulas for the graphs in Exercises 20–23.

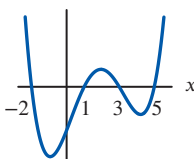
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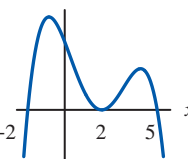
21.



22.



23.



In Exercises 24–26, choose the functions that are in the given family, assuming a , b , and c are positive constants.

$$\begin{aligned} f(x) &= \sqrt{x^4 + 16} & g(x) &= ax^{23} \\ h(x) &= -\frac{1}{5^{x-2}} & p(x) &= \frac{a^3 b^x}{c} \\ q(x) &= \frac{ab^2}{c} & r(x) &= -x + b - \sqrt{cx^4} \end{aligned}$$

24. Exponential 25. Quadratic 26. Linear

In Exercises 27–32, choose each of the families the given function is in, assuming a is a positive integer and b and c are positive constants.

- I. Exponential II. Power
III. Polynomial IV. Rational

27. $f(x) = \frac{ax}{b} + c$ 28. $g(x) = ax^2 + \frac{b}{x^2}$

29. $h(x) = b\left(\frac{x}{c}\right)^a$ 30. $k(x) = bx^a$

31. $j(x) = ax^{-1} + \frac{b}{x}$ 32. $l(x) = \left(\frac{a+b}{c}\right)^{2x}$

In Exercises 33–40, which of the following functions have the given property?

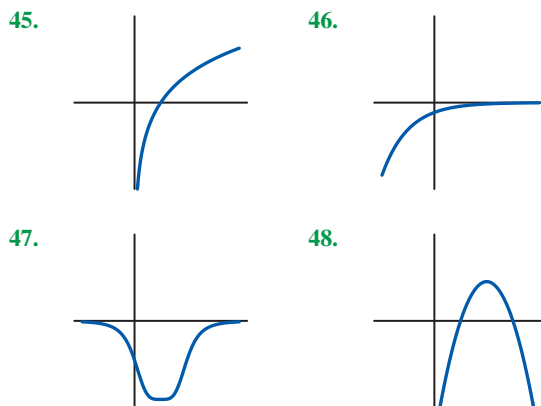
- I. $y = \frac{x^2 - 2}{x^2 + 2}$ II. $y = \frac{x^2 + 2}{x^2 - 2}$
III. $y = (x-1)(1-x)(x+1)^2$ IV. $y = x^3 - x$
V. $y = x - \frac{1}{x}$ VI. $y = (x^2 - 2)(x^2 + 2)$

33. A polynomial of degree 3.
34. A polynomial of degree 4.
35. A rational function that is not a polynomial.
36. Exactly two distinct zeros.
37. Exactly one vertical asymptote.
38. More than two distinct zeros.
39. Exactly two vertical asymptotes.
40. A horizontal asymptote.

In Exercises 41–44, which, if any, of $x = 1, 2, 3$ are zeros, and which, if any, are asymptotes of the function?

41. $(x-1)(x-2)(x-3)$ 42. $\frac{x-3}{(x-1)(x-2)}$
43. $\frac{(x-1)(x-3)}{x-2}$ 44. $\frac{1}{x-1} + \frac{1}{x-3}$

For Exercises 45–48, assuming the window is large enough to show end behavior, identify the graph as that of a rational function, exponential function or logarithmic function.



PROBLEMS

49. How many distinct roots can a polynomial of degree 5 have? (List all possibilities.) Sketch a possible graph for each case.

50. Find a calculator window in which the graphs of $f(x) = x^3 + 1000x^2 + 1000$ and $g(x) = x^3 - 1000x^2 - 1000$ appear indistinguishable.

51. A cubic polynomial with positive leading coefficient is shown in Figure 1.95 for $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$. What can be concluded about the total number of zeros of this function? What can you say about the location of each of the zeros? Explain.

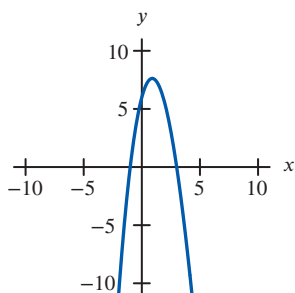


Figure 1.95

52. (a) If $f(x) = ax^2 + bx + c$, what can you say about the values of a , b , and c if:
- $(1, 1)$ is on the graph of $f(x)$?
 - $(1, 1)$ is the vertex of the graph of $f(x)$? (Hint: The axis of symmetry is $x = -b/(2a)$.)
 - The y -intercept of the graph is $(0, 6)$?
- (b) Find a quadratic function satisfying all three conditions.
53. A box of fixed volume V has a square base with side length x . Write a formula for the height, h , of the box in terms of x and V . Sketch a graph of h versus x .
54. A closed cylindrical can of fixed volume V has radius r .
- Find the surface area, S , as a function of r .
 - What happens to the value of S as $r \rightarrow \infty$?
 - Sketch a graph of S against r , if $V = 10 \text{ cm}^3$.
55. The DuBois formula relates a person's surface area s , in m^2 , to weight w , in kg, and height h , in cm, by

$$s = 0.01w^{0.25}h^{0.75}.$$

- What is the surface area of a person who weighs 65 kg and is 160 cm tall?
 - What is the weight of a person whose height is 180 cm and who has a surface area of 1.5 m^2 ?
 - For people of fixed weight 70 kg, solve for h as a function of s . Simplify your answer.
56. According to *Car and Driver*, an Alfa Romeo going at 70 mph requires 148 feet to stop.⁶⁰ Assuming that the stopping distance is proportional to the square of velocity, find the stopping distances required by an Alfa Romeo going at 35 mph and at 140 mph.

⁶⁰<http://www.caranddriver.com/alfa-romeo/4c>, accessed September 18, 2019.

⁶¹Niklas, K. and Enquist, B., "Invariant Scaling Relationships for Interspecific Plant Biomass Production Rates and Body Size", *PNAS*, Feb 27, 2001.

57. Poiseuille's Law gives the rate of flow, R , of a gas through a cylindrical pipe in terms of the radius of the pipe, r , for a fixed drop in pressure between the two ends of the pipe.

- Find a formula for Poiseuille's Law, given that the rate of flow is proportional to the fourth power of the radius.
- If $R = 400 \text{ cm}^3/\text{sec}$ in a pipe of radius 3 cm for a certain gas, find a formula for the rate of flow of that gas through a pipe of radius r cm.
- What is the rate of flow of the same gas through a pipe with a 5 cm radius?

58. A pomegranate is thrown from ground level straight up into the air at time $t = 0$ with velocity 64 feet per second. Its height at time t seconds is $f(t) = -16t^2 + 64t$. Find the time it hits the ground and the time it reaches its highest point. What is the maximum height?

59. The height of an object above the ground at time t is given by

$$s = v_0t - \frac{g}{2}t^2,$$

where v_0 is the initial velocity and g is the acceleration due to gravity.

- At what height is the object initially?
- How long is the object in the air before it hits the ground?
- When will the object reach its maximum height?
- What is that maximum height?

60. The rate, R , at which a population in a confined space increases is proportional to the product of the current population, P , and the difference between the carrying capacity, L , and the current population. (The carrying capacity is the maximum population the environment can sustain.)

- Write R as a function of P .
- Sketch R as a function of P .

■ In Problems 61–65, the length of a plant, L , is a function of its mass, M . A unit increase in a plant's mass stretches the plant's length more when the plant is small, and less when the plant is large.⁶¹ Assuming $M > 0$, decide if the function agrees with the description.

61. $L = 2M^{1/4}$

62. $L = 0.2M^3 + M^4$

63. $L = 2M^{-1/4}$

64. $L = \frac{4(M+1)^2 - 4}{(M+1)^2}$

65. $L = \frac{10(M+1)^2 - 1}{(M+1)^3}$

■ In Problems 66–68, find all horizontal and vertical asymptotes for each rational function.

66. $f(x) = \frac{5x-2}{2x+3}$

67. $f(x) = \frac{x^2+5x+4}{x^2-4}$

68. $f(x) = \frac{5x^3+7x-1}{x^3-27}$

69. For each function, fill in the blanks in the statements:

$f(x) \rightarrow \underline{\hspace{2cm}}$ as $x \rightarrow -\infty$,

$f(x) \rightarrow \underline{\hspace{2cm}}$ as $x \rightarrow +\infty$.

(a) $f(x) = 17 + 5x^2 - 12x^3 - 5x^4$

(b) $f(x) = \frac{3x^2 - 5x + 2}{2x^2 - 8}$

(c) $f(x) = e^x$

70. A rational function $y = f(x)$ is graphed in Figure 1.96. If $f(x) = g(x)/h(x)$ with $g(x)$ and $h(x)$ both quadratic functions, give possible formulas for $g(x)$ and $h(x)$.

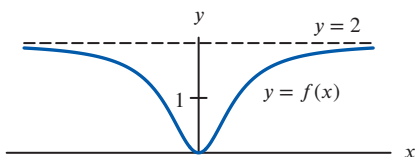


Figure 1.96

71. After running 3 miles at a speed of x mph, a man walked the next 6 miles at a speed that was 2 mph slower. Express the total time spent on the trip as a function of x . What horizontal and vertical asymptotes does the graph of this function have?

72. Which of the functions I–III meet each of the following descriptions? There may be more than one function for each description, or none at all.

(a) Horizontal asymptote of $y = 1$.

(b) The x -axis is a horizontal asymptote.

(c) Symmetric about the y -axis.

(d) An odd function.

(e) Vertical asymptotes at $x = \pm 1$.

I. $y = \frac{x-1}{x^2+1}$ II. $y = \frac{x^2-1}{x^2+1}$ III. $y = \frac{x^2+1}{x^2-1}$

73. Values of three functions are given in Table 1.24, rounded to two decimal places. One function is of the form $y = ab^t$, one is of the form $y = ct^2$, and one is of the form $y = kt^3$. Which function is which?

Table 1.24

t	$f(t)$	t	$g(t)$	t	$h(t)$
2.0	4.40	1.0	3.00	0.0	2.04
2.2	5.32	1.2	5.18	1.0	3.06
2.4	6.34	1.4	8.23	2.0	4.59
2.6	7.44	1.6	12.29	3.0	6.89
2.8	8.62	1.8	17.50	4.0	10.33
3.0	9.90	2.0	24.00	5.0	15.49

74. Use a graphing calculator or a computer to graph $y = x^4$ and $y = 3^x$. Determine approximate domains and ranges that give each of the graphs in Figure 1.97.

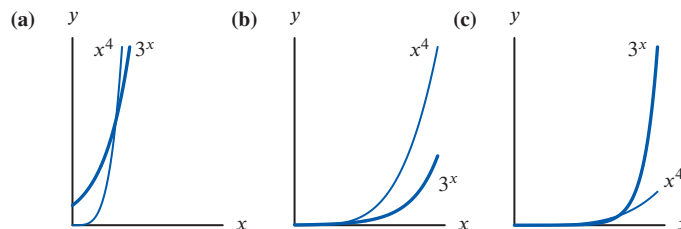


Figure 1.97

75. Consider the point P at the intersection of the circle $x^2 + y^2 = 2a^2$ and the parabola $y = x^2/a$ in Figure 1.98. If a is increased, the point P traces out a curve. For $a > 0$, find the equation of this curve.

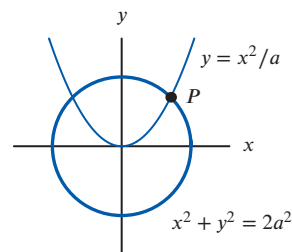


Figure 1.98

76. When an object of mass m moves with a velocity v that is small compared to the velocity of light, c , its energy is given approximately by

$$E \approx \frac{1}{2}mv^2.$$

If v is comparable in size to c , then the energy must be computed by the exact formula

$$E = mc^2 \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1 \right).$$

(a) Plot a graph of both functions for E against v for $0 \leq v \leq 5 \cdot 10^8$ and $0 \leq E \leq 5 \cdot 10^{17}$. Take $m = 1$ kg and $c = 3 \cdot 10^8$ m/sec. Explain how you can predict from the exact formula the position of the vertical asymptote.

(b) What do the graphs tell you about the approximation? For what values of v does the first formula give a good approximation to E ?

77. If $y = 100x^{-0.2}$ and $z = \ln x$, explain why y is an exponential function of z .

Strengthen Your Understanding

- In Problems 78–83, explain what is wrong with the statement.
78. The graph of a polynomial of degree 5 cuts the horizontal axis five times.
79. A fourth-degree polynomial tends to infinity as $x \rightarrow \infty$.
80. A rational function has a vertical asymptote.
81. $x^5 > x^3$ for $x > 0$
82. Every rational function has a horizontal asymptote.
83. A function cannot cross its horizontal asymptote.
- In Problems 84–89, give an example of:
84. A polynomial of degree 3 whose graph cuts the horizontal axis three times to the right of the origin.
85. A rational function with horizontal asymptote $y = 3$.
86. A rational function that is not a polynomial and that has no vertical asymptote.
87. A function that has a vertical asymptote at $x = -7\pi$.
88. A function that has exactly 17 vertical asymptotes.
89. A function that has a vertical asymptote which is crossed by a horizontal asymptote.
- Are the statements in Problems 90–93 true or false? Give an explanation for your answer.
90. Every polynomial of even degree has a least one real zero.
91. Every polynomial of odd degree has a least one real zero.
92. The composition of two quadratic functions is quadratic.
93. For $x > 0$ the graph of the rational function $f(x) = \frac{5(x^3 - x)}{x^2 + x}$ is a line.
94. List the following functions in order from smallest to largest as $x \rightarrow \infty$ (that is, as x increases without bound).
- (a) $f(x) = -5x$ (b) $g(x) = 10^x$
 (c) $h(x) = 0.9^x$ (d) $k(x) = x^5$
 (e) $l(x) = \pi^x$

1.7 INTRODUCTION TO LIMITS AND CONTINUITY

In this section we switch focus from families of functions to an intuitive introduction to *continuity* and *limits*. Limits are central to a full understanding of calculus.

The Idea of Continuity

Roughly speaking, a function is *continuous on an interval* if its graph has no breaks, jumps, or holes in that interval. A function is *continuous at a point* if nearby values of the independent variable give nearby values of the function. Most real-world phenomena are modeled using continuous functions.

The Graphical Viewpoint: Continuity on an Interval

A continuous function has a graph which can be drawn without lifting the pencil from the paper.

Example: The function $f(x) = 3x^3 - x^2 + 2x - 1$ is continuous on any interval. (See Figure 1.99.)

Example: The function $f(x) = 1/x$ is not defined at $x = 0$. It is continuous on any interval not containing 0. (See Figure 1.100.)

Example: A company rents cars for \$7 per hour or fraction thereof, so it costs \$7 for a trip of one hour or less, \$14 for a trip between one and two hours, and so on. If $p(x)$ is the price of trip lasting x hours, then its graph (in Figure 1.101) is a series of steps. This function is not continuous on any open interval containing a positive integer because the graph jumps at these points.

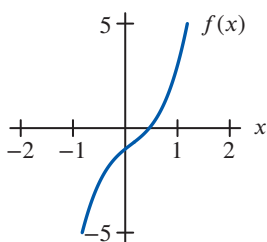


Figure 1.99: The graph of $f(x) = 3x^3 - x^2 + 2x - 1$

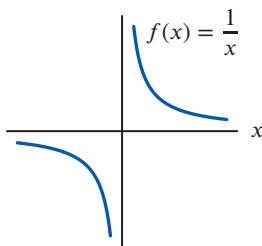


Figure 1.100: Not continuous on any interval containing 0

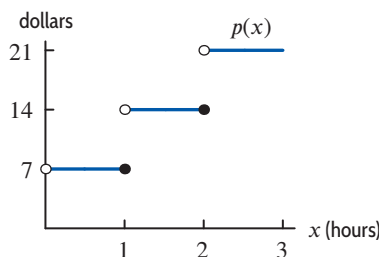


Figure 1.101: Cost of renting a car

The Numerical Viewpoint: Continuity at a Point

In practical work, continuity of a function at a point is important because it means that small errors in the independent variable lead to small errors in the value of the function. Conversely, if a function is not continuous at a point, a small error in input measurement can lead to an enormous error in output.

Example: Suppose that $f(x) = x^2$ and that we want to compute $f(\pi)$. Knowing f is continuous tells us that taking $x = 3.14$ should give a good approximation to $f(\pi)$, and that we can get as accurate an approximation to $f(\pi)$ as we want by using enough decimals of π .

Example: If $p(x)$ is the price of renting a car graphed in Figure 1.101, then $p(0.99) = p(1) = \$7$, whereas $p(1.01) = \$14$, because as soon as we pass one hour, the price jumps to \$14. So a small difference in time can lead to a significant difference in the cost. Hence p is not continuous at $x = 1$. As we see from its graph, this also means it is not continuous on any open interval including $x = 1$.

We express continuity at a point by saying that if $f(x)$ is continuous at $x = c$, the values of $f(x)$ approach $f(c)$ as x approaches c .

Example 1 Investigate the continuity of $f(x) = x^2$ at $x = 2$.

Solution From Table 1.25, it appears that the values of $f(x) = x^2$ approach $f(2) = 4$ as x approaches 2. Thus f appears to be continuous at $x = 2$.

Table 1.25 Values of x^2 near $x = 2$

x	1.9	1.99	1.999	2.001	2.01	2.1
x^2	3.61	3.96	3.996	4.004	4.04	4.41

Which Functions Are Continuous?

Most of the functions we have seen are continuous everywhere they are defined. For example, exponential functions, polynomials, and the sine and cosine are continuous everywhere. Rational functions are continuous on any interval in which their denominators are not zero. Functions created by adding, multiplying, or composing continuous functions are also continuous.⁶²

The Idea of a Limit

Continuity at a point describes behavior of a function *near* a point, as well as *at* the point. To find the value of a function at a point we can just evaluate the function there. To focus on what happens to the values of a function near a point, we introduce the concept of limit. First some notation:

We write $\lim_{x \rightarrow c} f(x) = L$ if the values of $f(x)$ approach L as x approaches c .

How should we find the limit L , or even know whether such a number exists? We look for trends in the values of $f(x)$ as x gets closer to c , but $x \neq c$. A graph from a calculator or computer often helps.

⁶²For more details, see Theorem 1.8 in Section 1.8.

Example 2 Use a graph to estimate $\lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \right)$. (Use radians.)

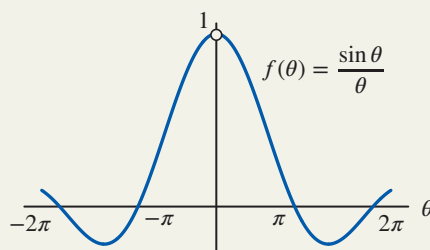


Figure 1.102: Find the limit as $\theta \rightarrow 0$

Solution Figure 1.102 shows that as θ approaches 0 from either side, the value of $\sin \theta / \theta$ appears to approach 1, suggesting that $\lim_{\theta \rightarrow 0} (\sin \theta / \theta) = 1$. Zooming in on the graph near $\theta = 0$ provides further support for this conclusion. Notice that $\sin \theta / \theta$ is undefined at $\theta = 0$.

Figure 1.102 suggests that $\lim_{\theta \rightarrow 0} (\sin \theta / \theta) = 1$, but to be sure we need to be more precise about words like “approach” and “close.”

Definition of Limit

By the beginning of the 19th century, calculus had proved its worth, and there was no doubt about the correctness of its answers. However, it was not until the work of the French mathematician Augustin Cauchy (1789–1857) that calculus was put on a rigorous footing. Cauchy gave a formal definition of the limit, similar to the following:

A function f is defined on an interval around c , except perhaps at the point $x = c$. We define the **limit** of the function $f(x)$ as x approaches c , written $\lim_{x \rightarrow c} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write

$$\lim_{x \rightarrow c} f(x) = L.$$

Note that this definition ensures a limit, if it exists, cannot have more than one value.

Finding Limits

As we saw in Example 2, we can often estimate the value of a limit from its graph. We can also estimate a limit by using a table of values. However, no matter how closely we zoom in on a graph at a point or how close to a point we evaluate a function, there are always points closer, so in using these techniques for estimating limits we are never completely sure we have the exact answer. In Example 4 and Section 1.9 we show how a limit can be found exactly using algebra.

Definition of Continuity

In Example 2 we saw that the values of $\sin \theta / \theta$ approach 1 as θ approaches 0. However, $\sin \theta / \theta$ is not continuous at $\theta = 0$ since its graph has a hole there; see Figure 1.102. This illustrates an important difference between limits and continuity: a limit is only concerned with what happens *near* a point but continuity depends on what happens *near* a point and *at* that point. We now give a more precise definition of continuity using limits.

The function f is **continuous** at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

In other words, $f(x)$ is as close as we want to $f(c)$ provided x is close enough to c . The function is **continuous on an interval** $[a, b]$ if it is continuous at every point in the interval.⁶³

The Intermediate Value Theorem

Continuous functions have many useful properties. For example, to locate the zeros of a continuous function, we can look for intervals where the function changes sign.

Example 3

What do the values in Table 1.26 tell you about the zeros of $f(x) = \cos x - 2x^2$?

Table 1.26

x	$f(x)$
0	1.00
0.2	0.90
0.4	0.60
0.6	0.11
0.8	-0.58
1.0	-1.46

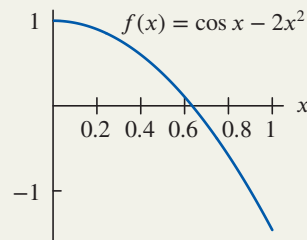


Figure 1.103: Zeros occur where the graph of a continuous function crosses the horizontal axis

Solution

Since $f(x)$ is the difference of two continuous functions, it is continuous. Since f is positive at $x = 0.6$ and negative at $x = 0.8$, and its graph has no breaks, the graph must cross the axis between these points. We conclude that $f(x) = \cos x - 2x^2$ has at least one zero in the interval $0.6 < x < 0.8$. Figure 1.103 suggests that there is only one zero in the interval $0 \leq x \leq 1$, but we cannot be sure of this from a graph or a table of values.

In the previous example, we concluded that $f(x) = \cos x - 2x^2$ has a zero between $x = 0$ and $x = 1$ because it changed from positive to negative without skipping values in between—in other words, because it is continuous. If it were not continuous, the graph could jump across the x -axis, changing sign but not creating a zero. For example, $f(x) = 1/x$ has opposite signs at $x = -1$ and $x = 1$, but no zeros for $-1 \leq x \leq 1$ because of the break at $x = 0$. (See Figure 1.100.)

More generally, the intuitive notion of continuity tells us that, as we follow the graph of a continuous function f from some point $(a, f(a))$ to another point $(b, f(b))$, then f takes on all intermediate values between $f(a)$ and $f(b)$. (See Figure 1.104.) The formal statement of this property is known as the Intermediate Value Theorem, and is a powerful tool in theoretical proofs in calculus.

Theorem 1.1: Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

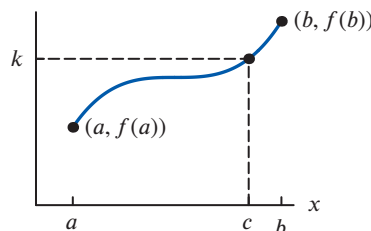


Figure 1.104: The Intermediate Value Theorem guarantees that a continuous f takes on the value k

⁶³If c is an endpoint of the interval, we can define continuity at $x = c$ using one-sided limits at c ; see Section 1.8.

Finding Limits Exactly Using Continuity and Algebra

The concept of limit is critical to a formal justification of calculus, so an understanding of limit outside the context of continuity is important. We have already seen how to use a graph or a table of values to estimate a limit. We now see how to find the exact value of a limit.

Limits of Continuous Functions

In Example 1, the limit of $f(x) = x^2$ at $x = 2$ is the same as $f(2)$. This is because $f(x)$ is a continuous function at $x = 2$, and this is precisely the definition of continuity:

Limits of Continuous Functions

If a function $f(x)$ is continuous at $x = c$, the limit is the value of $f(x)$ there:

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Thus, to find the limits for a continuous function: Substitute c .

Limits of Functions Which Are Not Continuous

If a function is not continuous at a point $x = c$ it is still possible for the limit to exist, but it can be hard to find. Sometimes such limits can be computed using algebra.

Example 4 Use a graph to estimate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$ and then use algebra to find the limit exactly.

Solution Evaluating $(x^2 - 9)/(x - 3)$ at $x = 3$ gives us $0/0$, so the function is undefined at $x = 3$, shown as a hole in Figure 1.105. However, we see that as x approaches 3 from either side, the value of $(x^2 - 9)/(x - 3)$ appears to approach 6, suggesting the limit is 6.

To find this limit exactly, we first use algebra to rewrite the function. We have

$$\frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{x - 3}.$$

Since $x \neq 3$ in the limit, we can cancel the common factor $x - 3$ to see

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} \frac{(x + 3)(x - 3)}{x - 3} = \lim_{x \rightarrow 3} (x + 3).$$

Since $x + 3$ is continuous, we have

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} = \lim_{x \rightarrow 3} (x + 3) = 3 + 3 = 6.$$

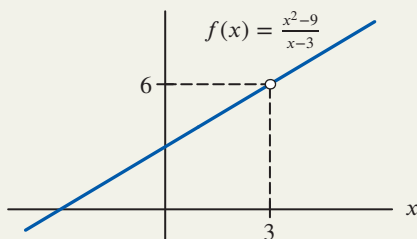


Figure 1.105: Find the limit as $x \rightarrow 3$

If a function is continuous at a point, then its limit must exist at that point. Example 4 illustrates that the existence of a limit is not enough to guarantee continuity.

More precisely, as we can see from Figure 1.105, the function $f(x) = (x^2 - 9)/(x - 3)$ is not continuous at $x = 3$ as there is a hole in its graph at that point. However, Example 4 shows that the limit $\lim_{x \rightarrow 3} f(x)$ does exist and is equal to 6. This is because for a function $f(x)$ to be continuous at a point $x = c$, the limit has to exist at $x = c$, the function has to be defined at $x = c$ and the limit has to equal the function. In this case, even though the limit does exist, the function does not have a value at $x = 3$, so it cannot be continuous there.

When Limits Do Not Exist

Whenever there is no number L such that $\lim_{x \rightarrow c} f(x) = L$, we say $\lim_{x \rightarrow c} f(x)$ does not exist. The following three examples show some of the ways in which a limit may fail to exist.

Example 5 Use a graph to explain why $\lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2}$ does not exist.

Solution As we see in Figure 1.106, the value of $f(x) = \frac{|x - 2|}{x - 2}$ approaches -1 as x approaches 2 from the left and the value of $f(x)$ approaches 1 as x approaches 2 from the right. This means if $\lim_{x \rightarrow 2} \frac{|x - 2|}{x - 2} = L$ then L would have to be both 1 and -1 . Since L cannot have two different values, the limit does not exist.

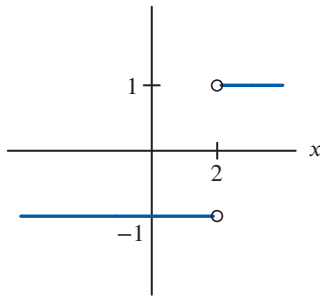


Figure 1.106: Graph of $f(x) = |x - 2|/(x - 2)$

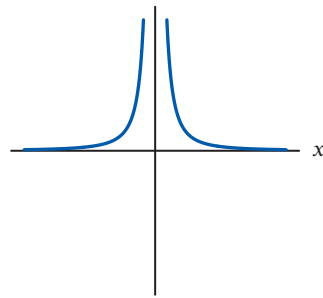


Figure 1.107: Graph of $g(x) = 1/x^2$

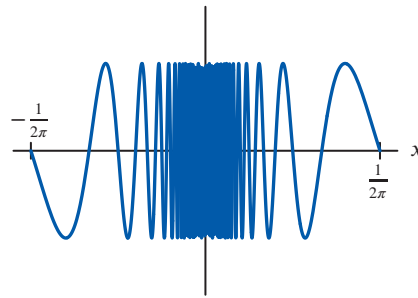


Figure 1.108: Graph of $h(x) = \sin(1/x)$

Example 6 Use a graph to explain why $\lim_{x \rightarrow 0} \frac{1}{x^2}$ does not exist.

Solution As x approaches zero, $g(x) = 1/x^2$ becomes arbitrarily large, so it cannot approach any finite number L . See Figure 1.107. Therefore we say $1/x^2$ has no limit as $x \rightarrow 0$.

Since $1/x^2$ gets arbitrarily large on both sides of $x = 0$, we can write $\lim_{x \rightarrow 0} 1/x^2 = \infty$. The limit still does not exist since it does not approach a real number L , but we can use limit notation to describe its behavior. This behavior may also be described as “diverging to infinity.”

Example 7 Explain why $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Solution The sine function has values between -1 and 1 . The graph of $h(x) = \sin(1/x)$ in Figure 1.108 oscillates more and more rapidly as $x \rightarrow 0$. There are x -values approaching 0 where $\sin(1/x) = -1$. There are also x -values approaching 0 where $\sin(1/x) = 1$. So if the limit existed, it would have to be both -1 and 1 . Thus, the limit does not exist.

Notice that in all three examples, the function is not continuous at the given point. This is because continuity requires the existence of a limit, so failure of a limit to exist at a point automatically means a function is not continuous there.

Summary for Section 1.7

- A function f is defined on an interval around c , except perhaps at the point $x = c$. We define the **limit** of the function $f(x)$ as x approaches c , written $\lim_{x \rightarrow c} f(x)$, to be a number L (if one exists) such that $f(x)$ is as close to L as we want whenever x is sufficiently close to c (but $x \neq c$). If L exists, we write

$$\lim_{x \rightarrow c} f(x) = L.$$

- The function f is **continuous** at $x = c$ if f is defined at $x = c$ and if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

In other words, $f(x)$ is as close as we want to $f(c)$ provided x is close enough to c . The function is **continuous on an interval** $[a, b]$ if it is continuous at every point in the interval.

- **Intermediate value theorem:** Suppose f is continuous on a closed interval $[a, b]$. If k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.
- **Limits of continuous functions:** If a function $f(x)$ is continuous at $x = c$, the limit is the value of $f(x)$ there:

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Thus, to find the limits for a continuous function: Substitute c .

Exercises and Problems for Section 1.7

EXERCISES

- Using Figure 1.109, find all values of x for which f is not continuous.
 - List the largest open intervals on which f is continuous.

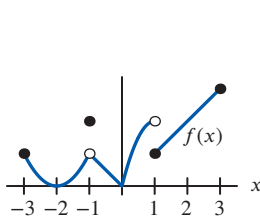


Figure 1.109

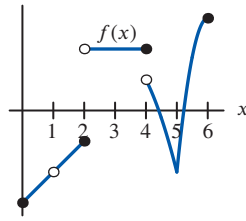


Figure 1.110

- Using Figure 1.110, find all values of x for which f is not continuous.
 - List the largest open intervals on which f is continuous.
- Use the graph of $f(x)$ in Figure 1.111 to give approximate values for the following limits (if they exist). If the limit does not exist, say so.

- (a) $\lim_{x \rightarrow -3} f(x)$ (b) $\lim_{x \rightarrow -2} f(x)$ (c) $\lim_{x \rightarrow -1} f(x)$
 (d) $\lim_{x \rightarrow 0} f(x)$ (e) $\lim_{x \rightarrow 1} f(x)$ (f) $\lim_{x \rightarrow 3} f(x)$

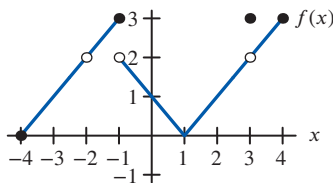


Figure 1.111

■ In Exercises 4–5, the graph of $y = f(x)$ is given.

- Give the x -values where $f(x)$ is not continuous.
- Does the limit of $f(x)$ exist at each x -value where $f(x)$ is not continuous? If so, give the value of the limit.

4.

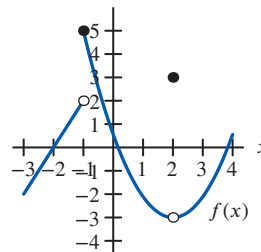


Figure 1.112

5.

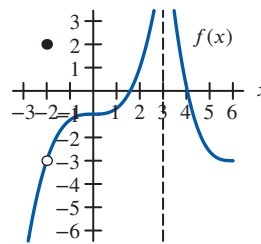


Figure 1.113

- Assume $f(x)$ is continuous on an interval around $x = 0$, except possibly at $x = 0$. What does the table of values suggest as the value of $\lim_{x \rightarrow 0} f(x)$? Does the limit definitely have this value?

x	-0.1	-0.01	0.01	0.1
$f(x)$	1.987	1.999	1.999	1.987

7. Assume $g(t)$ is continuous on an interval around $t = 3$, except possibly at $t = 3$. What does the table of values suggest as the value of $\lim_{t \rightarrow 3} g(t)$? Does the limit definitely have this value?

t	2.9	2.99	3.01	3.1
$g(t)$	0.948	0.995	1.005	1.049

■ In Exercises 8–9,

- (a) Make a table of values of $f(x)$ for $x = -0.1, -0.01, 0.01,$ and 0.1 .
 (b) Use the table to estimate $\lim_{x \rightarrow 0} f(x)$.

8. $f(x) = \frac{\sin(5x)}{x}$ 9. $f(x) = \frac{e^{3x} - 1}{x}$

10. Use a table of values to estimate $\lim_{x \rightarrow 1} (5 + \ln x)$.

■ In Exercises 11–16, is the function continuous on the interval?

11. $\frac{1}{x-2}$ on $[-1, 1]$ 12. $\frac{1}{x-2}$ on $[0, 3]$
 13. $\frac{1}{\sqrt{2x-5}}$ on $[3, 4]$ 14. $2x + x^{-1}$ on $[-1, 1]$
 15. $\frac{1}{\cos x}$ on $[0, \pi]$ 16. $\frac{e^{\sin \theta}}{\cos \theta}$ on $[-\frac{\pi}{4}, \frac{\pi}{4}]$

17. Are the following functions continuous? Explain.

(a) $f(x) = \begin{cases} x & x \leq 1 \\ x^2 & 1 < x \end{cases}$
 (b) $g(x) = \begin{cases} x & x \leq 3 \\ x^2 & 3 < x \end{cases}$

18. Let f be the function given by

$$f(x) = \begin{cases} 4 - x & 0 \leq x \leq 3 \\ x^2 - 8x + 17 & 3 < x < 5 \\ 12 - 2x & 5 \leq x \leq 6 \end{cases}$$

- (a) Find all values of x for which f is not continuous.

- (b) List the largest open intervals on which f is continuous.

■ In Exercises 19–22, show there is a number c , with $0 \leq c \leq 1$, such that $f(c) = 0$.

19. $f(x) = x^3 + x^2 - 1$ 20. $f(x) = e^x - 3x$
 21. $f(x) = x - \cos x$ 22. $f(x) = 2^x - 1/x$

■ In Exercises 23–28, use algebra to find the limit exactly.

23. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ 24. $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3}$
 25. $\lim_{x \rightarrow 1} \frac{x^2 + 4x - 5}{x - 1}$ 26. $\lim_{x \rightarrow 1} \frac{x^2 - 4x + 3}{x^2 + 3x - 4}$
 27. $\lim_{x \rightarrow 1} \frac{x^2 + 4}{x + 8}$ 28. $\lim_{h \rightarrow 0} \frac{(5 + h)^2 - 5^2}{h}$

■ For Exercises 29–30, find the value of the constant k such that

29. $\lim_{x \rightarrow 5} (kx + 10) = 20$ 30. $\lim_{x \rightarrow 2} \frac{(x + 6)(x - k)}{x^2 + x} = 4$

■ In Exercises 31–34 find k so that the function is continuous on any interval.

31. $f(x) = \begin{cases} kx & x \leq 3 \\ 5 & 3 < x \end{cases}$
 32. $f(x) = \begin{cases} kx & 0 \leq x < 2 \\ 3x^2 & 2 \leq x \end{cases}$
 33. $h(x) = \begin{cases} kx & 0 \leq x < 1 \\ x + 3 & 1 \leq x \leq 5 \end{cases}$
 34. $g(t) = \begin{cases} t + k & t \leq 5 \\ kt & 5 < t \end{cases}$

PROBLEMS

35. Which of the following are continuous functions of time?
 (a) The quantity of gas in the tank of a car on a journey between New York and Boston.
 (b) The number of students enrolled in a class during a semester.
 (c) The age of the oldest person alive.
36. An electrical circuit switches instantaneously from a 6-volt battery to a 12-volt battery 7 seconds after being turned on. Graph the battery voltage against time. Give formulas for the function represented by your graph. What can you say about the continuity of this function?
37. A stone dropped from the top of a cliff falls freely for 5 seconds before it hits the ground. Figure 1.114 shows

the speed $v = f(t)$ (in meters/sec) of the stone as a function of time t in seconds for $0 \leq t \leq 7$.

- (a) Is f continuous? Explain your answer in the context of the problem.
 (b) Sketch a graph of the height, $h = g(t)$, of the stone for $0 \leq t \leq 7$. Is g continuous? Explain.

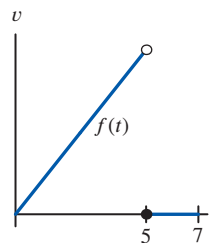


Figure 1.114

38. Beginning at time $t = 0$, a car undergoing a crash test accelerates for two seconds, maintains a constant speed for one second, and then crashes into a test barrier at $t = 3$ seconds.
- (a) Sketch a possible graph of $v = f(t)$, the speed of the car (in meters/sec) after t seconds, on the interval $0 \leq t \leq 4$.
- (b) Is the function f in part (a) continuous? Explain your answer in the context of this problem.
39. Discuss the continuity of the function g graphed in Figure 1.115 and defined as follows:

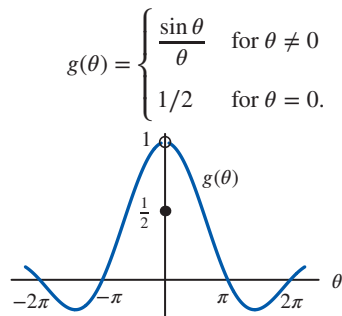


Figure 1.115

40. Is the following function continuous on $[-1, 1]$?

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

■ Estimate the limits in Problems 41–42 graphically.

41. $\lim_{x \rightarrow 0} \frac{|x|}{x}$ 42. $\lim_{x \rightarrow 0} x \ln |x|$

■ In Problems 43–48, use a graph to estimate the limit. Use radians unless degrees are indicated by $^\circ$.

43. $\lim_{\theta \rightarrow 0} \frac{\sin(2\theta)}{\theta}$ 44. $\lim_{\theta \rightarrow 0} \frac{\sin \theta^\circ}{\theta^\circ}$

45. $\lim_{h \rightarrow 0} \frac{e^h - 1}{h}$ 46. $\lim_{h \rightarrow 0} \frac{e^{5h} - 1}{h}$

47. $\lim_{h \rightarrow 0} \frac{2^h - 1}{h}$ 48. $\lim_{h \rightarrow 0} \frac{\cos(3h) - 1}{h}$

■ In Problems 49–56, find the limit.

49. $\lim_{x \rightarrow 4} \frac{2x^2 - 32}{x + 4}$ 50. $\lim_{x \rightarrow -4} \frac{2x^2 - 32}{x + 4}$

51. $\lim_{x \rightarrow 0} \frac{3^x - 1}{\sin x}$ 52. $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$

53. $\lim_{x \rightarrow -1} \frac{x^4 - x^2}{x + 1}$ 54. $\lim_{x \rightarrow 2} \frac{16 - 8x}{2x - 4}$

55. $\lim_{x \rightarrow 0} \frac{1 - 3^x}{e^x - 1}$ 56. $\lim_{x \rightarrow 2} \frac{4^x - 16}{2^x - 4}$

■ In Problems 57–62, find a value of k , if any, making $h(x)$ continuous on $[0, 5]$.

57. $h(x) = \begin{cases} k \cos x & 0 \leq x \leq \pi \\ 12 - x & \pi < x \end{cases}$

58. $h(x) = \begin{cases} kx & 0 \leq x \leq 1 \\ 2kx + 3 & 1 < x \leq 5. \end{cases}$

59. $h(x) = \begin{cases} k \sin x & 0 \leq x \leq \pi \\ x + 4 & \pi < x \leq 5. \end{cases}$

60. $h(x) = \begin{cases} e^{kx} & 0 \leq x < 2 \\ x + 1 & 2 \leq x \leq 5. \end{cases}$

61. $h(x) = \begin{cases} 0.5x & 0 \leq x < 1 \\ \sin(kx) & 1 \leq x \leq 5. \end{cases}$

62. $h(x) = \begin{cases} \ln(kx + 1) & 0 \leq x \leq 2 \\ x + 4 & 2 < x \leq 5. \end{cases}$

63. (a) Use Figure 1.116 to decide at what points $f(x)$ is not continuous.
- (b) At what points is the function $|f(x)|$ not continuous?

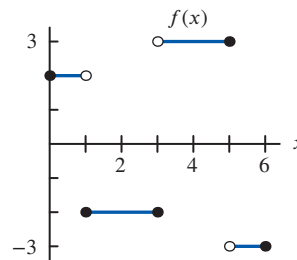


Figure 1.116

64. For t in months, a population, in thousands, is approximated by a continuous function

$$P(t) = \begin{cases} e^{kt} & 0 \leq t \leq 12 \\ 100 & t > 12. \end{cases}$$

- (a) What is the initial value of the population?
- (b) What must be the value of k ?
- (c) Describe in words how the population is changing.
65. A 0.6 ml dose of a drug is injected into a patient steadily for half a second. At the end of this time, the quantity, Q , of the drug in the body starts to decay exponentially at a continuous rate of 0.2% per second. Using formulas, express Q as a continuous function of time, t in seconds.

■ In Problems 66–69, at what values of x is the function not continuous? If possible, give a value for the function at each point of discontinuity so the function is continuous everywhere.

66. $f(x) = \frac{x^2 - 1}{x + 1}$ 67. $g(x) = \frac{x^2 - 4x - 5}{x - 5}$

68. $f(z) = \frac{z^2 - 11z + 18}{2z - 18}$ 69. $q(t) = \frac{-t^3 + 9t}{t^2 - 9}$

■ In Problems 70–73, is the function continuous for all x ? If not, say where it is not continuous and explain in what way the definition of continuity is not satisfied.

70. $f(x) = 1/x$

71. $f(x) = \begin{cases} |x|/x & x \neq 0 \\ 0 & x = 0 \end{cases}$

72. $f(x) = \begin{cases} x/x & x \neq 0 \\ 1 & x = 0 \end{cases}$

73. $f(x) = \begin{cases} 2x/x & x \neq 0 \\ 3 & x = 0 \end{cases}$

74. Graph three different functions, continuous on $0 \leq x \leq 1$, and with the values in the table. The first function is to have exactly one zero in $[0, 1]$, the second is to have at least two zeros in the interval $[0.6, 0.8]$, and the third is to have at least two zeros in the interval $[0, 0.6]$.

x	0	0.2	0.4	0.6	0.8	1.0
$f(x)$	1.00	0.90	0.60	0.11	-0.58	-1.46

75. Let $p(x)$ be a cubic polynomial with $p(5) < 0$, $p(10) > 0$, and $p(12) < 0$. What can you say about the number and location of zeros of $p(x)$?

76. (a) What does a graph of $y = e^x$ and $y = 4 - x^2$ tell you about the solutions to the equation $e^x = 4 - x^2$?

(b) Evaluate $f(x) = e^x + x^2 - 4$ at $x = -4, -3, -2, -1, 0, 1, 2, 3, 4$. In which intervals do the solutions to $e^x = 4 - x^2$ lie?

77. (a) Does $f(x)$ satisfy the conditions for the Intermediate Value Theorem on $0 \leq x \leq 2$ if

$$f(x) = \begin{cases} e^x & 0 \leq x \leq 1 \\ 4 + (x-1)^2 & 1 < x \leq 2 \end{cases}$$

(b) What are $f(0)$ and $f(2)$? Can you find a value of k between $f(0)$ and $f(2)$ such that the equation $f(x) = k$ has no solution? If so, what is it?

78. Let $g(x)$ be continuous with $g(0) = 3$, $g(1) = 8$, $g(2) = 4$. Use the Intermediate Value Theorem to explain why $g(x)$ is not invertible.

79. By graphing $y = (1+x)^{1/x}$, estimate $\lim_{x \rightarrow 0} (1+x)^{1/x}$. You should recognize the answer you get. What does the limit appear to be?

80. Investigate $\lim_{h \rightarrow 0} (1+h)^{1/h}$ numerically.

81. Let $f(x) = \sin(1/x)$.

(a) Find a sequence of x -values that approach 0 such that $\sin(1/x) = 0$.

[Hint: Use the fact that $\sin(\pi) = \sin(2\pi) = \sin(3\pi) = \dots = \sin(n\pi) = 0$.]

(b) Find a sequence of x -values that approach 0 such that $\sin(1/x) = 1$.

[Hint: Use the fact that $\sin(n\pi/2) = 1$ if $n = 1, 5, 9, \dots$.]

(c) Find a sequence of x -values that approach 0 such that $\sin(1/x) = -1$.

(d) Explain why your answers to any two of parts (a)–(c) show that $\lim_{x \rightarrow 0} \sin(1/x)$ does not exist.

Strengthen Your Understanding

■ In Problems 82–84, explain what is wrong with the statement.

82. For any function $f(x)$, if $f(a) = 2$ and $f(b) = 4$, the Intermediate Value Theorem says that f takes on the value 3 for some x between a and b .

83. If $f(x)$ is continuous on $0 \leq x \leq 2$ and if $f(0) = 0$ and $f(2) = 10$, the Intermediate Value Theorem says that $f(1) = 5$.

84. If $\lim_{x \rightarrow c} f(x)$ exists, then $f(x)$ is continuous at $x = c$.

■ In Problems 85–88, give an example of:

85. A function which is defined for all x and continuous everywhere except at $x = 15$.

86. A function to which the Intermediate Value Theorem does not apply on the interval $-1 \leq x \leq 1$.

87. A function that is continuous on $[0, 1]$ but not continuous on $[1, 3]$.

88. A function that is increasing but not continuous on $[0, 10]$.

■ Are the statements in Problems 89–91 true or false? Give an explanation for your answer.

89. If a function is not continuous at a point, then it is not defined at that point.

90. If f is continuous on the interval $[0, 10]$ and $f(0) = 0$ and $f(10) = 100$, then $f(c)$ cannot be negative for c in $[0, 10]$.

91. If $f(x)$ is not continuous on the interval $[a, b]$, then $f(x)$ must omit at least one value between $f(a)$ and $f(b)$.

1.8 EXTENDING THE IDEA OF A LIMIT

In Section 1.7, we introduced the idea of limit to describe the behavior of a function close to a point. We now extend limit notation to describe a function's behavior to values on only one side of a point and around an asymptote, and we extend limits to combinations of functions.

One-Sided Limits

When we write

$$\lim_{x \rightarrow 2} f(x),$$

we mean the number that $f(x)$ approaches as x approaches 2 *from both sides*. We examine values of $f(x)$ as x approaches 2 through values greater than 2 (such as 2.1, 2.01, 2.003) and values less than 2 (such as 1.9, 1.99, 1.994). If we want x to approach 2 only through values greater than 2, we write

$$\lim_{x \rightarrow 2^+} f(x)$$

for the number that $f(x)$ approaches (assuming such a number exists). Similarly,

$$\lim_{x \rightarrow 2^-} f(x)$$

denotes the number (if it exists) obtained by letting x approach 2 through values less than 2. We call $\lim_{x \rightarrow 2^+} f(x)$ a *right-hand limit* and $\lim_{x \rightarrow 2^-} f(x)$ a *left-hand limit*.

For the function graphed in Figure 1.117, we have

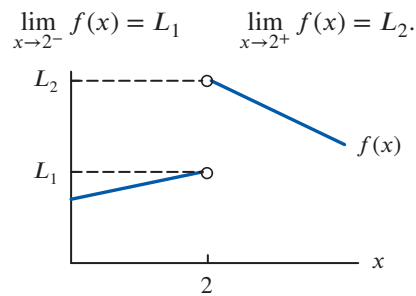


Figure 1.117: Left- and right-hand limits at $x = 2$

Observe that in this example $L_1 \neq L_2$; that is, $f(x)$ approaches different values as x approaches 2 from the left and from the right. Because of this, $\lim_{x \rightarrow 2} f(x)$ does not exist, since there is no single value that $f(x)$ approaches for all values of x close to 2.

Example 1

Find each of the following limits or explain why it does not exist:

(a) $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ (b) $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$ (c) $\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2}$

Solution

(a) In Example 5 of Section 1.7 we saw that $\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$ does not exist as it would have to take two different values. (See Figure 1.118.) However, it is still possible that the one-sided limits exist.

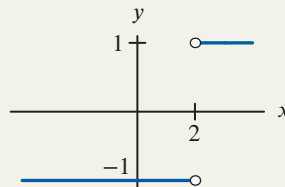


Figure 1.118: Limit of $y = |x-2|/(x-2)$ does not exist at $x = 2$

(b) To determine $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$, we look at the values of $|x-2|/(x-2)$ for values of x greater than 2. When $x > 2$,

$$\frac{|x-2|}{x-2} = \frac{x-2}{x-2} = 1.$$

So as x approaches 2 from the right, the value of $|x - 2|/(x - 2)$ is always 1. Therefore,

$$\lim_{x \rightarrow 2^+} \frac{|x - 2|}{x - 2} = 1.$$

(c) If $x < 2$, then

$$\frac{|x - 2|}{x - 2} = \frac{-(x - 2)}{x - 2} = -1.$$

So as x approaches 2 from the left, the value of $|x - 2|/(x - 2)$ is always -1 . Therefore,

$$\lim_{x \rightarrow 2^-} \frac{|x - 2|}{x - 2} = -1.$$

Limits and Asymptotes

We can use limit notation to describe the asymptotes of a function.

Horizontal Asymptotes and Limits

Sometimes we want to know what happens to $f(x)$ as x gets large, that is, the end behavior of f .

If $f(x)$ stays as close to a number L as we please when x is sufficiently large, then we write

$$\lim_{x \rightarrow \infty} f(x) = L.$$

Similarly, if $f(x)$ stays as close to L as we please when x is negative and has a sufficiently large absolute value, then we write

$$\lim_{x \rightarrow -\infty} f(x) = L.$$

The symbol ∞ does not represent a number. Writing $x \rightarrow \infty$ means that we consider arbitrarily large values of x . If the limit of $f(x)$ as $x \rightarrow \infty$ or $x \rightarrow -\infty$ is L , we say that the graph of f has $y = L$ as a *horizontal asymptote*.

Example 2 Investigate $\lim_{x \rightarrow \infty} \frac{1}{x}$ and $\lim_{x \rightarrow -\infty} \frac{1}{x}$.

Solution

A graph of $f(x) = 1/x$ in a large window shows $1/x$ approaching zero as x grows large in magnitude in either the positive or the negative direction (see Figure 1.119). This is as we would expect, since dividing 1 by larger and larger numbers yields answers which are closer and closer to zero. This suggests that

$$\lim_{x \rightarrow \infty} \frac{1}{x} = \lim_{x \rightarrow -\infty} \frac{1}{x} = 0,$$

and that $f(x) = 1/x$ has $y = 0$ as a horizontal asymptote as $x \rightarrow \pm\infty$.

There are many quantities in fields such as finance and medicine which change over time, t , and knowing the end behavior, or the limit of the quantity as $t \rightarrow \infty$, can be extremely important.

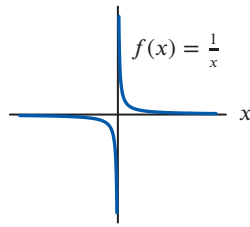


Figure 1.119: Limits describe the asymptotes of $f(x) = 1/x$

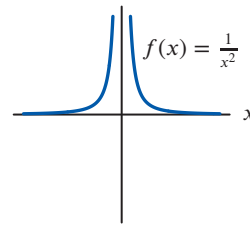


Figure 1.120: Vertical asymptote of $f(x) = 1/x^2$ at $x = 0$

Example 3

Total sales $P(t)$ of an app (in thousands) t months after the app was introduced is given in Table 1.27

and can be modeled by $P(t) = \frac{532}{1 + 869e^{-0.8t}}$.

- (a) Use Table 1.27 to estimate and interpret $\lim_{t \rightarrow \infty} P(t)$.
 (b) Find $\lim_{t \rightarrow \infty} P(t)$ using the model.

Table 1.27 Total sales of an app for increasing t

t (months)	8	12	16	20	24	28
$P(t)$ (sales in 1000s)	217.685	502.429	530.727	531.948	531.998	531.999

Solution

- (a) From Table 1.27, it appears that as t gets larger, $P(t)$ approaches 532. This makes sense as over time there is a limit on the number of people who are interested in the app. So the maximum potential sales for this app are estimated to be 532 thousand.
 (b) As t gets large, $e^{-0.8t}$ gets close to zero, so $P(t)$ gets close to $532/(1 + 0)$. This suggests

$$\lim_{t \rightarrow \infty} \frac{532}{1 + 869e^{-0.8t}} = 532.$$

Vertical Asymptotes and Limits

In Section 1.7, we wrote $\lim_{x \rightarrow 0} 1/x^2 = \infty$ because $1/x^2$ becomes arbitrarily large on both sides of zero; see Figure 1.120. Similarly, if a function becomes arbitrarily large in magnitude but negative on both sides of a point, we can say its limit at that point is $-\infty$. In each case, the limit does not exist since values of $1/x^2$ do not approach a real number; we are just able to use limit notation to describe its behavior.

We cannot say the same for $1/x$, even though it also has a vertical asymptote at $x = 0$, since the behavior of $1/x$ as x approaches 0 from the right does not match the behavior as x approaches 0 from the left. We can, however, use one-sided limit notation to describe this type of behavior; that is, we write $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$ and $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$.

Example 4

Describe the behavior of $f(t) = \frac{3 + 4t}{t + 2}$ near $t = -2$.

Solution

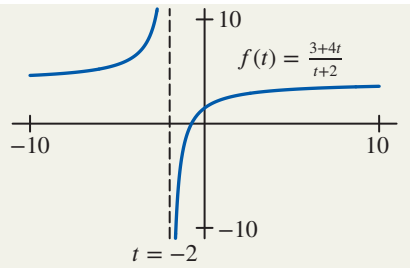
From Figure 1.121, we can see that as t approaches -2 from the right, $f(t)$ approaches $-\infty$. Therefore, we can say that

$$\lim_{t \rightarrow -2^+} f(t) = -\infty.$$

On the other hand, as t approaches -2 from the left, $f(t)$ gets arbitrarily large. Therefore,

$$\lim_{t \rightarrow -2^-} f(t) = \infty.$$

Since neither of the one-sided limits exists, the limit $\lim_{t \rightarrow -2} f(t)$ does not exist either.

Figure 1.121: Behavior of $f(t)$ near $t = -2$

Limits and Continuity for Combinations of Functions

The following properties of limits allow us to extend our knowledge of the limiting behavior of two functions to their sums and products, and sometimes to their quotients. These properties hold for both one- and two-sided limits, as well as limits at infinity (when $x \rightarrow \infty$ or $x \rightarrow -\infty$). They underlie many limit calculations, though we may not acknowledge them explicitly.

Theorem 1.2: Properties of Limits

Assuming all the limits on the right-hand side exist:

1. If b is a constant, then $\lim_{x \rightarrow c} (bf(x)) = b \left(\lim_{x \rightarrow c} f(x) \right)$.
2. $\lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$.
3. $\lim_{x \rightarrow c} (f(x)g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right)$.
4. $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$, provided $\lim_{x \rightarrow c} g(x) \neq 0$.
5. For any constant k , $\lim_{x \rightarrow c} k = k$.
6. $\lim_{x \rightarrow c} x = c$.

Since a function is continuous at a point only when it has a limit there, the properties of limits lead to similar properties of continuity for combinations of functions.

Theorem 1.3: Continuity of Sums, Products, and Quotients of Functions

If f and g are continuous on an interval and b is a constant, then, on that same interval,

1. $bf(x)$ is continuous.
2. $f(x) + g(x)$ is continuous.
3. $f(x)g(x)$ is continuous.
4. $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.

We prove the third of these properties.

Proof Let c be any point in the interval. We must show that $\lim_{x \rightarrow c} (f(x)g(x)) = f(c)g(c)$. Since $f(x)$ and $g(x)$ are continuous, we know that $\lim_{x \rightarrow c} f(x) = f(c)$ and $\lim_{x \rightarrow c} g(x) = g(c)$. So, by the third property of limits in Theorem 1.2,

$$\lim_{x \rightarrow c} (f(x)g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right) = f(c)g(c).$$

Since c was chosen arbitrarily, we have shown that $f(x)g(x)$ is continuous at every point in the interval.

Continuity also extends to compositions and inverses of functions.

Theorem 1.4: Continuity of Composite and Inverse Functions

If f and g are continuous, then

1. if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval.
2. if f has an inverse function f^{-1} , then f^{-1} is continuous.

Example 5

Use Theorems 1.3 and 1.4 to explain why the function is continuous.

(a) $f(x) = x^2 \cos x$ (b) $g(x) = \ln x$ (c) $h(x) = \sin(e^x)$

Solution

- (a) Since $y = x^2$ and $y = \cos x$ are continuous everywhere, by Theorem 1.3 their product $f(x) = x^2 \cos x$ is continuous everywhere.
- (b) Since $y = e^x$ is continuous everywhere it is defined and $\ln x$ is the inverse function of e^x , by Theorem 1.4, $g(x) = \ln x$ is continuous everywhere it is defined.
- (c) Since $y = \sin x$ and $y = e^x$ are continuous everywhere, by Theorem 1.4 their composition, $h(x) = \sin(e^x)$, is continuous.

Summary for Section 1.8

- **One-sided limits:** When we write $\lim_{x \rightarrow c} f(x)$, we mean the number that $f(x)$ approaches as x approaches c **from both sides**. If we want x to approach c only through values greater than c , we write $\lim_{x \rightarrow c^+} f(x)$ for the number that $f(x)$ approaches (assuming such a number exists). Similarly, $\lim_{x \rightarrow c^-} f(x)$ denotes the number (if it exists) obtained by letting x approach c through values less than c . We call $\lim_{x \rightarrow c^+} f(x)$ a **right-hand limit** and $\lim_{x \rightarrow c^-} f(x)$ a **left-hand limit**.
- **Horizontal asymptotes and limits:** If $f(x)$ stays as close to a number L as we please when x is sufficiently large, then we write $\lim_{x \rightarrow \infty} f(x) = L$. Similarly, if $f(x)$ stays as close to L as we please when x is negative and has a sufficiently large absolute value, then we write $\lim_{x \rightarrow -\infty} f(x) = L$. In either scenario, we say that the graph of f has $y = L$ as a **horizontal asymptote**.
- **Properties of limits:** Assuming all the limits on the right-hand side exist:
 - If b is a constant, then $\lim_{x \rightarrow c} (bf(x)) = b \left(\lim_{x \rightarrow c} f(x) \right)$.
 - $\lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$.
 - $\lim_{x \rightarrow c} (f(x)g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right)$.
 - $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$, provided $\lim_{x \rightarrow c} g(x) \neq 0$.
 - For any constant k , $\lim_{x \rightarrow c} k = k$.
 - $\lim_{x \rightarrow c} x = c$.
- **Continuity of sums, products, and quotients of functions:** If f and g are continuous on an interval and b is a constant, then, on that same interval,
 - $bf(x)$ is continuous.
 - $f(x) + g(x)$ is continuous.
 - $f(x)g(x)$ is continuous.
 - $f(x)/g(x)$ is continuous, provided $g(x) \neq 0$ on the interval.
- **Continuity of composite and inverse functions:**
 - if the composite function $f(g(x))$ is defined on an interval, then $f(g(x))$ is continuous on that interval.
 - if f has an inverse function f^{-1} , then f^{-1} is continuous.

Exercises and Problems for Section 1.8 Online Resource: Additional Problems for Section 1.8
EXERCISES

1. Use Figure 1.122 to find the limits or explain why they don't exist.

- (a) $\lim_{x \rightarrow -1^+} f(x)$ (b) $\lim_{x \rightarrow 0^-} f(x)$ (c) $\lim_{x \rightarrow 0} f(x)$
 (d) $\lim_{x \rightarrow -1^-} f(x)$ (e) $\lim_{x \rightarrow 1} f(x)$ (f) $\lim_{x \rightarrow 2^-} f(x)$

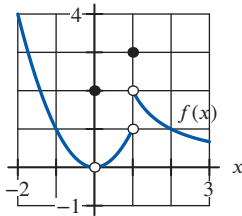


Figure 1.122

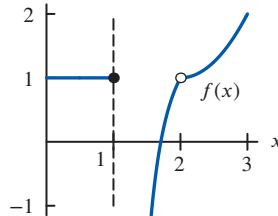


Figure 1.123

2. Use Figure 1.123 to estimate the following limits, if they exist.

- (a) $\lim_{x \rightarrow 1^-} f(x)$ (b) $\lim_{x \rightarrow 1^+} f(x)$ (c) $\lim_{x \rightarrow 1} f(x)$
 (d) $\lim_{x \rightarrow 2^-} f(x)$ (e) $\lim_{x \rightarrow 2^+} f(x)$ (f) $\lim_{x \rightarrow 2} f(x)$

3. Use Figure 1.124 to find each of the following or explain why they don't exist.

- (a) $f(-2)$ (b) $f(0)$ (c) $\lim_{x \rightarrow -4^+} f(x)$
 (d) $\lim_{x \rightarrow -2^-} f(x)$ (e) $\lim_{x \rightarrow -2^+} f(x)$ (f) $\lim_{x \rightarrow 0} f(x)$
 (g) $\lim_{x \rightarrow 2} f(x)$ (h) $\lim_{x \rightarrow 4^-} f(x)$

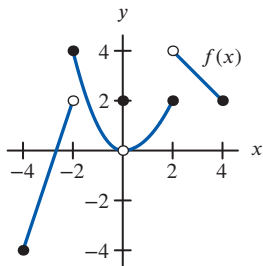


Figure 1.124

4. Use Figure 1.125 to find each of the following or explain why they don't exist.

- (a) $f(0)$ (b) $f(4)$ (c) $\lim_{x \rightarrow -2^-} f(x)$
 (d) $\lim_{x \rightarrow -2^+} f(x)$ (e) $\lim_{x \rightarrow -2} f(x)$ (f) $\lim_{x \rightarrow 0} f(x)$
 (g) $\lim_{x \rightarrow 2^-} f(x)$ (h) $\lim_{x \rightarrow 2^+} f(x)$ (i) $\lim_{x \rightarrow 2} f(x)$
 (j) $\lim_{x \rightarrow 4} f(x)$

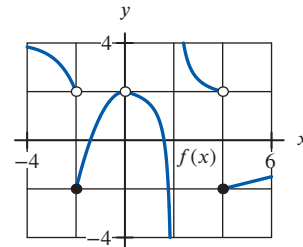


Figure 1.125

5. Use Figure 1.126 to estimate the following limits.

- (a) $\lim_{x \rightarrow \infty} f(x)$ (b) $\lim_{x \rightarrow -\infty} f(x)$

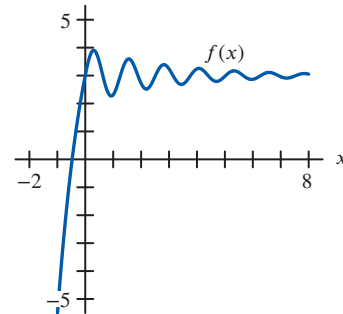


Figure 1.126

■ In Exercises 6–8, calculate the limit using the limit properties and $\lim_{x \rightarrow 2} f(x) = 7$, $\lim_{x \rightarrow 2} g(x) = -4$, $\lim_{x \rightarrow 2} h(x) = \frac{1}{2}$.

6. $\lim_{x \rightarrow 2} (f(x) - 2h(x))$ 7. $\lim_{x \rightarrow 2} (g(x))^2$
 8. $\lim_{x \rightarrow 2} \frac{f(x)}{g(x) \cdot h(x)}$

■ In Exercises 9–14, find the limit, given $\lim_{x \rightarrow c} f(x) = 27$ and $\lim_{x \rightarrow c} g(x) = 3$.

9. $\lim_{x \rightarrow c} (f(x) + g(x))$ 10. $\lim_{x \rightarrow c} (f(x) - g(x))$
 11. $\lim_{x \rightarrow c} (f(x)g(x))$ 12. $\lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)} \right)$
 13. $\lim_{x \rightarrow c} 10(f(x) - g(x))$ 14. $\lim_{x \rightarrow c} (g(x))^2$

15. Using Figures 1.127 and 1.128, estimate

- (a) $\lim_{x \rightarrow 1^-} (f(x) + g(x))$ (b) $\lim_{x \rightarrow 1^+} (f(x) + 2g(x))$
 (c) $\lim_{x \rightarrow 1^-} f(x)g(x)$ (d) $\lim_{x \rightarrow 1^+} \frac{f(x)}{g(x)}$

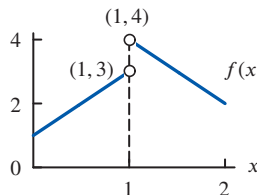


Figure 1.127

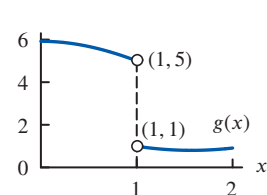


Figure 1.128

■ In Exercises 16–21, draw a possible graph of $f(x)$. Assume $f(x)$ is defined and continuous for all real x .

16. $\lim_{x \rightarrow \infty} f(x) = -\infty$ and $\lim_{x \rightarrow -\infty} f(x) = -\infty$
 17. $\lim_{x \rightarrow \infty} f(x) = -\infty$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$
 18. $\lim_{x \rightarrow \infty} f(x) = 1$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$
 19. $\lim_{x \rightarrow \infty} f(x) = -\infty$ and $\lim_{x \rightarrow -\infty} f(x) = 3$
 20. $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow -1} f(x) = 2$
 21. $\lim_{x \rightarrow 3} f(x) = 5$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$

■ In Exercises 22–34, find the limits using your understanding of the end behavior of each function.

22. $\lim_{x \rightarrow \infty} x^2$ 23. $\lim_{x \rightarrow -\infty} x^2$ 24. $\lim_{x \rightarrow -\infty} x^3$
 25. $\lim_{x \rightarrow \infty} x^3$ 26. $\lim_{x \rightarrow \infty} e^x$ 27. $\lim_{x \rightarrow \infty} e^{-x}$
 28. $\lim_{x \rightarrow \infty} 5^{-x}$ 29. $\lim_{x \rightarrow \infty} \sqrt{x}$ 30. $\lim_{x \rightarrow \infty} \ln x$
 31. $\lim_{x \rightarrow \infty} x^{-2}$ 32. $\lim_{x \rightarrow -\infty} x^{-2}$ 33. $\lim_{x \rightarrow -\infty} x^{-3}$
 34. $\lim_{x \rightarrow \infty} \left(\frac{1}{2}\right)^x$

■ In Exercises 35–40, give $\lim_{x \rightarrow -\infty} f(x)$ and $\lim_{x \rightarrow +\infty} f(x)$.

35. $f(x) = -x^4$
 36. $f(x) = 5 + 21x - 2x^3$
 37. $f(x) = x^5 + 25x^4 - 37x^3 - 200x^2 + 48x + 10$
 38. $f(x) = \frac{3x^3 + 6x^2 + 45}{5x^3 + 25x + 12}$
 39. $f(x) = 8x^{-3}$
 40. $f(x) = 25e^{0.08x}$

PROBLEMS

49. By graphing $y = (1 + 1/x)^x$, estimate $\lim_{x \rightarrow \infty} (1 + 1/x)^x$. You should recognize the answer you get.
 50. Investigate $\lim_{x \rightarrow \infty} (1 + 1/x)^x$ numerically.
 51. (a) Sketch $f(x) = e^{1/(x^2+0.0001)}$ around $x = 0$.
 (b) Is $f(x)$ continuous at $x = 0$? Use properties of continuous functions to confirm your answer.
 52. What does a calculator suggest about $\lim_{x \rightarrow 0^+} xe^{1/x}$? Does the limit appear to exist? Explain.

■ In Problems 53–62, evaluate $\lim_{x \rightarrow \infty}$ for the function.

53. $f(x) = \frac{x+3}{2-x}$ 54. $f(x) = \frac{\pi+3x}{\pi x-3}$
 55. $f(x) = \frac{x-5}{5+2x^2}$ 56. $f(x) = \frac{x^2+2x-1}{3+3x^2}$
 57. $f(x) = \frac{x^2+4}{x+3}$ 58. $f(x) = \frac{2x^3-16x^2}{4x^2+3x^3}$
 59. $f(x) = \frac{x^4+3x}{x^4+2x^5}$ 60. $f(x) = \frac{3e^x+2}{2e^x+3}$

41. Does $f(x) = \frac{|x|}{x}$ have right or left limits at 0? Is $f(x)$ continuous?

■ In Exercises 42–44, use algebra to evaluate $\lim_{x \rightarrow a^+} f(x)$, $\lim_{x \rightarrow a^-} f(x)$, and $\lim_{x \rightarrow a} f(x)$ if they exist. Sketch a graph to confirm your answers.

42. $a = 4, f(x) = \frac{|x-4|}{x-4}$
 43. $a = 2, f(x) = \frac{|x-2|}{x}$
 44. $a = 3, f(x) = \begin{cases} x^2 - 2, & 0 < x < 3 \\ 2, & x = 3 \\ 2x + 1, & 3 < x \end{cases}$

■ In Exercises 45–48, can $f(1)$ be defined to make $f(x)$ continuous at $x = 1$? If so, what is the value of $f(1)$?

45. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 2x & \text{if } x > 1 \end{cases}$
 46. $f(x) = \begin{cases} 4 - x^2 & \text{if } x < 1 \\ x + 1 & \text{if } x > 1 \end{cases}$
 47. $f(x) = \begin{cases} x - 1 & \text{if } x < 1 \\ 2 - 2x & \text{if } x > 1 \end{cases}$
 48. $f(x) = \begin{cases} \frac{1}{1-x} & \text{if } x < 1 \\ x & \text{if } x > 1 \end{cases}$

49. By graphing $y = (1 + 1/x)^x$, estimate $\lim_{x \rightarrow \infty} (1 + 1/x)^x$. You should recognize the answer you get.
 50. Investigate $\lim_{x \rightarrow \infty} (1 + 1/x)^x$ numerically.
 51. (a) Sketch $f(x) = e^{1/(x^2+0.0001)}$ around $x = 0$.
 (b) Is $f(x)$ continuous at $x = 0$? Use properties of continuous functions to confirm your answer.
 52. What does a calculator suggest about $\lim_{x \rightarrow 0^+} xe^{1/x}$? Does the limit appear to exist? Explain.

■ In Problems 53–62, evaluate $\lim_{x \rightarrow \infty}$ for the function.

53. $f(x) = \frac{x+3}{2-x}$ 54. $f(x) = \frac{\pi+3x}{\pi x-3}$
 55. $f(x) = \frac{x-5}{5+2x^2}$ 56. $f(x) = \frac{x^2+2x-1}{3+3x^2}$
 57. $f(x) = \frac{x^2+4}{x+3}$ 58. $f(x) = \frac{2x^3-16x^2}{4x^2+3x^3}$
 59. $f(x) = \frac{x^4+3x}{x^4+2x^5}$ 60. $f(x) = \frac{3e^x+2}{2e^x+3}$

61. $f(x) = \frac{2^{-x} + 5}{3^{-x} + 7}$ 62. $f(x) = \frac{2e^{-x} + 3}{3e^{-x} + 2}$

63. (a) Sketch the graph of a continuous function f with *all* of the following properties:
 (i) $f(0) = 2$
 (ii) $f(x)$ is decreasing for $0 \leq x \leq 3$
 (iii) $f(x)$ is increasing for $3 < x \leq 5$
 (iv) $f(x)$ is decreasing for $x > 5$
 (v) $f(x) \rightarrow 9$ as $x \rightarrow \infty$
 (b) Is it possible that the graph of f is concave down for all $x > 6$? Explain.

64. Sketch the graph of a function f with *all* of the following properties:
 (i) $f(0) = 2$ (ii) $f(4) = 2$
 (iii) $\lim_{x \rightarrow -\infty} f(x) = 2$ (iv) $\lim_{x \rightarrow 0} f(x) = 0$
 (v) $\lim_{x \rightarrow 2} f(x) = \infty$ (vi) $\lim_{x \rightarrow 4^-} f(x) = 2$
 (vii) $\lim_{x \rightarrow 4^+} f(x) = -2$

65. Sketch the graph of a function f with *all* of the following properties:

- (i) $f(-2) = 1$ (ii) $f(2) = -2$
 (iii) $f(3) = 3$ (iv) $\lim_{x \rightarrow -\infty} f(x) = -2$
 (v) $\lim_{x \rightarrow -1^-} f(x) = -\infty$ (vi) $\lim_{x \rightarrow -1^+} f(x) = \infty$
 (vii) $\lim_{x \rightarrow 2} f(x) = 1$ (viii) $\lim_{x \rightarrow 3^-} f(x) = 3$
 (ix) $\lim_{x \rightarrow 3^+} f(x) = 2$ (x) $\lim_{x \rightarrow \infty} f(x) = 1$

66. The graph of $f(x)$ has a horizontal asymptote at $y = -4$, a vertical asymptote at $x = 3$, and no other asymptotes.

- (a) Find a value of a such that $\lim_{x \rightarrow a} f(x)$ does not exist.
 (b) If $\lim_{x \rightarrow \infty} f(x)$ exists, what is its value?

67. A patient takes a 100 mg dose of a drug once daily for four days starting at time $t = 0$ (t in days). Figure 1.129 shows a graph of $Q = f(t)$, the amount of the drug in the patient's body, in mg, after t days.

- (a) Estimate and interpret each of the following:
 (i) $\lim_{t \rightarrow 1^-} f(t)$ (ii) $\lim_{t \rightarrow 1^+} f(t)$
 (b) For what values of t is f not continuous? Explain the meaning of the points of discontinuity.

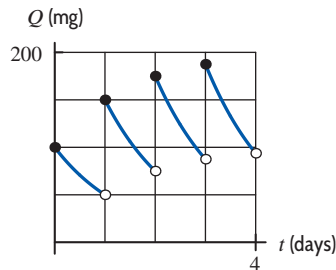


Figure 1.129

Strengthen Your Understanding

■ In Problems 75–76, explain what is wrong with the statement.

75. If $P(x)$ and $Q(x)$ are polynomials, $P(x)/Q(x)$ must be continuous for all x .

76. $\lim_{x \rightarrow 1} \frac{x-1}{|x-1|} = 1$

■ In Problems 77–78, give an example of:

77. A rational function that has a limit at $x = 1$ but is not continuous at $x = 1$.

78. A function $f(x)$ where

$$\lim_{x \rightarrow \infty} f(x) = 2 \text{ and } \lim_{x \rightarrow -\infty} f(x) = -2.$$

■ In Problems 79–83, let $\lim_{x \rightarrow 3} f(x) = 7$. Is the statement true or false? If it is true, explain how you know. If it is false, give a counterexample.

79. $\lim_{x \rightarrow 3} (xf(x)) = 21$.

68. If $p(x)$ is the function on page 62 giving the price of renting a car, explain why $\lim_{x \rightarrow 1} p(x)$ does not exist.

69. Evaluate $\lim_{x \rightarrow 3} \frac{x^2 + 5x}{x + 9}$ using the limit properties. State the property you use at each step.

70. Let $\lim_{x \rightarrow \infty} f(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = \infty$. Give possible formulas for $f(x)$ and $g(x)$ if

(a) $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$ (b) $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 3$

(c) $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$

71. (a) Rewrite $\frac{1}{x-5} - \frac{10}{x^2-25}$ in the form $f(x)/g(x)$ for polynomials $f(x)$ and $g(x)$.

(b) Evaluate the limit $\lim_{x \rightarrow 5} \left(\frac{1}{x-5} - \frac{10}{x^2-25} \right)$.

(c) Explain why you cannot use Property 4 of the limit properties to evaluate $\lim_{x \rightarrow 5} \left(\frac{1}{x-5} - \frac{10}{x^2-25} \right)$.

■ In Problems 72–73, modify the definition of limit on page 64 to give a definition of each of the following.

72. A right-hand limit

73. A left-hand limit

74. Use Theorem 1.2 on page 75 to explain why if f and g are continuous on an interval, then so are $f + g$, fg , and f/g (assuming $g(x) \neq 0$ on the interval).

80. If $g(3) = 4$, then $\lim_{x \rightarrow 3} (f(x)g(x)) = 28$.

81. If $\lim_{x \rightarrow 3} g(x) = 5$, then $\lim_{x \rightarrow 3} (f(x) + g(x)) = 12$.

82. If $\lim_{x \rightarrow 3} (f(x) + g(x)) = 12$, then $\lim_{x \rightarrow 3} g(x) = 5$.

83. If $\lim_{x \rightarrow 3} g(x)$ does not exist, then $\lim_{x \rightarrow 3} (f(x)g(x))$ does not exist.

■ In Problems 84–89, is the statement true or false? Assume that $\lim_{x \rightarrow 7^+} f(x) = 2$ and $\lim_{x \rightarrow 7^-} f(x) = -2$. Explain.

84. $\lim_{x \rightarrow 7} f(x)$ exists. 85. $\lim_{x \rightarrow 7} (f(x))^2$ exists.

86. $\frac{\lim_{x \rightarrow 7^+} f(x)}{\lim_{x \rightarrow 7^-} f(x)}$ exists.

87. The function $(f(x))^2$ is continuous at $x = 7$.

88. If $f(7) = 2$, the function $f(x)$ is continuous at $x = 7$.

89. If $f(7) = 2$, the function $(f(x))^2$ is continuous at $x = 7$.

■ In Problems 90–91, let $f(x) = (1/x) \sin(1/x)$. Is the statement true or false? Explain.

90. The function $f(x)$ has horizontal asymptote $y = 0$.

91. The function $f(x)$ has a vertical asymptote at $x = 0$.

1.9 FURTHER LIMIT CALCULATIONS USING ALGEBRA

Sections 1.7 and 1.8 are sufficient for the later chapters of this book. In this optional section we explore further algebraic calculations of limits.

Limits of Quotients

In calculus we often encounter limits of the form $\lim_{x \rightarrow c} f(x)/g(x)$ where $f(x)$ and $g(x)$ are continuous. There are three types of behavior for this type of limit:

- When $g(c) \neq 0$, the limit can be evaluated by substitution.
- When $g(c) = 0$ but $f(c) \neq 0$, the limit does not exist.
- When $g(c) = 0$ and $f(c) = 0$, the limit may or may not exist and can take any value.

We explore each these behaviors in more detail. When $g(c) \neq 0$, by Theorem 1.3, the limit can be found by substituting:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}.$$

When $g(c) = 0$, the situation is more complicated as substitution cannot be used.

Example 1 Evaluate the following limit or explain why it does not exist:

$$\lim_{x \rightarrow 3} \frac{x+1}{x-3}.$$

Solution If we try to evaluate at $x = 3$, we get $4/0$, which is undefined. Figure 1.130 shows that as x approaches 3 from the right, the function becomes arbitrarily large, and as x approaches 3 from the left, the function becomes arbitrarily large but negative, so this limit does not exist.

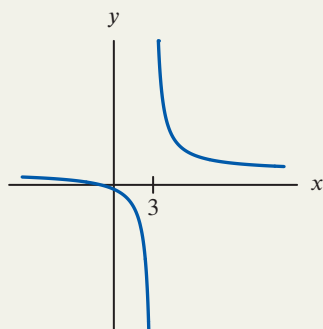


Figure 1.130: Limit of $y = (x+1)/(x-3)$ does not exist at $x = 3$

The limit in Example 1 does not exist because as x approaches 3, the denominator gets close to zero and the numerator gets close to 4. This means we are dividing a number close to 4 by a smaller and smaller number, resulting in a larger and larger number. This observation holds in general: for continuous functions, if $g(c) = 0$ but $f(c) \neq 0$, then $\lim_{x \rightarrow c} f(x)/g(x)$ does not exist.

Limits of the Form $0/0$ and Holes in Graphs

In Example 4 of Section 1.7 we saw that when both $f(c) = 0$ and $g(c) = 0$, so we have a limit of the form $0/0$, the limit can exist. We now explore limits of this form in more detail.

Example 2 Evaluate the following limit or explain why it does not exist:

$$\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}.$$

Solution If we try to evaluate at $x = 3$, we get $0/0$, which is undefined. Figure 1.131 suggests that as x approaches 3, the function gets close to 5, which suggests the limit is 5.

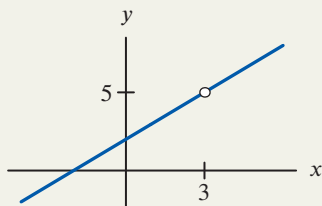


Figure 1.131: Graph of $y = (x^2 - x - 6)/(x - 3)$ is the same as the graph of $y = x + 2$ except at $x = 3$

This limit is similar to the one we saw in Example 4 of Section 1.7, so we check it algebraically using a similar method. Since the numerator factors as $x^2 - x - 6 = (x - 3)(x + 2)$ and $x \neq 3$ in the limit, we can cancel the common factor $x - 3$. We have:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x - 3)(x + 2)}{x - 3} && \text{Factoring the numerator} \\ &= \lim_{x \rightarrow 3} (x + 2) && \text{Canceling } (x - 3) \text{ since } x \neq 3 \\ &= 3 + 2 = 5 && \text{Substituting } x = 3 \text{ since } x + 2 \text{ is continuous} \end{aligned}$$

Even though $f(x) = x^2 - x - 6$ and $g(x) = x - 3$ approach 0 as x approaches 3, the limit in Example 2 exists and is equal to 5 because the values of $f(x)$ are approximately 5 times the value of $g(x)$ near $x = 3$.

Using a Function with the Same Values to Evaluate a Limit

The limit in Example 2 exists because provided $x \neq 3$ we have

$$\frac{x^2 - x - 6}{x - 3} = x + 2,$$

so their limits are the same. This means their graphs are identical except at $x = 3$ where the first has a hole, and the second passes through $(3, 5)$. The key point is that when two functions take the same values close to but not necessarily at $x = c$, then their limits are the same at $x = c$. We use this observation in the following example.

Example 3 Evaluate the following limits or explain why they don't exist:

$$(a) \lim_{h \rightarrow 0} \frac{(3 + h)^2 - 3^2}{h} \quad (b) \lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} \quad (c) \lim_{x \rightarrow 1} \frac{x - 1}{x^2 - 2x + 1}$$

Solution

(a) At $h = 0$, we get $0/0$, so the function is undefined. We calculate algebraically:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(3+h)^2 - 3^2}{h} &= \lim_{h \rightarrow 0} \frac{3^2 + 6h + h^2 - 3^2}{h} && \text{Expanding the numerator} \\ &= \lim_{h \rightarrow 0} \frac{6h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (6 + h) && \text{Canceling } h \text{ since } h \neq 0 \\ &= 6 + 0 = 6. && \text{Substituting } h = 0 \text{ since } 6 + h \text{ is continuous} \end{aligned}$$

The limit exists because $y = \frac{(3+h)^2 - 3^2}{h}$ has the same values as the continuous function $y = 6 + h$ except at $h = 0$. If we were to sketch their graphs, they would be identical except at $h = 0$ where the rational function has a hole.

(b) At $x = 4$, we have $0/0$, so the function is undefined. We decide to multiply in the numerator and denominator by $\sqrt{x} + 2$ because that creates a factor of $x - 4$ in the denominator:

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} &= \lim_{x \rightarrow 4} \left(\frac{x-4}{\sqrt{x}-2} \right) \left(\frac{\sqrt{x}+2}{\sqrt{x}+2} \right) && \text{Multiplying by 1 does not change the limit} \\ &= \lim_{x \rightarrow 4} \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{x}+2)} \\ &= \lim_{x \rightarrow 4} \frac{(x-4)(\sqrt{x}+2)}{x-4} && \text{Expanding the denominator} \\ &= \lim_{x \rightarrow 4} (\sqrt{x}+2) && \text{Canceling } (x-4) \text{ since } x \neq 4 \\ &= \sqrt{4} + 2 = 4. && \text{Substituting } x = 4 \text{ since } \sqrt{x} + 2 \text{ is continuous} \end{aligned}$$

Once again, we see that the limit exists because the function $y = \frac{x-4}{\sqrt{x}-2}$ has the same values as the continuous function $y = \sqrt{x} + 2$ except at $x = 4$. If we were to sketch their graphs, they would be identical except at $x = 4$ where the former has a hole.

(c) At $x = 1$, we get $0/0$, so the function is undefined. We try to calculate algebraically:

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x-1}{x^2 - 2x + 1} &= \lim_{x \rightarrow 1} \frac{x-1}{(x-1)^2} && \text{Factoring the denominator} \\ &= \lim_{x \rightarrow 1} \frac{1}{x-1} && \text{Canceling } (x-1) \text{ since } x \neq 1 \end{aligned}$$

So the function $y = \frac{x-1}{x^2 - 2x + 1}$ takes the same values as $y = \frac{1}{x-1}$ except at $x = 1$, so it must have the same limit. Since the limit of the denominator of $y = 1/(x-1)$ is 0 and the numerator is 1 at $x = 1$, this limit does not exist. The limit is of the same form as in Example 1.

Although the limit in part (c) of Example 3 did not exist, we used the same idea as in the other examples to show this. The values of $y = \frac{x-1}{x^2 - 2x + 1}$ are equal to the values of $y = \frac{1}{x-1}$ except at $x = 1$, so instead we considered the limit of $1/(x-1)$ at $x = 1$.

Finding Limits Using a New Variable

In order to evaluate the limit $\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}$ algebraically in Example 3, we needed an additional step to eliminate the square root in the denominator. After this step, the process was identical to the other examples. Another way to find this limit is to use a new variable to eliminate the square root.

Example 4 By letting $t = \sqrt{x}$, evaluate the limit

$$\lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2}$$

Solution If we let $t = \sqrt{x}$, we look for the limit as t approaches 2, since as x gets closer to 4, the value of t gets closer to 2. We calculate step by step algebraically:

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{x-4}{\sqrt{x}-2} &= \lim_{t \rightarrow 2} \frac{t^2-4}{t-2} && \text{Letting } t = \sqrt{x} \\ &= \lim_{t \rightarrow 2} \frac{(t-2)(t+2)}{t-2} && \text{Factoring the numerator} \\ &= \lim_{t \rightarrow 2} (t+2) && \text{Canceling } (t-2) \text{ since } t \neq 2 \\ &= 2+2 = 4. && \text{Evaluating the limit by substituting } t = 2 \text{ since } t+2 \text{ is continuous} \end{aligned}$$

Calculating Limits at Infinity

Algebraic techniques can also be used to evaluate limits at infinity and one-sided limits.

Example 5 Evaluate the limit

$$\lim_{y \rightarrow \infty} \frac{3+4y}{y+2}$$

Solution The limits of the numerator and the denominator do not exist as y grows arbitrarily large since both also grow arbitrarily large. However, dividing numerator and denominator by the highest power of y occurring enables us to see that the limit does exist:

$$\begin{aligned} \lim_{y \rightarrow \infty} \frac{3+4y}{y+2} &= \lim_{y \rightarrow \infty} \frac{3/y+4}{1+2/y} && \text{Dividing numerator and denominator by } y \\ &= \frac{0+4}{1+0} = 4 && \text{Since } 1/y \rightarrow 0 \text{ as } y \rightarrow \infty \end{aligned}$$

In Example 5, as y goes to infinity, the limit of neither the numerator nor of the denominator exists; however, the limit of the quotient does exist. The reason for this is that though both numerator and denominator grow without bound, the constants 3 and 2 are insignificant for large y -values. Thus, $3+4y$ behaves like $4y$ and $2+y$ behaves like y , so $(3+4y)/(y+2)$ behaves like $4y/y$.

The Squeeze Theorem

There are some limits for which the techniques we already have cannot be used. For example, if $f(x) = x^2 \cos(1/x)$, then $f(x)$ is undefined at $x = 0$. This means f is not continuous at $x = 0$, so we cannot evaluate the limit by substitution. Also, there is no obvious function $g(x)$ with the same values as $f(x)$ close to $x = 0$ that we could use to calculate the limit. From Figure 1.132, it looks as if the limit does exist and equals 0, so we need a new way to check this is indeed the limit.

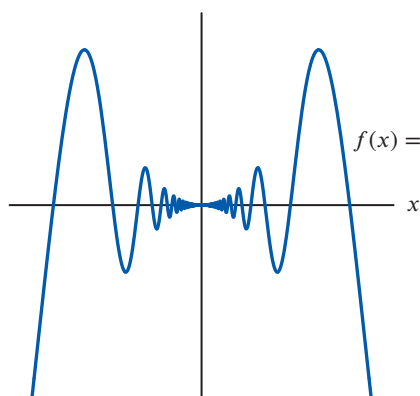


Figure 1.132: Find $\lim_{x \rightarrow 0} f(x)$

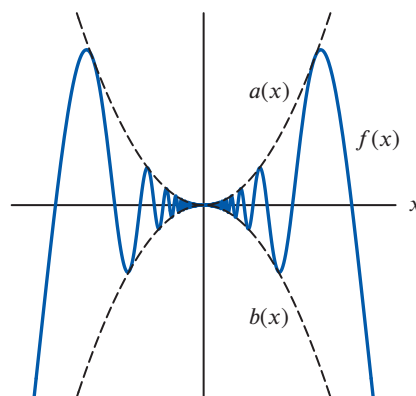


Figure 1.133: Graph of $f(x)$ is squeezed between $b(x) = -x^2$ and $a(x) = x^2$

From Figure 1.133, we can see that the graph of $a(x) = x^2$ is always above $f(x)$ and $b(x) = -x^2$ is always below $f(x)$, so $f(x)$ is always between $a(x)$ and $b(x)$. Since $a(x)$ and $b(x)$ both get closer and closer to 0 as x gets closer to 0, the values of $f(x)$ are “squeezed” between them and must have the same limit. Thus, we conclude $\lim_{x \rightarrow 0} f(x) = 0$.

Our calculation of this limit is an illustration of the Squeeze Theorem:

Theorem 1.5: The Squeeze Theorem

If $b(x) \leq f(x) \leq a(x)$ for all x close to $x = c$ except possibly at $x = c$, and $\lim_{x \rightarrow c} b(x) = L = \lim_{x \rightarrow c} a(x)$, then

$$\lim_{x \rightarrow c} f(x) = L.$$

The Squeeze Theorem can also be used to find limits at infinity.

Example 6

After driving over a speed bump at $t = 0$ seconds a car bounces up and down so the height of its body from the ground in inches is given by $h(t) = 7 + e^{-0.5t} \sin(2\pi t)$. Find and interpret $\lim_{t \rightarrow \infty} h(t)$.

Solution

Since $-1 \leq \sin(2\pi t) \leq 1$ for all t , we have

$$7 - e^{-0.5t} \leq h(t) \leq 7 + e^{-0.5t},$$

so we try the Squeeze Theorem with $a(t) = 7 + e^{-0.5t}$ and $b(t) = 7 - e^{-0.5t}$. We have:

$$\begin{aligned} \lim_{t \rightarrow \infty} a(t) &= \lim_{t \rightarrow \infty} (7 + e^{-0.5t}) = \lim_{t \rightarrow \infty} \left(7 + \frac{1}{e^{0.5t}} \right) && \text{Rewriting } e^{-0.5t} \text{ as } 1/e^{0.5t} \\ &= \lim_{t \rightarrow \infty} 7 = 7. && \text{Since } 1/e^{0.5t} \rightarrow 0 \text{ as } t \rightarrow \infty \end{aligned}$$

Similarly $\lim_{t \rightarrow \infty} b(t) = 7$, so by the Squeeze Theorem, $\lim_{t \rightarrow \infty} h(t) = 7$. This value makes sense as over time the shock absorbers of the car will lessen the bouncing until the height stabilizes to 7 inches.

Summary for Section 1.9

- There are three types of behavior for limits $\lim_{x \rightarrow c} f(x)/g(x)$ where $f(x)$ and $g(x)$ are continuous:
 - When $g(c) \neq 0$, the limit can be evaluated by substitution.
 - When $g(c) = 0$ but $f(c) \neq 0$, the limit does not exist.
 - When $g(c) = 0$ and $f(c) = 0$, the limit may or may not exist and can take any value. We use algebraic techniques to evaluate such limits.

- **The squeeze theorem:** If $b(x) \leq f(x) \leq a(x)$ for all x close to $x = c$ except possibly at $x = c$, and $\lim_{x \rightarrow c} b(x) = L = \lim_{x \rightarrow c} a(x)$, then

$$\lim_{x \rightarrow c} f(x) = L.$$

Exercises and Problems for Section 1.9

EXERCISES

- In Exercises 1–3, find the limit.

$$1. \lim_{x \rightarrow 0} \frac{3x^2}{x^2} \quad 2. \lim_{x \rightarrow 0} \frac{3x^2}{x} \quad 3. \lim_{x \rightarrow 0} \frac{3x^2}{x^4}$$

- For Exercises 4–23, use algebra to simplify the expression and find the limit.

$$4. \lim_{x \rightarrow 3} \frac{x^2 - 3x}{x - 3} \quad 5. \lim_{t \rightarrow 0} \frac{t^4 + t^2}{2t^3 - 9t^2}$$

$$6. \lim_{x \rightarrow 0} \frac{x^3 - 3x}{x\sqrt{2x+3}} \quad 7. \lim_{x \rightarrow -4} \frac{x+4}{2x^2 + 5x - 12}$$

$$8. \lim_{y \rightarrow 1} \frac{y^2 - 5y + 4}{y - 1} \quad 9. \lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x^2 - 3x + 2}$$

$$10. \lim_{t \rightarrow -2} \frac{2t^2 + 3t - 2}{t^2 + 5t + 6} \quad 11. \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 + x - 12}$$

$$12. \lim_{y \rightarrow -1} \frac{2y^2 + y - 1}{3y^2 + 2y - 1} \quad 13. \lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h}$$

$$14. \lim_{x \rightarrow -3} \frac{(x+5)^2 - 4}{x^2 - 9} \quad 15. \lim_{x \rightarrow \sqrt{3}} \frac{5x^2 - 15}{x^4 - 9}$$

$$16. \lim_{x \rightarrow 2} \frac{2/x - 1}{x - 2} \quad 17. \lim_{t \rightarrow 3} \frac{1/t - 1/3}{t - 3}$$

$$18. \lim_{t \rightarrow 0} \frac{1/(t+1) - 1}{t} \quad 19. \lim_{h \rightarrow 0} \frac{1/(4+h) - 1/4}{h}$$

$$20. \lim_{z \rightarrow 1} \frac{\sqrt{z} - 1}{z - 1} \quad 21. \lim_{h \rightarrow 0} \frac{\sqrt{9+h} - 3}{h}$$

$$22. \lim_{x \rightarrow 0} \frac{4^x - 1}{2^x - 1} \quad 23. \lim_{h \rightarrow 0} \frac{(1+h)^4 - 1}{h}$$

- In Exercises 24–26, for the given constant c and functions $f(x)$ and $g(x)$, answer the following:

(a) Is the limit $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ of the form $0/0$?

(b) Find $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$.

24. $c = 4$, $f(x) = 2x - 1$, $g(x) = x^{-2}$

25. $c = 1$, $f(x) = 2x^3 + x^2 - 3x$, $g(x) = x^2 + 3x - 4$

26. $c = 3$, $f(x) = x^2 - 3x$, $g(x) = \sqrt{x+1} - 2$

- In Exercises 27–34, use algebra to evaluate the limit.

$$27. \lim_{z \rightarrow \infty} \frac{5z^2 + 2z + 1}{2z^3 - z^2 + 9} \quad 28. \lim_{x \rightarrow \infty} \frac{x + 7x^2 - 11}{3x^2 - 2x}$$

$$29. \lim_{t \rightarrow \infty} \frac{4e^t + 3e^{-t}}{5e^t + 2e} \quad 30. \lim_{x \rightarrow \infty} \frac{2^{x+1}}{3^{x-1}}$$

$$31. \lim_{x \rightarrow \infty} \frac{2^{3x+2}}{3^{x+3}} \quad 32. \lim_{x \rightarrow \infty} xe^{-x}$$

$$33. \lim_{t \rightarrow \infty} (4e^t)(7e^{-t}) \quad 34. \lim_{t \rightarrow \infty} t^{-2} \cdot \sin t$$

- In Exercises 35–36, multiply by $\sqrt{4+h} + 2$ in numerator and denominator and use algebra to find the limit.

$$35. \lim_{h \rightarrow 0} \frac{\sqrt{4+h} - 2}{h} \quad 36. \lim_{h \rightarrow 0} \frac{1/\sqrt{4+h} - 1/2}{h}$$

37. Find $\lim_{x \rightarrow -1} f(x)$ if, for all x , $-4x+6 \leq f(x) \leq x^2-2x+7$.

38. Find $\lim_{x \rightarrow 0} f(x)$ if, for all x , $4 \cos(2x) \leq f(x) \leq 3x^2 + 4$.

39. Find $\lim_{x \rightarrow \infty} f(x)$ if, for $x > 0$,

$$\frac{4x^2 - 5}{x^2} \leq f(x) \leq \frac{4x^6 + 3}{x^6}.$$

PROBLEMS

- In Problems 40–51, find all values for the constant k such that the limit exists.

$$40. \lim_{x \rightarrow 4} \frac{x^2 - k^2}{x - 4} \quad 41. \lim_{x \rightarrow 1} \frac{x^2 - kx + 4}{x - 1}$$

$$42. \lim_{x \rightarrow -2} \frac{x^2 + 4x + k}{x + 2} \quad 43. \lim_{x \rightarrow 5} \frac{x^2 - kx + 5}{x^2 - 2x - 15}$$

$$44. \lim_{x \rightarrow 0} \frac{e^k + 2x - 8}{e^x - 1} \quad 45. \lim_{x \rightarrow 1} \frac{k^2 - 40x - 9}{\ln x}$$

$$46. \lim_{x \rightarrow \infty} \frac{x^2 + 3x + 5}{4x + 1 + x^k} \quad 47. \lim_{x \rightarrow -\infty} \frac{e^{2x} - 5}{e^{kx} + 3}$$

$$48. \lim_{x \rightarrow \infty} \frac{x^3 - 6}{x^k + 3} \quad 49. \lim_{x \rightarrow \infty} \frac{e^{kx} + 11}{e^{5x} - 3}$$

$$50. \lim_{x \rightarrow \infty} \frac{3^{kx} + 6}{3^{2x} + 4} \quad 51. \lim_{x \rightarrow -\infty} \frac{3^{kx} + 6}{3^{2x} + 4}$$

■ In Problems 52–57, use the indicated new variable to evaluate the limit.

52. $\lim_{y \rightarrow 4} \frac{\sqrt{y} - 2}{y - 4}$, let $t = \sqrt{y}$

53. $\lim_{x \rightarrow 9} \frac{x - \sqrt{x} - 6}{\sqrt{x} - 3}$, let $t = \sqrt{x}$

54. $\lim_{h \rightarrow 0} \frac{\sqrt{1+h} - 1}{h}$, let $t = \sqrt{1+h}$

55. $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x} - 1}{x - 1}$, let $t = \sqrt[3]{x}$

56. $\lim_{x \rightarrow 0} \frac{e^{3x} - e^{2x}}{e^x - 1}$, let $t = e^x$

57. $\lim_{x \rightarrow \infty} \frac{2e^{3x} - 1}{5e^{3x} + e^x + 1}$, let $t = e^x$

58. Use the Squeeze Theorem to prove $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$.

59. Use the Squeeze Theorem to prove $\lim_{x \rightarrow \infty} \frac{1}{x + e^{-x}} = 0$.

■ In Problems 60–63, use the Squeeze Theorem to calculate the limit.

60. $\lim_{x \rightarrow \infty} \frac{\cos^2 x}{2x + 1}$

61. $\lim_{x \rightarrow 0} x^4 \sin(1/x)$

62. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^3 + 1}}$

63. $\lim_{x \rightarrow \infty} \frac{1}{x + 2 \cos^2 x}$

■ In Problems 64–65, use the squeeze theorem to find the limit if it exists.

64. $\lim_{x \rightarrow 0} \left(x^2 \sin \left(\frac{1}{x} \right) + 1 \right)$

65. $\lim_{x \rightarrow 0} \sin \left(\frac{1}{x} \right) + 2$

66. Let $\lim_{x \rightarrow \infty} f(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 0$. Give possible formulas for $f(x)$ and $g(x)$ if

(a) $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$

(b) $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$

(c) $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \infty$

■ In Problems 67–70, for the given constant c and function $f(x)$, find a function $g(x)$ that has a hole in its graph at $x = c$ but $f(x) = g(x)$ everywhere else that $f(x)$ is defined. Give the coordinates of the hole.

67. $f(x) = x^2 + 1, c = 3$

68. $f(x) = x^2 + 1, c = 0$

69. $f(x) = \ln x, c = 1$

70. $f(x) = \sin x, c = \pi$

■ In Problems 71–76, for the given m and n , evaluate $\lim_{x \rightarrow 1} f(x)$ or explain why it does not exist, where

$$f(x) = \frac{(x-1)^n}{(x-1)^m}$$

71. $n = 3, m = 2$ 72. $n = 2, m = 3$ 73. $n = 2, m = 2$

74. n and m are positive integers with $n > m$.

75. n and m are positive integers with $m > n$.

76. n and m are positive integers with $m = n$.

77. For any $f(x)$, where $-\frac{1}{x} \leq f(x) \leq \frac{1}{x}$, find values of c and L for which the Squeeze Theorem can be applied.

Strengthen Your Understanding

■ In Problems 78–79, explain what is wrong with the statement.

78. If $f(x) = \frac{x^2 - 1}{x + 1}$ and $g(x) = x - 1$, then $f = g$.

79. If $f(1) = 0$ and $g(1) = 1$, then

$$\lim_{x \rightarrow 1} \frac{f(x)}{g(x)} = \frac{0}{1} = 0.$$

■ Are the statements in Problems 80–87 true or false? Explain.

80. If $b(x) \leq f(x) \leq a(x)$ and $\lim_{x \rightarrow 0} b(x) = -1, \lim_{x \rightarrow 0} a(x) = 1$, then $-1 \leq \lim_{x \rightarrow 0} f(x) \leq 1$.

81. If $0 \leq f(x) \leq a(x)$ and $\lim_{x \rightarrow 0} a(x) = 0$, then $\lim_{x \rightarrow 0} f(x) = 0$.

82. If $b(x) \leq f(x) \leq a(x)$ and $\lim_{x \rightarrow 0} b(x) = \lim_{x \rightarrow 0} f(x)$ then $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} a(x)$.

83. If $\lim_{x \rightarrow 0} g(x) = 0$ then $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \infty$.

84. If $\lim_{x \rightarrow 0} g(x) = 0$ and $\lim_{x \rightarrow 0} f(x) \neq 0$ then $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \infty$.

85. If $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)}$ exists, then $\lim_{x \rightarrow 0} f(x)$ exists and $\lim_{x \rightarrow 0} g(x)$ exists.

86. If $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)}$ exists, and $\lim_{x \rightarrow 0} g(x)$ exists then $\lim_{x \rightarrow 0} f(x)$ exists.

87. If $\lim_{x \rightarrow c^+} g(x) = 1$ and $\lim_{x \rightarrow c^-} g(x) = -1$ and $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ exists, then $\lim_{x \rightarrow c} f(x) = 0$.