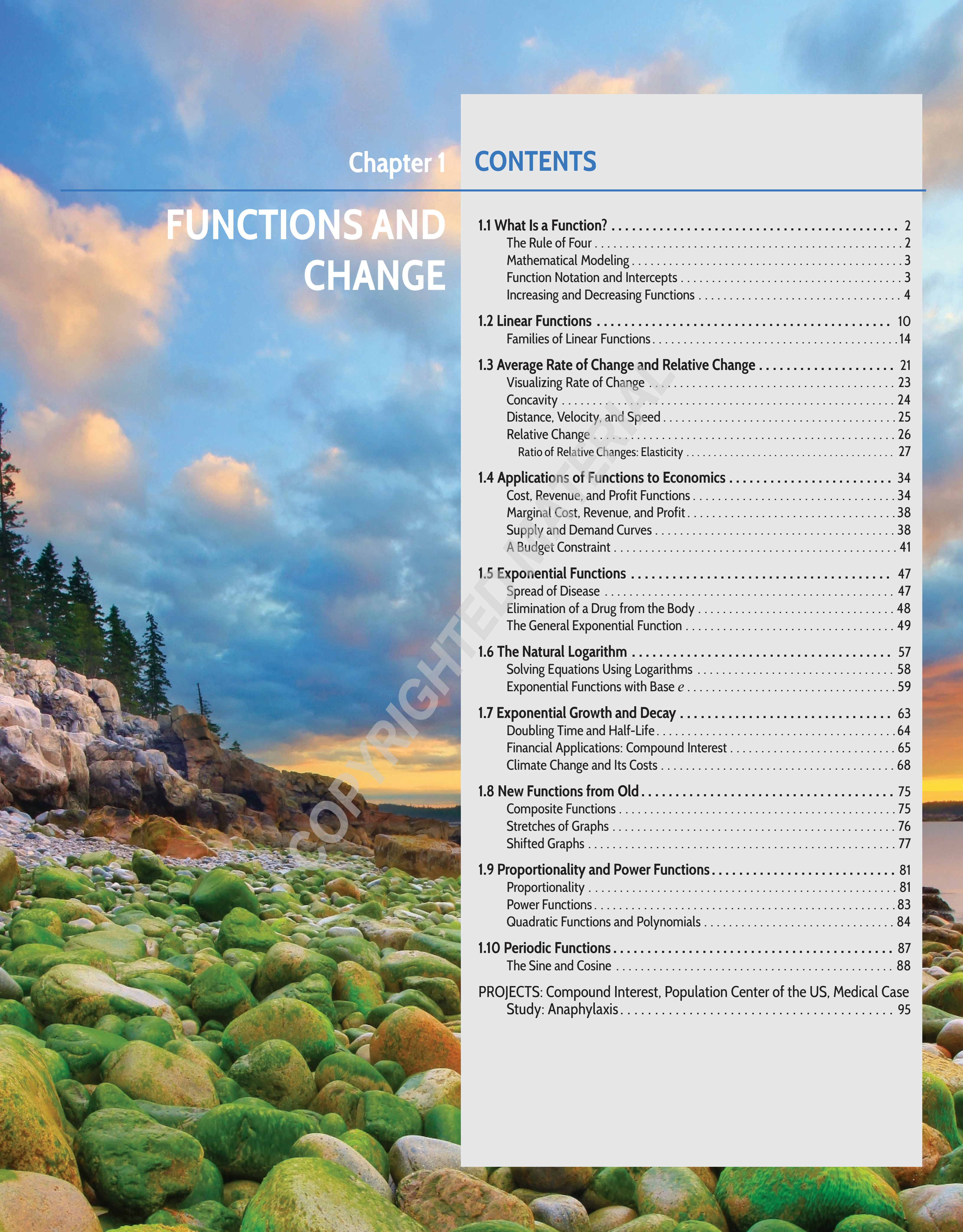


Chapter 1

FUNCTIONS AND CHANGE

CONTENTS

1.1 WHAT IS A FUNCTION?

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. In 2020, the Covid-19 pandemic spread around the globe. Table 1.1 shows the daily new Covid-19 cases confirmed in the US in January 2021.[1]

Table 1.1 *New Covid-19 cases confirmed each day in the US for the first ten days of 2021*

Jan date	1	2	3	4	5	6	7	8	9	10
New cases	153,394	299,786	208,074	183,406	233,654	253,993	277,068	292,105	262,337	213,161

You may not have thought of something so unpredictable as new Covid-19 cases as being a function, but it *is* a function of day. There is no formula for the daily Covid-19 cases, but the number of cases satisfies the definition of a function: Each day, t, has a unique number of new Covid-19 cases, N, associated with it—those reported that day.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the Covid-19 example, the domain is the set of t-values $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the range is the set of all N-values $\{153{,}394,\ 299{,}786,\ \ldots,\ 213{,}161\}$. We call the function f and write $N = f(t)$.

Some quantities, such as a day or date, are *discrete*, meaning they take only certain isolated values (days must be integers). Other quantities, such as time, are *continuous* since they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \le t \le b$ is called a *closed interval* and written $[a, b]$.

The set of numbers t such that $a < t < b$ is called an *open interval* and written (a, b).

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. The function giving the new Covid-19 cases in the US is represented by the graph in Figure 1.1 and by Table 1.1.

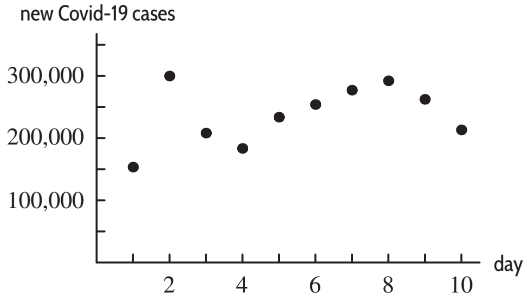


Figure 1.1: Daily new Covid-19 cases in the US, starting January 1, 2021

[1] JHU CSSE COVID-19 data at https://github.com/CSSEGISandData/COVID-19, accessed February 25, 2021.

Other functions arise naturally as graphs. Figure 1.2 contains electrocardiogram (EKG) pictures showing the heartbeat patterns of two patients, one normal and one not. Although it is possible to construct a formula to approximate an EKG function, this is seldom done. The pattern of repetitions is what a doctor needs to know, and these are more easily seen from a graph than from a formula. However, each EKG gives electrical activity as a function of time.

Functions, such as the EKG, whose graphs can be drawn without lifting pencil from paper are called *continuous functions*. See the Focus on Theory section on page 144 for details.

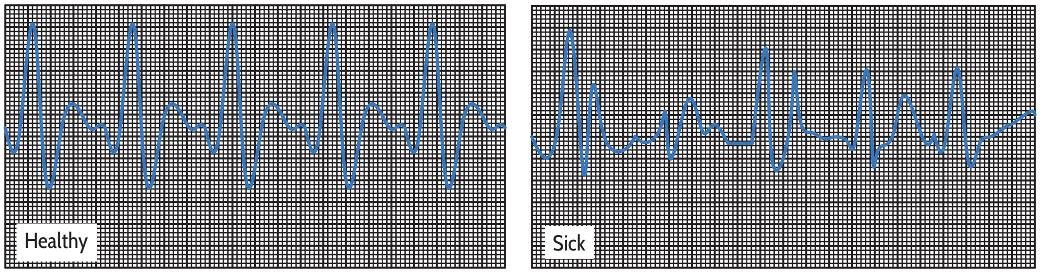


Figure 1.2: EKG readings on two patients

As another example of a function, consider the snowy tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed, to a fair degree of accuracy, using the formula

$$C = f(T) = 4T - 160.$$

Mathematical Modeling

A *mathematical model* is a mathematical description of a real situation. In this book we consider models that are functions, such as $C = f(T) = 4T - 160$.

Modeling almost always involves some simplification of reality. We choose which variables to include and which to ignore—for example, we consider the dependence of chirp rate on temperature, but not on other variables. The choice of variables is based on knowledge of the context (the biology of crickets, for example), not on mathematics. To test the model, we compare its predictions with observations.

In this book, we often model a situation that has a discrete domain with a continuous function whose domain is an interval of numbers. For example, the annual US gross domestic product (GDP) has a value for each year, $t = 0, 1, 2, 3, \ldots$. We may model it by a function of the form $G = f(t)$, with values for t in a continuous interval. In doing this, we expect that the values of $f(t)$ match the values of the GDP at the points $t = 0, 1, 2, 3, \ldots$, and that information obtained from $f(t)$ closely matches observed values.

A mathematical model can capture trends in the data to enable us to analyze and make predictions. A common way of finding a model is described in Appendix A.

Function Notation and Intercepts

We write $y = f(t)$ to express the fact that y is a function of t. The independent variable is t, the dependent variable is y, and f is the name of the function. The graph of a function has an *intercept* where it crosses the horizontal or vertical axis. Horizontal intercepts are also called the *zeros* of the function.

Example 1 (a) Graph the cricket chirp rate function, $C = f(T) = 4T - 160$.
(b) Solve $f(T) = 0$ and interpret the result.

Solution (a) The graph is in Figure 1.3.
(b) Solving $f(T) = 0$ gives the horizontal intercept:

$$\begin{aligned} 4T - 160 &= 0 \\ T &= \frac{160}{4} = 40. \end{aligned}$$

Thus at a temperature of 40°F, the chirp rate is zero.

For temperatures below 40°F, the model would predict negative values of C, so we conclude that the model does not apply for such temperature values.

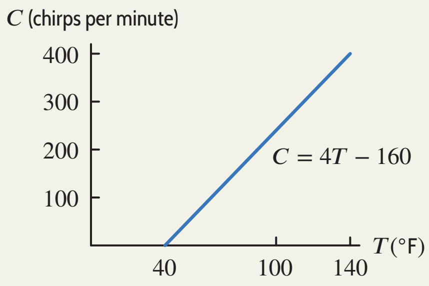


Figure 1.3: Cricket chirp rate as a function of temperature

Example 2 The value of a car, V, is a function of the age of the car, a, so $V = g(a)$, where g is the name we are giving to this function.

(a) Interpret the statement $g(5) = 15.975$ in terms of the value of a car if V is in thousands of dollars and a is in years.
(b) In the same units, the value of a Honda[2] is approximated by $g(a) = 24.975 - 1.8a$. Find and interpret the vertical and horizontal intercepts of the graph of this depreciation function g.

Solution (a) Since $V = g(a)$, the statement $g(5) = 15.975$ means $V = 15.975$ when $a = 5$. This tells us that the car is worth \$15,975 when it is 5 years old.
(b) Since $V = g(a)$, a graph of the function g has the value of the car on the vertical axis and the age of the car on the horizontal axis. The vertical intercept is the value of V when $a = 0$. It is $V = g(0) = 24.975$, so the Honda was valued at \$24,975 when new. The horizontal intercept is the value of a such that $g(a) = 0$, so

$$\begin{aligned} 24.975 - 1.8a &= 0 \\ a &= \frac{24.975}{1.8} = 13.875. \end{aligned}$$

After about 14 years, the Honda is predicted to have no value.

Increasing and Decreasing Functions

In the previous examples, the chirp rate increases with temperature, while the value of the Honda decreases with age. We express these facts by saying that f is an increasing function, while g is decreasing. See Figure 1.4. In general:

[2]Data obtained from the Kelley Blue Book, based on a 2020 Honda Accord sedan. www.kbb.com, accessed February 24, 2020.

A function f is **increasing** if the values of $f(x)$ increase as x increases.
A function f is **decreasing** if the values of $f(x)$ decrease as x increases.

The graph of an *increasing* function *climbs* as we move from left to right.
The graph of a *decreasing* function *descends* as we move from left to right.

Figure 1.4: Increasing and decreasing functions

Figure 1.1 of the daily Covid-19 cases in the US shows that a function can be increasing on one part of the domain and decreasing on another part of the domain.

Summary for Section 1.1

- **Definition of function**: a rule which takes numbers as inputs and assigns to each input exactly one output number.
- The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.
- **Function notation:** $y = f(t)$, where t is the **independent variable** and y is the **dependent variable**.
- The graph of a function has an **intercept** where it crosses the horizontal or vertical axis. Horizontal intercepts are also called the **zeros** of the function.
- A function f is **increasing** if the values of $f(x)$ increase as x increases.
- A function f is **decreasing** if the values of $f(x)$ decrease as x increases.

Problems for Section 1.1

1. Which graph in Figure 1.5 best matches each of the following stories?[3] Write a story for the remaining graph.
   (a) I had just left home when I realized I had forgotten my books, so I went back to pick them up.
   (b) Things went fine until I had a flat tire.
   (c) I started out calmly but sped up when I realized I was going to be late.

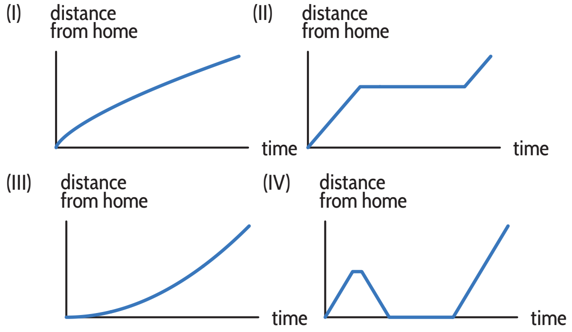


Figure 1.5

■ In Problems 2–5, use the description of the function to sketch a possible graph. Put a label on each axis and state whether the function is increasing or decreasing.

2. The height of a sand dune is a function of time, and the wind erodes away the sand dune over time.
3. The amount of carbon dioxide in the atmosphere is a function of time, and is going up over time.
4. The number of air conditioning units sold is a function of temperature, and goes up as the temperature goes up.
5. The noise level, in decibels, is a function of distance from the source of the noise, and the noise level goes down as the distance increases.
6. The population of Buffalo, NY, grew from 1900 to 1940, stayed approximately constant during the 1940s, and has decreased from 1950 onward. Graph the population as a function of years since 1900.

[3] Adapted from Jan Terwel, "Real Math in Cooperative Groups in Secondary Education", *Cooperative Learning in Mathematics*, ed. Neal Davidson, p. 234 (Reading: Addison Wesley, 1990).

In Problems 7–13, decide if the function, f, is increasing or decreasing.

7. An increase in the amount of rainfall r leads to an increase in the amount $G = f(r)$ of grass per hectare.

8. A drop in altitude h leads to an increase in atmospheric pressure $P = f(h)$.

9. A drop in air temperature T leads to a drop in available battery capacity $C = f(T)$ in an electric vehicle.

10. An increase in the mass m of a mammal increases the time $T = f(m)$ that blood takes to circulate once around the body.

11. An increase in ticket price P leads to a drop in attendance at a baseball game, $A = f(P)$.

12. An increase in the number of vehicles manufactured v leads to an increase in the total manufacturing cost, $C = f(v)$.

13. A decrease in the number of cars on the road c leads to a decrease in commuting time $T = f(c)$.

14. Let $W = f(t)$ represent wheat production in Argentina,[4] in millions of metric tons, where t is years since 2015. Interpret the statement $f(4) = 20$ in terms of wheat production.

15. The chirp rate, C, in chirps per minute, of the snowy tree cricket is given by $C = f(T) = 4T - 160$ where T is degrees Fahrenheit.

(a) Find an appropriate domain of f in the context of the model assuming a maximum temperature of 134°F, the highest recorded at a weather station.[5]
(b) Find the range of f on this domain.

16. The concentration of carbon dioxide, $C = f(t)$, in the atmosphere, in parts per million (ppm), is a function of years, t, since 2000.

(a) Interpret $f(19) = 415$ in terms of carbon dioxide.[6]
(b) What is the meaning of $f(22)$?

17. **(a)** The graph of $r = f(p)$ is in Figure 1.6. What is the value of r when p is 0? When p is 3?
(b) What is $f(2)$?

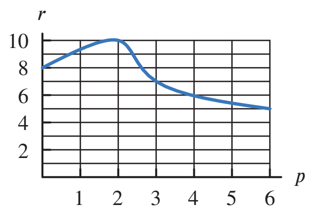


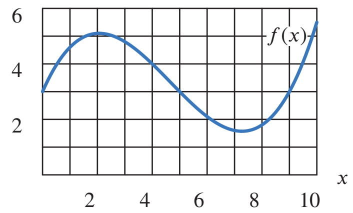
Figure 1.6

In Problems 18–22, for the function given, find $f(5)$.

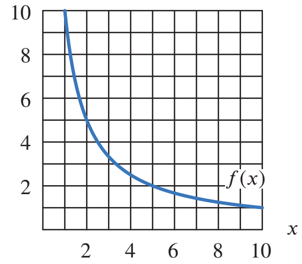
18. $f(x) = 2x + 3$

19. $f(x) = 10x - x^2$

20.



21.



22.

x	1	2	3	4	5	6	7	8
$f(x)$	2.3	2.8	3.2	3.7	4.1	5.0	5.6	6.2

23. Let $y = f(x) = x^2 + 2$.

(a) Find the value of y when x is zero.
(b) What is $f(3)$?
(c) What values of x give y a value of 11?
(d) Are there any values of x that give y a value of 1?

In Problems 24–27, the function $S = f(t)$ gives the average annual sea level, S, in meters above a fixed reference level, in Aberdeen, Scotland,[7] as a function of t, the number of years before 2020. Write a mathematical expression that represents the given statement.

24. In 2018 the average annual sea level in Aberdeen was 7.088 meters.

25. The average annual sea level in Aberdeen in 2020.

26. The average annual sea level in Aberdeen was the same in 1949 and 2000.

27. The average annual sea level in Aberdeen decreased by 11 millimeters from 2017 to 2018.

28. **(a)** A potato is put in an oven to bake at time $t = 0$. Which of the graphs in Figure 1.7 could represent the potato's temperature as a function of time?
(b) What does the vertical intercept represent in terms of the potato's temperature?

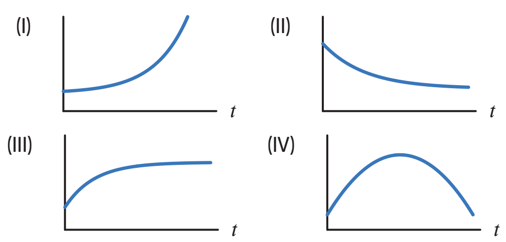


Figure 1.7

[4] www.indexmundi.com/agriculture, accessed February 25, 2020.
[5] http://www.guinnessworldrecords.com. Accessed January 2017.
[6] en.wikipedia.org, last accessed February 24, 2020.
[7] www.psmsl.org, accessed August 12, 2019.

29. Figure 1.8 shows the wheat crop yield, in tons, from one hectare of land as a function of the quantity of seeds planted, in kilograms.[8]

(a) Explain what the graph tells you about how the crop yield changes as the quantity of seeds planted increases.

(b) If a farmer has already planted 80 kg of seed, is it worth the cost of planting any additional seed?

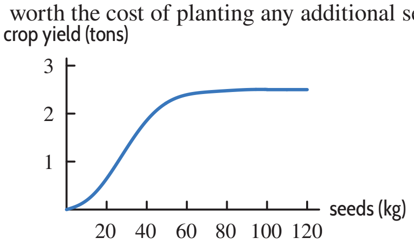


Figure 1.8

30. An object is put outside on a cold day at time $t = 0$ minutes. Its temperature, $H = f(t)$, in °C, is graphed in Figure 1.9.

(a) What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.

(b) Explain what the vertical intercept, a, and the horizontal intercept, b, represent in terms of temperature of the object and time outside.

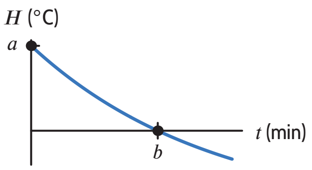


Figure 1.9

31. In the Andes mountains in Peru, the number, N, of species of bats is a function of the elevation, h, in feet above sea level, so $N = f(h)$.

(a) Interpret the statement $f(500) = 100$ in terms of bat species.

(b) What are the meanings of the vertical intercept, k, and horizontal intercept, c, in Figure 1.10?

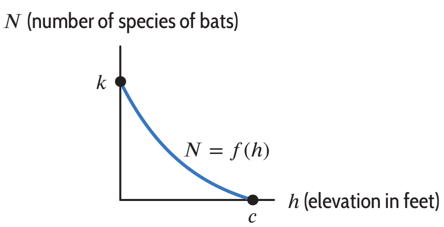


Figure 1.10

32. In tide pools on the New England coast, snails eat algae. Describe what Figure 1.11 tells you about the effect of snails on the diversity of algae.[9] Does the graph support the statement that diversity peaks at an intermediate number of snails per square meter?

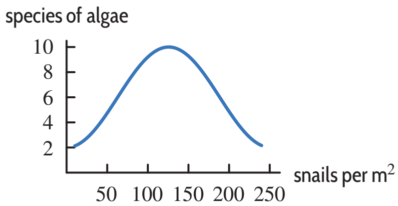


Figure 1.11

33. Figure 1.12 shows the amount of nicotine, $N = f(t)$, in mg, in a person's bloodstream as a function of the time, t, in hours, since the person finished smoking a cigarette.

(a) Estimate $f(3)$ and interpret it in terms of nicotine.

(b) About how many hours have passed before the nicotine level is down to 0.1 mg?

(c) What is the vertical intercept? What does it represent in terms of nicotine?

(d) If this function had a horizontal intercept, what would it represent?

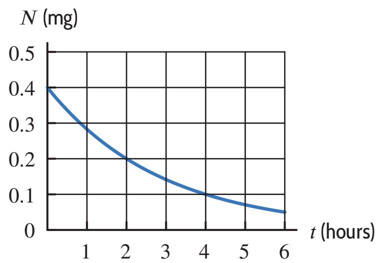


Figure 1.12

34. A deposit is made into an interest-bearing account. Figure 1.13 shows the balance, B, in the account t years later.

(a) What was the original deposit?

(b) Estimate $f(10)$ and interpret it.

(c) When does the balance reach \$5000?

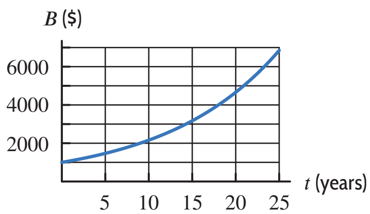


Figure 1.13

[8] http://www.fao.org/3/y5146e/y5146e07.htm, accessed September 12, 2019.

[9] M. L. Rosenzweig, *Species Diversity in Space and Time,* p. 343 (Cambridge: Cambridge University Press, 1995).

35. The use of CFCs (chlorofluorocarbons) has declined since the 1987 Montreal Protocol came into force to reduce the use of substances that deplete the ozone layer. World annual CFC consumption, $C = f(t)$, in million tons, is a function of time, t, in years since 1987. (CFCs are measured by the weight of ozone that they could destroy.)

(a) Interpret $f(10) = 0.2$ in terms of CFCs.[10]
(b) Interpret the vertical intercept of the graph of this function in terms of CFCs.
(c) Interpret the horizontal intercept of the graph of this function in terms of CFCs.

36. The range R of an electric vehicle, the average distance it can travel on a fully charged battery, depends on the outside temperature T. Figure 1.14 shows $R = f(T)$ for the 2014 Nissan Leaf.[11]

(a) What is the maximum range of the Leaf? At what temperature is it achieved?
(b) At what temperatures does an increase in temperature reduce the range of the Leaf? Is $f(T)$ increasing or decreasing for these temperatures?

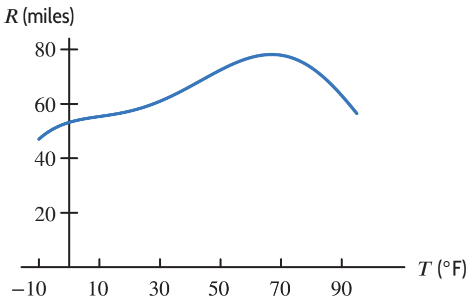


Figure 1.14

37. When the US Federal government spends more money than it takes in, the amount of the difference is called the deficit. In Figure 1.15, the function $D = f(t)$ gives a good approximation of the deficit from 2011 to 2015, where D is the deficit in billions of dollars and t is the number of years since 2010.

(a) Is the deficit increasing or decreasing over this time period?
(b) Interpret $f(4) = 485$ in terms of the deficit.
(c) In Figure 1.15, label the axes and put a dot at the point identified in part (b).
(d) If we extended the domain of this function to include a vertical intercept, what would the vertical intercept represent?
(e) If we extended the domain of this function to include a horizontal intercept, what would the horizontal intercept represent?

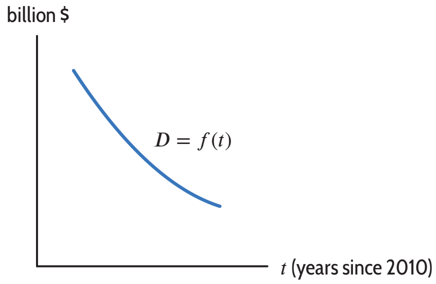


Figure 1.15

38. When a patient with a rapid heart rate takes a drug, the heart rate plunges dramatically and then slowly rises again as the drug wears off. Sketch the heart rate against time from the moment the drug is administered.

39. The gas mileage of a car (in miles per gallon) is highest when the car is going about 45 miles per hour and is lower when the car is going faster or slower than 45 mph. Graph gas mileage as a function of speed of the car.

40. After an injection, the concentration of a drug in a patient's body increases rapidly to a peak and then slowly decreases. Graph the concentration of the drug in the body as a function of the time since the injection was given. Assume that the patient has none of the drug in the body before the injection. Label the peak concentration and the time it takes to reach that concentration.

41. Financial investors know that, in general, the higher the expected rate of return on an investment, the higher the corresponding risk.

(a) Graph this relationship, showing expected return as a function of risk.
(b) On the figure from part (a), mark a point with high expected return and low risk. (Investors hope to find such opportunities.)

42. The number of sales per month, S, is a function of the amount, a (in dollars), spent on advertising that month, so $S = f(a)$.

(a) Interpret the statement $f(1000) = 3500$.
(b) Which of the graphs in Figure 1.16 is more likely to represent this function?
(c) What does the vertical intercept of the graph of this function represent, in terms of sales and advertising?

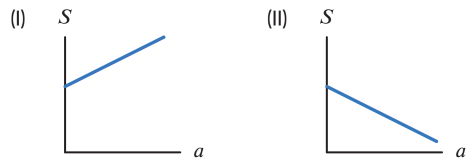


Figure 1.16

[10]The Worldwatch Institute, *Vital Signs 2007-2008* (New York: W.W. Norton & Company, 2007), p. 47.
[11]https://blog.ucsusa.org/dave-reichmuth/electric-cars-cold-weather-temperatures, accessed January 11, 2021.

43. Figure 1.17 shows fifty years of fertilizer use in the US, India, and the former Soviet Union.[12]

(a) Estimate fertilizer use in 1970 in the US, India, and the former Soviet Union.
(b) Write a sentence for each of the three graphs describing how fertilizer use has changed in each region over this 50-year period.

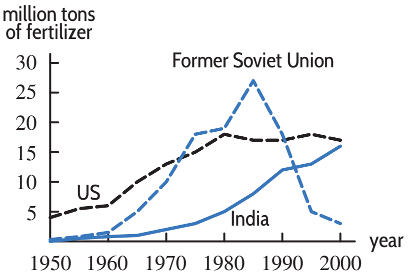


Figure 1.17

44. Figure 1.18 shows data from the Covid-19 outbreak in Switzerland[13] during the spring of 2020. Which graph is the graph of $P(t)$, the total number of cases confirmed up to day t, and which is the graph of $N(t)$, the number of new cases confirmed on day t? Explain.

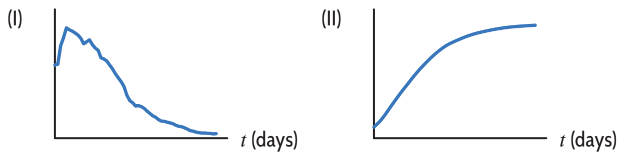


Figure 1.18

45. Let $P(t)$ be the total number of cases of an infectious disease outbreak confirmed up to and including day t. Let $N(t)$ be the number of new cases confirmed on day t. The outbreak ends 100 days after the first confirmed case is recorded at time $t = 0$. Over the course of the outbreak, 20,000 people contract the disease.

(a) What are the values of $P(100)$ and $N(100)$?
(b) What can we determine, if anything, about the value of $P(t)$ for $t > 100$?

46. Let $N(t)$ be the number of new Covid-19 cases in Iowa[14] confirmed on day t, where $t = 1$ is January 1, 2021. What does each statement mean in the context of the outbreak?

(a) $N(8) = 1419$
(b) $N(9) - N(8) = -898$
(c) $N(10) - N(9) = 581$

47. Let $N(t)$ be the number of new Covid-19 cases in Arizona[15] confirmed on day t, where $t = 1$ is February 1, 2021. What does each statement mean in the context of the outbreak?

(a) $N(3) = 2296$
(b) $N(4) - N(3) = 2121$
(c) $N(5) - N(4) = -591$

48. The six graphs in Figure 1.19 show frequently observed patterns of age-specific cancer incidence rates, in number of cases per 1000 people, as a function of age.[16] The scales on the vertical axes are equal.

(a) For each of the six graphs, write a sentence explaining the effect of age on the cancer rate.
(b) Which graph shows a relatively high incidence rate for children? Suggest a type of cancer that behaves this way.
(c) Which graph shows a brief decrease in the incidence rate at around age 50? Suggest a type of cancer that might behave this way.
(d) Which graph or graphs might represent a cancer that is caused by toxins which build up in the body over time? (For example, lung cancer.) Explain.

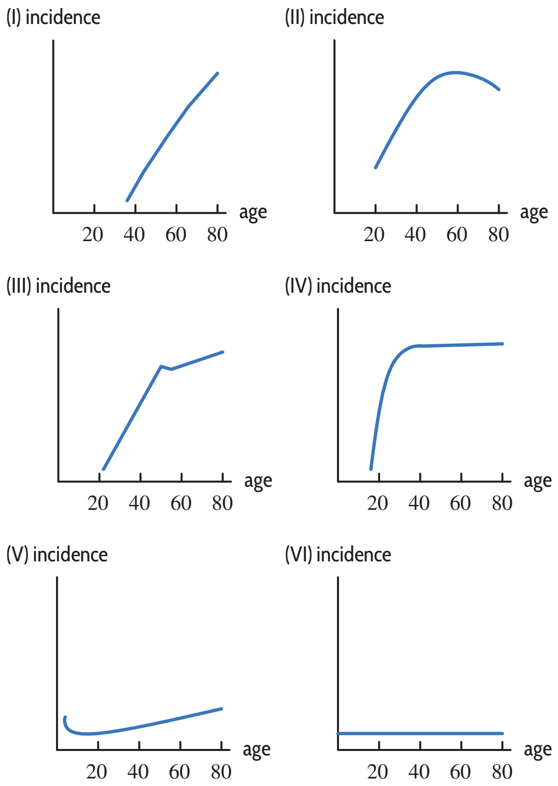


Figure 1.19

[12] The Worldwatch Institute, *Vital Signs 2001,* p. 32 (New York: W.W. Norton, 2001).
[13] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed May 23, 2020.
[14] JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed June 14, 2021.
[15] JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed February 24, 2021.
[16] Abraham M. Lilienfeld, *Foundations of Epidemiology*, p. 155 (New York: Oxford University Press, 1976).

49. Table 1.2 shows the average annual sea level, S, in meters, in Aberdeen, Scotland,[17] as a function of time, t, measured in years before 2014.

Table 1.2

t	0	25	50	75	100	125
S	7.071	7.083	6.990	6.964	6.985	6.900

(a) What was the average sea level in Aberdeen in 2014?

(b) In what year was the average sea level 7.083 meters? 6.985 meters?

(c) Table 1.3 gives the average sea level, S, in Aberdeen as a function of the year, x. Complete the missing values.

Table 1.3

x	1889	?	1939	1964	1989	2014
S	?	6.985	?	6.990	?	?

50. The range of an electric vehicle is the distance it will travel on a full charge. The advertised range may be different than the actual range, which depends on the air temperature. Figure 1.20 shows the actual range as a percent of the advertised range that electric vehicles achieve as a function of air temperature.[18]

(a) What percentage of the advertised range is an electric vehicle expected to achieve when the temperature is 14°F?

(b) At what temperatures can we expect an electric vehicle to achieve 80% of its advertised range?

(c) Is an electric vehicle ever expected to have a shorter range than the advertised range? If so, for which temperatures?

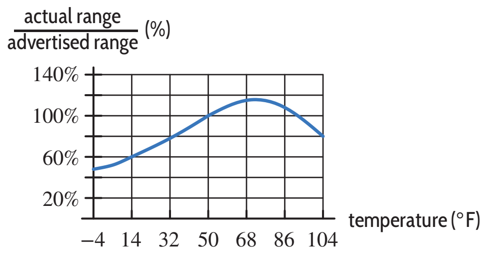


Figure 1.20

■ Problems 51–54 ask you to plot graphs based on the following story: "As I drove down the highway this morning, at first traffic was fast and uncongested, then it crept nearly bumper-to-bumper until we passed an accident, after which traffic flow went back to normal until I exited."

51. Driving speed against time on the highway

52. Distance driven against time on the highway

53. Distance from my exit vs time on the highway

54. Distance between cars vs distance driven on the highway

1.2 LINEAR FUNCTIONS

Probably the most commonly used functions are the *linear functions*, whose graphs are straight lines. The chirp-rate and the Honda-depreciation functions in the previous section are both linear. We now look at more examples of linear functions.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 20 centimeters every four years. Table 1.4 shows that the height started at 330 cm in 1900 and increased by the equivalent of 5 cm a year between 1900 and 1912. So the height was a linear function of time.

Table 1.4 *Winning height (approximate) for men's Olympic pole vault*

Year	1900	1904	1908	1912
Height (cm)	330	350	370	390

If y is the winning height in cm and t is the number of years since 1900, then y is predicted approximately by

$$y = f(t) = 330 + 5t.$$

[17] www.psmsl.org, accessed August 19, 2019.
[18] Graph adapted from https://www.geotab.com/blog/ev-range/, accessed January 11, 2021.

Since $y = f(t)$ increases with t, we see that f is an increasing function. The coefficient 5 tells us the rate, in cm per year, at which the height increases. This rate is the *slope* of the line in Figure 1.21. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{370 - 350}{8 - 4} = \frac{20}{4} = 5 \text{ cm/year.}$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

What about the constant 330? This represents the initial height in 1900, when $t = 0$. Geometrically, 330 is the intercept on the vertical axis.

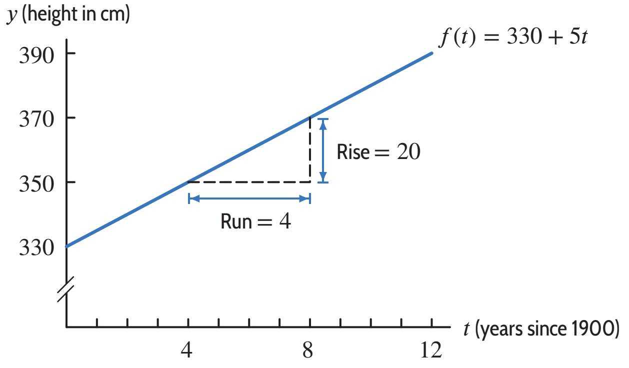


Figure 1.21: Olympic pole vault records

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it does not give a very good prediction. The formula $y = 330 + 5t$ predicts that the height in the Rio 2016 Olympics would be 910 cm, which is considerably higher than the actual value of 603 cm.[19] There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.4 is discrete, because it is given only at specific points (every four years). However, we have treated the variable t as though it were continuous, because the function $y = 330 + 5t$ makes sense for all values of t.

The graph in Figure 1.21 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

Example 1 If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.4t.$$

Explain the meaning of the intercept, 260, and the slope, -0.4, in terms of the world record time to run the mile and sketch the graph.

Solution The intercept, 260, tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.4, tells us that the world record decreased at a rate of about 0.4 seconds per year. See Figure 1.22.

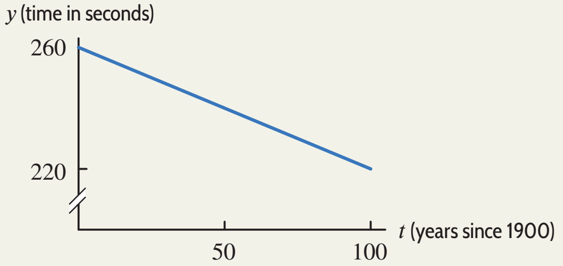


Figure 1.22: World record time to run the mile

[19] www.rio2016.com/en/athletics-standings-at-mens-pole-vault, accessed September 14, 2016.

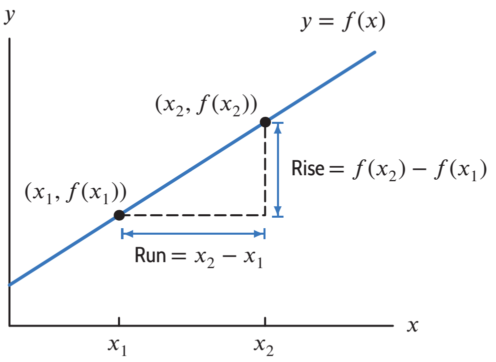


Figure 1.23: Difference quotient $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Slope and Rate of Change

We use the symbol Δ (the Greek letter capital delta) to mean "change in," so Δx means change in x and Δy means change in y. The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2, using the formula

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.23.) Since slope $= \Delta y/\Delta x$, the slope represents the *rate of change* of y with respect to x. The units of the slope are y-units over x-units.

Linear Functions in General

A **linear function** can be put in the *slope-intercept form*

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x.
- b is the **vertical intercept** or value of y when x is zero.

If the slope, m, is positive, then f is an increasing function. If m is negative, then f is decreasing.

Notice that if the slope, m, is zero, we have $y = b$, a horizontal line. For a line of slope m through the point (x_0, y_0), we have

$$\text{Slope} = m = \frac{y - y_0}{x - x_0}.$$

Therefore we can write the equation of the line in the *point-slope form*:

The equation of a line of slope m through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0).$$

Example 2 The solid waste generated each year in the cities of the US is increasing. The solid waste generated,[20] in millions of tons, was 262.11 in 2015 and 292.36 in 2018.

(a) Assuming that the amount of solid waste generated by US cities is a linear function of time, find a formula for this function by finding the equation of the line through these two points.
(b) Use this formula to predict the amount of solid waste generated in the year 2025.

Solution (a) We think of the amount of solid waste, in millions of tons W, as a function of year, t, and the two points are (2015, 262.11) and (2018, 292.36). The slope of the line is

$$m = \frac{\Delta W}{\Delta t} = \frac{292.36 - 262.11}{2018 - 2015} = \frac{30.25}{3} = 10.0833333 \text{ million tons/year.}$$

We use the point-slope form to find the equation of the line. We substitute the point (2015, 262.11) and the slope $m = 10.0833333$ into the equation:

$$\begin{aligned} W - W_0 &= m(t - t_0) \\ W - 262.11 &= 10.0833333(t - 2015) \\ W - 262.11 &= 10.0833333t - 20317.917 \\ W &= 10.0833333t - 20055.807. \end{aligned}$$

The equation of the line is $W = 10.0833333t - 20055.807$. Alternatively, we could use the slope-intercept form of a line to find the vertical intercept.

(b) To calculate solid waste predicted for the year 2025, we substitute $t = 2025$ into the equation of the line, $W = -20055.807 + 10.0833333t$, and calculate W:

$$W = -20055.807 + 10.0833333(2025) = 362.943.$$

The formula predicts that in the year 2025, there will be 362.943 million tons of solid waste.

Recognizing Data from a Linear Function: Values of x and y in a table could come from a linear function $y = b + mx$ if differences in y-values are constant for equal differences in x.

Example 3 Which of the following tables of values could represent a linear function?

x	0	1	2	3
$f(x)$	25	30	35	40

x	0	2	4	6
$g(x)$	10	16	26	40

t	20	30	40	50
$h(t)$	2.4	2.2	2.0	1.8

Solution Since $f(x)$ increases by 5 for every increase of 1 in x, the values of $f(x)$ could be from a linear function with slope $= 5/1 = 5$.

Between $x = 0$ and $x = 2$, the value of $g(x)$ increases by 6 as x increases by 2. Between $x = 2$ and $x = 4$, the value of y increases by 10 as x increases by 2. Since the differences in y-values are not constant for equal differences in x, $g(x)$ could not be a linear function.

Since $h(t)$ decreases by 0.2 for every increase of 10 in t, the values of $h(t)$ could be from a linear function with slope $= -0.2/10 = -0.02$.

[20] www.epa.gov, accessed March 1, 2021.

Example 4 The data in the following table lie on a line. Find formulas for each of the following functions, and give units for the slope in each case:

(a) q as a function of p

(b) p as a function of q

p (dollars)	5	10	15	20
q (tons)	100	90	80	70

Solution (a) If we think of q as a linear function of p, then q is the dependent variable and p is the independent variable. We can use any two points to find the slope. The first two points give

$$\text{Slope} = m = \frac{\Delta q}{\Delta p} = \frac{90 - 100}{10 - 5} = \frac{-10}{5} = -2.$$

The units are the units of q over the units of p, or tons per dollar.

To write q as a linear function of p, we use the equation $q = b + mp$. We know that $m = -2$, and we can use any of the points in the table to find b. Substituting $p = 10$, $q = 90$ gives

$$\begin{aligned} q &= b + mp \\ 90 &= b + (-2)(10) \\ 90 &= b - 20 \\ 110 &= b. \end{aligned}$$

Thus, the equation of the line is

$$q = 110 - 2p.$$

(b) If we now consider p as a linear function of q, then p is the dependent variable and q is the independent variable. We have

$$\text{Slope} = m = \frac{\Delta p}{\Delta q} = \frac{10 - 5}{90 - 100} = \frac{5}{-10} = -0.5.$$

The units of the slope are dollars per ton.

Since p is a linear function of q, we have $p = b + mq$ and $m = -0.5$. To find b, we substitute any point from the table, such as $p = 10$, $q = 90$, into this equation:

$$\begin{aligned} p &= b + mq \\ 10 &= b + (-0.5)(90) \\ 10 &= b - 45 \\ 55 &= b. \end{aligned}$$

Thus, the equation of the line is

$$p = 55 - 0.5q.$$

Alternatively, we could take our answer to part (a), that is $q = 110 - 2p$, and solve for p.

Appendix A shows how to fit a linear function to data that is not exactly linear.

Families of Linear Functions

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, represent a *family of functions*. All the functions in a family share certain properties—in this case, the graphs are lines. The constants m and b are called *parameters*. Figures 1.24 and 1.25 show graphs with several values of m and b. Notice that the greater the magnitude of m, the steeper the line.

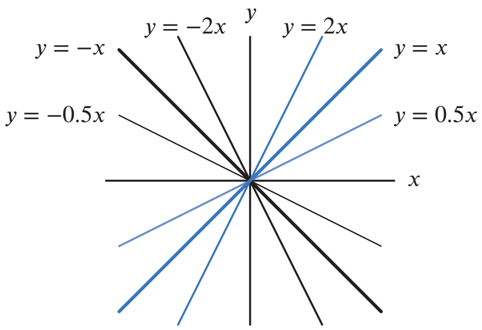


Figure 1.24: The family $y = mx$ (with $b = 0$)

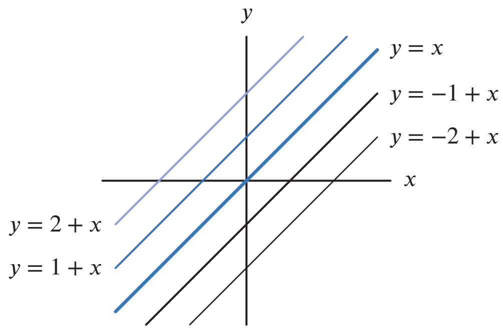


Figure 1.25: The family $y = b + x$ (with $m = 1$)

Summary for Section 1.2

- A **linear function** has values of y that change at a constant rate with the values of x.
- **Formula** for linear functions:

$$y = \underbrace{b}_{\text{Initial value}} + \underbrace{m}_{\text{Slope}} \cdot x$$

- The graph of a linear function is a **line**.
  - b is the **vertical intercept**, or ***y*-intercept**, and gives the value of y for $x = 0$.
  - m is the **slope** of the line and gives the rate of change of y with respect to x:

$$m = \frac{\Delta y}{\Delta x}.$$

- The equation of a line of slope m through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0)$$

- **Recognizing data from a linear function:** Values of x and y in a table could come from a linear function $y = b + mx$ if differences in y-values are constant for equal differences in x.
- Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, represent a **family of functions**.

Problems for Section 1.2

For Problems 1–4, find an equation for the line that passes through the given points.

1. $(0, 2)$ and $(2, 3)$
2. $(0, 0)$ and $(1, 1)$
3. $(-2, 1)$ and $(2, 3)$
4. $(4, 5)$ and $(2, -1)$

For Problems 5–8, determine the slope and the y-intercept of the line whose equation is given.

5. $7y + 12x - 2 = 0$
6. $3x + 2y = 8$
7. $12x = 6y + 4$
s
8. $-4y + 2x + 8 = 0$

9. Figure 1.26 shows four lines given by equation $y = b + mx$. Match the lines to the conditions on the parameters m and b.

   (a) $m > 0, b > 0$ **(b)** $m < 0, b > 0$
   (c) $m > 0, b < 0$ **(d)** $m < 0, b < 0$

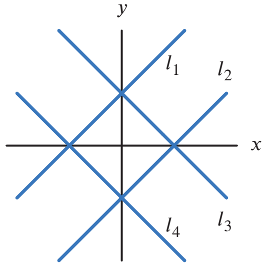


Figure 1.26

10. **(a)** Which two lines in Figure 1.27 have the same slope? Of these two lines, which has the larger y-intercept?
(b) Which two lines have the same y-intercept? Of these two lines, which has the larger slope?

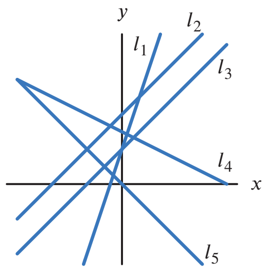


Figure 1.27

11. A city's population was 28,600 in the year 2015 and is growing by 750 people a year.

(a) Give a formula for the city's population, P, as a function of the number of years, t, since 2015.
(b) What is the population predicted to be in 2020?
(c) When is the population expected to reach 35,000?

12. A car rental company charges a daily fee of \$35 plus \$0.20 per mile driven. Find a formula for the daily charge, C, in dollars, as a function of the number of miles, m, driven that day.

13. A company rents cars at \$40 a day and 15 cents a mile. Its competitor's cars are \$50 a day and 10 cents a mile.

(a) For each company, give a formula for the cost of renting a car for a day as a function of the distance traveled.
(b) On the same axes, graph both functions.
(c) How should you decide which company is cheaper?

14. A phone company charges \$25 a month for unlimited talk, text and 2GB of data, and \$5 for every gigabyte of data above 2GB. Write an expression for the monthly phone charge, P, in dollars, as a function of the number of gigabytes of data, d, used that month.

15. Figure 1.28 shows the distance from home, in miles, of a person on a 5-hour trip.

(a) Estimate the vertical intercept. Give units and interpret it in terms of distance from home.
(b) Estimate the slope of this linear function. Give units, and interpret it in terms of distance from home.
(c) Give a formula for distance, D, from home as a function of time, t in hours.

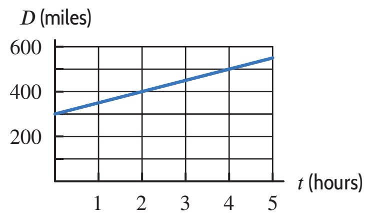


Figure 1.28

16. Which of the following tables could represent linear functions?

(a)

x	0	1	2	3
y	27	25	23	21

(b)

t	15	20	25	30
s	62	72	82	92

(c)

u	1	2	3	4
w	5	10	18	28

17. For each table in Problem 16 that could represent a linear function, find a formula for that function.

18. Facebook's annual ad revenue increased every year from 2011 through 2018, and each year the increase in revenue was larger than the year before.[21] Was the growth in revenue linear?

19. The top speed (in miles per hour) of the fastest[22] car, t years since 1949, is approximated by

$$S = 2.25t + 124.$$

(a) What is the vertical intercept of the line? What are its units and meaning in terms of cars?
(b) What is the slope of the line? What are its units and meaning in terms of cars?

20. Let t be in quarter-years after the last quarter of 2018. Zoom,[23] the video conferencing company, had quarterly revenue R through $t = 5$ of about

$$R = 44.6 + 15.5t \text{ million dollars}$$

Then, from $t = 6$ onward in 2020, quarterly revenue was about

$$R = 11.6 + 22.1t \text{ million dollars}$$

(a) Interpret the slope of each function.
(b) What might have caused the change in slope at the time it happened?

[21] https://i1.wp.com/www.mekkographics.com/wp-content/uploads/2019/03/Slide39.png
[22] These are the fastest *production* cars, meaning mass-produced, commercially available to the public, and able to be legally driven. Data from https://en.wikipedia.org/wiki/Production_car_speed_record, accessed January 9, 2021.
[23] news.alphastreet.com/infographic-zoom-video-communications-zm-q2-2021-earnings-results, accessed March 23, 2021.

21. Annual revenue R from McDonald's restaurants worldwide between 2015 and 2019 can be estimated by $R = 25.44 - 1.22t$, where R is in billion dollars and t is in years since January 1, 2015.[24]

(a) What is the slope of this function? Include units. Interpret the slope in terms of McDonald's revenue.
(b) What is the vertical intercept of this function? Include units. Interpret the vertical intercept in terms of McDonald's revenue.
(c) What annual revenue does the function predict for 2021?
(d) If revenue continues to decline at the same rate, when is annual revenue predicted to fall to 17 billion dollars?

Problems 22–24 use Figure 1.29 showing how the quantity, Q, of grass (kg/hectare) in different parts of Namibia depended on the average annual rainfall, r, (mm), in two different years.[25]

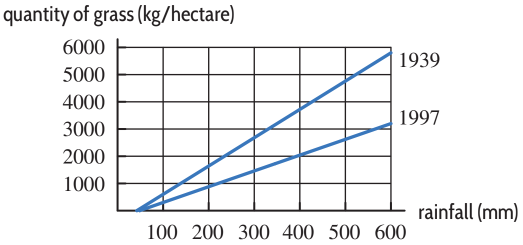


Figure 1.29

22. **(a)** For 1939, find the slope of the line, including units.
(b) Interpret the slope in this context.
(c) Find the equation of the line.

23. **(a)** For 1997, find the slope of the line, including units.
(b) Interpret the slope in this context.
(c) Find the equation of the line.

24. Which of the two functions in Figure 1.29 has the larger difference quotient $\Delta Q/\Delta r$? What does this tell us about grass in Namibia?

25. Figure 1.30 shows the 0–60 mph acceleration time, A (in seconds), for the best-accelerating cars[26] of the year, t years after 1955, where $A = -0.0566t + 6$.

(a) What are the units and meaning of the
(i) Slope? (ii) Vertical intercept?
(b) In what year is the 0–60 mph acceleration time predicted to drop to 1 second?

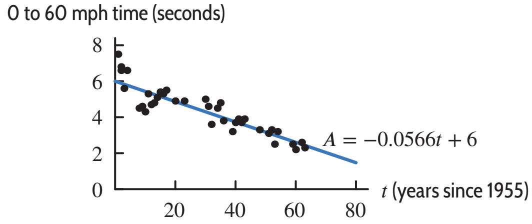


Figure 1.30

26. The amount of domestic waste, W, in millions of metric tons, dumped in landfills dropped from 22.42 in 2001 to 2.17 in 2019 in England.[27] Find a formula for W as a linear function of year t.

27. US net imports of crude oil and petroleum were decreasing between 2006 and 2019.[28] The trend is shown in Figure 1.31.

(a) Find the slope of the line. Include units.
(b) Write an equation for the line. What do your variables represent? What are their units?
(c) In what year does the model predict that the US will stop being a net importer of oil? Compare with the actual year this happened, 2020.[29]

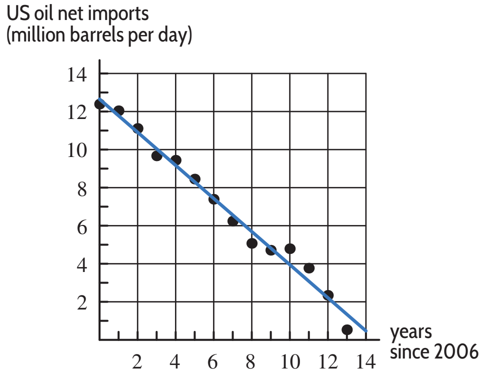


Figure 1.31

28. A company's pricing schedule in Table 1.5 is designed to encourage large orders. (A gross is 12 dozen.) Find a formula for:

(a) q as a linear function of p.
(b) p as a linear function of q.

Table 1.5

q (order size, gross)	3	4	5	6
p (price/dozen)	15	12	9	6

[24]Based on www.marketwatch.com/investing/stock/mcd/financials, accessed February 28, 2020.
[25]David Ward and Ben T. Ngairorue, "Are Namibia's Grasslands Desertifying?", *Journal of Range Management* 53, 2000, 138–144.
[26]Cars tested by *Car and Driver* Magazine, https://www.caranddriver.com/features/g15383525/car-and-driver-tested-the-quickest-cars-from-the-1950s-to-today/?slide=4, accessed January 9, 2021.
[27]www.statista.com, accessed April 6, 2021.
[28]https://www.eia.gov/energyexplained/oil-and-petroleum-products/imports-and-exports.php, accessed March 8, 2021.
[29]https://www.worldoil.com/news/2021/2/17/us-will-import-62-more-crude-by-2022-due-to-domestic-production-declines-says-eia, accessed March 8, 2021.

29. World milk production rose at an approximately constant rate between 2015 and 2019.[30] See Figure 1.32.

(a) Estimate the vertical intercept and interpret it in terms of milk production.
(b) Estimate the slope and interpret it in terms of milk production.
(c) Give an approximate formula for milk production, M, as a function of t.

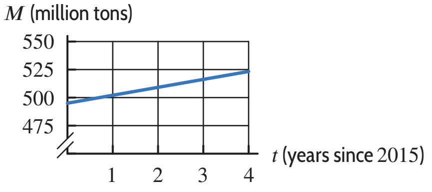


Figure 1.32

30. Marmots are large squirrels that hibernate in the winter and come out in the spring. Figure 1.33 shows the date (days after Jan 1) that they are first sighted each year in Colorado as a function of the average minimum daily temperature for that year.[31]

(a) Find the slope of the line, including units.
(b) What does the sign of the slope tell you about marmots?
(c) Use the slope to determine how much difference 6°C warming makes to the date of first appearance of a marmot.
(d) Find the equation of the line.

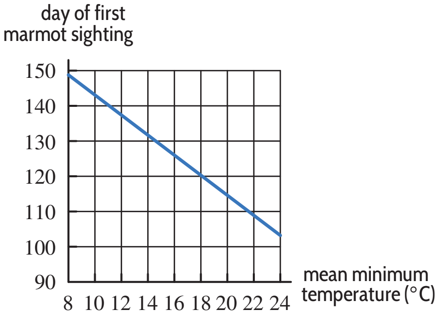


Figure 1.33

31. In Colorado spring has arrived when the bluebell first flowers. Figure 1.34 shows the date (days after Jan 1) that the first flower is sighted in one location as a function of the first date (days after Jan 1) of bare (snow-free) ground.[32]

(a) If the first date of bare ground is 140, how many days later is the first bluebell flower sighted?
(b) Find the slope of the line, including units.
(c) What does the sign of the slope tell you about bluebells?
(d) Find the equation of the line.

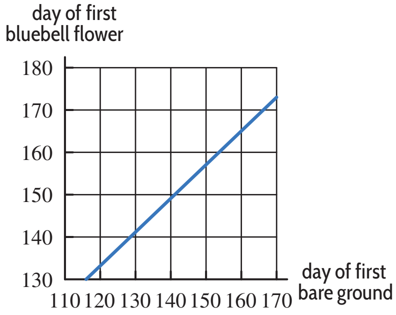


Figure 1.34

32. On March 5, 2015, Capracotta, Italy, received 256 cm (100.787 inches) of snow in 18 hours.[33]

(a) Assuming the snow fell at a constant rate and there were already 100 cm of snow on the ground, find a formula for $f(t)$, in cm, for the depth of snow as a function of t hours since the snowfall began on March 5.
(b) What are the domain and range of f?

33. For t in years since 1950, the quantity of carbon dioxide in the atmosphere, in ppm (parts per million), is predicted to be

$$g(t) = 300 + 1.4t.$$

(a) Find $g(10)$, $g(20)$ and $g(100)$.
(b) Suppose another model predicts the quantity of carbon dioxide to be

$$h(t) = 300 + 1.5t.$$

Find $h(10)$, $h(20)$ and $h(100)$.
(c) Do the functions $g(t)$ and $h(t)$ differ in their intercept or in their slope? Does this lead to a larger difference in values of $g(t)$ and $h(t)$ for small t or for large t? (Use your answers to parts (a) and (b) to decide.)

34. The approximate percentage of people, P, below the poverty level in the US[34] is given in Table 1.6.

(a) Find a formula for the percentage in poverty as a linear function of time in years since 2015.
(b) Use the formula to predict the percentage in poverty in 2019.
(c) What is the difference between the prediction and the actual percentage, 10.5%?

[30] www.statista.com/statistics/263952/production-of-milk-worldwide, accessed February 28, 2020.
[31] David W. Inouye, Billy Barr, Kenneth B. Armitage, and Brian D. Inouye, "Climate Change Is Affecting Altitudinal Migrants and Hibernating Species", *PNAS* 97, 2000, 1630–1633.
[32] David W. Inouye, Billy Barr, Kenneth B. Armitage, and Brian D. Inouye, "Climate Change Is Affecting Altitudinal Migrants and Hibernating Species", *PNAS* 97, 2000, 1630–1633.
[33] iceagenow.info, accessed April, 2015.
[34] Data based on: https://www.statista.com/statistics/200463/us-poverty-rate-since-1990/, accessed March 2, 2021.

Table 1.6

Year (since 2015)	0	1	2	3
P (percentage)	13.5	12.9	12.3	11.7

35. World grain production was 2048 million tons in 2005 and 2170 million tons in 2019, and has been increasing at an approximately constant rate.[35]

(a) Find a linear function for world grain production, P, in million tons, as a function of t, the number of years since 2005.
(b) Using units, interpret the slope in terms of grain production.
(c) Using units, interpret the vertical intercept in terms of grain production.
(d) According to the linear model, what is the predicted world grain production in 2020?
(e) According to the linear model, when is grain production predicted to reach 2250 million tons?

36. Global temperature increases are benchmarked to the average temperature between 1951 and 1980. In 2010, the global average temperature was 0.72°C above the benchmark. In 2020, it was 1.02°C above.[36]

(a) Assuming the difference between the global average temperature and the benchmark is a linear function of time, find a formula for this function.
(b) Use this formula to predict the difference in 2025.

37. Annual sales of vinyl records in the US have increased since 2006. Sales were 0.9 million in 2006 and 18.84 million in 2019.[37]

(a) Find a formula for annual sales, S, in millions of vinyl records, as a linear function of the number of years, t, since 2006.
(b) Give units for and interpret the slope and the vertical intercept of this function.
(c) Use the formula to predict vinyl record sales in 2025.

38. A \$25,000 vehicle depreciates \$2000 a year as it ages. Repair costs are \$1500 per year.

(a) Write formulas for each of the two linear functions at time t, value, $V(t)$, and repair costs to date, $C(t)$. Graph them.
(b) One strategy is to replace a vehicle when the total cost of repairs is equal to the current value. Find this time.
(c) Another strategy is to replace the vehicle when the value of the vehicle is some percent of the original value. Find the time when the value is 6%.

39. Search and rescue teams work to find lost hikers. Members of the search team separate and walk parallel to one another through the area to be searched. Table 1.7 shows the percent, P, of lost individuals found for various separation distances, d, of the searchers.[38]

(a) Explain how you know that the percent found, P, could be a linear function of separation distance, d.
(b) Find P as a linear function of d.
(c) What is the slope of the function? Give units and interpret the answer.
(d) What are the vertical and horizontal intercepts of the function? Give units and interpret the answers.

Table 1.7

Separation distance d (ft)	20	40	60	80	100
Approximate percent found, P	90	80	70	60	50

40. In a Washington town, the charge for commercial waste collection is \$694.55 for 5 tons and \$1098.32 for 8 tons of waste.

(a) Find a linear formula for the cost, C, of waste collection as a function of the weight, w, in tons.
(b) What is the slope of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.
(c) What is the vertical intercept of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.

41. Table 1.8 shows the pressure P, in torr,[39] at a depth of h meters below the surface of a lake.

(a) Explain why P could be a linear function of h.
(b) Find pressure, P, as a linear function of depth h and give practical interpretations of the slope and vertical intercept.
(c) Use part (b) to approximate the depth at which the pressure is twice that at the surface of the lake.

Table 1.8

h (m)	6	8	10	12	14	16
P (torr)	1201	1348	1495	1642	1789	1936

42. The number of species of coastal dune plants in Australia decreases as the latitude, in °S, increases. There are 34 species at 11°S and 26 species at 44°S.[40]

(a) Find a formula for the number, N, of species of coastal dune plants in Australia as a linear function of the latitude, l, in °S.

[35] www.world-grain.com/articles/13219-igc-lifts-2019-20-global-grain-production-forecast, accessed July 28, 2020.
[36] climate.nasa.gov, accessed March 23, 2021.
[37] https://www.statista.com/statistics/188822/lp-album-sales-in-the-united-states-since-2009/, accessed March 2, 2021.
[38] J. Wartes, *An Experimental Analysis of Grid Sweep Searching* (Explorer Search and Rescue, Western Region, 1974).
[39] A torr is a unit of pressure.
[40] M. L. Rosenzweig, *Species Diversity in Space and Time* (Cambridge: Cambridge University Press, 1995), p. 292.

(b) Give units for and interpret the slope and the vertical intercept of this function.
(c) Graph this function between $l = 11°$S and $l = 44°$S. (Australia lies entirely within these latitudes.)

43. Table 1.9 gives the maximum weight, w, in kilograms, of male American soldiers, aged between 21 and 27, for height, h, in centimeters.[41]

(a) How do you know that the data in this table could represent a linear function?
(b) Find weight, w, as a linear function of height, h. What is the slope of the line? What are the units for the slope?
(c) Find height, h, as a linear function of weight, w. What is the slope of the line? What are the units for the slope?

Table 1.9

h (cm)	172	176	180	184	188	192	196
w (kg)	76.264	80.032	83.8	87.568	91.336	95.104	98.872

■ Problems 44–49 concern the maximum heart rate (MHR), which is the maximum number of times a person's heart can safely beat in one minute. If MHR is in beats per minute and a is age in years, the formulas used to estimate MHR are

$$\text{For females: MHR} = 226 - a,$$

$$\text{For males: MHR} = 220 - a.$$

44. Which of the following is the correct statement?

(a) As you age, your maximum heart rate decreases by one beat per year.
(b) As you age, your maximum heart rate decreases by one beat per minute.
(c) As you age, your maximum heart rate decreases by one beat per minute per year.

45. Which of the following is the correct statement for a male and female of the same age?

(a) Their maximum heart rates are the same.
(b) The male's maximum heart rate exceeds the female's.
(c) The female's maximum heart rate exceeds the male's.

46. What can be said about the ages of a male and a female with the same maximum heart rate?

47. Recently[42] it has been suggested that a more accurate predictor of MHR for both males and females is given by

$$\text{MHR} = 206.9 - 0.67a.$$

(a) At what age do the old and new formulas give the same MHR for females? For males?
(b) Which of the following is true?
(i) The new formula predicts a higher MHR for young people and a lower MHR for older people than the old formula.
(ii) The new formula predicts a lower MHR for young people and a higher MHR for older people than the old formula.
(c) When testing for heart disease, doctors ask patients to walk on a treadmill while the speed and incline are gradually increased until their heart rates reach 85 percent of the MHR. For a 65-year-old male, what is the difference in beats per minute between the heart rate reached if the old formula is used and the heart rate reached if the new formula is used?

48. Experiments[43] suggest that the female MHR decreases by 2 beats per minute from birth to age 21, and by 5 beats per minute from birth to age 33. Is this consistent with MHR being approximately linear with age?

49. Experiments[44] suggest that the male MHR decreases by 6 beats per minute from birth to age 21, and by 11 beats per minute from birth to age 33. Is this consistent with MHR being approximately linear with age?

50. Let y be the percent increase in annual US national production during a year when the unemployment rate changes by u percent. (For example, $u = 2$ if unemployment increases from 4% to 6%.) Okun's law states that

$$y = 3.5 - 2u.$$

(a) What is the meaning of the number 3.5 in Okun's law?
(b) What is the effect on national production of a year when unemployment rises from 5% to 8%?
(c) What change in the unemployment rate corresponds to a year when production is the same as the year before?
(d) What is the meaning of the coefficient −2 in Okun's law?

51. An Australian[45] study found that, if other factors are constant (education, experience, etc.), taller people receive higher wages for the same work. The study reported a "height premium" for men of 3% of the hourly wage for a 10 cm increase in height; for women the height premium reported was 2%. We assume that

[41] Adapted from usmilitary.about.com, accessed March 29, 2015.
[42] www.verywellfit.com/maximum-heart-rate-1231221, accessed February 28, 2020.
[43] https://www.aqua-calc.com/calculate/maximum-heart-rate, accessed March 8, 2021.
[44] www.css.edu/users/tboone2/asep/May2002JEPonline.html, accessed March 8, 2021.
[45] "Study Finds Tall People at Top of Wages Ladder", *Yahoo News,* May 17, 2009.

hourly wages are a linear function of height. The slope is given by the height premium at the average hourly wage for that gender, AU\$29.40 for men and AU\$24.78 for women.

(a) The average hourly wage[46] for a 178 cm Australian man is AU\$29.40. Express the average hourly wage of an Australian man as a function of his height, x cm.

(b) The average hourly wage for a 164 cm Australian woman is AU\$24.78. Express the average hourly wage of an Australian woman as a function of her height, y cm.

(c) What is the difference in average hourly wages between men and women of height 178 cm?

(d) Is there a height for which men and women are predicted to have the same wage? If so, what is it?

1.3 AVERAGE RATE OF CHANGE AND RELATIVE CHANGE

Average Rate of Change

In the previous section, we saw that the height of the winning Olympic pole vault increased at an approximately constant rate of 5 cm/year between 1900 and 1912. Similarly, the world record for the mile decreased at an approximately constant rate of 0.4 seconds/year. We now see how to calculate rates of change when they are not constant.

Example 1 Table 1.10 shows the height of the winning pole vault at the Olympics[47] during the 1960s and more recently. Find the rate of change of the winning height between 1960 and 1968, and between 2008 and 2016. In which of these two periods did the height increase faster than during the period 1900–1912?

Table 1.10 *Winning height in men's Olympic pole vault (approximate)*

Year	1960	1964	1968	⋯	2008	2012	2016
Height (cm)	470	510	540	⋯	596	597	603

Solution From 1900 to 1912, the height increased by 5 cm/year. To compare the 1960s and the more recent period, we calculate

$$\begin{array}{c}\text{Average rate of change of height}\\ \text{1960 to 1968}\end{array} = \frac{\text{Change in height}}{\text{Change in time}} = \frac{540-470}{1968-1960} = 8.75 \text{ cm/year.}$$

$$\begin{array}{c}\text{Average rate of change of height}\\ \text{2008 to 2016}\end{array} = \frac{\text{Change in height}}{\text{Change in time}} = \frac{603-596}{2016-2008} = 0.88 \text{ cm/year.}$$

Thus, on average, the height was increasing more quickly during the 1960s than from 1900 to 1912, and was increasing more slowly from 2008 to 2016 than from 1900 to 1912.

In Example 1, the function does not have a constant rate of change (it is not linear). However, we can compute an *average rate of change* over any interval. The word average is used because the rate of change may vary within the interval. We have the following general formula.

> If y is a function of t, so $y = f(t)$, then
>
> $$\begin{array}{c}\textbf{Average rate of change} \text{ of } y\\ \text{between } t=a \text{ and } t=b\end{array} = \frac{\Delta y}{\Delta t} = \frac{f(b)-f(a)}{b-a}.$$
>
> The units of average rate of change of a function are units of y per unit of t.

The average rate of change of a linear function is the slope, and a function is linear if the average rate of change is the same on all intervals.

[46] Australian Fair Pay Commission, August 2007.

[47] www.statista.com/statistics/1100644/olympics-pole-vault-gold-medal-height-since-1896/, accessed March 4, 2020.

Example 2 Using Figure 1.35, estimate the average rate of change of the number of farms[48] in the US between 1970 and 1990.

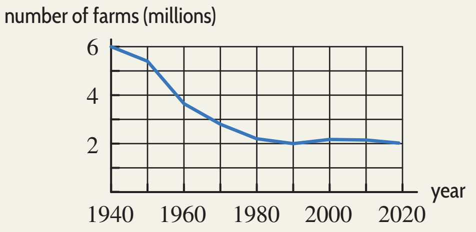


Figure 1.35: Number of farms in the US (in millions)

Solution Figure 1.35 shows that the number, N, of farms in the US was approximately 2.8 million in 1970 and approximately 2.0 million in 1990. If time, t, is in years, we have

$$\text{Average rate of change} = \frac{\Delta N}{\Delta t} = \frac{2.0 - 2.8}{1990 - 1970} = -0.04 \text{ million farms per year.}$$

The average rate of change is negative because the number of farms is decreasing. During this period, the number of farms decreased at an average rate of 0.04 million, or 40,000, per year.

We have looked at how an Olympic record and the number of farms change over time. In the next example, we look at average rate of change with respect to a quantity other than time.

Example 3 Polychlorinated biphenyl (PCB) is an industrial pollutant thought dangerous to wildlife. Table 1.11 shows data on PCB levels and the thickness of pelican egg shells.[49] (Thin shells break more easily.)

Find the average rate of change in the thickness of the shell as the PCB concentration changes from 87 ppm to 452 ppm. Give units and explain what the sign of the answer tells us.

Table 1.11 *Thickness of pelican eggshells and PCB concentration in the eggshells*

Concentration, c, in parts per million (ppm)	87	147	204	289	356	452
Thickness, h, in millimeters (mm)	0.44	0.39	0.28	0.23	0.22	0.14

Solution Since we are looking for the average rate of change of thickness with respect to change in PCB concentration, we have

$$\text{Average rate of change of thickness} = \frac{\text{Change in the thickness}}{\text{Change in the PCB level}} = \frac{\Delta h}{\Delta c} = \frac{0.14 - 0.44}{452 - 87}$$
$$= -0.00082 \frac{\text{mm}}{\text{ppm}}.$$

The units are thickness units (mm) over PCB concentration units (ppm), or millimeters over parts per million. The fact that the average rate of change is negative tells us that, for this data, the thickness of the eggshell decreases as the PCB concentration increases. The thickness of pelican eggs decreases by an average of 0.00082 mm for every additional part per million of PCB in the eggshell.

[48] *The World Almanac and Book of Facts, 2005*, p. 136 (New York).

[49] R. W. Risebrough, "Effects of Environmental Pollutants upon Animals Other than Man." *Proceedings of the 6th Berkeley Symposium on Mathematics and Statistics, VI*, p. 443–463 (Berkeley: University of California Press, 1972). PCBs were banned in the US in 1979.

Visualizing Rate of Change

For a function $y = f(x)$, the change in the value of the function between $x = a$ and $x = c$ is $\Delta y = f(c) - f(a)$. Since Δy is a difference of two y-values, it is represented by the vertical distance in Figure 1.36. The average rate of change of f between $x = a$ and $x = c$ is represented by the slope of the line joining the points A and C in Figure 1.37. This line is called the *secant line* between $x = a$ and $x = c$.

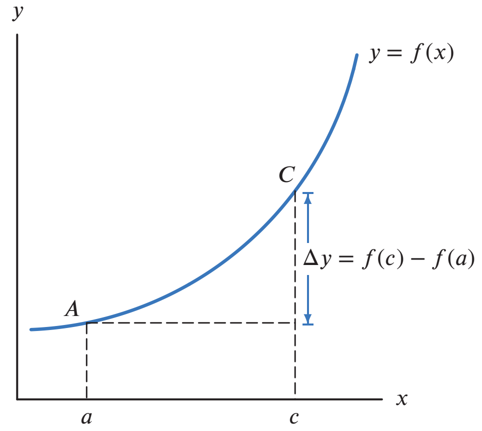


Figure 1.36: The change in a function is represented by a vertical distance

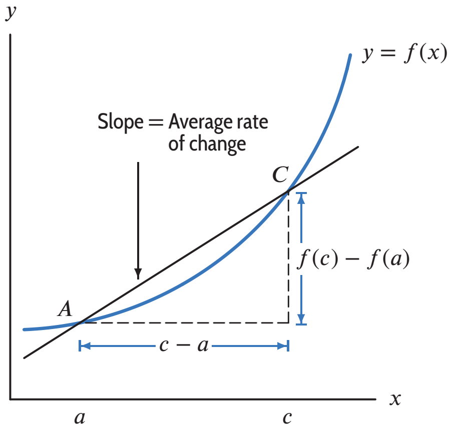


Figure 1.37: The average rate of change is represented by the slope of the line

Example 4 (a) Find the average rate of change of $y = f(x) = \sqrt{x}$ between $x = 1$ and $x = 4$.
(b) Graph $f(x)$ and represent this average rate of change as the slope of a line.
(c) Which is larger, the average rate of change of the function between $x = 1$ and $x = 4$ or the average rate of change between $x = 4$ and $x = 5$? What does this tell us about the graph of the function?

Solution (a) Since $f(1) = \sqrt{1} = 1$ and $f(4) = \sqrt{4} = 2$, between $x = 1$ and $x = 4$, we have

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}.$$

(b) A graph of $f(x) = \sqrt{x}$ is given in Figure 1.38. The average rate of change of f between 1 and 4 is the slope of the secant line between $x = 1$ and $x = 4$.

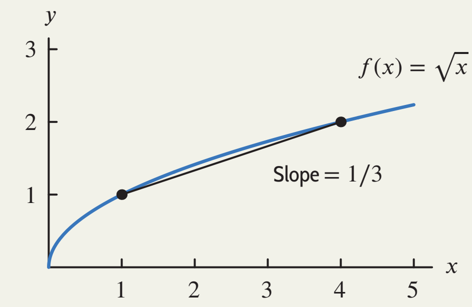


Figure 1.38: Average rate of change = Slope of secant line

(c) Since the secant line between $x = 1$ and $x = 4$ is steeper than the secant line between $x = 4$ and $x = 5$, the average rate of change between $x = 1$ and $x = 4$ is larger than it is between $x = 4$ and $x = 5$. The rate of change is decreasing. This tells us that the graph of this function is bending downward.

Concavity

We now look at the graphs of functions whose rates of change are increasing throughout an interval or decreasing throughout an interval.

Figure 1.38 shows a graph that is bending downward because the rate of change is decreasing. The graph in Figure 1.36 bends upward because the rate of change of the function is increasing. We make the following definitions.

> The graph of a function is **concave up** if it bends upward as we move left to right; the graph is **concave down** if it bends downward. (See Figure 1.39.) A line is neither concave up nor concave down.

Figure 1.39: Concavity of a graph

Example 5 Using Figure 1.40, estimate the intervals over which:

(a) The function is increasing; decreasing. (b) The graph is concave up; concave down.

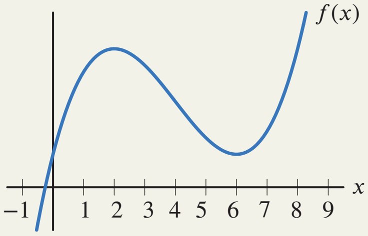


Figure 1.40

Solution

(a) The graph suggests that the function is increasing for $x < 2$ and for $x > 6$. It appears to be decreasing for $2 < x < 6$.

(b) The graph is concave down on the left and concave up on the right. It is difficult to tell exactly where the graph changes concavity, although it appears to be about $x = 4$. Approximately, the graph is concave down for $x < 4$ and concave up for $x > 4$.

Example 6 From the following values of $f(t)$, does f appear to be increasing or decreasing? Do you think its graph is concave up or concave down?

t	0	5	10	15	20	25	30
$f(t)$	12.6	13.1	14.1	16.2	20.0	29.6	42.7

Solution Since the given values of $f(t)$ increase as t increases, f appears to be increasing. As we read from left to right, the change in $f(t)$ starts small and gets larger (for constant change in t), so the graph is climbing faster. Thus, the graph appears to be concave up. Alternatively, plot the points and notice that a curve through these points bends up.

Distance, Velocity, and Speed

A grapefruit is thrown up in the air. The height of the grapefruit above the ground first increases and then decreases. See Table 1.12.

Table 1.12 *Height, y, of the grapefruit above the ground t seconds after it is thrown*

t (sec)	0	1	2	3	4	5	6
y (feet)	6	90	142	162	150	106	30

Example 7 Find the change and average rate of change of the height of the grapefruit during the first 3 seconds. Give units and interpret your answers.

Solution The change in height during the first 3 seconds is $\Delta y = 162 - 6 = 156$ ft. This means that the grapefruit goes up a total of 156 feet during the first 3 seconds. The average rate of change during this 3-second interval is $156/3 = 52$ ft/sec. During the first 3 seconds, the grapefruit is rising at an average rate of 52 ft/sec.

The average rate of change of height with respect to time is *velocity*. You may recognize the units (feet per second) as units of velocity.

$$\text{Average velocity} = \frac{\text{Change in distance}}{\text{Change in time}} = \begin{array}{c}\text{Average rate of change of distance}\\ \text{with respect to time}\end{array}$$

There is a distinction between *velocity* and *speed*. Suppose an object moves along a line. If we pick one direction to be positive, the velocity is positive if the object is moving in that direction and negative if it is moving in the opposite direction. For the grapefruit, upward is positive and downward is negative. Speed is the magnitude of velocity, so it is always positive or zero.

Example 8 Find the average velocity of the grapefruit over the interval $t = 4$ to $t = 6$. Explain the sign of your answer.

Solution Since the height is $y = 150$ feet at $t = 4$ and $y = 30$ feet at $t = 6$, we have

$$\text{Average velocity} = \frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta y}{\Delta t} = \frac{30 - 150}{6 - 4} = -60 \text{ ft/sec}.$$

The negative sign means the height is decreasing and the grapefruit is moving downward.

Example 9 A car travels away from home on a straight road. Its distance from home at time t is shown in Figure 1.41. Is the car's average velocity greater during the first hour or during the second hour?

Figure 1.41: Distance of car from home

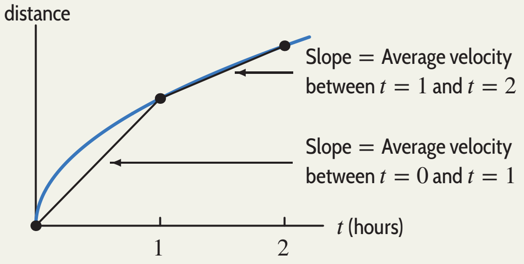


Figure 1.42: Average velocities of the car

Solution Average velocity is represented by the slope of a secant line. Figure 1.42 shows that the secant line between $t = 0$ and $t = 1$ is steeper than the secant line between $t = 1$ and $t = 2$. Thus, the average velocity is greater during the first hour.

Relative Change

Is a population increase of 1000 a significant change? It depends on the original size of the community. If the town of Coyote, NM, population 128, increases by 1000 people, the townspeople would definitely notice.[50] On the other hand, if New York City, population 8.5 million,[51] increases by 1000 people, almost no one will notice. To visualize the impact of the increase on the two different communities, we look at the change, 1000, as a fraction, or percentage, of the initial population. This percent change is called the *relative change*.

Example 10 If the population increases by 1000 people, find the relative change in the population for

(a) Coyote, NM (population 128)
(b) New York City (population 8,500,000)

Solution (a) The population increases by 1000 from 128 so

$$\text{Relative change} = \frac{\text{Change in population}}{\text{Initial population}} = \frac{1000}{128} = 7.813.$$

The population has increased by 781.3%, a significant increase.

(b) The population increases by 1000 from 8,500,000 so

$$\text{Relative change} = \frac{\text{Change in population}}{\text{Initial population}} = \frac{1000}{8{,}500{,}000} = 0.00012.$$

The population has increased by 0.012%, or less than one-tenth of one percent.

In general, when a quantity P changes from P_0 to P_1, we define

$$\text{Relative change in } P = \frac{\text{Change in } P}{P_0} = \frac{P_1 - P_0}{P_0}.$$

The relative change is a number, without units. It is often expressed as a percentage.

Example 11 Relative change can show the severity of a Covid-19 outbreak in a community. On November 22, 2020, there were 83 new Covid cases in Vermont (out of 3629 total cases) and 6587 new cases in Florida (out of 938,414 total cases). Find the relative change in infections in Vermont and in Florida.

Solution The change in infections in Vermont, 83, is much smaller than the change in Florida, 6387. Yet the relative change in Vermont is much greater than in Florida:

$$\begin{aligned}\text{Relative change in infections in Vermont} &= \frac{83}{3629} = 0.0229\\ \text{Relative change in infections in Florida} &= \frac{6587}{938414} = 0.0070.\end{aligned}$$

[50] suburbanstats.org/population/new-mexico/how-many-people-live-in-coyote, accessed March 3, 2020.
[51] worldpopulationreview.com/us-cities/new-york-city-population/, accessed March 3, 2020.

Example 12 A price increase can be significant or inconsequential depending on the item. In each of the following cases, find the relative change in price of a \$2 price increase; give your answer as a percent.

(a) A gallon of gas costing \$3.25 (b) A smartphone costing \$580

Solution (a) The change in the price is \$2 so we have

$$\text{Relative change in price of gas} = \frac{\text{Change in price}}{\text{Initial price}} = \frac{2}{3.25} = 0.615.$$

The price of gas has gone up 61.5%.

(b) We have

$$\text{Relative change in price of smartphone} = \frac{\text{Change in price}}{\text{Initial price}} = \frac{2}{580} = 0.003.$$

The price of the smartphone has gone up only 0.3%.

Relative change can be positive or negative, as we see in the following example.

Example 13 Find the relative change in the price of a \$75.99 pair of jeans if the sale price is \$52.99.

Solution The price has dropped from \$75.99 to \$52.99. We have

$$\text{Relative change} = \frac{52.99 - 75.99}{75.99} = \frac{-23}{75.99} = -0.303.$$

The price has been reduced by 30.3% for the sale.

Example 14 The number of sales per week of a \$75.99 pair of jeans is 25 pairs. The number of weekly sales goes up to 45 pairs when the price is reduced by 30.3%. Find the relative change in the weekly sales.

Solution The quantity sold goes up from 25 pairs of jeans to 45 pairs. We have

$$\text{Relative change} = \frac{45 - 25}{25} = \frac{20}{25} = 0.8.$$

The number of sales per week grows by 80%.

Ratio of Relative Changes: Elasticity

In Examples 13 and 14 we see that a 30.3% reduction in the price of a \$75.99 pair of jeans increased the weekly sales by 80%. If we are interested in the percent change in quantity sold as a result of a 1% change in price, we look at the ratio:

$$\left|\frac{\text{Relative change in quantity sold}}{\text{Relative change in price}}\right| = \left|\frac{0.8}{-0.303}\right| = 2.64.$$

So the number of pair of jeans sold increases by 2.64% when the price drops by 1%. This ratio is called the elasticity. It gives information about how the change in the price of a product effects the change in the quantity sold. Elasticity will be discussed in further detail in Section 4.6.

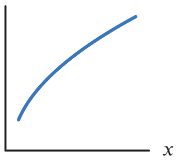
Summary for Section 1.3

- **Average rate of change** of $y = f(t)$ on $a \le t \le b$ is
$$\frac{\Delta y}{\Delta t} = \frac{f(b) - f(a)}{b - a}.$$
Average rate of change can be visualized as the **slope** of the line between the points on the graph of f from $t = a$ to $t = b$.
    - The **units** of average rate of change of a function are units of y per unit of t.
- **Concavity:** The graph of a function is **concave up** if it bends upward as we move left to right; the graph is **concave down** if it bends downward.
- Average velocity $= \dfrac{\text{Change in distance}}{\text{Change in time}} =$ Average rate of change of distance with respect to time.
- **Relative change**: When a quantity P changes from P_0 to P_1, we define
$$\text{Relative change in } P = \frac{\text{Change in } P}{P_0} = \frac{P_1 - P_0}{P_0}.$$
- **Elasticity** is equal to the ratio:
$$\left|\frac{\text{Relative change in quantity}}{\text{Relative change in price}}\right|$$

Problems for Section 1.3

In Problems 1–4, decide whether the graph is concave up, concave down, or neither.

1.



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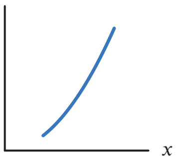

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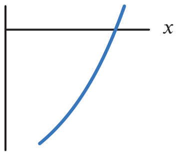
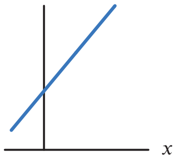


4.



5. Table 1.13 gives values of a function $w = f(t)$. Does this function appear to be increasing or decreasing? Does its graph appear to be concave up or concave down?

Table 1.13

t	0	4	8	12	16	20	24
w	100	58	32	24	20	18	17

6. Graph a function $f(x)$ which is increasing everywhere and concave up for negative x and concave down for positive x.

7. For which pairs of consecutive points in Figure 1.43 is the function graphed:

(a) Increasing and concave up?
(b) Increasing and concave down?
(c) Decreasing and concave up?
(d) Decreasing and concave down?

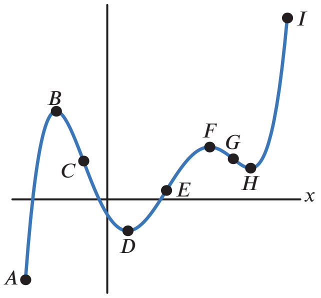


Figure 1.43

8. Find the average rate of change of $f(x) = 2x^2$ between $x = 1$ and $x = 3$.

9. Find the average rate of change of $f(x) = 3x^2 + 4$ between $x = -2$ and $x = 1$. Illustrate your answer graphically.

10. When a deposit of \$1000 is made into an account paying 2% interest, compounded annually, the balance, \$$B$, in the account after t years is given by $B = 1000(1.02)^t$. Find the average rate of change in the balance over the interval $t = 0$ to $t = 5$. Give units and interpret your answer in terms of the balance in the account.

■ In Problems 11–15, interpret the expression in terms of Arctic Sea ice extent, the area of sea covered by ice.[52] Let $E(x)$ and $F(t)$ be the Arctic Sea ice extent, both in millions of square kilometers, as a function of time, x, in years, since February 2000, and time, t, in days, since January 1, 2019.

11. $E(19) = 14.4$

12. $E(9) = 14.9$

13. $\dfrac{E(19) - E(9)}{19 - 9} = -0.04$

14. $F(31) = 14.11$

15. $\dfrac{F(59) - F(31)}{59 - 31} = 0.015$

16. Table 1.14 gives the net sales of The Gap, a clothing retailer.[53]

(a) Find the change in net sales between 2017 and 2019.
(b) Find the average rate of change in net sales between 2015 and 2019. Give units and interpret your answer.
(c) From 2015 to 2019, were there any one-year intervals during which the average rate of change was negative? If so, when?

Table 1.14 *Gap net sales, in billions of dollars*

Year	2015	2016	2017	2018	2019
Sales	15.80	15.52	15.86	16.58	16.38

17. Table 1.15 shows the number of bicyclists in the United States from 2006 to 2018, in millions.[54]

(a) Find the change in the number of bicyclists between 2006 and 2018. Give units.
(b) Find the average rate of change in the number of bicyclists between 2006 and 2018. Give units and interpret your answer in terms of the number of bicyclists.

Table 1.15 *Number of bicyclists in the United States, in millions*

Year	2006	2008	2010	2012	2014	2016	2018
Bicyclists	39.69	41.55	42.35	42.34	44.02	45.83	47.88

18. Table 1.16 shows total attendance at NFL football games.[55]

(a) Find the average rate of change in the attendance from 2015 to 2019. Give units.
(b) Find the annual change in the attendance for each year from 2015 to 2019. (Your answer should be four numbers.)
(c) Show that the average rate of change found in part (a) is the average of the four yearly changes found in part (b).

Table 1.16 *Attendance at NFL games, in millions*

Year	2015	2016	2017	2018	2019
Attendance	17.26	17.79	17.26	17.1	16.67

19. At time t in seconds, a particle's distance $s(t)$, in cm, from a point is given in the table. What is the average velocity of the particle from $t = 3$ to $t = 10$?

t	0	3	6	10	13
$s(t)$	0	72	92	144	180

20. Figure 1.44 shows a particle's distance from a point as a function of time, t. What is the particle's average velocity from $t = 0$ to $t = 3$?

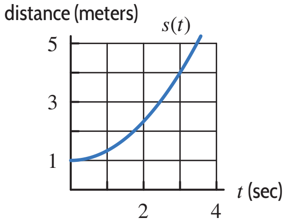


Figure 1.44

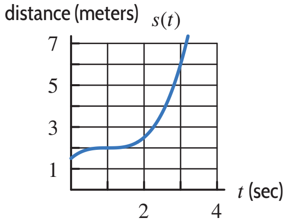


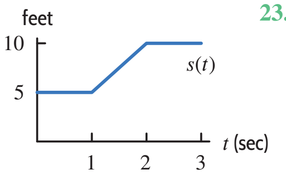
Figure 1.45

21. Figure 1.45 shows a particle's distance from a point as a function of time, t. What is the particle's average velocity from $t = 1$ to $t = 3$?

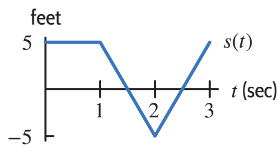
■ Problems 22–23 shows the distance, $s(t)$, in feet, of a scooter north of home at time t seconds. Find its average velocity between

(a) $t = 1$ and $t = 2$. **(b)** $t = 2$ and $t = 3$.

22.



23.



24. An observer tracks the distance a plane has rolled along the runway after touching down. Figure 1.46 shows this distance, x, in thousands of feet, as a function of time, t, in seconds since touchdown. Find the average velocity of the plane over the following time intervals:

(a) Between 0 and 20 seconds.
(b) Between 20 and 40 seconds.

[52] Sea ice extent definition and data values from nsidc.org/data/seaice_index/archives, accessed September 25, 2019.
[53] https://www.statista.com/statistics/242386/net-sales-of-the-gap-inc/, accessed March 11, 2021.
[54] www.statista.com/statistics/191204/participants-in-bicycling-in-the-us-since-2006, accessed January 28, 2021.
[55] http://www.statista.com/statistics/193420/regular-season-attendance-in-the-nfl-since-2006/, accessed March 3, 2020.

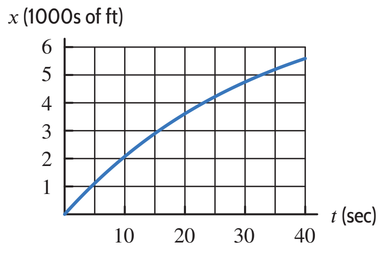


Figure 1.46

25. Figure 1.47 shows the total value of US imports, in trillions of dollars.[56]

(a) Was the value of the imports higher in 2005 or in 2020? Approximately how much higher?
(b) Estimate the average rate of change of US imports between 2005 and 2020. Give units and interpret your answer in terms of imports.

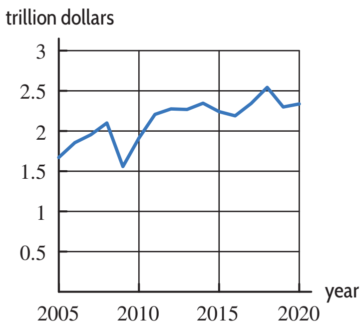


Figure 1.47

26. Figure 1.48 shows graphs of US advertising revenue (in billions of dollars) for print newspapers and for the internet[57] as a function of years since 1997.

(a) Which graph, A or B, is the graph of newspaper ad revenue? Explain how you know.
(b) Find the average rate of change of newspaper ad revenue and of internet ad revenue between 1997 and 2016. Include units and explain the signs.

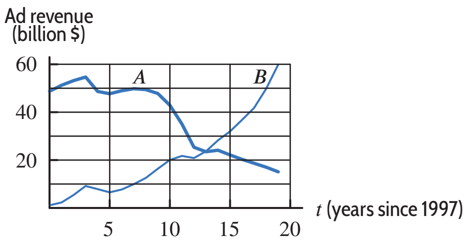


Figure 1.48

27. Starting from home, a car drives along a straight east/west highway. Table 1.17 gives the car's distance, d, east from home as a function of time, t.

(a) Find the average velocity of the car over the following intervals:
(i) $0 \le t \le 0.5$, (ii) $0.5 \le t \le 1$,
(iii) $3 \le t \le 3.5$, (iv) $3.5 \le t \le 4$.
(b) What is the average velocity of the car over the interval $0 \le t \le 5$? What does this tell you about the car's movement?

Table 1.17

t (hours)	0	0.5	1	2	3	3.5	4	5
d (miles)	0	10	40	70	65	25	0	0

28. Table 1.18 shows world population, P, in billions of people, world passenger automobile production, A, in millions of cars[58], and world cell phone subscribers, C, in billions of subscribers.[59]

(a) Find the average rate of change, with units, for each of P, A, and C between 2005 and 2015.
(b) Between 2005 and 2015, which increased faster:
(i) Population or the number of automobiles?
(ii) Population or the number of cell phone subscribers?

Table 1.18

Year	2005	2010	2015
P (billions)	6.51	6.92	7.34
A (millions)	59.7	73.3	90.8
C (billions)	2.71	5.27	7.15

29. Table 1.19 gives sales of Pepsico, which operates two major businesses: beverages (including Pepsi) and snack foods.[60]

(a) Find the change in sales between 2007 and 2018.
(b) Find the average rate of change in sales between 2007 and 2018. Give units and interpret your answer.

Table 1.19 *Pepsico sales, in billions of dollars*

Year	2007	2008	2009	2010	2011	2012
Sales	39.47	43.25	43.23	57.84	66.5	65.49
Year	2013	2014	2015	2016	2017	2018
Sales	66.42	66.68	63.06	62.8	63.53	64.66

[56] https://www.statista.com/statistics/218255/total-value-of-us-trade-in-goods-worldwide-since-2004/, accessed March 12, 2021.
[57] Based on 2017 research by David Flath using www.journalism.org/fact-sheet/newspapers, www.iab.net, and fred.stlouisfed.org/series/GDPDEF.
[58] *The World Almanac and Book of Facts, 2020,* p. 51.
[59] data.worldbank.org/indicator/IT.CEL.SETS.P2, accessed March 14, 2020.
[60] www.pepsico.com, accessed March 3, 2020.

30. Do you expect the average rate of change (in units per year) of each of the following to be positive or negative? Explain your reasoning.

(a) Number of acres of rain forest in the world.
(b) Population of the world.
(c) The number of pay phones in the US since 1995.
(d) Height of a sand dune that is being eroded.
(e) Cost of living in the US.

31. Table 1.20 shows the production of tobacco in the US.[61]

(a) What is the average rate of change in tobacco production between 2006 and 2019? Give units and interpret your answer in terms of tobacco production.
(b) During this thirteen-year period, are there any one-year intervals during which the average rate of change was positive? If so, when?

Table 1.20 *Tobacco production, in millions of pounds*

Year	2006	2007	2008	2009	2010	2011	2012
Production	728	788	801	823	718	598	763
Year	2013	2014	2015	2016	2017	2018	2019
Production	724	876	719	629	710	533	468

32. Figure 1.49 shows the length, L, in cm, of a sturgeon (a type of fish) as a function of the time, t, in years.[62]

(a) Is the function increasing or decreasing? Is the graph concave up or concave down?
(b) Estimate the average rate of growth of the sturgeon between $t = 5$ and $t = 15$. Give units and interpret your answer in terms of the sturgeon.

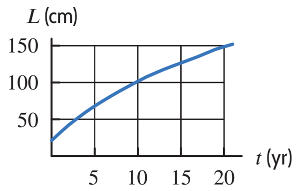


Figure 1.49

33. Table 1.21 shows the total US labor force, L. Find the average rate of change between 2000 and 2020; between 2000 and 2010; between 2010 and 2020. Give units and interpret your answers in terms of the labor force.[63]

Table 1.21 *US labor force, in millions of workers*

Year	2000	2005	2010	2015	2020
L	142.58	149.32	153.89	157.13	160.74

34. The total world marine catch[64] of fish, in metric tons, was 18.71 million in 1950 and 90.91 million in 2016. What was the average rate of change in the marine catch during this period? Give units and interpret your answer.

35. Table 1.22 gives the revenues, R, of General Motors.[65]

(a) Find the change in revenues between 2013 and 2019.
(b) Find the average rate of change in revenues between 2013 and 2019. Give units and interpret your answer.
(c) From 2013 to 2019, were there any one-year intervals during which the average rate of change was positive? If so, which?

Table 1.22 *GM revenues, billions of dollars*

Year	2013	2014	2015	2016	2017	2018	2019
R	138.79	137.96	135.73	149.18	145.59	147.05	137.2

36. The number of US households with cable television[66] was 104.1 million in in 2010 and 88.4 in 2019. Estimate the average rate of change in the number of US households with cable television during this 9-year period. Give units and interpret your answer.

37. Figure 1.50 shows the amount of nicotine $N = f(t)$, in mg, in a person's bloodstream as a function of the time, t, in hours, since the last cigarette.

(a) Is the average rate of change in nicotine level positive or negative? Explain.
(b) Find the average rate of change in the nicotine level between $t = 0$ and $t = 3$. Give units and interpret your answer in terms of nicotine.

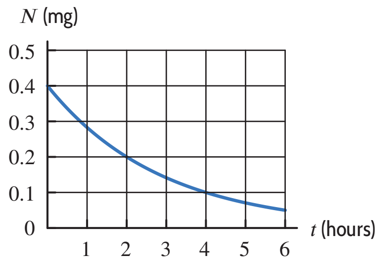


Figure 1.50

38. The population of the world[67] reached 1 billion in 1804, 2 billion in 1927, 3 billion in 1960, 4 billion in 1974, 5 billion in 1987, 6 billion in 1999, and 7 billion in 2012. Find the average rate of change of the population of the world, in people per minute, rounded to the nearest person, during each of these intervals (that is, from 1804 to 1927, 1927 to 1960, etc.).

[61] www.statista.com, accessed March 3, 2020.
[62] Data from L. von Bertalanffy, *General System Theory*, p. 177 (New York: Braziller, 1968).
[63] www.statista.com/statistics, accessed March 12, 2021.
[64] www.fao.org/state-of-fisheries-aquaculture/en/, accessed March 3, 2020.
[65] www.statista.com , accessed March 3, 2020.
[66] *The World Almanac and Book of Facts 2020*, p. 254 (New York).
[67] en.wikipedia.org/wiki/World_population_milestones, accessed March 31, 2021.

39. Table 1.23 shows the concentration, c, of creatinine in the bloodstream of a dog.[68]

(a) Including units, find the average rate at which the concentration is changing between the

(i) 6^{th} and 8^{th} minutes. (ii) 8^{th} and 10^{th} minutes.

(b) Explain the sign and relative magnitudes of your results in terms of creatinine.

Table 1.23

t (minutes)	2	4	6	8	10
c (mg/ml)	0.439	0.383	0.336	0.298	0.266

■ Problems 40–41 refer to Figure 1.51, which shows the contraction velocity of a muscle as a function of the load it pulls against.

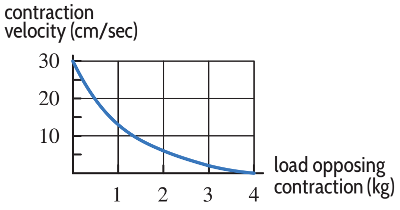


Figure 1.51

40. In terms of the muscle, interpret the

(a) Vertical intercept **(b)** Horizontal intercept

41. **(a)** With units, find the change in muscle contraction velocity when the load changes from 1 kg to 3 kg.

(b) Find the average rate of change in the contraction velocity between 1 kg and 3 kg. Give units.

42. Table 1.24 gives the sales, S, of Intel Corporation, a leading manufacturer of integrated circuits.[69]

(a) Find the change in sales between 2005 and 2020.

(b) Find the average rate of change in sales between 2005 and 2020. With units, interpret your answer.

Table 1.24 *Intel sales, in billions of dollars*

Year	2005	2010	2015	2020
S	38.8	43.6	55.36	77.87

43. Let $f(t)$ be the number of US billionaires in year t.

(a) Express the following statements[70] in terms of f.

(i) In 2001 there were 272 US billionaires.

(ii) In 2019 there were 607 US billionaires.

(b) Find the average yearly increase in the number of US billionaires from 2001 to 2019. Express this using f.

(c) Assuming the yearly increase remains constant, find a formula predicting the number of US billionaires in year t.

44. Figure 1.52 shows the position of an object at time t.

(a) Draw a line on the graph whose slope represents the average velocity between $t = 2$ and $t = 8$.

(b) Is average velocity greater between $t = 0$ and $t = 3$ or between $t = 3$ and $t = 6$?

(c) Is average velocity positive or negative between $t = 6$ and $t = 9$?

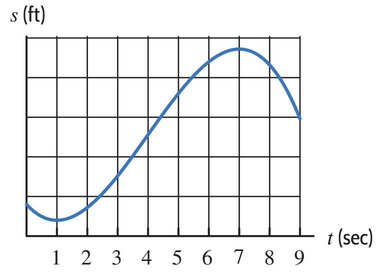


Figure 1.52

45. In an experiment, a lizard is encouraged to run as fast as possible. Figure 1.53 shows the distance run in meters as a function of the time in seconds.[71]

(a) If the lizard were running faster and faster, what would be the concavity of the graph? Does this match what you see?

(b) Estimate the average velocity of the lizard during this 0.8 second experiment.

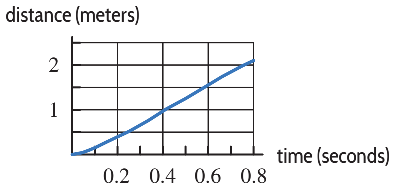


Figure 1.53

46. Values of $F(t)$, $G(t)$, and $H(t)$ are in Table 1.25. Which graph is concave up and which is concave down? Which function is linear?

Table 1.25

t	$F(t)$	$G(t)$	$H(t)$
10	15	15	15
20	22	18	17
30	28	21	20
40	33	24	24
50	37	27	29
60	40	30	35

[68] From M. R. Cullen, *Linear Models in Biology* (Chichester: Ellis Horwood, 1985).
[69] www.macrotrends.net/stocks/charts/INTC/intel/revenue, accessed January 28, 2021.
[70] www.forbes.com/billionaires, accessed August 15, 2019.
[71] Data from R. B. Huey and P. E. Hertz, "Effects of Body Size and Slope on the Acceleration of a Lizard," *J. Exp. Biol.*, Volume 110, 1984, p. 113-123.

47. Experiments suggest that the male maximum heart rate (the most times a male's heart can safely beat in a minute) decreases by 9 beats per minute during the first 21 years of his life, and by 26 beats per minute during the first 33 years.[72] If you model the maximum heart rate as a function of age, should you use a function that is increasing or decreasing? Concave up or concave down?

48. A car starts slowly and then speeds up. Eventually the car slows down and stops. Graph the distance that the car has traveled against time.

49. Between 1949 and 2017 the top speed of the fastest[73] car increased by approximately 2.25 mph per year. The fastest car in 2017 was the Koenigsegg Agera RS with a top speed of 277.9 mph.

(a) Predict the top speed of a car in 2016 and in 2020.
(b) The fastest car of 2020 was the SCC Tuatara at 316.1 mph.[74] What was the average rate of change in top speed between 2017 and 2020?
(c) Give two possible reasons that could explain the difference between your answer to part (b) and the 2.25 mph per year rate of change between 1949 and 2017.
(d) Using a rate of change of 2.25 mph per year, in what year is the top speed expected to reach 384 mph, half the speed of sound? What year is expected if the average rate of change from part (b) is used?

■ In Problems 50–53, find the relative, or percent, change.

50. B changes from 12,000 to 15,000

51. S changes from 400 to 450

52. W changes from 0.3 to 0.05

53. R changes from 50 to 47

■ In Problems 54–57, which relative change is bigger in magnitude? Justify your answer.

54. The change in the Dow Jones Industrial average from 164.6 to 77.9 in 1931; the change in the Dow Jones Industrial average from 13,261.8 to 8776.4 in 2008.

55. The change in the US population from 5.2 to 7.2 million from 1800 to 1810; the change in the US population from 151.3 to 179.3 million from 1950 to 1960.

56. An increase in class size from 5 to 10; an increase in class size from 30 to 50.

57. An increase in sales from \$100,000 to \$500,000; an increase in sales from \$20,000,000 to \$20,500,000.

58. Find the relative change of a population if it changes

(a) From 1000 to 2000 **(b)** From 2000 to 1000
(c) From 1,000,000 to 1,001,000

59. On Black Monday, October 28, 1929, the stock market on Wall Street crashed. The Dow Jones average dropped from 298.94 to 260.64 in one day. What was the relative change in the index?

60. On January 27, 2019, the cost to mail a letter in the US[75] was raised from 50 cents to 55 cents. Find the relative change in the cost.

61. The US Consumer Price Index (CPI) is a measure of the cost of living. The inflation rate is the average annual relative rate of change of the CPI. Use the data[76] in Table 1.26 to estimate the average inflation rate for each of the years 2014–2019.

Table 1.26

Year	2014	2015	2016	2017	2018	2019
CPI	236.736	237.017	240.007	245.120	251.107	255.657

62. During 2008 the US economy stopped growing and began to shrink. Table 1.27 gives quarterly data on the US Gross Domestic Product (GDP), which measures the size of the economy.[77]

(a) Estimate the relative growth rate (percent per year) at the first four times in the table.
(b) Economists often say an economy is in recession if the GDP decreases for two quarters in a row. Was the US in recession in 2008?

Table 1.27

t (years since 2008)	0	0.25	0.5	0.75	1.0
GDP (trillion dollars)	14.67	14.81	14.84	14.55	14.38

63. Let $P(t)$ be the total number of Covid-19 cases in Minnesota confirmed up to and including day t, where $t = 0$ is March 5, 2020.[78] Let $N(t)$ be the number of new cases confirmed on day t. What does each statement mean in the context of the outbreak?

(a) $P(37) - P(36) = 80$
(b) $(P(37) - P(36))/P(36) = 0.074 = 7.4\%$
(c) $(P(37) - P(36))/P(37) = 0.069 = 6.9\%$
(d) $P(35) + N(36) + N(37) = 1164$

[72] www.css.edu/users/tboone2/asep/May2002JEPonline.html, accessed January 4, 2005.
[73] These are the fastest *production* cars, meaning mass-produced, commercially available to the public, and able to be legally driven. Data from https://en.wikipedia.org/wiki/Production_car_speed_record, accessed on January 9, 2021.
[74] https://www.autoexpress.co.uk/best-cars/85932/what-fastest-car-world, accessed April 1, 2021.
[75] www.wikipedia.com, accessed March 3, 2020.
[76] www.usinflationcalculator.com, accessed March 3, 2020.
[77] www.bea.gov, accessed December 30, 2016.
[78] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed January 17, 2021.

64. Let $P(t)$ be the total number of Covid-19 cases in Maryland confirmed up to and including day t, where $t = 0$ is March 5, 2020.[79] What does each statement mean in the context of the outbreak? Which are average rates of change? Which are relative rates of change?

(a) $P(102) - P(95) = 4081$
(b) $(P(102) - P(95))/7 = 583$
(c) $(P(102) - P(95))/P(95) = 7.2\%$

65. Hydroelectric power is electric power generated by the force of moving water. Figure 1.54 shows[80] the annual percent growth in hydroelectric power consumption by the US industrial sector between 2008 and 2018.

(a) Find the largest time interval over which the percentage growth in the US consumption of hydroelectric power was an increasing function of time. Interpret what increasing means, practically speaking, in this case.
(b) Find the largest time interval over which the actual US consumption of hydroelectric power was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.

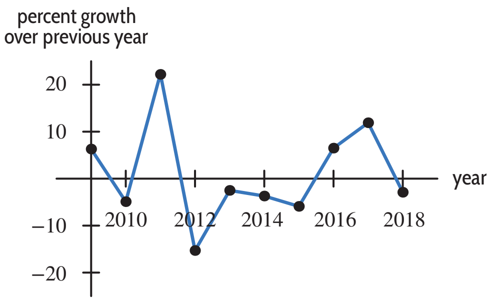


Figure 1.54

66. School organizations raise money by selling candy door-to-door. When the price is \$1 a school organization sells 2765 candies and when the price goes up to \$1.25 the quantity of candy sold drops down to 2440.

(a) Find the relative change in the price of candy.
(b) Find the relative change in the quantity of candy sold.
(c) Find and interpret the ratio

$$\left|\frac{\text{Relative change in quantity}}{\text{Relative change in price}}\right|.$$

67. An alternative to petroleum-based diesel fuel, biodiesel, is derived from renewable resources such as food crops, algae, and animal oils. The table shows the recent annual percent growth in US biodiesel exports.[81]

(a) Find the largest time interval over which the percentage growth in the US exports of biodiesel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.
(b) Find the largest time interval over which the actual US exports of biodiesel was an increasing function of time. Interpret what increasing means, practically speaking, in this case.

Year	2012	2013	2014	2015	2016	2017	2018
% growth over previous yr	69.9	53.0	−57.8	5.93	0.33	6.20	10.10

68. Solar panels are arrays of photovoltaic cells that convert solar radiation into electricity. The table shows the annual percent change in the US price per watt of a solar panel.[82]

(a) Find the largest time interval over which the percentage growth in the US price per watt of a solar panel was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.
(b) Find the largest time interval over which the actual price per watt of a solar panel was a decreasing function of time. Interpret what decreasing means, practically speaking, in this case.

Year	2005	2006	2007	2008	2009	2010
% growth over previous yr	6.7	9.7	−3.7	3.6	−20.1	−29.7

1.4 APPLICATIONS OF FUNCTIONS TO ECONOMICS

indexfunction!economic applications

In this section, we look at some of the functions of interest to decision-makers in a firm or industry.

The Cost Function

The **cost function**, $C(q)$, gives the total cost of producing a quantity q of some good.

[79] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed January 17, 2021.
[80] Yearly values have been joined with line segments to highlight trends in the data; however, values in between years should not be inferred from the segments. From www.eia.doe.gov, accessed August 19, 2019.
[81] www.eia.doe.gov, accessed August 19, 2019.
[82] We use the official price per peak watt, which uses the maximum number of watts a solar panel can produce under ideal conditions. From www.eia.doe.gov, accessed March 29, 2015.

What sort of function do you expect C to be? The more goods that are made, the higher the total cost, so C is an increasing function. Costs of production can be separated into two parts: the *fixed costs,* which are incurred even if nothing is produced, and the *variable costs*, which depend on how many units are produced.

An Example: Manufacturing Costs

Let's consider a company that makes phone cases. The factory and machinery needed to begin production are fixed costs, which are incurred even if no phone cases are made. The costs of labor and raw materials are variable costs since these quantities depend on how many phone cases are made. The fixed costs for this company are \$24,000 and the variable costs are \$7 per phone case. Then,

$$\begin{aligned}\text{Total costs for the company} &= \text{Fixed costs} + \text{Variable costs}\\ &= 24{,}000 + 7 \cdot \text{Number of phone cases,}\end{aligned}$$

so, if q is the number of phone cases produced,

$$C(q) = 24{,}000 + 7q.$$

This is the equation of a line with slope 7 and vertical intercept 24,000.

The variable cost for one additional unit is called the *marginal cost*. For a linear cost function, the marginal cost is the rate of change, or slope, of the cost function.

Example 1 Graph the cost function $C(q) = 24{,}000 + 7q$. Label the fixed costs and marginal cost.

Solution The graph of $C(q)$ is the line in Figure 1.55. The fixed costs are represented by the vertical intercept of 24,000. The marginal cost is represented by the slope of 7, which is the change in cost corresponding to a unit change in production.

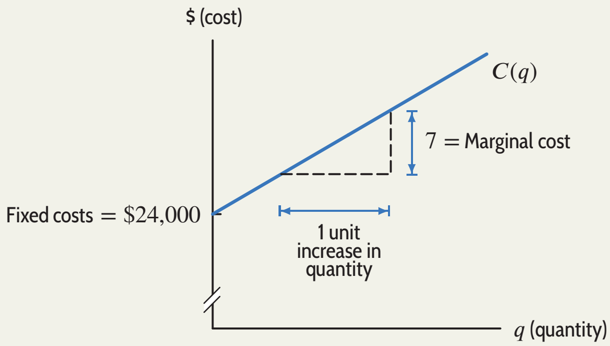


Figure 1.55: Cost function for the phone case manufacturer

If $C(q)$ is a linear cost function,

- Fixed costs are represented by the vertical intercept.
- Marginal cost is represented by the slope.

Example 2 In each case, draw a graph of a linear cost function satisfying the given conditions:

(a) Fixed costs are large but marginal cost is small.
(b) There are no fixed costs but marginal cost is high.

Solution

(a) The graph is a line with a large vertical intercept and a small slope. See Figure 1.56.

(b) The graph is a line with a vertical intercept of zero (so the line goes through the origin) and a large positive slope. See Figure 1.57. Figures 1.56 and 1.57 have the same scales.

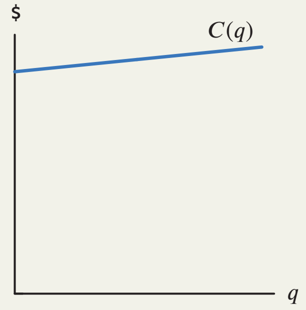


Figure 1.56: Large fixed costs, small marginal cost

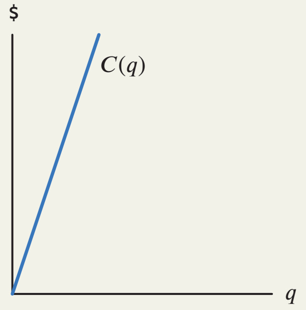


Figure 1.57: No fixed costs, high marginal cost

The Revenue Function

> The **revenue function**, $R(q)$, gives the total revenue received by a firm from selling a quantity, q, of some good.

If the good sells for a price of p per unit, and the quantity sold is q, then

$$\text{Revenue} = \text{Price} \cdot \text{Quantity}, \quad \text{so} \quad R = pq.$$

If the price does not depend on the quantity sold, so p is a constant, the graph of revenue as a function of q is a line through the origin, with slope equal to the price p.

Example 3 If phone cases sell for \$15 each, sketch the manufacturer's revenue function. Show the price of a phone case on the graph.

Solution Since $R(q) = pq = 15q$, the revenue graph is a line through the origin with a slope of 15. See Figure 1.58. The price is the slope of the line.

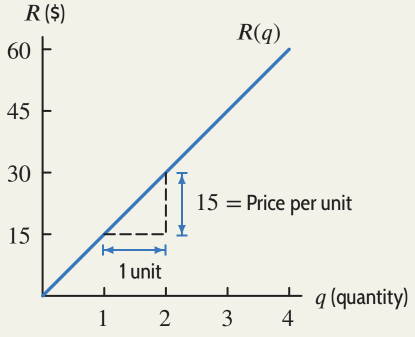


Figure 1.58: Revenue function for the phone case manufacturer

Example 4 Graph the cost function $C(q) = 24{,}000 + 7q$ and the revenue function $R(q) = 15q$ on the same axes. For what values of q does the company make money?

Solution The company makes money whenever revenues are greater than costs, so we find the values of q for which the graph of $R(q)$ lies above the graph of $C(q)$. See Figure 1.59.

We find the point at which the graphs of $R(q)$ and $C(q)$ cross:

$$\begin{aligned} \text{Revenue} &= \text{Cost} \\ 15q &= 24{,}000 + 7q \\ 8q &= 24{,}000 \\ q &= 3000. \end{aligned}$$

The company makes a profit if it produces and sells more than 3000 phone cases. The company loses money if it produces and sells fewer than 3000 phone cases.

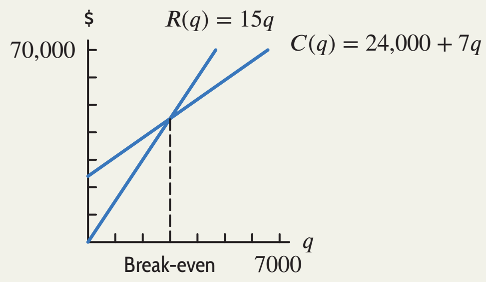


Figure 1.59: Cost and revenue functions for the phone case manufacturer: What values of q generate a profit?

The Profit Function

Decisions are often made by considering the profit, usually written[83] as π to distinguish it from the price, p. We have

$$\boxed{\text{Profit} = \text{Revenue} - \text{Cost} \quad \text{so} \quad \pi = R - C.}$$

The *break-even point* for a company is the point where the profit is zero and revenue equals cost. See Figure 1.59.

Example 5 Find a formula for the profit function of the phone case manufacturer. Graph it, marking the break-even point.

Solution Since $R(q) = 15q$ and $C(q) = 24{,}000 + 7q$, we have

$$\pi(q) = R(q) - C(q) = 15q - (24{,}000 + 7q) = -24{,}000 + 8q.$$

Notice that the negative of the fixed costs is the vertical intercept and the break-even point is the horizontal intercept. See Figure 1.60.

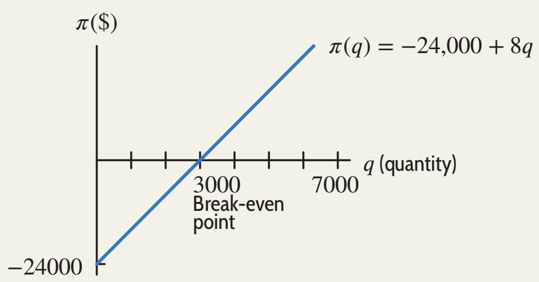


Figure 1.60: Profit for phone case manufacturer

[83]This π has nothing to do with the area of a circle, and merely stands for the Greek equivalent of the letter "p."

Example 6 (a) Using Table 1.28, estimate the break-even point for this company.
(b) Find the company's profit if 1000 units are produced.
(c) What price do you think the company is charging for its product?

Table 1.28 *Company's estimates of cost and revenue for a product*

q	500	600	700	800	900	1000	1100
$C(q)$, in \$	5000	5500	6000	6500	7000	7500	8000
$R(q)$, in \$	4000	4800	5600	6400	7200	8000	8800

Solution (a) The break-even point is the value of q for which revenue equals cost. Since revenue is below cost at $q = 800$ and revenue is greater than cost at $q = 900$, the break-even point is between 800 and 900. The values in the table suggest that the break-even point is closer to 800, as the cost and revenue are closer there. A reasonable estimate for the break-even point is $q = 830$.
(b) If the company produces 1000 units, the cost is \$7500 and the revenue is \$8000, so the profit is $8000 - 7500 = 500$ dollars.
(c) From the table, it appears that $R(q) = 8q$. This indicates the company is selling the product for \$8 each.

The Marginal Cost, Marginal Revenue, and Marginal Profit

Just as we used the term marginal cost to mean the rate of change, or slope, of a linear cost function, we use the terms *marginal revenue* and *marginal profit* to mean the rate of change, or slope, of linear revenue and profit functions, respectively. The term *marginal* is used because we are looking at how the cost, revenue, or profit change "at the margin," that is, by the addition of one more unit. For example, for the phone case manufacturer, the marginal cost is 7 dollars/item (the additional cost of producing one more item is \$7), the marginal revenue is 15 dollars/item (the additional revenue from selling one more item is \$15), and the marginal profit is 8 dollars/item (the additional profit from selling one more item is \$8).

The Depreciation Function

Suppose that the phone case manufacturer has a machine that costs \$20,000 and is sold ten years later for \$3000. We say the value of the machine *depreciates* from \$20,000 today to a resale value of \$3000 in ten years. The depreciation formula gives the value, $V(t)$, in dollars, of the machine as a function of the number of years, t, since the machine was purchased. We assume that the value of the machine depreciates linearly.

The value of the machine when it is new ($t = 0$) is \$20,000, so $V(0) = 20{,}000$. The resale value at time $t = 10$ is \$3000, so $V(10) = 3000$. We have

$$\text{Slope} = m = \frac{3000 - 20{,}000}{10 - 0} = \frac{-17{,}000}{10} = -1700 \text{ dollars per year.}$$

This slope tells us that the value of the machine is decreasing at a rate of \$1700 per year. Since $V(0) = 20{,}000$, the vertical intercept is 20,000, so

$$V(t) = 20{,}000 - 1700t \text{ dollars.}$$

Supply and Demand Curves

The quantity, q, of an item that is manufactured and sold depends on its price, p. As the price increases, manufacturers are usually willing to supply more of the product, whereas the quantity demanded by consumers falls.

The **supply curve**, for a given item, relates the quantity, q, of the item that manufacturers are willing to make per unit time to the price, p, for which the item can be sold.
The **demand curve** relates the quantity, q, of an item demanded by consumers per unit time to the price, p, of the item.

Economists often think of the quantities supplied and demanded as functions of price. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. (The reason for this state of affairs is that economists originally took price to be the dependent variable and put it on the vertical axis. Later, when the point of view changed, the axes did not.) Thus, typical supply and demand curves look like those shown in Figure 1.61.

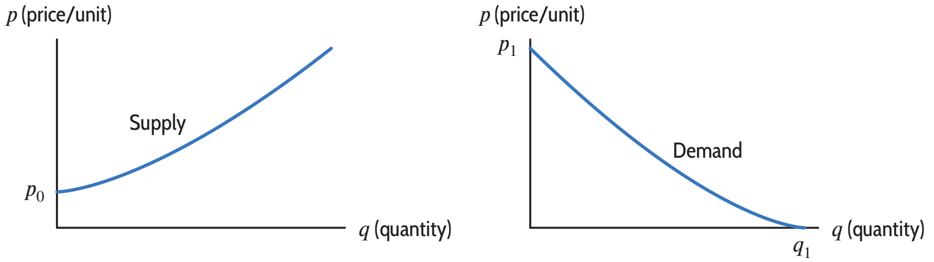


Figure 1.61: Supply and demand curves

Example 7 What is the economic meaning of the prices p_0 and p_1 and the quantity q_1 in Figure 1.61?

Solution The vertical axis corresponds to a quantity of zero. Since the price p_0 is the vertical intercept on the supply curve, p_0 is the price at which the quantity supplied is zero. In other words, for prices below p_0, the suppliers will not produce anything. The price p_1 is the vertical intercept on the demand curve, so it corresponds to the price at which the quantity demanded is zero. In other words, for prices above p_1, consumers buy none of the product.

The horizontal axis corresponds to a price of zero, so the quantity q_1 on the demand curve is the quantity demanded if the price were zero—the quantity that could be given away free.

Equilibrium Price and Quantity

If we plot the supply and demand curves on the same axes, as in Figure 1.62, the graphs cross at the *equilibrium point*. The values p^* and q^* at this point are called the *equilibrium price* and *equilibrium quantity*, respectively. It is assumed that the market naturally settles to this equilibrium point. (See Problem 29.)

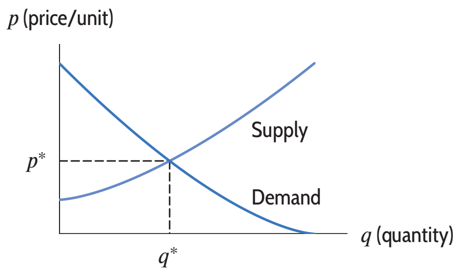


Figure 1.62: The equilibrium price and quantity

Example 8 Find the equilibrium price and quantity if

$$\text{Quantity supplied } = 3p - 50 \quad \text{and} \quad \text{Quantity demanded } = 100 - 2p.$$

Solution To find the equilibrium price and quantity, we find the point at which

$$\begin{aligned} \text{Supply } &= \text{Demand} \\ 3p - 50 &= 100 - 2p \\ 5p &= 150 \\ p &= 30. \end{aligned}$$

The equilibrium price is \$30. To find the equilibrium quantity, we use either the demand curve or the supply curve. At a price of \$30, the quantity produced is $100 - 2 \cdot 30 = 40$ items. The equilibrium quantity is 40 items. In Figure 1.63, the demand and supply curves intersect at $p^* = 30$ and $q^* = 40$.

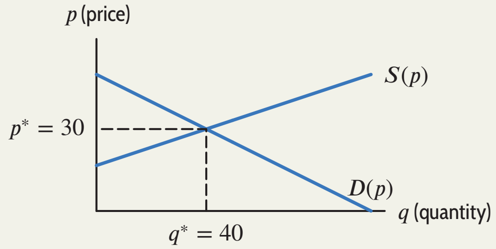


Figure 1.63: Equilibrium: $p^* = 30$, $q^* = 40$

The Effect of Taxes on Equilibrium

What effect do taxes have on the equilibrium price and quantity for a product? We distinguish between two types of taxes.[84] A *specific tax* is a fixed amount per unit of a product sold regardless of the selling price. This is the case with such items as gasoline, alcohol, and cigarettes. A specific tax is usually imposed on the producer. A *sales tax* is a fixed percentage of the selling price. Many cities and states collect sales tax on a wide variety of items. A sales tax is usually imposed on the consumer. We consider a specific tax now; a sales tax is considered in Problems 46 and 47.

Example 9 A specific tax of \$5 per unit is now imposed upon suppliers in Example 8. What are the new equilibrium price and quantity?

Solution The consumers pay p dollars per unit, but the suppliers receive only $p - 5$ dollars per unit because \$5 goes to the government as taxes. Since

$$\text{Quantity supplied } = 3(\text{Amount per unit received by suppliers}) - 50,$$

the new supply equation is

$$\text{Quantity supplied } = 3(p - 5) - 50 = 3p - 65;$$

the demand equation is unchanged:

$$\text{Quantity demanded } = 100 - 2p.$$

[84] Adapted from Barry Bressler, *A Unified Approach to Mathematical Economics*, p. 81–88 (New York: Harper & Row, 1975).

At the equilibrium price, we have

$$\begin{aligned} \text{Demand} &= \text{Supply} \\ 100 - 2p &= 3p - 65 \\ 165 &= 5p \\ p &= 33. \end{aligned}$$

The equilibrium price is \$33. The equilibrium quantity is 34 units, since the quantity demanded is $q = 100 - 2 \cdot 33 = 34$.

In Example 8, the equilibrium price was \$30; with the imposition of a \$5 tax in Example 9, the equilibrium price is \$33. Thus the equilibrium price increases by \$3 as a result of the tax. Notice that this is less than the amount of the tax. The consumer ends up paying \$3 more than if the tax did not exist. However the government receives \$5 per item. The producer pays the other \$2 of the tax, retaining \$28 of the price paid per item. Although the tax was imposed on the producer, some of the tax is passed on to the consumer in terms of higher prices. The tax has increased the price and reduced the number of items sold. See Figure 1.64. Notice that the taxes have the effect of moving the supply curve up by \$5 because suppliers have to be paid \$5 more to produce the same quantity.

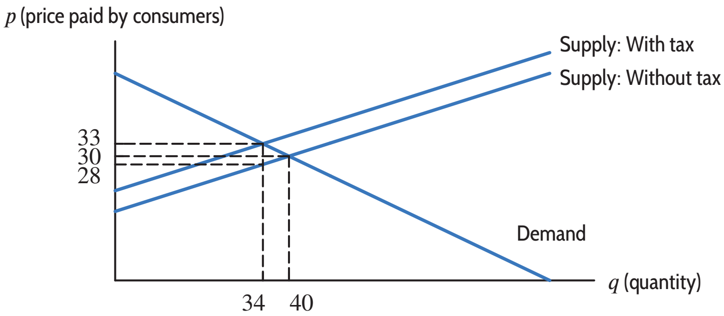


Figure 1.64: Specific tax shifts the supply curve, altering the equilibrium price and quantity

A Budget Constraint

An ongoing debate in the federal government concerns the allocation of money between defense and social programs. In general, the more that is spent on defense, the less that is available for social programs, and vice versa. Let's simplify the example to guns and butter. Assuming a constant budget, we show that the relationship between the number of guns and the quantity of butter is linear. Suppose that there is \$12,000 to be spent and that it is to be divided between guns, costing \$400 each, and butter, costing \$2000 a ton. Suppose the number of guns bought is g, and the number of tons of butter is b. Then the amount of money spent on guns is \$400g, and the amount spent on butter is \$2000b. Assuming all the money is spent,

$$\text{Amount spent on guns} + \text{Amount spent on butter} = \$12{,}000$$

or

$$400g + 2000b = 12{,}000.$$

Thus, dividing both sides by 400, we have

$$g + 5b = 30.$$

This equation is the budget constraint. Since the budget constraint can be written as

$$g = 30 - 5b,$$

the graph of the budget constraint is a line. See Figure 1.65.

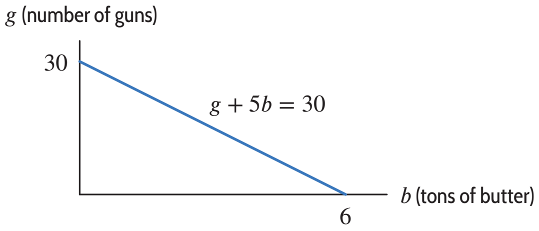


Figure 1.65: Budget constraint

Summary for Section 1.4

- The **cost function**, $C(q)$, gives the total cost of producing a quantity q of some good.
- The variable cost for one additional unit is called the **marginal cost**.
- If $C(q)$ is a linear cost function,
    - **Fixed costs** are represented by the vertical intercept.
    - **Marginal cost** is represented by the slope.
- The **revenue function**, $R(q)$, gives the total revenue received by a firm from selling a quantity, q, of some good.
    - **Marginal revenue** is the rate of change of the revenue function.
- The **break-even point** is the value of q for which revenue equals cost.
- Profit = Revenue − Cost
    - **Marginal profit** is the rate of change of the profit function.
- The **supply curve**, for a given item, relates the quantity, q, of the item that manufacturers are willing to make per unit time to the price, p, for which the item can be sold.
- The **demand curve** relates the quantity, q, of an item demanded by consumers per unit time to the price, p, of the item.
- The graphs of supply and demand cross at the **equilibrium point**, (p^*, q^*). The values p^* and q^* are called the equilibrium price and equilibrium quantity, respectively.
- **Taxes:** A **specific tax** is a fixed amount per unit of a product sold regardless of the selling price. A **sales tax** is a fixed percentage of the selling price.

Problems for Section 1.4

1. Figure 1.66 shows cost and revenue for a company.

(a) Approximately what quantity does this company have to produce to make a profit?
(b) Estimate the profit generated by 600 units.

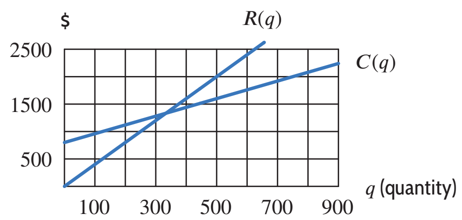


Figure 1.66

2. **(a)** Estimate the fixed costs and the marginal cost for the cost function in Figure 1.67.
(b) Estimate $C(10)$ and interpret it in terms of cost.

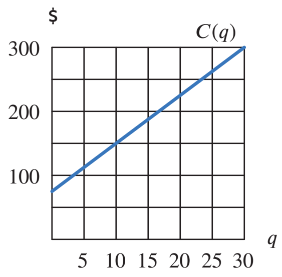


Figure 1.67

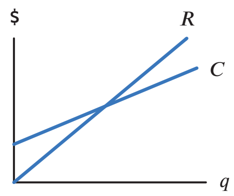


Figure 1.68

3. In Figure 1.68, which shows the cost and revenue functions for a product, label each of the following:

(a) Fixed costs **(b)** Break-even quantity

(c) Quantities at which the company:
(i) Makes a profit (ii) Loses money

4. Values of a linear cost function are in Table 1.29. What are the fixed costs and the marginal cost? Find a formula for the cost function.

Table 1.29

q	0	5	10	15	20
$C(q)$	5000	5020	5040	5060	5080

5. The cost C, in millions of dollars, of producing q items is given by $C = 5.7 + 0.002q$. Interpret the 5.7 and the 0.002 in terms of production. Give units.

6. **(a)** Give an example of a possible company where the fixed costs are zero (or very small).
(b) Give an example of a possible company where the marginal cost is zero (or very small).

7. Suppose that $q = f(p)$ is the demand curve for a product, where p is the selling price in dollars and q is the quantity sold at that price.

(a) What does the statement $f(12) = 60$ tell you about demand for this product?
(b) Do you expect this function to be increasing or decreasing? Why?

8. A company has cost and revenue functions, in dollars, given by $C(q) = 6000 + 10q$ and $R(q) = 12q$.

(a) Find the cost and revenue if the company produces 500 units. Does the company make a profit? What about 5000 units?
(b) Find the break-even point and illustrate it graphically.

9. The demand curve for a quantity q of a product is $q = 5500 - 100p$ where p is price in dollars. Interpret the 5500 and the 100 in terms of demand. Give units.

10. A demand curve is given by $75p + 50q = 300$, where p is the price of the product, in dollars, and q is the quantity demanded at that price. Find the p- and q-intercepts and interpret them in terms of consumer demand.

■ In Problems 11–14, give the cost, revenue, and profit functions.

11. An online seller of T-shirts pays \$500 to start up the website and \$6 per T-shirt, then sells the T-shirts for \$12 each.

12. A car wash operator pays \$35,000 for a franchise, then spends \$10 per car wash, which costs the consumer \$15.

13. A couple running a house-cleaning business invests \$5000 in equipment, and they spend \$15 in supplies to clean a house, for which they charge \$60.

14. A lemonade stand operator sets up a stand for free in front of a neighbor's house, makes 5 quarts of lemonade for \$4, then sells each 8-oz cup for 25 cents.

15. A company that makes patio chairs has fixed costs of \$5000 and variable costs of \$30 per chair. The company sells the chairs for \$50 each.

(a) Find formulas for the cost and revenue functions.
(b) Find the marginal cost and marginal revenue.
(c) Graph the cost and the revenue functions on the same axes.
(d) Find the break-even point.

16. Wildlife World Zoo in Phoenix, Arizona, charges an admission fee of \$23 per child visitor as well as an additional \$6.00 for each ride.[85]

(a) For one visitor, find the park's total revenue $R(n)$ as a function of the number of rides, n, taken.
(b) Find $R(2)$ and $R(8)$ and interpret your answers in terms of amusement park fees.

17. A company sells small, 20-page, hardcover photo books for \$30 plus \$1.50 per additional page or large, 20-page, softcover books for \$25 plus \$2 per additional page.

(a) For each format, find the price of a photo book as a function of the number of additional pages needed.
(b) Which format is cheaper for 20 additional pages?
(c) For what number of additional pages is the price of each format the same?

18. A company has cost function $C(q) = 4000 + 2q$ dollars and revenue function $R(q) = 10q$ dollars.

(a) What are the fixed costs for the company?
(b) What is the marginal cost?
(c) What price is the company charging for its product?
(d) Graph $C(q)$ and $R(q)$ on the same axes and label the break-even point, q_0. Explain how you know the company makes a profit if the quantity produced is greater than q_0.
(e) Find the break-even point q_0.

19. A movie theater has fixed costs of \$5000 per day and variable costs averaging \$11 per customer. The theater takes in an average revenue of \$16 per customer.

(a) How many customers per day does the theater need in order to make a profit?
(b) Find the cost and revenue functions and graph them on the same axes. Mark the break-even point.

20. A company producing jigsaw puzzles has fixed costs of \$6000 and variable costs of \$2 per puzzle. The company sells the puzzles for \$5 each.

(a) Find formulas for the cost function, the revenue function, and the profit function.
(b) Sketch a graph of $R(q)$ and $C(q)$ on the same axes. What is the break-even point, q_0, for the company?

[85] https://wildlifeworld.com/, accessed March 11, 2021.

21. Figure 1.69 shows revenue, R, as a function of quantity, q, of an item produced. Match each graph (I)–(IV) with one of the statements (a)–(d).

(a) Constant price
(b) Price increasing for small quantities and decreasing for large quantities
(c) Price increasing with quantity
(d) As the quantity increases, the price decreases

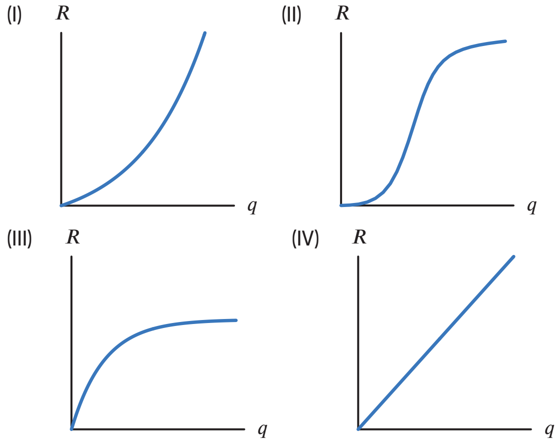


Figure 1.69

22. Production costs for manufacturing running shoes consist of a fixed overhead of $650,000 plus variable costs of $20 per pair of shoes. Each pair of shoes sells for $70.

(a) Find the total cost, $C(q)$, the total revenue, $R(q)$, and the total profit, $\pi(q)$, as a function of the number of pairs of shoes produced, q.
(b) Find the marginal cost, marginal revenue, and marginal profit.
(c) How many pairs of shoes must be produced and sold for the company to make a profit?

23. A $15,000 robot depreciates linearly to zero in 10 years.

(a) Find a formula for its value as a function of time.
(b) How much is the robot worth three years after it is purchased?

24. A $50,000 tractor has a resale value of $10,000 twenty years after it was purchased. Assume that the value of the tractor depreciates linearly from the time of purchase.

(a) Find a formula for the value of the tractor as a function of the time since it was purchased.
(b) Graph the value of the tractor against time.
(c) Find the horizontal and vertical intercepts, give units, and interpret them.

25. A new bus worth $100,000 in 2015 depreciates linearly to $25,000 in 2030.

(a) Find a formula for the value of the bus, V, as a function of time, t, in years since 2015.
(b) What is the value of the bus in 2020?
(c) Find and interpret the vertical and horizontal intercepts of the graph of the function.
(d) Assuming depreciation continues at the same linear rate, what is the domain of the function?

26. A corporate office provides the demand curve in Figure 1.70 to its ice cream shop franchises. At a price of $5.00 per scoop, 240 scoops per day can be sold.

(a) Estimate how many scoops could be sold per day at a price of $2.50 per scoop. Explain.
(b) Estimate how many scoops per day could be sold at a price of $7.50 per scoop. Explain.

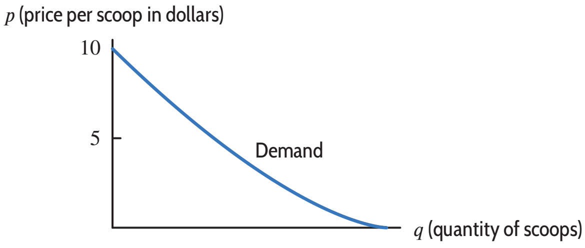


Figure 1.70

27. The table shows the cost of manufacturing various quantities of an item and the revenue obtained from their sale.

Quantity	0	10	20	30	40	50	60	70	80
Cost ($)	120	400	600	780	1000	1320	1800	2500	3400
Revenue ($)	0	300	600	900	1200	1500	1800	2100	2400

(a) What range of production levels appears to be profitable?
(b) Calculate the profit or loss for each of the quantities shown. Estimate the most profitable production level.

28. One of Tables 1.30 and 1.31 represents a supply curve; the other represents a demand curve.

(a) Which table represents which curve? Why?
(b) At a price of $155, approximately how many items would consumers purchase?
(c) At a price of $155, approximately how many items would manufacturers supply?
(d) Will the market push prices higher or lower than $155?
(e) What would the price have to be if you wanted consumers to buy at least 20 items?
(f) What would the price have to be if you wanted manufacturers to supply at least 20 items?

Table 1.30

p ($/unit)	182	167	153	143	133	125	118
q (quantity)	5	10	15	20	25	30	35

Table 1.31

p ($/unit)	6	35	66	110	166	235	316
q (quantity)	5	10	15	20	25	30	35

29. Figure 1.71 shows supply and demand for a product.

(a) What is the equilibrium price for this product? At this price, what quantity is produced?

(b) Choose a price above the equilibrium price—for example, $p = 12$. At this price, how many items are suppliers willing to produce? How many items do consumers want to buy? Use your answers to these questions to explain why, if prices are above the equilibrium price, the market tends to push prices lower (toward the equilibrium).

(c) Now choose a price below the equilibrium price—for example, $p = 8$. At this price, how many items are suppliers willing to produce? How many items do consumers want to buy? Use your answers to these questions to explain why, if prices are below the equilibrium price, the market tends to push prices higher (toward the equilibrium).

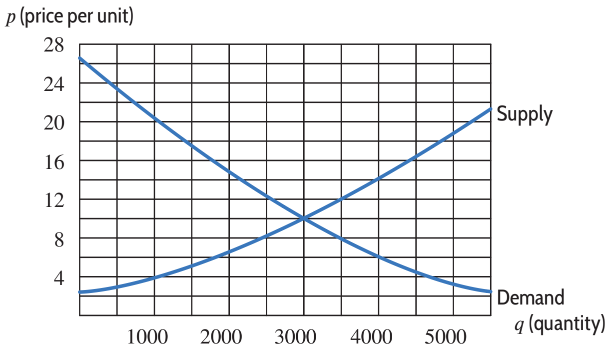


Figure 1.71

30. A company produces and sells shirts. The fixed costs are \$7000 and the variable costs are \$10 per shirt.

(a) Shirts are sold for \$25 each. Find cost and revenue as functions of the quantity of shirts, q.

(b) The company is considering changing the selling price of the shirts. Demand is $q = 4000 - 40p$, where p is price in dollars and q is the number of shirts. What quantity is sold at the current price of \$25? What profit is realized at this price?

(c) Use the demand equation to write cost and revenue as functions of the price, p. Then write profit as a function of price.

(d) Graph profit against price. Find the price that maximizes profits. What is this profit?

31. When the price, p, charged for a boat tour was \$25, the average number of passengers per week, N, was 500. When the price was reduced to \$20, the average number of passengers per week increased to 650. Find a formula for the demand curve, assuming that it is linear.

32. Table 1.32 gives data for the linear demand curve for a product, where p is the price of the product and q is the quantity sold every month at that price. Find formulas for the following functions. Interpret their slopes in terms of demand.

(a) q as a function of p. **(b)** p as a function of q.

Table 1.32

p (dollars)	16	18	20	22	24
q (tons)	500	460	420	380	340

33. The demand curve for a product is given by $q = 120{,}000 - 500p$ and the supply curve is given by $q = 1000p$ for $0 \leq q \leq 120{,}000$, where price, p, is in dollars.

(a) At a price of \$100, what quantity are consumers willing to buy and what quantity are producers willing to supply? Will the market push prices up or down?

(b) Find the equilibrium price and quantity. Does your answer to part (a) support the observation that market forces tend to push prices closer to the equilibrium price?

34. World consumption of silver, Q, in millions of ounces, to make jewelry and the price of silver, P, in dollars per ounce are given[86] in Table 1.33. Plot the value as a function of consumption. Sketch a possible demand curve.

Table 1.33

Year	2011	2012	2013	2014	2015
Q	162.2	159.2	187.1	193.5	202.6
P	35.12	31.15	23.79	19.08	15.68
Year	2016	2017	2018	2019	2020
Q	189.2	196.3	203.1	201.3	187.5
P	17.14	17.05	15.71	16.21	15.70

35. World supply of recycled silver, Q, in millions of ounces and the price of silver, P, in dollars per ounce, each year are given[87] in Table 1.34. Plot the value as a function of production. Sketch a possible supply curve.

Table 1.34

Year	2011	2012	2013	2014	2015
Q	232.9	216.0	192.7	174.9	166.5
P	35.12	31.15	23.79	19.08	15.68
Year	2016	2017	2018	2019	2020
Q	164.4	167.7	167.7	169.9	169.4
P	17.14	17.05	15.71	16.21	15.70

[86] www.silverinstitute.org/wp-content/uploads/2020/04/World-Silver-Survey-2020.pdf, accessed Februrary 3, 2021.
[87] www.silverinstitute.org/wp-content/uploads/2020/04/World-Silver-Survey-2020.pdf, accessed Februrary 3, 2021.

36. A car rental company has an annual budget of \$720,000 to spend on car replacements. Compact cars cost the company \$20,000 each to replace and SUVs cost \$30,000 each.

(a) What is the company's budget constraint equation? Let c be the number of compact cars and s the number of SUVs replaced.
(b) Find and interpret both intercepts of the graph of the equation.

37. A company has a total budget of \$500,000 and spends this budget on raw materials and personnel. The company uses m units of raw materials, at a cost of \$100 per unit, and hires r employees, at a cost of \$25,000 each.

(a) What is the equation of the company's budget constraint?
(b) Solve for m as a function of r.
(c) Solve for r as a function of m.

38. You have a budget of \$2000 for the year to cover your books and social outings. Books cost (on average) \$80 each and social outings cost (on average) \$20 each. Let b denote the number of books purchased per year and s denote the number of social outings in a year.

(a) What is the equation of your budget constraint?
(b) Graph the budget constraint. (It does not matter which variable you put on which axis.)
(c) Find the vertical and horizontal intercepts, and give a financial interpretation for each.

39. A bakery owner knows that customers buy a total of q cakes when the price, p, is no more than $p = d(q) = 20 - q/20$ dollars. She is willing to make and supply as many as q cakes at a price of $p = s(q) = 11 + q/40$ dollars each. The graphs of $d(q)$ and $s(q)$ are in Figure 1.72.

(a) Why, in terms of the context, is the slope of $d(q)$ negative and the slope of $s(q)$ positive?
(b) Is each of the ordered pairs (q, p) a solution to the inequality $p \leq 20 - q/20$? Interpret your answers in terms of the context.

$$(60, 18) \qquad (120, 12)$$

(c) Graph in the qp-plane the solution set of the system of inequalities $p \leq 20 - q/20$, $p \geq 11 + q/40$. What does this solution set represent in terms of the context?
(d) What is the rightmost point of the solution set you graphed in part (c)? Interpret your answer in terms of the context.

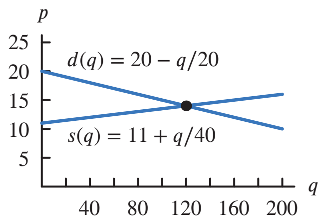


Figure 1.72

40. Linear supply and demand curves are shown in Figure 1.73, with price on the vertical axis.

(a) Label the equilibrium price p_0 and the equilibrium quantity q_0 on the axes.
(b) Explain the effect on equilibrium price and quantity if the slope, $\Delta p/\Delta q$, of the supply curve increases. Illustrate your answer graphically.
(c) Explain the effect on equilibrium price and quantity if the slope, $\Delta p/\Delta q$, of the demand curve becomes more negative. Illustrate your answer graphically.

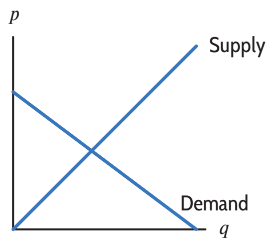


Figure 1.73

41. The demand for a product is given by $p = 90 - 10q$. Find the ratio

$$\left|\frac{\text{Relative change in demand}}{\text{Relative change in price}}\right|$$

if the price changes from $p = 50$ to $p = 51$. Interpret this ratio.

42. A demand curve has equation $q = 100 - 5p$, where p is price in dollars. A \$2 tax is imposed on consumers. Find the equation of the new demand curve. Sketch both curves.

43. A supply curve has equation $q = 4p - 20$, where p is price in dollars. A \$2 tax is imposed on suppliers. Find the equation of the new supply curve. Sketch both curves.

44. A tax of \$8 per unit is imposed on the supplier of an item. The original supply curve is $q = 0.5p - 25$ and the demand curve is $q = 165 - 0.5p$, where p is price in dollars. Find the equilibrium price and quantity before and after the tax is imposed.

45. The demand and supply curves for a product are given in terms of price, p, by

$$q = 2500 - 20p \quad \text{and} \quad q = 10p - 500.$$

(a) Find the equilibrium price and quantity. Represent your answers on a graph.
(b) A specific tax of \$6 per unit is imposed on suppliers. Find the new equilibrium price and quantity. Represent your answers on the graph.
(c) How much of the \$6 tax is paid by consumers and how much by producers?
(d) What is the total tax revenue received by the government?

46. The demand and supply curves are given by $q = 100 - 2p$ and $q = 3p - 50$, respectively; the equilibrium price is \$30 and the equilibrium quantity is 40 units. A sales tax of 5% is imposed on the consumer.

(a) Find the equation of the new demand and supply curves.
(b) Find the new equilibrium price and quantity.
(c) How much is paid in taxes on each unit? How much of this is paid by the consumer and how much by the producer?
(d) How much tax does the government collect?

47. Answer the questions in Problem 46, assuming that the 5% sales tax is imposed on the supplier instead of the consumer.

1.5 EXPONENTIAL FUNCTIONS

The function $f(x) = 2^x$, where the power is variable, is an *exponential function*. The number 2 is called the base. Exponential functions of the form $f(x) = k \cdot a^x$, where a is a positive constant, are used to represent many phenomena in the natural and social sciences.

Spread of Disease

In fall 2020, Arizona had a second wave of Covid-19 cases. Table 1.35 shows the total number of Covid-19 cases confirmed in the state counting from the start of September.[88] To see how the number of cases is growing, we look at the third column. If the number of cases had been growing linearly, all the numbers in the third column would be the same.

Table 1.35 *Arizona second-wave Covid cases, fall 2020*

Date	Cases	Change in cases
Oct 4	18,643	
		503
Oct 5	19,151	
		508
Oct 6	19,687	
		536
Oct 7	20,263	
		576
Oct 8	20,861	
		598
Oct 9	21,479	
		618
Oct 10	22,133	

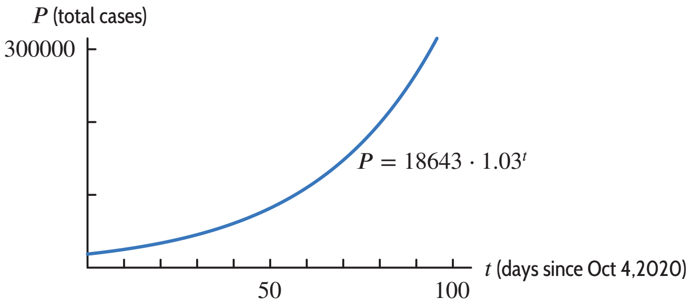


Figure 1.74: Covid cases in Arizona: Exponential growth

Suppose we divide each day's case total by the previous day's cases. For example,

$$\frac{\text{Covid cases confirmed by Oct 5}}{\text{Covid cases confirmed by Oct 4}} = \frac{19{,}151}{18{,}643} = 1.03$$

$$\frac{\text{Covid cases confirmed by Oct 6}}{\text{Covid cases confirmed by Oct 5}} = \frac{19{,}687}{19{,}151} = 1.03$$

$$\frac{\text{Covid cases confirmed by Oct 7}}{\text{Covid cases confirmed by Oct 6}} = \frac{20{,}263}{19{,}687} = 1.03$$

The fact that all three calculations give 1.03 shows the total number of cases grew by about 3% between October 4 and 5, *and* between October 5 and 6, *and* between October 6 and 7. Similar calculations for other dates show that the total Covid cases grew by a factor of approximately 1.03,

[88] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed December 22, 2020.

or 3%, every day for some time. Whenever we have a constant daily percent increase (here 3%), we have *exponential growth*. If t is the number of days since October 4, then

$$\begin{aligned}
&\text{When } t = 0, \quad \text{Projected cases} = 18{,}643 = 18643(1.03)^0.\\
&\text{When } t = 1, \quad \text{Projected cases} = 19{,}151 = 18643(1.03)^1.\\
&\text{When } t = 2, \quad \text{Projected cases} = 19{,}687 = 19151(1.03) = 18643(1.03)^2.\\
&\text{When } t = 3, \quad \text{Projected cases} = 20{,}263 = 19{,}687(1.03) = 18643(1.03)^3.
\end{aligned}$$

So P, the total number of cases confirmed by t days after October 4, is given by

$$P = 18643(1.03)^t.$$

Since the variable t is in the exponent, this is an exponential function. The base, 1.03, represents the factor by which the population grows each day and is called the *growth factor*. The formula held for some time, so we have the graph in Figure 1.74. Since P is the total number of people who have had Covid, including those who have recovered, the function is increasing. The value of P grows faster as time passes, so the graph is concave up.

Although P, the total number of Covid cases to date, takes only discrete values, we think of P as changing continuously as in Figure 1.74.

Elimination of a Drug from the Body

Now we look at a quantity that is decreasing instead of increasing. When a patient is given medication, the drug enters the bloodstream. The rate at which the drug is metabolized and eliminated depends on the particular drug. For the antibiotic ampicillin, approximately 40% of the drug is eliminated every hour. A typical dose of ampicillin is 250 mg. Suppose $Q = f(t)$, where Q is the quantity of ampicillin, in mg, in the bloodstream at time t hours since the drug was given. At $t = 0$, we have $Q = 250$. Since the quantity remaining at the end of each hour is 60% of the quantity remaining the hour before, we have

$$\begin{aligned}
f(0) &= 250\\
f(1) &= 250(0.6)\\
f(2) &= 250(0.6)(0.6) = 250(0.6)^2\\
f(3) &= 250(0.6)^2(0.6) = 250(0.6)^3.
\end{aligned}$$

So, after t hours,

$$Q = f(t) = 250(0.6)^t.$$

This function is called an *exponential decay* function. As t increases, the function values get arbitrarily close to zero. The t-axis is a *horizontal asymptote* for this function.

Notice the way the values in Table 1.36 are decreasing. Each additional hour a smaller quantity of drug is removed than the previous hour (100 mg the first hour, 60 mg the second, and so on). This is because as time passes, there is less of the drug in the body to be removed. Thus, the graph in Figure 1.75 bends upward. Compare this to the exponential growth in Figure 1.74, where each step upward is larger than the previous one. Notice that both graphs are concave up.

Table 1.36 *Value of decay function*

t (hours)	Q (mg)
0	250
1	150
2	90
3	54
4	32.4
5	19.4

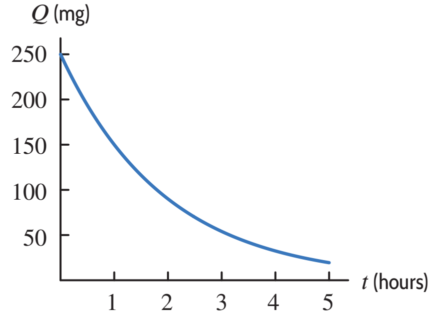


Figure 1.75: Drug elimination: Exponential decay

The General Exponential Function

Exponential growth is often described in terms of percent growth rates. The number of Covid cases in Arizona was growing at 3% per day, so it increases by a factor of $a = 1 + 0.03 = 1.03$ every day. Similarly, 40% of the ampicillin is removed every hour, so the quantity remaining decays by a factor of $a = 1 - 0.40 = 0.6$ each hour. We have the following general formulas.

> We say that P is an **exponential function** of t with base a if
>
> $$P = P_0 a^t,$$
>
> where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1. If $a > 1$, we have **exponential growth**; if $0 < a < 1$, we have **exponential decay**. The factor a is given by
>
> $$a = 1 + r$$
>
> where r is the decimal representation of the percent rate of change; r may be positive (for growth) or negative (for decay).

The largest possible domain for the exponential function is all real numbers,[89] provided $a > 0$.

Comparison Between Linear and Exponential Functions

Every exponential function changes at a constant percent, or *relative*, rate. For example, the number of Covid cases in Arizona increased approximately 3% per day. Every linear function changes at a constant absolute rate. For example, the Olympic pole vault record increased by 5 cm per year.

> A **linear** function has a constant rate of change.
> An **exponential** function has a constant percent, or relative, rate of change.

Example 1 The amount of adrenaline in the body can change rapidly. Suppose the initial amount is 15 mg. Find a formula for A, the amount in mg, at a time t minutes later if A is:

(a) Increasing by 0.4 mg per minute. (b) Decreasing by 0.4 mg per minute.
(c) Increasing by 3% per minute. (d) Decreasing by 3% per minute.

Solution (a) This is a linear function with initial quantity 15 and slope 0.4, so

$$A = 15 + 0.4t.$$

(b) This is a linear function with initial quantity 15 and slope -0.4, so

$$A = 15 - 0.4t.$$

(c) This is an exponential function with initial quantity 15 and base $1 + 0.03 = 1.03$, so

$$A = 15(1.03)^t.$$

(d) This is an exponential function with initial quantity 15 and base $1 - 0.03 = 0.97$, so

$$A = 15(0.97)^t.$$

[89] The reason we do not want $a \le 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 a^t = P_0 1^t = P_0$ is then a constant function.

Example 2 Wolves were once common in the western US. By the 1990s, the wolf population in Wyoming had been wiped out by hunters and wolves were put on the endangered species list. In 1995, wolves were reintroduced to Wyoming from Canada. Starting with 14 wolves, their number increased to 207 wolves in 2012.[90] Assuming the Wyoming wolf population was growing exponentially, find a function of the form $P = P_0 a^t$, where P is the population t years after 1995. What is the annual percent growth rate?

Solution We know that $P_0 = 14$ when $t = 0$. In 2012, when $t = 17$, we have $P = 207$. Substituting in $P = P_0 a^t$ gives an equation we can solve for a:

$$207 = 14a^{17}.$$

Dividing both sides by 14 gives

$$\frac{207}{14} = a^{17}.$$

Taking the 17th root of both sides, we get

$$a = (207/14)^{1/17} = 1.172.$$

Since $a = 1.172$, the Wyoming wolf population as a function of the number of years since 1995 is given by

$$P = 14(1.172)^t.$$

During this period, the population increased by about 17% per year.

Example 3 Suppose that $Q = f(t)$ is an exponential function of t. If $f(20) = 88.2$ and $f(23) = 91.4$:

(a) Find the base.
(b) Find the percent growth rate.
(c) Evaluate $f(25)$.

Solution

(a) Let $Q = Q_0 a^t$. Substituting $t = 20, Q = 88.2$ and $t = 23, Q = 91.4$ gives two equations for Q_0 and a:

$$88.2 = Q_0 a^{20} \quad \text{and} \quad 91.4 = Q_0 a^{23}.$$

Dividing the two equations enables us to eliminate Q_0:

$$\frac{91.4}{88.2} = \frac{Q_0 a^{23}}{Q_0 a^{20}} = a^3.$$

Solving for the base, a, gives

$$a = \left(\frac{91.4}{88.2}\right)^{1/3} = 1.012.$$

(b) Since $a = 1.012$, the percent growth rate is 1.2%.

(c) We want to evaluate $f(25) = Q_0 a^{25} = Q_0(1.012)^{25}$. First we find Q_0 from the equation

$$88.2 = Q_0(1.012)^{20}.$$

Solving gives $Q_0 = 69.5$. Thus,

$$f(25) = 69.5(1.012)^{25} = 93.6.$$

[90]Based on "Wyoming Drops Federal Protection of Gray Wolves", BBC, August 31, 2012, and *Wyoming Wolf Program Status Report*, US Fish and Wildlife Services, September, 2012.

The Family of Exponential Functions and the Number *e*

The formula $P = P_0 a^t$ gives a family of exponential functions with parameters P_0 (the initial quantity) and a (the base). The base tells us whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.76 and 1.77.) All members of the family $P = P_0 a^t$ are concave up if $P_0 > 0$.

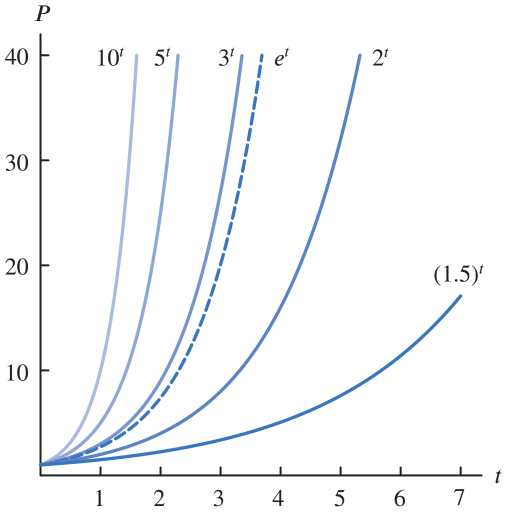


Figure 1.76: Exponential growth: $P = a^t$, for $a > 1$

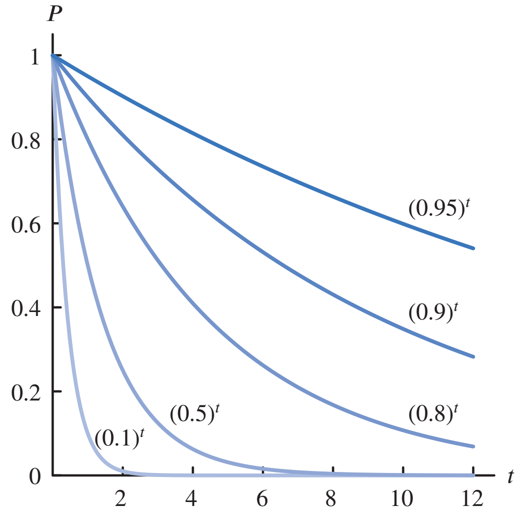


Figure 1.77: Exponential decay: $P = a^t$, for $0 < a < 1$

In practice the most commonly used base is the number $e = 2.71828\ldots$. The fact that most calculators have an e^x button is an indication of how important e is. Since e is between 2 and 3, the graph of $y = e^t$ in Figure 1.76 is between the graphs of $y = 2^t$ and $y = 3^t$.

The base e is used so often that it is called the natural base. At first glance, this is somewhat mysterious: What could be natural about using 2.71828... as a base? The full answer to this question must wait until Chapter 3, where you will see that many calculus formulas come out more neatly when e is used as the base. (See Appendix B for the relation to compound interest.)

Summary for Section 1.5

- We say that P is an **exponential function** of t with base a if

$$P = P_0 a^t.$$

  - P_0 is the initial quantity (when $t = 0$).
  - a is the factor by which P changes when t increases by 1.
- If $a > 1$, we have **exponential growth**.
- If $0 < a < 1$, we have **exponential decay**.
- The factor a is given by $a = 1 + r$ where r is the decimal representation of the percent rate of change. The value of r may be positive (for growth) or negative (for decay).
- **Comparison between linear and exponential functions:**
  - A **linear** function has a constant rate of change.
  - An **exponential** function has a constant percent, or relative, rate of change.
- The formula $P = P_0 a^t$ gives a **family of exponential functions** with parameters P_0 (initial quantity) and a (the base). In practice the most commonly used base is the number $e = 2.71828\ldots$.

Problems for Section 1.5

1. The following functions give the populations of four towns with time t in years.

(i) $P = 600(1.12)^t$ (ii) $P = 1{,}000(1.03)^t$
(iii) $P = 200(1.08)^t$ (iv) $P = 900(0.90)^t$

(a) Which town has the largest percent growth rate? What is the percent growth rate?
(b) Which town has the largest initial population? What is that initial population?
(c) Are any of the towns decreasing in size? If so, which one(s)?

2. Each of the following functions gives the amount of a substance present at time t. In each case, give the amount present initially (at $t = 0$), state whether the function represents exponential growth or decay, and give the percent growth or decay rate.

(a) $A = 100(1.07)^t$ **(b)** $A = 5.3(1.054)^t$
(c) $A = 3500(0.93)^t$ **(d)** $A = 12(0.88)^t$

3. Figure 1.78 shows $Q = 50(1.2)^t$, $Q = 50(0.6)^t$, $Q = 50(0.8)^t$, and $Q = 50(1.4)^t$. Match each formula to a graph.

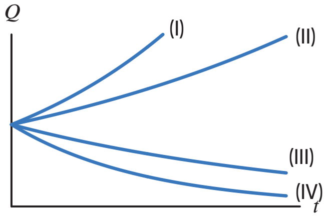


Figure 1.78

4. Figure 1.79 shows graphs of several cities' populations against time. Match each of the following descriptions to a graph and write a description to match each of the remaining graphs.

(a) The population increased at 5% per year.
(b) The population increased at 8% per year.
(c) The population increased by 5000 people per year.
(d) The population was stable.

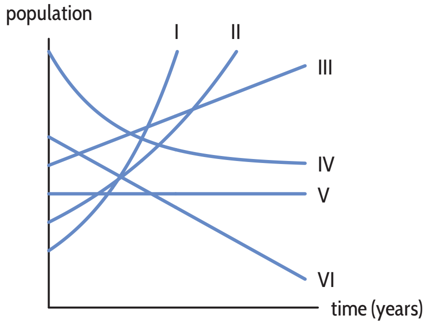


Figure 1.79

5. **(a)** The exponential functions in Figure 1.80 have b, d, q positive. Which of the constants a, c, and p must be positive?
(b) Which of the constants a, b, c, d, p, and q must be between 0 and 1?
(c) Which two of the constants a, b, c, d, p, and q must be equal?
(d) What information about the constants a and b does the point $(1, 1)$ provide?

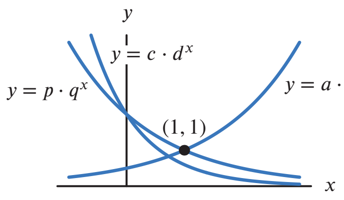
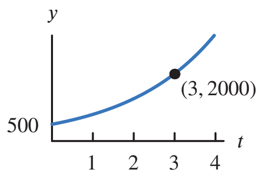


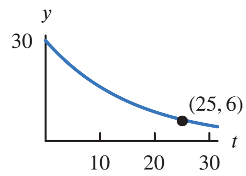
Figure 1.80

■ Give a possible formula for the functions in Problems 6–9.

6.



7.



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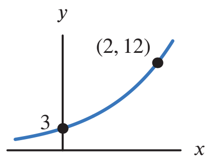
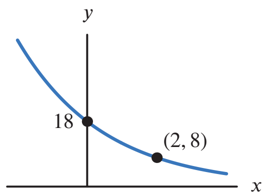


9.



10. The gross domestic product, G, of Switzerland[91] was 678.97 billion dollars in 2017. Give a formula for G (in billions of dollars) t years after 2017 if G increases by

(a) 1.09% per year
(b) 7 billion dollars per year

11. A town has a population of 1000 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.

(a) The population increases by 50 people a year.
(b) The population increases by 5% a year.

12. A product costs \$80 today. How much will the product cost in t days if the price is reduced by

(a) \$4 a day **(b)** 5% a day

[91] www.worldometers.info/gdp/switzerland-gdp/, accessed March 3, 2020.

13. An air-freshener starts with 30 grams and evaporates over time. In each of the following cases, write a formula for the quantity, Q grams, of air-freshener remaining t days after the start and sketch a graph of the function. The decrease is:

(a) 2 grams a day **(b)** 12% a day

14. World population[92] is about $P = 7.68(1.0103)^t$, with P in billions and t in years since 2020.

(a) What is the yearly percent rate of growth of the world population?
(b) What was the world population in 2020? What does this model predict for the world population in 2025?
(c) Use part (b) to find the projected average rate of change of the world population between 2020 and 2025.

15. A 50 mg dose of quinine is given to a patient to prevent malaria. Quinine leaves the body at a rate of 6% per hour.

(a) Find a formula for the amount, A (in mg), of quinine in the body t hours after the dose is given.
(b) How much quinine is in the body after 24 hours?
(c) Graph A as a function of t.
(d) Use the graph to estimate when 5 mg of quinine remains.

16. The consumer price index (CPI) for a given year is the amount of money in that year that has the same purchasing power as \$100 in 1983. At the start of 2020, the CPI was 258.8.[93] Write a formula for the CPI as a function of t years after 2020, assuming that the CPI increases by 2.5% every year.

17. Between 2005 and 2010, Brazil had a deforestation rate of about 0.33% per year.[94] (This is the rate at which land covered by forests is shrinking.) Assuming the rate continues, what percent of the land in Brazil covered by forests in 2010 will be forested in 2030?

18. The total number P of confirmed Covid-19 cases in Colombia[95] was 2979 on April 14, 2020, and for the next month the total grew at about 5.1% per day. Let $t = 0$ be April 14.

(a) Find a formula for the total number P of confirmed Covid-19 cases in Colombia t days after April 14.
(b) How many total confirmed cases by May 10, 2020 ($t = 26$) does your formula give?
(c) The actual number of cases on May 10, 2020 was 11,063. What is the percentage difference from the model's estimate?

19. Graph $y = 100e^{-0.4x}$. Describe what you see.

20. **(a)** Make a table of values for $y = e^x$ using $x = 0, 1, 2, 3$.
(b) Plot the points found in part (a). Does the graph look like an exponential growth or decay function?
(c) Make a table of values for $y = e^{-x}$ using $x = 0, 1, 2, 3$.
(d) Plot the points found in part (c). Does the graph look like an exponential growth or decay function?

■ For Problems 21–22, find a possible formula for the function represented by the data.

21.

x	0	1	2	3
$f(x)$	4.30	6.02	8.43	11.80

22.

t	0	1	2	3
$g(t)$	5.50	4.40	3.52	2.82

■ In Problems 23–24, find all the tables that have the given characteristic.

(A)

x	0	40	80	160
y	2.2	2.2	2.2	2.2

(B)

x	−8	−4	0	8
y	51	62	73	95

(C)

x	−4	−3	4	6
y	18	0	4.5	−2.25

(D)

x	3	4	5	6
y	18	9	4.5	2.25

23. y could be a linear function of x.

24. y could be an exponential function of x.

25. The table gives the number of North American houses (millions) with analog cable TV.[96]

(a) Plot the number of houses, H, in millions, with cable TV versus year, Y.
(b) Could H be a linear function of Y? Why or why not?
(c) Could H be an exponential function of Y? Why or why not?

Year	2010	2011	2012	2013	2014	2015
Houses	18.3	13	7.8	3.9	1	0.5

[92] www.indexmundi.com, accessed February 5, 2021.
[93] tradingeconomics.com/united-states/consumer-price-index-cpi, accessed March 4, 2020.
[94] data.worldbank.org, accessed March 4, 2020.
[95] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed May 12, 2020.
[96] http://www.statista.com, accessed May, 2015.

In Problems 26–29, a quantity P is an exponential function of time t. Use the given information about $P = P_0 a^t$ to:

(a) Find values for the parameters a and P_0.

(b) Give the initial quantity and the percent rate of growth or decay.

26. $P_0 a^3 = 75$ and $P_0 a^2 = 50$

27. $P_0 a^4 = 18$ and $P_0 a^3 = 20$

28. $P = 320$ when $t = 5$ and $P = 500$ when $t = 3$

29. $P = 1600$ when $t = 3$ and $P = 1000$ when $t = 1$

30. If the world's population[97] increased exponentially from 6.081 billion in 2000 to 7.504 billion in 2018 and continued to increase at the same percentage rate in 2018 and 2020, calculate what the world's population would have been in 2020. How does this compare to the Population Reference Bureau estimate of 7.77 billion in January, 2020? [98]

31. Facebook's annual ad revenue was 1.9 billion dollars in 2010 and increased an average of 54.5% per year from 2010 through 2017.[99]

(a) What was the revenue in 2017?

(b) The revenue was 80.9 billion dollars in 2020. Did growth in revenue continue to be exponential at the same rate through 2020?

32. The population of Nevada was 3.0 million in 2018 and 2.7 million in 2010. Assuming an exponential model:

(a) Write the population of Nevada in the form $N = N_0 a^t$, where N is the population of Nevada in millions, N_0 and a are constants, and t is time in years since 2010.

(b) What was the estimated population of Nevada in 2000?

33. The number of passengers using a railway fell from 190,205 to 174,989 during a 5-year period. Find the annual percentage decrease over this period.

34. The company that produces Cliffs Notes (abridged versions of classic literature) was started in 1958 with \$4000 and sold in 1998 for \$14,000,000. Find the annual percent increase in the value of this company over the 40 years.

35. Find a formula for the number of zebra mussels in a bay as a function of the number of years since 2014, given that there were 2700 at the start of 2014 and 3186 at the start of 2015.

(a) Assume that the number of zebra mussels is growing linearly. Give units for the slope of the line and interpret it in terms of zebra mussels.

(b) Assume that the number of zebra mussels is growing exponentially. What is the annual percent growth rate of the zebra mussel population?

36. In 2020, Arizona's second wave of Covid-19 infections grew at 3% a day; that is, the number of confirmed cases grew by a factor of $(1.03)^t$ over t days. Covid-19 confirmed cases in West Virginia in 2020 grew at 1.75% a day; that is, by a factor of $(1.0175)^t$ over t days.[100]

(a) By what factor did Arizona's cases grow over four months? How about West Virginia's cases?

(b) Notice that Arizona's daily percent growth rate is nearly double the daily percent growth rate in West Virginia. Is the 4-month growth factor for Arizona approximately double the 4-month growth factor for West Virginia?

37. Worldwide, wind energy[101] generating capacity, W, was 371.4 gigawatts at the end of 2014 and 591.1 gigawatts at the end of 2018. (Generating capacity is the maximum rate at which energy can be produced, measured in this problem in gigawatts.)

(a) Use the values given to write W, in gigawatts, as a linear function of t, the number of years since 2014.

(b) Use the values given to write W as an exponential function of t.

(c) Graph the functions found in parts (a) and (b) on the same axes. Label the given values.

(d) Use the functions found in parts (a) and (b) to predict the wind energy generating capacity in 2019. Generating capacity at the end of 2019 was 650.8 gigawatts. Which estimate is closer to the real value?

38. Which of the following tables of values could correspond to an exponential function, a linear function, or neither? For those which could correspond to an exponential or linear function, find a formula for the function.

(a)

x	$f(x)$
0	16
1	24
2	36
3	54
4	81

(b)

x	$g(x)$
0	14
1	20
2	24
3	29
4	35

(c)

x	$h(x)$
0	5.3
1	6.5
2	7.7
3	8.9
4	10.1

[97] www.indexmundi.com, accessed February 4, 2021.
[98] data.worldbank.org/indicator/SP.POP.TOTL, accessed March 4, 2020.
[99] https://i1.wp.com/www.mekkographics.com/wp-content/uploads/2019/03/Slide39.png, accessed May 10, 2021.
[100] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed January 10, 2021.
[101] wwindea.org/world-wind-capacity-at-650-gw accessed January 28, 2021.

39. **(a)** Which (if any) of the functions in the following table could be linear? Find formulas for those functions.
(b) Which (if any) of these functions could be exponential? Find formulas for those functions.

x	$f(x)$	$g(x)$	$h(x)$
−2	12	16	37
−1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

40. Determine whether each of the following tables of values could correspond to a linear function, an exponential function, or neither. For each table of values that could correspond to a linear or an exponential function, find a formula for the function.

(a)

x	$f(x)$
0	10.5
1	12.7
2	18.9
3	36.7

(b)

t	$s(t)$
−1	50.2
0	30.12
1	18.072
2	10.8432

(c)

u	$g(u)$
0	27
2	24
4	21
6	18

41. In 2021, there was a proposal before the US Congress to raise the national minimum wage to \$15, saying that minimum wage has not kept pace with inflation. The purchasing power of the minimum wage peaked in 1968 when it was \$1.60. Inflation has averaged 3.92% per year since 1968. Would raising the minimum wage to \$15 keep it on pace with inflation?

42. **(a)** Niki invested \$10,000 in the stock market. The investment was a loser, declining in value 10% per year each year for 10 years. How much was the investment worth after 10 years?
(b) After 10 years, the stock began to gain value at 10% per year. After how long will the investment regain its initial value (\$10,000)?

43. The total number of Covid-19 cases in the US[102] confirmed between March 10 and March 14, 2020 is in Table 1.37. Data shows that the daily percent growth rate remained approximately constant during the early days of the pandemic.
(a) Use the data from March 11–13 to find the daily percent growth rate of the total number of cases t days after March 10. Round to the nearest percent.
(b) Use part (a) to write a formula for the total number of cases as a function of t.
(c) There were 58,330 total cases on March 27 (t = 17). Compare this number to the formula's prediction for that day.
(d) There were 988,487 total cases on April 30 (t = 51). Compare this number to the prediction for that day.

Table 1.37

Day in March	10	11	12	13	14
Total Cases	444	606	813	1086	1419

44. Whooping cough was thought to have been almost wiped out by vaccinations. It is now known that the vaccination wears off, leading to an increase[103] in the number of cases, w, from 1248 in 1981 to 25,827 in 2004.
(a) With t in years since 1980, find an exponential function that fits this data.
(b) What does your answer to part (a) give as the average annual percent growth rate of the number of cases?
(c) On May 4, 2005, the *Arizona Daily Star* newspaper reported (correctly) that the number of cases had more than doubled between 2000 and 2004. Does your model confirm this report? Explain.

45. From 2013 through 2020 the price per kilowatt-hour of lithium ion batteries has dropped an average of 20% per year.[104] About how many years did it take for the price to be cut in half?

46. Aircraft require longer takeoff distances, called takeoff rolls, at high-altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0° C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

Elevation (ft)	Sea level	1000	2000	3000	4000
Takeoff roll (ft)	670	734	805	882	967

[102] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed April 26, 2020.
[103] www.cdc.gov/pertussis/surv-reporting/cases-by-year.html, accessed January 22, 2017.
[104] https://about.bnef.com/blog/battery-pack-prices-cited-below-100-kwh-for-the-first-time-in-2020-while-market-average-sits-at-137-kwh, accessed May 10, 2021.

47. On June 16, 2020, the New York Times wrote,[105] "The World Health Organization said last week that confirmed cases in Africa had doubled in 18 days to reach 200,000; the first 100,000 took 98 days."

- **(a)** Make an exponential model for the total number of Covid-19 cases in Africa during the initial 98 days. Use the fact that on the first day there was one confirmed case in Africa.[106]
- **(b)** How does your model's prediction for day 116 compare to the number 200,000 in the article?

■ Problems 48–49 concern biodiesel, a fuel derived from renewable resources such as food crops, algae, and animal oils. Table 1.38 shows the percent change over the previous year in US biodiesel consumption.[107]

Table 1.38

Year	2012	2013	2014	2015	2016	2017	2018
% change over previous year	1.47	58.95	−0.84	5.43	39.56	−4.80	−4.48

48. **(a)** According to the US Department of Energy, the US consumed 899 million gallons of biodiesel in 2012. Approximately how much biodiesel (in millions of gallons) did the US consume in 2013? In 2014?

- **(b)** Graph the points showing the annual US consumption of biodiesel, in millions of gallons of biodiesel, for the years 2012 to 2018. Label the scales on the horizontal and vertical axes.

49. **(a)** True or false: The annual US consumption of biodiesel grew exponentially from 2012 to 2018. Justify your answer without doing any calculations.

- **(b)** According to this data, in which year did US consumption have the greatest percentage
  - (a) Growth? (b) Decline?

50. Hydroelectric power is electric power generated by the force of moving water. Table 1.39 shows the percent change over the previous year in hydroelectric power generated in the US.[108]

- **(a)** According to the US Department of Energy, the US generated about 2.686 quadrillion[109] BTUs of hydroelectric power in 2013. Approximately how much hydroelectric power (in quadrillion BTUs) did the US consume in 2014? In 2015?
- **(b)** Graph the points showing the annual US consumption of hydroelectric power, in quadrillion BTUs, for the years 2013 to 2018. Label the scales on the horizontal and vertical axes.
- **(c)** According to this data, when did the largest yearly increase, in quadrillion BTUs, in the US consumption of hydroelectric power occur? What was this increase?

Table 1.39

Year	2013	2014	2015	2016	2017	2018
% change over previous year	−2.87	−3.42	−3.97	7.52	12.14	−2.84

■ Problems 51–52 concern wind power, which has been used for centuries to propel ships and mill grain. Modern wind power is obtained from windmills that convert wind energy into electricity. Figure 1.81 shows the annual percent growth in US wind power consumption[110] between 2013 and 2018. For example, the 20% growth rate in 2013 indicates there was 20% more wind power consumed in 2013 than in 2012.

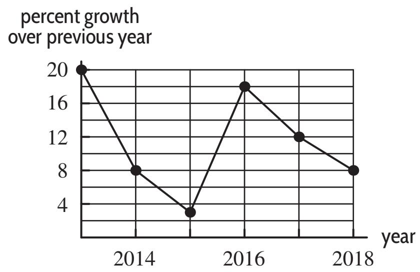


Figure 1.81

51. **(a)** According to the US Department of Energy, the US consumption of wind power was 1601 trillion BTUs in 2013. How much wind power did the US consume in 2014? In 2015?

- **(b)** Graph the points showing the annual US consumption of wind power, in trillion BTUs, for the years 2013 to 2018. Label the scales on the horizontal and vertical axes.
- **(c)** Based on this data, in what year did the largest yearly increase, in trillion BTUs, in the US consumption of wind power occur? What was this increase?

52. **(a)** According to Figure 1.81, in which years did the US consumption of wind power energy increase by at least 10% over the previous year? Decrease by at least 10%?

- **(b)** True or false: The US consumption of wind power energy grew by more than 40% from the beginning of 2015 to the end of 2018.

[105] https://www.nytimes.com/2020/06/16/world/africa/coronavirus-africa.html accessed January 10, 2021.
[106] https://github.com/owid/covid-19-data/tree/master/public/data, accessed March 26, 2021.
[107] www.eia.doe.gov, accessed August 20, 2019.
[108] From www.eia.doe.gov, accessed September 16, 2019.
[109] 1 quadrillion BTU=10^{15} BTU.
[110] Yearly values have been joined with line segments to highlight trends in the data. Actual values in between years should not be inferred from the segments. From www.eia.doe.gov, accessed September 18, 2019.

1.6 THE NATURAL LOGARITHM

If t is in years since 2000, the population[111] of Nevada (in millions) can be modeled by the function

$$P = f(t) = 1.998(1.023)^t,$$

How do we find when the population is projected to reach 4 million? We want to find the value of t for which

$$4 = f(t) = 1.998(1.023)^t.$$

We use logarithms to solve for a variable in an exponent.

Definition and Properties of the Natural Logarithm

We define the natural logarithm of x, written $\ln x$, as follows:

> The **natural logarithm** of x, written $\ln x$, is the power of e needed to get x. In other words,
>
> $$\ln x = c \quad \text{means} \quad e^c = x.$$
>
> The natural logarithm, sometimes written $\log_e x$, is called the *inverse function* of e^x.

For example, $\ln e^3 = 3$ since 3 is the power of e needed to give e^3. Similarly, $\ln(1/e) = \ln e^{-1} = -1$. A calculator gives $\ln 5 = 1.6094$, because $e^{1.6094} = 5$. However if we try to find $\ln(-7)$ on a calculator, we get an error message because e to any power is never negative or 0. In general,

> $\ln x$ is not defined if x is negative or 0.

To work with logarithms, we use the following properties:

> **Properties of the Natural Logarithm**
>
> 1. $\ln(AB) = \ln A + \ln B$
> 2. $\ln\left(\dfrac{A}{B}\right) = \ln A - \ln B$
> 3. $\ln(A^p) = p \ln A$
> 4. $\ln e^x = x$
> 5. $e^{\ln x} = x$
>
> In addition, $\ln 1 = 0$ because $e^0 = 1$, and $\ln e = 1$ because $e^1 = e$.

The graph of $f(x) = \ln x$ is in Figure 1.82. Observe that, for large x, the graph of $y = \ln x$ climbs very slowly as x increases. The x-intercept is $x = 1$, since $\ln 1 = 0$. For $x > 1$, the value of $\ln x$ is positive; for $0 < x < 1$, the value of $\ln x$ is negative.

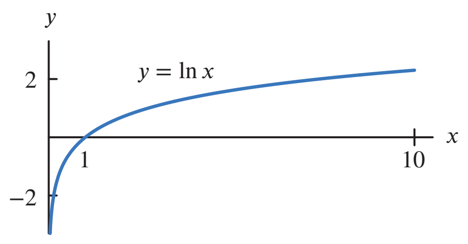


Figure 1.82: The natural logarithm function climbs very slowly

[111] census.gov, accessed March 17, 2021.

Solving Equations Using Logarithms

Natural logs can be used to solve for unknown exponents.

Example 1 Find t such that $3^t = 10$.

Solution First, notice that we expect t to be between 2 and 3, because $3^2 = 9$ and $3^3 = 27$. To find t exactly, we take the natural logarithm of both sides and solve for t:

$$\ln(3^t) = \ln 10.$$

The third property of logarithms tells us that $\ln(3^t) = t \ln 3$, so we have

$$\begin{aligned} t \ln 3 &= \ln 10 \\ t &= \frac{\ln 10}{\ln 3} \\ t &= 2.096. \end{aligned}$$

Example 2 We return to the question of when the population of Nevada reaches 4 million. To get an answer, we solve $4 = 1.998(1.023)^t$ for t, using logs.

Solution Dividing both sides of the equation by 1.998, we get

$$\frac{4}{1.998} = (1.023)^t.$$

Now take natural logs of both sides:

$$\ln\left(\frac{4}{1.998}\right) = \ln(1.023^t).$$

Using the fact that $\ln(1.023^t) = t \ln 1.023$, we get

$$\ln\left(\frac{4}{1.998}\right) = t \ln(1.023).$$

Solving this equation, we get

$$t = \frac{\ln(4/1.998)}{\ln(1.023)} = 30.526 \text{ years}.$$

Since $t = 0$ in 2000, this value of t corresponds to the year 2030.

Example 3 Find t such that $12 = 5e^{3t}$.

Solution It is easiest to begin by isolating the exponential, so we divide both sides of the equation by 5:

$$2.4 = e^{3t}.$$

Now take the natural logarithm of both sides:

$$\ln 2.4 = \ln(e^{3t}).$$

Since $\ln(e^x) = x$, we have

$$\ln 2.4 = 3t.$$

Solving for t, we get

$$t = \frac{\ln 2.4}{3} = 0.2918.$$

Exponential Functions with Base e

An exponential function with base a has formula

$$P = P_0 a^t.$$

For any positive number a, we can write $a = e^k$ where $k = \ln a$. Thus, the exponential function can be rewritten as

$$P = P_0 a^t = P_0(e^k)^t = P_0 e^{kt}.$$

If $a > 1$, then k is positive, and if $0 < a < 1$, then k is negative. We conclude:

Writing $a = e^k$, so $k = \ln a$, any exponential function can be written in two forms

$$P = P_0 a^t \quad \text{or} \quad P = P_0 e^{kt}.$$

- If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.
- If $k > 0$, we have exponential growth; if $k < 0$, we have exponential decay.
- k is called the *continuous* growth or decay rate.

The word continuous in continuous growth rate is used in the same way to describe continuous compounding of interest earned on money. (See Appendix A.)

Example 4

(a) Convert the function $P = 1000e^{0.05t}$ to the form $P = P_0 a^t$.
(b) Convert the function $P = 500(1.06)^t$ to the form $P = P_0 e^{kt}$.

Solution

(a) Since $P = 1000e^{0.05t}$, we have $P_0 = 1000$. We want to find a so that

$$1000a^t = 1000e^{0.05t} = 1000(e^{0.05})^t.$$

We take $a = e^{0.05} = 1.0513$, so the following two functions give the same values:

$$P = 1000e^{0.05t} \qquad \text{and} \qquad P = 1000(1.0513)^t.$$

So a continuous growth rate of 5% is equivalent to a growth rate of 5.13% per unit time.

(b) We have $P_0 = 500$ and we want to find k with

$$500(1.06)^t = 500(e^k)^t,$$

so we take

$$1.06 = e^k$$
$$k = \ln(1.06) = 0.0583.$$

The following two functions give the same values:

$$P = 500(1.06)^t \qquad \text{and} \qquad P = 500e^{0.0583t}.$$

So a growth rate of 6% per unit time is equivalent to a continuous growth rate of 5.83%.

Example 5

Sketch graphs of $P = e^{0.5t}$, a continuous growth rate of 50%, and $Q = 5e^{-0.2t}$, a continuous decay rate of 20%.

Solution The graph of $P = e^{0.5t}$ is in Figure 1.83. Notice that the graph is the same shape as the previous exponential growth curves: increasing and concave up. The graph of $Q = 5e^{-0.2t}$ is in Figure 1.84; it has the same shape as other exponential decay functions.

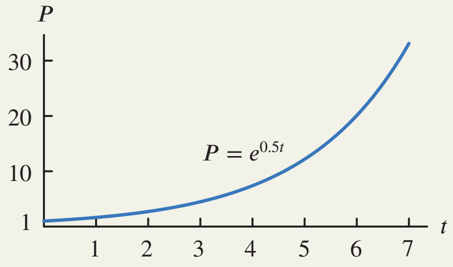


Figure 1.83: Continuous exponential growth function

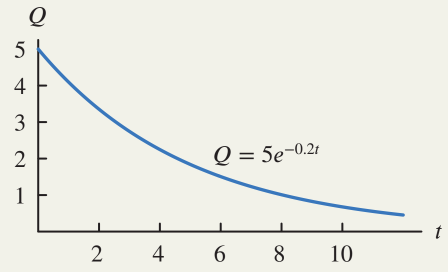


Figure 1.84: Continuous exponential decay function

Summary for Section 1.6

- The **natural logarithm** of x, written $\ln x$, is the power of e needed to get x. In other words,
$$\ln x = c \quad \text{means} \quad e^c = x.$$
  - $\ln x$ is not defined if x is negative or 0.
- **Properties of the natural logarithm**:
  - $\ln(AB) = \ln A + \ln B$
  - $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
  - $\ln(A^p) = p \ln A$
  - $\ln e^x = x$
  - $e^{\ln x} = x$
  - $\ln 1 = 0$ because $e^0 = 1$, and $\ln e = 1$ because $e^1 = e$.
- **Solving equations using logarithms**: Natural logs can be used to solve for unknown exponents.
- **Exponential functions with base e:** Writing $a = e^k$, so $k = \ln a$, any exponential function can be written in two forms
$$P = P_0 a^t \quad \text{or} \quad P = P_0 e^{kt}.$$
  - If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.
  - If $k > 0$, we have exponential growth; if $k < 0$, we have exponential decay.
  - k is called the **continuous** growth or decay rate.

Problems for Section 1.6

■ For Problems 1–20, solve for t using natural logarithms.

1. $10 = 2^t$
2. $5^t = 7$
3. $2 = (1.02)^t$
4. $130 = 10^t$
5. $10 = e^t$
6. $100 = 25(1.5)^t$
7. $50 = 10 \cdot 3^t$
8. $5 = 2e^t$
9. $e^{3t} = 100$
10. $10 = 6e^{0.5t}$
11. $40 = 100e^{-0.03t}$
12. $a = b^t$
13. $B = Pe^{rt}$
14. $2P = Pe^{0.3t}$
15. $7 \cdot 3^t = 5 \cdot 2^t$
16. $5e^{3t} = 8e^{2t}$
17. $Ae^{2t} = Be^t$
18. $2e^t - 5 = 0$
19. $0 = 7 - 3e^t$
20. $Pe^{4t} - Qe^{-t} = 0$

■ Problems 21–24 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

21. $P = 5(1.07)^t$
22. $P = 7.7(0.92)^t$
23. $P = 15e^{-0.06t}$
24. $P = 3.2e^{0.03t}$

■ In Problems 25–28, write the function in the form $P = P_0 a^t$. Which represent exponential growth and which represent exponential decay?

25. $P = 15e^{0.25t}$
26. $P = 2e^{-0.5t}$
27. $P = P_0 e^{0.2t}$
28. $P = 7e^{-\pi t}$

■ In Problems 29–32, put the functions in the form $P = P_0 e^{kt}$.

29. $P = 15(1.5)^t$

30. $P = 10(1.7)^t$

31. $P = 174(0.9)^t$

32. $P = 4(0.55)^t$

■ In Problems 33–34, a quantity P is an exponential function of time t. Use the given information about $P = P_0 e^{kt}$ to:
(a) Find values for the parameters k and P_0.
(b) Give the initial quantity and the continuous percent rate of growth or decay.

33. $P = 140$ when $t = 3$ and $P = 100$ when $t = 1$

34. $P = 40$ when $t = 4$ and $P = 50$ when $t = 3$

35. **(a)** What is the continuous percent growth rate for $P = 100e^{0.06t}$, with time, t, in years?
(b) Write this function in the form $P = P_0 a^t$. What is the annual percent growth rate?

36. **(a)** What is the annual percent decay rate for $P = 25(0.88)^t$, with time, t, in years?
(b) Write this function in the form $P = P_0 e^{kt}$. What is the continuous percent decay rate?

37. What annual percent growth rate is equivalent to a continuous percent growth rate of 8%?

38. What continuous percent growth rate is equivalent to an annual percent growth rate of 10%?

39. The following formulas give the populations of four different towns, A, B, C, and D, with t in years from now.

$$P_A = 600e^{0.08t} \qquad P_B = 1000e^{-0.02t}$$
$$P_C = 1200e^{0.03t} \qquad P_D = 900e^{0.12t}$$

(a) Which town is growing fastest (that is, has the largest percentage growth rate)?
(b) Which town is the largest now?
(c) Are any of the towns decreasing in size? If so, which one(s)?

40. A 2008 study of 300 oil fields producing a total of 84 million barrels per day reported that daily production was decaying at a continuous rate of 9.1% per year.[112] Find the estimated production in these fields in 2025 if the decay continues at the same rate.

41. A city's population is 1000 and growing at 5% a year.
(a) Find a formula for the population at time t years from now assuming that the 5% per year is an:
(i) Annual rate (ii) Continuous annual rate
(b) In each case in part (a), estimate the population of the city in 10 years.

42. The population, P, in millions, of Nicaragua[113] was 5.97 million in 2016 and growing at an annual rate of 0.99%. Let t be time in years since 2016.
(a) Express P as a function in the form $P = P_0 a^t$.
(b) Express P as an exponential function using base e.
(c) Compare the annual and continuous growth rates.

43. The gross world product is $W = 123.3(1.032)^t$, where W is in trillions of dollars and t is years since 2016.[114] Find a formula for gross world product using a continuous growth rate.

44. The population of the world[115] can be represented by $P = 7.68(1.0103)^t$, where P is in billions of people and t is years since 2020. Find a formula for the population of the world using a continuous growth rate.

45. A fishery stocks a pond with 1000 young trout. The number of trout t years later is given by $P(t) = 1000e^{-0.5t}$.
(a) How many trout are left after six months? After 1 year?
(b) Find $P(3)$ and interpret it in terms of trout.
(c) At what time are there 100 trout left?
(d) Graph the number of trout against time, and describe how the population is changing. What might be causing this?

46. The Hershey Company is the largest US producer of chocolate. In 2019, annual net sales were 7.986 billion dollars and were increasing at a continuous rate of 2.5% per year.[116]
(a) Write a formula for annual net sales, S, as a function of time, t, in years since 2019.
(b) Estimate annual net sales in 2022.
(c) Use a graph to estimate the year in which annual net sales are expected to pass 10 billion dollars and check your estimate using logarithms.

47. During a recession a firm's revenue declines continuously so that the revenue, R (measured in millions of dollars), in t years' time is given by $R = 5e^{-0.15t}$.
(a) Calculate the current revenue and the revenue in two years' time.
(b) After how many years will the revenue decline to \$2.7 million?

[112] International Energy Agency, *World Energy Outlook*, 2008.
[113] www.cia.gov/library/publications/the-world-factbook, accessed December 30, 2016.
[114] www.indexmundi.com/world, accessed March 4, 2020.
[115] www.indexmundi.com/world/population_growth_rate.html, accessed February 4, 2021.
[116] www.thehersheycompany.com, accessed March 4, 2020.

48. The population of a city is 50,000 in 2014 and is growing at a continuous rate of 4.5% per year.

(a) Give the population of the city as a function of the number of years since 2014. Sketch a graph of the population against time.
(b) What will be the city's population in the year 2020?
(c) Calculate the time for the population of the city to reach 100,000. This is called the doubling time of the population.

49. For children and adults with diseases such as asthma, the number of respiratory deaths per year increases by 0.33% when pollution particles increase by a microgram per cubic meter of air.[117]

(a) Write a formula for the number of respiratory deaths per year as a function of quantity of pollution in the air. (Let Q_0 be the number of deaths per year with no pollution.)
(b) What quantity of air pollution results in twice as many respiratory deaths per year as there would be without pollution?

50. The concentration of the car exhaust fume nitrous oxide, NO_2, in the air near a busy road is a function of distance from the road. The concentration decays exponentially at a continuous rate of 2.54% per meter.[118] At what distance from the road is the concentration of NO_2 half what it is on the road?

51. With time, t, in years since the start of 1980, textbook prices have increased at 6.7% per year while inflation has been 3.3% per year.[119] Assume both rates are continuous growth rates.

(a) Find a formula for $B(t)$, the price of a textbook in year t if it cost $\$B_0$ in 1980.
(b) Find a formula for $P(t)$, the price of an item in year t if it cost $\$P_0$ in 1980 and its price rose according to inflation.
(c) A textbook cost \$50 in 1980. When is its price predicted to be double the price that would have resulted from inflation alone?

52. In 2017, the populations of China and India were approximately 1.379 and 1.282 billion people,[120] respectively. However, the annual population growth rate of China was 0.41% while the population of India was growing by 1.17% each year. If these growth rates remain constant, when will the population of India exceed that of China?

53. In 2010, there were about 246 million vehicles (cars and trucks) and about 308.7 million people in the US.[121] The number of vehicles grew 15.5% over the previous decade, while the population has been growing at 9.7% per decade. If the growth rates remain constant, when will there be, on average, one vehicle per person?

54. A quantity is growing exponentially: $P = P_0 e^{kt}$, with P_0 and k constant.

(a) Show that a plot of $y = \ln P$ against t, with t on the horizontal axis, is a line.
(b) What is the slope of the line?
(c) What is its vertical intercept?

55. Use the results of Problem 54. Assume Florida's total number of Covid-19 cases confirmed up to day t grow exponentially: $P(t) = P_0 e^{kt}$, with P_0 and k constant.

(a) Estimate the continuous growth rate in Florida's Covid total cases in the 70 days between Nov 15, 2020 and Jan 24, 2021. Use Figure 1.85, which shows $\ln(P(t))$ with t in days since Nov 15.[122]
(b) Estimate the initial value of P.

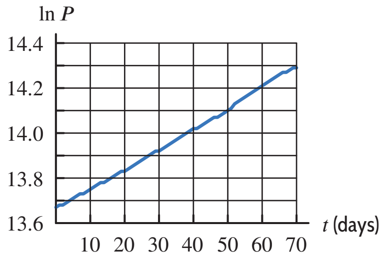


Figure 1.85

56. Use the results of Problem 54. Let P be the total number of Covid-19 cases in South Africa[123] confirmed up to day t, where $t = 0$ is March 18, 2020. Figure 1.86 shows $\ln P$ plotted against t.

(a) Estimate the continuous growth rates before and after the bend in the graph at about $t = 10$ days.
(b) What might have happened to lead to the bend in the graph?

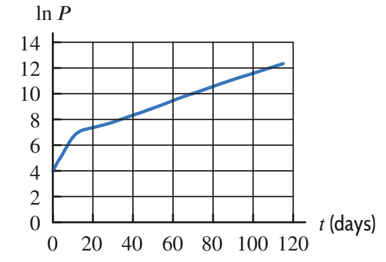


Figure 1.86

[117] R. D. Brook, B. Franklin, W. Cascio, Y. Hong, G. Howard, M. Lipsett, R. Luepker, M. Mittleman, J. Samet, and S. C. Smith, "Air Pollution and Cardiovascular Disease", *Circulation,* 109, pp. 2655–2671 (2004).
[118] P. Rickwood, and D. Knight, "The Health Impacts of Local Traffic Pollution on Primary School Age Children", *State of Australian Cities 2009 Conference Proceedings*, www.be.unsw.edu.au, accessed September 12, 2019.
[119] Data from "Textbooks Headed for Ash Heap of History", http://educationtechnews.com, Vol. 5, 2010.
[120] www.indexmundi.com, accessed September 12, 2019.
[121] http://www.autoblog.com/2010/01/04/report-number-of-cars-in-the-u-s-dropped-by-four-million-in-20/ and http://2010.census.gov/news/releases/operations/cb10-cn93.html, accessed February, 2012.
[122] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed January 10, 2021.
[123] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed July 5, 2020.

1.7 EXPONENTIAL GROWTH AND DECAY

Many quantities in nature change according to an exponential growth or decay function of the form $P = P_0 e^{kt}$, where P_0 is the initial quantity and k is the continuous growth or decay rate.

Example 1 The Environmental Protection Agency (EPA) investigated a spill of radioactive iodine. The radiation level at the site was about 2.4 millirems/hour (four times the maximum acceptable limit of 0.6 millirems/hour), so the EPA ordered an evacuation of the surrounding area. The level of radiation from an iodine source decays at a continuous hourly rate of $k = -0.004$.

(a) What was the level of radiation 24 hours later?
(b) Find the number of hours until the level of radiation reached the maximum acceptable limit, and the inhabitants could return.

Solution (a) The level of radiation, R, in millirems/hour, at time t, in hours since the initial measurement, is given by

$$R = 2.4e^{-0.004t},$$

so the level of radiation 24 hours later was

$$R = 2.4e^{(-0.004)(24)} = 2.18 \text{ millirems per hour.}$$

(b) A graph of $R = 2.4e^{-0.004t}$ is in Figure 1.87. The maximum acceptable value of R is 0.6 millirems per hour, which occurs at approximately $t = 350$. Using logarithms, we have

$$\begin{aligned} 0.6 &= 2.4e^{-0.004t} \\ 0.25 &= e^{-0.004t} \\ \ln 0.25 &= -0.004t \\ t &= \frac{\ln 0.25}{-0.004} = 346.57. \end{aligned}$$

The inhabitants will not be able to return for 346.57 hours, or about 15 days.

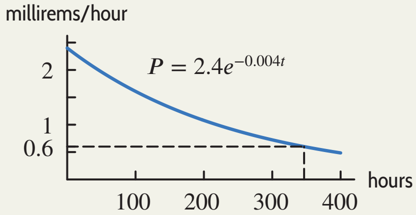


Figure 1.87: The level of radiation from radioactive iodine

Example 2 The population of Kenya[124] was 19.1 million in 1984 and 53.8 million in 2020. Assuming the population increases exponentially, find a formula for the population of Kenya as a function of time.

Solution We measure the population, P, in millions and time, t, in years since 1984. We can express P in terms of t using the continuous growth rate k by

$$P = P_0 e^{kt} = 19.1e^{kt},$$

where $P_0 = 19.1$ is the initial value of P. We find k using the fact that $P = 53.8$ when $t = 36$:

$$53.8 = 19.1e^{k \cdot 36}.$$

[124] www.worldometers.info/world-population/kenya-population/, accessed March 4, 2020.

Divide both sides by 18.9, giving
$$\frac{53.8}{19.1} = e^{36k}.$$
Take natural logs of both sides:
$$\ln\left(\frac{53.8}{19.1}\right) = \ln(e^{36k}).$$
Since $\ln(e^{36k}) = 36k$, this becomes
$$\ln\left(\frac{53.8}{19.1}\right) = 36k.$$
We get
$$k = \frac{1}{36}\ln\left(\frac{53.8}{19.1}\right) = 0.029,$$
and therefore
$$P = 18.9e^{0.029t}.$$
Since $k = 0.029 = 2.9\%$, the population of Kenya was growing at a continuous rate of 2.9% per year.

Doubling Time and Half-Life

Every exponential growth function has a constant doubling time and every exponential decay function has a constant half-life.

> The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.
> The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.

Example 3 Show algebraically that every exponentially growing function has a fixed doubling time.

Solution Consider the exponential function $P = P_0a^t$. For any base a with $a > 1$, there is a positive number d such that $a^d = 2$. We show that d is the doubling time. If the population is P at time t, then at time $t + d$, the population is
$$P_0a^{t+d} = P_0a^ta^d = (P_0a^t)(2) = 2P.$$
So, no matter what the initial quantity and no matter what the initial time, the size of the population is doubled d time units later.

Example 4 The release of chlorofluorocarbons used in air conditioners and household sprays (hair spray, shaving cream, etc.) destroys the ozone in the upper atmosphere. In the 1990s, the quantity of ozone, Q, was decaying exponentially at a continuous rate of 0.29% per year.[125] At this rate of decay, what is the half-life of ozone? In other words, at this rate, how long would it take for half the ozone to disappear?

Solution If Q_0 is the initial quantity of ozone and t is in years, then
$$Q = Q_0e^{-0.0029t}.$$

[125] Based on https://en.wikipedia.org/wiki/Ozone_depletion, accessed March 22, 2021.

We want to find the value of t making $Q = Q_0/2$, so

$$\frac{Q_0}{2} = Q_0 e^{-0.0029t}.$$

Dividing both sides by Q_0 and taking natural logs gives

$$\ln\left(\frac{1}{2}\right) = -0.0029t,$$

so

$$t = \frac{\ln(1/2)}{-0.0029} = 239 \text{ years}.$$

Half the present atmospheric ozone would be gone in 239 years.

Financial Applications: Compound Interest

We deposit \$100 in a bank paying interest at a rate of 2% per year. How much is in the account at the end of the year? This depends on how often the interest is compounded. If the interest is paid into the account *annually*, that is, only at the end of the year, then the balance in the account after one year is \$102. However, if the interest is paid twice a year, then 1% is paid at the end of the first six months and 1% at the end of the year. Slightly more money is earned this way, since the interest paid early in the year will earn interest during the rest of the year. This effect is called *compounding*.

In general, the more often interest is compounded, the more money is earned (although the increase may not be large). What happens if interest is compounded more frequently, such as every minute or every second? The benefit of increasing the frequency of compounding becomes negligible beyond a certain point. When that point is reached, we find the balance using the number e and we say that the interest per year is *compounded continuously*. If we have deposited \$100 in an account paying 2% interest per year compounded continuously, the balance after one year is $100e^{0.02} = \$102.02$. Compounding is discussed further in Appendix B. In general:

> An amount P_0 is deposited in an account paying interest at a rate of r per year. Let P be the balance in the account after t years.
> - If interest is compounded annually, then $P = P_0(1+r)^t$.
> - If interest is compounded continuously, then $P = P_0 e^{rt}$, where $e = 2.71828\ldots$.

We write P_0 for the initial deposit because it is the value of P when $t = 0$. Note that for a 1.5% interest rate, $r = 0.015$. If a rate is continuous, we will say so explicitly.

Example 5 A bank advertises an interest rate of 2% per year. If you deposit \$10,000, how much is in the account 4 years later if the interest is compounded (a) Annually? (b) Continuously?

Solution
(a) For annual compounding, $P = P_0(1+r)^t = 10{,}000(1.02)^4 = \$10{,}824.32$.
(b) For continuous compounding, $P = P_0 e^{rt} = 10{,}000e^{0.02\cdot 4} = \$10{,}832.87$. As expected, the amount in the account 4 years later is larger if the interest is compounded continuously (\$10,832.87) than if the interest is compounded annually (\$10,824.32).

Example 6 If \$10,000 is deposited in an account paying interest at a rate of 1.5% per year, compounded continuously, how long does it take for the balance in the account to reach \$12,000?

Solution Since interest is compounded continuously, we use $P = P_0 e^{rt}$ with $r = 0.015$ and $P_0 = 10{,}000$. We want to find the value of t for which $P = 12{,}000$. The equation is

$$12{,}000 = 10{,}000e^{0.015t}.$$

Now divide both sides by 10,000, then take logarithms and solve for t:

$$\begin{aligned} 1.2 &= e^{0.015t} \\ \ln(1.2) &= \ln(e^{0.015t}) \\ \ln(1.2) &= 0.015t \\ t &= \frac{\ln(1.2)}{0.015} = 12.1548. \end{aligned}$$

It takes about 12.2 years for the balance in the account to reach \$12,000.

Example 7

(a) Calculate the doubling time, D, for interest rates of 1%, 2%, 3%, and 4% per year, compounded annually.

(b) Use your answers to part (a) to check that an interest rate of i% gives a doubling time approximated for small values of i by

$$D \approx \frac{70}{i} \text{ years.}$$

This is the "Rule of 70" used by bankers: To compute the approximate doubling time of an investment, divide 70 by the percent annual interest rate.

Solution

(a) We find the doubling time for an interest rate of 1% per year using the formula $P = P_0(1.01)^t$ with t in years. To find the value of t for which $P = 2P_0$, we solve

$$\begin{aligned} 2P_0 &= P_0(1.01)^t \\ 2 &= (1.02)^t \\ \ln 2 &= \ln(1.01)^t \\ \ln 2 &= t\ln(1.01) \qquad \text{(using the third property of logarithms)} \\ t &= \frac{\ln 2}{\ln 1.01} = 69.661 \text{ years.} \end{aligned}$$

With an annual interest rate of 1%, it takes about 70 years for an investment to double in value. Similarly, we find the doubling times for 2%, 3%, and 4% in Table 1.40.

Table 1.40 *Doubling time as a function of interest rate*

i (% annual growth rate)	1	2	3	4
D (doubling time in years)	69.661	35.003	23.45	17.673

(b) We compute $(70/i)$ for $i = 1, 2, 3, 4$. The results are shown in Table 1.41.

Table 1.41 *Approximate doubling time as a function of interest rate: Rule of* 70

i (% annual growth rate)	1	2	3	4
$(70/i)$ (Approximate doubling time in years)	70.000	35.000	23.333	17.500

Comparing Tables 1.40 and 1.41, we see that the quantity $(70/i)$ gives a reasonably accurate approximation to the doubling time, D, for the small interest rates we considered.

Present and Future Value

Many business deals involve payments in the future. For example, when a car is bought on credit, payments are made over a period of time. Being paid \$100 in the future is clearly worse than being paid \$100 today for many reasons. If we are given the money today, we can do something else with it—for example, put it in the bank, invest it somewhere, or spend it. Thus, even without considering inflation, if we are to accept payment in the future, we would expect to be paid more to compensate for this loss of potential earnings.[126] The question we consider now is, how much more?

To simplify matters, we consider only what we would lose by not earning interest; we do not consider the effect of inflation. Let's look at some specific numbers. Suppose we deposit \$100 in an account that earns 7% interest per year compounded annually, so that in a year's time we have \$107. Thus, \$100 today is worth \$107 a year from now. We say that the \$107 is the *future value* of the \$100, and that the \$100 is the *present value* of the \$107. In general, we say the following:

- The **future value**, B, of a payment, P, is the amount to which the P would have grown if deposited today in an interest-bearing bank account.
- The **present value**, P, of a future payment, B, is the amount that would have to be deposited in a bank account today to produce exactly B in the account at the relevant time in the future.

Due to the interest earned, the future value is larger than the present value. The relation between the present and future values depends on the interest rate, as follows.

Suppose B is the *future value* of P and P is the *present value* of B.
If interest is compounded annually at a rate r for t years, then

$$B = P(1+r)^t, \qquad \text{or equivalently,} \qquad P = \frac{B}{(1+r)^t}.$$

If interest is compounded continuously at a rate r for t years, then

$$B = Pe^{rt}, \qquad \text{or equivalently,} \qquad P = \frac{B}{e^{rt}} = Be^{-rt}.$$

The rate, r, is sometimes called the *discount rate*. The present value is often denoted by PV and the future value by FV.

Example 8 In 2007, a few months after their college graduation, three students with an initial loan of \$375,000 started the Sweetgreen salad chain.[127]

Assume the loan was invested at 4% interest compounded continuously and measure time in years from when the loan was taken.

(a) Assume the first expenditure was \$100,000, made a year after they got the loan and their second expenditure was \$50,000, made one year later. How much money remained invested right after the second expenditure was made? Give your answer to the nearest dollar.
(b) Assume the first scale-up was planned for two years after the second expenditure and was expected to cost \$280,000. Do the founders have enough money? How much money remains from the original loan at that time? Give your answer to the nearest dollar.
(c) What is the present value of the amount remaining before the scale-up?

[126] This is referred to as the time value of money.
[127] https://en.wikipedia.org/wiki/Sweetgreen. Accessed January 12, 2020.

Solution With time t in years since the loan was taken:

(a) Working in thousands of dollars:

$$\begin{aligned}
\text{Amount invested right before first expenditure } &= 375e^{0.4(1)}\\
\text{Amount remaining right after first expenditure } &= 375e^{0.04} - 100\\
\text{Amount remaining right before second expenditure } &= (375^{0.04} - 100)e^{0.04(1)}\\
\text{Amount remaining right after second expenditure } &= (375e^{0.04} - 100)e^{0.04} - 50 = 252.152.
\end{aligned}$$

There is \$252,152 invested right after the second expenditure.

(b) After the second expenditure there is \$252,152 in the account. So the value two years later is

$$\text{Amount remaining right before scale-up } = 252{,}152e^{0.04(2)} = 273{,}153 \text{ dollars}$$

Thus, the founders would have \$273,153 in the account right before the scale-up. The founders would need \$280,000 − 273,153 = \$6847 more.

(c) Since the scale-up was at $t = 4$, we have

$$\text{Present value } = 273{,}153e^{-0.04(4)} = 232{,}766.$$

Thus if invested at 4% interest compounded continually, \$232,766 would grow to \$273,153 in four years' time.

Climate Change and Its Costs

On September 23, 2019, Greta Thunberg, a Swedish high school student, spoke to the UN calling on world leaders to address climate change. Although there is controversy about climate change, temperatures and sea levels are in fact rising in much of the world and many people think climate change is responsible for flooding, droughts, and fires.

Faced by rising water levels, some communities are considering building sea walls. By one estimate, Florida needs to spend \$76 billion on sea walls by 2040.[128] To protect New York City, the mayor proposes extending the shoreline of lower Manhattan at a cost of \$10 billion.[129] While the costs are huge, the costs of damage are also enormous. Hurricane Sandy, which hit New York in 2012, caused \$19 billion in damage in the city alone.

To compare today's costs with future damages, we use present and future values, which allow us to compare payments at widely different times. We can think of Florida as having two options: Build a sea wall and avoid flooding, or not build a sea wall, invest the \$76 billion, and pay for damages when they occur in 2040.[130] If invested, the amount Florida would have available to pay for damages is the *future value* of the \$76 billion.

Example 9 What is the future value in 2040 of \$76 billion in 2020 if the interest rate is 2% compounded annually?

Solution Since 2040 is 20 years after 2020, we have

$$\text{Future value in 2040 } = 76(1.02)^{20} = 112.932 \text{ billion dollars.}$$

Florida could expect its investment to be worth \$112.932 billion in 2040.

[128] https://www.climatecosts2040.org. Accessed January 9, 2020.

[129] https://www.bloomberg.com/news/articles/2019-03-14/de-blasio-seeks-to-flood-proof-lower-manhattan-by-adding-land. Accessed January 9, 2020.

[130] We are considering a very simplified model: All the damages occur in 2040 and we ignore the human and social costs.

Example 10 Suppose we expect the damage in 2040 to be \$100 billion. How much would Florida have to invest in 2020 at 2% annual interest to be able to pay for the damage?

Solution We want the present value of the \$100 billion in 2020. It is

$$\text{Present value in 2020} = \frac{100}{(1.02)^{20}} = 67.297 \text{ billion dollars.}$$

To prepare for the \$100 billion expenditure, Florida would need to invest \$67.297 billion in 2020 at 2% annual interest.

Example 11 Find the present value in 2020 of a disaster costing \$200 billion in 2050. Use a 1% per year continuous interest rate.

Solution Since 2050 is 30 years in the future, we have

$$\text{Present value} = 200e^{-0.01\cdot 30} = 148.164 \text{ billion dollars.}$$

Summary for Section 1.7

- The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.
- The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced to one half of its starting quantity.
- An amount P_0 is deposited in an account paying interest at a rate of r per year. Let P be the balance in the account after t years.
    - If interest is compounded **annually**, then $P = P_0(1+r)^t$.
    - If interest is compounded **continuously**, then $P = P_0e^{rt}$, where $e = 2.71828\ldots$.
- The **future value**, B, of a payment, P, is the amount to which the P would have grown if deposited today in an interest-bearing bank account.
- The **present value**, P, of a future payment, B, is the amount that would have to be deposited in a bank account today to produce exactly B in the account at the relevant time in the future.
    - If interest is compounded annually at a rate r for t years, then

$$B = P(1+r)^t, \quad \text{or equivalently,} \quad P = \frac{B}{(1+r)^t}.$$

    - If interest is compounded continuously at a rate r for t years, then

$$B = Pe^{rt} \text{ or equivalently,} \quad P = \frac{B}{e^{rt}} = Be^{-rt}.$$

    - The rate, r, is sometimes called the **discount rate**. The present value is often denoted by PV and the future value by FV.

Problems for Section 1.7

For Problems 1–2, find k such that $p = p_0 e^{kt}$ has the given doubling time.

1. 10
2. 0.4

3. Figure 1.88 shows the balances in two bank accounts. Both accounts pay the same interest rate, but one compounds continuously and the other compounds annually. Which curve corresponds to which compounding method? What is the initial deposit in each case?

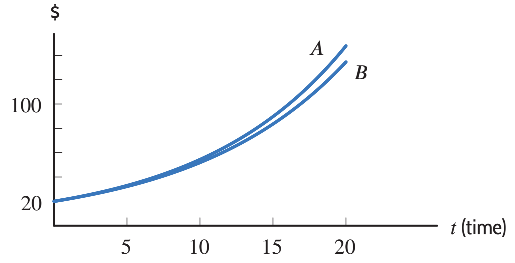


Figure 1.88

4. The exponential function $y(x) = Ce^{\alpha x}$ satisfies the conditions $y(0) = 2$ and $y(1) = 1$. Find the constants C and α. What is $y(2)$?

5. Suppose \$1000 is invested in an account paying interest at a rate of 1.5% per year. How much is in the account after 8 years if the interest is compounded

   (a) Annually? **(b)** Continuously?

6. If you deposit \$10,000 in an account earning interest at an 8% annual rate compounded continuously, how much money is in the account after five years?

7. If you need \$20,000 in your bank account in 6 years, how much must be deposited now? The interest rate is 10%, compounded continuously.

8. If \$15,000 is deposited in an account paying 1.5% interest per year, compounded continuously, how long will it take for the balance to reach \$20,000?

9. If a bank pays 1.25% per year interest compounded continuously, how long does it take for the balance in an account to double?

10. Find the doubling time of a quantity that is increasing by 7% per year.

11. Persistent organic pollutants (POPs) are a serious environmental hazard. Figure 1.89 shows their natural decay over time t, in years, in human fat.[131]

   (a) How long does it take for the concentration to decrease from 100 units to 50 units?

   (b) How long does it take for the concentration to decrease from 50 units to 25 units?

   (c) Explain why your answers to parts (a) and (b) suggest that the decay may be exponential.

   (d) Find an exponential function that models concentration, C, as a function of t.

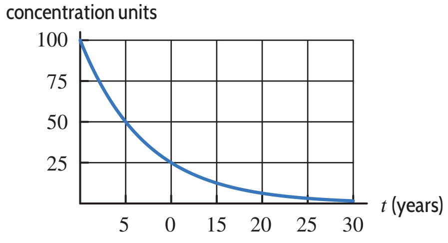


Figure 1.89

12. The half-life of nicotine in the blood is 2 hours. A person absorbs 0.4 mg of nicotine by smoking a cigarette. Fill in the table with the amount of nicotine remaining in the blood after t hours. Estimate the length of time until the amount of nicotine is reduced to 0.04 mg.

t (hours)	0	2	4	6	8	10
Nicotine (mg)	0.4					

13. World wind energy generating[132] capacity, W, was 591 gigawatts by the end of 2018 and has been increasing at a continuous rate of approximately 9.2% per year. Assume this rate continues. (Generating capacity is the maximum amount of power generated per unit time.)

   (a) Give a formula for W, in gigawatts, as a function of time, t, in years since the end of 2018.

   (b) When is wind capacity predicted to pass 1000 gigawatts?

14. Oil consumption in China grew exponentially[133] from 11.986 million barrels per day in 2015 to 13.525 million barrels per day in 2018. Assuming exponential growth continues at the same rate, what will oil consumption be in 2025?

15. From January 2014 to January 2019 the number $N(t)$ of Wikipedia articles[134] was approximated by $N(t) = N_0 e^{0.054t}$, where t is the number of days after January 1, 2014. Find the doubling time for the number of Wikipedia articles during this period.

16. You want to invest money for your child's education in a certificate of deposit (CD). You want it to be worth \$12,000 in 10 years. How much should you invest if the CD pays interest at a 9% annual rate compounded

   (a) Annually? **(b)** Continuously?

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[131] K.C. Jones, P. de Voogt, "Persistent Organic Pollutants (POPs): State of the Science," *Environmental Pollution* 100, 1999, pp. 209–221.

[132] en.wikipedia.org/wiki/Wind_power_by_country, accessed March 4, 2020.

[133] Based on www.ceicdata.com, accessed September, 2019.

[134] en.wikipedia.org/wiki/Wikipedia:Size_of_Wikipedia#Annual_growth_rate_for_the_English_Wikipedia, accessed March 4, 2020.

17. According to the EPA, sales of electronic devices in the US doubled between 1997 and 2009, when 438 million electronic devices sold.[135]

(a) Find an exponential function, $S(t)$, to model sales in millions since 1997.
(b) What was the annual percentage growth rate between 1997 and 2009?

18. In 2017, the world's population reached 7.41 billion[136] and was increasing at a rate of 1.1% per year. Assume that this growth rate remains constant. (In fact, the growth rate has been decreasing since 2008.)

(a) Write a formula for the world population (in billions) as a function of the number of years since 2017.
(b) Estimate the population of the world in the year 2025.
(c) Sketch world population as a function of years since 2017. Use the graph to estimate the doubling time of the population of the world.

19. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.
(b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.
(c) Use logarithms to find the half-life of caffeine.

20. A population, currently 200, is growing at 5% per year.

(a) Write a formula for the population, P, as a function of time, t, years in the future.
(b) Graph P against t.
(c) Estimate the population 10 years from now.
(d) Use the graph to estimate the doubling time of the population.

21. Air pressure, P, decreases exponentially with height, h, above sea level. If P_0 is the air pressure at sea level and h is in meters, then

$$P = P_0 e^{-0.00012h}.$$

(a) At the top of Denali, height 6194 meters (about 20,320 feet), what is the air pressure, as a percent of the pressure at sea level?
(b) The maximum cruising altitude of an ordinary commercial jet is around 12,000 meters (about 39,000 feet). At that height, what is the air pressure, as a percent of the sea level value?

22. The antidepressant fluoxetine (or Prozac) has a half-life of about 3 days. What percentage of a dose remains in the body after one day? After one week?

23. A firm decides to increase output at a constant relative rate from its current level of 20,000 to 30,000 units during the next five years. Calculate the annual percent rate of increase required to achieve this growth.

24. The half-life of a radioactive substance is 12 days. There are 10.32 grams initially.

(a) Write an equation for the amount, A, of the substance as a function of time.
(b) When is the substance reduced to 1 gram?

25. One of the main contaminants of a nuclear accident, such as that at Chernobyl, is strontium-90, which decays exponentially at a rate of approximately 2.4% per year.

(a) Write the percent of strontium-90 remaining, P, as a function of years, t, since the nuclear accident. [Hint: 100% of the contaminant remains at $t = 0$.]
(b) Graph P against t.
(c) Estimate the half-life of strontium-90.
(d) After the Chernobyl disaster, it was predicted that the region would not be safe for human habitation for 100 years. Estimate the percent of original strontium-90 remaining at this time.

26. The number of people living with HIV increased worldwide approximately exponentially from 2.5 million in 1985 to 36.7 million in 2015.[137] (HIV is the virus that causes AIDS.)

(a) Give a formula for the number of people living with HIV, H (in millions), as a function of years, t, since 1985. Use the form $H = H_0 e^{kt}$. Graph this function.
(b) What was the yearly continuous percent change in the number of people living with HIV between 1985 and 2015?

27. **(a)** Figure 1.90 shows exponential growth. Starting at $t = 0$, estimate the time for the population to double.
(b) Repeat part (a), but this time start at $t = 3$.
(c) Pick any other value of t for the starting point, and notice that the doubling time is the same no matter where you start.

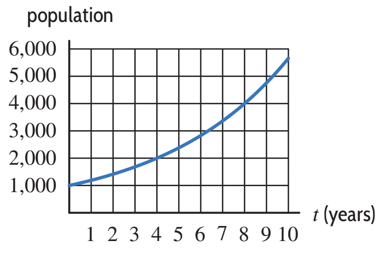

Figure 1.90

[135] http://www.epa.gov/osw/conserve/materials/ecycling/docs/summarybaselinereport2011.pdf, accessed March, 2015.
[136] www.indexmundi.com, accessed August 20, 2019.
[137] www.unaids.org/en/resources/fact-sheet, accessed March 4, 2020.

28. An exponentially growing animal population numbers 500 at time $t = 0$; two years later, it is 1500. Find a formula for the size of the population in t years and find the size of the population at $t = 5$.

29. If the quantity of a substance decreases by 4% in 10 hours, find its half-life.

30. Pregnant women metabolize some drugs at a slower rate than the rest of the population. The half-life of caffeine is about 4 hours for most people. In pregnant women, it is 10 hours.[138] (This is important because caffeine, like all psychoactive drugs, crosses the placenta to the fetus.) If a pregnant woman and her husband each have a cup of coffee containing 100 mg of caffeine at 8 am, how much caffeine does each have left in the body at 10 pm?

31. In 1923, koalas were introduced on Kangaroo Island off the coast of Australia. In 1996, the population was 5000. By 2005, the population had grown to 27,000.

(a) Assuming exponential growth, find the (continuous) rate of growth of the koala population between 1996 and 2005.
(b) Find a formula for the koala population as a function of the number of years since 1996.
(c) Use your formula to estimate the population in 2020 and compare it to the actual population of 50,000 koalas before the 2019–2020 bushfires, which killed most of the koalas on the island.[139]

32. In the Netherlands, a total of 337,245 Covid-19 cases had been confirmed by November 1, 2020. With t in days since November 1, the total number of cases, $P(t)$, was growing at 3.4% per day.[140]

(a) Find the total number of cases expected in the Netherlands by December 31, 2020 ($t = 60$) if the growth rate remained constant.
(b) Find a new growth factor a making the number of total cases on December 31, 2020 half of the number predicted in part (a).

33. The total number P of confirmed Covid-19 cases in Colombia[141] was 2979 on April 14, 2020 and growing at 5.1% per day. At this rate, after how many days would the total number of cases equal the total population of Colombia, about 50 million?[142]

34. The total number P of confirmed Covid-19 cases in Colombia[143] was growing at about 5.1% per day starting on April 14, 2020. At that rate, how long does it take P to

(a) Double?
(b) Increase by a factor of 10?

35. In spring 2020, the total number of Covid-19 cases in the US was growing exponentially with a doubling time of about 3 days.[144]

(a) Someone says that since the number of cases grows by 100% in 3 days, the daily growth rate is $100\%/3 = 33.33\%$. Explain why this is not correct by computing the 3-day growth factor of a quantity that grows 33.33% per day.
(b) Find the daily growth rate of a quantity that doubles every 3 days.

36. In the spring of 2021, the number of cases of a new Covid-19 variant from the UK were discovered to be doubling every 10 days in the US.[145] How long did it take for the number of cases to multiply by 10?

37. The population of the US was 282.2 million in 2000 and 327.2 million in 2018.[146] Assuming exponential growth,

(a) In what year is the population expected to go over 350 million?
(b) What population is predicted for the 2020 census?

38. In 2020, the world's population[147] was 7.68 billion, and the population is projected[148] to reach approximately 9 billion by the year 2037. What annual growth rate is projected?

39. A picture supposedly painted by Vermeer (1632–1675) contains 99.5% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.

40. Food bank usage in Britain grew rapidly from 2015 to 2019. The number of emergency three-day food parcels given by the Trussell Trust, in millions, is estimated to be $N(t) = 1.109e^{0.12t}$, where t is the number of years since 2015.[149]

(a) What does the 1.109 represent in this context? Give units.

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[138] From Robert M. Julien, *A Primer of Drug Action*, 7th ed., p. 159 (New York: W. H. Freeman, 1995).
[139] sagov-env.maps.arcgis.com/apps/Cascade/index.html?appid=ef29c850d8e34e98a8307dbdfe0aaaab, accessed March 27, 2021.
[140] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed February 26, 2021.
[141] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed May 12, 2020.
[142] https://en.wikipedia.org/wiki/Colombia, accessed March 4, 2021.
[143] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed May 12, 2020.
[144] Smoothed from JHU CSSE COVID-19 data at github.com/CSSEGISandData/COVID-19, accessed June 10, 2020.
[145] www.medrxiv.org/content/10.1101/2021.02.06.21251159v1.full.pdf+html and www.webmd.com/lung/news/20210208/uk-covid-variant-doubles-every-10-days-in-the-us, accessed February 11, 2021.
[146] data.worldbank.org, accessed September 10, 2019.
[147] www.indexmundi.com, accessed February 4, 2021.
[148] www.worldometers.info, accessed February 4, 2021.
[149] Estimates for the Trussell Trust: www.trusselltrust.org/stats, accessed August, 2019.

(b) What is the continuous growth rate of parcels per year?
(c) What is the annual percent growth rate of parcels per year?
(d) Using only your answer for part (c), decide if the doubling time is more or less than 1 year.

41. In November 2010, a "tiger summit" was held in St. Petersburg, Russia.[150] In 1900, there were 100,000 wild tigers worldwide; in 2010 the number was 3200.

(a) Assuming the tiger population has decreased exponentially, find a formula for $f(t)$, the number of wild tigers t years since 1900.
(b) Between 2000 and 2010, the number of wild tigers decreased by 40%. Is this percentage larger or smaller than the decrease in the tiger population predicted by your answer to part (a)?

42. Tiny marine organisms reproduce at different rates. Phytoplankton doubles in population twice a day, but foraminifera doubles every five days. If the two populations are initially the same size and grow exponentially, how long does it take for

(a) The phytoplankton population to be double the foraminifera population.
(b) The phytoplankton population to be 1000 times the foraminifera population.

43. The world population was 6.9 billion at the end of 2010 and is predicted to reach 9 billion by the end of 2050.[151]

(a) Assuming the population is growing exponentially, what is the continuous growth rate per year?
(b) The United Nations celebrated the "Day of 6 Billion" on October 12, 1999, and "Day of 7 Billion" on October 31, 2011. Using the growth rate in part (a), when is the "Day of 8 Billion" predicted to be?

44. The table gives the number of alternative fuel vehicles in the US[152] running on E85, a fuel that is up to 85% plant-derived ethanol,.

(a) By what percent did the number of E85-powered vehicles grow from 2012 to 2017?
(b) Assuming exponential growth of the form Ae^{rt}, where t is years since 2012, find r and A, and fill in the missing table values.
(c) Use this exponential model to estimate the number of E85-powered vehicles in the US in 2011.

Year	2012	2013	2014	2015	2016	2017
E85 vehicles	302,341	?	?	?	?	393,553

45. Find the future value in 8 years of a \$10,000 payment today, if the interest rate is 3% per year compounded continuously.

46. Find the future value in 15 years of a \$20,000 payment today, if the interest rate is 3.8% per year compounded continuously.

47. Find the present value of an \$8000 payment to be made in 5 years. The interest rate is 4% per year compounded continuously.

48. Find the present value of a \$20,000 payment to be made in 10 years. Assume an interest rate of 3.2% per year compounded continuously.

49. Interest is compounded annually. Consider the following choices of payments to you:

Choice 1: \$1500 now and \$3000 one year from now

Choice 2: \$1900 now and \$2500 one year from now

(a) If the interest rate on savings were 5% per year, which would you prefer?
(b) Is there an interest rate that would lead you to make a different choice? Explain.

50. The island of Manhattan was sold for \$24 in 1626. Suppose the money had been invested in an account which compounded interest continuously.

(a) How much money would be in the account in the year 2012 if the yearly interest rate was
(i) 5%? (ii) 7%?
(b) If the yearly interest rate was 6%, in what year would the account be worth one billion dollars?

51. (a) Use the Rule of 70 to predict the doubling time of an investment which is earning 8% interest per year.
(b) Find the doubling time exactly, and compare your answer to part (a).

52. A business associate who owes you \$3000 offers to pay you \$2800 now, or else pay you three yearly installments of \$1000 each, with the first installment paid now. If you use only financial reasons to make your decision, which option should you choose? Justify your answer, assuming a 3% interest rate per year, compounded continuously.

53. A person is to be paid \$2000 for work done over a year. Three payment options are being considered. Option 1 is to pay the \$2000 in full now. Option 2 is to pay \$1000 now and \$1000 in a year. Option 3 is to pay the full \$2000 in a year. Assume an annual interest rate of 5% a year, compounded continuously.

(a) Without doing any calculations, which option is the best option financially for the worker? Explain.
(b) Find the future value, in one year's time, of all three options.
(c) Find the present value of all three options.

[150] "Tigers Would be Extinct in Russia if Unprotected," Yahoo! News, November 21, 2010.
[151] "Reviewing the Bidding on the Climate Files", in About Dot Earth, *New York Times*, November 19, 2010.
[152] www.eia.gov, accessed September 18, 2019.

54. A company is considering whether to buy a new machine, which costs $97,000. The cash flows (adjusted for taxes and depreciation) that would be generated by the new machine are given in the following table:

Year	1	2	3	4
Cash flow	$50,000	$40,000	$25,000	$20,000

(a) Find the total present value of the cash flows. Treat each year's cash flow as a lump sum at the end of the year and use an interest rate of 7.5% per year, compounded annually.

(b) Based on a comparison of the cost of the machine and the present value of the cash flows, would you recommend purchasing the machine?

55. In 2017, Wil Myers and Dexter Fowler started baseball contracts reported as having almost the same value, but paying out differently.[153] Table 1.42 shows the actual cash earned in millions of dollars. Assume they get paid on January 1 of each year.

(a) If they expect an annual rate of return on investments of 4% per year, what is the present value of each contract on January 1, 2017? Whose contract is more valuable?

(b) Suppose instead they expect to earn 13%, approximately the rate of return of the stock market for 2017 to 2021. What are the present values? Does it change whose contract is more valuable?

Table 1.42

Year	2017	2018	2019	2020	2021	2022
Myers	17	2	3	7.4	20	20
Fowler	16.5	16.5	16.5	7.4	1.8	0

56. You win $38,000 in the state lottery to be paid in two installments—$19,000 now and $19,000 one year from now. A friend offers you $36,000 in return for your two lottery payments. Instead of accepting your friend's offer, you take out a one-year loan at an interest rate of 8.25% per year, compounded annually. The loan will be paid back by a single payment of $19,000 (your second lottery check) at the end of the year. Which is better, your friend's offer or the loan?

57. You are considering whether to buy or lease a machine whose purchase price is $12,000. Taxes on the machine will be $580 due in one year, $464 due in two years, and $290 due in three years. If you buy the machine, you expect to be able to sell it after three years for $5,000. If you lease the machine for three years, you make an initial payment of $2650 and then three payments of $2650 at the end of each of the next three years. The leasing company will pay the taxes. The interest rate is 7.75% per year, compounded annually. Should you buy or lease the machine? Explain.

58. You are buying a car that comes with a one-year warranty and are considering whether to purchase an extended warranty for $375. The extended warranty covers the two years immediately after the one-year warranty expires. You estimate that the yearly expenses that would have been covered by the extended warranty are $150 at the end of the first year of the extension and $250 at the end of the second year of the extension. The interest rate is 5% per year, compounded annually. Should you buy the extended warranty? Explain.

■ Problems 59–62 involve a report in the Texas Observer[154] that estimated that annual storm-related losses along the Gulf Coast attributable to climate change, particularly sea-level rise, could increase over the next decade by up to $222 million in 2030, and as much as $650 million in 2050.

59. Calculate the present value in 2020 of a single payment which would be necessary to cover the additional cost in 2030 assuming an interest rate of 2% compounded annually.

60. Calculate the present value in 2020 of a single payment which would be necessary to cover the additional cost in 2050 assuming an interest rate of 1.5% compounded continuously.

61. What interest rate would be needed for the single deposit of $150 million in 2020 to cover the additional costs in 2030 assuming interest is compounded annually?

62. What interest rate would be needed for the single deposit of $300 million in 2020 to cover the additional costs in 2050 assuming interest is compounded continuously?

63. The Congressional Budget Office (CBO) estimates that annual costs due to hurricane damage in coastal regions will increase to $39 billion by 2075, with half this cost attributed to climate change.[155] Calculate the present value in 2020 of a single-sum payment to cover the cost of the damage attributed to climate change in 2075, assuming an annual interest rate of 2%.

64. By 2070, some neighborhoods in Boston are projected to flood, resulting in $1.4 billion in damages.[156]

(a) How much money, in billions, should Boston invest in 2020 at 1% annual interest to be able to pay for the flood damage?

(b) The mayor proposed $0.28 billion be spent addressing climate change in 2020. How much of the $1.4 billion would not be covered by that amount in 2070?

[153] https://www.spotrac.com/mlb, accessed March 22, 2021. The nominal values of the contracts were $83 and $82.5 million.
[154] July 28, 2015, www.texasobserver.org/climate-change-in-texas-will-cost-businesses-billions, accessed January 10, 2020.
[155] www.thebalance.com/hurricane-damage-economic-costs-4150369, accessed January 10, 2020.
[156] https://e360.yale.edu/features/who-will-pay-for-the-huge-costs-of-holding-back-rising-seas

65. The Outer Banks are a string of linked islands off the coast of North Carolina. Rising sea levels are eroding the beach and threatening the only road. Avon, a small community, is deciding how to pay for "beach nourishing," which builds up the beach with sand, but has to be repeated every 5 years. If the nourishings cost \$12 million each and the first is in one year, what is the present value of a sequence of four beach nourishings?[157] Assume the interest rate is 1.5%, compounded annually.

66. Lourdes Hospital in Binghamton, NY, is near the Susquehanna River, which floods. In 2006, an unusually big flood closed the hospital and caused \$20 million of damage. In 2010, concrete flood walls costing \$7 million were built around the hospital. In 2011, the river flooded again and the walls held.[158] Assume that the river floods every five years (so 2011, 2016, 2021, etc.) and that the damage done without the wall is \$2.5 million for each flood.

(a) If, instead of building a wall, the \$7 million had been invested in 2010 at 2% continuous interest per year, what would the balance be in 2011 after paying the damages from that year's flood?

(b) At the 2% interest rate, when has the wall paid off financially? (Do not include the huge human costs of the flooding.) That is, for how many five-year floods would the \$7 million investment cover the damages?

67. California has had buildings floating on the sea for several decades. Since they adapt to sea level rise, such buildings may have a promising future. A large floating house is projected to have expenses of \$2400 a year; repairing the floating structure incurs a one-time cost of \$150,000. A buyer for such a house wants to know the present value at the time of purchase of the repairs and four years of expenses.[159] The costs for the repairs are due at the time of purchase and the expenses are due starting one year after the purchase. The interest rate is 1.2% compounded annually.

68. In 2006, the University of British Columbia, Canada, built a floating boathouse, which they use for sports and rent out for events.[160] The additional cost to make the building float was \$700,000. If the rental income pays off the \$700,000 in 5 years, how much do they earn each year? Assume rental income comes in at one moment during the year, starting one year after the building was finished. The interest rate is 1% per year, compounded continuously. Give an answer to the nearest dollar.

1.8 NEW FUNCTIONS FROM OLD

We have studied linear and exponential functions and the logarithm function. In this section, we learn how to create new functions by composing, stretching, and shifting functions we already know.

Composite Functions

A drop of water falls onto a paper towel. The area, A of the circular damp spot is a function of r, its radius, which is a function of time, t. We know $A = f(r) = \pi r^2$; suppose $r = g(t) = t + 1$. By substitution, we express A as a function of t:

$$A = f(g(t)) = \pi(t + 1)^2.$$

The function $f(g(t))$ is a "function of a function," or a *composite function*, in which there is an *inside function* and an *outside function*. To find $f(g(2))$, we first add one ($g(2) = 2+1 = 3$) and then square and multiply by π. We have

$$f(g(2)) = \pi(2 + 1)^2 = \underbrace{\pi 3^2}_{\text{First calculation}} = \underbrace{9\pi}_{\text{Second calculation}}.$$

The inside function is $t + 1$ and the outside function is squaring and multiplying by π. In general, the inside function represents the calculation that is done first and the outside function represents the calculation done second.

Example 1 If $f(t) = t^2$ and $g(t) = t + 2$, find

(a) $f(t + 1)$ (b) $f(t) + 3$ (c) $f(t + h)$ (d) $f(g(t))$ (e) $g(f(t))$

[157] https://news.yahoo.com/tiny-town-big-decision-willing-121315761.html, accessed March 15, 2021.

[158] floodbreak.com/wp-content/uploads/2012/08/FEMA_FloodMitigationBestPractice_Lourdes-Hospital_Detail-copy.pdf, accessed March 15, 2021.

[159] https://www.marincounty.com/sausalito/floating-homes/ and www.zillow.com, accessed March 18, 2021. Cost estimate based on K. Wright, J. Koo, and A. Belden, *Enhancing Resilience in Boston*, 2015.

[160] https://sportfacilities.ubc.ca/ubc-boathouse/history/, accessed March 17, 2021. Cost estimate based on K. Wright, J. Koo, and A. Belden, *Enhancing Resilience in Boston*, 2015.

Solution

(a) Since $t+1$ is the inside function, $f(t+1) = (t+1)^2$.
(b) Here 3 is added to $f(t)$, so $f(t)+3 = t^2+3$.
(c) Since $t+h$ is the inside function, $f(t+h) = (t+h)^2$.
(d) Since $g(t) = t+2$, substituting $t+2$ into f gives $f(g(t)) = f(t+2) = (t+2)^2$.
(e) Since $f(t) = t^2$, substituting t^2 into g gives $g(f(t)) = g(t^2) = t^2+2$.

Example 2 If $f(x) = e^x$ and $g(x) = 5x+1$, find (a) $f(g(x))$ (b) $g(f(x))$

Solution

(a) Substituting $g(x) = 5x+1$ into f gives $f(g(x)) = f(5x+1) = e^{5x+1}$.
(b) Substituting $f(x) = e^x$ into g gives $g(f(x)) = g(e^x) = 5e^x+1$.

Example 3 Using the following table, find $g(f(0))$, $f(g(0))$, $f(g(1))$, and $g(f(1))$.

x	0	1	2	3
$f(x)$	3	1	−1	−3
$g(x)$	0	2	4	6

Solution

To find $g(f(0))$, we first find $f(0) = 3$ from the table. Then we have $g(f(0)) = g(3) = 6$. For $f(g(0))$, we must find $g(0)$ first. Since $g(0) = 0$, we have $f(g(0)) = f(0) = 3$. Similar reasoning leads to $f(g(1)) = f(2) = -1$ and $g(f(1)) = g(1) = 2$.

We can write a composite function using a new variable u to represent the value of the inside function. For example,

$$y = (t+1)^4 \quad \text{is the same as} \quad y = u^4 \quad \text{with} \quad u = t+1.$$

Other expressions for u, such as $u = (t+1)^2$, with $y = u^2$, are also possible.

Example 4 Use a new variable u for the inside function to express each of the following as a composite function:
(a) $y = \ln(3t)$ (b) $w = 5(2r+3)^2$ (c) $P = e^{-0.03t}$

Solution

(a) We take the inside function to be $3t$, so $y = \ln u$ with $u = 3t$.
(b) We take the inside function to be $2r+3$, so $w = 5u^2$ with $u = 2r+3$.
(c) We take the inside function to be $-0.03t$, so $P = e^u$ with $u = -0.03t$.

Stretches of Graphs

If the demand function is linear, the graph of a possible revenue function $R = f(p)$ is in Figure 1.91. What does the graph of $R = 3f(p)$ look like? The factor 3 in the function $R = 3f(p)$ stretches each $f(p)$ revenue value by multiplying it by 3. See Figure 1.92. If c is positive, the graph of $R = cf(p)$ is the graph of $R = f(p)$ stretched or shrunk vertically by c units. If c is negative, the function no longer makes sense as a revenue function, but we can still draw the graph. What does the graph of $R = -2f(p)$ look like? The factor -2 in the function $R = -2f(p)$ stretches $f(p)$ by multiplying by 2 and reflecting it across the x-axis. See Figure 1.92.

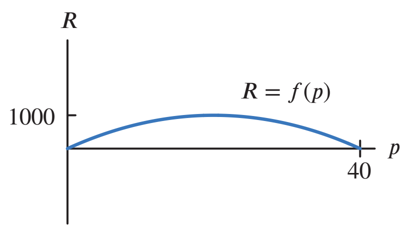


Figure 1.91: Graph of $f(p)$

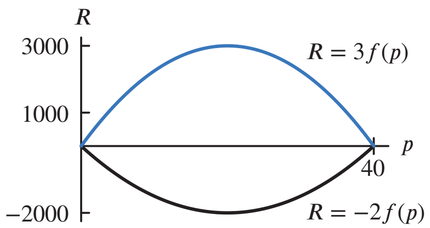


Figure 1.92: Multiples of the function $f(p)$

Multiplying a function by a constant, c, stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$). A negative sign (if $c < 0$) reflects the graph across the x-axis, in addition to shrinking or stretching.

Shifted Graphs

Consider the function $y = x^2 + 4$. The y-coordinates for this function are exactly 4 units larger than the corresponding y-coordinates of the function $y = x^2$. So the graph of $y = x^2 + 4$ is obtained from the graph of $y = x^2$ by adding 4 to the y-coordinate of each point, that is, by moving the graph of $y = x^2$ up 4 units. (See Figure 1.93.)

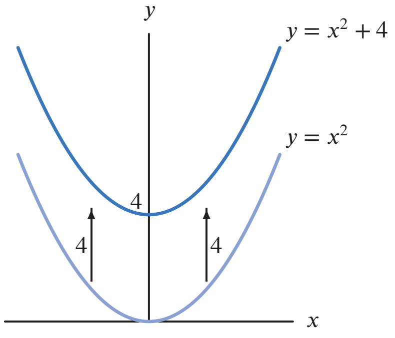


Figure 1.93: Vertical shift: Graphs of $y = x^2$ and $y = x^2 + 4$

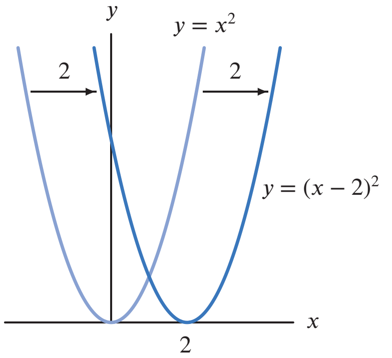


Figure 1.94: Horizontal shift: Graphs of $y = x^2$ and $y = (x - 2)^2$

A graph can also be shifted to the left or to the right. In Figure 1.94, we see that the graph of $y = (x - 2)^2$ is the graph of $y = x^2$ shifted to the right 2 units. In general,

- The graph of $y = f(x) + k$ is the graph of $y = f(x)$ moved up k units (down if k is negative).
- The graph of $y = f(x - k)$ is the graph of $y = f(x)$ moved to the right k units (to the left if k is negative).

Example 5 (a) A cost function, $C(q)$, for a company is shown in Figure 1.95. The fixed cost increases by \$1000. Sketch a graph of the new cost function.
(b) A supply curve, S, for a product is given in Figure 1.96. A new factory opens and produces 100 units of the product no matter what the price. Sketch a graph of the new supply curve.

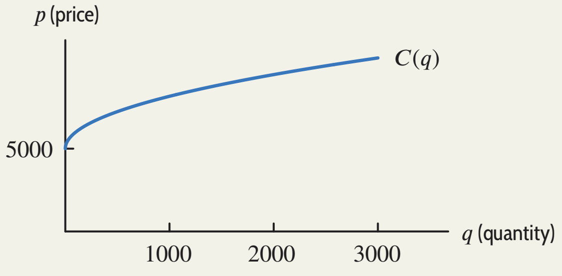


Figure 1.95: A cost function

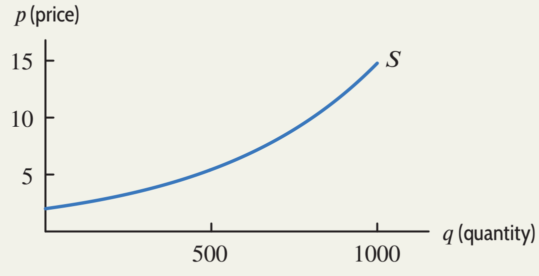


Figure 1.96: A supply function

Solution

(a) For each quantity, the new cost is \$1000 more than the old cost. The new cost function is $C(q)+1000$, whose graph is the graph of $C(q)$ shifted vertically up 1000 units. (See Figure 1.97.)

(b) To see the effect of the new factory, look at an example. At a price of 10 dollars, approximately 800 units are currently produced. With the new factory, this amount increases by 100 units, so the new amount produced is 900 units. At each price, the quantity produced increases by 100, so the new supply curve is S shifted horizontally to the right by 100 units. (See Figure 1.98.)

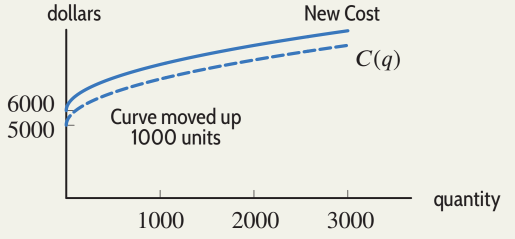


Figure 1.97: New cost function (original curve dashed)

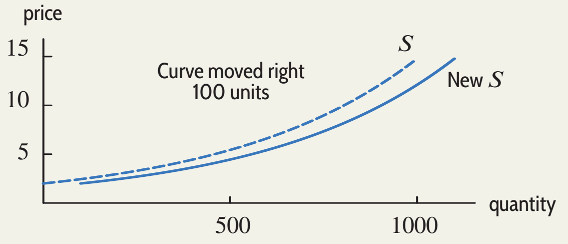


Figure 1.98: New supply curve (original curve dashed)

Summary for Section 1.8

- **Composition of functions:** The function $f(g(t))$ is a "function of a function," or a **composite function**, in which there is an **inside function** and an **outside function**.
- **Stretches of graphs:** Multiplying a function by a constant, c,
  - stretches the graph vertically (if $c > 1$) or
  - shrinks the graph vertically (if $0 < c < 1$).
  - A negative sign (if $c < 0$) reflects the graph across the x-axis, in addition to shrinking or stretching.
- **Shifted graphs:**
  - The graph of $y = f(x)+k$ is the graph of $y = f(x)$ moved up k units (down if k is negative).
  - The graph of $y = f(x-k)$ is the graph of $y = f(x)$ moved to the right k units (to the left if k is negative).

Problems for Section 1.8

In Problems 1–3, find the following:

(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(f(x))$

1. $f(x) = 5x - 1$ and $g(x) = 3x + 2$
2. $f(x) = x - 2$ and $g(x) = x^2 + 8$
3. $f(x) = 3x$ and $g(x) = e^{2x}$
4. Let $f(x) = x^2$ and $g(x) = 3x - 1$. Find the following:

   (a) $f(2) + g(2)$ **(b)** $f(2) \cdot g(2)$
   (c) $f(g(2))$ **(d)** $g(f(2))$

5. For $g(x) = x^2 + 2x + 3$, find and simplify:

   (a) $g(2 + h)$ **(b)** $g(2)$
   (c) $g(2 + h) - g(2)$

6. If $f(x) = x^2 + 1$, find and simplify:

   (a) $f(t + 1)$ **(b)** $f(t^2 + 1)$ **(c)** $f(2)$
   (d) $2f(t)$ **(e)** $(f(t))^2 + 1$

For the functions f and g in Problems 7–10, find

(a) $f(g(1))$ **(b)** $g(f(1))$ **(c)** $f(g(x))$
(d) $g(f(x))$ **(e)** $f(t)g(t)$

7. $f(x) = x^2$, $g(x) = x + 1$
8. $f(x) = \sqrt{x + 4}$, $g(x) = x^2$
9. $f(x) = e^x$, $g(x) = x^2$
10. $f(x) = 1/x$, $g(x) = 3x + 4$
11. Use Table 1.43 to find:

    (a) $f(g(1))$ **(b)** $g(f(1))$ **(c)** $f(g(4))$
    (d) $g(f(4))$ **(e)** $f(g(6))$ **(f)** $g(f(6))$

Table 1.43

x	1	2	3	4	5	6
$f(x)$	5	4	3	3	4	5
$g(x)$	6	5	4	3	2	1

12. Use Table 1.44 to find:

(a) $f(g(0))$ (b) $f(g(1))$ (c) $f(g(2))$
(d) $g(f(2))$ (e) $g(f(3))$

Table 1.44

x	0	1	2	3	4	5
$f(x)$	10	6	3	4	7	11
$g(x)$	2	3	5	8	12	15

13. Make a table of values for each of the following functions using Table 1.44:

(a) $f(x)+3$ (b) $f(x-2)$ (c) $5g(x)$
(d) $-f(x)+2$ (e) $g(x-3)$ (f) $f(x)+g(x)$

14. Use the variable u for the inside function to express each of the following as a composite function:

(a) $y=(5t^2-2)^6$ (b) $P=12e^{-0.6t}$
(c) $C=12\ln(q^3+1)$

15. Use the variable u for the inside function to express each of the following as a composite function:

(a) $y=2^{3x-1}$ (b) $P=\sqrt{5t^2+10}$
(c) $w=2\ln(3r+4)$

■ Simplify the quantities in Problems 16–19 using $m(z)=z^2$.

16. $m(z+1)-m(z)$ **17.** $m(z+h)-m(z)$

18. $m(z)-m(z-h)$ **19.** $m(z+h)-m(z-h)$

■ For Problems 20–25, use the graphs in Figure 1.99.

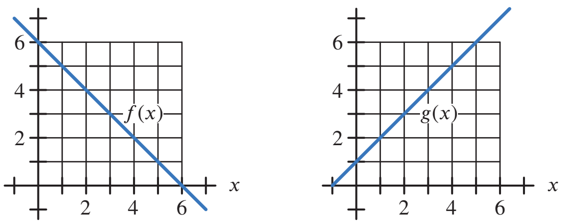


Figure 1.99

20. Estimate $f(g(1))$. **21.** Estimate $g(f(1))$.

22. Estimate $f(g(4))$. **23.** Estimate $g(f(4))$.

24. Estimate $f(f(2))$. **25.** Estimate $g(g(2))$.

■ For Problems 26–29, use the graphs in Figure 1.100.

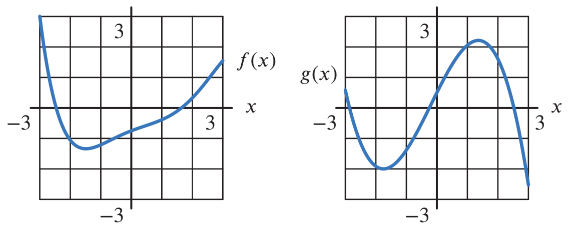


Figure 1.100

26. Estimate $f(g(1))$. **27.** Estimate $g(f(2))$.

28. Estimate $f(f(1))$. **29.** Estimate $f(g(3))$.

30. Using Table 1.45, create a table of values for $f(g(x))$ and for $g(f(x))$.

Table 1.45

x	−3	−2	−1	0	1	2	3
$f(x)$	0	1	2	3	2	1	0
$g(x)$	3	2	2	0	−2	−2	−3

31. A tree of height y meters has, on average, B branches, where $B=y-1$. Each branch has, on average, n leaves, where $n=2B^2-B$. Find the average number of leaves on a tree as a function of height.

■ In Problems 32–35, use Figure 1.101 to estimate the function value or explain why it cannot be done.

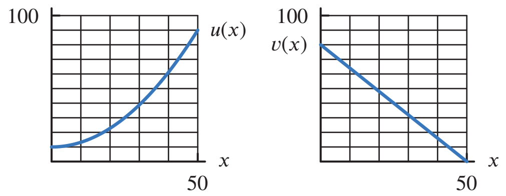


Figure 1.101

32. $u(v(10))$ **33.** $u(v(40))$

34. $v(u(10))$ **35.** $v(u(40))$

36. The Heaviside step function, H, is graphed in Figure 1.102. Graph the following functions.

(a) $2H(x)$ (b) $H(x)+1$ (c) $H(x+1)$
(d) $-H(x)$ (e) $H(-x)$

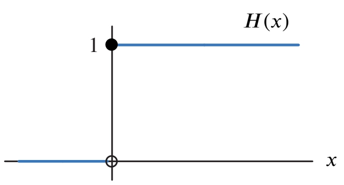


Figure 1.102

■ In Problems 37–42, use Figure 1.103 to graph the function.

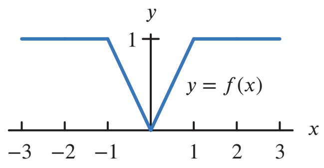


Figure 1.103

37. $y = f(x) + 1$

38. $y = f(x - 2)$

39. $y = 3f(x)$

40. $y = f(x + 1) - 2$

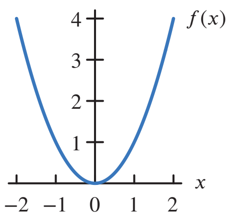
41. $y = -f(x) + 3$

42. $y = -2f(x - 1)$

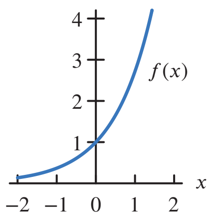
■ For the functions f in Problems 43–45, graph:

(a) $f(x + 2)$ (b) $f(x - 1)$ (c) $f(x) - 4$
(d) $f(x + 1) + 3$ (e) $3f(x)$ (f) $-f(x) + 1$

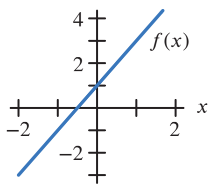
43.



44.



45.



■ In Problems 46–51, use Figure 1.104 to graph the function.

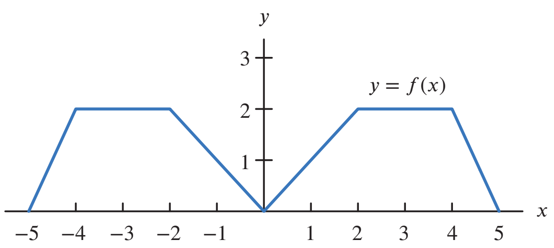


Figure 1.104

46. $y = f(x) + 2$

47. $y = 2f(x)$

48. $y = f(x - 1)$

49. $y = -3f(x)$

50. $y = 2f(x) - 1$

51. $y = 2 - f(x)$

52. Figure 1.105 shows the concentration of a drug in the body, $f(t) = a - be^{kt}$, over time, t:

(a) What does the constant a represent in this context?
(b) What is the sign of the constant a?
(c) What is the relation between the constants a and b?
(d) What is the sign of the constant k?

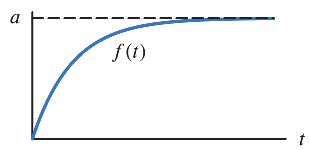


Figure 1.105

53. Morphine, a pain-relieving drug, is administered to a patient intravenously starting at 8 am. The drug saturation curve $Q = f(t)$ in Figure 1.106 gives the quantity, Q, of morphine in the blood t hours after 8 am.

(a) Draw the drug saturation curve $Q = g(t)$ if the IV line is started at noon instead of 8 am.
(b) Is g one of the following transformations of f: vertical shift, vertical stretch, horizontal shift, horizontal stretch? If so, which?
(c) Write $g(t)$ in terms of the function f.

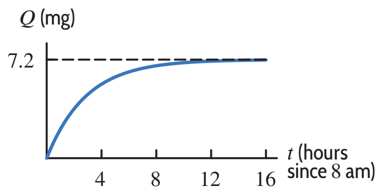


Figure 1.106

54. (a) Write an equation for a graph obtained by vertically stretching the graph of $y = x^2$ by a factor of 2, followed by a vertical upward shift of 1 unit. Sketch it.
(b) What is the equation if the order of the transformations (stretching and shifting) in part (a) is interchanged?
(c) Are the two graphs the same? Explain the effect of reversing the order of transformations.

■ In Problems 55–58 the functions $r = f(t)$ and $V = g(r)$ give the radius and the volume of a commercial hot air balloon being inflated for testing. The variable t is in minutes, r is in feet, and V is in cubic feet. The inflation begins at $t = 0$. In each case, give a mathematical expression that represents the given statement.

55. The volume of the balloon t minutes after inflation began.

56. The volume of the balloon if its radius were twice as big.

57. The time that has elapsed when the radius of the balloon is 30 feet.

58. The time that has elapsed when the volume of the balloon is 10,000 cubic feet.

59. Cyanide is used in solution to isolate gold in a mine.[161] This may result in contaminated groundwater near the mine, which needs to be filtered. Table 1.46 gives the concentration, $c(t)$ (in parts per million), of cyanide in the groundwater, where t is in years since 2017.

(a) Find an exponential model for $c(t)$.
(b) Use the model in part (a) to find the number of years it takes for the cyanide concentration to fall to 10 ppm.
(c) If the filtering process removing the cyanide is sped up so that the new model is $D(t) = c(2t)$, find $D(t)$.
(d) If the cyanide removal was started three years earlier, but run at the speed of part (a), find a new model, $E(t)$.

Table 1.46

t (years)	0	1	2
$c(t)$ (ppm)	25.0	21.8	19.01

60. **(a)** Let $P(t)$ be the total number of cases confirmed up to and including day t of an infectious disease outbreak. If every confirmed case goes into quarantine immediately for exactly 10 days, why is the number of people in quarantine on day t given by

$$\text{People in quarantine} = P(t) - P(t-10)?$$

(b) How does the formula in part (a) change if only a fraction r of total cases goes into quarantine and the length of the quarantine is s days?

61. Let $N(t)$ be the number of new Covid-19 cases in a community confirmed on day t. Let $h(t)$ be the number of Covid-19 patients who spent the preceding night in the hospital. In the US during May 2020, 14% of all newly confirmed cases were hospitalized.[162] Assume all hospital admissions are on the day of diagnosis and each patient stays s days. Explain why

$$h(t+1) = h(t) + 0.14N(t) - 0.14N(t-s).$$

■ Johns Hopkins University reports Covid-19 statistics daily for every US state and many countries in the world.[163] In Problems 62–63, we are concerned with these statistics for some region. Each one is a function of time t, in days:

- $P(t)$: Cumulative Confirmed Cases: total number of cases confirmed up to and including day t.
- $N(t)$: Daily New Cases: number of cases first confirmed on day t.
- $R(t)$: Cumulative Recovered or Removed: total number of people who were confirmed to have Covid-19 but have recovered or have passed away up to and including day t.

62. **(a)** What is the meaning of $P(t-1)$? Answer in words.
(b) How would you figure out $N(t)$ if you knew $P(t)$ and $P(t-1)$? Write an equation connecting these three quantities. Explain why it is true.
(c) What is the meaning of $N(t-1)$? Of $N(t-2)$? Answer in words.

63. Many communities are interested in the active cases, $A(t)$, the number of people who are infected and have not recovered on day t. This gives them an idea of how many people may be infectious on a given day and of the hospital capacity they may need.

(a) How can you find $A(t)$ from $P(t)$ and one of the other functions? Give a formula for $A(t)$. Explain.
(b) Use the fact that most Covid-19 cases last for 14 days and assume that people are infectious for the whole 14 days. Write a formula for $A(t)$ in terms of the function $N(t)$ (and no other functions).

1.9 PROPORTIONALITY AND POWER FUNCTIONS

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, if apples are \$1.40 a pound, we say the price you pay, p dollars, is proportional to the weight you buy, w pounds, because

$$p = f(w) = 1.40w.$$

As another example, the area, A, of a circle is proportional to the square of the radius, r:

$$A = f(r) = \pi r^2.$$

[161] www.waterboards.ca.gov, accessed September 16, 2019.
[162] Reported by the CDC at https://www.cdc.gov/mmwr/volumes/69/wr/mm6924e2.htm.
[163] JHU CSSE COVID-19 data at https://github.com/CSSEGISandData/COVID-19.

We say y is (directly) **proportional** to x if there is a nonzero constant k such that

$$y = kx.$$

This k is called the constant of proportionality.

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v, at which you make a 50-mile trip is inversely proportional to the time, t, taken, because v is proportional to $1/t$:

$$v = 50\left(\frac{1}{t}\right) = \frac{50}{t}.$$

Notice that if y is directly proportional to x, then the magnitude of one variable increases (decreases) when the magnitude of the other increases (decreases). If, however, y is inversely proportional to x, then the magnitude of one variable increases when the magnitude of the other decreases.

Example 1 The heart mass of a mammal is proportional to its body mass.[164]

(a) Write a formula for heart mass, H, as a function of body mass, B.
(b) A human with a body mass of 70 kilograms has a heart mass of 0.42 kilograms. Use this information to find the constant of proportionality.
(c) Estimate the heart mass of a horse with a body mass of 650 kg.

Solution

(a) Since H is proportional to B, for some constant k, we have

$$H = kB.$$

(b) We use the fact that $H = 0.42$ when $B = 70$ to solve for k:

$$\begin{aligned} H &= kB \\ 0.42 &= k(70) \\ k &= \frac{0.42}{70} = 0.006. \end{aligned}$$

(c) Since $k = 0.006$, we have $H = 0.006B$, so the heart mass of the horse is given by

$$H = 0.006(650) = 3.9 \text{ kilograms.}$$

Example 2 The period of a pendulum, T, is the amount of time required for the pendulum to make one complete swing. For small swings, the period, T, is approximately proportional to the square root of l, the pendulum's length. So

$$T = k\sqrt{l} \quad \text{where } k \text{ is a constant.}$$

Notice that T is not directly proportional to l, but T is proportional to $\sqrt{l}$.

Example 3 An object's weight, w, is inversely proportional to the square of its distance, r, from the earth's center. So, for some constant k,

$$w = \frac{k}{r^2}.$$

Here w is not inversely proportional to r, but to r^2.

[164] K. Schmidt-Nielson, *Scaling—Why is Animal Size So Important*? (Cambridge: CUP, 1984).

Power Functions

In each of the previous examples, one quantity is proportional to the power of another quantity. We make the following definition:

> We say that $Q(x)$ is a **power function** of x if $Q(x)$ is proportional to a constant power of x. If k is the constant of proportionality, and if p is the power, then
> $$Q(x) = k \cdot x^p.$$

For example, the function $H = 0.006B$ is a power function with $p = 1$. The function $T = k\sqrt{l} = kl^{1/2}$ is a power function with $p = 1/2$, and the function $w = k/r^2 = kr^{-2}$ is a power function with $p = -2$.

Example 4 Which of the following are power functions? For those which are, write the function in the form $y = kx^p$, and give the coefficient k and the exponent p.

(a) $y = \dfrac{5}{x^3}$ (b) $y = \dfrac{2}{3x}$ (c) $y = \dfrac{5x^2}{2}$

(d) $y = 5 \cdot 2^x$ (e) $y = 3\sqrt{x}$ (f) $y = (3x^2)^3$

Solution

(a) Since $y = 5x^{-3}$, this is a power function with $k = 5$ and $p = -3$.
(b) Since $y = (2/3)x^{-1}$, this is a power function with $k = 2/3$ and $p = -1$.
(c) Since $y = (5/2)x^2$, this is a power function with $k = 5/2 = 2.5$ and $p = 2$.
(d) This is not a power function. It is an exponential function.
(e) Since $y = 3x^{1/2}$, this is a power function with $k = 3$ and $p = 1/2$.
(f) Since $y = 3^3 \cdot (x^2)^3 = 27x^6$, this is a power function with $k = 27$ and $p = 6$.

Graphs of Power Functions

The graph of $y = x^2$ is shown in Figure 1.107. It is decreasing for negative x and increasing for positive x. Notice that it is bending upward, or concave up, for all x. The graph of $y = x^3$ is shown in Figure 1.108. Notice that it is bending downward, or concave down for negative x and bending upward, or concave up for positive x. The graph of $y = \sqrt{x} = x^{1/2}$ is shown in Figure 1.109. Notice that the graph is increasing and concave down.

Since x^2 increases without bound as x increases, we often say that it tends to infinity as x approaches infinity, which we write in symbols as

$$x^2 \to \infty \quad \text{as} \quad x \to \infty.$$

Since x^3 decreases without bound as x decreases, we write

$$x^3 \to -\infty \quad \text{as} \quad x \to -\infty.$$

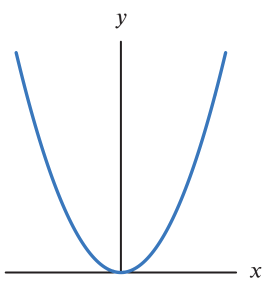


Figure 1.107: Graph of $y = x^2$

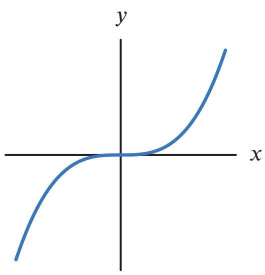


Figure 1.108: Graph of $y = x^3$

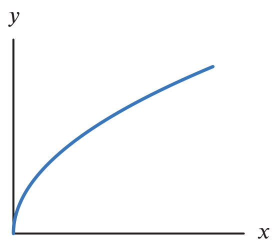


Figure 1.109: Graph of $y = x^{1/2}$

Example 5 If N is the average number of species found on an island and A is the area of the island, observations have shown[165] that N is approximately proportional to the cube root of A. Write a formula for N as a function of A and describe the shape of the graph of this function.

Solution For some positive constant k, we have

$$N = k\sqrt[3]{A} = kA^{1/3}.$$

It turns out that the value of k depends on the region of the world in which the island is found. The graph of N against A (for $A > 0$) has a shape similar to the graph in Figure 1.109. It is increasing and concave down. Thus, larger islands have more species on them (as we would expect), but the increase slows as the island gets larger.

The function $y = x^0 = 1$ has a graph that is a horizontal line. For negative powers, rewriting

$$y = x^{-1} = \frac{1}{x} \quad \text{and} \quad y = x^{-2} = \frac{1}{x^2}$$

makes it clear that as $x > 0$ increases, the denominators increase and the functions decrease. The graphs of $y = x^{-1}$ and $y = x^{-2}$ have both the x- and y-axes as asymptotes. (See Figure 1.110.)

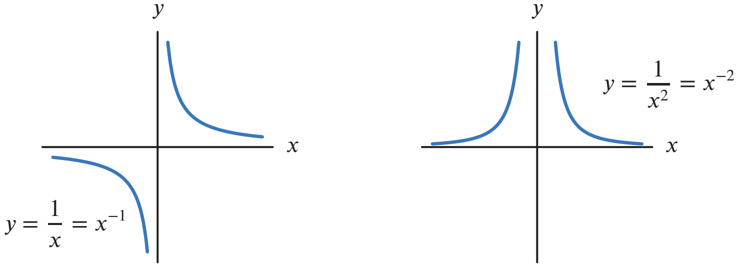


Figure 1.110: Graphs of negative powers of x

Quadratic Functions and Polynomials

Sums of power functions with nonnegative integer exponents are called *polynomials*. With a_n, a_{n-1}, $\ldots$, a_0 constants, polynomials are functions of the form

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Here, n is a nonnegative integer, called the *degree* of the polynomial, and a_n is a nonzero number called the *leading coefficient*. We call $a_n x^n$ the *leading term*.

If $n = 2$, the polynomial is called *quadratic* and has the form $ax^2 + bx + c$ with $a \neq 0$. The graph of a quadratic polynomial is a parabola. It opens up if the leading coefficient a is positive and opens down if a is negative.

Example 6 A company finds that the average number of people attending a concert is 75 if the price is \$50 per person. At a price of \$35 per person, the average number of people in attendance is 120.

(a) Assume that the demand curve is a line. Write the demand, q, as a function of price, p.
(b) Use your answer to part (a) to write the revenue, R, as a function of price, p.
(c) Use a graph of the revenue function to determine what price should be charged to obtain the greatest revenue.

Solution (a) Two points on the line are $(p, q) = (50, 75)$ and $(p, q) = (35, 120)$. The slope of the line is

$$m = \frac{120 - 75}{35 - 50} = \frac{45}{-15} = -3 \text{ people/dollar.}$$

[165] *Scientific American*, p. 112 (September, 1989).

To find the vertical intercept of the line, we use the slope and one of the points:

$$75 = b + (-3)(50)$$
$$225 = b.$$

The demand function is $q = 225 - 3p$.

(b) Since $R = pq$ and $q = 225 - 3p$, we see that $R = p(225 - 3p) = 225p - 3p^2$.

(c) The revenue function is the quadratic polynomial graphed in Figure 1.111. The maximum revenue occurs at $p = 37.5$. Thus, the company maximizes revenue by charging \$37.50 per person.

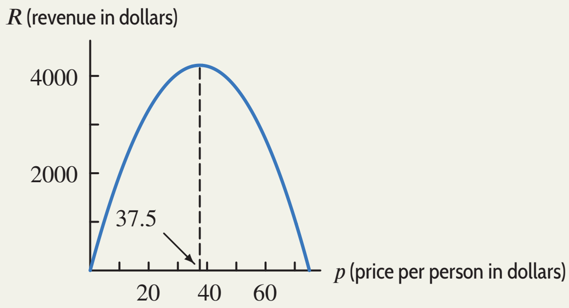


Figure 1.111: Revenue function for concert ticket sales

Summary for Section 1.9

- We say y is (directly) **proportional** to x if there is a nonzero constant k such that $y = kx$. This k is called the **constant of proportionality**.
- We also say that one quantity is **inversely proportional** to another if one is proportional to the reciprocal of the other.
- **Power functions:** We say that $Q(x)$ is a **power function** of x if $Q(x)$ is proportional to a constant power of x. If k is the constant of proportionality and if p is the power, then $Q(x) = k \cdot x^p$.
- **Quadratic functions and polynomials:** Sums of power functions with nonnegative integer exponents are called *polynomials*. With $a_n, a_{n-1}, \dots, a_0$ constants, polynomials are functions of the form
$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$
  - n is a nonnegative integer, called the **degree** of the polynomial.
  - a_n is a nonzero number called the **leading coefficient**.
  - $a_n x^n$ is the **leading term**.
  - If $n = 2$, the polynomial is called **quadratic** and has the form $ax^2 + bx + c$ with $a \neq 0$.

Problems for Section 1.9

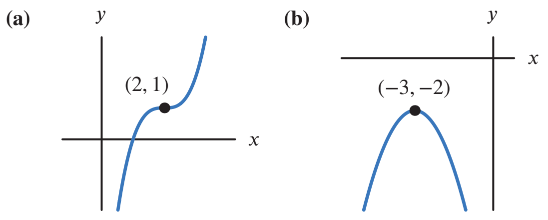
In Problems 1–12, determine whether or not the function is a power function. If it is a power function, write it in the form $y = kx^p$ and give the values of k and p.

1. $y = \dfrac{x}{5}$
2. $y = 5\sqrt{x}$
3. $y = \dfrac{8}{x}$
4. $y = \dfrac{3}{x^2}$
5. $y = 2^x$
6. $y = \dfrac{3}{8x}$
7. $y = (3x^5)^2$
8. $y = \dfrac{5}{2\sqrt{x}}$
9. $y = 3 \cdot 5^x$
10. $y = \dfrac{2x^2}{10}$
11. $y = (5x)^3$
12. $y = 3x^2 + 4$

In Problems 13–16, write a formula representing the function.

13. The strength, S, of a beam is proportional to the square of its thickness, h.
14. The energy, E, expended by a swimming dolphin is proportional to the cube of the speed, v, of the dolphin.
15. The average velocity, v, for a trip over a fixed distance, d, is inversely proportional to the time of travel, t.
16. The gravitational force, F, between two bodies is inversely proportional to the square of the distance d between them.

17. Use shifts of power functions to find a possible formula for each of the graphs:



18. The surface area of a mammal, S, satisfies the equation $S = kM^{2/3}$, where M is the body mass and the constant of proportionality k depends on the body shape of the mammal. A human of body mass 70 kilograms has surface area 18,600 cm^2. Find the constant of proportionality for humans. Find the surface area of a human with body mass 60 kilograms.

19. The number of species of lizards, N, found on an island off Baja California is proportional to the fourth root of the area, A, of the island.[166] Write a formula for N as a function of A. Graph this function. Is it increasing or decreasing? Is the graph concave up or concave down? What does this tell you about lizards and island area?

20. The blood mass of a mammal is proportional to its body mass. A rhinoceros with body mass 3000 kilograms has blood mass of 150 kilograms. Find a formula for the blood mass of a mammal as a function of the body mass and estimate the blood mass of a human with body mass 70 kilograms.

21. Kleiber's Law states that the metabolic needs (such as calorie requirements) of a mammal are proportional to its body weight raised to the 0.75 power.[167] Surprisingly, the daily diets of mammals conform to this relation well. Assuming Kleiber's Law holds:

(a) Write a formula for C, daily calorie consumption, as a function of body weight, W.
(b) Sketch a graph of this function. (You do not need scales on the axes.)
(c) If a human weighing 150 pounds needs to consume 1800 calories a day, estimate the daily calorie requirement of a horse weighing 700 lbs and of a rabbit weighing 9 lbs.
(d) On a per-pound basis, which animal requires more calories: a mouse or an elephant?

22. Allometry is the study of the relative size of different parts of a body as a consequence of growth. In this problem, you will check the accuracy of an allometric equation: the weight of a fish is proportional to the cube of its length.[168] Table 1.47 relates the weight, y, in gm, of plaice (a type of fish) to its length, x, in cm. Does this data support the hypothesis that (approximately) $y = kx^3$? If so, estimate the constant of proportionality, k.

Table 1.47

x	y	x	y	x	y
33.5	332	37.5	455	41.5	623
34.5	363	38.5	500	42.5	674
35.5	391	39.5	538	43.5	724
36.5	419	40.5	574		

23. Biologists estimate that the number of animal species of a certain body length is inversely proportional to the square of the body length.[169] Write a formula for the number of animal species, N, of a certain body length as a function of the length, L. Are there more species at large lengths or at small lengths? Explain.

24. The specific heat, s, of an element is the number of calories of heat required to raise the temperature of one gram of the element by one degree Celsius. Use the following table to decide if s is proportional or inversely proportional to the atomic weight, w, of the element. If so, find the constant of proportionality.

Element	Li	Mg	Al	Fe	Ag	Pb	Hg
w	6.9	24.3	27.0	55.8	107.9	207.2	200.6
s	0.92	0.25	0.21	0.11	0.056	0.031	.033

25. The circulation time of a mammal (that is, the average time it takes for all the blood in the body to circulate once and return to the heart) is proportional to the fourth root of the body mass of the mammal.

(a) Write a formula for the circulation time, T, in terms of the body mass, B.
(b) If an elephant of body mass 5230 kilograms has a circulation time of 148 seconds, find the constant of proportionality.
(c) What is the circulation time of a human with body mass 70 kilograms?

[166] M. L. Rosenzweig, *Species Diversity in Space and Time*, p. 143 (Cambridge: Cambridge University Press, 1995).
[167] S. Strogatz, "Math and the City," *The New York Times,* May 20, 2009. Kleiber originally estimated the exponent as 0.74; it is now believed to be 0.75.
[168] Adapted from R. J. H. Beverton and S. J. Holt, "On the Dynamics of Exploited Fish Populations", *Fishery Investigations*, Series II, 19, 1957.
[169] *US News & World Report*, August 18, 1997, p. 79.

26. Zipf's Law, developed by George Zipf in 1949, states that in a given country, the population of a city is inversely proportional to the city's rank by size in the country.[170] Assuming Zipf's Law:

(a) Write a formula for the population, P, of a city as a function of its rank, R.
(b) If the constant of proportionality k is 300,000, what is the approximate population of the largest city (rank 1)? The second largest city (rank 2)? The third largest city?
(c) Answer the questions of part (b) if $k = 6$ million.
(d) Interpret the meaning of the constant of proportionality k in this context.

27. The infrastructure needs of a region (for example, the number of miles of electrical cable, the number of miles of roads, the number of gas stations) depend on its population. Cities enjoy economies of scale.[170] For example, the number of gas stations is proportional to the population raised to the power of 0.77.

(a) Write a formula for the number, N, of gas stations in a city as a function of the population, P, of the city.
(b) If city A is 10 times bigger than city B, how do their number of gas stations compare?
(c) Which is expected to have more gas stations per person, a town of 10,000 people or a city of 500,000 people?

28. A sporting goods wholesaler finds that when the price of a product is \$25, the company sells 500 units per week. When the price is \$30, the number sold per week decreases to 460 units.

(a) Find the demand, q, as a function of price, p, assuming that the demand curve is linear.
(b) Use your answer to part (a) to write revenue as a function of price.
(c) Graph the revenue function in part (b). Find the price that maximizes revenue. What is the revenue at this price?

29. A health club has cost and revenue functions given by $C = 10{,}000 + 35q$ and $R = pq$, where q is the number of annual club members and p is the price of a one-year membership. The demand function for the club is $q = 3000 - 20p$.

(a) Use the demand function to write cost and revenue as functions of p.
(b) Graph cost and revenue as a function of p, on the same axes. (Note that price does not go above \$170 and that the annual costs of running the club reach \$120,000.)
(c) Explain why the graph of the revenue function has the shape it does.
(d) For what prices does the club make a profit?
(e) Estimate the annual membership fee that maximizes profit. Mark this point on your graph.

1.10 PERIODIC FUNCTIONS

What Are Periodic Functions?

Many functions have graphs that oscillate, resembling a wave. Figure 1.112 shows the number of new housing construction starts (one-family units) in the US, 2016–2019, where t is time in quarter-years.[171] Notice that few new homes begin construction during the first quarter of a year (January, February, and March), whereas many new homes are begun in the second quarter (April, May, and June).

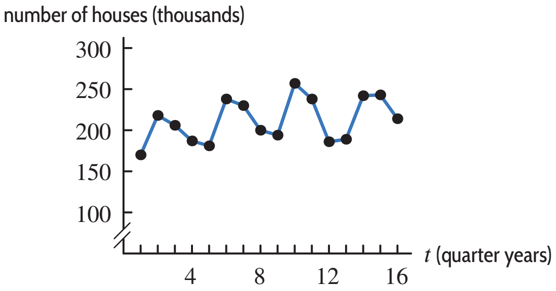


Figure 1.112: New housing construction starts, 2016–2019

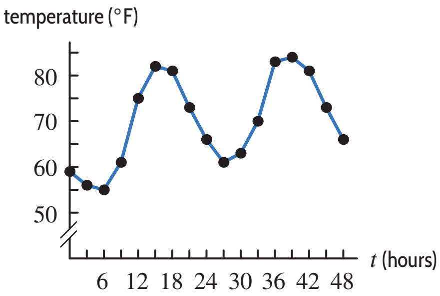


Figure 1.113: Temperature in Phoenix after midnight March 5th, 2020

Let's look at another example. Figure 1.113 is a graph of the temperature (in °F) in Phoenix, AZ, in hours after midnight, March 5, 2020. Notice that the maximum is in the afternoon and the minimum is in the early morning.[172] Again, the graph looks like a wave.

[170] S. Strogatz, "Math and the City," *The New York Times,* May 20, 2009.
[171] www.census.gov/construction/nrc, accessed July 28, 2020.
[172] https://weather.com/weather/hourbyhour, accessed March 11, 2020.

Functions whose values repeat at regular intervals are called *periodic*. Many processes, such as the number of housing starts or the temperature, are approximately periodic. The water level in a tidal basin, the blood pressure in a heart, retail sales in the US, and the position of air molecules transmitting a musical note are also all periodic functions of time.

Amplitude and Period

Periodic functions repeat exactly the same cycle forever. If we know one cycle of the graph, we know the entire graph.

For any periodic function of time:
- The **amplitude** is half the difference between its maximum and minimum values.
- The **period** is the time for the function to execute one complete cycle.

Example 1 Estimate the amplitude and period of the new housing starts function shown in Figure 1.112.

Solution Figure 1.112 is not exactly periodic, since the maximum and minimum are not the same for each cycle. Nonetheless, the minimum is about 175, and the maximum is about 250. The difference between them is 75, so the amplitude is about $\frac{1}{2}(75) = 37.5$ thousand houses.

The wave completes a cycle between $t = 1$ and $t = 5$, so the period is $t = 4$ quarter-years, or one year. The business cycle for new housing construction is one year.

Example 2 Figure 1.114 shows the temperature in an unopened freezer. Estimate the temperature in the freezer at 12:30 and at 2:45.

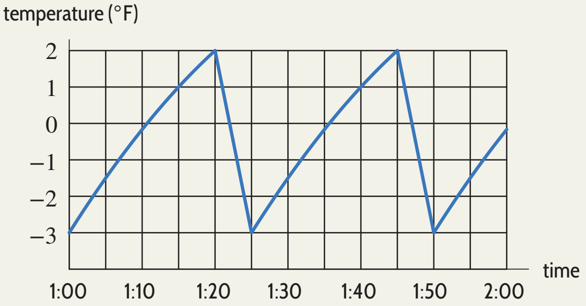


Figure 1.114: Oscillating freezer temperature. Estimate the temperature at 12:30 and 2:45

Solution The maximum and minimum values each occur every 25 minutes, so the period is 25 minutes. The temperature at 12:30 should be the same as at 12:55 and at 1:20, namely, 2°F. Similarly, the temperature at 2:45 should be the same as at 2:20 and 1:55, or about −1.5°F.

The Sine and Cosine

Many periodic functions are represented using the *sine* and *cosine* functions, written "sin" and "cos". *Warning*: A calculator can be in either "degree" or "radian" mode. Here, we always use radians.

Graphs of the Sine and Cosine

The graphs of the sine and the cosine functions are periodic; see Figures 1.115 and 1.116. Notice that the graph of the cosine function is the graph of the sine function, shifted $\pi/2$ to the left.

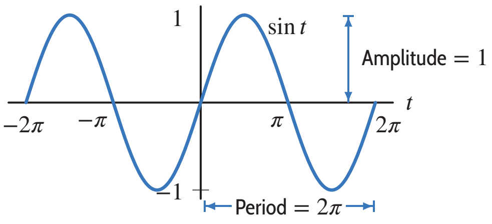


Figure 1.115: Graph of $\sin t$

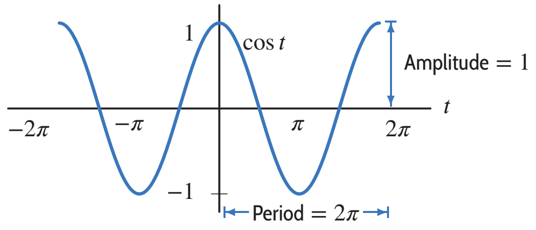


Figure 1.116: Graph of $\cos t$

The maximum and minimum values of $\sin t$ are $+1$ and -1, so the amplitude of the sine function is 1. The graph of $y = \sin t$ completes a cycle between $t = 0$ and $t = 2\pi$; the rest of the graph repeats this portion. The period of the sine function is 2π.

Example 3 Use a graph of $y = 3\sin 2t$ to estimate the amplitude and period of this function.

Solution In Figure 1.117, the waves have a maximum of $+3$ and a minimum of -3, so the amplitude is 3. The graph completes one complete cycle between $t = 0$ and $t = \pi$, so the period is π.

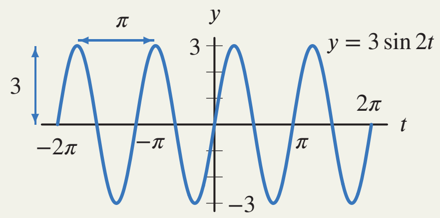


Figure 1.117: The amplitude is 3 and the period is π

Example 4 Explain how the graphs of each of the following functions differ from the graph of $y = \sin t$.

(a) $y = 6\sin t$ (b) $y = 5 + \sin t$ (c) $y = \sin\left(t + \frac{\pi}{2}\right)$

Solution

(a) The graph of $y = 6\sin t$ is in Figure 1.118. The maximum and minimum values are $+6$ and -6, so the amplitude is 6. This is the graph of $y = \sin t$ stretched vertically by a factor of 6.
(b) The graph of $y = 5+\sin t$ is in Figure 1.119. The maximum and minimum values of this function are 6 and 4, so the amplitude is $(6 - 4)/2 = 1$. The amplitude (or size of the wave) is the same as for $y = \sin t$, since this is a graph of $y = \sin t$ shifted up 5 units.
(c) The graph of $y = \sin(t + \pi/2)$ is in Figure 1.120. This has the same amplitude, namely 1, and period, namely 2π, as the graph of $y = \sin t$. It is the graph of $y = \sin t$ shifted $\pi/2$ units to the left. (In fact, this is the graph of $y = \cos t$.)

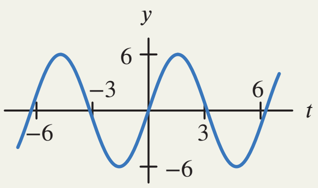


Figure 1.118: Graph of $y = 6\sin t$

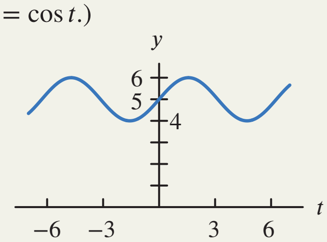


Figure 1.119: Graph of $y = 5 + \sin t$

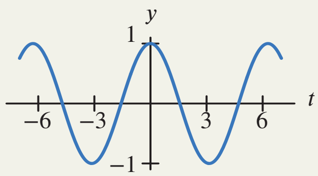


Figure 1.120: Graph of $y = \sin(t + \frac{\pi}{2})$

Families of Curves: The Graph of $y = A\sin(Bt)$

The constants A and B in the expression $y = A\sin(Bt)$ are called *parameters*. We can study families of curves by varying one parameter at a time and studying the result.

Example 5

(a) Graph $y = A\sin t$ for several positive values of A. Describe the effect of A on the graph.
(b) Graph $y = \sin(Bt)$ for several positive values of B. Describe the effect of B on the graph.

Solution

(a) From the graphs of $y = A\sin t$ for $A = 1, 2, 3$ in Figure 1.121, we see that A is the amplitude.

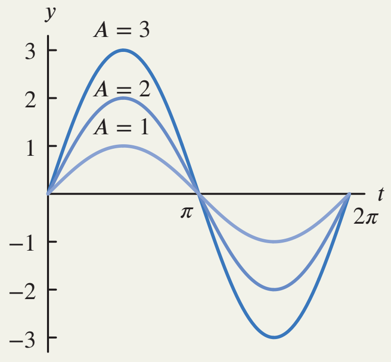


Figure 1.121: Graphs of $y = A\sin t$ with $A = 1, 2, 3$

(b) The graphs of $y = \sin(Bt)$ for $B = \frac{1}{2}$, $B = 1$, and $B = 2$ are shown in Figure 1.122. When $B = 1$, the period is 2π; when $B = 2$, the period is π; and when $B = \frac{1}{2}$, the period is 4π. The parameter B affects the period of the function. The graphs suggest that the larger B is, the shorter the period. In fact, the period is $2\pi/B$.

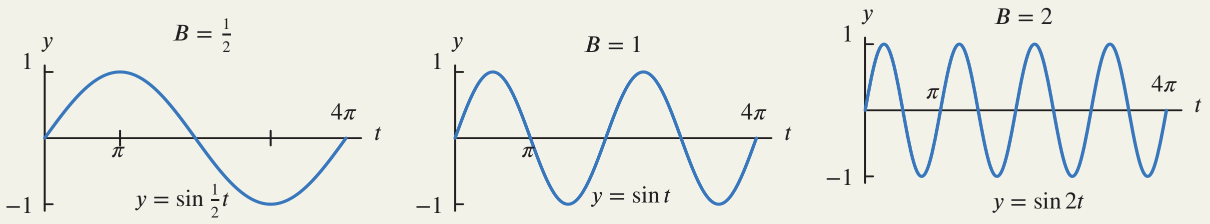
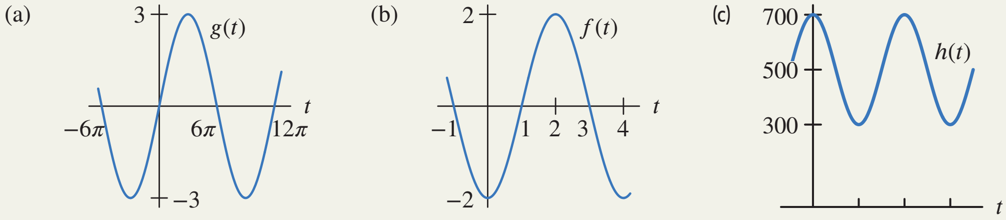


Figure 1.122: Graphs of $y = \sin(Bt)$ with $B = \frac{1}{2}, 1, 2$

In Example 5, the amplitude of $y = A\sin(Bt)$ was determined by the parameter A, and the period was determined by the parameter B. In addition, the oscillations may take place around a midline, given by the vertical shift, C. In general, we have

> The functions $y = A\sin(Bt) + C$ and $y = A\cos(Bt) + C$ are periodic with
>
> $$\text{Amplitude} = |A|, \qquad \text{Period} = \frac{2\pi}{|B|}, \qquad \text{Vertical shift (midline)} = C$$

Example 6 Find possible formulas for the following periodic functions.



Solution

(a) This function looks like a sine function of amplitude 3, so $g(t) = 3\sin(Bt)$. Since the function executes one full oscillation between $t = 0$ and $t = 12\pi$, when t changes by 12π, the quantity Bt changes by 2π. This means $B \cdot 12\pi = 2\pi$, so $B = 1/6$. Therefore, $g(t) = 3\sin(t/6)$ has the graph shown.

(b) This function looks like an upside-down cosine function with amplitude 2, so $f(t) = -2\cos(Bt)$. The function completes one oscillation between $t = 0$ and $t = 4$. Thus, when t changes by 4, the quantity Bt changes by 2π, so $B \cdot 4 = 2\pi$, or $B = \pi/2$. Therefore, $f(t) = -2\cos(\pi t/2)$ has the graph shown.

(c) This function looks like a cosine function. The maximum is 700 and the minimum is 300, so the amplitude is $\frac{1}{2}(700 - 300) = 200$. The height halfway between the maximum and minimum is 500, so the cosine curve has been shifted up 500 units, so $h(t) = 500 + 200\cos(Bt)$. The period is 12, so $B \cdot 12 = 2\pi$. Thus, $B = \pi/6$. The function $h(t) = 500 + 200\cos(\pi t/6)$ has the graph shown.

Example 7 On September 29, 2019, high tide in Portland, Maine was at midnight.[173] The height of the water (measured in feet above the mean level of the low tide) in the harbor is a periodic function, since it oscillates between high and low tide. If t is in hours since midnight, the height (in feet) is approximated by the formula

$$y = 4.9 + 6.3\cos\left(\frac{\pi}{6}t\right).$$

(a) Graph this function from $t = 0$ to $t = 24$.
(b) What was the water level at high tide?
(c) When was low tide, and what was the water level at that time?
(d) What is the period of this function, and what does it represent in terms of tides?
(e) What is the amplitude of this function, and what does it represent in terms of tides?

Solution

(a) See Figure 1.123.
(b) The water level at high tide was 11.2 feet (given by the y-intercept on the graph).
(c) Low tide occurs at $t = 6$ (6 am) and at $t = 18$ (6 pm). The water level at this time is -1.4 feet, that is 1.4 feet below the mean level of the low tide.
(d) The period is 12 hours and represents the interval between successive high tides or successive low tides. Of course, there is something wrong with the assumption in the model that the period is 12 hours. If so, the high tide would always be at noon or midnight, instead of progressing slowly through the day, as it in fact does. The interval between successive high tides actually averages about 12 hours 25 minutes, which could be taken into account in a more precise mathematical model.
(e) The maximum is 11.2, and the minimum is -1.4, so the amplitude is $(11.2-(-1.4))/2 = 12.6/2$, which is 6.3 feet. This represents half the difference between the depths at high and low tide.

[173] me.usharbors.com/monthly-tides, accessed March 12, 2020.

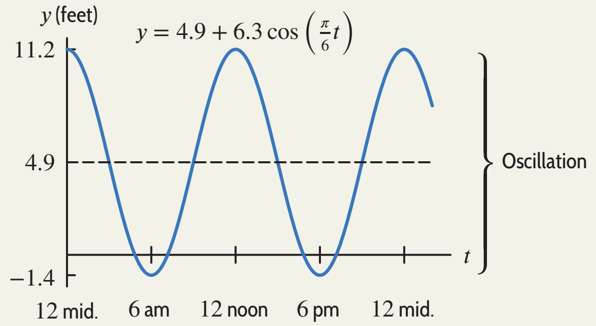


Figure 1.123: Graph of the function approximating the height of the water above mean low tide in Portland, Maine on September 29, 2019

Summary for Section 1.10

- **Periodic functions:** Functions whose values repeat at regular intervals are called **periodic**. For any periodic function of time:
  - The **amplitude** is half the difference between its maximum and minimum values.
  - The **period** is the time for the function to execute one complete cycle.
- **The sine and cosine:** The functions $y = A\sin(Bt) + C$ and $y = A\cos(Bt) + C$ are periodic with

$$\text{Amplitude} = |A|, \qquad \text{Period} = \frac{2\pi}{|B|}, \qquad \text{Vertical shift (midline)} = C$$

Problems for Section 1.10

In Problems 1–6, graph the function. What is the amplitude and period?

1. $y = 3\sin x$

2. $y = 4\cos 2x$

3. $y = -3\sin 2\theta$

4. $y = 3\sin 2x$

5. $y = 5 - \sin 2t$

6. $y = 4\cos\left(\frac{1}{2}t\right)$

7. Figure 1.124 shows quarterly US beer production during the period 2017 to 2019. Quarter 1 reflects production during the first three months of each year, etc.[174]

(a) Explain why a periodic function should be used to model these data.
(b) Approximately when does the maximum occur? The minimum? Why does this make sense?
(c) What are the period and amplitude for these data?

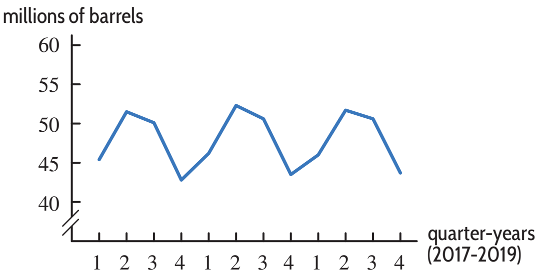


Figure 1.124

8. Sketch a possible graph of sales of sunscreen in the northeastern US over a 3-year period, as a function of months since January 1 of the first year. Explain why your graph should be periodic. What is the period?

9. The following table shows values of a periodic function $f(x)$. The maximum value attained by the function is 5.

(a) What is the amplitude of this function?
(b) What is the period of this function?
(c) Find a formula for this periodic function.

x	0	2	4	6	8	10	12
$f(x)$	5	0	−5	0	5	0	−5

10. Coober Pedy is a town in Australia that mines most of the world's opals (precious gemstones). Because of the scorching heat, much of the population lives in underground "dugouts." The Coober Pedy monthly high temperatures are shown in Figure 1.125, with $H(n)$ the temperature in month n, where $n = 1$ is January.[175] We fit the data with a function,

$$H(n) = A\cos(Bn) + C.$$

(a) What does the value of C represent and what is its value?

[174] www.ttb.gov/statistics, accessed March 10, 2020.
[175] https://en.wikipedia.org/wiki/Coober_Pedy, accessed May 10, 2020.

(b) What does the value of A represent and what is its value?
(c) What is the value of B?
(d) Graph the function.

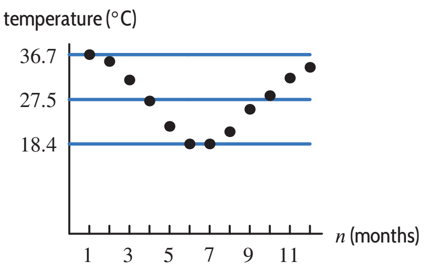


Figure 1.125

11. A person breathes in and out every three seconds. The volume of air in the person's lungs varies between a minimum of 2 liters and a maximum of 4 liters. Which of the following is the best formula for the volume of air in the person's lungs as a function of time?

(a) $y = 2 + 2\sin\left(\frac{\pi}{3}t\right)$ **(b)** $y = 3 + \sin\left(\frac{2\pi}{3}t\right)$

(c) $y = 2 + 2\sin\left(\frac{2\pi}{3}t\right)$ **(d)** $y = 3 + \sin\left(\frac{\pi}{3}t\right)$

12. Values of a function are given in the following table. Explain why this function appears to be periodic. Approximately what are the period and amplitude of the function? Assuming that the function is periodic, estimate its value at $t = 15$, at $t = 75$, and at $t = 135$.

t	20	25	30	35	40	45	50	55	60
$f(t)$	1.8	1.4	1.7	2.3	2.0	1.8	1.4	1.7	2.3

13. Average daily high temperatures in Ottawa, the capital of Canada, range from a low of −6° Celsius on January 1 to a high of 26° Celsius on July 1 six months later. See Figure 1.126. Find a formula for H, the average daily high temperature in Ottawa, in °C, as a function of t, the number of months since January 1.

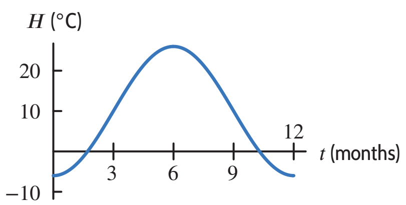


Figure 1.126

14. Figure 1.127 shows the levels of the hormones estrogen and progesterone during the monthly ovarian cycles in females.[176] Is the level of both hormones periodic? What is the period in each case? Approximately when in the monthly cycle is estrogen at a peak? Approximately when in the monthly cycle is progesterone at a peak?

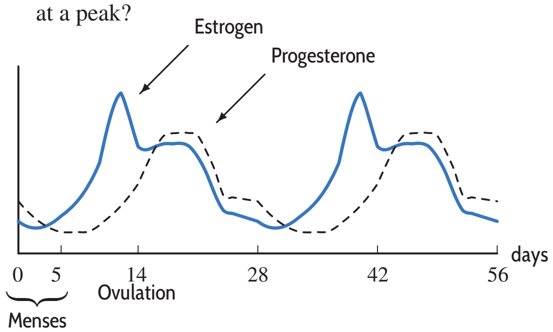


Figure 1.127

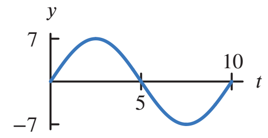
15. Delta Cephei is one of the most visible stars in the night sky. Its brightness has periods of 5.4 days, the average brightness is 4.0 and its brightness varies by ±0.35. Find a formula that models the brightness of Delta Cephei as a function of time, t, with $t = 0$ at peak brightness.

16. Most breeding birds in the northeast US migrate elsewhere during the winter. The number of bird species in an Ohio forest preserve oscillates between a high of 28 in June and a low of 10 in December.[177]

(a) Graph the number of bird species in this preserve as a function of t, the number of months since June. Include at least three years on your graph.
(b) What are the amplitude and period of this function?
(c) Find a formula for the number of bird species, N, as a function of the number of months, t since June.

■ In Problems 17–28, find a possible formula for the graph.

17.



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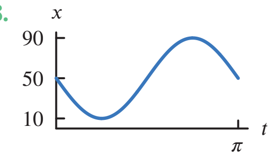

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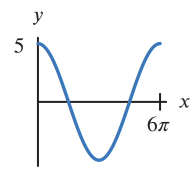

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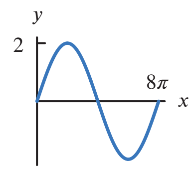

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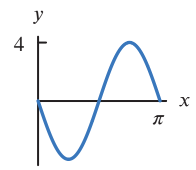

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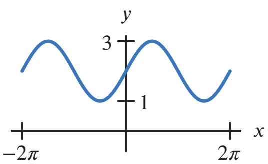
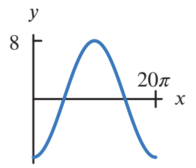


[176] Robert M. Julien, *A Primer of Drug Action*, Seventh Edition, p. 360 (W. H. Freeman and Co., New York: 1995).
[177] M. L. Rosenzweig, *Species Diversity in Space and Time,* p. 71 (Cambridge: Cambridge University Press, 1995).

23.



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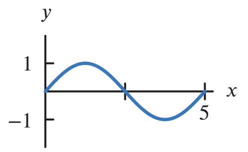

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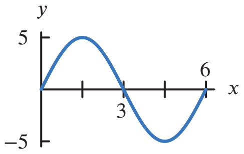

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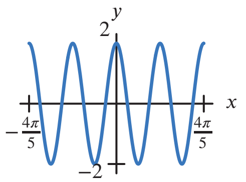

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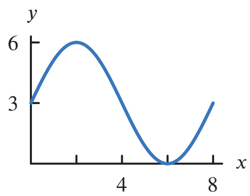

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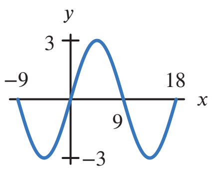

■ In Problems 29–32, graph the given function on the axes in Figure 1.128.

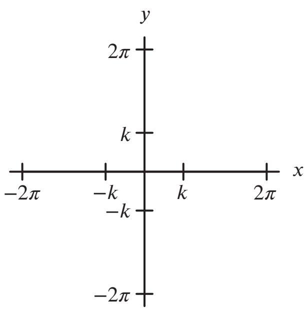

Figure 1.128

29. $y = k \sin x$

30. $y = -k \cos x$

31. $y = k(\cos x) + k$

32. $y = k(\sin x) - k$

33. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, y meters, is given as a function of time, t, in hours since midnight by

$$y = D + A\cos(B(t - C)).$$

(a) What is the physical meaning of D?
(b) What is the value of A?
(c) What is the value of B? Assume the time between successive high tides is 12.4 hours.
(d) What is the physical meaning of C?

34. The depth of water in a tank oscillates once every 6 hours. If the smallest depth is 5.5 feet and the largest depth is 8.5 feet, find a possible formula for the depth in terms of time in hours.

35. The desert temperature, H, oscillates daily between 40°F at 5 am and 80°F at 5 pm. Write a possible formula for H in terms of t, measured in hours from 5 am.

36. Table 1.48 gives values for $g(t)$, a periodic function.

(a) Estimate the period and amplitude for this function.
(b) Estimate $g(34)$ and $g(60)$.

Table 1.48

t	0	2	4	6	8	10	12	14
$g(t)$	14	19	17	15	13	11	14	19
t	16	18	20	22	24	26	28	
$g(t)$	17	15	13	11	14	19	17	

37. In Figure 1.129, the blue curve shows monthly mean carbon dioxide (CO_2) concentration, in parts per million (ppm) at Mauna Loa Observatory, Hawaii, as a function of t, in months, since December 2013. The black curve shows the monthly mean concentration adjusted for seasonal CO_2 variation.[178]

(a) Approximately how much did the monthly mean CO_2 increase between December 2013 and December 2018?
(b) Find the average monthly rate of increase of the monthly mean CO_2 between December 2013 and December 2018. Use this information to find a linear function that approximates the black curve.
(c) The seasonal CO_2 variation between December 2013 and December 2018 can be approximated by a sinusoidal function of the form $A \sin Bt$. What is the approximate period of the function? What is its amplitude? Give a formula for the function.
(d) The blue curve may be approximated by a function of the form $h(t) = f(t) + g(t)$, where $f(t)$ is sinusoidal and $g(t)$ is linear. Using your work in parts (b) and (c), find a possible formula for $h(t)$. Graph $h(t)$ using the scale in Figure 1.129.

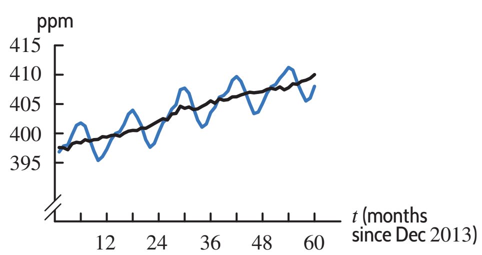

Figure 1.129

[178] www.esrl.noaa.gov/gmd/ccgg/trends, accessed March, 2019. Monthly means joined by segments to highlight trends.

PROJECTS FOR CHAPTER ONE

1. **Compound Interest**

The newspaper article below is from *The New York Times*, May 27, 1990. Fill in the three blanks. (For the first blank, assume that daily compounding is essentially the same as continuous compounding. For the last blank, assume the interest has been compounded yearly, and give your answer in dollars. Ignore the occurrence of leap years.)

213 Years After Loan, Uncle Sam Is Dunned

By LISA BELKIN

Special to The New York Times

SAN ANTONIO, May 26 — More than 200 years ago, a wealthy Pennsylvania merchant named Jacob DeHaven lent $450,000 to the Continental Congress to rescue the troops at Valley Forge. That loan was apparently never repaid.

So Mr. DeHaven's descendants are taking the United States Government to court to collect what they believe they are owed. The total: ___ in today's dollars if the interest is compounded daily at 6 percent, the going rate at the time. If compounded yearly, the bill is only ___.

Family Is Flexible

The descendants say that they are willing to be flexible about the amount of a settlement and that they might even accept a heartfelt thank you or perhaps a DeHaven statue. But they also note that interest is accumulating at ___ a second.

2. **Population Center of the US**

Since the opening up of the West, the US population has moved westward. To observe this, we look at the "population center" of the US, which is the point at which the country would balance if it were a flat plate with no weight, and every person had equal weight. In 1790 the population center was east of Baltimore, Maryland. It has been moving westward ever since, and in 2020 it was estimated to be in Hartville, Missouri.[179] During the second half of the 20th century, the population center has moved about 40 miles west every 10 years.

(a) Let us measure position westward from Hartville along the line running through Baltimore. For the years since 2020, express the approximate position of the population center as a function of time in years from 2020.

(b) The distance from Baltimore to Hartville is a bit over 1000 miles. Could the population center have been moving at roughly the same rate for the last two centuries?

(c) Could the function in part (a) continue to apply for the next five centuries? Why or why not? [Hint: You may want to look at a map. Note that distances are in air miles and are not driving distances.]

3. **Medical Case Study: Anaphylaxis**[180]

During surgery, a patient's blood pressure was observed to be dangerously low. One possible cause is a severe allergic reaction called *anaphylaxis*. A diagnosis of anaphylaxis is based in part on a blood test showing the elevation of the serum *tryptase*, a molecule released by allergic cells. In anaphylaxis, the concentration of tryptase in the blood rises rapidly and then decays back to baseline in a few hours.

However, low blood pressure from an entirely different cause (say from a heart problem) can also lead to an elevation in tryptase. Before diagnosing anaphylaxis, the medical team needs to make sure that the observed tryptase elevation is the result of an allergy problem, not a heart problem. To do this, they need to know the peak level reached by the serum tryptase. The normal range for the serum tryptase is 0–15 ng/ml (nanograms per milliliter). Mild to moderate elevations from low blood pressure are common, but if the peak were three times the normal maximum (that is, above 45 ng/ml), then a diagnosis of anaphylaxis would be made.

The surgeons who resuscitated this patient ran two blood tests to measure T_r, the serum tryptase concentration; the results are in Table 1.49. Use the test results to estimate the peak

[179] https://en.wikipedia.org/wiki/Mean_center_of_the_United_States_population, accessed March 23, 2021.
[180] From David E. Sloane, M.D., drawing from an actual episode in his clinic.

serum tryptase level at the time of surgery, assuming that tryptase decays exponentially. Did this patient experience anaphylaxis?

Table 1.49 *Serum tryptase levels*

t, hours since surgery	4	19.5
T_r, concentration in ng/ml	37	13