CHAPTER 1

MAXWELL'S EQUATIONS, CONSTITUTIVE RELATIONS, WAVE EQUATION, AND POLARIZATION

1.1 Introductory Comments

In 1873, James Clerk Maxwell presented a field theory based mathematical framework for the laws governing electromagnetic (EM) phenomena at large (or macroscopic) scales. His field theory was developed by unifying, into a single body, a set of mathematical equations deduced earlier (through experimental observations) by Coulomb, Gauss, Oersted, Faraday, Ampère, and others, after he had introduced a new displacement current quantity into Ampère's original equation. This displacement current term played a key role in Maxwell's theoretical prediction of the existence of EM waves even before they were produced and detected experimentally. This displacement current term also facilitated Maxwell's unification of the different mathematical laws of electromagnetism into a single, consistent framework. In its modern form, this mathematical framework for EM fields consists of four compact vector equations, in either differential or integral form; these four are known as Maxwell's equations. The compact form of Maxwell's equations in vector notation was originally given by Oliver Heaviside. One notes that Maxwell's equations need to be supplemented by a few additional equations, namely the constitutive relations, to provide information and simplification that is useful for systematically including the presence of matter into these equations in terms of its electric/magnetic properties. It is remarkable that these Maxwell's equations have withstood the test of time; indeed, Maxwell's equations, together with the constitutive relations and the Lorentz law of force (for a charged particle moving in an EM field), are known to accurately predict all classical EM phenomena at macroscopic scales (i.e., at scales large in comparison to atomic dimensions and atomic charges).

From a historical perspective, it is interesting that Coulomb's law for the force between a pair of electric charges, and likewise the force between a pair of magnets, as well as Biot-Savart's law for the force between a pair of current carrying conductors (based on an extension of Oersted's and Ampère's work), respectively, all behave inversely proportional to the square of the distance separating the pair, as is exactly true of Newton's law for the gravitational force of attraction between two masses. Thus, initially it appeared that electromagnetism would develop along the Newtonian concept of "instantaneous action-at-a-distance" as for gravitation. However, Faraday and Gauss viewed the forces of electromagnetism as being based on the concept of "lines of force" or "flux"; the latter constitutes a field concept for electromagnetism that was eventually accepted after it was put on a firm footing, via mathematics, by Maxwell. Furthermore, Maxwell showed not only the existence of EM waves through his field theory based laws for electromagnetism, but he also demonstrated that EM waves travel with a finite velocity

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that equals the velocity of light. The latter finite velocity, therefore, directly requires a finite propagation time for the wave to travel from its source to a distant receiver; clearly, this time delay associated with EM wave propagation was therefore in stark contrast to the Newtonian instantaneous action-at-a-distance concept. Also, EM waves were seen to have all the characteristics of optical reflection and transmission at material boundaries. Furthermore, Maxwell was able to relate optical properties of materials with their electrical properties. Thus, Maxwell showed that light and EM waves were one and the same phenomena; in so doing, Maxwell unified the areas of electromagnetism and optics into a single EM field theory. Somewhat later, in 1905, Einstein's special theory of relativity, which was developed to describe particle dynamics at high velocities (that can be an appreciable fraction of the velocity of light), required the speed of light to be invariant; i.e., to be independent of any inertial frame of reference in which it is measured. In this regard, it is noted that the laws of Newtonian mechanics are only invariant with respect to the Galilean velocity transformation between different inertial reference frames in relative motion. The Galilean transformation assumes a "universal" or absolute time so that time, but not the speed of light, is invariant in the Galilean transformation. It turns out that Maxwell's equations, which predict that EM waves propagate with the speed of light, are not invariant with respect to the Galilean transformation. However, Lorentz had already demonstrated earlier that Maxwell's equations remained invariant with respect to a different transformation, namely that which later came to be known, following Einstein's theory of relativity, as the Lorentz transformation. In particular, it is noted that the special theory of relativity developed by Einstein was based on the equations of velocity transformation which he obtained by invoking the constancy of the speed of light and certain symmetries; the latter transformation was found to be the same as that of Lorentz. No experimental observations have been found to disprove Einstein's relativity theory. It was therefore necessary to modify Newtonian mechanics rather than Maxwell's equations to be consistent with Einstein's theory of relativity in order to predict relativistic effects occurring for high speed particles. Maxwell's equations being already invariant within the Lorentz transformation thus needed no relativistic modifications. It is of course true that at sufficiently low velocities (that are very small compared to the velocity of light), the Lorentz transformation automatically reduces to the Galilean transformation; thus, Newtonian mechanics, although not relativistically correct, becomes valid at sufficiently low velocities.

It may also be mentioned in the passing that Einstein later developed a field theory for gravitation in 1916; in this sense, Einstein did to gravitation what Faraday, Gauss, and Maxwell did to electromagnetism. About the same time as Einstein's relativity theory and following it, the ideas of quantum mechanics began to develop rapidly during the 1900s. It was noted by physicists that, at very short wavelengths, light exhibits quantum effects. The latter property of light was specifically based on Planck's work in quantum theory and on Einstein's photo electric emission theory. In particular, as per quantum mechanics, light energy propagates in discrete bundles or light quanta called photons. Nevertheless, it is now well known that at sufficiently low frequencies, light (i.e., an EM field) exhibits a wave character since the energy of each photon which is directly proportional to the wave frequency, is thus relatively small (for low frequencies). Hence, the more classical, macroscopic EM wave phenomena is seen to result from a statistical average over a large number of photons required to produce the energy generated by conventional EM sources in the classical regime. On the other hand, at microscopic scales and sufficiently high frequencies, the existence of a single photon becomes distinguishable in experiments. It is thus found that light, or EM phenomena, exhibits a dual nature, namely it simultaneously has the characteristics of particles (or photons) and of waves. Modifications of the classical ideas for EM fields to account for relativistic and quantum effects forms a branch of physics referred to as relativistic quantum electrodynamics. Classical EM field theory, or classical electrodynamics, to which this book is restricted, may be seen as the limit of quantum electrodynamics for the propagation of low energy/momentum fields, thus requiring a statistical averaging over a large number of photons. As stated earlier, Maxwell's equations have been, up to the present time, found to be complete for describing all classical, macroscopic, EM field behavior. Although the present book deals primarily with classical EM theory suitable for engineering graduate students, the above somewhat brief historical discussion is included to highlight the truly powerful scientific impact that Maxwell's EM field theory has had in incorporating optics within EM; in the development of Einstein's special theory of relativity via the Lorentz transformations which the EM fields already satisfy; and also through its involvement with the developments in relativistic quantum electrodynamics.

Maxwell's equations have had a profound impact on society through their role in paving the way for many major technological inventions, some of which are enumerated below. In particular, Hertz performed experiments during 1887-1891 which clearly demonstrated that EM waves could certainly be produced and detected, thereby strongly confirming Maxwell's theory. Maxwell's equations and the production of EM waves have led to the development of EM-powered transmission lines, the radio, microwave communications, television, weather radar, satellite imagery of the earth via synthetic aperture radar, ground penetrating radars (GPRs), microwave ovens, magnetic resonance imaging (MRI), modern satellite and wireless mobile communications, radio frequency identification (RFID), transmission and reception of signals for global navigation satellite systems (GNSSs), and global positioning systems (GPSs), among others.

The development which follows will be restricted to the analysis of EM radiation, propagation, scattering and diffraction in linear time-invariant media at macroscopic scales, and the international system of units (SI units) will be employed.

One may begin by recognizing the basic source of EM fields to be the electric charge. Assuming that an amount of charge Q (in Coulombs) is distributed in some volume V, one can then define a corresponding macroscopic scalar, time-dependent volume charge density denoted by boldface $\boldsymbol{\rho}_v(\bar{r}, t)$ in Coulombs per cubic meter (C/m^3), within V via the following relationship:

$$Q = \int_V \boldsymbol{\rho}_v \left(\bar{r}, t\right) dv, \tag{1.1}$$

where $\bar{r}$ is the position vector at an observation point in V and t denotes time.

At microscopic (atomic/subatomic) scales, the fields and sources in media exhibit rapid space-time fluctuations. Therefore, at macroscopic scales, it is convenient to average the microscopic quantities over appropriate space-time intervals. Thus, $\boldsymbol{\rho}_v$ in (1.1), which is evaluated at any point $\bar{r}$ in ΔV, is given by the following space-time averages:

$$\boldsymbol{\rho}_v \left(\bar{r}, t\right) \approx \frac{1}{\Delta V \Delta t} \int_{\Delta V} \int_{t-\Delta T/2}^{t+\Delta T/2} \boldsymbol{\rho}'_v \left(\bar{r} + \bar{r}', t'\right) dt' dv', \tag{1.2}$$

where $\boldsymbol{\rho}'_v$ is the microscopic charge density at any dv' within ΔV, and $\bar{r} + \bar{r}'$ denotes the position vector at dv' as shown in Figure 1.1. The ΔV must be big enough so as to contain a sufficiently large number of samples or atomic particles, but small enough to resolve the overall significant spatial variation of $\boldsymbol{\rho}_v$ as a function of $\bar{r}$. Also, ΔT must be big enough to average over a large number of temporal fluctuations

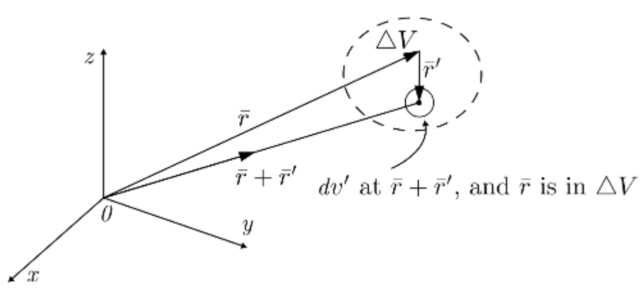

Figure 1.1 Space-time averaging of microscopic sources and fields over ΔV and ΔT intervals to obtain macroscopic values of sources and fields.

of $\rho_v\left(\bar{r}+\bar{r}', t'\right)$, but small enough to resolve the overall significant temporal variation of ρ_v as a function of t.

The electric field intensity, $\bar{\mathcal{E}}(\bar{r}, t)$ (in volts/meter) at the macroscopic scales is similarly given by a space-time averaging of the corresponding microscopic values, $\bar{\mathcal{E}}'\left(\bar{r}+\bar{r}', t\right)$, within ΔV and ΔT, namely

$$\bar{\mathcal{E}}\left(\bar{r}, t\right) \approx \frac{1}{\Delta V \Delta t} \int_{\Delta V} \int_{t-\Delta T/2}^{t+\Delta T/2} \bar{\mathcal{E}}'\left(\bar{r}+\bar{r}', t'\right) dt' dv'. \tag{1.3}$$

It is noted that the interaction of EM waves with matter can be analyzed in a far simpler manner at macroscopic scales where it becomes possible to treat matter as a continuum using the above space-time averages. Such a continuum model also allows a wide class of practical problems to be analyzed in a highly accurate manner.

For the case of single frequency or time-harmonic variation of the EM fields and sources, one can select ΔV and ΔT so that $\Delta V \ll \lambda^3$ and $\Delta T \ll 1/f$, where λ is the wavelength of the EM fields and f is their temporal frequency. Here λ is in meters (m), and f is in Hertz (Hz).

A fundamental postulate in EM theory is that charge is always conserved; i.e., it is assumed that charge can never be created or destroyed. However, equal amounts of positive and negative charge can be made to appear by separation or made to disappear by recombination. Also, the charge is assumed to have the same value whether it is in motion or at rest. Any net motion of charges along a specific path constitutes a current $\mathcal{I}$ (in amperes or amps). In particular, the current, $\mathcal{I}$, which flows across any surface S is defined to exactly equal the rate at which charge crosses that surface. Hence, the rate of decrease of charge within a volume, V, bounded by a closed mathematical surface S_V constitutes an outward flow of current $\mathcal{I}$, across that surface via conservation of charge. Therefore, $\mathcal{I}$ constitutes a transfer of incremental charge ΔQ from the region internal to V into the region external to it in an incremental time Δt (as $\Delta t \to 0$); i.e.,

$$\mathcal{I} = \lim_{\Delta t \to 0}\left[-\frac{\Delta Q}{\Delta t}\right] = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_v\left(\bar{r}, t\right) dv, \tag{1.4}$$

where Q can be either positive or negative in (1.4). One can also define a volume current density $\bar{\mathcal{J}}_v$ (in amperes/m^2) at any point to be equal in magnitude to the charge which crosses per unit area of a surface in unit time at that point; the direction $\bar{\mathcal{J}}_v$ is along the motion of charge across the surface at that point. The current, $\mathcal{I}$, flowing across S_V due to decrease of charge within V bounded by S_V (as shown in Figure 1.2) can thus be expressed as

$$\mathcal{I} = \oint_{S_v} \bar{\mathcal{J}}_v\left(\bar{r}, t\right) \cdot \hat{n} ds, \tag{1.5}$$

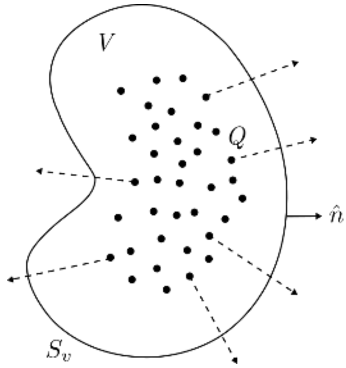

Figure 1.2 Rate of decrease of Q in V constitutes a flow of current $\mathcal{I}$.

where $\hat{n}$ is the outward unit normal vector to S_v.

From (1.4) and (1.5),

$$\oint_{S_v} \overline{\mathcal{J}}_v \left(\bar{r}, t\right) \cdot \hat{n} ds = -\frac{d}{dt} \int_V \rho_v \left(\bar{r}, t\right) dv. \tag{1.6}$$

An application of the divergence theorem to the left-hand side (LHS) of (1.6), and the assumption that S_V is nonmoving so that the time derivative operator in (1.6) can be interchanged with the volume integral operation on the right-hand side (RHS) of (1.6), respectively, leads to

$$\int_V \left[\nabla \cdot \overline{\mathcal{J}}_v + \frac{\partial \rho_v}{\partial t}\right] dv = 0. \tag{1.7}$$

Since (1.7) is true for any V, it follows that the integrand must vanish, i.e.,

$$\nabla \cdot \overline{\mathcal{J}}_v \left(\bar{r}, t\right) = -\frac{\partial \rho_v}{\partial t} \left(\bar{r}, t\right). \tag{1.8}$$

The above relation is referred to as the equation of continuity of current; it is based directly on the principle of conservation of charge as seen from the above development. Clearly, (1.8) implies that any sudden appearance or disappearance of charge must always be accompanied by a corresponding and simultaneous flow of electric current, and that a time-varying charge density, ρ_v, is not independent of $\overline{\mathcal{J}}_v$, respectively. The latter leads one to conclude that time-varying EM fields can thus be found from a knowledge of $\overline{\mathcal{J}}_v(\bar{r}, t)$ alone since $\overline{\mathcal{J}}_v$ and ρ_v are related via (1.8). Equation (1.8) constitutes a fundamental equation of EM theory.

1.2 Maxwell's Equations

As mentioned previously, Maxwell's equations combine into a single framework, the separate laws of electromagnetism deduced experimentally by Gauss, Faraday, and Ampère (with Maxwell's modification to include the displacement current), respectively, together with an analog of Gauss' law for magnetic fields. Such a single system of laws also shows their inter-relationships in a clear fashion and consists of a set of four basic equations which can be summarized as follows.

First, Gauss' law in integral form states that the total electric flux leaving a closed mathematical surface S_V must exactly equal the total charge Q enclosed within the volume, V, bounded by S_V. The electric flux is defined in terms of an electric flux density vector, $\overline{\mathcal{D}}(\bar{r}, t)$ (in Coulombs/m^2). Therefore, Gauss' theorem yields

$$\oint_{S_V} \overline{\mathcal{D}} \left(\bar{r}, t\right) \cdot \hat{n} ds = Q = \int_V \rho_v \left(\bar{r}, t\right) dv, \tag{1.9}$$

where ρ_v is the charge density corresponding to Q in V. From the divergence theorem, (1.9) becomes

$$\int_V \left[\nabla \cdot \overline{\mathcal{D}} - \rho_v\right] dv = 0, \quad \text{for any } V, \tag{1.10}$$

so that one obtains Gauss' law in differential form at any point as

$$\nabla \cdot \overline{\mathcal{D}} \left(\bar{r}, t\right) = \rho_v \left(\bar{r}, t\right). \tag{1.11}$$

Second, Faraday's law of induction states that the electromotive force (emf) induced along any closed mathematical path, C, is given by the negative of the total time rate of change of the magnetic flux passing through the area S_C enclosed by C, i.e.,

$$\oint_C \overline{\mathcal{E}} \cdot \overline{dl} = -\frac{d}{dt} \iint_{S_C} \overline{\mathcal{B}} \cdot \overline{ds}, \tag{1.12}$$

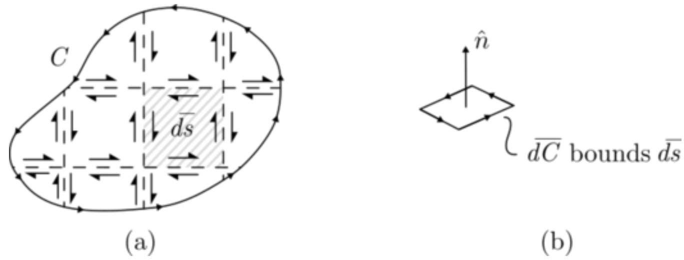

Figure 1.3 (a) The contour C bounds the surface S_C and (b) element $\overline{ds}$ of S_C. Here S_C is subdivided into smaller incremental areas $\overline{ds}$. Let $\overline{dC}$ be the contour which bounds $\overline{ds}$. The arrows show the CCW sense in which the path $\overline{dC}$ encloses the area $\overline{ds}$. Also, $\overline{ds} = \hat{n}\, ds$ where $\hat{n}$ is the unit normal to $\overline{ds}$.

where $\overline{\mathcal{E}}$ is defined as the electric field intensity (in Volts/m), and the line integral in the above equality constitutes the emf induced along C. It is noted that the contour C is taken in the counter clockwise (CCW) sense so that region bounded by C lies on the left as one moves along C. Furthermore, the unit normal vector $\hat{n} = \frac{\overline{ds}}{|\overline{ds}|}$, to an incremental area $\overline{ds}$, which lies within the surface area S_c bounded by C is given by the right-hand rule, where the thumb points along $\hat{n}$ while the remaining fingers are closed in the CCW sense around $\overline{dC}$; here $\overline{dC}$ is an incremental CCW closed path enclosing $\overline{ds}$ shown in Figure 1.3. Also, $\overline{\mathcal{B}}$ (in Webers/m^2) above denotes the magnetic flux density vector. From Stokes' theorem and the assumption of a nonmoving boundary, C, one can express the above equality as

$$\iint_{S_C} \left[\nabla \times \overline{\mathcal{E}} + \frac{\partial \overline{\mathcal{B}}}{\partial t}\right] \cdot \hat{n}\, ds = 0, \quad \text{for any } S_C, \tag{1.13}$$

so that one obtains Faraday's law in differential form at any point, namely

$$\nabla \times \overline{\mathcal{E}}\left(\overline{r}, t\right) = -\frac{\partial \overline{\mathcal{B}}}{\partial t}. \tag{1.14}$$

Third, the Maxwell-Ampère law states that the line integral of the magnetic field intensity, $\overline{\mathcal{H}}$ (in amps/m), around any closed mathematical path, C, exactly equals the total current crossing the surface area S_C enclosed by C; thus,

$$\oint_C \overline{\mathcal{H}} \cdot \overline{dl} = \iint_{S_c} \left[\overline{\mathcal{J}}_v + \frac{\partial \overline{\mathcal{D}}}{\partial t}\right] \cdot \overline{ds}, \tag{1.15}$$

where $\overline{\mathcal{J}}_v$ is related to motion of charges, and $\frac{\partial \overline{\mathcal{D}}}{\partial t}$ represents Maxwell's displacement current density. Employing Stokes' theorem to the above equation yields

$$\iint_{S_C} \left[\nabla \times \overline{\mathcal{H}} - \left(\overline{\mathcal{J}}_v + \frac{\partial \overline{\mathcal{D}}}{\partial t}\right)\right] \cdot \hat{n}\, ds = 0, \quad \text{for any } S_C, \tag{1.16}$$

thereby leading to the differential form of Maxwell-Ampère law valid at any point as

$$\nabla \times \overline{\mathcal{H}} = \overline{\mathcal{J}}_v + \frac{\partial \overline{\mathcal{D}}}{\partial t}. \tag{1.17}$$

Finally, based on experimental evidence that isolated magnetic charges do not exist, or that $\overline{\mathcal{B}}$ is always solenoidal, leads to the fourth law expressed as follows:

$$\oint_{S_V} \overline{\mathcal{B}} \cdot \hat{n}\, ds = 0. \tag{1.18}$$

In other words, the net outflow of the total magnetic flux from any closed surface S_V containing magnetic poles is zero because they occur only in opposite pairs so that the total magnetic charge within the volume, V, bounded by S_V vanishes. It follows that as much magnetic flux leaving S_V (due to positive magnetic poles or magnetic charges in V) also enters S_V (due to negative magnetic poles in V whose value is exactly equal and opposite to the positive ones in V), thus leaving a net outflow of magnetic flux from S_V to be zero as indicated by (1.18). From an application of the divergence theorem to the above equation, one obtains

$$\int_V \nabla \cdot \overline{\mathcal{B}}\, dv = 0, \quad \text{for any } V, \tag{1.19}$$

or

$$\nabla \cdot \overline{\mathcal{B}}\left(\overline{r}, t\right) = 0. \tag{1.20}$$

One is referred to Appendix A for a summary of vector identities and integral theorems including the Divergence and Stokes Theorems which are used above.

An additional fundamental relationship in EM theory is the Lorentz law of force, which predicts the force on a moving charge in an EM field; it is given by

$$\overline{\mathcal{F}}_v = \rho_v \left[\overline{\mathcal{E}} + \overline{v} \times \overline{\mathcal{B}}\right]. \tag{1.21}$$

In the above, $\overline{\mathcal{F}}_v$ (in Newtons/m^3) represents the volume force density that is associated with the volume charge density ρ_v which moves with a velocity $\overline{v}$ in the presence of the EM fields $\left(\overline{\mathcal{E}}, \overline{\mathcal{B}}\right)$. The total force $\overline{\mathcal{F}}$ is given by $\int_V \overline{\mathcal{F}}_v\, dv$, where the integration is over the volume V containing the charges which define ρ_v. Hence, $\overline{\mathcal{F}} = Q\left[\overline{\mathcal{E}} + \overline{v} \times \overline{\mathcal{B}}\right]$ if $\overline{\mathcal{E}}$ and $\overline{\mathcal{B}}$ are uniform in V which contains Q.

The expressions in (1.11), (1.14), (1.17) and (1.20), respectively, taken together constitute the system of four Maxwell's equations.

It is evident that the four Maxwell's equations consist of two divergence and two curl equations; in this regard, it may be mentioned that generally the divergence of a vector field provides its scalar source density, while the curl of a vector field provides its vector source density.

The $\overline{\mathcal{J}}_v$ term in Maxwell's equation (1.17) can be decomposed as

$$\overline{\mathcal{J}}_v = \overline{\mathcal{J}}_{vi} + \overline{\mathcal{J}}_{vc}, \tag{1.22}$$

in which $\overline{\mathcal{J}}_{vi}$ represents the impressed source current density, and it is the primary source of the EM fields. The $\overline{\mathcal{J}}_{vi}$ typically results from the conversion of other forms of energy (such as chemical, mechanical, etc.) into electrical energy to serve as a generator, or a primary source, for the EM fields. On the other hand, $\overline{\mathcal{J}}_{vc}$ is produced by the motion of charges which results when the latter are acted on by the EM fields that are produced originally by $\overline{\mathcal{J}}_{vi}$. In particular, $\overline{\mathcal{J}}_{vc}$ could represent the conduction current due to the presence of an EM field within a conducting medium. Also, $\overline{\mathcal{J}}_{vc}$ could represent convection or a diffusion current that again results from an application of an EM field to charges in empty space, or to a semiconductor, respectively. Convection currents can as well be produced by a moving charged medium, or by the motion of an object with static charge. The diffusion current in a semiconductor is a transient effect due to charge (electron and hole) migration away from the region of same initial charge concentration. Other material or media-related effects not explicitly described in $\overline{\mathcal{J}}_{vc}$ are contained implicitly in the fields $\overline{\mathcal{E}}$, $\overline{\mathcal{D}}$, $\overline{\mathcal{B}}$ and $\overline{\mathcal{H}}$; the latter will be discussed in Section 1.3.

In many instances, the conducting medium is linear and isotropic, with negligible dispersion (see Section 1.3 for discussion on dispersion) so that one can employ the point form of Ohm's law, namely $\overline{\mathcal{J}}_{vc} = \sigma\overline{\mathcal{E}}$, where σ is the conductivity of the medium. In the case of convection currents, the $\overline{\mathcal{J}}_{vc}$ can be expressed as $\overline{\mathcal{J}}_{vc} = \rho_v\overline{v}$, where $\overline{v}$ is the velocity of ρ_v under the action of the EM field produced by $\overline{\mathcal{J}}_{vi}$. For semiconducting materials, $\overline{\mathcal{J}}_{vc} = -q\left[\mathcal{D}_e\nabla n_e - \mathcal{D}_h\nabla n_h\right]$, where q is the amount of charge undergoing migration with the diffusion constants $\mathcal{D}_e$ and $\mathcal{D}_h$ for electrons and holes, respectively, and with ∇n_e and ∇n_h representing the gradients of the electron and hole volume densities. In a simple two-carrier system consisting of equal densities of positive and negative charge, as is the case for a solid electric conductor, one can write $\overline{\mathcal{J}}_{vc} = \rho_v^+ \overline{v}^+ + \rho_v^- \overline{v}^- = |\rho_v^+|\,\overline{v}^+ - |\rho_v^-|\,\overline{v}^-$. For a conductor, $|\rho_v^+| = |\rho_v^-|$ with $\rho_v^+ + \rho_v^- = 0$. Furthermore, $\overline{\mathcal{J}}_{vc} \approx -|\rho_v^-|\,\overline{v}^-$ since $\overline{v}^+ \approx 0$ for this situation due to the fact that the positive charges are composed of protons, which are about 2000 times heavier than the electrons and thus almost immobile. It is noted that $\overline{v}^\pm$ represents the net or drift velocity of the charge density $\rho_v^\pm$; clearly, a random motion of charges does not constitute a current, and there must be a net flow of charge along a specific path to define current flow as mentioned previously. Finally, it is noted that, in general, one must sum up all possible types of $\overline{\mathcal{J}}_{vc}$ to obtain the total value of $\overline{\mathcal{J}}_v$ in (1.22) and hence in (1.17).

One can also define surface and line currents in addition to volume currents. The volume current density, $\overline{\mathcal{J}}_v$, at any point in volume V can be depicted as in Figure 1.4, where a current $\Delta\mathcal{I}$ passes through an elemental cross-sectional area ΔS of a surface S at that point within a volume V. In particular

$$\Delta\mathcal{I} \approx \overline{\mathcal{J}}_v \cdot \hat{n}\Delta S,$$

where $\hat{n}$ is a unit normal vector to ΔS at which $\overline{\mathcal{J}}_v$ is defined; thus,

$$\overline{\mathcal{J}}_v \cdot \hat{n} = \lim_{\Delta S \to 0} \frac{\Delta\mathcal{I}}{\Delta S}. \tag{1.23}$$

One can describe a surface current density $\overline{\mathcal{J}}_S$ at any point on a surface S due to a flow of current per unit length at that point. In particular, if a current $\Delta\mathcal{I}$ flows across an element of length ΔL at any point on S, then

$$\Delta\mathcal{I} = \overline{\mathcal{J}}_S \cdot \hat{n}_L \Delta L,$$

where $\hat{n}_L$ is normal to the incremental length ΔL as in Figure 1.5. Thus,

$$\overline{\mathcal{J}}_S \cdot \hat{n}_L = \lim_{\Delta L \to 0} \frac{\Delta\mathcal{I}}{\Delta L}. \tag{1.24}$$

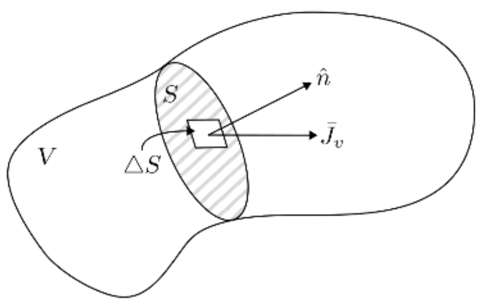

Figure 1.4 Volume current density $\overline{\mathcal{J}}_v$ at any point in V defined by a flow of current per unit cross-sectional area of a surface S within some volume V.

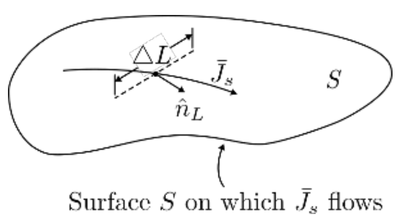

Figure 1.5 Surface current density $\overline{\mathcal{J}}_S$ at any point on S flowing on a surface.

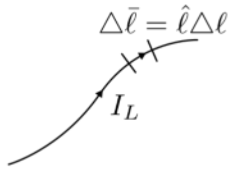

Figure 1.6 Line current $\mathcal{I}_L$ flowing along $\Delta\overline{\ell}$ at any point on an arbitrary curved path; here $\Delta\overline{\ell} = \hat{\ell}\,\Delta\,\ell$ and $\hat{\ell}$ is tangent to the curvilinear path at each point along that path.

The units of $\overline{\mathcal{J}}_S$ are amperes/m, whereas those of $\overline{\mathcal{J}}_v$ are amperes/m^2, as defined earlier.

A line current, $\mathcal{I}_L$ (as shown in Figure 1.6) is simply defined as a current (in amperes) which is restricted to flow only along a line which may be straight or even arbitrarily curved.

It is interesting to note that the Maxwell's equation in (1.20) can be derived from the other Maxwell's equation in (1.14) by taking the divergence of the latter, namely,

$$\nabla \cdot \nabla \times \overline{\mathcal{E}} = -\nabla \cdot \frac{\partial \overline{\mathcal{B}}}{\partial t}, \tag{1.25}$$

and since the divergence of the curl of any vector is identically zero one obtains

$$0 = -\frac{\partial}{\partial t}\left(\nabla \cdot \overline{\mathcal{B}}\right), \tag{1.26}$$

where the space and time derivatives have been interchanged since $\overline{\mathcal{B}}$ is assumed to have continuous derivatives. The preceding equation indicates that $\nabla \cdot \overline{\mathcal{B}}$ is everywhere constant in time. Assume that $\overline{\mathcal{B}}$ was nonexistent prior to the time (in the finitely remote past) when it was originally generated; hence, this constant had to be zero then, and because this constant does not change with time, it must therefore remain zero for all time. Consequently, from (1.26) one obtains (1.20), namely

$$\nabla \cdot \overline{\mathcal{B}} = 0.$$

Likewise, Maxwell's equation, (1.11) can be obtained from Maxwell's equation (1.17) and the equation of continuity in (1.8), respectively. In particular, the divergence of (1.17) yields

$$0 = \nabla \cdot \overline{\mathcal{J}}_v + \frac{\partial}{\partial t}\nabla \cdot \overline{\mathcal{D}}, \tag{1.27}$$

upon assuming continuous derivatives for $\overline{\mathcal{D}}$. From (1.8), the above becomes

$$\frac{\partial}{\partial t}\left[-\rho_v + \nabla \cdot \overline{\mathcal{D}}\right] = 0.$$

One concludes that $\nabla \cdot \overline{\mathcal{D}} - \rho_v$ = constant with respect to time. Again, using arguments that ρ_v and $\overline{\mathcal{D}}$ had to be zero in the finitely remote past, one can show that the value of this constant must always be zero. Therefore, one obtains (1.11), namely

$$\nabla \cdot \overline{\mathcal{D}} = \rho_v.$$

Clearly, one can also deduce the continuity equation of (1.8) from a combination of Maxwell's equations (1.11) and (1.17) after taking the divergence of the latter.

Problem 1.1

Derive the equation of continuity from two of the four Maxwell's equations and the use of the divergence theorem.

It is noted, from the above Problem 1.1, that one could not have obtained the continuity equation of (1.8) as mentioned above by combining Maxwell's equations (1.11) and (1.17) if the displacement current term $\frac{\partial \overline{\mathcal{D}}}{\partial t}$ had been missing in (1.17); also without the displacement current, it is not possible to show the mathematical existence of EM waves as mentioned earlier. The above discussion serves to indicate the interrelationships between Maxwell's equations, and also between those equations and the continuity equation. However, it is also clear from above that only the two Maxwell's curl equations (namely, (1.14) and (1.17)) are independent, since the remaining two Maxwell's divergence equations (of (1.11) and (1.20)) can be deduced from the curl equations and the continuity equation. Consequently, one cannot in general solve for the four unknown field vectors, $\overline{\mathcal{E}}$, $\overline{\mathcal{D}}$, $\overline{\mathcal{B}}$, and $\overline{\mathcal{H}}$ from only two independent Maxwell's curl equations as explained below in Section 1.3. Therefore, it becomes necessary to develop additional equations, referred to as the constitutive relations that help to relate $\overline{\mathcal{D}}$ and $\overline{\mathcal{B}}$, respectively, to the remaining fields ($\overline{\mathcal{E}}$ and $\overline{\mathcal{H}}$). Furthermore, if there is a conduction current $\overline{\mathcal{J}}_{vc}$ present in the total $\overline{\mathcal{J}}_v$ of (1.14), then it is necessary to have one more constitutive relation for the additional unknown $\overline{\mathcal{J}}_{vc}$ which expresses $\overline{\mathcal{J}}_{vc}$ in terms of the fields $\overline{\mathcal{E}}$ and $\overline{\mathcal{H}}$.

1.3 Constitutive Relations

Maxwell's equations presented earlier are valid for general media; they are summarized below for later convenience, namely

$$\nabla \times \overline{\mathcal{H}} - \frac{\partial \overline{\mathcal{D}}}{\partial t} = \overline{\mathcal{J}}_{vi} + \overline{\mathcal{J}}_{vc}, \tag{1.28}$$

$$\nabla \times \overline{\mathcal{E}} + \frac{\partial \overline{\mathcal{B}}}{\partial t} = 0, \tag{1.29}$$

$$\nabla \cdot \overline{\mathcal{D}} = \rho_{vi} + \rho_{vc}, \tag{1.30}$$

$$\nabla \cdot \overline{\mathcal{B}} = 0. \tag{1.31}$$

The impressed electric current source density $\overline{\mathcal{J}}_{vi}$, which is the primary source of the electromagnetic fields, satisfies the continuity equation given by

$$\nabla \cdot \overline{\mathcal{J}}_{vi} = -\frac{\partial \rho_{vi}}{\partial t}. \tag{1.32}$$

Therefore, it is clear from the divergence of (1.28) and the use of (1.30) together with (1.32) that, as expected, the current density $\overline{\mathcal{J}}_{vc}$ satisfies its own continuity condition:

$$\nabla \cdot \overline{\mathcal{J}}_{vc} = -\frac{\partial \rho_{vc}}{\partial t}. \tag{1.33}$$

It is noted, as mentioned previously, that only (1.28) and (1.29) are independent because (1.30) and (1.31) can be derived from them. The $\overline{\mathcal{J}}_{vi}$ is of course assumed to be a known quantity; thus, ρ_{vi} is also assumed to be known via (1.32). One also notes that the material medium effects are partly contained in $\overline{\mathcal{J}}_{vc}$ and ρ_{vc} as indicated in (1.28) and (1.30) of Maxwell's equations; their interrelation being shown in (1.33). The other material media effects are contained implicitly in (1.28)-(1.31), and they result from dielectric and magnetic effects in such media as will be discussed next.

The solution to Maxwell's equations for conducting media in (1.28)-(1.31) in general requires one to solve for the five unknowns $\overline{\mathcal{E}}$, $\overline{\mathcal{D}}$, $\overline{\mathcal{B}}$, $\overline{\mathcal{H}}$, and $\overline{\mathcal{J}}_{vc}$, which therefore cannot be found from just the two independent Maxwell's curl equations (1.28) and (1.29). Obviously, three additional independent equations are required to determine all the unknowns as indicated previously. In order to arrive at these additional independent relations, it is useful to first write Maxwell's equations for empty (or free) space for which it is found experimentally that the following constitutive relations hold, namely

$$\overline{\mathcal{D}} = \epsilon_0 \overline{\mathcal{E}}, \tag{1.34}$$

$$\overline{\mathcal{B}} = \mu_0 \overline{\mathcal{H}}. \tag{1.35}$$

The ϵ_0 and μ_0 appearing in (1.34) and (1.35) are given by $\epsilon_0 = 8.854\times10^{-12}$ Farads/m and $\mu_0 = 4\pi\times10^{-7}$ Henrys/m. Such extremely simple linear constitutive relations for empty space (or free space) also remain valid for the static (non time-varying) field case. Also, ρ_{vc} and hence $\overline{\mathcal{J}}_{vc}$ must be zero for empty space. Consequently, (1.28) through (1.33) become the following for empty space:

$$\frac{\nabla \times \overline{\mathcal{B}}}{\mu_0} - \epsilon_0 \frac{\partial \overline{\mathcal{E}}}{\partial t} = \overline{\mathcal{J}}_{vi}, \tag{1.36}$$

$$\nabla \times \overline{\mathcal{E}} + \frac{\partial \overline{\mathcal{B}}}{\partial t} = 0, \tag{1.37}$$

$$\epsilon_0 \nabla \cdot \overline{\mathcal{E}} = \rho_{vi}, \tag{1.38}$$

$$\nabla \cdot \overline{\mathcal{B}} = 0, \tag{1.39}$$

$$\nabla \cdot \overline{\mathcal{J}}_{vi} = -\frac{\partial \rho_{vi}}{\partial t}. \tag{1.40}$$

It may be mentioned that the velocity of light in free space, denoted here by c, happens to be related to (μ_0, ϵ_0) by $c = 1/\sqrt{\mu_0 \epsilon_0}$, where ϵ_0 and μ_0 are the same as in (1.34) and (1.35). EM waves can be shown to travel in free space with the same velocity, c. Furthermore, EM waves travel without the need of a medium for propagation, and they exhibit the same reflection and transmission properties at boundaries as light waves do. It was thus confirmed by Maxwell that light is an EM phenomenon as stated earlier.

When matter is introduced into an EM field which originally exists in free space, then this field changes because of the additional field which is created by the secondary sources, defined here as $\overline{\mathcal{J}}_{vm}$, $\overline{\mathcal{J}}_{vc}$, ρ_{vm}, and ρ_{vc}, that are induced within the material medium. Consequently, one may replace the entire effects of the material medium by these equivalent impressed sources $\overline{\mathcal{J}}_{vm}$, $\overline{\mathcal{J}}_{vc}$, ρ_{vm}, and ρ_{vc} which now exist in free space, i.e., in the absence of the material medium, as shown in Figure 1.7(a) and (b); i.e., these equivalent impressed sources, whose values are identical to the original material induced sources, generate the same effects as the material medium which they have replaced. It is noted that the terminology "impressed sources" used here implies that if these were turned off, then only free space would

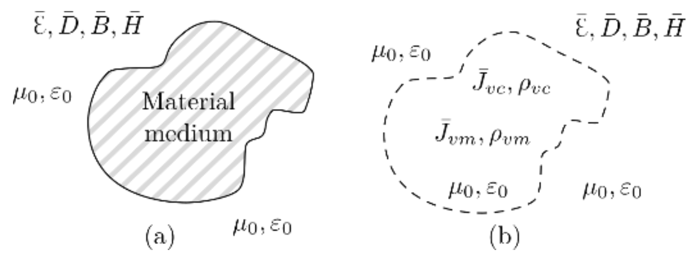

Figure 1.7 Replacing a material body in free space by a set of equivalent impressed sources in free space. (a) Original situation and (b) equivalent situation.

remain in their place. In particular, the equivalent impressed sources $\left(\overline{\mathcal{J}}_{vm}, \rho_{vm}\right)$ arise from dielectric and magnetic effects in the material which are implicitly present in (1.28)-(1.33), while $\left(\overline{\mathcal{J}}_{vc}, \rho_{vc}\right)$ that have been introduced previously arise specifically from conduction effects in the material. One may now account for the presence of a material body by rewriting free space Maxwell's equations (1.36) through (1.39), and the equation of continuity, to include the presence of these additional equivalent impressed sources ($\overline{\mathcal{J}}_{vm}$, $\overline{\mathcal{J}}_{vc}$, ρ_{vm}, and ρ_{vc}) which exist in free space with the material removed[1]; thus, the fields $\overline{\mathcal{E}}$, $\overline{\mathcal{D}}$, $\overline{\mathcal{B}}$, and $\overline{\mathcal{H}}$ corresponding to the equivalent free space situation in Figure 1.7(b) must now satisfy the following:

$$\frac{\nabla \times \overline{\mathcal{B}}}{\mu_0} - \epsilon_0 \frac{\partial \overline{\mathcal{E}}}{\partial t} = \overline{\mathcal{J}}_{vi} + \overline{\mathcal{J}}_{vm} + \overline{\mathcal{J}}_{vc}, \tag{1.41}$$

$$\nabla \times \overline{\mathcal{E}} + \frac{\partial \overline{\mathcal{B}}}{\partial t} = 0, \tag{1.42}$$

$$\epsilon_0 \nabla \cdot \overline{\mathcal{E}} = \rho_{vi} + \rho_{vm} + \rho_{vc}, \tag{1.43}$$

$$\nabla \cdot \overline{\mathcal{B}} = 0, \tag{1.44}$$

$$\nabla \cdot \overline{\mathcal{J}}_{vi} = -\frac{\partial \rho_{vi}}{\partial t}, \tag{1.45}$$

$$\nabla \cdot \overline{\mathcal{J}}_{vc} = -\frac{\partial \rho_{vc}}{\partial t}. \tag{1.46}$$

Note that (1.45) is identical to (1.40) and remains valid for the primary sources $\overline{\mathcal{J}}_{vi}$ and ρ_{vi}, while (1.46) follows directly (as seen in (1.33)) from the conservation of charge associated with conduction (or free) electrons in the material body. It is important to note that if convective or free charges are present "outside the material," in addition to the conduction effects within the material, then they must be added separately as $\overline{\mathcal{J}}_{fc}$ into the right side of (1.41) and as ρ_{fc} into the right side of (1.43), with the understanding that $\nabla \cdot \overline{\mathcal{J}}_{fc} = -\partial \rho_{fc}/\partial t$ is true. From here on, $\overline{\mathcal{J}}_{vc}$ will be assumed to include only conduction effects in materials and not any external convective or other nonmaterial free charge motion that should be contained in $\overline{\mathcal{J}}_{fc}$ and ρ_{fc}; the latter may be used together with the Lorentz law of force.

Problem 1.2

Show that (1.47) is true. Also show that if a convective charge density ρ_{fc} is present, then the corresponding current $\overline{\mathcal{J}}_{fc}$ resulting from the motion of ρ_{fc} satisfies $\nabla \cdot \overline{\mathcal{J}}_{fc} = -\partial \rho_{fc}/\partial t$.

Taking the divergence of (1.41) and employing (1.43), (1.45), and (1.46) leads to

$$\nabla \cdot \overline{\mathcal{J}}_{vm} = -\frac{\partial \rho_{vm}}{\partial t}. \tag{1.47}$$

One may define an electric polarization vector, $\overline{\mathcal{P}}$ related to ρ_{vm} by

$$\nabla \cdot \overline{\mathcal{P}} \equiv -\rho_{vm}, \tag{1.48}$$

where it is implied that ρ_{vm} is the scalar source density of $\overline{\mathcal{P}}$. From (1.47) and (1.48), one arrives at

$$\nabla \cdot \left(\overline{\mathcal{J}}_{vm} - \frac{\partial \overline{\mathcal{P}}}{\partial t} \right) = 0,$$

[1]This development is based in part on some handwritten notes provided to the authors by Prof. R.G. Kouyoumjian.

and hence, the quantity within the brackets of the preceding equation can be expressed as the curl of some vector $\overline{\mathcal{M}}$, namely

$$\overline{\mathcal{J}}_{vm} - \frac{\partial \overline{\mathcal{P}}}{\partial t} \equiv \nabla \times \overline{\mathcal{M}}, \tag{1.49}$$

because the divergence of a curl is always zero. In (1.49), the $\overline{\mathcal{M}}$ is defined as a magnetic polarization vector whose vector source densities are $\overline{\mathcal{J}}_{vm}$ and $-\frac{\partial \overline{\mathcal{P}}}{\partial t}$, respectively. If one incorporates (1.48) and (1.49) into (1.41) and (1.43), then (1.41) through (1.46) become

$$\nabla \times \left(\frac{\overline{\mathcal{B}}}{\mu_0} - \overline{\mathcal{M}} \right) - \frac{\partial}{\partial t} \left(\epsilon_0 \overline{\mathcal{E}} + \overline{\mathcal{P}} \right) = \overline{\mathcal{J}}_{vi} + \overline{\mathcal{J}}_{vc}, \tag{1.50}$$

$$\nabla \times \overline{\mathcal{E}} + \frac{\partial \overline{\mathcal{B}}}{\partial t} = 0, \tag{1.51}$$

$$\nabla \cdot \left(\epsilon_0 \overline{\mathcal{E}} + \overline{\mathcal{P}} \right) = \rho_{vi} + \rho_{vc}, \tag{1.52}$$

$$\nabla \cdot \overline{\mathcal{B}} = 0, \tag{1.53}$$

$$\nabla \cdot \overline{\mathcal{J}}_{vi} = -\frac{\partial \rho_{vi}}{\partial t}, \tag{1.54}$$

$$\nabla \cdot \overline{\mathcal{J}}_{vc} = -\frac{\partial \rho_{vc}}{\partial t}. \tag{1.55}$$

Equations (1.50) through (1.55) are valid for material media and must therefore be identical to (1.28) through (1.33), respectively, which are known to hold true for general media. Hence, a direct comparison of (1.50) with (1.28), and of (1.52) with (1.30) leads, via inspection, to the following constitutive relations, namely

$$\overline{\mathcal{D}} = \epsilon_0 \overline{\mathcal{E}} + \overline{\mathcal{P}}, \tag{1.56}$$

and

$$\overline{\mathcal{B}} = \mu_0 \left(\overline{\mathcal{H}} + \overline{\mathcal{M}} \right). \tag{1.57}$$

The media encountered in practice are generally passive; therefore, $\overline{\mathcal{P}}$ and $\overline{\mathcal{M}}$ vanish in the absence of an EM field. The $\overline{\mathcal{P}}$ responds to $\overline{\mathcal{E}}$ for strongly dielectric materials, where $\overline{\mathcal{M}}$ can be ignored for this case because such materials are weakly diamagnetic or paramagnetic. In particular, $\overline{\mathcal{P}}$ results from the motion of the bound charges within the atoms and molecules comprising the dielectric; this response cannot be instantaneous because of the inertial effect resulting from the mass of the charges. These time delayed responses due to inertial effects in all practical cases are also referred to as dispersion effects in materials. For a linear, isotropic, dielectric material, the $\overline{\mathcal{P}}$ can be expressed by a convolution integral,

$$\overline{\mathcal{P}} \left(\overline{r}, t \right) = \int_{-\infty}^{t} \mathcal{X}_e \left(\overline{r}, t - \tau \right) \overline{\mathcal{E}} \left(\overline{r}, \tau \right) d\tau, \tag{1.58}$$

in which $\mathcal{X}_e$ is defined as the electrical polarizability of the material and $\mathcal{X}_e = 0$ for $\tau > t$ by the causality condition. Note that $\overline{\mathcal{E}}$ in (1.58) represents the total electric field and it includes the original applied field together with the resulting secondary field produced by the material medium. Thus, the unknown $\overline{\mathcal{D}}$ of (1.56) is now related to the unknown $\overline{\mathcal{E}}$ via (1.58) provided $\mathcal{X}_e$ can be found (usually through measurements).

Only for the ideal case of a nondispersive lossless dielectric one assumes that is responds instantaneously to the EM field in which case one may write

$$\mathcal{X}_e \left(\overline{r}, t \right) = X_e \left(\overline{r} \right) \delta \left(t \right), \tag{1.59}$$

where $X_e(\overline{r})$ is not time-dependent, so that (1.58) becomes via (1.59) the following:

$$\overline{\mathcal{P}}\left(\overline{r},t\right) = X_e\left(\overline{r}\right)\overline{\mathcal{E}}\left(\overline{r},t\right). \tag{1.60}$$

It is clear that $\mathcal{X}_e(\overline{r}, t-t')$ of (1.58) is simply the response, $\overline{\mathcal{P}}$, to a time impulsive ($\delta(\tau - t')$) field $\overline{\mathcal{E}}(\overline{r}, \tau)$. Here δ is the Dirac delta (impulse) function (see Appendix C).

In the case of magnetic materials, diamagnetic effects are caused by induced magnetic moments that tend to oppose an externally applied EM field, whereas paramagnetic effects result from the alignment of magnetic moments when there is an applied EM field. The diamagnetic effects result in a material when the magnetic moments resulting from electron orbits cancel those resulting from electron spin in the absence of any applied field. In paramagnetic materials, these two effects do not cancel, but the atoms are randomly oriented so that once again the average magnetic moment is zero in the absence of any external field. Examples of diamagnetic materials are many inert gases (e.g., hydrogen and helium), as well as bismuth, germanium, silicon, graphite, copper, gold, sulfur, sodium chloride, etc. On the other hand, potassium, oxygen, tungsten, yttrium oxide, neodymium oxide, and many other rare earth elements as well as some of their salts constitute paramagnetic materials. Finally, ferromagnetic, anti-ferromagnetic and super-paramagnetic materials exhibit strong magnetic moments even in the absence of an applied EM field. It is noted that ferromagnetic and anti-ferromagnetic phenomena are highly nonlinear. The magnetic polarization response $\overline{\mathcal{M}}$ due to an applied field in an isotropic magnetic material can be expressed in a fashion similar to that in (1.58) as

$$\overline{\mathcal{M}}\left(\overline{r},t\right) = \int_{-\infty}^{t} \mathcal{X}_m\left(\overline{r},t-\tau\right)\overline{\mathcal{H}}\left(\overline{r},\tau\right) d\tau, \tag{1.61}$$

where $\mathcal{X}_m$ is the magnetic polarizability of the material and again $\overline{\mathcal{H}}$ in (1.61) represents the total field. Thus, the unknown $\overline{\mathcal{B}}$ in (1.57) is now related to the unknown $\overline{\mathcal{H}}$ via (1.61) provided $\mathcal{X}_m$ is known (usually through measurements).

Also, the presence of conductivity in a material medium requires one to express $\overline{\mathcal{J}}_{vc}$ in terms of $\overline{\mathcal{E}}$ in the material as

$$\overline{\mathcal{J}}_{vc}\left(\overline{r},t\right) = \int_{-\infty}^{t} \boldsymbol{\sigma}_e\left(\overline{r},t-\tau\right)\overline{\mathcal{E}}\left(\overline{r},\tau\right) d\tau, \tag{1.62}$$

for an isotropic conductor, where boldface $\boldsymbol{\sigma}_e$ is a scalar, time-dependent conductivity response. Thus, $\overline{\mathcal{J}}_{vc}$ is related to the unknown $\overline{\mathcal{E}}$ in (1.62) in terms of $\boldsymbol{\sigma}_e$ which again must be known (usually via measurements). Only for a linear, isotropic, and highly nondispersive conductor one can approximate (1.62) by Ohm's law as

$$\overline{\mathcal{J}}_{vc}\left(\overline{r},t\right) \approx \sigma\left(\overline{r}\right)\overline{\mathcal{E}}\left(\overline{r},t\right), \tag{1.63}$$

where $\boldsymbol{\sigma}_e(\overline{r}, t) = \sigma(\overline{r})\delta(t)$.

One can likewise approximate isotropic, highly nondispersive dielectric and magnetic media by

$$\overline{\mathcal{D}}\left(\overline{r},t\right) = \epsilon\left(\overline{r}\right)\overline{\mathcal{E}}\left(\overline{r},t\right), \tag{1.64}$$

and

$$\overline{\mathcal{B}}\left(\overline{r},t\right) = \mu\left(\overline{r}\right)\overline{\mathcal{H}}\left(\overline{r},t\right), \tag{1.65}$$

where ϵ and μ are the permittivity and permeability, respectively, of such a medium. Materials for which (1.63), (1.64), and (1.65) are valid are referred to as simple media. Most media which exhibit negligible magnetic effects have $\mu \approx \mu_0$. Strongly dielectric media are generally negligibly magnetic.

In case of a general linear medium which may be bi-anisotropic, one has the following constitutive relations:

$$\overline{\mathcal{P}}\left(\overline{r},t\right) = \int_{-\infty}^{t} \left[\overline{\overline{\mathcal{X}}}_e\left(\overline{r},t-\tau\right)\cdot\overline{\mathcal{E}}\left(\overline{r},\tau\right) + \overline{\overline{\mathcal{X}}}_{em}\left(\overline{r},t-\tau\right)\cdot\overline{\mathcal{H}}\left(\overline{r},\tau\right)\right] d\tau, \tag{1.66}$$

$$\overline{\mathcal{M}}\left(\bar{r},t\right)=\int_{-\infty}^{t}\left[\overline{\overline{\mathcal{X}}}_{m}\left(\bar{r},t-\tau\right)\cdot\overline{\mathcal{H}}\left(\bar{r},\tau\right)+\overline{\overline{\mathcal{X}}}_{me}\left(\bar{r},t-\tau\right)\cdot\overline{\mathcal{E}}\left(\bar{r},\tau\right)\right]d\tau, \tag{1.67}$$

and

$$\overline{\mathcal{J}}_{vc}\left(\bar{r},t\right)=\int_{-\infty}^{t}\left[\overline{\overline{\sigma}}_{e}\left(\bar{r},t-\tau\right)\cdot\overline{\mathcal{E}}\left(\bar{r},\tau\right)+\overline{\overline{\sigma}}_{m}\left(\bar{r},t-\tau\right)\cdot\overline{\mathcal{H}}\left(\bar{r},\tau\right)\right]d\tau. \tag{1.68}$$

The double bars on $\overline{\overline{\mathcal{X}}}$ and $\overline{\overline{\sigma}}$ denote dyadic quantities which are needed to describe anisotropic and bi-anisotropic effects on materials. A dyad is a special case of a tensor. Basically, the transformation of a vector $\overline{A}$ in a given coordinate system to another vector $\overline{B}$ in the same system is expressed as $\overline{B}=\overline{\overline{T}}\cdot\overline{A}$, where $\overline{\overline{T}}$ is referred to as the transformation dyadic. Some key dyadic relationships are summarized in Appendix A.

When $\overline{\overline{\mathcal{X}}}_{em}$, $\overline{\overline{\mathcal{X}}}_{me}$, and $\overline{\overline{\sigma}}_{m}$ are absent in (1.66)-(1.68), the latter reduce to those valid for only anisotropic (rather than bi-anisotropic) materials, whereas if all the dyads $\overline{\overline{\mathcal{X}}}$ and $\overline{\overline{\sigma}}$ in (1.66)-(1.68) reduce to scalars $\mathcal{X}$ and σ, then one obtains relations for bi-isotropic materials of which the chiral medium is a special case. If $\overline{\overline{\mathcal{X}}}_{e}$, $\overline{\overline{\mathcal{X}}}_{m}$, and $\overline{\overline{\sigma}}_{e}$ are scalars and the coupling terms $\overline{\overline{\mathcal{X}}}_{em}$ and $\overline{\overline{\mathcal{X}}}_{me}$, and the term $\overline{\overline{\sigma}}_{m}$ are zero, then the constitutive relations reduce to those for an isotropic medium. For anisotropic media, the presence of a dyadic $\overline{\overline{\mathcal{X}}}_{e}$ indicates that $\overline{\mathcal{P}}$, and hence, $\overline{\mathcal{D}}$ is not perfectly aligned (i.e., not parallel) with $\overline{\mathcal{E}}$, likewise, the presence of a dyadic $\overline{\overline{\mathcal{X}}}_{m}$ indicates that $\overline{\mathcal{M}}$, and hence, $\overline{\mathcal{B}}$ is not parallel to $\overline{\mathcal{H}}$, etc. Many crystals, as well as magnetized plasmas exhibit anisotropic effects. Some artificial materials are also anisotropic. Bi-anisotropic materials are also referred to as magneto electric because of the cross-coupling between electric and magnetic field effects that are present in the constitutive relations for such materials. Some examples of magneto electric materials include anti-ferromagnetic chromium oxide and ferromagnetic gallium iron oxide. In addition, bi-anisotropic effects can arise in problems involving moving media.

As one might expect, the rather complicated forms of the constitutive relationships presented above for the time domain case, in terms of convolution integrals, simplify considerably in the frequency domain.

In conclusion, one notes that the two Maxwell's curl equations along with the three constitutive relations in (1.63)-(1.65) for simple media (or those in (1.56)-(1.68) for the most general media) thus allows one to solve for the five unknowns $\overline{\mathcal{E}}$, $\overline{\mathcal{D}}$, $\overline{\mathcal{B}}$, $\overline{\mathcal{H}}$, and $\overline{\mathcal{J}}_{vc}$.

1.4 Frequency Domain Fields

The analysis of EM problems can often be simplified by separating or suppressing the time variation of the fields and sources using the Fourier (or Laplace) transform as is commonly done in the analysis of electric circuits. Thus, the space-time dependence of the fields and sources is converted to a space-frequency dependence in which the temporal angular frequency, ω is defined by the Fourier transform of the time function, namely

$$\overline{f}_{G}\left(\bar{r},\omega\right)=\int_{-\infty}^{\infty}\overline{\mathcal{G}}\left(\bar{r},t\right)\,e^{-j\omega t}\,dt, \tag{1.69}$$

where $\overline{f}_G(\bar{r},\omega)$ denotes the Fourier transform of $\overline{\mathcal{G}}(\bar{r},t)$. The above transformation from the time (t) domain to the frequency (ω) domain allows one to replace all the time derivative operators, $\frac{\partial}{\partial t}$, in Maxwell's equations and the continuity equations, by the algebraic operator ($j\omega$), as well as to replace all time convolution integral operations such as $\int_{-\infty}^{t}\overline{\overline{\mathcal{H}}}(\bar{r},t-\tau)\cdot\overline{\mathcal{G}}(\bar{r},\tau)d\tau$ by an algebraic product $\overline{\overline{f}}_H(\bar{r},\omega)\cdot\overline{f}_G(\bar{r},\omega)$ in which $\overline{\overline{f}}_H(\bar{r},\omega)$ and $\overline{f}_G(\bar{r},\omega)$ are the Fourier transforms of $\overline{\overline{\mathcal{H}}}(\bar{r},t)$ and $\overline{\mathcal{G}}(\bar{r},t)$, respectively. It is noted that here the dyadic quantity $\overline{\overline{\mathcal{H}}}(\bar{r},t)$ is not to be confused with the vector magnetic field $\overline{\mathcal{H}}(\bar{r},t)$. The transforms $\overline{\overline{f}}_H$ and $\overline{f}_G$ are complex valued. If $\overline{\overline{f}}_H$ and $\overline{f}_G$ are known, then the corresponding time-dependent functions $\overline{\overline{\mathcal{H}}}(\bar{r},t)$ and $\overline{\mathcal{G}}(\bar{r},t)$ can be found via an inverse Fourier transformation. For example,

$\overline{\mathcal{G}}(\overline{r},t)$ is given by the inverse Fourier transformation shown below:

$$\overline{\mathcal{G}}\left(\overline{r},t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}_G\left(\overline{r},\omega\right)\, e^{j\,\omega\, t}\, d\omega, \tag{1.70}$$

where the Fourier kernal $e^{j\omega t}$ of the inverse transformation above is the complex conjugate of the one in (1.69) for the direct transform. The $\overline{\overline{\mathcal{H}}}(\overline{r},t)$ can be found similarly from $\overline{\overline{f}}_H(\overline{r},\omega)$. Therefore, one can first transform Maxwell's equations from the time domain to the frequency domain. Next, one can solve Maxwell's equations in the frequency domain because that often simplifies the method of solution, and finally the time-dependent EM fields can be found by a Fourier inversion of the frequency domain solution. Of course, it is assumed that the Fourier transform in (1.69) exists; such as assumption is generally valid for physically real EM fields and sources. One notes that the inverse Fourier transform of (1.70) converges to the actual time-dependent $\overline{\mathcal{G}}(\overline{r},t)$ at every point where it is continuous in time, whereas it converges to the arithmetic mean at points, where $\overline{\mathcal{G}}(\overline{r},t)$ exhibits a temporal discontinuity. The integrals in (1.69) and (1.70) constitute a Fourier transform pair. Since $\overline{\mathcal{G}}(\overline{r},t)$ and $\overline{\overline{\mathcal{H}}}(\overline{r},t)$ above are taken to be real functions of time, it follows from (1.70) that

$$\overline{f}_G\left(\overline{r},\omega\right) = \overline{f}^{\,*}_G\left(\overline{r},-\omega\right);\quad \overline{\overline{f}}_H\left(\overline{r},\omega\right) = \overline{\overline{f}}^{\,*}_H\left(\overline{r},-\omega\right), \tag{1.71}$$

where the asterisk (*) on mathematical quantities refers to the complex conjugate operator.

It is also possible to write (1.70) as follows:

$$\begin{aligned}\overline{\mathcal{G}}\left(\overline{r},t\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}_G\left(\overline{r},\omega\right) e^{j\omega t} d\omega = \frac{1}{2\pi}\left[\int_{-\infty}^{0} \overline{f}_G\left(\overline{r},\omega\right) e^{j\omega t} d\omega + \int_{0}^{\infty} \overline{f}_G\left(\overline{r},\omega\right) e^{j\omega t} d\omega\right] \\ &= \frac{1}{2\pi}\left[-\int_{\infty}^{0} \overline{f}_G\left(\overline{r},-\xi\right) e^{-j\xi t} d\xi + \int_{0}^{\infty} \overline{f}_G\left(\overline{r},\omega\right) e^{j\omega t} d\omega\right],\end{aligned}$$

where the variable ω has been changed to $-\xi$ (with $d\omega = -d\xi$) in the integral over $-\infty < \omega \le 0$. Since ξ is just a dummy variable of integration, the above becomes

$$\overline{\mathcal{G}}\left(\overline{r},t\right) = \frac{1}{2\pi}\left[\int_{0}^{\infty}\left(\overline{f}_G\left(\overline{r},-\omega\right) e^{-j\omega t} + \overline{f}_G\left(\overline{r},\omega\right) e^{j\omega t}\right) d\omega\right].$$

Next replacing $\overline{f}_G(\overline{r},-\omega)$ by $\overline{f}^{\,*}_G(\overline{r},\omega)$ via (1.71) yields

$$\overline{\mathcal{G}}\left(\overline{r},t\right) = \frac{1}{\pi}\,\mathrm{Re}\left[\int_{0}^{\infty} \overline{f}_G\left(\overline{r},\omega\right)\, e^{j\omega t} d\omega\right].$$

The preceding expression is sometimes found to be useful as an alternative to (1.70). It is noted that the above result is valid only for real-time functions $\overline{\mathcal{G}}(\overline{r},t)$, since $\overline{f}_G(\overline{r},-\omega) = \overline{f}^{\,*}_G(\overline{r},\omega)$ has been utilized. In practice, the function $\overline{\mathcal{G}}(\overline{r},t)$ is typically real valued.

There are some time-dependent functions of practical interest, such as the harmonic function and the Dirac delta function, that do not appear to satisfy the conditions for the existence of the transform in (1.69) in the classical sense. However, they may be treated as a limiting sequence of transformable functions within the framework of the theory of generalized functions, or distribution theory, thus providing a formal justification for their transforms.

The Fourier transform of the electric field intensity $\overline{\mathcal{E}}(\overline{r},t)$ is defined via (1.69) as follows:

$$\overline{f}_e\left(\overline{r},\omega\right) = \int_{-\infty}^{\infty} \overline{\mathcal{E}}\left(\overline{r},t\right)\, e^{-j\,\omega\, t}\, dt. \tag{1.72}$$

Likewise, the transform of the current and charge densities is defined by

$$\overline{f}_J\left(\overline{r},\omega\right) = \int_{-\infty}^{\infty} \overline{\mathcal{J}}\left(\overline{r},t\right)\, e^{-j\,\omega\,t}\, dt, \tag{1.73}$$

and

$$f_\rho\left(\overline{r},\omega\right) = \int_{-\infty}^{\infty} \rho\left(\overline{r},t\right)\, e^{-j\,\omega\,t}\, dt, \tag{1.74}$$

respectively. Also, $\overline{f}_h(\overline{r},\omega)$ denotes the transform of the magnetic field intensity, $\overline{\mathcal{H}}(\overline{r},t)$, etc.

A case of practical interest is the time-harmonic or continuous-wave (cw) field which oscillates with a single angular frequency $\omega = 2\pi f$ (in which f = frequency in Hertz (Hz)), namely

$$\overline{\mathcal{E}}\left(\overline{r},t\right) = \mathrm{Re}\,\overline{E}\left(\overline{r},\omega\right)\, e^{j\,\omega\,t}, \tag{1.75}$$

where Re denotes the real part of a complex quantity, and $\overline{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$, with $E_x = |E_x|e^{j\phi_x}$; $E_y = |E_y|e^{j\phi_y}$; and $E_z = |E_z|e^{j\phi_z}$ in which ϕ_x, ϕ_y, and ϕ_z above are the phase of E_x, E_y, and E_z, respectively. Also $|\overline{E}(\overline{r},\omega)|$ is the real magnitude of $\overline{E}(\overline{r},\omega)$ defined by

$$\left|\overline{E}\left(\overline{r},\omega\right)\right| = \left|\sqrt{\overline{E}\left(\overline{r},\omega\right)\cdot\overline{E}^{*}\left(\overline{r},\omega\right)}\,\right|, \tag{1.76}$$

which of course, yields $|\overline{E}| = \left|\sqrt{|E_x|^2 + |E_y|^2 + |E_z|^2}\right|$. It is noted that the values of ϕ_x, ϕ_y, and ϕ_z are generally different from each other, and that if and only if $\phi_x = \phi_y = \phi_z \equiv \Phi$, then (1.75) can be written simply as $\overline{\mathcal{E}}(\overline{r},t) = \mathrm{Re}\,|\overline{E}|e^{j\Phi}e^{j\,\omega\,t}\,\hat{u}$, where $\overline{E} = \hat{u}\,|\overline{E}|\,e^{j\,\Phi}$ in this very special case, with $\hat{u}$ being a unit vector in the direction of $\overline{E}$.

It is clear that for the single frequency (or monochromatic) field in (1.75) one may show, via (1.72), that

$$\overline{f}_e\left(\overline{r},\omega'\right) = 2\,\pi\,\overline{E}\,\left(\overline{r},\omega'\right)\left[\frac{\delta\left(\omega'-\omega\right)+\delta\left(\omega'+\omega\right)}{2}\right], \tag{1.77}$$

so that incorporating (1.77) into the inverse transform (see (1.70)) yields

$$\overline{\mathcal{E}}\left(\overline{r},t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\,\pi\,\overline{E}\left(\overline{r},\omega'\right)\left[\frac{\delta\left(\omega'-\omega\right)+\delta\left(\omega'+\omega\right)}{2}\right] e^{j\,\omega'\,t}\, d\omega', \tag{1.78}$$

which in turn leads to (1.75) via the sifting property of the delta function, namely $\int_{-\infty}^{\infty} f(\omega')\,\delta(\omega' \pm \omega)\,d\omega' = f(\mp\omega)$ provided $(f \mp \omega)$ is continuous at $(\omega' = \pm\omega)$ (otherwise, it is an arithmetic mean value of the jump discontinuity in $f(\pm\omega)$ at $(\omega' = \pm\omega)$). Also, it is implied that $\overline{E}(\overline{r},\omega) = \overline{E}^{*}(\overline{r},-\omega)$ in deriving (1.75) from (1.78); this is consistent with the requirement given earlier in (1.71).

For the single-frequency (monochromatic) case, also known as the time-harmonic case, indicated in (1.75), the equations for classical electrodynamics with the assumed $e^{j\,\omega\,t}$ time convention suppressed, become the following:

$$\nabla\times\overline{E}\left(\overline{r},\omega\right) = -j\,\omega\,\overline{B}\left(\overline{r},\omega\right), \tag{1.79}$$

$$\nabla\times\overline{H}\left(\overline{r},\omega\right) = \overline{J}\left(\overline{r},\omega\right) + j\,\omega\,\overline{D}\left(\overline{r},\omega\right), \tag{1.80}$$

$$\nabla\cdot\overline{D}\left(\overline{r},\omega\right) = \rho_v\left(\overline{r},\omega\right), \tag{1.81}$$

$$\nabla\cdot\overline{B}\left(\overline{r},\omega\right) = 0, \tag{1.82}$$

$$\nabla \cdot \overline{J}_v\left(\overline{r},\omega\right) = -j\,\omega\,\rho_v\left(\overline{r},\omega\right), \tag{1.83}$$

$$\overline{D}\left(\overline{r},\omega\right) = \epsilon_0\,\overline{E}\left(\overline{r},\omega\right) + \overline{P}\left(\overline{r},\omega\right), \tag{1.84}$$

$$\overline{P}\left(\overline{r},\omega\right) = \overline{\overline{X}}_e\left(\overline{r},\omega\right)\cdot\overline{E}\left(\overline{r},\omega\right) + \overline{\overline{X}}_{em}\left(\overline{r},\omega\right)\cdot\overline{H}\left(\overline{r},\omega\right), \tag{1.85}$$

$$\overline{B}\left(\overline{r},\omega\right) = \mu_0\left(\overline{H}\left(\overline{r},\omega\right) + \overline{M}\left(\overline{r},\omega\right)\right), \tag{1.86}$$

$$\overline{M}\left(\overline{r},\omega\right) = \overline{\overline{X}}_m\left(\overline{r},\omega\right)\cdot\overline{H}\left(\overline{r},\omega\right) + \overline{\overline{X}}_{me}\left(\overline{r},\omega\right)\cdot\overline{E}\left(\overline{r},\omega\right), \tag{1.87}$$

$$\overline{J}_v\left(\overline{r},\omega\right) = \overline{J}_{vi}\left(\overline{r},\omega\right) + \overline{J}_{vc}\left(\overline{r},\omega\right), \tag{1.88}$$

$$\rho_v\left(\overline{r},\omega\right) = \rho_{vi}\left(\overline{r},\omega\right) + \rho_{vc}\left(\overline{r},\omega\right), \tag{1.89}$$

$$\overline{J}_{vc}\left(\overline{r},\omega\right) = \overline{\overline{\sigma}}_e\left(\overline{r},\omega\right)\cdot\overline{E}\left(\overline{r},\omega\right) + \overline{\overline{\sigma}}_m\left(\overline{r},\omega\right)\cdot\overline{H}\left(\overline{r},\omega\right). \tag{1.90}$$

The $\overline{J}_{vc}$ in (1.88) above is assumed to result only from conduction effects in the material as described by (1.90). It is convenient to combine (1.84) and (1.85) into a single equation given by

$$\overline{D}\left(\overline{r},\omega\right) = \overline{\overline{\epsilon}}\left(\overline{r},\omega\right)\cdot\overline{E}\left(\overline{r},\omega\right) + \overline{\overline{\alpha}}\left(\overline{r},\omega\right)\cdot\overline{H}\left(\overline{r},\omega\right), \tag{1.91}$$

where $\overline{\overline{\epsilon}} = \epsilon_0\,\overline{\overline{I}} + \overline{\overline{X}}_e$ and $\overline{\overline{\alpha}} = \overline{\overline{X}}_{em}$. Likewise, one can combine (1.86) and (1.87) into

$$\overline{B}\left(\overline{r},\omega\right) = \overline{\overline{\mu}}\left(\overline{r},\omega\right)\cdot\overline{H}\left(\overline{r},\omega\right) + \overline{\overline{\beta}}\left(\overline{r},\omega\right)\cdot\overline{E}\left(\overline{r},\omega\right), \tag{1.92}$$

with $\overline{\overline{\mu}} = \mu_0\left(\overline{\overline{I}} + \overline{\overline{X}}_m\right)$ and $\overline{\overline{\beta}} = \mu_0\,\overline{\overline{X}}_{me}$. Here, $\overline{\overline{I}}$ is the identity dyad, where $\overline{A}\cdot\overline{\overline{I}} = \overline{\overline{I}}\cdot\overline{A} = \overline{A}$ with $\overline{A}$ being any vector. For isotropic media (1.91), (1.92), and (1.90) simplify to

$$\overline{D}\left(\overline{r},\omega\right) = \epsilon\left(\overline{r},\omega\right)\,\overline{E}\left(\overline{r},\omega\right), \tag{1.93}$$

$$\overline{B}\left(\overline{r},\omega\right) = \mu\left(\overline{r},\omega\right)\,\overline{H}\left(\overline{r},\omega\right), \tag{1.94}$$

$$\overline{J}_{vc}\left(\overline{r},\omega\right) = \sigma\left(\overline{r},\omega\right)\,\overline{E}\left(\overline{r},\omega\right), \tag{1.95}$$

because $\overline{\overline{\epsilon}} = \epsilon\,\overline{\overline{I}}$ (with $\overline{\overline{\alpha}} = 0$), $\overline{\overline{\mu}} = \mu\,\overline{\overline{I}}$ (with $\overline{\overline{\beta}} = 0$), and $\overline{\overline{\sigma}}_e = \sigma\,\overline{\overline{I}}$ (with $\overline{\overline{\sigma}}_m = 0$) in the isotropic case. It is noted that the constitutive parameters for dispersive media are frequency-dependent. For lossy media, the constitutive parameters will become complex. However, the real and imaginary parts of these parameters are related, i.e., they are not independent of each other; the latter is referred to as the Kramers-Kronig relation, and it is developed below in Section 1.5 for the isotropic case in which $X_e = X_e' - jX_e''$, where X_e' and X_e'' are both real. The minus sign in front of the imaginary part X_e'' is required in the $e^{+j\,\omega\,t}$ time convention chosen here for a single-frequency (or monochromatic) field as in (1.75). The latter choice of this negative sign corresponding to $-j$ in front of X_e'' for an $e^{+j\,\omega\,t}$ time variation will be discussed in Section 1.6.

It is noted that for all time-harmonic fields in this book, the $e^{+j\omega t}$ time convention will be assumed and suppressed, except in Chapter 11, where an $e^{-i\omega t}$ time convention is utilized for convenience ($i = j = \sqrt{-1}$).

Exactly the same form of the equations as those given above ((1.79)-(1.90)) are obtained in the case of an arbitrary time dependence for the fields and sources after the time dependence is transformed to the frequency (or ω) domain. Thus, (1.79) and (1.80) become $\nabla\times\overline{f}_E = -j\,\omega\,\overline{f}_B$ and $\nabla\times\overline{f}_H = \overline{f}_J + j\,\omega\,\overline{f}_D$, respectively, for the case of an arbitrary time dependence, where $\overline{f}_E$ and $\overline{f}_H$ denote the Fourier transforms of $\overline{\mathcal{E}}$ and $\overline{\mathcal{H}}$. However, for the sake of notational convenience, $\overline{f}_E$, $\overline{f}_H$, $\overline{f}_J$, $\overline{f}_D$, $\overline{f}_B$, and f_ρ will henceforth be simply denoted by $\overline{E}$, $\overline{H}$, $\overline{J}$, $\overline{D}$, $\overline{B}$, and ρ, respectively, so that the very same notation used for the results in (1.79)-(1.90) corresponding to the single-frequency or time-harmonic case will now also be employed for convenience in the case of the frequency domain representation of the fields and sources which exhibit an arbitrary time dependence. Most of the material to follow will deal with the analysis of time-harmonic EM problems; however, there are a few instances where the transient behavior of EM fields resulting from a pulsed excitation, or from the motion of a charged particle, are also analyzed.

1.5 Kramers-Kronig Relationship

The $\overline{\overline{X}}_e(\omega)$ of (1.85) reduces to a scalar $X_e(\omega)$ for an isotropic medium; in this isotropic case, it becomes more convenient to demonstrate that the real and imaginary parts of X_e are related. Hence, it suffices to know only one of these, because the other can be found from the relationship between the two which is referred to as the Kramers-Kronig relationship. The latter relationship is developed below. One begins with the assumption that $\mathcal{X}_e(t)$ is causal (or $\mathcal{X}_e(t) = 0$ for $t < 0$), which is true for physical media, and that $\mathcal{X}_e(t)$ is Fourier transformable, i.e., $X_e(\omega)$ exists, where

$$X_e(\omega) = \int_{-\infty}^{\infty} \mathcal{X}_e(t)\, e^{-j\omega t} dt \equiv X_e'(\omega) - jX_e''(\omega). \tag{1.96}$$

It follows from above that

$$X_e(\omega) = \int_{-\infty}^{\infty} \mathcal{X}_e(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} \mathcal{X}_e(t) \sin \omega t \, dt, \tag{1.97}$$

which in turn leads to

$$X_e'(\omega) = \int_{-\infty}^{\infty} \mathcal{X}_e(t) \cos \omega t \, dt; \quad X_e''(\omega) = \int_{-\infty}^{\infty} \mathcal{X}_e(t) \sin \omega t \, dt. \tag{1.98}$$

From the preceding, it is readily evident that $X_e(\omega) = X_e^*(-\omega)$, since (1.98) implies that

$$X_e'(\omega) = X_e'(-\omega); \quad X_e''(\omega) = -X_e''(-\omega). \tag{1.99}$$

The preceding relations ensure that $\mathcal{X}_e(t)$ is real valued. The above comments also remain true for $X_m = X_m' - jX_m''$.

To relate X_e' with X_e'', one can first analytically continue $X_e(\omega)$ (which is defined above for real ω, over the range $|\omega| < \infty$) into the lower half of the complex ω plane. Note that the transform of $\mathcal{X}_e(t)$ given at the top indicates that $X_e(\omega)$ is analytic for Im $\omega < 0$ (or the lower half ω plane) and hence, is free of any singularities there due to the fact that $\mathcal{X}_e(t)$ is causal; also, the Fourier transform integral of $\mathcal{X}_e(t)$ in (1.96) is assumed to exist and hence, it converges as $e^{-j\omega t} \to 0$ for $t > 0$ and Im $\omega < 0$. In addition, it is important to note, as a consequence of the Riemann-Lebesgue lemma, that for any integrable function $\mathcal{X}_e(t)$, the $X_e(\omega)$ given in (1.96) vanishes for $\omega \to \infty$; i.e., $\lim_{\omega \to \infty} \int_{-\infty}^{\infty} \mathcal{X}_e(t)\, e^{-j\omega t} dt \to 0$; this behavior holds for all physical problems. Hence, one may consider the following integral, I, which is defined in the lower half of a complex ω' plane by

$$I \equiv \oint_C \frac{X_e(\omega')}{\omega' - \omega} d\omega'. \tag{1.100}$$

The contour of integration C in (1.100) is given by $C = C_1 + C_\delta + C_2 + C_\infty$ (as shown in Figure 1.8). Here C_1 goes from $-\infty$ to $\omega - \delta$; C_2 goes from $\omega + \delta$ to ∞, and C_δ is a semicircular contour of radius δ indented below $\omega' = \omega$, and C_∞ is a semicircular contour at $\omega' \to \infty$ in the lower half ω' plane. Since $X_e(\omega')$ is analytic for Im $\omega' < 0$, there are no singularities contained within C, it follows from Cauchy's theorem that

$$I = 0 = \int_{C_1 + C_2} \frac{X_e(\omega')}{\omega' - \omega} d\omega' + \int_{C_\delta} \frac{X_e(\omega')}{\omega' - \omega} d\omega' + \int_{C_\infty} \frac{X_e d\omega'}{\omega' - \omega}. \tag{1.101}$$

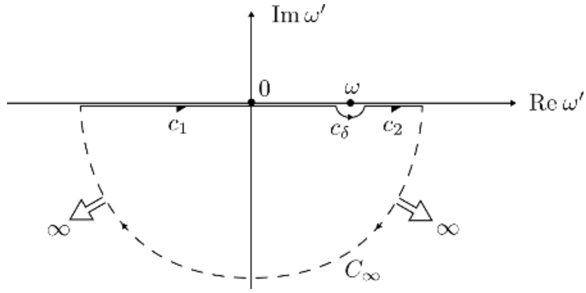

Figure 1.8 Contour of integration for I in (1.100).

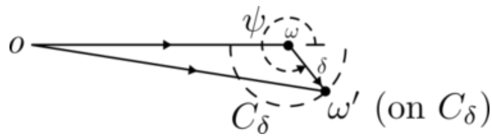

Figure 1.9 Semicircular integration on C_δ around the pole at ω in the complex ω' plane.

From Jordan's lemma, the last integral over C_∞ vanishes for Im $\omega' < 0$ because $X_e \to 0$ as $\omega' \to \infty$, via the Riemann-Lebesgue lemma as indicated earlier. In order to evaluate the integral over C_δ, one introduces the transformation:

$$\omega' - \omega = \delta\, e^{j\psi} \Rightarrow d\omega' = j\,\delta\, e^{j\psi}\, d\psi, \tag{1.102}$$

where δ is the radius of the semicircular indentation C_δ, and ψ is the angle measured around the point ω as shown in Figure 1.9.

It is of interest to evaluate C_δ in the limit as $\delta \to 0$. From above,

$$\lim_{\delta\to 0}\left[\int_{C_1+C_2}\frac{X_e(\omega')}{\omega'-\omega}d\omega' + j\int_{\pi}^{2\pi}\frac{X_e(\omega+\delta e^{j\psi})}{\delta e^{j\psi}}\cdot \delta e^{j\psi}d\psi\right] = 0,$$

or

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{X_e(\omega')}{\omega'-\omega}d\omega' = -j\,\pi\, X_e(\omega), \tag{1.103}$$

where PV denotes the Cauchy principal value of the integral over $C_1 + C_2$ as $\delta \to 0$, which excludes the point $\omega' = \omega$ along its path. It follows directly from above that

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{X_e'(\omega') - jX_e''(\omega')}{\omega'-\omega}d\omega' = -j\pi\left[X_e'(\omega) - jX_e''(\omega)\right], \tag{1.104}$$

or

$$\frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{X_e'(\omega')}{\omega'-\omega}d\omega' - \frac{j}{\pi}⨍_{-\infty}^{\infty}\frac{X_e''(\omega')}{\omega'-\omega}d\omega' = -jX_e'(\omega) - X_e''(\omega), \tag{1.105}$$

where the symbol on the integral sign in (1.105) denotes the PV part of the integral which excludes the point $\omega' = \omega$. Equating the real and imaginary parts on both sides of the above equation yields the desired relations:

$$X_e'(\omega) = \frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{X_e''(\omega')}{\omega'-\omega}d\omega'; \quad X_e''(\omega) = -\frac{1}{\pi}⨍_{-\infty}^{\infty}\frac{X_e'(\omega')}{\omega'-\omega}d\omega', \tag{1.106}$$

which are referred to as the Kramers-Kronig relations. These equations are directly related to the Hilbert transform, which in turn indicates that $\mathcal{X}_e(t)$ is causal in time if $X_e(\omega)$ satisfies (1.106); the latter is true for all physical media. The preceding statement is consistent with the fact that the Kramers-Kronig relations result from the requirements of analyticity of $X(\omega)$ for $\mathrm{Im}\,\omega < 0$; the Fourier inversion of such an $X(\omega)$ yields causality, namely, $\mathcal{X}(t) = 0$ for $t < 0$. Causality implies that there can be no material response before the source of the EM signal is turned on.

Since the dielectric constant of any material medium is given by the relation (see below (1.91))

$$\epsilon(\omega) = \epsilon_0 + X_e(\omega) = \epsilon_0 + X_e'(\omega) - j\,X_e''(\omega). \tag{1.107}$$

It therefore follows that

$$\epsilon(\omega) = \epsilon'(\omega) - j\epsilon''(\omega), \tag{1.108}$$

and, from (1.107), one obtains

$$\epsilon'(\omega) = \epsilon_0 + X_e'(\omega); \quad \epsilon''(\omega) = X_e''(\omega), \tag{1.109}$$

or, in terms of the Kramers-Kronig relations,

$$\epsilon'(\omega) = \epsilon_0 + \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\epsilon''(\omega')}{\omega' - \omega}\, d\omega'; \quad \epsilon''(\omega) = -\frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\epsilon'(\omega') - \epsilon_0}{\omega' - \omega}\, d\omega. \tag{1.110}$$

Similar relations can be obtained for $X_m(\omega) = X_m'(\omega) - jX_m''(\omega)$ and

$$\mu(\omega) = \mu'(\omega) - j\,\mu''(\omega). \tag{1.111}$$

For an anisotropic medium, one encounters a dyadic $\overline{\overline{X}}$ instead of a scalar X (where X represents X_e or X_m, etc.) as utilized above to develop the Kramers-Kronig relationship. However, for the anisotropic case, one can treat each $(ik)^{th}$ scalar component of $\overline{\overline{X}}$, denoted here by X_{ik}, in the same manner as done for X_e above. Thus, $X_{ik} = X_{ik}' - jX_{ik}''$, where X_{ik}' and X_{ik}'' satisfy the Kramers-Kronig relationship.

Problem 1.3

It is given that $X_e'(\omega) = X_e'(0)\frac{(\nu^2-\omega^2)}{(\nu^2+\omega^2)^2}$. Find X_e'' from the given X_e' via the Kramers-Kronig relations indicated above. Show that $X_e''(\omega) = X_e'(0)\frac{2\nu^3\omega}{(\nu^2+\omega^2)^2}$. Plot $X_e'(\omega)/X_e'(0)$ and $X_e''(\omega)/X_e'(0)$ as a function of ω/ν in the range $.01 < \frac{\omega}{\nu} < 10^2$.

1.6 Vector and Scalar Wave Equations

1.6.1 Vector Wave Equations for EM Fields

In this section, Maxwell's equations will be decoupled to arrive at separate differential equations for $\overline{E}$ and $\overline{H}$. These resulting vector differential equations are referred to as vector wave equations for $\overline{E}$ and $\overline{H}$, respectively. In rectangular coordinates, these vector wave equations reduce to simpler scalar wave equations for each rectangular field component of $\overline{E}$ or $\overline{H}$, as indicated in the next Section 1.6.2.

For any anisotropic, inhomogeneous media, Maxwell's curl equations are given via (1.91) and (1.92), together with $\overline{\overline{\alpha}} = 0 = \overline{\overline{\beta}}$, as

$$\nabla \times \overline{E}\left(\overline{r}\right) = -j\,\omega\,\overline{\overline{\mu}}\left(\overline{r}\right) \cdot \overline{H}\left(\overline{r}\right), \tag{1.112}$$

and

$$\nabla \times \overline{H}\left(\overline{r}\right) = j\,\omega\,\overline{\overline{\epsilon}}\left(\overline{r}\right) \cdot \overline{E}\left(\overline{r}\right) + \overline{J}_v\left(\overline{r}\right). \tag{1.113}$$

Pre-multiplying (1.112) by $\left(\overline{\overline{\mu}}^{-1}\cdot\right)$, then taking the curl of both sides and making use of (1.113) allows one to obtain a decoupled equation in terms of only the electric field, $\overline{E}$, namely

$$\nabla \times \left(\overline{\overline{\mu}}^{-1}\left(\overline{r}\right) \cdot \nabla \times \overline{E}\left(\overline{r}\right)\right) - \omega^2\,\overline{\overline{\epsilon}}\left(\overline{r}\right) \cdot \overline{E}\left(\overline{r}\right) = -j\,\omega\,\overline{J}_v\left(\overline{r}\right). \tag{1.114}$$

Likewise, pre-multiplying (1.113) by $\left(\overline{\overline{\epsilon}}^{-1}\cdot\right)$, then taking the curl and utilizing (1.112) yields a decoupled equation for just the magnetic field, $\overline{H}$ as

$$\nabla\times\left(\overline{\overline{\epsilon}}^{-1}\left(\overline{r}\right)\cdot\nabla\times\overline{H}\left(\overline{r}\right)\right)-\omega^2\overline{\overline{\mu}}\left(\overline{r}\right)\cdot\overline{H}\left(\overline{r}\right)=\nabla\times\left(\overline{\overline{\epsilon}}^{-1}\cdot\overline{J}_v\right). \tag{1.115}$$

The relations in (1.114) and (1.115) are referred to as the vector wave equations for $\overline{E}$ and $\overline{H}$, respectively, because, as will be seen later, their solutions constitute EM waves propagating in anisotropic and inhomogeneous media. In the event that the medium is isotropic and inhomogeneous, one may replace the dyadic permeability and permittivity $\overline{\overline{\mu}}(\overline{r})$ and $\overline{\overline{\epsilon}}(\overline{r})$ in (1.114) and (1.115) simply by their scalar values $\mu(\overline{r})$ and $\epsilon(\overline{r})$, respectively.

1.6.2 Scalar Wave Equations for EM Fields

For the present, one may consider a source-free isotropic, homogeneous medium for simplicity in demonstrating a wave solution to (1.114) (and similarly for (1.115)) for this special case. Thus, letting $\overline{\overline{\mu}}(\overline{r})$ and $\overline{\overline{\epsilon}}(\overline{r})$ be replaced by μ and ϵ, respectively, where μ and ϵ are now constants in (1.114) and (1.115), together with $\overline{J}_v=0$ for the source-free condition, yields

$$\nabla\times\nabla\times\overline{E}\left(\overline{r}\right)-k^2\overline{E}=0, \tag{1.116}$$

and

$$\nabla\times\nabla\times\overline{H}\left(\overline{r}\right)-k^2\overline{E}=0, \tag{1.117}$$

with

$$k^2\equiv\omega^2\,\mu\,\epsilon. \tag{1.118}$$

The k in (1.118) is defined as the wave number of the medium. From the vector identity $\nabla\times\nabla\times=\nabla(\nabla\bullet)-\nabla^2$ and the fact that $\nabla\cdot\overline{E}=0$ and $\nabla\cdot\overline{H}=0$ for a source-free, homogeneous, isotropic medium (see (1.81), (1.82), (1.93), and (1.94) together with $\rho_v=0$ via (1.83) for the source-free case when $\overline{J}_v=0$), (1.116) and (1.117) reduce to what are referred to as the source-free vector Helmholtz's wave equations, namely,

$$\left(\nabla^2+k^2\right)\overline{E}=0, \tag{1.119}$$

and

$$\left(\nabla^2+k^2\right)\overline{H}=0. \tag{1.120}$$

The $\overline{E}$ and $\overline{H}$ above can be expressed in terms of their rectangular components defined along the usual right-handed triad of unit vectors, $\hat{x}$, $\hat{y}$, and $\hat{z}$ as

$$\overline{E}=E_x\,\hat{x}+E_y\,\hat{y}+E_z\,\hat{z};\quad \overline{H}=H_x\,\hat{x}+H_y\,\hat{y}+H_z\,\hat{z}. \tag{1.121}$$

Incorporating (1.121) into (1.119) and (1.120) yields

$$\left(\nabla^2+k^2\right)E_{x,y,z}=0;\quad\left(\nabla^2+k^2\right)H_{x,y,z}=0, \tag{1.122}$$

since the vanishing of $\left(\nabla^2+k^2\right)\left(\overline{E};\overline{H}\right)$ as required by the right-hand side of (1.119) and (1.120) can take place only if each rectangular component of the field satisfies the conditions in (1.122) due to the fact that $\hat{x}$, $\hat{y}$, and $\hat{z}$ are constant unit vectors whose directions are independent of the observer at $\overline{r}=x\hat{x}+y\hat{y}+z\hat{z}$, where (x,y,z) are the observation coordinates which are arbitrary (see Figure 1.11(a)). The relations in (1.122) are referred to as the scalar source-free Helmholtz's wave equations for the components of $\overline{E}$ and $\overline{H}$, respectively. It will be shown in the next subsection that the general solutions of (1.122) are called plane waves which propagate at the speed of light in the isotropic, homogeneous medium.

1.7 Separable Solutions of the Source-Free Wave Equation in Rectangular Coordinates and for Isotropic Homogeneous Media. Plane Waves

It is convenient to write (1.122) as

$$\left(\nabla^2 + k^2\right) E_\xi = 0; \quad \xi = x \text{ or } y \text{ or } z. \tag{1.123}$$

The objective here is to solve (1.119), or (1.122), and hence solve (1.123) for the electric field. It is assumed, via separation of variables, that each vector component of $\overline{E}$, namely E_x or E_y, or E_z can be expressed as

$$E_\xi(x, y, z, k) = \mathbf{X}(x)\,\mathbf{Y}(y)\,\mathbf{Z}(z), \tag{1.124}$$

where $\mathbf{X}$ depends only on x, $\mathbf{Y}$ depends only on y, and $\mathbf{Z}$ is a function of only z. Also, $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ can depend on k which for now is fixed for a given ω; hence, the k dependence is not shown explicitly. A solution for E_ξ, and therefore, for $\overline{E}$, can be found once the functional forms of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are known. If the right side of (1.124) is a solution to (1.123), then one requires that

$$\left(\nabla^2 + k^2\right) \mathbf{X}(x)\,\mathbf{Y}(y)\,\mathbf{Z}(z) = 0. \tag{1.125}$$

It follows that

$$\mathbf{YZ}\,\frac{d^2\mathbf{X}}{dx^2} + \mathbf{XZ}\,\frac{d^2\mathbf{Y}}{dy^2} + \mathbf{XY}\,\frac{d^2\mathbf{Z}}{dz^2} = k^2\,\mathbf{XYZ} = 0. \tag{1.126}$$

One may next divide (1.126) by $\mathbf{XYZ}$ for a nontrivial solution. Thus,

$$\frac{1}{\mathbf{X}}\frac{d^2\mathbf{X}}{dx^2} + \frac{1}{\mathbf{Y}}\frac{d^2\mathbf{Y}}{dy^2} + \frac{1}{\mathbf{Z}}\frac{d^2\mathbf{Z}}{dz^2} = -k^2. \tag{1.127}$$

Since x, y, and z can be made to vary arbitrarily, it is clear that each of the three terms on the left-hand side of (1.127) can also be made to vary arbitrarily. The latter implies that each of the terms on the left of (1.127) must vary independently of the other two remaining terms and yet their sum must always add up to the same constant, namely to $(-k^2)$ as shown by the right side of (1.127). The latter is possible if each term on the left of (1.127) is a constant. Thus,

$$\frac{1}{\mathbf{X}(x)}\frac{d^2\mathbf{X}(x)}{dx^2} \equiv -k_x^2; \quad \frac{1}{\mathbf{Y}(y)}\frac{d^2\mathbf{Y}(y)}{dy^2} \equiv -k_y^2; \quad \frac{1}{\mathbf{Z}(z)}\frac{d^2\mathbf{Z}(z)}{dz^2} \equiv -k_z^2. \tag{1.128}$$

The k_x^2, k_y^2, and k_z^2 in (1.128) are known as separation constants, which allow one to solve the fully three dimensional (3-D) partial differential equation of (1.123) in terms of three, simpler, 1-D second-order linear differential equation problems of (1.128), using separation of variables as assumed in (1.124), subject to the separation condition, based on (1.127) and (1.128), that

$$k_x^2 + k_y^2 + k_z^2 = k^2. \tag{1.129}$$

The general form of the solution to each of the 1-D equations of (1.128) can be expressed for convenience in terms of a single variable ξ (where $\xi = x$, y, or z as in (1.123)), namely:

$$\frac{1}{U(\xi)}\frac{d^2U(\xi)}{d\xi^2} + k_\xi^2 U(\xi) = 0, \tag{1.130}$$

with $k_\xi = k_x$, k_y, or k_z, respectively. A solution to $U(\xi)$ is of the form

$$U(\xi) = c_1\, e^{-jk_\xi \xi} + c_2\, e^{+jk_\xi \xi} \tag{1.131}$$

or

$$U(\xi) = c_3 \sin(k_\xi \xi) + c_4 \cos(k_\xi \xi), \tag{1.132}$$

via the Euler formula $e^{\pm j k_\xi \xi} = \cos(k_\xi \xi) \pm j \sin(k_\xi \xi)$. The c_1 and c_2 in (1.131) (and likewise c_3 and c_4 in (1.132)) are constants with respect to the variable ξ. For unbounded regions, namely for $|\xi| < \infty$, the exponential representation in (1.131) is generally more convenient to use, while that involving sin and cos functions in (1.132) is often more useful for bounded regions.

In unbounded regions, the $\mathbf{X}(x)$, $\mathbf{Y}(y)$, and $\mathbf{Z}(z)$ of (1.124) can be expressed via (1.131) as

$$\mathbf{X}(x) = A_\xi^\pm \, e^{\pm j\, k_x x}; \quad \mathbf{Y}(y) = B_\xi^\pm \, e^{\pm j\, k_y y}; \quad \mathbf{Z}(z) = C_\xi^\pm \, e^{\pm j\, k_z z}, \tag{1.133}$$

where $A_\xi^\pm$, $B_\xi^\pm$, and $C_\xi^\pm$ are constants that do not depend on the coordinates (x, y, z). The subscript ξ on these constants is used to simply indicate that they are different for each E_ξ (i.e., for E_x, E_y, and E_z). Both solutions in (1.131) or (1.132) are expressed compactly for now as a single set of terms in (1.133). From (1.124) and (1.133), it is clear that E_ξ becomes (with $\xi = x$, or y, or z)

$$E_\xi = \begin{cases} E_x = A_x^\pm \, B_x^\pm \, C_x^\pm \, e^{\pm j\,(k_x x + k_y y + k_z z)} \\ E_y = A_y^\pm \, B_y^\pm \, C_y^\pm \, e^{\pm j\,(k_x x + k_y y + k_z z)} \\ E_z = A_z^\pm \, B_z^\pm \, C_z^\pm \, e^{\pm j\,(k_x x + k_y y + k_z z)} \end{cases}. \tag{1.134}$$

Let

$$E_{0\xi} \equiv A_\xi^\pm \, B_\xi^\pm \, C_\xi^\pm. \tag{1.135}$$

Also, one may define a vector wavenumber, $\overline{k}$, by

$$\overline{k} = k_x \, \hat{x} + k_y \, \hat{y} + k_z \, \hat{z}. \tag{1.136}$$

Noting that the vector, $\overline{r}$, to the observation point is

$$\overline{r} = x \, \hat{x} + y \, \hat{y} + z \, \hat{z}, \tag{1.137}$$

one may then express (1.134) together with (1.135)-(1.137) as

$$E_\xi = E_{0\,\xi} \; e^{-j\,\overline{k} \cdot \overline{r}}, \tag{1.138}$$

where the symbols $\pm$ in (1.134) and (1.135) are dropped in (1.138) because one can allow k_x, k_y, and k_z to each separately have both positive and negative values to cover all possible cases in (1.134), respectively. Note that $E_{0\xi}$ is still independent of (x, y, z); however, it may depend on ω (or $\overline{k}$) in general. From (1.138)

$$\overline{E} = \hat{x} \, E_x + \hat{y} \, E_y + \hat{z} \, E_z = \left(\hat{x} \, E_{0x} + \hat{y} \, E_{0y} + \hat{z} \, E_{0z} \right) e^{-j\,\overline{k} \cdot \overline{r}},$$

or

$$\overline{E} = \overline{E}_0 \, e^{-j\,\overline{k} \cdot \overline{r}}, \tag{1.139}$$

in which

$$\overline{E}_0 = \hat{x} \, E_{0x} + \hat{y} \, E_{0y} + \hat{z} \, E_{0z}. \tag{1.140}$$

The time-harmonic field $\overline{\mathcal{E}}(\overline{r}, t)$ corresponding to the $\overline{E}(\overline{r}, \omega)$ of (1.139) is given as usual by

$$\begin{array}{c} \overline{\mathcal{E}} \left(\overline{r}, t \right) = \mathrm{Re} \, \overline{E} \left(\overline{r}, \omega \right) \; e^{j\omega t} \\ \text{or} \\ \overline{\mathcal{E}} \left(\overline{r}, t \right) = \mathrm{Re} \, \overline{E}_0 \left(\omega \right) \; e^{j \left(\omega t - \overline{k} \cdot \overline{r} \right)} \end{array}. \tag{1.141}$$

The time-dependent quantity $\overline{\mathcal{E}}$ in (1.141), and its counterpart $\overline{E}$ in the frequency domain, is said to be a single-frequency plane wave field which propagates in the isotropic, homogeneous medium characterized by the parameters (μ, ϵ).

One may verify that $\overline{\mathcal{E}}$ (and hence $\overline{E}$) is a propagating wave field by allowing the observer to move in the same direction, and with the same speed as the wave. Thus, if the observer travels with the wave, then the observer must see the same value of the argument $\left(\omega t - \overline{k} \cdot \overline{r}\right)$ corresponding to that wave field, for all time. Therefore, $\left(\omega t - \overline{k} \cdot \overline{r}\right)$ must remain a constant as the observer location $\overline{r}$ moves with time (t) to keep in step with the waves, namely

$$\omega t - \overline{k} \cdot \overline{r} = \mathrm{K} \;\; \text{(constant in time)}. \tag{1.142}$$

Differentiating the preceding equation with respect to t yields

$$\omega = \overline{k} \cdot \frac{\overline{dr}}{dt} = k \left(\hat{k} \cdot \frac{\overline{dr}}{dt} \right) = k \left(\hat{k} \cdot \hat{r} \right) \frac{dr}{dt}. \tag{1.143}$$

The phase velocity dr/dt along $\hat{r}$ is defined as v_{pr} in the direction of observation $\overline{r}$. Thus, while the actual wave propagation direction is $\hat{k}$, the component of the phase velocity of the wave in the direction $\hat{r}$ is given by (1.143) as

$$v_{pr} \equiv \frac{dr}{dt} = \frac{\omega}{k} \cdot \frac{1}{\hat{k} \cdot \hat{r}}. \tag{1.144}$$

Here $\overline{k}$ is assumed to be a real vector for the present discussion. The term "phase velocity" comes from setting the time-phase equal to a constant as in (1.142). Since $\omega = 2\pi f$ (where f = wave frequency in Hz) and $k = \omega\sqrt{\mu\epsilon}$, the v_{pr} in (1.144) becomes for $\hat{r} = \hat{k}$, the phase velocity v_{pk} in the direction of actual wave propagation ($\hat{r} = \hat{k}$), namely

$$v_{pk} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \equiv c. \tag{1.145}$$

Also,

$$\lambda = \frac{c}{f}, \tag{1.146}$$

where λ is the wavelength associated with the frequency f of the wave as in Figure 1.10, and c is the speed of light in the medium. It is seen that λ is the value of the distance that is traversed by the wave to undergo a phase change of 2π, i.e., $\overline{k} \cdot \overline{r} = 2\pi$ in the exponent of (1.139), provided $\overline{r}$ is chosen to be oriented along the wave propagation direction, $\hat{k}$, so that $\overline{r} = r\,\hat{k}$ (or $\hat{r} = \hat{k}$). (See Figure 1.11.)

The values of v_{pr} along any of its rectangular component directions $\hat{\xi}$ ($\hat{x}$ or $\hat{y}$ or $\hat{z}$) is given by $\hat{r} = \hat{\xi}$ in (1.144)

$$v_{p\xi} = \frac{\omega}{k_\xi} = \frac{\omega}{k\left(\hat{k} \cdot \hat{\xi}\right)}, \tag{1.147}$$

in which k_ξ is k_x, k_y, or k_z, are as defined in (1.130) and (1.131) and the corresponding velocity components are v_{px}, v_{py} or v_{pz}, respectively. One notes that in some cases, k_ξ may become complex, as is

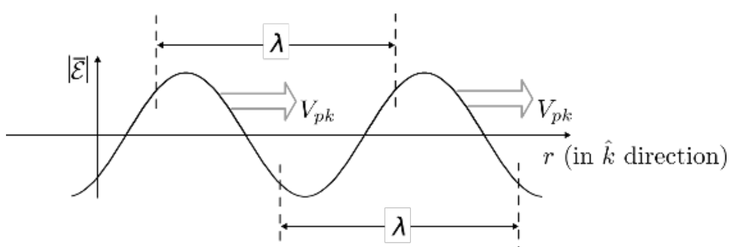

Figure 1.10 Wavelength λ associated with $\overline{\mathcal{E}}$ (or $\overline{E}$).

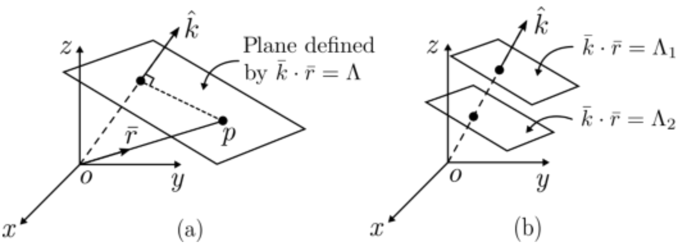

Figure 1.11 Geometrical description of plane wavefront or phasefronts. (a) Surface of constant phase (or equivalent surface) and (b) plane phasefronts propagating in $\hat{k}$ direction.

true for a lossy medium when k in (1.129) is complex. It is also possible in the lossless case, in some instances, for k_ξ to be purely imaginary. In general, if k_ξ is complex, then one can write (1.131) in terms of

$$k_\xi = \beta_\xi - j\,\alpha_\xi, \tag{1.148}$$

where α_ξ and β_ξ are real and positive for now. As will be seen, the minus sign on the right side of (1.148) guarantees that the wave decays in the direction of propagation for an $e^{j\,\omega t}$ time convention which is assumed and suppressed $\left(\overline{\mathcal{E}} = \text{Re}\,\overline{E}\,e^{j\,\omega t}\right)$.

One can therefore write the exponential form of the solution in (1.131) in terms of (1.148) as follows:

$$U(\xi) = c_1\,e^{-\alpha_\xi\,\xi}\,e^{-j\,\beta_\xi\,\xi} + c_2\,e^{\alpha_\xi\,\xi}\,e^{+j\,\beta_\xi\,\xi}. \tag{1.149}$$

If any k_ξ becomes complex valued as in (1.148), then the phase velocity component of (1.147) associated with k_ξ is now ω/β_ξ, since only β_ξ of (1.148) contributes to the wave phase, whereas α_ξ modifies just the wave amplitude alone through the factor $e^{\mp\alpha_\xi\,\xi}$ for $\xi \gtrless 0$ with $\alpha_\xi \geq 0$.

The field in (1.139) is defined as a uniform plane wave field, because it can be shown that the surfaces of constant phase (or equiphase surfaces) or phasefronts are planes and $\overline{E}_0$ is constant or uniform on the phasefront. Since $\overline{E} = \overline{E}_0\,e^{-j\,\overline{k}\cdot\overline{r}}$ in which $\overline{E}_0$ is independent of $\overline{r}$, only the term $e^{-j\,\overline{k}\cdot\overline{r}}$ contributes to phase variation with $\overline{r}$; hence, the surfaces of constant phase are defined by

$$\overline{k}\cdot\overline{r} = \Lambda, \tag{1.150}$$

where Λ is a constant.

Clearly (1.150) defines a plane; i.e., all points, P, which satisfy (1.150) for a given constant, Λ, must lie on a plane as shown in Figure 1.11(a). Actually, all surfaces which are perpendicular to $\hat{k}$ define planes such as $\hat{k}\cdot\overline{r} = \Lambda_1$, $\hat{k}\cdot\overline{r} = \Lambda_2$, etc., where Λ_1 and Λ_2 are different constants. Thus, as the wave propagates along the direction $\hat{k}$, the planes of constant phase (or phasefronts/wavefronts) of the wave can be defined at each point along the wave propagation path; this is best seen when $\hat{r}$ is chosen to be $\hat{k}$, so that the observer moves along the plane wave front or phase front with the velocity $\overline{v}_{pk} = \hat{k}\frac{\omega}{k}$ in a lossless medium.

It is noted that if the observer location, $\overline{r}$, is kept fixed, i.e., if the observer is stationary, then $\left(\omega t - \overline{k}\cdot\overline{r}\right)$ in (1.141) is no longer constant relative to the observer as t increases, and in this case, any given phasefront of the wave simply moves past the stationary observer.

Some additional properties of the plane wave field, which are very important for field calculations, are deduced as follows. Since $\nabla\cdot\overline{E} = 0$ in the present case,

$$\nabla\cdot\overline{E} = 0 \Rightarrow \nabla\cdot\overline{E}_0\,e^{-j\,\overline{k}\cdot\overline{r}} = 0, \tag{1.151}$$

or

$$\overline{E}_0\cdot\nabla\,e^{-j\,\overline{k}\cdot\overline{r}} = 0 = -j\,\overline{k}\cdot\overline{E}_0\,e^{-j\,\overline{k}\cdot\overline{r}}. \tag{1.152}$$

Thus,

$$\hat{k} \cdot \overline{E}_0 = 0; \quad \hat{k} \cdot \overline{E} = 0. \tag{1.153}$$

It is evident from (1.153) that the complex plane wave amplitude $\overline{E}_0$, and hence, $\overline{E}$ itself, is always perpendicular to the direction of wave propagation, $\hat{k}$. The vector orientation of $\overline{E}_0$, and therefore of $\overline{E}$, is defined as the POLARIZATION of $\overline{E}$ which is always transverse to $\hat{k}$ for a plane wave. The magnetic field intensity, $\overline{H}$, associated with the plane wave field $\overline{E}$ is given via

$$\overline{H} = \frac{j}{\omega\mu} \nabla \times \overline{E} = \frac{j}{\omega\mu} \nabla \times \overline{E}_0\, e^{-j\,\overline{k}\cdot\overline{r}} \tag{1.154}$$

or

$$\overline{H} = \frac{-j}{\omega\mu} \overline{E}_0 \times \nabla\, e^{-j\,\overline{k}\cdot\overline{r}} = -\frac{\overline{E}_0 \times \overline{k}}{\omega\mu}\, e^{-j\,\overline{k}\cdot\overline{r}}, \tag{1.155}$$

i.e.,

$$\overline{H} = \frac{1}{Z} \hat{k} \times \overline{E} \tag{1.156}$$

where $\overline{k} = \hat{k}\omega\sqrt{\mu\epsilon}$ has been used. In the above, Z is defined as the plane wave impedance (in ohms) in the isotropic, homogeneous medium. In particular,

$$Z = \sqrt{\frac{\mu}{\epsilon}} = (Y)^{-1}, \tag{1.157}$$

with Y being the wave admittance. It follows from (1.156) that for a plane wave,

$$\overline{E} = -Z\,\hat{k} \times \overline{H}. \tag{1.158}$$

The key properties of plane waves may be summarized by stating that they propagate in the $\hat{k}$ direction with a velocity $\overline{v}_{pk} = \hat{k}\frac{\omega}{k}$, and the plane wave fields $\overline{E}$ and $\overline{H}$ are polarized transverse to each other as well as to the direction of propagation, $\hat{k}$, respectively. Hence, a plane wave field is a transverse EM wave (or a TEM wave). Also, the power density in a plane wave is transported along $\hat{k}$. The power density in an EM wave at any instant of time (t), which is denoted here by $\overline{\mathcal{P}}$, is defined as follows:

$$\overline{\mathcal{P}} = \text{instantaneous power density} \equiv \overline{\mathcal{E}} \times \overline{\mathcal{H}}, \tag{1.159}$$

which for a time-harmonic variation becomes

$$\begin{aligned} \overline{\mathcal{P}}\left(\overline{r}, t\right) &= \overline{\mathcal{E}}\left(\overline{r}, t\right) \times \overline{\mathcal{H}}\left(\overline{r}, t\right) = \mathrm{Re}\left(\overline{E}\, e^{j\,\omega t}\right) \times \mathrm{Re}\left(\overline{H}\, e^{j\,\omega t}\right) \\ &= \mathrm{Re}\left(\overline{E}_0\, e^{j(\omega t - \overline{k}\cdot\overline{r})}\right) \times \mathrm{Re}\left(\frac{1}{Z}\,\hat{k} \times \overline{E}_0\, e^{j\,(\omega t - \overline{k}\cdot\overline{r})}\right). \end{aligned} \tag{1.160}$$

The units of $\overline{\mathcal{P}}$ are in watts/m^2. The power $\Delta\,\overline{\mathcal{P}}$ crossing an elemental area $\Delta\,\overline{S}$ of a planar phasefront is given by $\Delta\,\overline{\mathcal{P}} = \overline{\mathcal{P}} \cdot \Delta\,\overline{S}$, where $\Delta\,\overline{S} = \hat{k}\,\Delta\,S$.

One notes that $\overline{\mathcal{P}}$ is also referred to as the instantaneous Poynting vector. The time average power density, $\overline{P}_{\text{avg}}$, for a continuous EM wave or a time-harmonic wave is given by

$$\overline{P}_{\text{avg}}\left(\overline{r}, \omega\right) = \frac{1}{T}\int_0^T \overline{\mathcal{P}}\left(\overline{r}, t\right)\, dt, \tag{1.161}$$

where $T = 2\pi/\omega$ is the time period. Note that $T = 1/f$ in which $\omega = 2\pi f$ and f is the wave frequency. Thus,

$$\overline{P}_{\text{avg}} = \frac{1}{T}\int_0^T \left(\mathrm{Re}\ \overline{E}\, e^{j\,\omega t} \times \mathrm{Re}\ \overline{H}\, e^{j\,\omega t}\right)\, dt. \tag{1.162}$$

Since,

$$\text{Re}\left(\overline{E}\,e^{j\omega t}\right) \times \text{Re}\left(\overline{H}\,e^{j\omega t}\right) = \frac{\overline{E}\,e^{j\omega t} + \overline{E}^*\,e^{-j\omega t}}{2} \times \frac{\overline{H}\,e^{j\omega t} + \overline{H}^*\,e^{-j\omega t}}{2}, \tag{1.163}$$

one obtains

$$\overline{P}_{\text{avg}} = \frac{1}{2}\text{Re}\left(\overline{E}\times\overline{H}^*\right) + \int_0^T \frac{1}{2T}\text{Re}\left[\left(\overline{E}\times\overline{H}\right)e^{j2\omega t}\right]dt, \tag{1.164}$$

and the first term on the right of the above equation is independent of time, while the second behaves as $e^{j2\omega t}$ (or more like $\cos[2\omega t + \psi]$ in time), and hence, its integral vanishes since its time average is zero. Thus, it is clear that (1.162) reduces to

$$\overline{P}_{\text{avg}} = \frac{1}{2}\text{Re}\,\overline{E}\times\overline{H}^*. \tag{1.165}$$

Equation (1.165) further simplifies for the case of a plane wave as follows:

$$\overline{P}_{\text{avg}} = \frac{1}{2}\text{Re}\left[\overline{E}\times\frac{\hat{k}\times\overline{E}^*}{Z}\right],$$

or

$$\overline{P}_{\text{avg}} = \hat{k}\,\frac{|\overline{E}|^2}{2Z}, \tag{1.166}$$

which indicates that the time average power density at any point on the plane wavefront (or phasefront or equiphase surface) is transported along $\hat{k}$.

Problem 1.4

Consider a plane wave with an electric field $\overline{E}^i = \hat{x}\,A_0\,e^{-j\,k_0\,z}$, where $k_0 = \omega\sqrt{\mu_0\,\epsilon_0}$, which propagates in free space. A_0 is a known constant. If this plane wave is incident on an isotropic, homogeneous half space at $z = 0$, which is characterized by the electrical parameters (μ, ϵ), then part of the incident plane wave is reflected back into the free space region ($z < 0$) as shown in Figure 1.12.

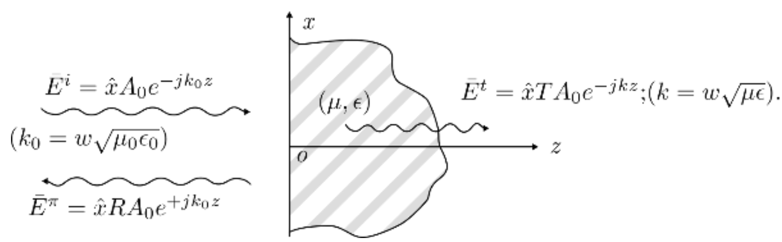

Figure 1.12 A plane wave $\overline{E}^i$ incident from free space ($z < 0$) impinges on a material half space to create a reflected field (in $z < 0$) and a transmitted field (in $z > 0$).

Let $\overline{E}^r$ denote the reflected plane wave field with an amplitude RA_0, where R is the reflection coefficient. There is also a total plane wave field in the material half space; it is referred to as the transmitted field in $z > 0$, with a transmission coefficient T. Show that the time average power density in the reflected and transmitted plane wave is given by $-\frac{\hat{z}}{2Z_0}|R|^2\,|A_0|^2$ and $\frac{\hat{z}}{2Z}\,|T|^2\,|A_0|^2$, respectively, where $Z_0 = \sqrt{\mu_0/\epsilon_0}$ and $Z = \sqrt{\mu/\epsilon}$. Also, show that $|R|^2 + \frac{Z_0}{Z}\,|T|^2 = 1$, if $R = \frac{Z-Z_0}{Z+Z_0}$ and $T = \frac{Z}{Z_0}(1 - R)$.

Thus, is the power per unit area conserved in the above interaction of an incident plane wave with a material half space?

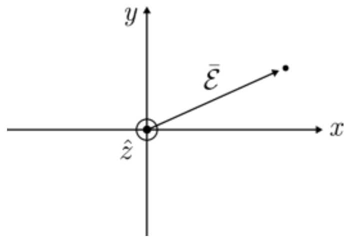

Figure 1.13 Coordinate system for the plane wave which propagates in the $+\hat{z}$ direction.

1.8 Polarization of Plane Waves, Poincaré Sphere, and Stokes Parameters

1.8.1 Polarization States

Consider a uniform plane wave field, traveling in the $+\hat{z}$ direction, in a linear, isotropic, and homogeneous medium. Let the electric field associated with this plane wave be expressed as follows:

$$\overline{\mathcal{E}}(z,t) = \text{Re}\left[\left(\hat{x}\,E_x + \hat{y}\,E_y\right)\,e^{j\,(\omega\,t - k\,z)}\right], \tag{1.167}$$

where the coordinates (x, y, z) are shown in Figure 1.13. Also, the phasor quantities E_x and E_y can be written in terms of their magnitude and phase by

$$E_x = |E_x|\,e^{j\,\delta_x}; \tag{1.168}$$

$$E_y = |E_y|\,e^{j\,\delta_y}. \tag{1.169}$$

From (1.168) and (1.169) above,

$$\overline{\mathcal{E}}(z,t) = \text{Re}\left[\left(\hat{x}\,|E_x|\,e^{j\,\delta_x} + \hat{y}\,|E_y|\,e^{j\,\delta_y}\right)\,e^{j(\omega\,t - k\,z)}\right]. \tag{1.170}$$

Let P, the polarization ratio, be defined as

$$P \equiv \frac{E_y}{E_x} = |P|\,e^{j\,\delta}; \tag{1.171}$$

$$|P| = \frac{|E_y|}{|E_x|}; \tag{1.172}$$

$$\delta \equiv \delta_y - \delta_x. \tag{1.173}$$

For later convenience, one can also define

$$\tan\gamma \equiv \frac{|E_y|}{|E_x|}; \tag{1.174}$$

$$|P| = \tan\gamma; \tag{1.175}$$

$$P = \tan\gamma\,e^{j\,\delta}. \tag{1.176}$$

Thus, (1.170) becomes

$$\overline{\mathcal{E}}(z,t) = \text{Re}\left[|E_x|\,e^{j\,\delta_x}\left(\hat{x}\,e^{j\,(\omega t - k\,z)} + \hat{y}\,|P|\,e^{j\,(\omega t - k\,z + \delta)}\right)\right], \tag{1.177}$$

or directly from (1.170), one can write

$$\overline{\mathcal{E}}(z,t) = \hat{x}\,|E_x|\,\cos\left(\Omega + \delta_x\right) + \hat{y}\,|E_y|\cos\left(\Omega + \delta_y\right), \tag{1.178}$$

where

$$\Omega \equiv \omega t - k z. \tag{1.179}$$

Both of the above forms in (1.177) and (1.178) are useful as will be seen below.

Let $\overline{\mathcal{E}} = \hat{x}\,\mathcal{E}_x + \hat{y}\,\mathcal{E}_y$, where

$$\mathcal{E}_x = |E_x| \cos\left(\Omega + \delta_x\right); \quad \mathcal{E}_y = |E_y| \cos\left(\Omega + \delta_y\right). \tag{1.180}$$

It follows that

$$\frac{\mathcal{E}_x}{|E_x|} = \cos\left(\Omega + \delta_x\right) = \cos\Omega\,\cos\delta_x - \sin\Omega\,\sin\delta_x, \tag{1.181}$$

and

$$\frac{\mathcal{E}_y}{|E_y|} = \cos\left(\Omega + \delta_y\right) = \cos\Omega\,\cos\delta_y - \sin\Omega\,\sin\delta_y. \tag{1.182}$$

Based on the above information, it is of interest to determine the curve traced by the tip of the electric field vector, $\overline{\mathcal{E}}(z,t)$, in the x-y plane (at a given position z) as time changes. To this end, one may multiply (1.181) by $\sin\delta_y$, and multiply (1.182) by $\sin\delta_x$, and then take their difference to arrive at

$$\begin{aligned}\frac{\mathcal{E}_x}{|E_x|}\sin\delta_y - \frac{\mathcal{E}_y}{|E_y|}\sin\delta_x &= \cos\Omega\,\cos\delta_x\,\sin\delta_y - \sin\Omega\,\sin\delta_x\,\sin\delta_y \\ &\quad - \cos\Omega\,\cos\delta_y\,\sin\delta_x + \sin\Omega\,\sin\delta_x\,\sin\delta_y \\ &= \cos\Omega\,\sin\left(\delta_y - \delta_x\right) = \cos\Omega\,\sin\delta.\end{aligned} \tag{1.183}$$

Similarly, one can show that

$$\frac{\mathcal{E}_x}{|E_x|}\cos\delta_y - \frac{\mathcal{E}_y}{|E_y|}\cos\delta_x = \sin\Omega\,\sin\delta. \tag{1.184}$$

Next, squaring (1.183) and (1.184), respectively, and then adding yields

$$\begin{aligned}&\frac{\mathcal{E}_x^2}{|E_x|^2}\left(\sin^2\delta_y + \cos^2\delta_y\right) + \frac{\mathcal{E}_y^2}{|E_y|^2}\left(\sin^2\delta_x + \cos^2\delta_x\right) - \frac{2\mathcal{E}_x\,\mathcal{E}_y}{|E_x|\,|E_y|}\left(\cos\delta_y\cos\delta_x + \sin\delta_y\sin\delta_x\right) \\ &= \sin^2\delta\left(\cos^2\Omega + \sin^2\Omega\right),\end{aligned}$$

or

$$\frac{\mathcal{E}_x^2}{|E_x|^2} + \frac{\mathcal{E}_y^2}{|E_y|^2} - \frac{2\mathcal{E}_x\,\mathcal{E}_y}{|E_x|\,|E_y|}\cos\delta = \sin^2\delta. \tag{1.185}$$

The above is an equation of a conic, which can be verified to be an ellipse. Thus, the most general state of polarization of an arbitrary plane wave field of (1.167;1.170) is elliptical; i.e., the tip of the vector $\overline{\mathcal{E}}$ traces an ellipse for an observer located at any z = constant plane.

It is useful to consider some special cases as follows.

Case (a). $\delta = -\pi/2$ and $|P| > 1$: From (1.177) with an observer located at $z = 0$ for convenience, one obtains

$$\overline{\mathcal{E}}(0,t) = \hat{x}\,|E_x|\cos\omega t + \hat{y}\,|E_y|\sin\omega t. \tag{1.186}$$

As t increases, the tip of the electric vector in the above equation traces an ellipse which rotates in the right-hand sense; i.e., if the thumb points in the $\hat{z}$ direction, namely, the wave propagation direction, then the remaining fingers of the right-hand will curl in the direction of the rotation of the field vector. Hence, this field is defined to have a right-handed elliptical polarization (RHEP). The right-handed ellipse is

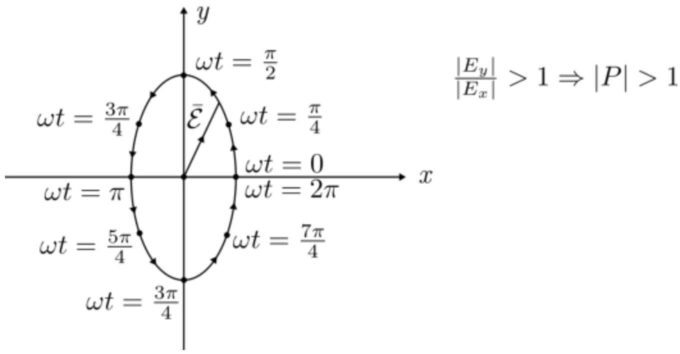

Figure 1.14 Right-handed elliptical polarization (RHEP) with major axis along the y-coordinate. Note that $z = 0$ in the above illustration.

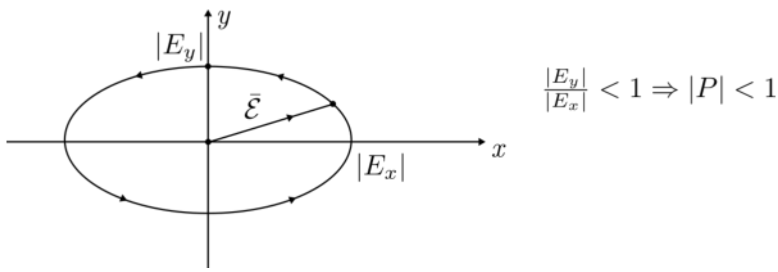

Figure 1.15 RHEP with major axis along the x-coordinate. Note that $z = 0$ in the above illustration.

traced in Figure 1.14, with t increasing from 0 to $2\pi/\omega$, or as ωt varies from 0 to 2π, in the $z = 0$ plane. Since $|P| > 1$, the major axis of the ellipse is along the y coordinate.

It is clear from (1.186) and $\overline{\mathcal{E}} = \hat{x}\mathcal{E}_x + \hat{y}\mathcal{E}_y$ that

$$\frac{\mathcal{E}_x}{|E_x|} = \cos \omega t; \tag{1.187}$$

$$\frac{\mathcal{E}_y}{|E_y|} = \sin \omega t, \tag{1.188}$$

from which it follows that the equation for the ellipse in Figure 1.14 is

$$\frac{\mathcal{E}_x^2}{|E_x|^2} + \frac{\mathcal{E}_y^2}{|E_y|^2} = 1, \tag{1.189}$$

which is the same as the one given in (1.185) for the general case after it is specialized to $\delta = -\pi/2$.

Case (b). $\delta = -\pi/2$ and $|P| < 1$: This case is essentially the same as case (a), except the right-handed elliptical polarization (RHEP) is described by an ellipse whose major axis now sits along the x direction, because $|P| < 1$. The RHEP results because, as in case (a), $\delta = -\pi/2$. The ellipse traced by the tip of the electric vector as a function of time with $z = 0$ is shown in Figure 1.15 for this case.

Case (c). $\delta = -\pi/2$ and $|P| = 1$: For this case, the electric field vector traces out a circle because the major and minor axes of the ellipse become identical in their dimensions with $|P| = |E_y|/|E_x| = 1$. The circle is of course traced out in a right-handed sense because once again, $\delta = -\pi/2$. The resulting state of polarization is thus named right-handed circular polarization (RHCP) for obvious reasons. Figure 1.16 illustrates the circle traced by the tip of the electric vector for this RHCP case with $z = 0$.

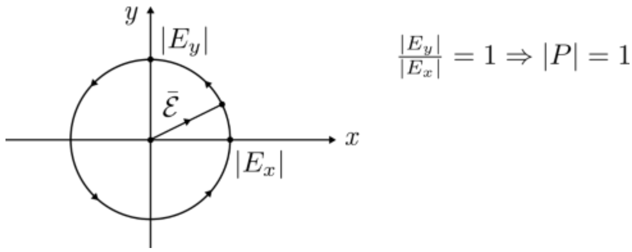

Figure 1.16 RHCP case.

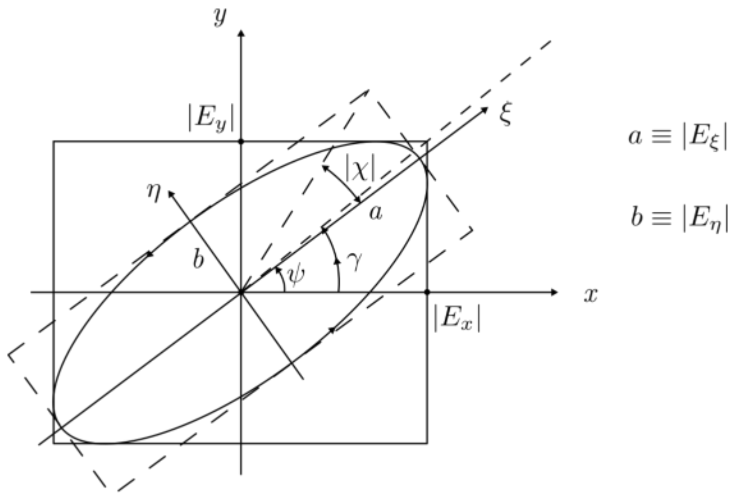

Figure 1.17 Polarization ellipse with a tilt angle ψ. Note that $|\mathcal{X}|$ is shown above.

Case (d). $\delta = +\pi/2$ and $|P| > 1$: In this case, the tip of the electric vector traces an ellipse which is identical to that in Figure 1.14, except that the electric vector now rotates in an opposite sense to the one in Figure 1.14 for $z = 0$. Hence, the polarization in this case is defined to be a left-handed elliptical polarization (LHEP). Thus, if the thumb of the left-hand points in the $\hat{z}$ direction, or the wave propagation direction, then the remaining fingers of the left-hand will curl in the direction of rotation of the field vector. Clearly, the left-handedness of the ellipse comes from $\delta = +\pi/2$ (instead of $\delta = -\pi/2$ in case (a)).

Case (e). $\delta = +\pi/2$ and $|P| < 1$: This case is identical to case (b) above except that the ellipse is now left-handed at $z = 0$. Thus, this case leads to LHEP for the electric field vector.

Case (f). $\delta = +\pi/2$ and $|P| = 1$: Once again, this case is identical to that in case (c), except that the circular polarization is now left-handed at $z = 0$. Thus this case constitutes a left-handed circular polarization (LHCP).

1.8.2 General Elliptical Polarization

If δ is odd multiples of $\pi/2$, then one obtains the polarizations indicated in cases (a)-(f) above (in which the principal axes of the ellipse are aligned with the (x, y) coordinates). If δ is an integer multiple of π, then the field is linearly polarized. If δ is arbitrary, then the polarization state at $z = 0$ is described by a tilted ellipse based on (1.185); such a tilted ellipse is shown in Figure 1.17. The tilt angle of the ellipse is denoted by ψ. When $\delta = [\mp\pi/2]$, and $\psi = \pi/2$ or 0, then one obtains the special cases discussed above. The parameters (γ, δ), where γ was defined previously in (1.174) (and of course, δ was defined in (1.173)), must be known to completely describe the state of polarization. Generally, $|E_x|$ and $|E_y|$ as well as δ ($= \delta_y - \delta_x$) are available from a known expression for $\overline{\mathcal{E}}$. It is also possible to describe the state of polarization in terms of the tilt angle (ψ), and $\mathcal{X}$, respectively. Note that $\mathcal{X}$ here is not to be confused with the medium parameters $\mathcal{X}_e$ or $\mathcal{X}_m$ introduced previously. The polarization parameters ($\psi, \mathcal{X}$) can be found in terms of the given parameters (γ, δ). It is noted that ($\psi, \mathcal{X}$) are more natural to deal with for the tilted ellipse which represents the most general state of polarization. Indeed one may find ($\psi, \mathcal{X}$) from a diagonalization of (1.185) in the (x, y) system to the principal axes (ξ, η) of the tilted ellipse via a matrix approach. However, a somewhat lengthier algebraic procedure, but which contains simpler operations, as in [1, 2], will be followed here to relate ($\psi, \mathcal{X}$) to (γ, δ). This procedure begins by expressing the field $\overline{\mathcal{E}}$ in terms of its components $\mathcal{E}_\xi$ and $\mathcal{E}_\eta$ as

$$\overline{\mathcal{E}} = \hat{\xi}\,\mathcal{E}_\xi + \hat{\eta}\,\mathcal{E}_\eta, \tag{1.190}$$

with

$$\mathcal{E}_\xi = \mathrm{Re}\, E_\xi\, e^{j\,\Omega}; \tag{1.191}$$

$$\mathcal{E}_\eta = \mathrm{Re}\, E_\eta\, e^{j\,\Omega}, \tag{1.192}$$

where Ω has been defined earlier in (1.179). Let

$$E_\xi = |E_\xi|\, e^{j\,\delta_\xi}; \tag{1.193}$$

$$E_\eta = |E_\eta|\, e^{j\,\delta_\eta}. \tag{1.194}$$

Since (ξ, η) are aligned with the principal axes of the ellipse, this is the same as that which happens for the ellipse in cases (a) and (b), or cases (d) and (e) (pertaining to $\psi = \pi/2$ or $\psi = 0$), respectively, in which the ellipses are aligned with the (x, y) coordinates. Thus, one can relate ($\delta_\eta - \delta_\xi$) of (1.193;1.194) to $\delta = \delta_y - \delta_x = \mp\pi/2$ as follows:

$$\delta_\xi \equiv \delta_0; \tag{1.195}$$

$$\delta_\eta \equiv \delta_0 \mp \frac{\pi}{2}, \tag{1.196}$$

so that $\delta_\eta - \delta_\xi = \mp\pi/2$. Again, $\delta_\eta - \delta_\xi = \mp\pi/2$ refers to the $\begin{bmatrix}\text{RHEP}\\\text{LHEP}\end{bmatrix}$ cases as does $\delta = \mp\pi/2$. From (1.190), (1.191;1.192), (1.193;1.194), and (1.195;1.196),

$$\mathcal{E}_\xi = \mathrm{Re}\,|E_\xi|\, e^{j\,(\Omega+\delta_0)};$$

$$\mathcal{E}_\eta = \mathrm{Re}\,|E_\eta|\, e^{j\left(\Omega+\delta_0\mp\frac{\pi}{2}\right)},$$

or

$$\mathcal{E}_\xi = |E_\xi|\,\cos\left(\Omega + \delta_0\right); \tag{1.197}$$

$$\mathcal{E}_\eta = |E_\eta|\,\cos\left(\Omega + \delta_0 \mp \frac{\pi}{2}\right), \tag{1.198}$$

and hence, $\overline{\mathcal{E}}$ in the (ξ, η) coordinates is given as follows:

$$\overline{\mathcal{E}} = \hat{\xi}\,|E_\xi|\,\cos\left(\Omega + \delta_0\right) \pm \hat{\eta}\,|E_\eta|\sin\left(\Omega + \delta_0\right). \tag{1.199}$$

The $\pm$ signs in (1.199) relate to $\delta_\eta - \delta_\xi = \mp\pi/2$ corresponding to $\begin{bmatrix}\text{RHEP}\\\text{LHEP}\end{bmatrix}$ states of polarization. Note that the lengths of the principal axes (a, b) of the ellipse in Figure 1.17 relate to the $|E_\xi|$ and $|E_\eta|$, namely, $a \equiv |E_\xi|$ and $b \equiv |E_\eta|$. Also, note that the angle, ψ, relates ($\mathcal{E}_\xi, \mathcal{E}_\eta$) to ($\mathcal{E}_x, \mathcal{E}_y$) through the transformations:

$$\mathcal{E}_\xi = \mathcal{E}_x \cos\psi + \mathcal{E}_y \sin\psi, \tag{1.200}$$

$$\mathcal{E}_\eta = -\mathcal{E}_x \sin\psi + \mathcal{E}_y \cos\psi. \tag{1.201}$$

Now from (1.180)-(1.182), (1.197), and (1.198), the above two transformations, respectively, become

$$\begin{aligned}|E_\xi|\,(\cos\Omega\cos\delta_0 - \sin\Omega\sin\delta_0) =& |E_x|\,(\cos\Omega\cos\delta_x - \sin\Omega\sin\delta_x)\cos\psi \\ &+ |E_y|\,(\cos\Omega\cos\delta_y - \sin\Omega\sin\delta_y)\sin\psi,\end{aligned} \tag{1.202}$$

$$\begin{aligned}\pm|E_\eta|\,(\sin\Omega\cos\delta_0 + \cos\Omega\sin\delta_0) =& -|E_x|\,(\cos\Omega\cos\delta_x - \sin\Omega\sin\delta_x)\sin\psi \\ &+ |E_y|\,(\cos\Omega\cos\delta_y - \sin\Omega\sin\delta_y)\cos\psi.\end{aligned} \tag{1.203}$$

Equating coefficients of $\sin\Omega$ and $\cos\Omega$ on both sides of (1.202) and (1.203), to remove the Ω dependent terms as a part of the simplification, yields:

$$|E_\xi|\cos\delta_0 = |E_x|\cos\delta_x\cos\psi + |E_y|\cos\delta_y\sin\psi, \tag{1.204}$$

$$|E_\xi|\sin\delta_0 = |E_x|\sin\delta_x\cos\psi + |E_y|\sin\delta_y\sin\psi, \tag{1.205}$$

$$\pm|E_\eta|\cos\delta_0 = |E_x|\sin\delta_x\sin\psi - |E_y|\sin\delta_y\cos\psi, \tag{1.206}$$

$$\pm|E_\eta|\sin\delta_0 = -|E_x|\cos\delta_x\sin\psi + |E_y|\cos\delta_y\cos\psi. \tag{1.207}$$

Squaring (1.204) and (1.205), respectively, and then summing the two gives

$$|E_\xi|^2 = |E_x|^2\cos^2\psi + |E_y|^2\sin^2\psi + |E_x|\,|E_y|\sin 2\psi\cos\delta. \tag{1.208}$$

Likewise,

$$|E_\eta|^2 = |E_x|^2\sin^2\psi + |E_y|^2\cos^2\psi - |E_x|\,|E_y|\sin 2\psi\cos\delta, \tag{1.209}$$

is obtained by squaring (1.206) and (1.207), respectively, and then adding them. Next, summing (1.208) and (1.209) yields

$$a^2 + b^2 = |E_\xi|^2 + |E_\eta|^2 = |E_x|^2 + |E_y|^2. \tag{1.210}$$

Multiplying (1.204) and (1.206) gives

$$\begin{aligned}\pm ab\cos^2\delta_0 =& \pm|E_\xi|\,|E_\eta|\cos^2\delta_0 \\ =& |E_x|^2\frac{\sin 2\delta_x}{2}\frac{\sin 2\psi}{2} - |E_y|^2\frac{\sin 2\delta_y}{2}\frac{\sin 2\psi}{2} \\ &- |E_x|\,|E_y|\sin\delta_y\cos\delta_x\cos^2\psi + |E_x|\,|E_y|\sin\delta_x\cos\delta_y\sin^2\psi.\end{aligned} \tag{1.211}$$

Likewise, multiplying (1.205) and (1.207) gives

$$\begin{aligned}\pm ab\sin^2\delta_0 =& \pm|E_\xi|\,|E_\eta|\sin^2\delta_0 \\ =& -|E_x|^2\frac{\sin 2\delta_x}{2}\frac{\sin 2\psi}{2} + |E_y|^2\frac{\sin 2\delta_y}{2}\frac{\sin 2\psi}{2} \\ &+ |E_x|\,|E_y|\sin\delta_x\cos\delta_y\cos^2\psi - |E_x|\,|E_y|\sin\delta_y\cos\delta_x\sin^2\psi.\end{aligned} \tag{1.212}$$

Adding (1.211) and (1.212) leads to

$$\begin{aligned}\pm ab = \pm|E_\xi|\,|E_\eta| =& 2|E_x|\,|E_y|(\sin\delta_x\cos\delta_y)(\cos^2\psi + \sin^2\psi) - 2|E_x|\,|E_y|(\sin\delta_y\cos\delta_x)(\cos^2\psi + \sin^2\psi) \\ =& 2|E_x|\,|E_y|\sin(\delta_x - \delta_y),\end{aligned}$$

or

$$\pm ab = \pm|E_\xi|\,|E_\eta| = -2|E_x|\,|E_y|\sin\delta. \tag{1.213}$$

Next, dividing (1.206) by (1.204) yields

$$\pm\frac{b}{a} = \pm\frac{|E_\eta|}{|E_\xi|} = \frac{|E_x|\sin\delta_x\sin\psi - |E_y|\sin\delta_y\cos\psi}{|E_x|\cos\delta_x\cos\psi + |E_y|\cos\delta_y\sin\psi}. \tag{1.214}$$

Similarly, dividing (1.207) by (1.205) yields

$$\pm\frac{b}{a} = \pm\frac{|E_\eta|}{|E_\xi|} = \frac{-|E_x|\cos\delta_x\sin\psi + |E_y|\cos\delta_y\cos\psi}{|E_x|\sin\delta_x\cos\psi + |E_y|\sin\delta_y\sin\psi}. \tag{1.215}$$

Equating (1.214) and (1.215) allows one to eliminate $\pm b/a$ and express everything in terms of $|E_x|$, $|E_y|$, $\delta = \delta_y - \delta_x$ and ψ as follows:

$$(|E_x|^2 - |E_y|^2)\sin 2\psi = 2|E_x|\,|E_y|\cos\delta\cos 2\psi,$$

or

$$\tan 2\psi = \frac{2|E_x|\,|E_y|\cos\delta}{|E_x|^2 - |E_y|^2}. \tag{1.216}$$

The preceding result can also be expressed in terms of γ of (1.174) after dividing the numerator and denominator on the right side of (1.216) by $|E_x|^2$, namely

$$\tan 2\psi = \frac{2\tan\gamma}{1-\tan^2\gamma}\cos\delta, \tag{1.217}$$

or

$$\tan 2\psi = \tan 2\gamma\cos\delta. \tag{1.218}$$

The above relationship allows one to find ψ from a knowledge of (γ, δ). It is also necessary to find $\mathcal{X}$ in terms of (γ, δ); hence, one divides (1.213) by (1.210) to obtain

$$\mp\frac{ab}{a^2+b^2} = \frac{|E_x|\,|E_y|\sin\delta}{|E_x|^2+|E_y|^2} = \frac{\left(\frac{|E_y|}{|E_x|}\right)\sin\delta}{1+\left(\frac{|E_y|}{|E_x|}\right)^2},$$

or

$$\mp\frac{2ab}{a^2+b^2} = \frac{2\tan\gamma}{1+\tan^2\gamma}\sin\delta. \tag{1.219}$$

It is useful to introduce the ellipticity angle $\mathcal{X}$ as shown in Figure 1.17. While $\mathcal{X}$ lies in the range $0 \le \mathcal{X} \le \pi/4$ in Figure 1.17, it is convenient to also allow $\mathcal{X}$ to take on negative values so that $-\pi/4 \le \mathcal{X} \le 0$, for the following reason. In particular, one may write

$$\frac{b}{a} \equiv \mp\tan\mathcal{X} = \mp\tan|\mathcal{X}|\,; \quad \begin{bmatrix} -\pi/4 \le \mathcal{X} \le 0 \\ 0 \le \mathcal{X} \le \pi/4 \end{bmatrix}, \tag{1.220}$$

since $b/a \ge 0$ always (with $a \ge 0$ and $b \ge 0$). Therefore, the introduction of $\mp$ signs in (1.220) corresponding to $\begin{bmatrix} -\pi/4\le\mathcal{X}\le 0 \\ 0\le\mathcal{X}\le\pi/4 \end{bmatrix}$; i.e., for $\mathcal{X} \lesseqgtr 0$, can now allow one to easily distinguish between the $\begin{bmatrix}\text{right-handed}\\ \text{left-handed}\end{bmatrix}$ states of wave polarization, because

$$\mp\frac{b}{a} = \mp\frac{|E_\eta|}{|E_\xi|} = \mp\tan|\mathcal{X}| \Rightarrow \begin{bmatrix}\text{RHEP}\\ \text{LHEP}\end{bmatrix} \text{ for } \begin{bmatrix} -\pi/4 \le \mathcal{X} \le 0 \\ 0 \le \mathcal{X} \le \pi/4 \end{bmatrix}; \tag{1.221}$$

i.e., the polarization ellipse in Figure 1.17 is traced out in the $\begin{bmatrix}\text{RHEP}\\ \text{LHEP}\end{bmatrix}$ sense based on the $\mp$ sign in $(\mp b/a)$ of (1.221). The fact that $\begin{bmatrix}-\pi/4\le\mathcal{X}\le 0\\ 0\le\mathcal{X}\le\pi/4\end{bmatrix}$ corresponds to $\begin{bmatrix}\text{RHEP}\\ \text{LHEP}\end{bmatrix}$ is seen more clearly on the Poincaré sphere, which will be introduced later on to vividly describe the state of polarization of the fields of antennas in their far zone. The definition of the far zone region of an antenna is presented in Chapter 5.

It is noted that (b/a) in (1.221) is referred to as the axial ratio of the ellipse.

Thus, (1.219) becomes

$$\frac{2\left(\mp\frac{b}{a}\right)}{1+\left(\mp\frac{b}{a}\right)^2} = \frac{2\tan\gamma}{1+\tan^2\gamma}\sin\delta, \tag{1.222}$$

which in turn reduces via (1.220) to

$$\frac{2\tan\mathcal{X}}{1+\tan^2\mathcal{X}} = \frac{2\tan\gamma}{1+\tan^2\gamma}\sin\delta, \tag{1.223}$$

or

$$\sin 2\mathcal{X} = \sin 2\gamma\ \sin\delta. \tag{1.224}$$

Equations (1.218) and (1.224) provide the required information to find $(\psi, \mathcal{X})$ from (γ, δ).

1.8.3 Decomposition of a Polarization State into Circularly Polarized Components

It is quite useful to note that any $\hat{z}$-propagating plane wave field $\overline{\mathcal{E}} = \text{Re}\,\overline{E}\,e^{j\Omega}$ of (1.167), where $\overline{E} = \hat{x}E_x + \hat{y}E_y$, can be decomposed into a set of RHCP and LHCP plane wave fields. Thus, if one defines $\hat{R}_C \equiv \frac{\hat{x}-j\hat{y}}{\sqrt{2}}$ and $\hat{L}_C \equiv \frac{\hat{x}+j\hat{y}}{\sqrt{2}}$, then one can write $\overline{E}$ as the following sum, namely

$$\overline{E} = \frac{1}{\sqrt{2}}\left[\hat{R}_C\,E_R + \hat{L}_C\,E_L\right]; \quad \begin{bmatrix} E_R = E_x + jE_y \\ E_L = E_x - jE_y \end{bmatrix}. \tag{1.225}$$

In the above, $E_R\hat{R}_C$ is the RHCP component of $\overline{E}$ with amplitude E_R, while $E_L\hat{L}_c$ is the LHCP component of $\overline{E}$ with amplitude E_L, respectively. It is clear that $\hat{R}_C$ and $\hat{L}_C$ are right and left circularly-polarized plane wave fields of unit amplitude.

It follows directly from (1.225) above that an elliptically polarized field, $\overline{E}$, can be decomposed into its right- and left-handed circularly polarized components $\overline{E}_R$ and $\overline{E}_L$, respectively, where $\overline{E}_R = E_R\hat{R}_C$ and $\overline{E}_L = E_L\hat{L}_C$. One can thus define a parameter Q as follows:

$$Q \equiv \frac{E_L}{E_R} = \frac{|E_L|e^{j\psi_L}}{|E_R|e^{j\psi_R}},$$

where $E_{L,R} = |E_L, E_R|e^{j\psi_{L,R}}$. It is easily seen via (1.225) that

$$Q = \frac{E_L}{E_R} = \frac{E_x - jE_y}{E_x + jE_y} = \frac{1-jP}{1+jP} = -\frac{jP-1}{jP+1},$$

in which P has been defined previously in (1.171). The preceding relationship is analogous to $\Gamma = \frac{(Z/Z_0)-1}{(Z/Z_0)+1}$ for a transmission line (X-line), where Γ is the reflection coefficient, and Z is the impedance associated with the line having a characteristic impedance of Z_0. Specifically, $(-Q)$ is analogous to the voltage wave reflection coefficient, and jP is analogous to the normalized line impedance Z/Z_0. Often, it is required to find the length of a X-line to obtain a certain driving point impedance Z/Z_0 using $\Gamma = \frac{Z/Z_0-1}{Z/Z_0+1}$ or $\frac{Z}{Z_0} = \frac{1+\Gamma}{1-\Gamma}$. A solution to the latter transcendental expression is quite tedious;

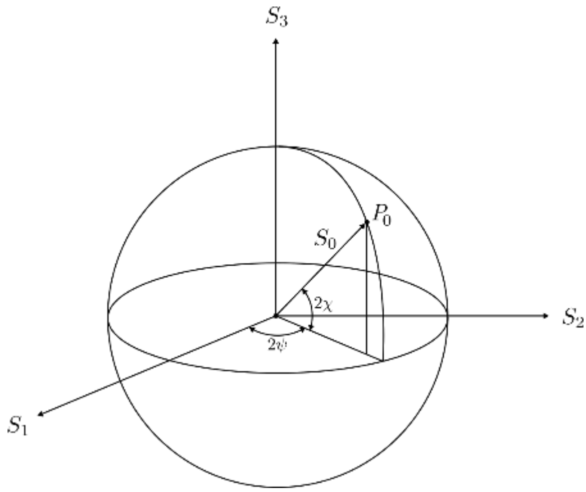

Figure 1.18 Poincaré sphere with coordinate axes S_1, S_2, and S_3. The radius of the sphere is S_0 $\left(= \sqrt{S_1^2 + S_2^2 + S_3^2}\right)$. The polarization at any point P_0 on the Poincaré sphere is given by the coordinates $\left(S_0, 2\mathcal{X}, 2\psi\right)$.

however, it can be easily facilitated by the use of the Smith Chart which provides a graphical solution. The Smith Chart provides a transformation between Z/Z_0 and Γ. The simplest type of impedance transformation is due to a section of uniform X-line, which causes a phase change at a constant amplitude of Γ. Rumsey (part 1 of [3]) analogously noted that the simplest type of polarization transformation results from the rotation of an antenna producing an elliptical polarization, which causes a phase change at a constant amplitude of Q; thus, Rumsey employed the Smith Chart for polarization calculations to study the transmission between elliptically polarized antennas. A different approach for analyzing the latter problem was developed by Sinclair [4].

1.8.4 Poincaré Sphere for Describing Polarization States

The state of polarization can also be described visually by a point P_0 on the surface of an appropriate sphere, named the Poincaré sphere, as shown in Figure 1.18. The coordinates of the point P_0 on the sphere are given by the Stokes parameters which in turn depend on $(\psi, \mathcal{X})$. One recalls that $(\psi, \mathcal{X})$ may be found from the given parameters (γ, δ) of the wave via (1.218) and (1.224) derived earlier. The Stokes parameters for time-harmonic fields are defined as follows:

$$S_0 \equiv |E_x|^2 + |E_y|^2; \tag{1.226}$$

$$S_1 \equiv |E_x|^2 - |E_y|^2; \tag{1.227}$$

$$S_2 \equiv 2\,|E_x|\,|E_y|\,\cos\delta; \tag{1.228}$$

$$S_3 \equiv 2\,|E_x|\,|E_y|\,\sin\delta. \tag{1.229}$$

From (1.226), it is clear that the Stokes parameter S_0 is directly proportional to the total time average power density in the wave, since $\overline{P}_{\text{avg}}$ is given by $\frac{1}{2}\,\text{Re}\,\overline{E} \times \overline{H}^* = \frac{1}{2Z}\,\hat{k}\,|\overline{E}|^2$, or

$$\overline{P}_{\text{avg}} = \frac{\hat{k}}{2Z}\left[|E_x|^2 + |E_y|^2\right] = \frac{\hat{k}}{2Z}\,S_0, \tag{1.230}$$

where $\hat{k} = \hat{z}$ in the present instance. The rectangular coordinates (S_1, S_2, S_3) at point P_0 on the Poincaré sphere can be shown to be related to its corresponding Poincaré spherical polar coordinates $(S_0, 2\mathcal{X}, 2\psi)$

by

$$S_1 = S_0 \cos 2\mathcal{X} \cos 2\psi. \tag{1.231}$$

$$S_2 = S_0 \cos 2\mathcal{X} \sin 2\psi. \tag{1.232}$$

$$S_3 = S_0 \sin 2\mathcal{X}. \tag{1.233}$$

It follows from (1.231), (1.232), and (1.233) that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2. \tag{1.234}$$

It is clear from (1.234) that any three of the four parameters S_0, S_1, S_2, and S_3 are independent. The angles (2ψ) and $(2\mathcal{X})$ in Figure 1.18 are more natural on the Poincaré sphere as they correspond to the spherical polar angles. The alternative set of angles (2γ) and (2δ) are shown in Figure 1.19 for specifying the same state of polarization at P_0.

Different points on the Poincaré sphere constitute different polarization states as may be visualized from Figure 1.20. The upper hemisphere contains left-handed elliptical polarization (LHEP) while the lower hemisphere contains the right-handed elliptical polarization (RHEP). The point at the north pole

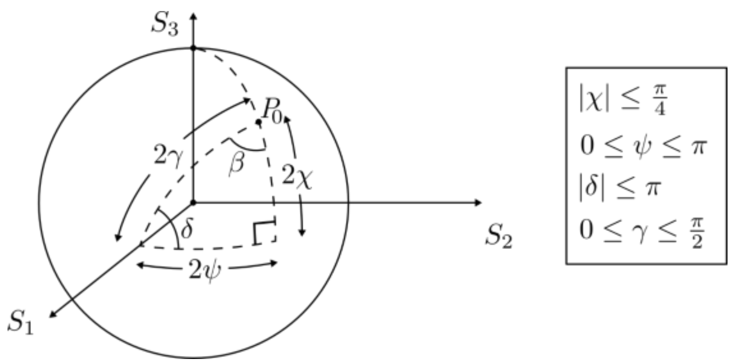

Figure 1.19 Angles γ and δ on the Poincaré sphere.

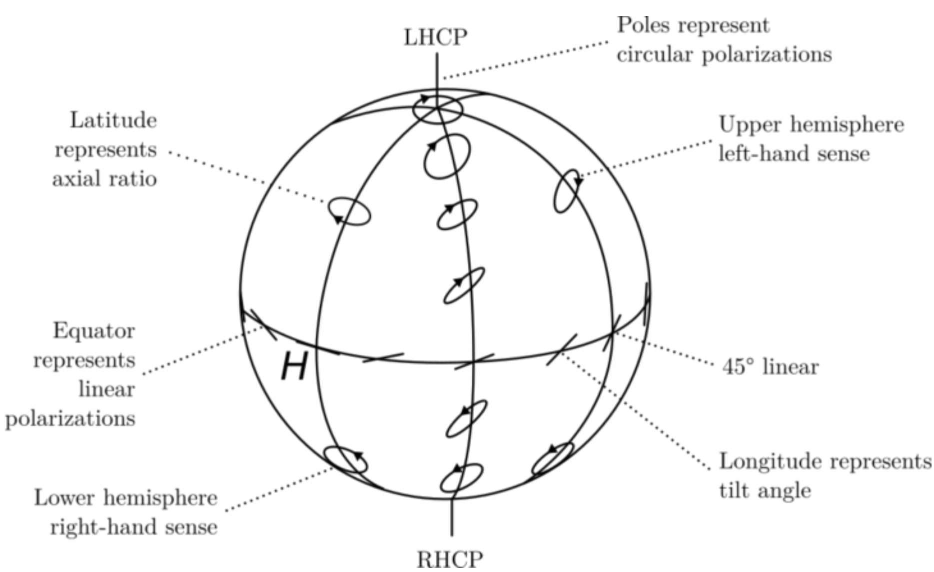

Figure 1.20 Polarization representation on the Poincaré sphere. (Source: [5] ©1978 IEEE.)

is circularly polarized in the left-handed sense or is LHCP, while at the south pole it is RHCP. Points on the equator are linearly polarized. Linear polarization results when $\delta = 0, \pm\pi$, etc.

It is not possible for an antenna to radiate EM waves with the same state of polarization in all directions in space. One may track the changes in polarization with direction in space via a curve traced by P_0 on the Poincaré sphere. Thus, it is noted that the polarization state at P_0 is related to the direction of propagation, $\hat{k}$ in space. Finally, the Stokes parameters are often used to describe the state of polarization of quasi monochromatic partially polarized waves (generated by stellar sources, for example) as discussed elsewhere [6]-[8].

Problem 1.5

Consider a plane wave with an electric field $\overline{E} = (\hat{x}E_x + \hat{y}E_y)\, e^{-jkz}$, where $E_x = |E_x| = 1$ and $E_y = |E_y|\, e^{j\frac{\pi}{4}} = 2\, e^{j\frac{\pi}{4}}$ with k fixed.

(a) Find the polarization ratio, P, and the angles δ and γ for this wave.
(b) Describe the state of polarization of this wave using the Poincaré sphere.
(c) What is the polarization state of this wave?
(d) Plot the tip of the $\overline{E}$ vector as a function of ωt at $z = 0$.
(e) Plot the vector $\overline{E}$ for discrete values of $\omega t = 0$; $\frac{\pi}{4}$; $\frac{\pi}{2}$; $\frac{3\pi}{4}$; π; $\frac{5\pi}{4}$; $\frac{3\pi}{2}$; $\frac{7\pi}{4}$; and 2π, respectively, and indicate the corresponding values of z in order for the observer to move with the wave (with velocity $\frac{\omega}{k}\hat{z}$ such that $(\omega t - kz)$ stays fixed. Show that the distance moved in z for $0 \le \omega t < 2\pi$ corresponds to one wavelength (λ). Thus, the wave polarization rotates once for every wavelength (λ) of travel.

Problem 1.6

Consider a LCP plane wave field $\overline{E} = \overline{A}(\omega)e^{-jkz}$ with $k = \frac{\omega}{c}$.

(a) Find the time-dependent field $\overline{\mathcal{E}}(z,t)$ from $\overline{E}(z,\omega) = \overline{A}(\omega)e^{-j\frac{\omega}{c}z}$ if $\overline{A}(\omega) = \hat{x} + j\hat{y}$, and if $\overline{E}(z,\omega)$ is a monochromatic, or single-frequency field with $\omega = \omega_c$. Use the relation

$$\overline{\mathcal{E}}(z,t) = \mathrm{Re}\, \overline{E}(z,\omega_c)e^{+j\omega_c t}$$

to show that $\overline{\mathcal{E}}(z,t) = \hat{x}\cos\omega_c\left(t - \frac{z}{c}\right) - \hat{y}\sin\omega_c\left(t - \frac{z}{c}\right)$.

(b) Repeat part (a), but obtain $\overline{\mathcal{E}}(z,t)$ via a Fourier inversion integral. Note that in this case, one may use $\overline{f}_e(z,\omega)$ given by (1.77), namely

$$\overline{f}_e(z,\omega') = 2\pi\overline{E}(z,\omega')\left[\frac{\delta(\omega'-\omega_c) + \delta(\omega'+\omega_c)}{2}\right],$$

and then use (1.78) to obtain $\overline{\mathcal{E}}$ as follows:

$$\overline{\mathcal{E}}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}_e(z,\omega')e^{j\omega' t}d\omega'.$$

(Note: $\overline{E}(z,\omega') = (\hat{x} + j\hat{y})e^{-j\frac{\omega'}{c}z}$ in the expression for $\overline{f}_e(z,\omega')$.) One may invoke the relationship $\overline{E}(z,\omega_c) = \overline{E}^*(z,-\omega_c)$ to obtain a real valued field $\overline{\mathcal{E}}(z,t)$ in the time domain. Show that the result is the same as that obtained in part (a).

(c) Find $\overline{\mathcal{E}}(z,t)$ from $\overline{E}(z,\omega) = (\hat{x} + j\hat{y})e^{-j\frac{\omega}{c}z}$ for $\omega > 0$, if ω is a variable. Use the inverse Fourier integral to obtain $\overline{\mathcal{E}}(z,t)$. Note that $\overline{E}(z,-\omega) = E^*(z,\omega)$ to obtain a real valued $\overline{\mathcal{E}}(z,t)$. One may thus employ the result given below (1.71) for $\overline{\mathcal{G}}(\overline{r},t)$, which is obtained via a Fourier inversion of $\overline{f}_G(\overline{r},\omega)$. Note that $\int_0^\infty \cos\omega\left(t-\frac{z}{c}\right)d\omega = \pi\delta\left(t-\frac{z}{c}\right)$, while $\int_{-\infty}^{\infty}\cos\omega\left(t-\frac{z}{c}\right)d\omega = 2\pi\delta\left(t-\frac{z}{c}\right)$. Also, $\int_{-\infty}^{\infty} e^{j\omega\left(t-\frac{z}{c}\right)}d\omega = 2\pi\delta\left(t-\frac{z}{c}\right)$.

(d) Repeat part (c) if $\overline{E}(z,\omega) = \overline{A}(\omega)e^{-j\frac{\omega}{c}z}$; $\overline{A}(\omega) = (\hat{x}+j\hat{y})P(\omega)$, where $P(\omega)$ is a very narrow pulse function of unit amplitude and width $\Delta\omega$ centered at $\omega=\omega_c$.

1.9 Phase and Group Velocity

Consider a single-frequency signal which propagates as a plane wave with the standard form:

$$\overline{\mathcal{E}}(\overline{r},t) = \mathrm{Re}\,\overline{A}\,e^{j\left(\omega_c t-\overline{k}(\omega_c)\cdot\overline{r}\right)};\quad \overline{\mathcal{E}}(0,t) = \mathrm{Re}\,\overline{A}e^{j\omega_c t}, \tag{1.235}$$

where, for convenience, $\overline{A}$ is assumed to be a real constant with $\overline{A}\cdot\hat{k}=0$, and the signal carrier frequency, $f_c \equiv \frac{\omega_c}{2\pi}$, where $k(\omega_c) = \omega_c\sqrt{\mu\,\epsilon}$, in a lossless isotropic homogeneous medium. The above signal propagates with a phase velocity $v_{pk} = \frac{\omega_c}{k(\omega_c)}$, or $v_{pk} = \frac{1}{\sqrt{\mu(\omega_c)\,\epsilon(\omega_c)}}$, as indicated previously in (1.145).

If the above single-frequency wave in (1.235) is now modulated, as depicted in Figure 1.21, by a signal, or time function, $f(t)$, with a frequency spectrum $F(\omega)$ that is centered around the carrier angular frequency ω_c, then the information or the signal ($f(t)$) can be shown to propagate essentially as a wavepacket. This wavepacket is a narrow group of plane waves with frequencies within the narrow band centered at ω_c (or f_c), and it propagates along $\hat{k}$ with a wavepacket velocity or group velocity, v_{gk}. This v_{gk} is generally different from v_{pk} as shown below. Only the plane wave phase of the carrier (at f_c) is seen to propagate with the phase velocity, v_{pk}, in the $\hat{k}$ direction. In particular, the carrier wave modulated by $f(t)$ is assumed to propagate as a plane wave field with a form similar to that in (1.235) at $\overline{r}=0$ as

$$\overline{\mathcal{E}}(\overline{r}=0,t) = \overline{A}\,f(t)\cos\omega_c t = \mathrm{Re}\,\overline{A}\,f(t)\,e^{j\,\omega_c\,t}. \tag{1.236}$$

The only difference between (1.235) above and (1.236) is that the modulation signal (or signal information) $f(t)$ is now included in the latter equation. Next, the Fourier transformation of $\overline{\mathcal{E}}(\overline{r}=0,t)$ yields the corresponding frequency domain field, $\overline{E}(\overline{r}=0,\omega)$ of the modulated carrier as follows:

$$\overline{E}(\overline{r}=0,\omega) = \int_{-\infty}^{\infty}\overline{\mathcal{E}}(t,\overline{r}=0)\,e^{-j\,\omega\,t}dt, \tag{1.237}$$

which from (1.236) becomes

$$\overline{E}(\overline{r}=0,\omega) = \overline{A}\int_{-\infty}^{\infty} f(t)\left[\frac{e^{-j\,(\omega-\omega_c)\,t}+e^{-j\,(\omega+\omega_c)\,t}}{2}\right]dt. \tag{1.238}$$

For $\overline{r}\neq 0$, one must include a plane wave propagation factor $e^{-j\,\overline{k}(\omega)\cdot\overline{r}}$, for each ω in (1.238), to obtain

$$\overline{E}(\overline{r},\omega) = \overline{E}\,(\overline{r}=0,\omega)\,e^{-j\,\overline{k}(\omega)\cdot\overline{r}};$$

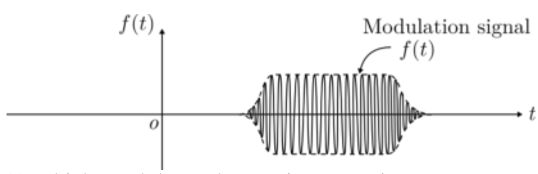

Figure 1.21 Signal $f(t)$ which modulates the carrier operating at $f = f_c = \omega_c/2\pi$. The signal is assumed to be narrow band in frequency.

thus, (1.238) becomes for $\bar{r} \neq 0$ the following:

$$\overline{E}(\bar{r},\omega) = \overline{A}\, e^{-j\,\bar{k}(\omega)\cdot\bar{r}} \left[\frac{F(\omega-\omega_c)+F(\omega+\omega_c)}{2}\right], \tag{1.239}$$

where $F(\omega)$ is the Fourier transform of $f(t)$. As indicated earlier, $F(\omega)$ is a narrow frequency band signal; i.e., $F(\omega \mp \omega_c)$ in (1.239) possesses a narrow bandwidth centered around $\omega = \pm\omega_c$. The inverse transform of $\overline{E}(\bar{r},\omega)$ yields $\overline{\mathcal{E}}(\bar{r},t)$ as

$$\overline{\mathcal{E}}(\bar{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{E}(\bar{r},\omega)\, e^{j\,\omega\, t} d\omega, \tag{1.240}$$

which via (1.239) and (1.240) yields:

$$\overline{\mathcal{E}}(\bar{r},t) = \overline{A}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left[\frac{F(\omega-\omega_c)\, e^{+j\left(\omega t-\bar{k}(\omega)\cdot\bar{r}\right)} + F(\omega+\omega_c)\, e^{+j\left(\omega t-\bar{k}(\omega)\cdot\bar{r}\right)}}{2}\right]\right]. \tag{1.241}$$

Since F is assumed to be narrow band, the exponent $\bar{k}(\omega)\cdot\bar{r}$ in the factor $e^{j(\omega t-\bar{k}(\omega)\cdot\bar{r})}$ can be expanded in a Taylor series about $\omega = \pm\omega_c$ pertaining to the $F(\omega \mp \omega_c)$ terms. Retaining only the first two terms of the Taylor expansion for ω close to $\pm\omega_c$ yields:

$$\bar{k}(\omega)\cdot\bar{r} \approx \bar{k}(\pm\omega_c)\cdot\bar{r} + \left.\frac{\partial\bar{k}(\omega)}{\partial\omega}\right|_{\omega=\pm\omega_c}(\omega\mp\omega_c)\cdot\bar{r}. \tag{1.242}$$

Thus, from (1.242) one notes that in the case of a narrow band signal centered about $\omega = \pm\omega_c$, the wavenumber $k(\omega) = \omega\sqrt{\mu(\omega)\,\epsilon\,(\omega)}$ behaves highly linearly with ω, so that $\bar{k}'(\omega) = (\partial\bar{k})/(\partial\omega)$ is very slowly varying or nearly constant at $\omega = \pm\omega_c$. It is also noted that the approximation in (1.242) is also valid for a medium that is very weakly dispersive, where $\mu(\omega) \approx \mu(\pm\omega_c)$ and $\epsilon(\omega) \approx \epsilon(\pm\omega_c)$ so that once again, $\bar{k}'(\omega) \approx$ constant at $\omega = \pm\omega_c$. For a plane wave $\bar{k}(\pm\omega_c) = k(\pm\omega_c)\hat{k} = \pm\omega_c\sqrt{\mu(\omega_c)\,\epsilon\,(\omega_c)}\,\hat{k}$ with $\mu(-\omega_c) = \mu(\omega_c)$ and $\epsilon(-\omega_c) = \epsilon(\omega_c)$, respectively, in a lossless medium via (1.99) since μ and ϵ are real in this case. Hence, it follows that

$$\bar{k}(\pm\omega_c) = \pm\bar{k}(\omega_c), \tag{1.243}$$

and

$$\bar{k}'(\pm\omega_c) = \bar{k}'(\omega_c). \tag{1.244}$$

The relationship in (1.244) results from the fact that $\bar{k}'(\omega) \equiv \frac{\partial\bar{k}}{\partial\omega} = \hat{k}\frac{\partial[\omega\sqrt{\mu(\omega)\epsilon(\omega)}]}{\partial\omega}$ is an even function of ω with $\bar{k}(\omega)$ being an odd function of ω as in (1.243). Thus, incorporating (1.242)-(1.244) into (1.241) yields:

$$\begin{aligned}\overline{\mathcal{E}}(\bar{r},t) \approx& \overline{A}\, e^{j\left(\omega_c t-\bar{k}(\omega_c)\cdot\bar{r}\right)}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\frac{F(\omega-\omega_c)}{2}\, e^{+j(\omega-\omega_c)\left[t-\bar{k}'(\omega_c)\cdot\bar{r}\right]}\right]\\ &+\overline{A}\, e^{-j\left(\omega_c t-\bar{k}(\omega_c)\cdot\bar{r}\right)}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,\frac{F(\omega+\omega_c)}{2}\, e^{+j(\omega+\omega_c)\left[t-\bar{k}'(\omega_c)\cdot\bar{r}\right]}\right].\end{aligned} \tag{1.245}$$

Note that (1.241) with $\bar{r} = 0$ is the same as (1.236). Now, with a change of variables $\omega \mp \omega_c = \zeta$ with $d\omega = d\zeta$ in (1.245), one can recognize the inverse Fourier transform $1/(2\pi)\int_{-\infty}^{\infty} d\omega\, \frac{F(\omega\mp\omega_c)}{2}\, e^{+j(\omega\mp\omega_c)\left[t-\bar{k}'(\omega_c)\cdot\bar{r}\right]}$ to be $\frac{1}{2}f(t-\bar{k}'(\omega_c)\cdot\bar{r})$ so that (1.245) reduces to:

$$\overline{\mathcal{E}}(\bar{r},t) \approx \overline{A}\, f\left(t-\bar{k}'(\omega_c)\cdot\bar{r}\right)\cos\left(\omega_c t-\bar{k}(\omega_c)\cdot\bar{r}\right), \tag{1.246}$$

or

$$\overline{\mathcal{E}}(\overline{r},t) \approx \mathrm{Re}\,\overline{A}\, f\left(t - \overline{k}'(\omega_c)\cdot\overline{r}\right)\, e^{j\,(\omega_c t - \overline{k}(\omega_c)\cdot\overline{r})}. \tag{1.247}$$

From (1.247) and (1.236), it is clear that a narrow band signal $f(t)$ of (1.236) propagates without distortion as a narrow wavepacket but with a time delay of $t_0 \equiv \overline{k}'(\omega_c)\cdot\overline{r}$. The velocity with which the wavepacket actually propagates is defined as the group velocity $\overline{v}_{gk} = \hat{k}v_{gk}$. The group velocity v_{gr} along any observation direction $\hat{r}$ is given by

$$v_{gr} = \frac{r}{t_0} = \frac{r}{\overline{k}'(\omega_c)\cdot\overline{r}} = \frac{r}{\overline{k}'(\omega_c)\cdot\hat{r}r},$$

or

$$v_{gr} = \frac{1}{\hat{r}\cdot\frac{\partial\overline{k}}{\partial\omega}} = \frac{1}{\frac{\partial k}{\partial\omega}}\cdot\left(\frac{1}{\hat{r}\cdot\hat{k}}\right). \tag{1.248}$$

If $\hat{r} = \hat{k}$, then one obtains the group velocity v_{gk} in the direction of actual wave propagation ($\hat{k}$), namely:

$$v_{gk} = \frac{1}{\left.\frac{\partial\overline{k}}{\partial\omega}\right|_{\omega=\omega_c}\cdot\hat{k}} = \frac{1}{\left.\frac{\partial k}{\partial\omega}\right|_{\omega=\omega_c}} = \frac{1}{k'(\omega_c)}.$$

On the other hand, the phase velocity of the wavepacket in the direction of wave propagation $\hat{k}$ is obtained either via (1.245) or (1.247) as the usual quantity v_{pk} from $e^{j(\omega_c t - \overline{k}(\omega_c)\cdot\overline{r})}$, namely:

$$v_{pk} = \frac{\omega_c}{k(\omega_c)} = c; \quad k(\omega_c) = \omega_c\sqrt{\mu(\omega_c)\,\epsilon(\omega_c)}. \tag{1.249}$$

From (1.144) and (1.145), it is obvious that $v_{pr} \geq v_{pk}(= c)$. Thus, v_{pr} can exceed the speed of light c in the medium. The group velocity v_{gk} can never exceed the speed of light. It can be verified that the energy velocity, which is the velocity of energy transport in the direction of the Poynting vector, is always the same as v_{gk}. It may also be remarked that for a narrow band signal in a lossy medium one can analytically continue (1.247) to obtain

$$\overline{\mathcal{E}}(\overline{r},t) \approx \mathrm{Re}\,\overline{A}\, f\left(t - \overline{k}'(\omega_c)\cdot\overline{r}\right)\, e^{-\alpha(\omega_c)\hat{p}\cdot\overline{r}}\, e^{j\,(\omega_c t - \beta(\omega_c)\,\hat{p}\cdot\overline{r})}, \tag{1.250}$$

where $\overline{k}(\omega)$ is now complex; namely, $\overline{k}(\omega) = [\beta(\omega) - j\alpha(\omega)]\hat{p}$ for a plane wave propagating with a propagation constant or wavenumber β in the direction $\hat{p}$ so that $\overline{\beta} = \beta\hat{p}$, with the attenuation constant α accounting for loss. The term $e^{-\alpha\hat{p}\cdot\overline{r}}$ indicates that the signal information $f(t - \overline{k}'(\omega_c)\cdot\overline{r})$ is attenuated due to loss as it propagates. It is noted that the minus sign between the real and imaginary parts of a complex valued $\overline{k}(\omega)$, namely $\overline{k}(\omega) = (\beta(\omega) - j\alpha(\omega))\hat{p}$ as shown above, yields waves which decay in the direction of propagation $\hat{p}$ for the $e^{+j\omega t}$ time convention that has been assumed unless otherwise specified. This minus sign mentioned in the preceding is consistent with the corresponding minus sign on the right side of (1.96) which has been chosen for the same reason. In this case $v_{gk} = 1/(\beta'(\omega_c))$ and $v_{pk} = \omega_c/(\beta(\omega_c))$, where $\beta'(\omega_c) \equiv [\partial\beta/\partial\omega]$ evaluated at $\omega = \omega_c$.

It is useful to plot the behavior of ω vs. $k(\omega)$ for narrow band and broad band signals as indicated in Figure 1.22.

Problem 1.7

If $k(\omega) = \sqrt{k_0^2(\omega) - \eta^2}$ with $k_0(\omega) = \omega\sqrt{\mu(\omega)\,\epsilon(\omega)}$ in which $\mu(\omega) \equiv \mu_0$ and $\epsilon(\omega) \equiv \epsilon_0$ with μ_0 and ϵ_0 being constants, then show that as $\omega \to \infty$ the $k(\omega) \to k_0(\omega)$ if η is a constant. Consider a field which

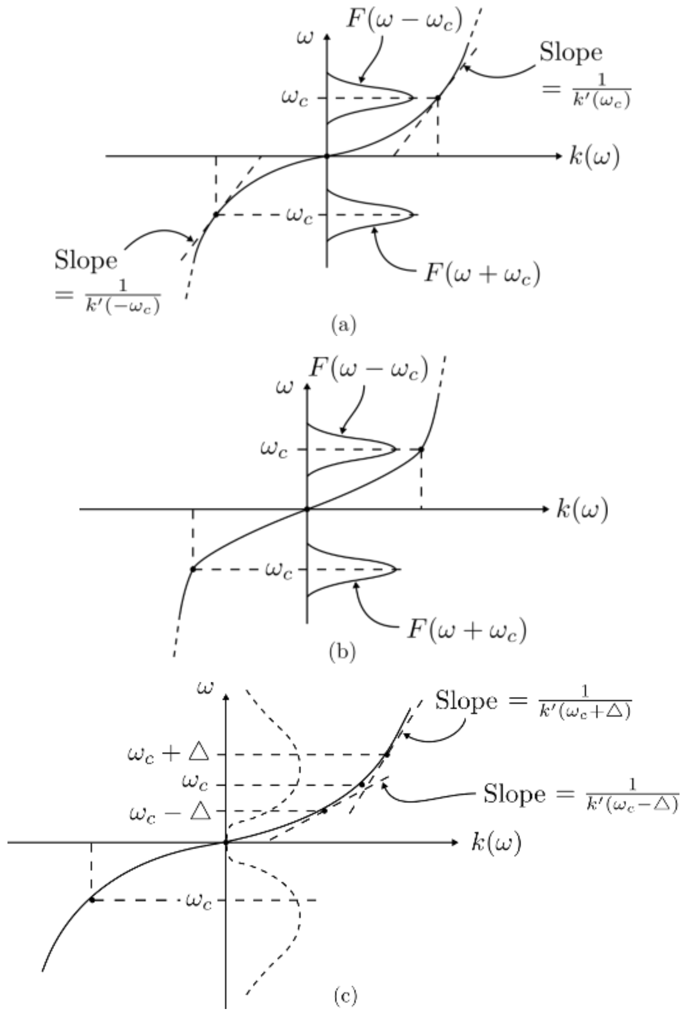

Figure 1.22 Plots of ω vs. $(1/k'(\omega))$. (a) The slowly varying $(1/k'(\omega))$ provides a relatively well-defined signal velocity near $\omega = \omega_c$; thus, the narrow band signal will propagate without distortion. (b) Here, $(1/k'(\omega))$ is rapidly varying near $\omega = \omega_c$; therefore, the signal will propagate with distortion even though it is narrow band. (c) In this case, the signal velocity is slower for $\omega = \omega_c - \Delta$ and faster for $\omega = \omega_c + \Delta$ and therefore different frequency components of the broad band signal propagate with different velocities causing signal distortion.

propagates as $e^{-jk(\omega)z}$ along $\hat{z}$, then find the v_{gk} and v_{pk} for this wave for $k_0^2 > \eta^2$. If $\omega \to \infty$, then what are v_{gk} and v_{pk} in this limiting case? Plot ω vs. $k(\omega)$ only for $k_0^2(\omega) \geq \eta^2$.

Problem 1.8

In the event that the frequency-domain representation of the signal given by $F(\omega \pm \omega_c)$ is not sufficiently narrow band in (1.239) above, then it is necessary to include additional terms in the Taylor expansion of (1.242). If one includes the next term in the Taylor expansion of (1.242) about $\omega = \pm\omega_c$, then one obtains

$$\overline{k}(\omega) \cdot \overline{r} \approx \overline{k}(\pm\omega_c) \cdot \overline{r} + \left.\frac{\partial \overline{k}(\omega)}{\partial \omega}\right|_{\omega=\pm\omega_c} (\omega \mp \omega_c) \cdot \overline{r} + \frac{1}{2} \left.\frac{\partial^2 \overline{k}(\omega)}{\partial \omega^2}\right|_{\omega=\pm\omega_c} (\omega \mp \omega_c)^2 \cdot \overline{r}.$$

Show that when the above expansion is utilized in (1.241), the third (or quadratic) term gives rise to signal distortion as it propagates.

Problem 1.9

Consider a narrow band frequency domain signal centered about $\omega = \pm\omega_c$ with $|\omega| >> \omega_c$ given by the Gaussian function $F(\omega \pm \omega_c) = e^{-[(\omega\pm\omega_c)^2/\omega]}$.

Show that the corresponding time-dependent electric field $\overline{\mathcal{E}}(\overline{r}, t)$ given by the expression (1.237) has the form:

$$\overline{\mathcal{E}}(\overline{r}, t) = \mathrm{Re}\,\overline{A}\left[\sqrt{\pi}\,W\,e^{-\frac{1}{4}W^2\left(t-\overline{k}'(\omega_c)\cdot\hat{z}\,z\right)}\right],$$

for the present case. This field propagates as a Gaussian wavepacket with a group velocity $\left[k'(\omega_c)\right]^{-1}$. Identify W.

From Problem 1.8, it is evident that the result in (1.246) is strictly valid for quasi-monochromatic, or narrow band signals. The description in (1.246) thus breaks down for signals which do not possess a sufficiently narrow frequency bandwidth; also (1.242) and hence (1.246) eventually lose validity as $\overline{r}$ keeps increasing resulting in a loss of signal integrity. However, within its region of validity, (1.246) is very useful for introducing the concept of a wavepacket and signal or group velocity.

1.10 Separable Solutions of the Source-Free Wave Equation in Cylindrical and Spherical Coordinates and for Isotropic Homogeneous Media

The source-free vector Helmholtz's wave equation of the form $\left(\nabla^2 + k^2\right)\left[\overline{E}; \overline{H}\right] = 0$, for the EM fields $\left(\overline{E}, \overline{H}\right)$ in an isotropic, homogeneous medium, can be scalarized in cylindrical, and spherical coordinates, via the use of an appropriate pair of vector potential functions as will be shown later in Chapter 7 and Chapter 10. Here, the resultant scalarized Helmholtz's equations will be presented for the case of cylindrical and spherical coordinates, and a formal solution to these equations in terms of corresponding cylindrical and spherical wave functions, respectively, will be obtained via the method of separation of variables.

1.10.1 Source-Free Cylindrical Wave Solutions

The scalarized source-free wave equation in cylindrical coordinates (ρ, ϕ, z) is given as

$$\left(\nabla^2 + k^2\right)\,\psi = 0, \tag{1.251}$$

where ∇^2 in cylindrical coordinates is given by

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}, \tag{1.252}$$

and $\psi = \psi(\rho, \phi, z)$, is the wavefunction in cylindrical coordinates in which ρ is the radial coordinate, ϕ is the azimuthal coordinate and z is the axial coordinate. The corresponding unit vectors $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$, which are taken to be positive in the increasing ρ, ϕ, and z coordinate directions, respectively, represent an orthogonal triad where $\hat{\rho}\times\hat{\phi} = \hat{z}$; $\hat{\phi}\times\hat{z} = \hat{\rho}$; $\hat{z}\times\hat{\rho} = \hat{\phi}$.

One may express the cylindrical wave solution, ψ, in terms of separation of variables as

$$\psi(\rho, \phi, z) = R(\rho)\,\Phi(\phi)\,Z(z). \tag{1.253}$$

One may incorporate the assumed solution (1.253) into (1.251), via (1.252), to arrive at

$$\Phi\, Z \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) + R\, Z \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + R\,\Phi \frac{\partial^2 Z}{\partial z^2} + k^2\, R\,\Phi\, Z = 0. \tag{1.254}$$

For a nontrivial solution, one can divide (1.254) by $\psi = R\,\Phi\, Z$ to obtain

$$\frac{1}{R}\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{d^2 \Phi}{d\phi^2} + \frac{1}{Z}\frac{d^2 Z}{d z^2} + k^2 = 0. \tag{1.255}$$

Since $\frac{1}{Z} Z''$ is independent of (ρ, ϕ), with primes denoting derivatives with respect to its variable, and since the coordinates (ρ, ϕ, z) can vary independently of each other in (1.255) it follows that $\frac{Z''}{Z}$ must be independent of z, i.e., $\frac{Z''}{Z}$ must be a constant denoted here by $-k_z^2$, namely $\frac{Z''}{Z} = -k_z^2$, or

$$\frac{d^2 Z}{d z^2} + k_z^2\, Z = 0. \tag{1.256}$$

From (1.255) and (1.256), one obtains

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2 \Phi}{d\phi^2} + \left(k^2 - k_z^2\right)\rho^2 = 0. \tag{1.257}$$

Now, Φ''/Φ in (1.257) can vary with ϕ only and (ρ, ϕ) can change independently of ϕ while still requiring (1.257) to be satisfied; the latter requirement leads to $(\Phi''/\Phi) = -p^2$ or

$$\frac{d^2 \Phi}{d\phi^2} + p^2\, \Phi = 0, \tag{1.258}$$

where p, like k_z, is also a separation constant. Now, from (1.257) and (1.258), one obtains Bessel's differential equation:

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \left[\left(k_\rho \rho\right)^2 - p^2\right] R = 0. \tag{1.259}$$

One notes that (1.256), (1.258), and (1.259) are the three resultant separated 1-D differential equations which arise from the original 3-D scalar Helmholtz's wave equation of (1.251) in cylindrical coordinates, subject to the separation condition

$$k_\rho^2 \equiv k^2 - k_z^2, \tag{1.260}$$

which has been utilized in (1.257) to arrive at (1.259). The solutions to the second-order differential equations in (1.256) and (1.258) have been obtained previously in terms of exponential (or alternatively in terms of trigonometric) functions, namely, from (1.256) one may write

$$Z(z) = c_1\, e^{-j\, k_z\, z} + c_2\, e^{+j\, k_z\, z}, \tag{1.261a}$$

or

$$Z(z) = A\, \cos k_z\, z + B\, \sin k_z\, z, \tag{1.261b}$$

for real k_z, or

$$Z\,(z) = C z + D, \tag{1.261c}$$

for $k_z = 0$. One can analytically continue (1.261a) and (1.261b) to imaginary or complex values of k_z; in the latter case, (1.261a) leads to

$$Z\,(z) = C_1\, e^{-(\alpha + j\,\beta)\, z} + C_2\, e^{+(\alpha + j\,\beta)\, z}, \tag{1.261d}$$

with $k_z \equiv \beta - j\,\alpha$, where α and β are real constants. On the other hand, if $k_z = -j\,\alpha$ with $\beta = 0$, then one can also obtain a solution of the form in (1.261b) except that the trigonometric functions $\cos k_z z$ and $\sin k_z z$ therein are now replaced by $\cosh(\alpha\, z)$ and $-j\ \sinh(\alpha\, z)$, respectively.

Also, following the solution to (1.256) given by (1.261a) or (1.261b), one may write the solutions to (1.258) in a similar fashion, namely

$$\Phi\,(\phi) = C_3\, e^{-j\,p\,\phi} + C_4\, e^{+j\,p\,\phi}, \tag{1.262a}$$

or

$$\Phi\,(\phi) = C_5\ \cos p\,\phi + C_6\ \sin p\,\phi. \tag{1.262b}$$

The C_1, C_2, C_3, C_4, C_5, and C_6 are all constants that must be determined from boundary conditions in any given problem.

The two independent solutions to the radial (ρ) differential equation of (1.259) is given by the functions $J_p\,(k_\rho\rho)$ and $J_{-p}\,(k_\rho\rho)$, where [9]

$$J_p\,\left(k_\rho\rho\right) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\,(k_\rho\rho)^{2\ell+p}}{\ell\,!\,(\ell+p)\,!\,2^{2\ell+p}}, \tag{1.263a}$$

and

$$J_{-p}\,\left(k_\rho\rho\right) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell\,(k_\rho\rho)^{2\ell-p}}{\ell\,!\,(\ell-p)\,!\,2^{2\ell-p}}. \tag{1.263b}$$

If $p = m$, an integer, then $J_{-m}\,(k_\rho\rho) = (-1)^m\, J_m(k_\rho\rho)$ so that only J_m, or only J_{-m}, remains independent. Another independent solution of Bessel's differential equation can be expressed as

$$N_p\,(k_\rho\rho) = \frac{J_p(k_\rho\rho)\ \cos p\,\pi - J_{-p}(k_\rho\rho)}{\sin p\,\pi}. \tag{1.264a}$$

Also, if $p = m$, then

$$N_m\,(k_\rho\rho) = \lim_{p\to m}\ N_p\,(k_\rho\rho). \tag{1.264b}$$

Thus, $J_p\,(k_\rho\rho)$ and $N_p\,(k_\rho\rho)$ are two independent solutions to Bessel's differential equation in (1.259) valid for any p whether or not $p = m$ (integer). It is also possible to define two additional forms of independent solutions to Bessel's differential equations by combining $J_p\,(k_\rho\rho)$ and $N_p\,(k_\rho\rho)$ as follows:

$$H_p^{(1)}\,(k_\rho\rho) = J_p\,(k_\rho\rho) + j\,N_p\,(k_\rho\rho), \tag{1.265a}$$

and

$$H_p^{(2)}\,(k_\rho\rho) = J_p\,(k_\rho\rho) - j\,N_p\,(k_\rho\rho). \tag{1.265b}$$

Hence, J_p, N_p, $H_p^{(1)}$, and $H_p^{(2)}$, each individually constitute independent solutions of Bessel's differential equation in (1.259); as a result, a linear combination of any two of these four solutions provide a complete solution to (1.259). The J_p and N_p are referred to as Bessel and Neumann functions, respectively. Also $H_p^{(1)}$ and $H_p^{(2)}$ are referred to as Hankel functions of the first and second kind, respectively. It follows from the above remarks that a complete solution to (1.259) can be expressed as

$$R\,(\rho) = C_7\,J_p\,(k_\rho\rho) + C_8\,H_p^{(2)}\,(k_\rho\rho), \tag{1.266a}$$

or

$$R\,(\rho) = D_7\,H_p^{(1)}\,(k_\rho\rho) + D_8\,H_p^{(2)}\,(k_\rho\rho), \tag{1.266b}$$

where C_7, C_8, D_7, and D_8 are constants. From (1.265a) and (1.265b), it directly follows that

$$J_p\,(k_\rho\rho) = \frac{1}{2}\,\left[H_p^{(1)}\,(k_\rho\rho) + H_p^{(2)}\,(k_\rho\rho)\right], \tag{1.267a}$$

and

$$N_p\left(k_\rho\rho\right) = \frac{1}{2}\left[H_p^{(1)}\left(k_\rho\rho\right) - H_p^{(2)}\left(k_\rho\rho\right)\right]. \tag{1.267b}$$

For angular 2π wave periodicity in ϕ, it is clear that $p = m$, where m is an integer. In later chapters, p will be allowed to have fractional values in regions defined by angular sectors in azimuth, and also to have complex values in some situations, respectively. For integer values of p in which $p = m$ (integer), the small and large argument forms of the cylindrical functions are as follows:

$$J_m\left(k_\rho\rho\right) \approx \frac{\left(k_\rho\rho\right)^m}{2^m\, m\,!};\;\; \text{as } k_\rho\rho \to 0 \text{ and } m > 0, \tag{1.268a}$$

and

$$J_o\left(k_\rho\rho\right) \to 1;\;\; \text{as } k_\rho\rho \to 0. \tag{1.268b}$$

$$N_m\left(k_\rho\rho\right) \approx \frac{-2^m\,(m-1)\,!}{\pi\left(k_\rho\rho\right)^m};\;\; \text{as } k_\rho\rho \to 0 \text{ and } m > 0, \tag{1.268c}$$

and

$$N_o\left(k_\rho\rho\right) \to -\frac{2}{\pi}\log\left(\frac{2}{\gamma\, k_\rho\rho}\right);\;\; \text{as } k_\rho\rho \to 0. \text{ Also, } \gamma = 1.781. \tag{1.268d}$$

Likewise,

$$\overset{(1)}{H_m^{(2)}}\left(k_\rho\rho\right) \approx \left\{\begin{array}{ll} \dfrac{\left(k_\rho\rho\right)^m}{2^m\, m\,!} \mp j\,\dfrac{2^m\,(m-1)\,!}{\pi\left(k_\rho\rho\right)^m}; & \text{as } k_\rho\rho \to 0 \text{ and } m > 0 \\ 1 \mp j\,\frac{2}{\pi}\,\log\left(\dfrac{2}{\pi\, k_\rho\rho}\right); & \text{as } k_\rho\rho \to 0 \text{ and } m = 0 \end{array}\right\}, \tag{1.268e}$$

where the symbol $\approx$ implies "approximately equal to." Also,

$$J_m\left(k_\rho\rho\right) \sim \sqrt{\frac{2}{\pi\, k_\rho\rho}}\,\cos\left(k_\rho\rho - \frac{m\,\pi}{2} - \frac{\pi}{4}\right);\;\; \left(k_\rho\rho\right) \to \infty, \tag{1.269a}$$

$$N_m\left(k_\rho\rho\right) \sim \sqrt{\frac{2}{\pi\, k_\rho\rho}}\,\sin\left(k_\rho\rho - \frac{m\,\pi}{2} - \frac{\pi}{4}\right);\;\; \left(k_\rho\rho\right) \to \infty, \tag{1.269b}$$

$$\overset{(1)}{H_m^{(2)}}\left(k_\rho\rho\right) \sim \sqrt{\frac{\mp 2\,j}{\pi\, k_\rho\rho}}\; j^{\mp m}\, e^{\pm j\, k_\rho\rho};\;\; \left(k_\rho\rho\right) \to \infty, \tag{1.269c}$$

where the symbol $\sim$ implies "asymptotically equal to." Additionally, a useful orthogonality relation is

$$\int_0^a d\rho\rho J_n\left(\lambda_{nm}\frac{\rho}{a}\right) J_n\left(\lambda_{nq}\frac{\rho}{a}\right) = \frac{a^2}{2}\, J_{n+1}^2\left(\lambda_{nm}\right)\,\delta_{mq}, \tag{1.269d}$$

where

$$\delta_{mq} = \left[\begin{array}{ll} 1, & m = q \\ 0, & m \neq q \end{array}\right].$$

One notes that only the $J_m\left(k_\rho\rho\right)$ type solution is bounded at $\rho = 0$; thus, it alone must be used as the solution to (1.259) in any finite region containing the origin ($\rho = 0$). It is also noted that for an $e^{\pm j\,\omega\, t}$ time convention which pertains to a harmonic (single-frequency $f = \omega/2\pi$) wave, the $\overset{(1)}{H_m^{(2)}}\left(k_\rho\rho\right)$

represents a radially (ρ) propagating cylindrical wave function of the type $e^{\pm j\,k_\rho \rho}$ as $\rho \to \infty$. Hence, in an unbounded region excluding the origin, one must use only $H_m^{\substack{(1)\\(2)}}(k_\rho \rho)$ for the outgoing wave solution to (1.259) with an assumed $e^{\mp j\,\omega t}$ time convention. In a finite region excluding the origin, one may express the solution either as $R(\rho) = C J_p(k_\rho \rho) + D N_p(k_\rho \rho)$ or as $R(\rho) = P_1\, H_p^{(1)}(k_\rho \rho) + P_2\, H_p^{(2)}(k_\rho \rho)$, which for $e^{+j\omega t}$ explicitly represents both incoming and outgoing radially ($\hat{\rho}$) propagating fields via $H_p^{(1)}$ and $H_p^{(2)}$, respectively, with C, P_1, and P_2 being constants.

Additional properties of cylindrical functions may be found in texts on mathematical functions [9].

In summary, a general solution to the source-free scalar wave equation of (1.251) in cylindrical coordinates may be expressed as

$$\psi(\rho, \phi, z) = \left[A_1 J_p(k_\rho \rho) + B_1 H_p(k_\rho \rho)\right]\left[A_2 e^{-jp\phi} + B_2 e^{jp\phi}\right]\left[A_3 e^{-jk_z z} + B_3 e^{jk_z z}\right],$$

where the constants (A_i, B_i) with $i = 1, 2, 3$, must be determined by enforcing the boundary conditions specified for any given problem.

1.10.2 Source-Free Spherical Wave Solutions

In spherical coordinates (r, θ, ϕ), the ∇^2 operator is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}, \tag{1.270}$$

which may be incorporated into the scalar Helmholtz's wave equation:

$$\left(\nabla^2 + k^2\right)\psi = 0. \tag{1.271}$$

The $\psi = \psi(r, \theta, \phi)$ in (1.271) represents a wave solution in spherical coordinates, in which r represents the radial coordinate, ϕ is once again the azimuthal coordinate as for the cylindrical case, and θ is the elevation or polar coordinate (with respect to the polar axis z). Using separation of variables, one may write the spherical wave solution to (1.270) as

$$\psi(r, \theta, \phi) = R(r)L(\theta)\Phi(\phi), \tag{1.272}$$

where R depends only on r, L only on θ and Φ only on ϕ. Incorporating the assumed solution of (1.272) into (1.271) using (1.270), and multiplying by r^2, leads to

$$L\Phi\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dL}{d\theta}\right) + \frac{RL}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} + k^2r^2RL\Phi = 0. \tag{1.273}$$

For a nontrivial solution, one can divide (1.273) by (1.272), and multiplying by $\sin^2\theta$ to obtain

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{L}\frac{d}{d\theta}\left(\sin\theta\frac{dL}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + k^2r^2\sin^2\theta = 0. \tag{1.274}$$

As before, one may separate the ϕ variation via

$$\frac{\Phi''}{\Phi} = -p^2, \tag{1.275}$$

where, again, p is a separation constant. Incorporating (1.275) into (1.274) yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{L\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dL}{d\theta}\right) - \frac{p^2}{\sin^2\theta} + k^2r^2 = 0. \tag{1.276}$$

One can separate the θ variation via a separation constant ν as

$$\frac{1}{L\,\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{dL}{d\theta}\right)-\left(\frac{p^2}{\sin^2\theta}\right)=-\nu\,(\nu+1), \tag{1.277}$$

where $\nu(\nu+1)$ is chosen so that (1.277) may be identified with associated Legendre's differential equation in θ, namely:

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{dL}{d\theta}\right)+\left[\nu\,(\nu+1)-\frac{p^2}{\sin^2\theta}\right]\,L=0. \tag{1.278}$$

Next from (1.276) and (1.278) one obtains

$$\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right)+\left[k^2\,r^2-\nu\,(\nu+1)\right]\,R=0, \tag{1.279}$$

which is related to Bessel's differential equation. The solution to (1.275) is given as before by

$$\Phi\,(\phi)=A_1\,e^{-j\,p\,\phi}+B_1\,e^{-j\,p\,\phi}, \tag{1.280}$$

where A_1 and B_1 are constants. For angularly 2π periodic solutions in ϕ, it is clear that $p=m$, where m is an integer.

The second-order associated Legendre's differential equation in (1.277) can be put in an alternative form via the transformation

$$u=\cos\theta, \tag{1.281}$$

which yields

$$\left(1-u^2\right)\,\frac{d^2L}{du^2}-2\,u\,\frac{dL}{du}+\left[\nu\,(\nu+1)-\frac{p^2}{1-u^2}\right]\,L=0. \tag{1.282}$$

In the spherical coordinate system, the region $0\leq\theta\leq\pi$ is of interest; this region corresponds to $-1\leq u\leq 1$. When $p=0$, one obtains Legendre's equation:

$$\left(1-u^2\right)\,\frac{d^2L(u)}{du^2}-2\,u\,\frac{dL(u)}{du}+\nu\,(\nu+1)\,L(u)=0, \tag{1.283}$$

which has a solution given by

$$L(u)=A_2\,P_\nu(u)+B_2\,P_\nu(-u), \tag{1.284}$$

where $P_\nu(u)$ and $P_\nu(-u)$ are two independent solutions of (1.283), and A_2 and B_2 are constants. If $\nu=n$, where n is an integer, then

$$P_n(-u)=(-1)^n\,P_n(u);\quad n=\text{integer}. \tag{1.285}$$

For $\nu=n$, an integer, it is clear from (1.285) that $P_n(u)$ and $P_n(-u)$ are no longer two independent solutions of Legendre's equation. However, another form of an independent solution of Legendre's equation referred to as Legendre's solution of the second kind denoted by $Q_\nu(u)$ is defined by

$$Q_\nu\,(u)=\frac{\pi}{2}\,\frac{P_\nu\,(u)\,\cos\nu\,\pi-P_\nu\,(-u)}{\sin\nu\,\pi}, \tag{1.286}$$

which for $\nu=n$ (integer) provides

$$Q_n\,(u)=\lim_{\nu\to n}Q_{\,\nu}\,(u). \tag{1.287}$$

Thus, for $\nu=n$, one can write the solution to (1.283) as a linear combination of the two independent solutions $P_n(u)$ and $Q_n(u)$, namely:

$$L(u)=A_3\,P_n\,(u)+B_3\,Q_n\,(u), \tag{1.288}$$

where A_3 and B_3 are constants.

It is noted that $P_\nu(u)$ is given

$$P_\nu(u) = \sum_{\ell=0}^{N} \frac{(-1)^\ell \ (\nu+\ell)!}{(\ell !)^2 \ (\nu-\ell)!} \left(\frac{1-u}{2}\right)^\ell - \frac{\sin \nu\pi}{\pi} \sum_{\ell=N+1}^{\infty} \frac{(\ell-1-\nu)!\ (\ell+\nu)!}{(\ell !)^2} \left(\frac{1-u}{2}\right)^\ell, \quad (1.289)$$

where N is the nearest integer which satisfies $N \le \upsilon$ [10]. When $\nu = n$ (integer) one obtains the following via (1.289)

$$P_n(u) = \sum_{\ell=0}^{N} \frac{(-1)^\ell \ (2n-2\ell)!}{2^n \ell !\ (n-\ell)!\ (n-2\ell)!} \ u^{n-2\ell}, \quad (1.290)$$

where $N = n/2$ or $(n-1)/2$, whichever is an integer [10].

A procedure for generating the Legendre polynomials, $P_n(u)$, is given in a convenient form by Rodrigues' formula, namely

$$P_n(u) = \frac{1}{2^n\, n!} \frac{d^n}{d\, u^n} \ (u^2-1)^n. \quad (1.291)$$

It is noted that $Q_n(u)$ can be expressed as

$$Q_n\,(u) = P_n\,(u) \left[\frac{1}{2} \ \log \ \frac{1+u}{1-u} - \left(\sum_{q=1}^{n} \frac{1}{q}\right)\right] + \sum_{\ell=1}^{n} \frac{(-1)^\ell \ (n+\ell)!}{(\ell !)^2 \ (n-\ell)!} \left(\frac{1-u}{2}\right)^\ell \left[\sum_{q=1}^{\ell} \frac{1}{q}\right]. \quad (1.292)$$

From (1.292), it is clear that $Q_n(u)$ become infinite at $u = \pm 1$ corresponding to $\theta = 0$ and $\theta = \pi$.

Next, one considers the associated Legendre's equation for $L(u)$ in (1.282), when $p \neq 0$, which has two linearly independent solutions $P_\nu^p(u)$ and $P_\nu^p(-u)$, namely

$$L\,(u) = A_4 \ P_\nu^p\,(u) + B_4 \ P_\nu^p\,(-u), \quad (1.293)$$

with A_4 and B_4 being constants. When $(p+u)$ becomes an integer, then $P_\nu^p(u)$ and $P_\nu^p(-u)$ are no longer independent; instead, P_ν^p and $Q_\nu^p(u)$ constitute the required linearly independent solutions in the latter case.

In the special case when $\nu = n$ (integer) and $p = m$ (integer), then the linearly independent solutions of (1.282) are $P_n^m(u)$ and $Q_n^m(u)$; i.e.,

$$L\,(u) = A_5 \ P_n^m\,(u) + B_5 \ Q_n^m\,(u), \quad (1.294)$$

with A_5 and B_5 being constants. The associated Legendre polynomials of the first and second kind denoted by P_n^m and Q_n^m, respectively, can be found via the well-known relations as follows:

$$P_n^m\,(u) = (-1)^m\,(1-u^2)^{m/2}\, \frac{d^m\, P_n(u)}{du^m}, \quad (1.295)$$

and

$$Q_n^m\,(u) = (-1)^m\,(1-u^2)^{m/2}\, \frac{d^m\, Q_n(u)}{du^m}. \quad (1.296)$$

In the more general case, where ν and p are not integers, the $P_\nu^p(u)$ and $Q_\nu^p(u)$ are given in terms of a hypergeometric function [9].

It is important to note that in the special case of $\nu = n$ and $p = m$, where n and m are integers, $P_n^m(u) = 0$ for $m > n$.

The solution to (1.279) is given by

$$R\,(r) = C_3 \ j_\nu\,(kr) + C_4 \ n_\nu\,(kr), \quad (1.297a)$$

with C_3 and C_4 being constants, or

$$R(r) = C_5\ h_\nu^{(1)}(kr) + C_6\ h_\nu^{(2)}(kr), \tag{1.297b}$$

with C_5 and C_6 being constants. Also one can choose to write

$$R(r) = C_7\ j_\nu(kr) + C_8\ h_\nu^{(2)}(kr), \tag{1.297c}$$

where C_7 and C_8 are constants. The results for $R(r)$ in (1.297a)-(1.297c) are all valid solutions since any two of the spherical Bessel, Neumann, and Hankel functions of the first and second kind, namely, j_ν, n_ν, $h_\nu^{(1)}$, and $h_\nu^{(2)}$, respectively, are linearly independent solutions and are known to be related to the corresponding cylindrical functions introduced earlier via [10]

$$\begin{bmatrix} j_\nu(kr) \\ n_\nu(kr) \\ h_\nu^{(1)}(kr) \\ h_\nu^{(2)}(kr) \end{bmatrix} = \sqrt{\frac{\pi}{2kr}} \cdot \begin{bmatrix} J_{\nu+\frac{1}{2}}(kr) \\ N_{\nu+\frac{1}{2}}(kr) \\ H^{(1)}_{\nu+\frac{1}{2}}(kr) \\ H^{(2)}_{\nu+\frac{1}{2}}(kr) \end{bmatrix}. \tag{1.298}$$

It is noted that there is no explicitly direct relationship between the separation constants ν and p introduced above.

Additional information on the functions introduced above, as the solutions to the scalar wave equation in spherical coordinates, may be found in [9].

A general solution to the source-free scalar wave equation of (1.271) in spherical coordinates may be expressed as

$$\psi(r,\theta,\phi) = \left[A_1\, j_n(kr) + B_1 h_n^{(2)}(kr)\right]\left[A_2 P_n^m(\cos\theta) + B_2 Q_n^m(\cos\theta)\right]\left[A_3 e^{-jm\phi} + B_3 e^{jm\phi}\right],$$

when $\upsilon = n$ and $p = m$, with n and m being integers. The constants (A_i, B_i), where $i = 1, 2, 3$, must as usual be found by enforcing the boundary conditions for any given problem.

A useful orthogonality relation is

$$\int_{-1}^{1} du\, P_n^m(u)\, P_\ell^m(u) = \frac{2(n+m)!}{(2n+1)(n-m)!}\,\delta_{\ell n}. \tag{1.299}$$

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