

Part I
TENSOR THEORY

COPYRIGHTED MATERIAL

Preliminaries

1.1 Introduction

Some quantities are associated with their magnitude and direction, but certain quantities are associated with two or more directions. Such a quantity is called a tensor, e.g., the stress at a point of an elastic solid is an example of a tensor which depends on two directions: one is normal and the other is that of force on the area. Tensor comes from the word tension.

In this chapter, we discuss the notation of systems of different orders, which are applied in the theory of determinants, symbols, and summation conventions. Also, results on some matrices and determinants are discussed because they will be used frequently later on.

1.2 Systems of Different Orders

Let us consider the two quantities, a_1, a_1 or a^1, a^2 , which are represented by a_i or a^i , respectively, for $i = 1, 2$. In such cases, the expressions a_i, a^i, a_{ij}, a^{ij} , and a_j^i are called *systems*. In each value of a_i and a^i are called *systems of first order* and each value of a_{ij}, a^{ij} , and a_j^i is called a *double system* or *system of second order*, of which $a_{12}, a_{22}, a^{23}, a^{13}$, and a_4^3 are called their respective *components*. Similarly, we have *systems of the third order* that depend on three indices shown as $a_{ijk}, a^{ikl}, a_{ijm}, a^{ijn}$, and a_{jk}^i and each number of their respective components are 8.

In a system of order zero, it is shown that the quantity has no index, such as a . The upper and lower indices of a system are called its indices of *contravariance* and *covariance*, respectively. For a system of A_k^{ij} , i and j are indices of a *contravariant* and k is of *covariance*. Accordingly, the system A_j^i is called a *contravariant system*, A_{klm} is called a *covariant system*, and is A_j^i called a *mixed system*.

1.3 Summation Convention Certain Index

If in some expressions a certain index occurs twice, this means that this expression is summed with respect to that index for all admissible values of the index.

Thus, the linear form $\sum_{i=1}^4 a_i x_i$ has an index, i , occurring in it twice. We will omit the summation symbol Σ and write $a_i x_i$ to mean $a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$. In order to avoid Σ , we shall make use of a convention used by A. Einstein which is accordingly called the *Einstein Summation Convention* or *Summation Convention*.

Of course, the range of admissible values of the index, 1 to 4 in this case, must be specified. If the symbol i has a range of values from 1 to 3 and j ranges from 1 to 4, the expression

$$a_{ij} x_j, \quad (i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4) \quad (1.1)$$

represents three linear forms:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + a_{14} x_4 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + a_{24} x_4 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 + a_{34} x_4 \end{aligned} \quad (1.2)$$

Here, index i is the identifying (free) index and since index j , occurs twice, it is the summation index.

We shall adopt this convention throughout the chapters and take the sum whenever a letter appears in a term once in a subscript and once in superscript or if the same two indices are in subscript or are in superscript.

Example 1.3.1. Express the sum $\sum_{i=1}^3 \sum_{j=1}^3 a_{ij} u^i v^j$.

$$\begin{aligned} \text{Solution: } a_{ij} u^i v^j &= \sum_{i=1}^3 \sum_{j=1}^3 a_{ij} u^i v^j = \sum_{i=1}^3 (a_{i1} u^i v^1 + a_{i2} u^i v^2 + a_{i3} u^i v^3) \\ &= \sum_{i=1}^3 a_{i1} u^i v^1 + \sum_{i=1}^3 a_{i2} u^i v^2 + \sum_{i=1}^3 a_{i3} u^i v^3 \\ &= (a_{11} u^1 v^1 + a_{21} u^2 v^1 + a_{31} u^3 v^1) + (a_{12} u^1 v^2 + a_{22} u^2 v^2 + a_{32} u^3 v^2) \\ &\quad + (a_{13} u^1 v^3 + a_{23} u^2 v^3 + a_{33} u^3 v^3) \end{aligned}$$

1.3.1 Dummy Index

The summation (or dummy) index can be changed at will. Thus, Equation (1.1) can be written in the form $a_{ik}x_k$ if k has the same range of values as j .

We will assume that the summation and identifying indices have ranges of value from 1 to n .

Thus, $a_i x_i$ will represent a linear form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_n x_n$$

For example, $\sum_{i=1}^n \sum_{k=1}^n a_{ik} x^i x^k$ can be written as $a_{ik} x^i x^k$ and here, i and k both are dummy indexes.

So, any dummy index can be replaced by any other index with a range of the same numbers.

1.3.2 Free Index

If in an expression an index is not a dummy, i.e., it is not repeated twice, then it is called a *free index*. For example, for $a_{ij} x^j$, the index j is dummy, but index i is free.

1.4 Kronecker Symbols

A particular system of second order denoted by δ_j^i $i, j=1, 2, \dots, n$, is defined as

$$\begin{aligned} \delta_j^i &= 1 \text{ for } i = j \\ &= 0 \text{ for } i \neq j \end{aligned} \tag{1.3}$$

Such a system is called a *Kronecker symbol* or *Kronecker delta*.

For example, δ_j^i , by summation convention is expressed as

$$\delta_i^i = \delta_i^1 + \delta_i^2 + \dots + \delta_i^n = (\delta_1^1) + (\delta_2^2) + \dots + \delta_n^n = 1 + 1 + \dots + 1 = n$$

We shall now consider some *properties* of this system.

Property 1.4.1. If x^1, x^2, \dots, x^n are independent variables, then

$$\begin{aligned} \frac{\partial x^i}{\partial x^j} &= 1 \text{ for } i = j \\ &= 0 \text{ for } i \neq j \end{aligned}$$

Hence, $\frac{\partial x^i}{\partial x^j} = \delta_j^i$ (1.4)

Property 1.4.2. From the summation convention, we get

$$\delta_i^i = \delta_1^1 + \delta_2^2 + \dots + \delta_n^n = 1 + 1 + \dots + 1 = n$$

Similarly, $\delta_{ii} = \delta^i_i = n$

Property 1.4.3. From the definition of δ^{ij} , taken as an element of unit matrix I, we have

$$I = (\delta^{ij}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Its determinant is $|\delta^{ij}| = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{vmatrix} = 1$

Property 1.4.4.

$$\begin{aligned} \text{Again, } \delta_j^i a^j &= \delta_1^i a^1 + \delta_2^i a^2 + \dots + \delta_i^i a^i + \dots + \delta_n^i a^n \\ &= 0 + 0 + \dots + 1.a^i + 0 + \dots + .. = a^i \end{aligned} \tag{1.5}$$

$$\text{Similarly, } \delta_j^i a_i = a_j \quad (1.6)$$

$$\begin{aligned} \delta_j^i a_{ik} &= \delta_j^1 a_{1k} + \delta_j^2 a_{2k} + \cdots \delta_j^n a_{jk} \\ &+ \cdots + \delta_j^n a_{nk} = a_{jk} \\ &= 0 + 0 \dots + 1 \cdot a_{jk} + 0 \dots + 0 = a_{jk} \end{aligned}$$

$$\text{Similarly, } \delta_j^i a^{jk} = a^{ik} \quad (1.7)$$

Property 1.4.5.

$$\delta_j^i \delta_k^j = \delta_1^i \delta_k^1 + \delta_2^i \delta_k^2 + \cdots \dots + \delta_n^i \delta_k^n = 1 \cdot \delta_k^1 + \delta_k^2 + \cdots + \delta_k^n = \delta_k^i$$

$$\text{Also, by definition, } \delta_j^i \delta_k^j = \frac{\partial x^i}{\partial x^j} \frac{\partial x^j}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta_k^i$$

In particular, when $i = k$, we get $\delta_j^i \delta_i^j = \delta_i^i = n$

Remark 1.4.1. If we multiply x^k by δ_k^i , we simply replace index k of x^k with index i and for this reason, δ_k^i is called a *substitution factor*.

Example 1.4.1. Evaluate (a) $\delta_j^i \delta_l^j$ and (b) $\delta_j^i \delta_l^j \delta_k^l$ where the indices take all values from 1 to n .

$$\text{Solution: (a) We have } \delta_j^i \delta_l^j = \delta_1^i \delta_l^1 + \delta_2^i \delta_l^2 + \cdots + \delta_n^i \delta_l^n \quad (1.8a)$$

$$\text{Now, } \delta_1^i \delta_l^1 = \delta_1^1 \delta_l^1 + \delta_1^2 \delta_l^1 + \cdots + \delta_1^n \delta_l^1 = 1 \cdot \delta_l^1 = \delta_l^1$$

Similarly, other terms of i are $\delta_l^2, \delta_l^3, \dots, \delta_l^n$.

$$\therefore \delta_j^i \delta_l^j = \delta_1^i \delta_l^1 + \delta_2^i \delta_l^2 + \cdots + \delta_n^i \delta_l^n = \delta_l^1 + \delta_l^2 + \delta_l^3 + \cdots + \delta_l^n = \delta_l^i \quad (1.8b)$$

$$\text{(b) } \delta_j^i \delta_l^j \delta_k^l = \delta_l^i \delta_k^l = \delta_k^i \text{ by 1.8b}$$

Example 1.4.2. If x^i and y^j are independent coordinates of a point, it is shown that

$$\frac{\partial x^j}{\partial y^k} \frac{\partial y^k}{\partial x^i} = \delta_i^j$$

Solution: The partial derivative of ϕ in two coordinate systems are different and are connected by the following formula of Differential Calculus:

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial y^1} \frac{\partial y^1}{\partial x^i} + \frac{\partial \phi}{\partial y^2} \frac{\partial y^2}{\partial x^i} + \cdots + \frac{\partial \phi}{\partial y^n} \frac{\partial y^n}{\partial x^i} = \frac{\partial \phi}{\partial y^k} \frac{\partial y^k}{\partial x^i}$$

In particular, when $\phi = x^j$, we have
$$\frac{\partial x^j}{\partial x^i} = \frac{\partial x^j}{\partial y^k} \frac{\partial y^k}{\partial x^i} \quad (1.9a)$$

Since x^j is independent of x^i , $\frac{\partial x^j}{\partial x^i} = 0$ when $j \neq i$

$$= 1 \text{ for } i = j \quad (1.9b)$$

Hence, the result follows from (1.9a) and (1.9b).

1.5 Linear Equations

Let us consider n linear equations such that

$$a_{ij}x^j = b_i, \quad (1.10a)$$

where x^1, x^2, \dots, x^n are n unknown variables.

Let us consider:

For the expansion of $\det |a_{ij}|$ in terms of cofactors we have

$$a_{ij}A^{jk} = a\delta_i^k \quad (1.10b)$$

where $a = |a_{ij}|$ and the cofactor of a_{ij} is A^{ij} .

We can derive Cramer's Rule for the solution of the system of n linear equations:

Now, multiplying both sides of (1.10a) by A^{ij} , we get

$$A^{ij}a_{ij}x^j = b_iA^{ij}$$

by (1.10b), we get, $ax^j = b_iA^{ij}$.

From here, we can easily get

$$x^j = \frac{b_i A^{ij}}{a}, \text{ where } a = |a_{ij}| \neq 0.$$

Example 1.5.1. Show that $a_{ij}A^{kj} = a\delta_i^k$, where a is a determinant a_{ij} , $a = |a_{ij}|$ of order 3 and A^{ij} are cofactors of a_{ij} .

Solution: By expansion of determinants, we have:

$$a_{11}A^{11} + a_{12}A^{12} + a_{13}A^{13} = a$$

$$a_{11}A^{2j} + a_{12}A^{2j} + a_{13}A^{2j} = 0$$

$$a_{11}A^{3j} + a_{12}A^{3j} + a_{13}A^{3j} = 0$$

Which can be written as $a_{1j}A^{1j} = a$, $a_{1j}A^{2j} = 0$ and $a_{1j}A^{3j} = 0$ [we know $a_{ij}A^{ij} = a$].

Similarly, we have $a_{2j}A^{1j} = 0$, $a_{2j}A^{2j} = a$ and $a_{2j}A^{3j} = 0$

$a_{3j}A^{1j} = 0$, $a_{3j}A^{2j} = 0$ and $a_{3j}A^{3j} = a$.

Using Kronecker Delta Notation, these can be combined into a single equation:

$$a_{1j}A^{kj} = a\delta_1^k \quad a_{2j}A^{kj} = a\delta_2^k \quad a_{3j}A^{kj} = a\delta_3^k \quad [\delta_j^i = 1, \text{ when } i = j \\ = 0, \text{ when } j \neq i].$$

All nine of these equations can be combined into $a_{ij}A^{kj} = a\delta_i^k$.

1.6 Results on Matrices and Determinants of Systems

It is known that if the range of the indices of a system of second order are from 1 to n , the number of components is n^2 . Systems of second order are organized into three types: a^{ij} , a_{ij} , a_j^i and their matrices,

(a^{ij}) , (a_{ij}) , (a_j^i) :

$$\begin{bmatrix} a^{11} & a^{12} \dots & a^{1n} \\ a^{21} & a^{22} \dots & a^{2n} \\ a^{n1} & a^{n2} \dots & a^{nn} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ a_{n1} & a_{n2} \dots & a_{nn} \end{bmatrix} \text{ and } \begin{bmatrix} a_1^1 & a_2^1 \dots & a_n^1 \\ a_1^2 & a_2^2 \dots & a_n^2 \\ a_1^n & a_2^n \dots & a_n^n \end{bmatrix}$$

each of which is an $n \times n$ matrix.

We shall now establish the following results:

Property 1.6.1. If $a_j^i b_p^j = c_p^i$, then $(a_j^i)(b_p^j) = (c_p^i)$ and $|a_j^i||b_p^j| = |c_p^i|$.

Proof: We shall prove this result by taking the range of the indices from 1 to 2, but the results hold, in general, when they range from 1 to n .

We get $a_j^i b_p^j = a_1^i b_p^1 + a_2^i b_p^2$. Hence, $c_p^i = a_1^i b_p^1 + a_2^i b_p^2$.

$$\begin{aligned} \therefore \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix} &= (a_1^i)(b_p^1) + (a_2^i)(b_p^2) \\ &= \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} b_1^1 & 0 \\ 0 & b_2^1 \end{pmatrix} + \begin{pmatrix} a_2^1 & a_2^2 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{pmatrix} \\ &= \begin{pmatrix} a_1^1 b_1^1 & a_1^1 b_2^1 \\ a_2^1 b_1^1 & a_2^1 b_2^1 \end{pmatrix} + \begin{pmatrix} a_2^1 b_1^2 & a_2^1 b_2^2 \\ a_2^2 b_1^2 & a_2^2 b_2^2 \end{pmatrix} = \begin{pmatrix} a_1^1 b_1^1 + a_2^1 b_1^2 & a_1^1 b_2^1 + a_2^1 b_2^2 \\ a_2^1 b_1^1 + a_2^2 b_1^2 & a_2^1 b_2^1 + a_2^2 b_2^2 \end{pmatrix} \\ &= \begin{pmatrix} a_1^1 & a_2^1 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix} = (a_j^i)(b_p^j) \end{aligned}$$

Taking the determinant of both sides, we get $|a_j^i||b_p^j| = |c_p^i|$, as we know $|AB| = |A||B|$.

Property 1.6.2. If $a_{ij} b^{ik} = c_j^k$, then, $(b^{ik})^T(a_{ij}) = (c_j^k)$ and $|a_{ij}||b^{ik}| = |c_j^k|$, where $(b^{ik})^T$ is the transpose of (c_j^k)

Proof: We have $a_{ij} b^{ik} = a_{1j} b^{ik} + a_{2j} b^{ik}$, hence, $c_j^k = a_{1j} b^{1k} + a_{2j} b^{2k}$.

Therefore,

$$\begin{aligned} \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix} &= \begin{pmatrix} a_{11} b^{11} + a_{21} b^{21} & a_{12} b^{11} + a_{22} b^{21} \\ a_{11} b^{12} + a_{21} b^{22} & a_{12} b^{12} + a_{22} b^{22} \end{pmatrix} \\ &= \begin{pmatrix} b^{11} & b^{21} \\ b^{12} & b^{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (b^{ik})^T(a_{ij}) \end{aligned}$$

Taking determinants of both sides, we get $|c_j^k| = |b^{ikT}||a_{ij}| = |b^{ik}||a_{ij}|$ (since $|A^T| = |A|$).

Property 1.6.3. Let the cofactor of the element a_j^i in the determinant $|a_j^i|$ be denoted by A_j^i . Then, by summation convention we have

$$a_j^i A_k^j = a_1^i A_k^1 + a_2^i A_k^2 + \dots + a_n^i A_k^n = \delta_k^i |a_j^i| = \delta_k^i a \quad \text{and}$$

$$a_j^i A_j^k = a_j^1 A_1^k + a_j^2 A_2^k + \dots + a_j^n A_n^k = \delta_k^i |a_j^i| = \delta_j^k a, \text{ where } a = |a_j^i|.$$

If the cofactor of a_{ij} is represented by A^{kj} , it is expressed by the equation:

$$a_{ij} A^{kj} = a \delta_i^k.$$

If we divide the cofactor A^{kj} of the element of a_{kj} by the value a of the determinant, we form the *normalized cofactor*, represented by:

$$b^{kj} = \frac{1}{a} A^{kj}.$$

The above equation becomes

$$a_{ij} b^{kj} = \delta_i^k$$

Property 1.6.4. Let us consider a system of n linear equations:

$$a_j^i x^j = b^i, \quad i, j = 1, 2, \dots, n$$

for n unknown x^i , where $|a_j^i| \neq 0$

$$A_i^k a_j^i x^j = b^i A_i^k, \text{ where } A_i^k \text{ is cofactor of } a_j^i.$$

$ax^j = b^i A_i^k \therefore x^j = \frac{b^i A_i^k}{a}$, which is called *Cramer's Rule*, for the solution of n linear equations.

Property 1.6.5. Considering the transformation $z^i = z^i(y^k)$ and $y^i = y^i(x^k)$, let N function $z^i(y^k)$ be of independent N variables of y^k so that $\left| \frac{\partial z^i}{\partial y^k} \right| \neq 0$.

Here, N equation $z^i = z^i(y^k)$ is solvable for the z 's in terms terms of y^i 's.

Similarly, $y^i = y^i(x^k)$ is a solution of y^i in terms of x^i 's so that $\left| \frac{\partial y^i}{\partial x^k} \right| \neq 0$.

Now, we have by the chain rule of differentiation that

$$\frac{\partial z^i}{\partial x^k} = \frac{\partial z^i}{\partial y^1} \frac{\partial y^1}{\partial x^k} + \frac{\partial z^i}{\partial y^2} \frac{\partial y^2}{\partial x^k} + \dots + \frac{\partial z^i}{\partial y^N} \frac{\partial y^N}{\partial x^k} = \frac{\partial z^i}{\partial y^j} \frac{\partial y^j}{\partial x^k}.$$

Taking the determinant, we get

$$\left| \frac{\partial z^i}{\partial x^k} \right| = \left| \frac{\partial z^i}{\partial y^j} \right| \left| \frac{\partial y^j}{\partial x^k} \right| = \left| \frac{\partial z^i}{\partial y^j} \right| \left| \frac{\partial y^j}{\partial x^j} \right| \tag{1.11}$$

Considering a particular case in which $z^i = x^i$, Equation (1.5) becomes

$$\left| \frac{\partial x^i}{\partial x^k} \right| = \left| \frac{\partial x^i}{\partial y^j} \right| \left| \frac{\partial y^j}{\partial x^j} \right| \quad \text{or} \quad \left| \delta_k^i \right| = \left| \frac{\partial x^i}{\partial y^j} \right| \left| \frac{\partial y^j}{\partial x^j} \right|$$

Or

$$1 = \left| \frac{\partial x^i}{\partial y^j} \right| \left| \frac{\partial y^j}{\partial x^j} \right|$$

$$\Rightarrow \left| \frac{\partial x^i}{\partial y^j} \right| = \frac{1}{\left| \frac{\partial y^j}{\partial x^j} \right|}.$$

This implies that the Jacobian of Direct Transformation is the reciprocal of the Jacobian of Inverse Transformation.

1.7 Differentiation of a Determinant

Consider the determinant $|a_j^i| = a$ and let the element a_j^i be a function of x_1, x_2, \dots, x_n , etc. Let A_j^i be the cofactor of a_j^i of $\det a$.

Then, the derivative of a with respect to x_i is given by

$$\begin{aligned} \frac{\partial a}{\partial x_1} &= \left| \begin{array}{c} \frac{\partial a_1^1}{\partial x_1} \frac{\partial a_2^1}{\partial x_1} \dots \frac{\partial a_n^1}{\partial x_1} \\ a_1^2 a_2^2 \dots a_n^2 \\ \dots \\ a_1^n a_2^n \dots a_n^n \end{array} \right| + \left| \begin{array}{c} a_1^1 a_2^1 \dots a_n^1 \\ \frac{\partial a_1^2}{\partial x_1} \frac{\partial a_2^2}{\partial x_1} \dots \frac{\partial a_n^2}{\partial x_1} \\ \dots \\ a_1^n a_2^n \dots a_n^n \end{array} \right| + \dots + \left| \begin{array}{c} a_1^2 a_2^2 \dots a_n^2 \\ a_1^2 a_2^2 \dots a_n^2 \\ \dots \\ \frac{\partial a_1^n}{\partial x_1} \frac{\partial a_2^n}{\partial x_1} \dots \frac{\partial a_n^n}{\partial x_1} \end{array} \right| \\ &= \left(\frac{\partial a_1^1}{\partial x_1} A_1^1 + \frac{\partial a_2^1}{\partial x_1} A_1^2 + \dots + \frac{\partial a_n^1}{\partial x_1} A_1^n \right) + \left(\frac{\partial a_1^2}{\partial x_1} A_2^1 + \frac{\partial a_2^2}{\partial x_1} A_2^2 + \dots + \frac{\partial a_n^2}{\partial x_1} A_2^n \right) \\ &\quad + \dots + \left(\frac{\partial a_1^n}{\partial x_1} A_n^1 + \frac{\partial a_2^n}{\partial x_1} A_n^2 + \dots + \frac{\partial a_n^n}{\partial x_1} A_n^n \right) \\ &= \frac{\partial a_i^1}{\partial x_1} A_i^1 + \frac{\partial a_i^2}{\partial x_1} A_i^2 + \dots + \frac{\partial a_i^n}{\partial x_1} A_i^n = \frac{\partial a_i^j}{\partial x_1} A_j^i \end{aligned}$$

Therefore, in general, we can write $\frac{\partial a}{\partial x_p} = \frac{\partial a_i^j}{\partial x_p} A_j^i$.

1.8 Examples

Example 1.8.1. Write the terms contained in $S = a_{ij} x^i x^j$ taking $n = 3$.

Solution: Since the index i (or j) occurs both in subscript and superscript, we first sum on i from 1 to 3, then on j from 1 to 3.

$$\begin{aligned} S &= a_{1j} x^1 x^j + a_{2j} x^2 x^j + a_{3j} x^3 x^j \\ &= (a_{11} x^1 x^1 + a_{12} x^1 x^2 + a_{13} x^1 x^3) + (a_{21} x^2 x^1 + a_{22} x^2 x^2 + a_{23} x^2 x^3) \\ &\quad + (a_{31} x^3 x^1 + a_{32} x^3 x^2 + a_{33} x^3 x^3) \\ &= a_{11} (x^1)^2 + a_{22} (x^2)^2 + a_{33} (x^3)^2 + (a_{12} + a_{21}) x^1 x^2 + (a_{13} + a_{31}) x^1 x^3 + (a_{32} + a_{23}) x^2 x^3 \end{aligned}$$

Example 1.8.2. Express the sum of $\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 a_{ijk} x^i x^j x^k$.

Solution: Here, the number of terms is $3^3 = 27$.

Since the index i (or j or k) occurs both in subscript and superscript, we first sum on i from 1 to 3, then on each term of its 3 terms we sum j from 1 to 3. This results in 9 terms. Then, on each of the 9 terms we sum k from 1 to 3, which results in 27 terms. Like the last example, we sum

$$\begin{aligned}
 S &= (a_{1jk} x^1 x^j x^k + a_{2jk} x^2 x^j x^k + a_{3jk} x^3 x^j x^k) \\
 &= (a_{11k} x^1 x^1 x^k + a_{12k} x^1 x^2 x^k + a_{13k} x^1 x^3 x^k) \\
 &+ \cdots + (a_{31k} x^3 x^1 x^k + a_{32k} x^3 x^2 x^k + a_{33k} x^3 x^3 x^k) \\
 &= [(a_{111} x^1 x^1 x^1 + a_{122} x^1 x^2 x^2 + a_{133} x^1 x^3 x^3) + (a_{121} x^1 x^2 x^1 + a_{122} x^1 x^2 x^2 + a_{123} x^1 x^2 x^3) \\
 &+ \cdots \cdots \cdots + (\cdots \cdots + a_{32k} x^3 x^2 x^k + [a_{331} x^3 x^3 x^1 + a_{332} x^3 x^3 x^2 + a_{333} x^3 x^3 x^3) \\
 &= [a_{111} (x^1)^3 + a_{222} (x^2)^3 + a_{333} (x^3)^3 + [a_{123} + a_{132} + a_{231} + a_{213} + a_{321} + a_{312}] x^1 x^2 x^3 \\
 &+ [a_{112} + a_{121} + a_{211}] (x^1)^2 x^2 + [a_{113} + a_{131} + a_{311}] (x^1)^2 x^3 + [a_{221} + \dots] (x^2)^2 x^1 \\
 &+ [a_{223} + \dots] (x^2)^2 x^3 + [a_{331} + \dots] (x^3)^2 x^1 + [a_{332} + \dots] (x^3)^2 x^2.
 \end{aligned}$$

Example 1.8.3. If f is a function of n variables x^i , write the differential of f .

Solution: Since $f = f(x^1, x^2, \dots, x^n)$,

from calculus, we have $df = \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^2} dx^2 + \cdots + \frac{\partial f}{\partial x^n} dx^n = \frac{\partial f}{\partial x^i} dx^i$.

Example 1.8.4. (a) If $a_{pq} x^p x^q = 0$ for all values of the independent variables x^1, x^2, \dots, x^n and a_{pq} 's are constant, show that $a_{ij} + a_{ji} = 0$.

(b) If $a_{pqr} x^p x^q x^r = 0$ for all values of the independent variables x^1, x^2, \dots, x^n and a_{pqr} 's are constant, show that $a_{kij} + a_{kji} + a_{ikj} + a_{jki} + a_{ijk} + a_{jik} = 0$.

Solution: Differentiating: $a_{pq} x^p x^q = 0$ (1.12a)

with respect to x^i

$$\begin{aligned}
 a_{pq} x^q \frac{\partial}{\partial x^i} (x^p) + a_{pq} x^p \frac{\partial}{\partial x^i} (x^q) &= 0 \\
 \text{or } a_{pq} x^q \delta_i^p + a_{pq} x^p \delta_i^q &= 0 \tag{1.12b}
 \end{aligned}$$

$$\text{or } a_{pi} x^p + a_{iq} x^q = 0$$

Differentiating (1.12b), with respect to x^j , we get

$$\begin{aligned} a_{pi} \frac{\partial}{\partial x^j} (x^p) + a_{iq} \frac{\partial}{\partial x^j} (x^q) &= 0 \\ \text{or } a_{pi} \delta_j^p + a_{iq} \delta_j^q &= 0 \\ \text{or } a_{ji} + a_{ij} &= 0. \end{aligned}$$

(b) Differentiating
with respect to x^i

$$a_{pqr} x^p x^q x^r = 0$$

$$\begin{aligned} a_{pqr} x^q x^r \frac{\partial}{\partial x^i} (x^p) + a_{pqr} x^p x^r \frac{\partial}{\partial x^i} (x^q) + a_{pqr} x^p x^q \frac{\partial}{\partial x^i} (x^r) &= 0 \\ \text{or, } a_{pqr} x^q x^r \delta_i^p + a_{pqr} x^p x^r \delta_i^q + a_{pqr} x^p x^q \delta_i^r &= 0 \\ \text{or, } a_{iqr} x^q x^r + a_{pir} x^p x^r + a_{pqi} x^p x^q &= 0 \end{aligned}$$

Differentiating with respect to x^j , we get

$$\begin{aligned} a_{iqr} \frac{\partial}{\partial x^j} (x^q) x^r + a_{pir} x^p \frac{\partial}{\partial x^j} (x^r) + a_{pqi} \frac{\partial}{\partial x^j} (x^p) x^q \\ + a_{iqr} (x^q) \frac{\partial}{\partial x^j} (x^r) + a_{pir} \frac{\partial}{\partial x^j} (x^p) (x^r) + a_{pqi} (x^p) \frac{\partial}{\partial x^j} (x^q) &= 0 \\ \text{or } a_{iqr} \delta_j^q x^r + a_{pir} x^p \delta_j^r + a_{pqi} \delta_j^p x^q + a_{iqr} (x^q) \delta_j^r + a_{pir} \delta_j^p (x^r) + a_{pqi} (x^p) \delta_j^q &= 0 \\ \text{or } a_{ijr} x^r + a_{pij} x^p + a_{jq i} x^q + a_{iqj} (x^q) + a_{jir} (x^r) + a_{pji} (x^p) &= 0 \end{aligned}$$

Differentiating in the same way, with respect to x^k we get

$$\text{or } a_{ijk} + a_{kij} + a_{jki} + a_{ikj} + a_{jik} + a_{kji} = 0.$$

Example 1.8.5. If a_j^i is a double system such that $a_k^i a_j^k = \delta_j^i$, show that $|a_j^i| = \pm 1$.

Solution: We have $a_k^i a_j^k = \delta_j^i$, taking determinant $|a_k^i a_j^k| = |\delta_j^i|$,

$$\begin{aligned} \text{Or, } |a_k^i| |a_j^k| &= |\delta_j^i| \text{ or, } \begin{vmatrix} a_1^1 a_2^1 \dots a_n^1 & | & a_1^1 a_2^1 \dots a_n^1 \\ a_1^2 a_2^2 \dots n & | & a_1^2 a_2^2 \dots n \\ \dots & | & \dots \\ a_1^n a_2^n \dots a_n^n & | & a_1^n a_2^n \dots a_n^n \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 0 \dots 0 \\ 0 & 1 & 0 \dots 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots 1 \end{vmatrix} \quad [\text{since det. Of } \delta_j^i \text{ is unit } \det = |I|] \end{aligned}$$

$$\text{Or } \begin{vmatrix} a_1^1 a_2^1 \dots a_n^1 \\ a_1^2 a_2^2 \dots n \\ \dots \\ a_1^n a_2^n \dots a_n^n \end{vmatrix}^2 = 1, \quad \Rightarrow |a_j^i|^2 = 1 \Rightarrow |a_j^i| = \pm 1$$

Example 1.8.6. If a_j^i is a double system such that $a_k^i a_j^k = \delta_j^i$, show that either $|a_k^i - \delta_k^i| = 0$ or $|a_k^i + \delta_k^i| = 0$.

Solution: From above result $|a_j^i|^2 = 1$

$$\begin{aligned} \Rightarrow |a_j^i|^2 &= |\delta_j^i| \Rightarrow |a_j^i|^2 - |\delta_j^i|^2 = 0 \\ \Rightarrow (|a_j^i| - |\delta_j^i|)(|a_j^i| + |\delta_j^i|) &= 0 \Rightarrow (|a_j^i| - |\delta_j^i|)(|a_j^i| + |\delta_j^i|) = 0 \\ \Rightarrow \text{either } |a_j^i| - |\delta_j^i| &= 0 \text{ or } |a_j^i| + |\delta_j^i| = 0 \end{aligned}$$

Example 1.8.7. If $|a_j^i| = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix}$ and $|b_j^i| = \begin{vmatrix} b_1^1 & a_2^1 & b_3^1 \\ b_1^2 & b_2^2 & b_3^2 \\ b_1^3 & b_2^3 & b_3^3 \end{vmatrix}$, show that

$$|a_j^i||b_m^k| = |c_j^i| \quad \text{where } c_m^i = a_p^i a_m^p \quad (1.13a)$$

Solution: $|a_j^i||b_m^k| = |a_j^i| e_{kms} b_1^k b_2^m b_3^s$ [we know, $|a_m^k| = e_{ijk} a_1^i a_2^j a_3^k$] ... (α)
1.13b

$$\begin{aligned} &= (e_{kms} |a_j^i|) b_1^k b_2^m b_3^s \\ &= (e_{ijt} a_k^i a_m^j a_s^t) b_1^k b_2^m b_3^s \quad [|a_j^i| e_{pqr} = e_{ijk} a_p^i a_q^j a_r^k] \\ &= e_{ijt} (a_k^i b_1^k) (a_m^j b_2^m) (a_s^t b_3^s) \\ &= e_{ijt} c_1^i c_2^j c_3^t = |c_j^i| \quad \text{[By (1.13a) and by (1.13b)]} \end{aligned}$$

The above result can be stated as $|a_j^i||b_m^k| = |a_p^i a_m^p|$. It is the result of the multiplication of two determinants of the third order.

1.9 Exercises

1. Write out in full the following expression.
 - (a) $\delta_j^i a^j$ (b) $\delta_{ij} x^i x^j$ (c) $\delta_{ij} \delta^{jk}$ (d) δ_i^i (e) $a^i = \frac{\partial x^i}{\partial y^j} b^j$ (f) $\delta_j^i \delta_i^j \delta_k^l \delta_l^k$
2. Expand the following using the summation convention.
 - a) $A_{ij} B^{jk}$ (b) $a_j^i x^j$ (c) $\frac{\partial}{\partial x^k} (\sqrt{g} b^k)$
3. Prove the following.
 - a) $\delta_{ij} a_{jk} = a_{ik}$ (b) $\delta_{ij} a_{ij} = a_{ik}$ (c) $\delta_{ij} \delta_{jk} a_{nm} = a_{im}$ (d) $\delta_{ij} \delta^{jk} = \delta_i^k$.
4. Show that $\frac{\partial}{\partial x^i} (a_p x^p) = a_i$ for all values of independent variables, x^1, x^2, \dots, x^n , and where x^p 's are constants.
5. Calculate
 - a) $\frac{\partial}{\partial x^k} (a_{ij} x^j)$
 - b) $\frac{\partial}{\partial x^p} (a_{ijk} x^i x^j x^k)$

6. Using the relation $\frac{\partial x^r}{\partial x^s} \delta_s^r$, show that

$$\frac{\partial}{\partial x^p} (a_{ij} x^i x^j) = (a_{ip} + a_{pi}) x^i$$

7. Express each of the following sums using the summation convention:

(a) $\sum_{i=1}^n \sum_{j=1}^n u^i v^j$

(b) $\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n a_{ijk} x^i x^j x^k$

8. Evaluate each of the following (range of indices 1 to n):

(a) $\delta_j^i A_i^{kl}$ (b) $\delta_j^i \delta_i^k B^{jl}$ (c) $a^i a_j \delta_i^j$ (d) $\delta_j^i \delta_i^j \delta_k^l \delta_l^k$

9. (a) If $x^i = a_p^i y^p$ and $y^j = b_q^j z^q$, show that $x^i = a_q^i b_p^q z^p$.

(b) If $x^i = a_p^i y^p$ and $z^i = b_q^i y^q$, show that $z^i = a_q^p b_p^i y^q$.

10. If y^i are n independent functions of variables x^i and z^i are n independent functions of y^i and if $u^i = v^j \frac{\partial x^i}{\partial y^j}$ and

$$v^i = w^j \frac{\partial y^i}{\partial z^j} \quad (i, j = 1, 2, \dots, n),$$

then show that $u^i = w^j \frac{\partial}{\partial z^j} x^i$.

11. If $a = |a_j^i|$ and b_i^j is a^{-1} times the cofactor of a_j^i in the determinant of a_j^i , show that $b_i^j a_j^i = \delta_i^j$.

12. Prove that $|a_j^i| |b_l^k| = |c_j^i|$ where $c_i^j = a_p^j b_l^p$.