1

Nonlinear Systems Analysis

In this chapter, we give a basic mathematical introduction to analysis of nonlinear dynamical systems, with a focus on Lyapunov stability analysis.

Stability analysis is known as one of the main challenges in the study of dynamical systems. The responses of the system against perturbations can demonstrate the quality of stability in the system. Considering the behavior of the system around an equilibrium point, different scenarios are expected for the trajectories of the system under perturbations. Accordingly, different notions of stability arise. In this regard, Lyapunov analysis provides an efficient framework for analyzing the stability of nonlinear dynamical systems. Moreover, it can aid in designing feedback controllers as well as analyzing the closed-loop system. In this chapter, we briefly review some basic concepts of stability and discuss several well-known Lyapunov stability theorems in a self-contained manner. For detailed discussions, we refer the readers to Haddad and Chellaboina [2011] and Khalil [2002].

1.1 Notation

We introduce a minimal set of mathematical notation that will be used in this book.

- $\mathbb{R}$: The set of real numbers.
- $\mathbb{R}_{\geq 0}$: The set of nonnegative real numbers.
- $\mathbb{Z}$: The set of all integers.
- $\mathbb{Z}_{\geq 0}$: The set of nonnegative integers.
- $|x|$: The absolute value of a real number.
- $\mathbb{R}^n$: The n-dimensional Euclidean space.
- $\|x\|$: The 2-norm (or Euclidean norm) of a vector $x \in \mathbb{R}^n$, defined by

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}.$$

Model-Based Reinforcement Learning: From Data to Continuous Actions with a Python-based Toolbox, First Edition. Milad Farsi and Jun Liu.

- $A \backslash B$: The set difference between the set A and the set B, defined by

$$A \backslash B = \{x \mid x \in A,\ x \notin B\}.$$

- $\|x\|_A$: The distance from a point $x \in \mathbb{R}^n$ to a set $A \subseteq \mathbb{R}^n$, i.e.

$$\|x\|_A = \inf_{y \in A} \|x - y\|.$$

- $B_r(A)$: The set consisting of all points with distance less than or equal to r from the set $A \subseteq \mathbb{R}^n$, i.e.

$$B_r(A) = \left\{x :\ \|x\|_A \le r\right\}.$$

  If $A = \{x\}$, where $x \in \mathbb{R}^n$, $B_r(A)$ reduces to $B_r(x)$. If $x = 0$, we simply write it as B_r.
- $\frac{\partial V}{\partial x}$: The gradient for a function $V :\ \mathbb{R}^n \to \mathbb{R}$ defined by $\frac{\partial V}{\partial x} = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}\right)$.
- $\frac{\partial \Phi}{\partial x}$: The Jacobian matrix for a function $\Phi :\ \mathbb{R}^n \to \mathbb{R}^p$ defined by

$$\frac{\partial \Phi}{\partial x} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial x} \\ \vdots \\ \frac{\partial \Phi_p}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial x_1} & \cdots & \frac{\partial \Phi_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial \Phi_p}{\partial x_1} & \cdots & \frac{\partial \Phi_p}{\partial x_n} \end{bmatrix}.$$

1.2 Nonlinear Dynamical Systems

We consider nonlinear dynamical systems of the form

$$\dot{x}(t) = f(t, x(t), w(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in W \subseteq \mathbb{R}^p$ is a disturbance signal, $t_0 \in \mathbb{R}_{\ge 0}$ is the initial time, and x_0 is the initial state. Let $D \subseteq \mathbb{R}^n$ be an open set and $J \subseteq \mathbb{R}$ be an open interval. We assume that $f :\ J \times D \times W \to \mathbb{R}^n$ satisfies the basic regularity conditions such that, for any $(t_0, x_0) \in J \times D$ and any "well-behaved" input signal $w :\ J \to W$, there exists some interval $J_0 \subseteq J$ containing t_0 and a unique local solution $x :\ J_0 \to \mathbb{R}^n$ such that (1.1) is satisfies for all $t \in J_0$.

1.2.1 Remarks on Existence, Uniqueness, and Continuation of Solutions

Remark 1.1 ***(Basic regularity assumptions)*** The dependence of f in t sometimes comes from obtaining f from another function $F(x, u, w)$, where u is a control input and w is a disturbance input. For input signals (either control or disturbance signals), they are usually assumed to be piecewise continuous. In this setting, we assume that $f(t, x, w)$ is piecewise continuous in t and continuous in x and w to ensure existence of a local solution, for a given piecewise continuous input $w(\cdot)$. This is known as *Peano's existence theorem*. If we further assume that f is locally Lipschitz continuous in x, i.e. for each bounded set $K \subseteq J \times D \times W$, there exists a constant L, such that

$$\|f(t,x,w) - f(t,y,w)\| \leq L\,\|x-y\|\,, \quad \forall (t,x,w),(t,y,w) \in K,$$

then every solution to (1.1) is also unique. Existence and uniqueness under the Lipschitz continuity assumption is often known as *Picard's existence theorem*.

In a more general setting, we are sometimes required to consider input signals that are only measurable with regard to (w.r.t.) t. If we allow the input signals to measurable functions, then we need to relax the condition on f to be measurable w.r.t. t and allow $w(\cdot)$ to be locally essentially bounded, i.e. a measurable function that is "almost" equal to a function that is bounded on a neighborhood of every point. Here, "almost" means almost everywhere, i.e. except on a set of zero measure. *Carathéodory's existence theorem* deals with existence and uniqueness of solutions under this setting.

Remark 1.2 ***(Continuation of solutions)*** A local solution to (1.1) can be extended to its *maximum interval of existence* $J^* \subseteq J$. This interval can be shown to be open relative to J. Consider the special case of $J = \mathbb{R}$ and $D = \mathbb{R}^n$. Then the maximum interval of existence is $\mathbb{R}$, unless the solution *blows up* in finite time, i.e. the solution becomes unbounded as time approaches the boundary of the interval of existence. Solutions that are defined on $\mathbb{R}$ are called *global solutions*. If a solution is defined on $[t_0, \infty)$, we say it is *forward complete*. Similarly, if it is defined on $(-\infty, t_0]$, we say it is *backward complete*. Hence, a usual way to show global existence and completeness of solutions is to show that solutions remain bounded on bounded intervals.

We refer the readers to the Appendix of Sontag [2013] for a precise mathematical treatment of the basic theory of nonlinear systems of Ordinary Differential Equations (ODEs) with inputs (see also [Haddad and Chellaboina, 2011; Khalil, 2002, 2015]).

1.3 Lyapunov Analysis of Stability

Stability is a central notion in systems and control. We define stability for solutions of (1.1] with respect to a compact invariant set as follows.

Definition 1.1 A set $A \subseteq \mathbb{R}^n$ is said to be **(forward) invariant set** for system (1.1), if it is nonempty and, for any $x_0 \in A$, all solutions of (1.1) starting from $x(t_0) = x_0$ stay in A for all $t \geq t_0$.

Definition 1.2 Let $A \subseteq \mathbb{R}^n$ be a compact invariant set for system (1.1). We say that A is **Uniformly Asymptotically Stable** (**UAS**) for system (1.1), if the following two conditions hold:

1. (Uniform stability) For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $\|x_0\|_A \leq \delta$, then $\|x(t)\|_A \leq \varepsilon$ for all $t \geq t_0$, where $x(t)$ is any solution of (1.1) starting from $x(t_0) = x_0$.
2. (Uniform attraction) There exists some $\rho > 0$ such that, for any $\eta > 0$, there exists some $T = T(\rho, \eta) \geq 0$ such that $\|x(t)\|_A \leq \eta$, whenever $\|x_0\|_A \leq \rho$ and $t \geq t_0 + T$, where $x(t)$ is any solution of (1.1) starting from $x(t_0) = x_0$.

We say that A is **Globally Uniformly Asymptotically Stable** (**GUAS**) for system (1.1), if the above conditions hold for δ chosen such that $\lim_{\varepsilon\to\infty}\delta(\varepsilon) = \infty$ and any $\rho > 0$.

We also introduce a special case of asymptotic stability as follows.

Definition 1.3 Let $A \subseteq \mathbb{R}^n$ be a compact invariant set for system (1.1). We say that A is **Uniformly Exponentially Stable** (**UES**) for system (1.1), if then there exist positive constants ρ, k, and c such that

$$\|x(t)\|_A \leq k\|x_0\|_A e^{-c(t-t_0)}, \quad \forall t \geq t_0, \forall \|x_0\|_A \leq \rho,$$

holds for all solutions of (1.1). It is said to be **Globally Uniformly Exponentially Stable** (**GUES**) if the above holds for all $x_0 \in \mathbb{R}^n$.

Remark 1.3 Note that, when $A = \{0\}$ and $x = 0$ is an equilibrium point for system (1.1), i.e. $f(t, 0, w) = 0$ for all $t \geq 0$ and $w \in W$, the above notions of stability reduce to the corresponding notions of stability for an equilibrium.

We present a standard Lyapunov theorem on stability analysis of system (1.1) with respect to a compact invariant set.

Definition 1.4 Let $D \subseteq \mathbb{R}^n$ be an open set and A be a compact set contained in D. We say that a function $V : D \to \mathbb{R}$ is **positive definite with respect to** A if $V(x) = 0$ for all $x \in A$ and $V(x) > 0$ for all $x \in D \backslash A$. Similarly, we say that $V : D \to \mathbb{R}$ is negative definite with respect to A, if $-V$ is positive definite with respect to A.

Definition 1.5 We say that a function $W : \mathbb{R}^n \to \mathbb{R}$ is **radially unbounded** if $W(x) \to \infty$ as $\|x\| \to \infty$.

Theorem 1.1 ***(Lyapunov theorem for stability)*** *Let A be a compact invariant set of system (1.1). Let $D \subseteq \mathbb{R}^n$ be an open set containing A and $V : [0, \infty) \times D \to \mathbb{R}$*

be a continuously differentiable function. Suppose that there exist continuous functions W_i $(i = 1,2,3)$ *that are defined on* D *and positive definite with respect to* A *such that*

$$W_1(x) \le V(t,x) \le W_2(x), \quad \forall x \in D, \quad \forall t \ge 0, \tag{1.2}$$

and

$$\frac{dV}{dt} + \frac{dV}{dx} f(t,x,w) \le -W_3(x), \quad \forall x \in D \backslash A, \quad \forall t \ge 0, \quad \forall w \in W. \tag{1.3}$$

Then A *is UAS for system (1.1). If the above conditions hold for* $D = \mathbb{R}^n$ *and* W_1 *is radially unbounded, then* A *is GUAS for system (1.1).*

Proof: (**Uniform stability**) Fix an arbitrary $\varepsilon > 0$. Without loss of generality, assume that ε is sufficiently small such that $B_\varepsilon(A) \subseteq D$. Let $c < \min_{\{x:\, \|x\|_A = \varepsilon\}} W_1(x)$. Then the set

$$W_1^c := \left\{ x \in B_\varepsilon(A) : \;\; W_1(x) \le c \right\}$$

is contained in the interior of $B_\varepsilon(A)$. Define

$$W_2^c := \left\{ x \in B_\varepsilon(A) : \;\; W_2(x) \le c \right\}.$$

Then $W_2^c \subseteq W_1^c$. Pick $\delta \in (0, \varepsilon)$ such that $B_\delta(A) \subseteq W_2^c$. This is always possible because $W_2(x)$ is continuous on D and $W_2(x) = 0$ for all $x \in A$. Hence, for sufficiently small $\delta > 0$, $W_2(x) \le c$ for all $x \in B_\delta(A)$, which implies $B_\delta(A) \subseteq W_2^c$. We claim that solutions of (1.1) starting from any initial state $x_0 \in B_\delta(A)$ and any initial time $t_0 \ge 0$ will remain in $W_1^c \subseteq B_\varepsilon(A)$ for all $t \ge t_0$. This would imply uniform stability.

Pick any $x_0 \in B_\delta(A) \subseteq W_2^c$. Let $x(t)$ be any solution of (1.1) satisfying $x(t_0) = x_0$. Then

$$V(t_0, x(t_0)) = V(t_0, x_0) \le W_2(x_0) \le c.$$

We have

$$\frac{dV(t,x(t))}{dt} = \frac{\partial V}{\partial t} + \frac{dV}{dx} f(t, x(t), w(t)) \le -W_3(x(t)) \le 0 \tag{1.4}$$

for all $t \ge t_0$, provided that $x(t)$ remains in D. To escape $B_\varepsilon(A)$, the solution needs to cross the boundary of $B_\varepsilon(A)$, on which $W_1(x)) > c$. This implies that $V(t, x(t)) \ge W_1(x(t)) > c$ for some t. This is impossible, since (1.4) implies that $V(t, x(t))$ is non-increasing for $x(t) \in D$.

(**Uniform attraction**) From the proof of uniform stability, we have shown that, for every $\epsilon > 0$ such that $B_\varepsilon(A) \subseteq D$, there exists some $\delta > 0$ such that solutions of (1.1) starting from $B_\delta(A)$ will stay in $B_\varepsilon(A)$ for all $t \ge t_0$. We fix some choice of ϵ and δ for the following argument. Let $\rho = \delta$. Fix any $\eta \in (0, \varepsilon)$ (without loss of generality). We show that there exists $T = T(\eta)$ such that any solution $x(t)$ of (1.1) starting in $B_\delta(A)$ will reach and stay in $B_\eta(A)$ for all $t \geq t_0 + T$. By the argument of uniform stability again, there exists some $\delta' = \delta'(\eta) > 0$ such that solutions of (1.1) starting in $B_{\delta'}(A)$ will stay in $B_\eta(A)$ for all future time. Hence, we only need to show that solutions starting in $B_\delta(A)$ will reach $B_{\delta'}(A)$ within some finite time T.

Let $\lambda = \min_{\{x:\ \delta' \leq \|x\|_A \leq \varepsilon\}} W_3(x) > 0$. Let $c = \max_{x \in B_\varepsilon(A)} W_2(x)$. Choose $T > \frac{c}{\lambda}$. Let $x(t)$ be any solution of (1.1) starting from $x(t_0) = x_0 \in B_\rho(A)$. Without loss of generality, assume $x_0 \notin B_{\delta'}(A)$. Otherwise, $x(t) \in B_\eta(A)$ for all $t \geq t_0$.

Suppose that $x(t)$ never reaches $B_{\delta'}(A)$ on $[t_0, t_0 + T]$. That is, $x(t) \in \left\{x : \delta' \leq \|x\|_A \leq \varepsilon\right\}$ for all $t \in [t_0, t_0 + T]$. Then we have

$$\begin{aligned}
&W_1(x(t_0 + T)) \\
&\quad \leq V(t_0 + T, x(t_0 + T)) \\
&\quad = V(t_0, x_0) + \int_{t_0}^{t_0+T} \frac{dV(s, x(s))}{ds} ds \\
&\quad = V(t_0, x_0) + \int_{t_0}^{t_0+T} \left[\frac{\partial V}{\partial t}(s, x(s)) + \frac{dV}{dx}(s, x(s)) f(s, x(s), w(s))\right] ds \\
&\quad \leq V(t_0, x_0) - \int_{t_0}^{t_0+T} W_3(x(s)) ds \\
&\quad \leq W_2(x_0) - \lambda T \leq c - \lambda T < 0,
\end{aligned} \tag{1.5}$$

which is a contradiction because $W_1(x)$ cannot be negative. Hence, $x(t)$ must have reached $B_{\delta'}(A)$ for some $t \in [t_0, t_0 + T]$. It follows that $x(t) \in B_\eta(A)$ for all $t \geq t_0 + T$.

(**Global stability**) When $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, we can pick $\delta = \delta(\varepsilon)$ for uniform stability such that $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$. This is because as $c \to \infty$, as $\varepsilon \to \infty$, and W_2^c can contain any given $B_\delta(A)$ for c sufficiently large. ■

The next theorem deals with exponential stability.

Theorem 1.2 ***(Lyapunov theorem for exponential stability)*** *Let A be a compact invariant set of system (1.1). Let $D \subseteq \mathbb{R}^n$ be an open set containing A and $V : [0, \infty) \times D \to \mathbb{R}$ be a continuously differentiable function. Suppose that there exist constants c_i $(i = 1, 2, 3)$ and p such that*

$$c_1 \|x\|_A^p \leq V(t, x) \leq c_2 \|x\|_A^p, \quad \forall x \in D, \quad \forall t \geq 0, \tag{1.6}$$

and

$$\frac{dV}{dt} + \frac{dV}{dx} f(t, x, w) \leq -c_3 \|x\|_A^p, \quad \forall x \in D \backslash A, \quad \forall t \geq 0, \quad \forall w \in W. \tag{1.7}$$

Then A is UES for system (1.1). If the above conditions hold for $D = \mathbb{R}^n$, then A is GUES for system (1.1).

Proof: By Theorem 1.1, for any $\varepsilon > 0$ such that $B_\varepsilon(A) \subseteq D$, there exists some $\delta \in (0, \varepsilon)$ such that solutions starting from $B_\delta(A)$ remain in $B_\varepsilon(A)$ for all $t \geq t_0$.

Let $x(t)$ be a solution. By the condition (1.7), we have

$$\begin{aligned}\frac{dV(t,x(t))}{dt} &= \frac{\partial V}{\partial t}(t,x(t)) + \frac{dV}{dx}(t,x(t))f(t,x(t),w(t))\\ &\leq -c_3 \|x(t)\|_A^p\\ &\leq -\frac{c_3}{c_2}V(t,x(t)),\end{aligned}$$

which implies

$$V(t,x(t)) \leq V(t_0,x_0)e^{-\frac{c_3}{c_2}(t-t_0)}.$$

By condition (1.6), the above inequality implies

$$\begin{aligned}\|x(t)\|_A &\leq \left[\frac{V(t,x(t))}{c_1}\right]^{\frac{1}{p}}\\ &\leq \left[\frac{V(t_0,x_0)e^{-\frac{c_3}{c_2}(t-t_0)}}{c_1}\right]^{\frac{1}{p}}\\ &\leq \left[\frac{c_2\|x(t)\|_A^p\, e^{-\frac{c_3}{c_2}(t-t_0)}}{c_1}\right]^{\frac{1}{p}} = \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}}\|x_0\|_A e^{-\frac{c_3}{c_2 p}(t-t_0)}, \quad t \geq t_0,\end{aligned}$$

provided that $x_0 \in B_\delta(A)$. This shows that A is UES for system (1.1). If $D = \mathbb{R}^n$, then the above inequality holds for all $x_0 \in \mathbb{R}^n$, which implies that A is GUES. ∎

1.4 Stability Analysis of Discrete Time Dynamical Systems

In many situations, we also need to consider discrete time dynamical systems. For example, in control applications, a discrete time dynamical system arises when we compute the control inputs at discrete time instants. Discrete time dynamical systems can also be obtained as time discretization of continuous time dynamical systems. Such models are more amenable to sequential decision-making.

Consider a discrete time dynamical system of the form

$$x(t+1) = f(t,x(t),w(t)), \quad x(t_0) = x_0, \tag{1.8}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $w(t) \in W \subseteq \mathbb{R}^p$ is a disturbance signal, $t_0 \in \mathbb{Z}_{\geq 0}$ is the initial time, and x_0 is the initial state. In other words, compared with a continuous time dynamical system, the time t now lies in $\mathbb{Z}$ and the dynamics are described by the difference Eq. (1.8), instead of a differential equation. Given an initial condition $x(t_0) = x_0$, a sequence $\{w(t)\}_{t=t_0}^{\infty}$, a **solution** to (1.8) is a sequence $\{x(t)\}_{t=t_0}^{\infty}$ that satisfies (1.8). For solutions to be defined for all $t \geq t_0$, we assume that $f : \mathbb{R}_{\geq 0} \times D \times W \to D$.

We present similar results on Lyapunov analysis of discrete time dynamical systems of the form (1.8). The definitions are almost identical with that for continuous time systems.

Definition 1.6 A set $A \subseteq \mathbb{R}^n$ is said to be **(forward) invariant set** for system (1.8), if it is nonempty and, for any $x_0 \in A$, all solutions of (1.8) starting from $x(t_0) = x_0$ stay in A for all $t \geq t_0$.

Definition 1.7 Let $A \subseteq \mathbb{R}^n$ be a compact invariant set for system (1.8). We say that A is UAS for system (1.8), if the following two conditions hold:

1. (Uniform stability) For every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, if $\|x_0\|_A \leq \delta$, then $\|x(t)\|_A \leq \varepsilon$ for all $t \geq t_0$, where $x(t)$ is any solution of (1.8) starting from $x(t_0) = x_0$.
2. (Uniform attraction) There exists some $\rho > 0$ such that, for any $\eta > 0$, there exists some $T = T(\rho, \eta) \geq 0$ such that $\|x(t)\|_A \leq \eta$, whenever $\|x_0\|_A \leq \rho$ and $t \geq t_0 + T$, where $x(t)$ is any solution of (1.8) starting from $x(t_0) = x_0$.

We say that A is GUAS for system (1.8), if the above conditions hold for δ chosen such that $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$ and any $\rho > 0$.

Definition 1.8 Let $A \subseteq \mathbb{R}^n$ be a compact invariant set for system (1.8). We say that A is UES for system (1.8), if there exist positive constants ρ, k, and $\lambda \in (0,1)$ such that

$$\|x(t)\|_A \leq k\|x_0\|_A \lambda^{t-t_0}, \quad \forall t \geq t_0, \forall \|x_0\|_A \leq \rho,$$

holds for all solutions of (1.8). It is said to be GUES if the above holds for all $x_0 \in \mathbb{R}^n$.

Lyapunov functions can also be used to analyze the stability of discrete time systems.

Theorem 1.3 ***(Lyapunov theorem for stability)*** *Let A be a compact invariant set of system (1.8). Let $D \subseteq \mathbb{R}^n$ be an open set containing A and $V : [0, \infty) \times D \to \mathbb{R}$ be a continuous function. Suppose that there exist continuous functions W_i $(i = 1, 2, 3)$ that are defined on D and positive definite with respect to A such that*

$$W_1(x) \leq V(t, x) \leq W_2(x), \quad \forall x \in D, \quad \forall t \geq 0, \tag{1.9}$$

and

$$V(t+1, f(t,x,w)) - V(t,x) \le -W_3(x), \quad \forall x \in D, \quad \forall t \ge 0, \quad \forall w \in W. \tag{1.10}$$

Then A is UAS for system (1.8). If the above conditions hold for $D = \mathbb{R}^n$, and W_1 is radially unbounded, then A is GUAS for system (1.8).

Proof: The proof follows almost verbatim the proof of Theorem 1.1, with (1.4) replaced by

$$V(t+1, x(t+1)) - V(t, x(t)) \le -W_3(x(t)) \le 0, \tag{1.11}$$

and (1.5) replaced by

$$\begin{aligned} & W_1(x(t_0+T)) \le V(t_0+T, x(t_0+T)) \\ & = V(t_0, x_0) + \sum_{s=t_0}^{t_0+T-1} [V(s+1, x(s+1)) - V(s, x(s))] \\ & \le V(t_0, x_0) - \sum_{s=t_0}^{t_0+T-1} W_3(x(s))ds \\ & \le W_2(x_0) - \lambda T \le c - \lambda T < 0. \end{aligned} \tag{1.12}$$

The conclusion follows. ■

The next theorem concerns exponential stability of discrete time dynamical systems.

Theorem 1.4 ***(Lyapunov theorem for exponential stability)*** *Let A be a compact invariant set of system (1.8). Let $D \subseteq \mathbb{R}^n$ be an open set containing A and $V : [0,\infty) \times D \to \mathbb{R}$ be a continuous function. Suppose that there exist positive constants c_i ($i = 1,2,3$) and p such that $c_3 \in (0,1)$,*

$$c_1 \|x\|_A^p \le V(t,x) \le c_2 \|x\|_A^p, \quad \forall x \in D, \quad \forall t \ge 0, \tag{1.13}$$

and

$$V(t+1, f(t,x,w)) \le c_3 V(t,x), \quad \forall x \in D, \quad \forall t \ge 0, \quad \forall w \in W. \tag{1.14}$$

Then A is UES for system (1.8). If the above conditions hold for $D = \mathbb{R}^n$ and W_1 is radially unbounded, then A is GUES for system (1.8).

Proof: Similar to the proof of Theorem 1.2, we can obtain

$$V(t, x(t)) \le V(t_0, x_0) c_3^{t-t_0}.$$

which implies

$$
\begin{aligned}
\|x(t)\|_A &\leq \left[\frac{V(t, x(t))}{c_1}\right]^{\frac{1}{p}} \\
&\leq \left[\frac{V(t_0, x_0) c_3^{t-t_0}}{c_1}\right]^{\frac{1}{p}} \\
&\leq \left[\frac{c_2 \, \|x(t)\|_A^p \, c_3^{t-t_0}}{c_1}\right]^{\frac{1}{p}} = \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}} \|x_0\|_A \left(c_3^{1/p}\right)^{t-t_0}, \quad t \geq t_0.
\end{aligned}
$$

The conclusion follows. ■

1.5 Summary

The material of this chapter is standard Lyapunov stability analysis for nonlinear systems. For more comprehensive treatment and further reading, readers are referred to Khalil [2002] and Haddad and Chellaboina [2011]. Here we present the stability analysis with respect to a compact set, which is often the case when the system is perturbed by a nonvanishing uncertainty. We also discuss stability analysis of discrete time systems, which will be used in a later chapter for analyzing a time discretization of a learned dynamic model.

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