1

The Physics of Waves

The solution of the difficulty is that the two mental pictures which experiment lead us to form — the one of the particles, the other of the waves — are both incomplete and have only the validity of analogies which are accurate only in limiting cases.

Heisenberg

1.1 Introduction

The properties of waves are central to the study of optics. As we will see, light (or more properly, electromagnetic radiation) has both particle and wave properties. These *complementary* aspects are a result of quantum mechanics, and prior to the early 1900s, there were two schools of thought. Newton postulated that light consists of particles, while contemporaries Huygens and Hooke promoted a wave theory of light. The matter seemed settled with Young's important double-slit experiment offering clear experimental evidence that light is a wave. Maxwell's sweeping theory of electromagnetism finally provided a deep and complete description of electromagnetic waves that we consider in detail in Chapter 2. Although current theories of optics include both wave and particle descriptions, the wave picture still forms the bedrock of most optical technology. In this chapter, we will outline some general properties that apply to traveling waves of all types.

1.2 One-Dimensional Wave Equation

Mechanical waves travel within elastic media whose material properties provide restoring forces that result in oscillation. When a guitar string is plucked, it is displaced away from its equilibrium position, and the mechanical energy of this disturbance subsequently propagates along the string as traveling waves. In this case, the waves are *transverse*, meaning that the displacement of the medium (the string) is perpendicular to the direction of energy travel. Acoustic waves in a gas are *longitudinal*, meaning that the gas molecules are displaced back and forth along the direction of energy flow as regions of high and low pressure are created along the wave.

As we shall see in Chapter 2, *electromagnetic waves* are transverse but differ from mechanical waves in that they do not require an elastic medium. Rather, they propagate as

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disturbances in the *electromagnetic field*. Mechanical waves and electromagnetic waves are both examples of *classical waves*, which can be described with classical physics.[1]

Consider a mechanical wave, and let $\Psi(x,t)$ describe a disturbance of the medium away from its equilibrium condition. For a transverse wave along a horizontal string, $\Psi(x,t)$ represents a vertical displacement along the y-axis. For a longitudinal acoustic wave traveling horizontally through air, $\Psi(x,t)$ might represent deviations away from ambient pressure along the x-axis. In any case, we refer to $\Psi(x,t)$ as the *wavefunction*. Since $\Psi(x,t)$ depends only on the single spatial coordinate x, it is a *one-dimensional wavefunction*.

All classical mechanical waves can be described by the *differential wave equation*

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2} \tag{1.1}$$

where v is the wave speed and $\Psi(x,t)$ is the wavefunction. To demonstrate that a given function $\Psi(x,t)$ describes a classical wave, it is necessary only to show that this function satisfies Equation 1.1. Conversely, any physical system that can be shown to be described by Equation 1.1 must necessarily involve classical traveling waves.

Equation 1.1 is an example of a class of mathematical equations known as *differential equations*. When taking a partial derivative with respect to a given parameter, the remaining parameters are treated as constants.

Example 1.1

Show that

$$\Psi(x,t) = (x+vt)^2$$

is a solution to the differential wave equation. Assume that v is a constant.

Solution

We must show that $\Psi(x,t)$ solves Equation 1.1:

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2}$$

To do this, we must take the indicated derivatives. Begin with the partial derivative with respect to x:

$$\frac{\partial\Psi(x,t)}{\partial x} = \frac{\partial}{\partial x}(x+vt)^2 = 2(x+vt)\frac{\partial}{\partial x}(x+vt) = 2(x+vt)(1) = 2(x+vt)$$

Now take the second partial derivative with respect to x:

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial\Psi}{\partial x}\right] = \frac{\partial}{\partial x}[2(x+vt)] = 2$$

Next, take the partial derivatives of $\Psi(x,t)$ with respect to t:

$$\frac{\partial\Psi}{\partial t} = 2(x+vt)\frac{\partial}{\partial t}(x+vt) = 2(x+vt)(v) = 2v(x+vt)$$

$$\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial}{\partial t}[2v(x+vt)] = \frac{\partial}{\partial t}\left[2vx+2v^2t\right] = 2v^2$$

Substitute these results into the differential wave equation

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = 2$$

$$\frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2} = \frac{1}{v^2}\left(2v^2\right) = 2$$

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1 The term classical refers to the physics that came before the modern theories of relativity and quantum mechanics.

Thus

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2}$$

$\Psi(x,t)$ solves the differential wave equation, and thus we have shown that it is a traveling wave.

1.3 General Solutions to the 1D Wave Equation

Consider a plot of $\Psi(x,t)$ vs. x. Data for such a plot could be provided by an array of measuring devices, such as an array of pressure sensors arranged linearly to record the pressure amplitude of a passing acoustic wave. A plot of $\Psi(x,t)$ vs. x at a particular time t represents a *snap shot* of the wave as it passes by an array of measuring devices.

There are many possible shapes for this amplitude. Perhaps a sinusoidal function comes to mind, with distinct periodic crests and troughs. However, such waves are not the most general solution to Equation 1.1, as you can determine by comparing the sound of your voice as you hum or sing a specific musical note (if you can!) with the sound that your hands make when you clap them together. We will refer to the sound of a clap as a *pulse*.

We will show below that the most general solutions to Equation 1.1 may be expressed as follows:

$$\Psi(x,t) = f(x - vt) \tag{1.2}$$

$$\Psi(x,t) = g(x + vt) \tag{1.3}$$

where the functions f and g represent any function that has finite second derivatives and the parameters x, v, and t all occur explicitly within the function as $x - vt$ or $x + vt$.

As an example, consider the function

$$\Psi(x,t) = \frac{A}{1 + (x - vt)^2} \tag{1.4}$$

where A is a constant. Equation 1.4 represents a peaked function whose maximum is located at points given by $x = vt$. A plot of this wavefunctions at two different times is shown in Figure 1.1.

To show that Equation 1.4 represents a traveling wave, we could simply check to see if it solves the differential wave equation. It is more elegant, however, to show this for any differentiable functions given by Equations 1.2 and 1.3. Begin with the function $f(x - vt)$, and define a new variable u given by $u = x - vt$. Differentiate $f(u)$ using the *chain rule*:

$$\frac{\partial f(u)}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$$

In this case, $u = x - vt$, so the derivative of u with respect to x is just 1. Thus,

$$\frac{\partial f(u)}{\partial x} = \frac{\partial f}{\partial u}$$

and

$$\frac{\partial^2 f(u)}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x} = \frac{\partial^2 f}{\partial u^2} \tag{1.5}$$

The time derivative of $f(u)$ is given by

$$\frac{\partial f(u)}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} = -v\frac{\partial f}{\partial u}$$

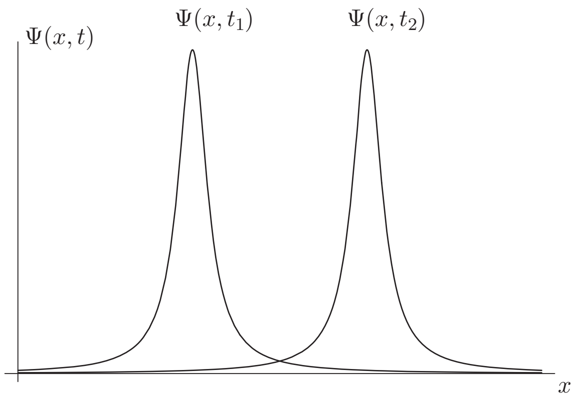

Figure 1.1 The traveling pulse of Equation 1.4, shown at two different times $t_1 < t_2$.

and

$$\frac{\partial^2 f(u)}{\partial t^2} = \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial t} = v^2\frac{\partial^2 f}{\partial u^2} \tag{1.6}$$

Substitute the results of Equations 1.5 and 1.6 into the differential wave equation:

$$\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\left(v^2\frac{\partial^2 f}{\partial u^2}\right) = \frac{\partial^2 f}{\partial u^2} = \frac{\partial^2 f}{\partial x^2}$$

Thus Equation 1.1 is satisfied by $f(u) = f(x - vt)$. In a similar way, you may show that $g(x + vt)$ also solves Equation 1.1 (see Problem 1.7). Thus $f(x - vt)$ and $g(x + vt)$ both solve the differential wave equation and therefore represent traveling waves.

In summary, we have shown that functions of x and t with finite second derivatives and with explicit occurrences of x and t that can be grouped as $x - vt$ or $x + vt$ are solutions to the differential wave equation and thus represent traveling waves. In particular, the function of Equation 1.4 fits this requirement and is therefore a traveling wave. Of course, you can also demonstrate this by substituting Equation 1.4 directly into Equation 1.1 (see Example 1.2).

We now show that the function $f(x - vt)$ represents a *forward-traveling wave*. Consider values of x and t determined by $x - vt = constant$. In Equation 1.4, choosing the value zero for this constant locates the peak of the pulse. As time proceeds, the specific value of x that satisfies this equation changes according to

$$dx - v\,dt = 0$$

or

$$\frac{dx}{dt} = +v$$

Thus, $f(x - vt)$ propagates in the positive x direction with velocity $+v$ and is therefore a forward-traveling wave. In a similar way, you can show that $g(x + vt)$ represents a backward-traveling wave (see Problem 1.8).

Example 1.2
Show explicitly that Equation 1.4 satisfies the differential wave equation.

Solution
We must substitute

$$\Psi(x,t) = \frac{A}{1 + (x - vt)^2}$$

into the differential wave equation

$$\frac{\partial^2\Psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi(x,t)}{\partial t^2}$$

Begin by differentiating with respect to x:

$$\frac{\partial\Psi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{A}{1+(x-vt)^2}\right) = \frac{(-1)\,A\,[2\,(x-vt)]}{\left[1+(x-vt)^2\right]^2}$$

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{2A[2\,(x-vt)]^2}{\left[1+(x-vt)^2\right]^3} - \frac{2A}{\left[1+(x-vt)^2\right]^2}$$

Now differentiate with respect to t:

$$\frac{\partial\Psi}{\partial t} = -\frac{A\,[2\,(x-vt)\,(-v)]}{\left[1+(x-vt)^2\right]^2} = (-v)\left[-\frac{A\,[2\,(x-vt)]}{\left[1+(x-vt)^2\right]^2}\right]$$

$$\frac{\partial^2\Psi}{\partial t^2} = \left(v^2\right)\left[\frac{2A[2\,(x-vt)]^2}{\left[1+(x-vt)^2\right]^3} - \frac{2A}{\left[1+(x-vt)^2\right]^2}\right]$$

Thus

$$\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial^2\Psi}{\partial x^2}$$

and $\Psi(x,t)$ is a traveling wave.

If you can cast $\Psi(x,t)$ in the form of Equation 1.2 or 1.3, you do not have to compute the derivatives explicitly to show that $\Psi(x,t)$ is a solution to Equation 1.1.

1.4 Harmonic Traveling Waves

According to the results of Section 1.3, any function described by Equation 1.2 or 1.3 represents a traveling wave. In particular, *harmonic functions* (i.e. sines and cosines) with the appropriate arguments solve the differential wave equation. Thus, the following function represents a traveling wave:

$$\Psi(x,t) = A\sin\frac{2\pi}{\lambda}(x \mp vt) \tag{1.7}$$

where A is the wave amplitude and λ is defined below.

Harmonic functions are periodic with a period of 2π radians:

$$\sin(\theta \pm 2\pi) = \sin(\theta)$$

$$\cos(\theta \pm 2\pi) = \cos(\theta)$$

The $\Psi(x,t)$ given in Equation 1.7 is periodic in both space and time coordinates. The term λ represents the *spatial period*:

$$\Psi(x+\lambda,t) = A\sin\frac{2\pi}{\lambda}(x+\lambda \mp vt) = A\sin\left[\frac{2\pi}{\lambda}(x \mp vt) + 2\pi\right] = \Psi(x,t)$$

The parameter λ is also called the *wavelength*. It has SI units of meters.

Let T represent the *temporal period*, i.e. the time required for one cycle. The SI unit of T is seconds (s), but it is convenient to use s/cycle as a reminder of what T represents. Since Equation 1.7 is periodic in T, we have

$$\Psi(x, t+T) = A \sin \frac{2\pi}{\lambda} [x \mp v(t+T)] = A \sin \left[\frac{2\pi}{\lambda}(x \mp vt) \mp \frac{2\pi}{\lambda} vT\right]$$

T represents the temporal period provided that

$$\frac{vT}{\lambda} = 1 \tag{1.8}$$

or

$$v = \frac{\lambda}{T} \tag{1.9}$$

Thus, a periodic classical wave travels one wavelength λ in one temporal period T. It is customary to define the wave *frequency* as

$$f = \frac{1}{T} \tag{1.10}$$

The units of f are cycles/s (SI unit: s^{-1}), often referred to as Hertz (Hz). In terms of frequency, Equation 1.9 becomes

$$f\lambda = v \tag{1.11}$$

A plot of $\Psi(x, t)$ vs. x is shown in Figure 1.2(a). Figure 1.2(b) shows a plot of $\Psi(x, t)$ vs. t. A plot such as this could be obtained from data provided by a measuring device located at a particular value of x.

It is customary to define the *propagation constant* as follows:

$$k = \frac{2\pi}{\lambda} \tag{1.12}$$

This quantity is also sometimes referred to as the *wavenumber*. Since k converts meters to radians, the units are rad/m (SI unit: m^{-1}). We may rewrite Equation 1.7 as

$$\Psi(x, t) = A \sin k(x \mp vt)$$

Similarly, we can define the *angular frequency*:

$$\omega = kv = \frac{2\pi}{\lambda} v = 2\pi f \tag{1.13}$$

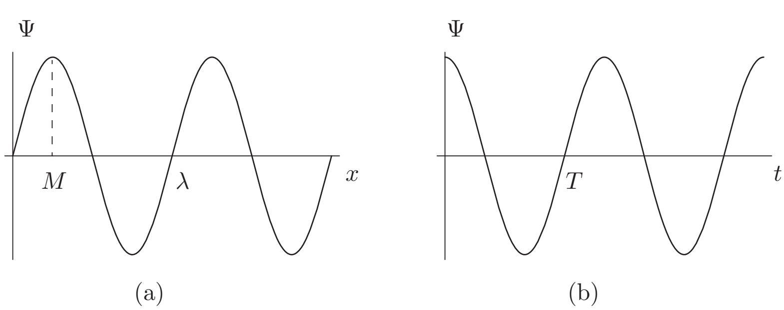

Figure 1.2 Plots of a harmonic wavefunction. (a) A plot of $\Psi(x, t)$ vs. position x. (b) A plot of $\Psi(x, t)$ vs. time using data recorded by a single measuring device located at M in (a).

Since ω converts time to radians, the units are rad/s (SI unit: s^{-1}). Note that, according to the last result,

$$\frac{\omega}{k} = v \tag{1.14}$$

In terms of k and ω, Equation 1.7 becomes

$$\Psi(x,t) = A\sin(kx \mp \omega t)$$

Collectively, the terms $kx \mp \omega t$ are called the *phase* of the harmonic traveling wave. We may also include an explicit value of the phase when x and t are zero by specifying the *initial phase* ϕ:

$$\Psi(x,t) = A\sin(kx \mp \omega t + \phi) \tag{1.15}$$

The units of ϕ are radians (SI unit: dimensionless).

1.5 The Principle of Superposition

The differential wave equation is a linear differential equation. If a differential equation is linear, then the principle of superposition holds: if Ψ_1 and Ψ_2 individually satisfy the differential equation, then $\Psi_1 + \Psi_2$ is also a solution. We may demonstrate this explicitly for Equation 1.1 given that

$$\frac{\partial^2\Psi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi_1}{\partial t^2}$$

and

$$\frac{\partial^2\Psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi_2}{\partial t^2}$$

Then

$$\frac{\partial^2\left(\Psi_1+\Psi_2\right)}{\partial x^2} = \frac{\partial^2\Psi_1}{\partial x^2} + \frac{\partial^2\Psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi_1}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2\Psi_2}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2\left(\Psi_1+\Psi_2\right)}{\partial t^2}$$

The proof may be extended to any number of solutions.

The principle of superposition provides the basis for our subsequent study of interference and diffraction in Chapters 7 and 8.

1.5.1 Periodic Traveling Waves

It is certainly possible for a function to be periodic but not harmonic. In fact, it is very common for waves to be periodic, but it is physically impossible for any classical wave to be perfectly harmonic. We will gain more insights into this interesting fact in Chapter 7. For now, suffice it to say that no laser is perfectly monochromatic, nor does any tuning fork emit a perfectly precise musical note. An example of a periodic function that is not harmonic is illustrated in Figure 1.3.

1.5.2 Linear Independence

As we will discuss in Chapter 7, a periodic wave with arbitrary profile such as that shown in Figure 1.3 may be represented as a linear combination of traveling harmonic waves.

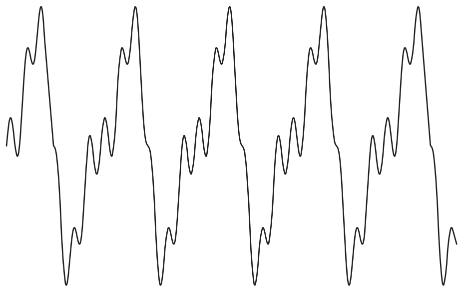

Figure 1.3 A periodic wave that is not harmonic.

However, a forward-traveling periodic wave must be constructed of forward-traveling harmonic waves, and backward-traveling periodic waves are represented as linear combinations of backward-traveling harmonic waves. Backward and forward-traveling waves are *linearly independent*, meaning that one cannot be obtained as a linear combination of the other. Thus, the most general solution of the one-dimensional differential wave equation is a linear combination of both solutions given by Equations 1.2 and 1.3.

1.6 Complex Numbers and the Complex Representation

Complex numbers can provide algebraic shortcuts that will prove very convenient as we continue our discussion of classical optics. In this section, we provide a quick overview of the properties of complex numbers and a few of their algebraic features that we will find most useful.

Complex numbers include the concept of the *imaginary* number i:

$$i = \sqrt{-1} \tag{1.16}$$

Clearly, the square root of a negative number has no counterpart within the set of all *real* numbers. A complex number z has both a real part and an imaginary part:

$$z = x + iy \tag{1.17}$$

where x is the *real part* of z and y is the *imaginary part* of z. Complex numbers may be visualized in the *complex plane*, as shown in Figure 1.4.

In the *Cartesian representation*, a complex number is plotted with coordinates (x, y); thus the horizontal axis is called the *real axis*, and the vertical axis is called the *imaginary axis*.

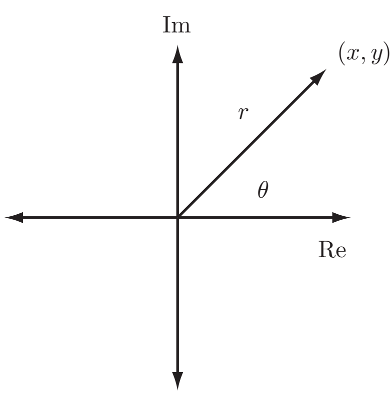

Figure 1.4 The complex plane.

Many of the features of complex numbers that we will find most useful result from the *Euler relation*:

$$e^{i\theta} = \cos\,\theta + i\,\sin\,\theta \tag{1.18}$$

See Example 1.3 for a derivation of this important formula. We may use the Euler relation to express any complex number z in *polar form*. In Figure 1.4, let

$$x = r\cos\theta \tag{1.19}$$

$$y = r\sin\theta \tag{1.20}$$

where

$$r = \sqrt{x^2 + y^2} \tag{1.21}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{1.22}$$

Using these relations, we may represent z as

$$z = x + iy = r\cos\theta + i\,(r\sin\theta) \tag{1.23}$$

Thus, according to Equation 1.18,

$$z = r\,e^{i\theta} \tag{1.24}$$

In the polar representation, r is called the *magnitude* of z, and θ is called the *phase* of z. The following examples are often useful:

$$e^{\pm i(2\pi)} = 1; \quad e^{\pm i(\pi)} = -1; \quad e^{i\frac{\pi}{2}} = i; \quad e^{i\frac{3\pi}{2}} = -i \tag{1.25}$$

The *complex conjugate* z^* of a complex number z is obtained by inverting the sign on each occurrence of i. For example, in the Cartesian representation where $z = x + iy$, z^* is given by

$$z^* = x - iy \tag{1.26}$$

In the polar representation where $z = r\,e^{i\theta}$, z^* is given by

$$z^* = r\,e^{-i\theta} \tag{1.27}$$

1.6.1 Complex Algebra

Let $z_1 = x_1 + i\,y_1 = r_1\,e^{i\theta_1}$ and $z_2 = x_2 + i\,y_2 = r_2\,e^{i\theta_2}$ with r_1, r_2, θ_1, and θ_2 defined by Equations 1.21 and 1.22. The addition of complex numbers is easiest in the Cartesian representation:

$$z_1 + z_2 = \left(x_1 + iy_1\right) + \left(x_2 + iy_2\right) = x_1 + x_2 + i\left(y_1 + y_2\right) \tag{1.28}$$

Notice that complex numbers add just like vectors in a two-dimensional real space. To add or subtract two complex numbers that are given in the polar representation, first convert to Cartesian form using the Euler relation (Equation 1.23).

Let Re[z] and Im[z] denote the real and imaginary parts of z. In the Cartesian form, Re[z] $= x$ and Im[z] $= y$. In the polar form, Re[z] $= r\,\cos\theta$ and Im[z] $= r\,\sin\theta$. In either case,

$$\mathrm{Re}[z] = \frac{z + z^*}{2} \tag{1.29}$$

$$\mathrm{Im}[z] = \frac{z - z^*}{2i} \tag{1.30}$$

According to the Euler relation,

$$\cos\ \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{1.31}$$

$$\sin\ \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.32}$$

Let λ be any real number. Scalar multiplication is defined by

$$\lambda z = \lambda\,(x + iy) = \lambda x + i\,(\lambda y) \tag{1.33}$$

To subtract two complex numbers, let $\lambda = -1$:

$$z_1 - z_2 = \left(x_1 + iy_1\right) - \left(x_2 + iy_2\right) = x_1 - x_2 + i\left(y_1 - y_2\right) \tag{1.34}$$

Complex multiplication is straightforward in either representation. Using the Cartesian representation gives

$$z_1\, z_2 = \left(x_1 + iy_1\right)\left(x_2 + iy_2\right) = x_1x_2 - y_1y_2 + i\left(x_1y_2 + x_2y_1\right) \tag{1.35}$$

where we used the fact that $i^2 = -1$. In the polar representation,

$$z_1z_2 = \left(r_1e^{i\theta_1}\right)\left(r_2e^{i\theta_2}\right) = r_1r_2e^{i\left(\theta_1+\theta_2\right)} \tag{1.36}$$

Multiplication of a complex number by its complex conjugate gives the square of the magnitude. In the Cartesian form,

$$z\,z^* = (x + iy)\,(x - iy) = x^2 + y^2 \tag{1.37}$$

and in the polar form,

$$z\,z^* = \left(r\,e^{i\theta}\right)\left(r\,e^{-i\theta}\right) = r^2 \tag{1.38}$$

which, according to Equation 1.21, agrees with the previous result. We will also refer to the magnitude of z as $|z|$:

$$z\,z^* = |z|^2$$

Division is straightforward in the polar representation:

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \tag{1.39}$$

In the Cartesian representation, division is a bit more involved if we wish to express the result in the Cartesian form

$$\begin{aligned}\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} &= \left(\frac{x_1 + iy_1}{x_2 + iy_2}\right)\left(\frac{x_2 - iy_2}{x_2 - iy_2}\right) \\ &= \left(\frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)\end{aligned} \tag{1.40}$$

This can also be written as

$$\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{z_1z_2^*}{\left|z_2\right|^2} \tag{1.41}$$

Example 1.3
Show that

$$e^{i\theta} = \cos\ \theta + i\ \sin\ \theta$$

Solution

We will demonstrate this relation using a *Taylor series*[2]:

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0} \left(x - x_0\right) + \frac{1}{2!}\left.\frac{d^2f}{dx^2}\right|_{x_0} \left(x - x_0\right)^2 + \frac{1}{3!}\left.\frac{d^3f}{dx^3}\right|_{x_0} \left(x - x_0\right)^3 + \cdots$$

where x_0 is the *expansion point*. The accuracy of this expansion depends on how close x_0 is to the *evaluation point* x.

Let us expand the function $f(x) = \sin x$ in a Taylor series about the point $x_0 = 0$. Begin by computing each of the indicated derivatives evaluated at the expansion point:

$$\frac{df}{dx} = \frac{d}{dx}\sin x = \cos x \qquad \left.\frac{df}{dx}\right|_{x_0=0} = 1$$

Similarly,

$$\left.\frac{d^2f}{dx^2}\right|_0 = 0 \qquad \left.\frac{d^3f}{dx^3}\right|_0 = -1 \qquad \left.\frac{d^4f}{dx^4}\right|_0 = 0 \qquad \left.\frac{d^3f}{dx^3}\right|_0 = 1$$

and so on. Substitution into the Taylor series formula gives

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

In a similar way, one finds

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

as the reader should verify. Substitution of $x = i\theta$ with θ measured in radians gives

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \cdots \\ &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \frac{i^7\theta^7}{7!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

Taking the real part or the imaginary part of $e^{i\theta}$ gives the corresponding harmonic function.

1.6.2 The Complex Representation of Harmonic Waves

We will often find it useful to represent harmonic traveling waves in the *complex representation*:

$$\Psi(x, t) = Ae^{i(kx\mp\omega t+\theta)} \tag{1.42}$$

According to the Euler relation, this is equal to

$$\Psi(x, t) = Ae^{i(kx\mp\omega t+\theta)} = A\cos(kx \mp \omega t + \theta) + iA\sin(kx \mp \omega t + \theta) \tag{1.43}$$

The rules of complex algebra just outlined will often simplify the trigonometry of harmonic traveling waves. For classical traveling waves, the complex representation is not necessary,

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2 See, for example, Hass and Weir (2017).

but only a convenience. Techniques for using this representation will be illustrated as we proceed.

1.7 The Three-Dimensional Wave Equation

In Cartesian coordinates, the extension of Equation 1.1 to include three dimensions is made in the obvious way:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{1.44}$$

where $\Psi = \Psi(x, y, z, t)$ is now understood to be a function of time and all three spatial coordinates.

Equation 1.44 can be written in a more general form using the Laplacian operator:

$$\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{1.45}$$

where, in Cartesian coordinates, the *Laplacian* is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.46}$$

Equation 1.45 is the *coordinate independent* representation of the three-dimensional wave equation. The wave equation in curvilinear coordinate systems is obtained by using the explicit form of the Laplacian for that system of coordinates.

In three dimensions, it is important to use a *right-handed* Cartesian coordinate system. According to the *right-hand rule*, $\hat{\imath} \times \hat{\jmath} = \hat{k}$ where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the unit normal vectors along the x, y, and z axes.[3] A right-handed Cartesian coordinate system is illustrated in Figure 1.5.

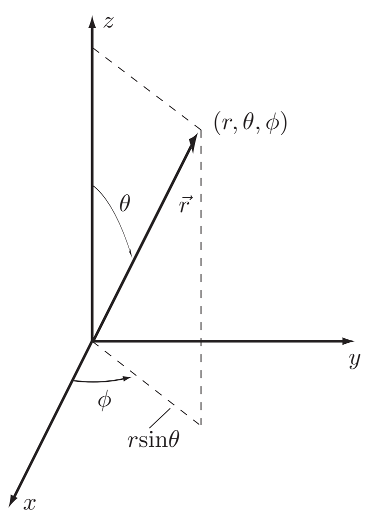

Figure 1.5 Spherical coordinates illustrated within a right-handed Cartesian coordinate system.

3 When used with vector quantities, the symbol $\times$ denotes a *vector cross product*. For details, see any introductory physics text (e.g. (Halliday et al., 2018)).

1.7.1 Spherical Coordinates

Figure 1.5 illustrates the spherical coordinate system, where the three spatial coordinates are (r, θ, ϕ).[4] The coordinate r is the magnitude of the position vector $\vec{r}$, θ measures the angle between $\vec{r}$ and the positive z-axis, and ϕ measures the angle between the positive x-axis and the projection of $\vec{r}$ onto the x–y plane. The transformation equations between spherical and Cartesian coordinates are

$$\begin{aligned} x &= r\sin\theta\cos\phi & r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\theta\sin\phi & \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ z &= r\cos\theta & \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \tag{1.47}$$

The volume element is obtained by incrementing each coordinate

$$d\tau = (dr)\,(rd\theta)\,(r\sin\theta\; d\phi) = r^2\sin\theta\; dr\; d\theta\; d\phi \tag{1.48}$$

In spherical coordinates, the Laplacian is given by[5]

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \tag{1.49}$$

Substitution of the Laplacians given by Equation 1.46 or 1.49 into the coordinate independent version of the differential wave equation given by Equation 1.45 yields very different looking but entirely equivalent differential equations. The choice of which coordinate system to use will typically be dictated by the *symmetry* of the problem. Problems with rectangular symmetry are typically easier to solve in Cartesian coordinates, while problems with spherical symmetry are more easily solved in the spherical coordinate system. A traveling wave solution that satisfies the differential wave equation in any particular coordinate system will always solve the wave equation when expressed in any other coordinate system.

As a final note, it is customary to use the notation $\Psi(\vec{r}, t)$ to represent a wavefunction in an arbitrary coordinate system. Thus, in Cartesian coordinates, $\Psi(\vec{r}, t)$ is interpreted as $\Psi(x, y, z, t)$, while in spherical coordinates, $\Psi(\vec{r}, t)$ represents $\Psi(r, \theta, \phi, t)$. Using this notation, the coordinate independent differential wave equation of Equation 1.45 becomes

$$\nabla^2\Psi\left(\vec{r}, t\right) = \frac{1}{v^2}\frac{\partial^2\Psi\left(\vec{r}, t\right)}{\partial t^2} \tag{1.50}$$

1.7.2 Three-Dimensional Plane Waves

Consider an acoustic wave traveling through air. Pressure is a property of space, and the pressure variations that define the acoustic wave are defined throughout a region. In a three-dimensional *plane wave*, the properties of the medium are constant over any *plane* oriented normal to the direction of propagation. To describe such a wave, we define the propagation vector $\vec{k}$ with magnitude $\frac{2\pi}{\lambda}$ and direction given by the wave propagation. The corresponding plane wave is given by

$$\Psi(x, y, z, t) = A\sin\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right) \tag{1.51}$$

---

4 Mathematicians typically interchange the roles of θ and ϕ in spherical coordinates. The convention adopted here is more common in physics.

5 See, for example, Boas (1983).

A plane has Cartesian symmetry,[6] and in Cartesian coordinates, $\vec{k}$ is given by

$$\vec{k} = k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k} \tag{1.52}$$

The position vector in Cartesian coordinates is given by

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} \tag{1.53}$$

For example, in a plane harmonic sound wave traveling in the positive x-direction, the value of the air pressure is constant over any plane that is parallel to the y–z plane:

$$\Psi(x, y, z, t) = A\sin(kx - \omega t + \varphi) \tag{1.54}$$

This expression only *looks* one-dimensional; the value of $A\sin(kx - \omega t + \varphi)$ at any point on the x-axis determines the value of $\Psi(x, y, z, t)$ at all points on a plane parallel to the y–z plane located at x.

In Figure 1.6 it is seen that $\vec{k}$ is normal to planes defined by $\vec{k}\cdot\vec{r} = const$. It is convenient to define *wavefronts* located at points where the phase of Equation 1.51 is equal to integer multiples of 2π. Thus, a three-dimensional plane wave may be visualized as a train of wavefronts separated by one wavelength λ and moving with the wave speed v.

In the complex representation, harmonic plane waves are given by

$$\Psi(x, y, z, t) = Ae^{i\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)} \tag{1.55}$$

It is instructive to show explicitly that the plane wave in Equation 1.55 solves the differential wave equation. Plane waves have rectangular symmetry, so we will use Cartesian coordinates:

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \tag{1.56}$$

Using Equations 1.52 and 1.53 for $\vec{k}$ and $\vec{r}$, we find that

$$\Psi(x, y, z, t) = Ae^{i\left(\vec{k}\cdot\vec{r} - \omega t + \varphi\right)} = Ae^{i(k_x x + k_y y + k_z z - \omega t + \varphi)}$$

Thus,

$$\frac{\partial\Psi}{\partial x} = ik_x\Psi$$
$$\frac{\partial^2\Psi}{\partial x^2} = -k_x^2\Psi$$

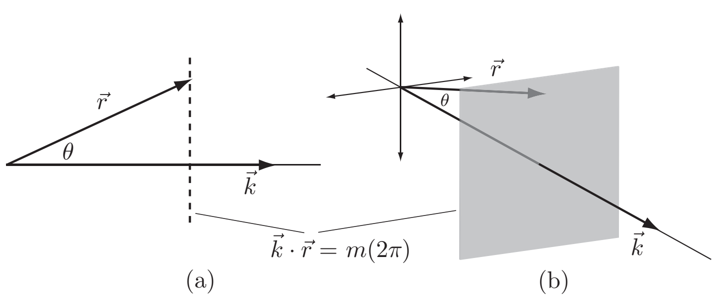

Figure 1.6 (a), (b) A wavefront defined by $\vec{k}\cdot\vec{r} = m(2\pi)$, where m is an integer. Vectors $\vec{k}$ and $\vec{r}$ do not have the same units.

6 An object with Cartesian symmetry can be oriented so that it is unaffected by reflections about x, y, or z axes. It is always simpler to use a coordinates system with the same symmetry as the system being studied.

with similar expressions for derivatives with respect to y and z. Thus

$$\nabla^2\Psi = -\left(k_x^2 + k_y^2 + k_z^2\right)\Psi = -k^2\Psi$$

The time derivatives are given by

$$\frac{\partial\Psi}{\partial t} = -i\omega\Psi$$
$$\frac{\partial^2\Psi}{\partial t^2} = -\omega^2\Psi$$

Substitution into the wave equation gives

$$\nabla^2\Psi = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2} \Rightarrow -k^2\Psi = \frac{1}{v^2}\left(-\omega^2\right)\Psi$$

This will be true provided

$$v = \frac{\omega}{k}$$

in agreement with Equation 1.14.

1.7.3 Spherical Waves

Spherical waves are waves with wavefronts that are spherical in shape. Examples include waves that emanate from an isotropic point source or are converging to a point. An isotropic source emits waves symmetrically in all directions.

In spherical coordinates, spherical waves have no dependence on the angular coordinates θ and ϕ, so the derivatives with respect to these coordinates are zero. In the notation introduced by Equation 1.50, this means that

$$\Psi(\vec{r}, t) = \Psi(r, t)$$

In other words, $\Psi(r, t)$ depends only on the spherical coordinate r and the time t. Derivatives of such a function with respect to θ and ϕ give zero, leading to a simplified version of the Laplacian for spherical coordinates (Equation 1.49):

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \tag{1.57}$$

Using this Laplacian, we obtain the *spherically symmetric differential wave equation*

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r,t)}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2} \tag{1.58}$$

An important solution to this equation is the *harmonic spherical wave*

$$\Psi(r, t) = \frac{A}{r}\sin\left(kr \mp \omega t + \varphi\right) \tag{1.59}$$

If the minus sign is chosen, the wave emanates from an isotropic source located at $r = 0$. A plus sign represents a wave that is converging to the point $r = 0$. The amplitude $\frac{A}{r}$ of the wave is not constant, since it depends on the radial coordinate r. For a wave emanating from a point, the amplitude decreases as the wave travels. Waves converging to a point have amplitudes that increase as time increases.

In the complex representation, Equation 1.59 becomes

$$\psi(r, t) = \frac{A}{r}e^{i(kr\mp\omega t+\varphi)} \tag{1.60}$$

Example 1.4
Show that Equation 1.59 solves the spherically symmetric differential wave equation.

Solution
We must show that Equation 1.58

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r,t)}{\partial r}\right)=\frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2}$$

is solved by Equation 1.59

$$\Psi(r,t)=\frac{A}{r}\sin(kr\mp\omega t+\varphi)$$

Begin by taking the derivative of $\Psi(r,t)$ with respect to r:

$$\frac{\partial\Psi}{\partial r}=-\frac{A}{r^2}\sin(kr\mp\omega t+\varphi)+\frac{A}{r}(k)\cos(kr\mp\omega t+\varphi)$$

To evaluate the left-hand side of the wave equation, multiply this last result by r^2, take another derivative with respect to r, and then divide by r^2:

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r,t)}{\partial r}\right)&=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{A}{r^2}\sin(kr\mp\omega t+\varphi)+\frac{A}{r}(k)\cos(kr\mp\omega t+\varphi)\right)\right]\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left[-A\sin(kr\mp\omega t+\varphi)+rAk\cos(kr\mp\omega t+\varphi)\right]\\&=\frac{1}{r^2}\left[-rAk^2\sin(kr\mp\omega t+\varphi)\right]\\&=-k^2\left[\frac{A}{r}\sin(kr\mp\omega t+\varphi)\right]\\&=-k^2\Psi\end{aligned}$$

The time derivatives on the right-hand side of the wave equation are given by

$$\begin{aligned}\frac{\partial\Psi}{\partial t}&=\frac{\partial}{\partial t}\left(\frac{A}{r}\sin(kr\mp\omega t+\varphi)\right)=\mp\omega\frac{A}{r}\cos(kr\mp\omega t+\varphi)\\\frac{\partial^2\Psi}{\partial t^2}&=-\omega^2\Psi\end{aligned}$$

Substitution into the wave equation gives

$$-k^2\Psi=\frac{1}{v^2}\left(-\omega^2\Psi\right)$$

which is satisfied since $v=\frac{\omega}{k}$.

Problems

Problems with † have solutions in Appendix E.

Section 1.2

1.1 Use the differential wave equation (Equation 1.1) to show that

$$\Psi(x,t)=(x+vt)$$

is a traveling wave.

1.2 Use the differential wave equation (Equation 1.1) to show that

$$\Psi(x, t) = (x - vt)$$

is a traveling wave.

1.3 Use the differential wave equation (Equation 1.1) to show that

$$\Psi(x, t) = (x - vt)^2$$

is a traveling wave.

1.4 †Use the differential wave equation (Equation 1.1) to show that

$$\Psi(x, t) = a(x - bt)^2$$

is a traveling wave. Assume that a and b are all constants and that the quantity bt has the same units as x.

Section 1.3

1.5 †Show that

$$\Psi(x, t) = Ae^{-(a^2x^2+b^2t^2+2\,a\,b\,x\,t)}$$

is a traveling wave, and find the wave speed and direction of propagation. Assume that A, a, and b are all constants and that a and b have units that make the quantity in the exponential function unitless.

1.6 Show that

$$\Psi(x, t) = Ae^{-(a^2x^2+b^2t^2-2\,a\,b\,x\,t)}$$

is a traveling wave, and find the wave speed and direction of propagation. Assume that A, a, and b are all constants and that a and b have units that make the quantity in the exponential function unitless.

1.7 †Show that $g(x + vt)$ in Equation 1.3 is a solution of the one-dimensional differential wave equation.

1.8 †Show that $g(x + vt)$ in Equation 1.3 represents a wave that travels in the negative-x direction.

Section 1.4

1.9 †Show explicitly (by direct substitution) that the function

$$\Psi(x, t) = A\sin\frac{2\pi}{\lambda}(x \mp vt)$$

solves the one-dimensional wave equation.

1.10 †Show explicitly that the function

$$\Psi(x, t) = A\cos(kx \mp \omega t + \phi)$$

solves the one-dimensional wave equation.

1.11 †The speed of sound is 343 m/s in air and 1493 m/s in water. Find the wavelength emitted by a transducer that oscillates at 1000 Hz in both air and water. Assume that the frequency of the oscillator is the same in both media. In each case, find k and ω.

1.12 A harmonic traveling wave is given by $\Psi(x, t) = A\sin(60x - 9000t)$. Find the wave speed, frequency, angular frequency, and direction of propagation.

1.13 †A harmonic traveling wave is given by $\Psi(z, t) = A\sin(50z + 3000t)$. Find the wave speed, frequency, angular frequency, and direction of propagation.

1.14 A harmonic traveling wave is given by $\Psi(y, t) = A\sin(1000y - 3000t)$. Find the wave speed, frequency, angular frequency, and direction of propagation.

Section 1.5

1.15 †Show explicitly that the function

$$\Psi(x, t) = A\sin(kx - \omega t + \phi) + B\sin(kx + \omega t + \phi)$$

solves the one-dimensional wave equation.

1.16 Show explicitly that the function

$$\Psi(x, t) = A\sin(kx - \omega t + \phi) - B\cos(kx + \omega t + \phi)$$

solves the one-dimensional wave equation.

Section 1.6

1.17 †The *small-angle approximation* is often useful for avoiding tedious algebra: for small θ, $\sin\theta \approx \theta$. Check the approximation for $\theta = 5°$, $10°$, and $20°$.

1.18 †Show that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

1.19 †Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

1.20 †Express the following numbers in polar form: $4 + 5i$, $-4 - 5i$, $4 - 5i$, $-4 + 5i$. In each case, graph the number on the complex plane.

1.21 †Express the following numbers in polar form: $3 + 5i$, $-2 - 6i$, $5 - 4i$, $-3 + 8i$. In each case, graph the number on the complex plane.

1.22 Express the following numbers in polar form: $1 + i$, $-1 - i$, $1 - i$, $-1 + i$. In each case, graph the number on the complex plane.

1.23 Express the following numbers in polar form: $5 + 4i$, $-5 - 4i$, $5 - 4i$, $-5 + 4i$. In each case, graph the number on the complex plane.

1.24 Express the following numbers in polar form: $5 + 3i$, $-6 - 2i$, $4 - 5i$, $-8 + 3i$. In each case, graph the number on the complex plane.

1.25 Find the real and imaginary parts of $z = e^{\pi\, i}$.

1.26 Find the real and imaginary parts of $z = 10\, e^{\frac{\pi}{2}\, i}$.

1.27 Find the real and imaginary parts of $z = 10\, e^{-\frac{\pi}{2}\, i}$.

1.28 †Let $z_1 = 10\, e^{0.5\, i}$ and $z_2 = 20\, e^{-0.8\, i}$. Find the real part, imaginary part, and polar form of as follows:
(a) $z_1 + z_2$
(b) $z_1 - z_2$
(c) $z_1\, z_2$
(d) z_1/z_2

1.29 Let $z_1 = 5\, e^{1.5\, i}$ and $z_2 = 2\, e^{-1.8\, i}$. Find the real part, imaginary part, and polar form of as follows:
(a) $z_1 + z_2$
(b) $z_1 - z_2$
(c) $z_1\, z_2$
(d) z_1/z_2

1.30 †For each of the following complex numbers, find the real part and the imaginary part, and express the number in polar form:
(a) $z = (4 + 5i)^2$
(b) $z = 5(1 + i)e^{i\pi/6}$

1.31 For each of the following complex numbers, find the real part and the imaginary part, and express the number in polar form:
(a) $z = (5 + 4i)^2$
(b) $z = 10(1 + i)e^{i\pi/3}$

Section 1.7

1.32 †Show explicitly that the following functions are solutions to the spherically symmetric wave equation

(a) $\Psi(r, t) = \dfrac{A}{r} \cos{(kr \mp \omega t + \varphi)}$

(b) $\Psi(r, t) = \dfrac{A}{r} e^{i(kr \mp \omega t + \varphi)}$

1.33 Show explicitly that the following function is a solution to the spherically symmetric wave equation

$$\Psi(r, t) = \frac{A}{r} \sin{(kr \mp \omega t + \varphi)}$$

1.34 †Find the equation for a plane electromagnetic wave whose propagation vector $\vec{k}$ is parallel to a line in the x–z plane that is 30° measured counterclockwise from the positive x-axis. Assume that $\vec{k}$ lies in the first quadrant of the x–z plane.

1.35 Find the equation for a plane electromagnetic wave whose propagation vector $\vec{k}$ is parallel to a line in the x–z plane that is 60° measured counterclockwise from the positive x-axis. Assume that $\vec{k}$ lies in the first quadrant of the x–z plane.

Additional Problems

1.36 †Determine the direction of propagation of the following harmonic traveling waves:
(a) $\Psi(z,t) = A\sin(kz - \omega t)$
(b) $\Psi(y,t) = A\cos(\omega t - ky)$
(c) $\Psi(x,t) = A\cos(\omega t + kx)$
(d) $\Psi(x,t) = A\cos(-\omega t - kx)$

1.37 Determine the direction of propagation of the following harmonic traveling waves:
(a) $\Psi(z,t) = A\cos(ky + \omega t + \pi/4)$
(b) $\Psi(y,t) = A\cos(\omega t - kz - \pi/2)$
(c) $\Psi(x,t) = A\cos(\omega t + kx + \pi)$
(d) $\Psi(x,t) = A\cos(-\omega t - kx - \pi)$

1.38 †Show that the *Gaussian* wave $\Psi(x,t) = Ae^{-a(bx-ct)^2}$ is a solution to the one-dimensional wave equation. If $a = 5.00$, $b = 10.0$, and $c = 100$, with $a(bx - ct)^2$ unitless, determine the wave speed.

1.39 Show that the *Gaussian* wave $\Psi(x,t) = Ae^{-a(bx+ct)^2}$ is a solution to the one-dimensional wave equation. If $a = 10.0$, $b = 5.00$, and $c = 500$, with $a(bx + ct)^2$ unitless, determine the wave speed.

1.40 †Sketch or plot the following wavefunction at times $t = 0$, $t = 0.5$ s, and $t = 1.0$ s:

$$\Psi(x,t) = \frac{1.00}{1 + (x + 10t)^2}$$

1.41 Sketch or plot the following wavefunction at times $t = 0$, $t = 0.5$ s, and $t = 1.0$ s:

$$\Psi(x,t) = \frac{10.0}{1 + (x - 20t)^2}$$

1.42 †A 1D harmonic traveling wave that travels in the $-y$ direction has amplitude of 10 (unitless), wavelength of 10.0 m, period of 2.0 s, and initial phase π. Using the complex representation, find an expression for this wave that uses angular frequency and propagation constant.

1.43 A 1D harmonic traveling wave that travels in the $+z$ direction has amplitude of 100 (unitless), wavelength of 10^{-6} m, speed of 3.00×10^8 m/s, and initial phase $-\pi$. Using the complex representation, find an expression for this wave that uses angular frequency and propagation constant.

1.44 †Light from a helium–neon laser has a wavelength of 633 nm and a wave speed of 3.00×10^8 m/s. Find the frequency, period, angular frequency, and wavenumber for this light.

1.45 Light from a green laser pointer has a wavelength of 532 nm and a wave speed of 3.00×10^8 m/s. Find the frequency, period, angular frequency, and wavenumber for this light.

1.46 †Consider a harmonic wave given by

$$\Psi(x,t) = U(x,y,z)e^{-i\omega t}$$

where $U(x,y,z)$ is called the *complex amplitude*. Show that U satisfies the *Helmholtz equation*

$$\left(\nabla^2 + k^2\right) U(x,y,z) = 0$$

where k is the propagation constant.

1.47 †Show that a complex number divided by its complex conjugate gives a result whose magnitude is one.

1.48 †Show that $z = \frac{\sqrt{2}}{2}(1+i)$ is a square root of i. Find another one.

1.49 †Find the real and imaginary parts of as follows:
(a) $z = \left(2e^{i\frac{\pi}{4}}\right)^3$
(b) $z = (2+3i)^3$

1.50 Find the real and imaginary parts of as follows:
(a) $z = \left(3e^{i\frac{\pi}{6}}\right)^3$
(b) $z = (3+2i)^3$

1.51 †Hyperbolic sines and cosines are defined as follows: $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$. Show that $\sin(ix) = i\sinh(x)$ and $\cos(ix) = \cosh(x)$.

1.52 †Find the Taylor series expansions for hyperbolic sine and hyperbolic cosine.

1.53 †Consider a vector $\vec{v}$, and let a be the angle between $\vec{v}$ and the positive x-axis, b be the angle between $\vec{v}$ and the positive y-axis, and c be the angle between $\vec{v}$ and the positive z-axis. Define the *direction cosines* α, β, and γ as follows:

$$\alpha = \cos a = \frac{v_x}{|v|}$$

$$\beta = \cos b = \frac{v_y}{|v|}$$

$$\gamma = \cos c = \frac{v_z}{|v|}$$

(a) Show that $\alpha^2 + \beta^2 + \gamma^2 = 1$.
(b) Show that the function

$$\Psi(x,y,z,t) = Ae^{i\left[k(\alpha x + \beta y + \gamma z) \mp \omega t\right]}$$

is a three-dimensional plane wave that solves the differential wave equation in Cartesian coordinates.

1.54 †The Laplacian in cylindrical coordinates (ρ, φ, z) is given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

A *cylindrical wave* has a wavefront that is constant on a cylinder. In other words, it does not depend upon φ or z. Show that

$$\Psi = \frac{A}{\sqrt{\rho}} e^{i(k\rho \mp \omega t)}$$

approximately solves the differential wave equation in cylindrical coordinates for values of ρ that are sufficiently large.

1.55 †Show that the spherically symmetric wave equation can be written as

$$\frac{\partial^2}{\partial r^2}\left[r\Psi(r,t)\right] = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left[r\Psi(r,t)\right]$$

which is a *linear* wave equation in the quantity $r\Psi(r,t)$. Show that Equations 1.59 and 1.60 are solutions.

1.56 †*Photons* are particles of light with energy and momentum given by

$$E = hf$$

$$p = \frac{h}{\lambda}$$

where f is the light frequency, λ is the light wavelength, and h is *Planck's constant*: $h = 6.626 \times 10^{-34}$ J · s. Show that for photons, $E = cp$.