1

Basic System Analysis

1.1 Introduction

The twentieth century began with the electric power industry in its infancy; Thomas Edison and Nikola Tesla were locked in battle with Edison advocating direct current (dc) and Tesla alternating current (ac). The century ended with the electric power industry expanding rapidly from the traditional power generation, transmission, and utilization into propulsion of air, ground, and sea transportation. The advent of the computer and the silicon-controlled rectifier in the mid-1900s brought about an expansion of the power area to include the smart-grid, microgrids, efficient and robust electric drives, more electric aircraft, ships, and land vehicles. A growth which is likely to continue into the foreseeable future.

Before the advent of the computer, engineers were essentially limited to steady-state analysis and therefore unable to conveniently deal with the analytical challenges of the expanding power industry. This chapter sets forth some of the basic concepts and analysis tools that are part of the present-day power and electric drives area. Although not inclusive, the material covered in this chapter is representative and common to most disciplines of the power area.

1.2 Phasor Analysis and Power Calculations

Since the early twentieth century, we have lived in an ac world. Thanks to George Westinghouse and Nikola Tesla, power systems are predominately ac; power is generated by large ac generators, transmitted by high-voltage transmission lines, and transformed to a low voltage and distributed to homes and factories. The evolution of the ac power system brought about many engineering challenges and, as we look back, it is difficult to comprehend how these problems were solved without a computer. Even steady-state ac-circuit analysis posed a problem until

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the early 1900s when Charles Stienmetz, who was a less flamboyant colleague of Edison and Tesla, came up with the concept of what is now known as phasors. Some may consider the phasor a casualty of the computer age along with the slide rule. It is, however, still a very useful means for understanding and portraying the steady-state performance of electric machines, power systems, and electric drives. Moreover, the phasor concept provides a means of visualizing sinusoidal variations from different frames of reference, and in Chapter 2 we will find that the voltage and current phasors combined with Tesla's rotating magnetic field provides a straightforward means of analyzing and portraying the steady-state operation of ac machines.

The phasor can be established by expressing a steady-state sinusoidal variable as

$$F_a(t) = F_p \cos \theta_{ef} \tag{1.1}$$

where the a subscript is used here to denote sinusoidal quantities. The sinusoidal variations are expressed as cosines, capital letters are used to denote steady-state quantities, and F_p is the peak value of the sinusoidal variation. Generally, F or f represents voltage (V or v) or current (I or i) in circuit analysis, but it could be any sinusoidal variable. For steady-state conditions, θ_{ef} may be written as

$$\theta_{ef}(t) = \omega_e t + \theta_{ef}(0) \tag{1.2}$$

where ω_e is the electrical angular velocity in rad/s (2π times the frequency) and $\theta_{ef}(0)$ is the time-zero position of the electrical variable. Substituting (1.2) into (1.1) yields

$$F_a(t) = F_p \cos \left[\omega_e t + \theta_{ef}(0)\right] \tag{1.3}$$

Now, Euler's identity is

$$e^{j\alpha} = \cos \alpha + j \sin \alpha \tag{1.4}$$

and since we are expressing the sinusoidal variation as a cosine, (1.3) may be written as

$$F_a(t) = \text{Re}\left\{F_p e^{j[\omega_e t + \theta_{ef}(0)]}\right\} \tag{1.5}$$

where Re is shorthand for the "real part of." Equations (1.3) and (1.5) are equivalent. Let us rewrite (1.5) as

$$F_a(t) = \text{Re}\left\{F_p e^{j\theta_{ef}(0)} e^{j\omega_e t}\right\} \tag{1.6}$$

We need to take a moment to define what is referred to as the root-mean-square (rms) of a sinusoidal variation. In particular, the rms value is defined as

$$F = \left(\frac{1}{T}\int_0^T F_a^2(t)dt\right)^{\frac{1}{2}} \tag{1.7}$$

where F is the rms value of $F_a(t)$ and T is the period of the sinusoidal variation. It is left to the reader to show that the rms value of (1.3) is $F_p/\sqrt{2}$. Therefore, we can express (1.6) as

$$F_a(t) = \mathrm{Re}\left[\sqrt{2}Fe^{j\theta_{ef}(0)}e^{j\omega_e t}\right] \tag{1.8}$$

By definition, the phasor representing $F_a(t)$, which is denoted with a raised tilde, is

$$\tilde{F}_a = Fe^{j\theta_{ef}(0)} \tag{1.9}$$

which is a complex number. The reason for using the rms value as the magnitude of the phasor will be addressed later in this section. Equation (1.6) may now be written as

$$F_a(t) = \mathrm{Re}\left[\sqrt{2}\tilde{F}_a e^{j\omega_e t}\right] \tag{1.10}$$

A shorthand notation for (1.9) is

$$\tilde{F}_a = F\underline{/\theta_{ef}(0)} \tag{1.11}$$

Equation (1.11) is commonly referred to as the *polar form* of the phasor. The *Cartesian* form is

$$\tilde{F}_a = F\cos\theta_{ef}(0) + jF\sin\theta_{ef}(0) \tag{1.12}$$

When using phasors to calculate steady-state voltages and currents, we think of the phasors as being stationary at $t = 0$; however, we know from (1.10) that a phasor is related to the instantaneous value of the sinusoidal quantity it represents. In other words, the real projection of the phasor $\tilde{F}_a$ rotating counterclockwise at ω_e is the instantaneous value of $F_a(t)/\sqrt{2}$. Thus, with $\theta_{ef}(0) = 0$ in (1.3)

$$F_a(t) = \sqrt{2}F\cos\omega_e t \tag{1.13}$$

the phasor representing (1.13) is

$$\tilde{F}_a = Fe^{j0} = F\underline{/0^\circ} = F + j0 \tag{1.14}$$

For

$$\begin{aligned} F_a(t) &= \sqrt{2}F\sin\omega_e t \\ &= \sqrt{2}F\cos(\omega_e t - 90^\circ) \end{aligned} \tag{1.15}$$

the phasor is

$$\tilde{F}_a = Fe^{-j\pi/2} = F\underline{/-90^\circ} = 0 - jF \tag{1.16}$$

We will use degrees and radians interchangeably when expressing phasors. Although there are several ways to arrive at (1.16) from (1.15), it is helpful to ask yourself where must the rotating phasor be positioned at time zero so that, when it rotates counterclockwise at ω_e, its real projection is $(1/\sqrt{2})F_p \sin \omega_e t$? It follows that a phasor of amplitude F positioned at 90° represents $-\sqrt{2}F \sin \omega_e t$.

In other words, we are viewing a sinusoidal variation as the real projection in the real-imaginary plane of a rotating line equal in magnitude to the positive peak value ($\sqrt{2}F$) of the variation and rotating at the electrical angular velocity of the sinusoidal variation. Since we are dealing with a steady-state variation, we can stop the rotation at any time and view it as a fixed line, but knowing full well that it, in fact, represents a sinusoidal variations and to represent the sinusoidal variation we must rotate it counterclockwise at ω_e and take the real projection. Please understand that if we ran at ω_e in unison with the rotating $\sqrt{2}F$ line it would appear as a constant to us. Therefore, this is no different than stopping the phasor at some arbitrary time zero; but realizing that it actually represents a sinusoidal variation. We'll talk more about this important aspect as we go along. See Example 1A.

To show the facility of the phasor in the analysis of steady-state performance of ac circuits and devices, it is useful to consider a series circuit consisting of a resistance, an inductance L and a capacitance C. Thus, using uppercase letters to indicate steady-state variables

$$V_a = RI_a + L\frac{dI_a}{dt} + \frac{1}{C}\int I_a dt \tag{1.17}$$

Throughout the text, we will use either R or r to represent resistance. For steady-state operation, let

$$V_a = \sqrt{2}V \cos[\omega_e t + \theta_{ev}(0)] \tag{1.18}$$

$$I_a = \sqrt{2}I \cos[\omega_e t + \theta_{ei}(0)] \tag{1.19}$$

where we have dropped the functional notation, and the subscript a helps to distinguish the instantaneous value from the rms value of the steady-state variables. The steady-state voltage equation may be obtained by substituting (1.18) and (1.19) into (1.17), whereupon we can write

$$\begin{aligned}\sqrt{2}V \cos[\omega_e t + \theta_{ev}(0)] = R\sqrt{2}I \cos[\omega_e t + \theta_{ei}(0)] \\ + \omega_e L\sqrt{2}I \cos\left[\omega_e t + \frac{1}{2}\pi + \theta_{ei}(0)\right] \\ + \frac{1}{\omega_e C}\sqrt{2}I \cos\left[\omega_e t - \frac{1}{2}\pi + \theta_{ei}(0)\right]\end{aligned} \tag{1.20}$$

The second term on the right-hand side of (1.20), which is $L\frac{dI_a}{dt}$, can be written

$$\omega_e L\sqrt{2}I \cos\left[\omega_e t + \frac{1}{2}\pi + \theta_{ei}(0)\right] = \omega_e L \mathrm{Re}\left[\sqrt{2}Ie^{j\frac{1}{2}\pi}e^{j\theta_{ei}(0)}e^{j\omega_e t}\right] \tag{1.21}$$

Since $\widetilde{I}_a = Ie^{j\theta_{ei}(0)}$, from (1.21), we can write

$$L\widetilde{\frac{dI_a}{dt}} = \omega_e L e^{j\frac{1}{2}\pi}\widetilde{I}_a \tag{1.22}$$

Since $e^{j\frac{1}{2}\pi} = j$, (1.22) may be written

$$L\widetilde{\frac{dI_a}{dt}} = j\omega_e L\widetilde{I}_a \tag{1.23}$$

If we follow a similar procedure, we can show that

$$\frac{1}{C}\widetilde{\int I_a dt} = -j\frac{1}{\omega_e C}\widetilde{I}_a \tag{1.24}$$

Differentiation of a steady-state sinusoidal variable rotates the phasor counterclockwise by $\frac{1}{2}\pi$ or j; integration rotates the phasor clockwise by $\frac{1}{2}\pi$ or $-j$.

The steady-state voltage equation given by (1.20) can now be written in phasor form as

$$\widetilde{V}_a = \left[R + j\left(\omega_e L - \frac{1}{\omega_e C}\right)\right]\widetilde{I}_a \tag{1.25}$$

We can express (1.25) compactly as

$$\widetilde{V}_a = Z\widetilde{I}_a \tag{1.26}$$

where the impedance, Z, is a complex number; it is not a phasor. It may be expressed as

$$Z = R + j(X_L - X_C) \tag{1.27}$$

where $X_L = \omega_e L$ is the inductive reactance and $X_C = \frac{1}{\omega_e C}$ is the capacitive reactance. We should be careful here. Some prefer to write (1.27) as $R + jX$ where X is $X_L + X_C$ and let X_C be negative. This is essentially a matter of choice and does not change the end result. We will deal primarily with X_L and not X_C, therefore, this will have little impact on our work; nevertheless, since some authors will use a negative X_C we should make the reader aware of this difference.

It is appropriate to discuss the notation that will be used throughout the text. When an equation is written with the variables in lowercase letters it is valid for transient and steady state. If the variables are written with uppercase letters as in (1.17), the equation is a function of time and valid for instantaneous steady-state conditions. Equations (1.26) and (1.27) are phasor equations representing steady-state sinusoidal variables and are written in uppercase letters with an over tilde.

1.2.1 Power and Reactive Power

The instantaneous steady-state power is

$$P = V_a I_a$$
$$= \sqrt{2}V\cos[\omega_e t + \theta_{ev}(0)]\sqrt{2}I\cos[\omega_e t + \theta_{ei}(0)] \tag{1.28}$$

where V and I are rms values. After some manipulation, we can write (1.28) as

$$P = VI\cos[\theta_{ev}(0) - \theta_{ei}(0)] + VI\cos[2\omega_e t + \theta_{ev}(0) + \theta_{ei}(0)] \tag{1.29}$$

The instantaneous steady-state power given by (1.29) varies about an average value at a frequency of $2\omega_e$. That is, the second term of (1.29) has a zero average value and the average power P_{ave} may be written

$$P_{\text{ave}} = \left|\tilde{V}_a\right| \left|\tilde{I}_a\right| \cos[\theta_{ev}(0) - \theta_{ei}(0)] \tag{1.30}$$

where $\left|\tilde{V}_a\right|$ and $\left|\tilde{I}_a\right|$ are V and I, respectively, which are the magnitudes of the phasors (rms value), $\theta_{ev}(0) - \theta_{ei}(0)$ is referred to as the *power factor angle* φ_{pf}, and $\cos[\theta_{ev}(0) - \theta_{ei}(0)]$ is the *power factor*. Power is in watts. If current is assumed positive in the direction of voltage drop, then (1.30) is positive if power is consumed and negative if power is generated. It is interesting to point out that in going from (1.28) to (1.29), the coefficient of the two right-hand terms is $\frac{1}{2}(\sqrt{2}V\sqrt{2}I)$ or one-half the product of the peak values of the sinusoidal variables. Therefore, it was considered more convenient to use the rms values for the phasors, whereupon average steady-state power could be calculated by the product of the magnitude of the voltage and current phasors as given by (1.30).

The reactive power is defined as

$$Q = \left|\tilde{V}_a\right| \left|\tilde{I}_a\right| \sin\left[\theta_{ev}(0) - \theta_{ei}(0)\right] \tag{1.31}$$

The units of Q are var (volt–ampere reactive). An inductance is said to absorb reactive power where the current lags the voltage by 90° and Q is positive. In the case of a capacitor, where the current leads the voltage by 90°, Q is supplied and is negative. Actually, Q is a measure of the interchange of energy stored in the electric (capacitor) and magnetic (inductor) fields. However, unlike instantaneous real power, the average value of instantaneous reactive power is zero. We'll talk more about reactive power later.

Example 1A ***Phasor analysis***

The parameters of a series RLC circuit are $R = 6\ \Omega$, $L = 20$ mH, $C = 1 \times 10^3\ \mu$F. The 60-Hz applied voltage is $V_a = 155.6\cos\omega_e t$. Calculate $\tilde{I}_a$, P_{ave}, Q and draw the phasor diagram and the sinusoidal variations as viewed running at counterclockwise with the phasor representing maximum V_a ($\sqrt{2}V\underline{/0°}$). From the expression of V_a

$$\tilde{V}_a = 110\underline{/0°}\ \text{V} \tag{1A.1}$$

Now, $\omega_e = 2\pi f = 2\pi \times 60 = 377$ rad/s and

$$Z = R + j(X_L - X_C)$$
$$= R + j\left(\omega_e L - \frac{1}{\omega_e C}\right)$$
$$= 6 + j\left(377 \times 20 \times 10^{-3} - \frac{1}{377 \times 1 \times 10^{-3}}\right) = 7.73\underline{/39.1^\circ}\ \Omega \qquad (1A.2)$$

$$\tilde{I}_a = \frac{\tilde{V}_a}{Z} = \frac{110\underline{/0^\circ}}{7.73\underline{/39.1^\circ}} = 14.2\underline{/-39.1^\circ}\ \mathrm{A} \qquad (1A.3)$$

$$P_{\mathrm{ave}} = \left|\tilde{V}_a\right| \left|\tilde{I}_a\right| \cos\varphi_{pf} \qquad (1A.4)$$

where

$$\varphi_{pf} = \theta_{ev}(0) - \theta_{ei}(0)$$
$$= 0 - (-39.1^\circ) = 39.1^\circ \qquad (1A.5)$$

$$P_{\mathrm{ave}} = 110 \times 14.2 \cos 39.1^\circ$$
$$= 1212.2\ \mathrm{W} \qquad (1A.6)$$

$$Q = \left|\tilde{V}_a\right| \left|\tilde{I}_a\right| \sin\varphi_{pf}$$
$$= 110 \times 14.2 \sin 39.1^\circ = 985.1\ \mathrm{vars} \qquad (1A.7)$$

The phasor diagram is shown in Figure 1A.1.

SP1.2.1 Express the instantaneous steady-state power for Example 1A. [Substitute into (1.29)]

SP1.2.2 Redraw the phasor diagram shown in Figure 1A.1 showing $jX_L\tilde{I}_a$ and $-jX_C\tilde{I}_a$ as individual voltages. [Show $jX_L\tilde{I}_a$ and then from the terminus of $jX_L\tilde{I}_a$ show $-jX_C\tilde{I}_a$]

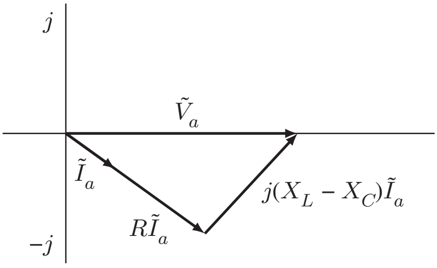


Figure 1A.1 Phasor diagram.

SP1.2.3 We know that $P_{\text{ave}} = |\widetilde{I}|^2 R$ does $Q = |\widetilde{I}_a|^2 X_L - |\widetilde{I}_a|^2 X_C$? [Yes]

SP1.2.4 If $\widetilde{V} = 1\underline{/0^\circ}$ V and $\widetilde{I} = 1\underline{/180^\circ}$ A in the direction of the voltage drop, calculate Z and P_{ave}. Is power generated or consumed? [$(-1 + j0)$ Ω, 1 W, generated]

SP1.2.5 Express the instantaneous power for 60 Hz voltage, $\widetilde{V}_a = 1\underline{/0^\circ}$, applied to a resistive circuit, $\widetilde{I}_a = 1\underline{/0^\circ}$. [$1 + \cos 754t$]

SP1.2.6 Repeat SP1.2.5 for (a) an inductance, $\widetilde{I}_a = I_L\underline{/-90^\circ}$ and (b) a capacitance, $\widetilde{I}_a = I_C\underline{/90^\circ}$. [(a) $I_L \cos(754t - 90^\circ)$, (b) $I_C \cos(754t + 90^\circ)$]

1.3 Elementary Magnetic Circuits

Electric machines and transformers, which are the backbone of the power industry, are electromagnetic systems. Therefore, magnetic circuits and magnetic coupling play a major role in power and drives systems, and it is necessary to establish the principles of magnetic systems sufficiently to convey the basic operation of the electromagnetic devices considered in later chapters. We will attempt to do this without becoming too involved.

An elementary magnetic circuit is shown in Figure 1.1. It consists of a ferromagnetic member (core) with a coil of wire of N turns wound on it and an air gap of length x. The ferromagnetic member could be iron, nickel, cobalt, or steel, for example. The voltage equation of the electric circuit may be written

$$v = ri + e \tag{1.32}$$

where r is the total resistance of the circuit, v is the source voltage, e is the voltage induced in the coil according to Faraday's law, and i is the current flowing in the circuit. The current flowing through the coil causes a magnetomotive force (mmf), which produces flux in the magnetic circuit denoted as Φ_m and Φ_l in Figure 1.1,

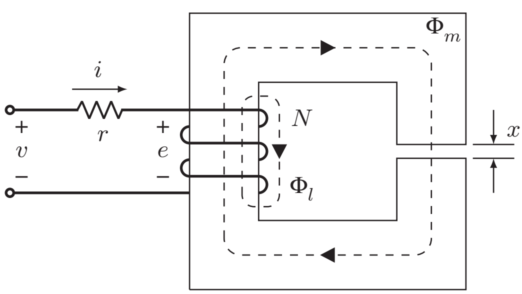


Figure 1.1 Elementary magnetic circuit with one air gap.

much as an electromotive force (emf) or source voltage produces current in an electric circuit.

There are arrows associated with the dashed lines representing the flux paths in Figure 1.1. These arrows indicate the assumed positive direction of flux which is determined from the assumed positive direction of current by the so-called "right-hand" rule. If you grasp the coil with your right hand with your fingers in the assumed direction of positive current flow around the coil, your thumb will point in the direction of positive flux. Or, imagine grasping a turn of the coil with thumb in the assumed direction of positive current. If you ungrasp your fingers, they will point in the direction of positive flux.

The total flux, Φ, that travels through (links) all of the turns N is

$$\Phi = \Phi_l + \Phi_m \tag{1.33}$$

where Φ_l is the equivalent flux that links all the turns of the coil but does not traverse the ferromagnetic member and Φ_m is referred to as the magnetizing flux that transverses the ferromagnetic member and links all of the turns of the coil. The leakage flux Φ_l is shown by one streamline in Figure 1.1, it represents the aggregate of the flux that occurs around each wire of the coil, traveling partially in the ferromagnetic material and partially in air.

The concept of magnetic poles can be used to advantage to explain the operation of electromechanical devices. The poles can be established by considering the magnetic circuit shown in Figure 1.1. In particular, to locate the north and south poles of an electromechanical device for the assumed positive direction of coil current, place yourself on the member that has the coil (windings). Use the right-hand rule to establish the direction of positive flux Φ_m for the assumed positive direction of coil current, where positive flux issues from the member with the coil into the air is the assumed north pole. The south pole is where the positive flux returns from the air to the member with the coil. In the case of the magnetic circuit shown in Figure 1.1, with the assumed direction of positive current, a north pole exists over the upper face of the air gap and a south pole over the lower face.

A magnetically linear circuit behaves much as a resistive electric circuit. According to Ohm's law, the current i in a resistive circuit is equal to the applied emf or applied voltage divided by the resistance r. In the case of a magnetic circuit, the total flux Φ is equal to the mmf, which is in ampere turns Ni, divided by the equivalent reluctance $\mathcal{R}$ of the magnetic circuit. Thus, in (1.33)

$$\Phi_l = \frac{Ni}{\mathcal{R}_l} \tag{1.34}$$

$$\Phi_m = \frac{Ni}{\mathcal{R}_i + \mathcal{R}_g} \tag{1.35}$$

where $\mathcal{R}_l$ is the reluctance of the leakage flux path, $\mathcal{R}_i$ is the reluctance of the ferromagnetic member, and $\mathcal{R}_g$ is the reluctance of the air gap. Although the leakage reluctance is generally determined by test or an involved calculation, $\mathcal{R}_i$ and $\mathcal{R}_g$ may be calculated from

$$\mathcal{R} = \frac{\ell}{\mu_r \mu_0 A} \tag{1.36}$$

where ℓ is the length of the flux path, A is the cross sectional area of the flux path, μ_0 is the permeability of free space ($4\pi \times 10^{-7}$ Wb/A m or H/m), and μ_r is the permeability relative to free space. The units of reluctance are (henry^{-1}) or H^{-1}. In the case of the air gap, the relative permeability is considered to be unity ($\mu_{rg} \approx 1$), for the ferromagnetic member typical μ_{ri} values may be as high as 4000 depending upon the type of ferromagnetic material. It follows that the reluctance of the ferromagnetic member is much less than the reluctance of either the reluctance of the air gap or the reluctance of the leakage path since it is partially in air. In fact, the reluctance of the ferromagnetic member is generally neglected when an air gap is present as in the case of an electric machine. The magnetic equivalent circuit shown in Figure 1.2 may be helpful to visualize the flux paths of the magnetic system shown in Figure 1.1.

Before proceeding, there are a couple things that we should talk about. We have arrived at (1.34) and (1.35) by exploiting the similarities between a resistive electric circuit and a linear magnetic circuit. Although this approach is straightforward and easy to follow, let's take a minute to apply basic laws of magnetically linear systems that we learned in early physics courses to justify (1.34) and (1.35). Ampere's law states that the line integral of the field intensity (field strength) H about a closed path is equal to the net current enclosed within the closed path of integration. Now, for a two dimensional magnetically linear system

$$B = \mu H \tag{1.37}$$

where B is the flux density, μ is the permeability, and H is the field strength. If we assume the flux Φ is uniform over the cross-sectional surface area of the magnetic path, then

$$B = \frac{\Phi}{A} \tag{1.38}$$

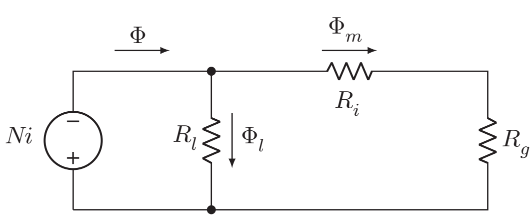


Figure 1.2 Magnetic equivalent circuit for magnetic system shown in Figure 1.1.

where A is the cross-sectional area. Thus, H may be expressed

$$H = \frac{\Phi}{A\mu} \tag{1.39}$$

Now, if H is integrated over the closed path through iron and air gap that encloses the total turns of the coil and assuming H is the same over this path, then the mmf, or Ni is

$$Ni = \int_0^{\ell} \frac{\Phi}{A\mu} d\xi \tag{1.40}$$

and

$$Ni = \frac{\Phi}{A\mu}\ell \tag{1.41}$$

Thus Φ may be expressed

$$\begin{aligned}\Phi &= \frac{Ni}{\frac{\ell}{A\mu_r\mu_0}} \\ &= \frac{Ni}{\mathcal{R}}\end{aligned} \tag{1.42}$$

where we have substituted $\mu_r\mu_0$ for μ. We now understand the difficulty in determining the reluctance to the leakage flux; however, the reluctance of the iron and air gap can be calculated quite accurately for a magnetically linear system.

We now see that (1.34) is obtained by applying Ampere's law about the leakage flux Φ_ℓ path and (1.35) is obtained by applying Ampere's law about the magnetizing flux Φ_m path. In regard to (1.35), since for a magnetically linear system the reluctance of an air gap is much larger than the reluctance of the iron, the mmf drop across the iron is generally neglected and (1.35) could be written

$$\Phi_m = \frac{Ni}{\mathcal{R}_g} \tag{1.43}$$

where the total mmf is dropped across the air gap which is commonly referred to as the air-gap mmf. We, however, will not neglect $\mathcal{R}_i$ in the following development for the sake of completeness.

The total flux linking all the turns of the coil may be expressed from (1.33) as

$$\begin{aligned}\lambda &= N\Phi \\ &= N(\Phi_l + \Phi_m)\end{aligned} \tag{1.44}$$

Substituting (1.34) and (1.35) into (1.44) yields the flux linkage as

$$\lambda = \left(\frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_i + \mathcal{R}_g}\right) i \tag{1.45}$$

It is now time to get back to the electric circuit. Faraday's law tells us that the voltage induced in the coil, e in Figure 1.1, may be expressed

$$e = \frac{d\lambda}{dt} \tag{1.46}$$

Note that the coil resistance has been lumped outside the coil in the circuit shown in Figure 1.1. If we now substitute (1.45) into (1.46) and then substitute the result into (1.32) for e, we obtain

$$v = ri + \frac{d}{dt}\left[\left(\frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_i + \mathcal{R}_g}\right)i\right] \tag{1.47}$$

Since we are dealing exclusively with a magnetically linear system, the reluctances are assumed to be constant; therefore, (1.47) becomes

$$v = ri + L\frac{di}{dt} \tag{1.48}$$

where L is the self-inductance which is

$$\begin{aligned} L &= \frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_i + \mathcal{R}_g} \\ &= L_l + L_m \end{aligned} \tag{1.49}$$

where L_l is the leakage inductance and L_m is the magnetizing inductance. Since inductance (units of H) is turns (no units) squared divided by reluctance, the unit of reluctance is H^{-1}. Moreover, the use of an inductance in the analysis signifies that a magnetically linear system is being assumed.

Let us rewrite (1.45) as

$$\lambda = Li \tag{1.50}$$

Here we see a linear relationship between flux linkages and current. The λ versus i relationship shown in Figure 1.3 is often referred to as the magnetization curve.

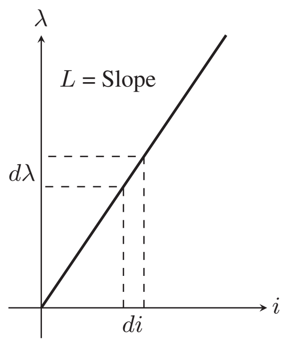


Figure 1.3 λi characteristic of a magnetically linear system.

1.3.1 Field Energy and Coenergy

Since power is the rate of energy transfer, the energy entering the magnetic field from the electrical system (Figure 1.1) may be expressed

$$W_e = \int ei\,dt \tag{1.51}$$

As previously mentioned, the resistance of the coil is lumped external to the coil; therefore, W_e is the energy transferred only to the field of the magnetic system from the electrical system. Thus, the energy stored in the field which is referred to as the field energy is

$$\begin{aligned} W_f &= W_e \\ &= \int i\frac{d\lambda}{dt}dt = \int i\,d\lambda \\ &= \frac{1}{2}Li^2 \end{aligned} \tag{1.52}$$

Therefore, the energy stored in the field (W_f) for a given λ or i is the area to the left of the magnetization curve shown in Figure 1.2. Also, since, for the magnetic system shown in Figure 1.1, $L_l \ll L_m$ and since $\mathcal{R}_i \ll \mathcal{R}_g$, most of the field energy is stored in the air gap.

The area to the right of the magnetization curve is referred to as the coenergy; it is expressed as

$$W_c = \int \lambda\,di \tag{1.53}$$

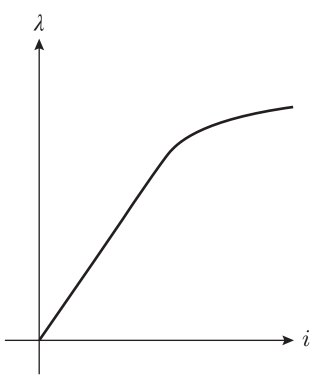
It has no physical meaning; however, it is often used as a convenience in expressing the force and torque, since it is easier to work with λ in terms of i rather than i in terms of λ. Also, we see from Figure 1.3 that

$$\lambda i = W_f + W_c \tag{1.54}$$

Clearly, $W_f = W_c$ for a magnetically linear system.

We will not take saturation into account in our analysis in this text. To do so would greatly complicate our work compared to treating all electromechanical devices as magnetically linear and little is sacrificed in portraying their salient operating characteristics. Nevertheless, magnetically nonlinear systems should be mentioned in passing. In this regard, Figure 1.4 is a plot of the flux linkage λ versus the current i for a magnetically nonlinear system. The actual characteristic is dependent on the type of ferromagnetic material. The knee of the curve occurs due

Figure 1.4 λ versus i for a magnetically nonlinear magnetic system.



to the saturation of the ferromagnetic member. We will assume that the λ versus i characteristic is a single-valued function in that there is only one value of λ for a given value of i, whereupon hysteresis is neglected. When a ferromagnetic material is subjected to a magnetic field, its magnetic characteristics tend to aid in producing flux. This continues at a near-linear rate until this rate begins to decrease causing the knee of the magnetization curve. As the strength of the applied field continues to increase (by increasing i), the rate will continue to decrease and ultimately the magnetic characteristic of the ferromagnetic material approaches that of air. It is clear that after the knee of the magnetization curve the field energy W_f is no longer equal to the coenergy W_c.

Example 1B ***Inductance calculations of a magnetic circuit***
The parameters of the magnetic circuit shown in Figure 1.1 are: $r = 1\ \Omega$, $N = 100$ turns, $v = 10$ V, $\ell_i = 40$ cm, $x = 3$ mm, $A_i = A_g = 40$ cm^2, $\mu_0 = 4\pi \times 10^{-7}$ H/m, $\mu_{ri} = 1000$. Determine (a) the steady-state total flux Φ, assume $\mathcal{R}_l = 6.77 \times 10^7$ H^{-1}; (b) the leakage inductance L_l, the magnetizing inductance L_m, and the self-inductance L; (c) express the energy from the source to the system following a step applied voltage.

(a) It would be helpful to refer to Figure 1.2. Therein,

$$\begin{aligned}\mathcal{R}_i &= \frac{\ell_i}{\mu_{ri}\mu_0 A_i}\\ &= \frac{40 \times 10^{-2}}{(1000)\left(4\pi \times 10^{-7}\right)\left(4 \times 10^{-4}\right)} = 7.95 \times 10^5\ \ \text{H}^{-1}\end{aligned} \tag{1B.1}$$

$$\begin{aligned}\mathcal{R}_g &= \frac{x}{\mu_{rg}\mu_0 A_g}\\ &= \frac{3 \times 10^{-3}}{(1)\left(4\pi \times 10^{-7}\right)\left(4 \times 10^{-4}\right)} = 5.97 \times 10^6\ \ \text{H}^{-1}\end{aligned} \tag{1B.2}$$

$$\begin{aligned}\Phi_m &= \frac{Ni}{\mathcal{R}_i + \mathcal{R}_g}\\ &= \frac{100\left(\frac{10}{1}\right)}{7.95 \times 10^5 + 5.97 \times 10^6} = 1.48 \times 10^{-4}\ \text{Wb}\end{aligned} \tag{1B.3}$$

$$\begin{aligned}\Phi_l &= \frac{Ni}{\mathcal{R}_l}\\ &= \frac{100\left(\frac{10}{1}\right)}{6.77 \times 10^7} = 1.48 \times 10^{-5}\ \text{Wb}\end{aligned} \tag{1B.4}$$

$$\begin{aligned}\Phi &= \Phi_l + \Phi_m\\ &= 1.48 \times 10^{-5} + 1.48 \times 10^{-4} = 1.63 \times 10^{-4}\ \text{Wb}\end{aligned} \tag{1B.5}$$

(b) $$L_l = \frac{N^2}{\mathcal{R}_l} = \frac{(100)^2}{6.77 \times 10^7} = 0.148 \text{ mH} \tag{1B.6}$$

$$L_m = \frac{N^2}{\mathcal{R}_m} = \frac{(100)^2}{(7.96 \times 10^5) + (5.97 \times 10^5)} = 1.48 \text{ mH} \tag{1B.7}$$

$$L = L_l + L_m = (0.148 \times 10^{-3}) + (1.48 \times 10^{-3}) = 1.63 \text{ mH} \tag{1B.8}$$

(c) The expression for the total energy from the source, W_E, is

$$W_E = \int vi\, dt \tag{1B.9}$$

From (1.32), which can now be written

$$v = ri + L\frac{di}{dt} \tag{1B.10}$$

Substituting (1B.10) into (1B.9) yields

$$\begin{aligned} W_E &= \int ri^2 dt + \int L\frac{di}{dt} i\, dt \\ &= r\int i^2 dt + \int L\, i\, di \\ &= r\int i^2 dt + \frac{1}{2}Li^2 \\ &= r\int i^2 dt + \frac{1}{2}(L_l + L_m)i^2 \end{aligned} \tag{1B.11}$$

The first term on the right side of (1B.11) is the energy dissipated in the resistor, while the second term is the energy stored in the inductance (magnetic field).

Since the expression for the current after the step voltage is applied is

$$i = \frac{v}{r}(1 - e^{-t/\tau}) \tag{1B.12}$$

where τ is the time constant L/r, the energy supplied by the source may be expressed

$$\begin{aligned} W_E &= r\int \left[\frac{v}{r}(1 - e^{-t/\tau})\right]^2 dt + \frac{1}{2}Li^2 \\ &= r\left(\frac{v}{r}\right)^2 \int (1 - 2e^{-t/\tau} + e^{-2t/\tau})\, dt + \frac{1}{2}Li^2 \\ &= \frac{v^2}{r}\left[t + 2\tau e^{-t/\tau} - \frac{\tau}{2}e^{-2t/\tau}\right] + \frac{1}{2}L\left(\frac{v}{r}\right)^2(1 - e^{-t/\tau})^2 \end{aligned} \tag{1B.13}$$

For $t \gg \tau$,

$$W_E \cong \frac{v^2}{r}t + \frac{1}{2}L\left(\frac{v}{r}\right)^2 \tag{1B.14}$$

It is interesting to note that the energy stored in the inductance converges to a constant; however, the resistor continues to dissipate energy at a rate of v^2/r.

SP1.3.1 Determine two changes in the magnetic circuit shown in Figure 1.1 that would result in interchanging the position of the poles. [Reverse the direction of positive current or reverse the sense of the winding]

SP1.3.2 Express (a) W_f and (b) W_c if $L = 1$ H and the current is $I = I_p \cos \omega_e t$. [(a) $\frac{1}{2}I_p^2\left(\frac{1}{2} + \frac{1}{2}\cos 2\omega_e t\right)$, (b) same]

SP1.3.3 Re-draw the λ versus i plot in Figure 1.3. Indicate the change in the slope as the length of the air-gap increases. [slope decreases]

SP1.3.4 Consider Figure 1.1. Assume the reluctance of the iron may be neglected. Express the energy stored in the air gap in terms of its air-gap mmf (mmf_{ag}) and its reluctance $\mathcal{R}_g$. [$\frac{(\text{mmf}_{ag})^2}{2\mathcal{R}_g}$]

1.4 Stationary Coupled Circuits – The Transformer

Faraday's law tells us that a voltage is induced in an electric circuit due to a change of flux linkages. In the case of stationary circuits, such as the transformer, a change of flux linkages occurs due to time-varying currents. Therefore, in an ac power system, we can change the level of the voltage and current. That is, power can be generated at convenient voltage and current levels and transmitted at a high voltage with a low current to minimize transmission losses and then lower the voltage to a safe level at the point of usage.

Flux linkages of an electric circuit can also be changed, and thus voltages induced, due to relative motion between magnetic fields. This allows power to be generated at the source where mechanical motion is converted to electric power (generator action). At the load, electric power is often converted back to mechanical motion (motor action). The work of Faraday and later the inventions and development of the ac power system by Tesla and Westinghouse doomed Edison's dc power system in the very early twentieth century. We'll start our analytical journey into the world of Faraday and Tesla with the transformer.

1.4.1 Magnetically Linear Transformer

Two coupled circuits are shown in Figure 1.5. Actually, all we have done is "closed up" the air gap of Figure 1.1 and added a second coil. About all we have to concern ourselves with is how to handle the coupling between coils since in the previous section we became familiar with leakage flux and magnetizing flux. The flux linkages are

$$\lambda_1 = N_1\Phi_1 \tag{1.55}$$

$$\lambda_2 = N_2\Phi_2 \tag{1.56}$$

where Φ_1 and Φ_2 are the total flux linking N_1 and N_2, respectively. From Figure 1.5

$$\begin{aligned}\Phi_1 &= \Phi_{l1} + \Phi_{m1} + \Phi_{m2}\\ &= \frac{N_1 i_1}{\mathcal{R}_{l1}} + \frac{N_1 i_1}{\mathcal{R}_m} + \frac{N_2 i_2}{\mathcal{R}_m}\end{aligned} \tag{1.57}$$

$$\begin{aligned}\Phi_2 &= \Phi_{l2} + \Phi_{m2} + \Phi_{m1}\\ &= \frac{N_2 i_2}{\mathcal{R}_{l2}} + \frac{N_2 i_2}{\mathcal{R}_m} + \frac{N_1 i_1}{\mathcal{R}_m}\end{aligned} \tag{1.58}$$

Mutual coupling occurs since Φ_{m2}, which is the flux created by current flowing in coil N_2, links coil N_1. Likewise, Φ_{m1}, which is the flux created by i_1 current flowing in coil N_1, links coil N_2.

Substituting (1.57) into (1.55) and (1.58) into (1.56) yields

$$\lambda_1 = \frac{N_1^2}{\mathcal{R}_{l1}} i_1 + \frac{N_1^2}{\mathcal{R}_m} i_1 + \frac{N_1 N_2}{\mathcal{R}_m} i_2 \tag{1.59}$$

$$\lambda_2 = \frac{N_2^2}{\mathcal{R}_{l2}} i_2 + \frac{N_2^2}{\mathcal{R}_m} i_2 + \frac{N_2 N_1}{\mathcal{R}_m} i_1 \tag{1.60}$$

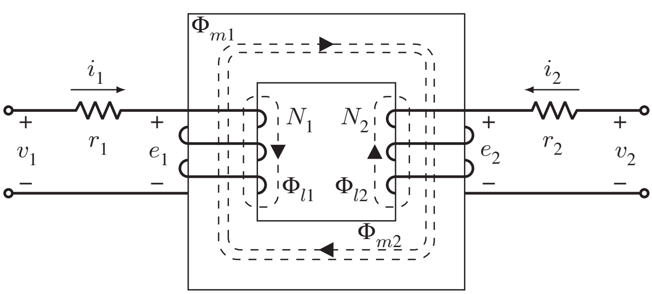


Figure 1.5 Magnetically coupled circuits.

The coefficients of the first two terms of (1.59) and (1.60) are the self-inductances L_{11} and L_{22}, respectively, that is

$$L_{11} = \frac{N_1^2}{\mathcal{R}_{l1}} + \frac{N_1^2}{\mathcal{R}_m} = L_{l1} + L_{m1} \tag{1.61}$$

$$L_{22} = \frac{N_2^2}{\mathcal{R}_{l2}} + \frac{N_2^2}{\mathcal{R}_m} = L_{l2} + L_{m2} \tag{1.62}$$

The coefficient of the last term of (1.59) and (1.60) is the mutual inductance

$$L_{12} = L_{21} = \frac{N_1 N_2}{\mathcal{R}_m} \tag{1.63}$$

We can now write λ_1 and λ_2 as

$$\lambda_1 = L_{11} i_1 + L_{12} i_2 \tag{1.64}$$

$$\lambda_2 = L_{22} i_2 + L_{21} i_1 \tag{1.65}$$

It is important to note that L_{m1}, L_{m2}, L_{12}, and L_{21} have a common term $\mathcal{R}_m$ which is the reluctance of the mutual path around the ferromagnetic core. This fact will allow us to derive a very useful equivalent circuit; however, before getting started on that, let us go back to (1.59) and (1.60). The mutual inductance can be positive or negative in these equations depending upon the relative directions of Φ_{m1} and Φ_{m2}. If the assumed positive directions of the currents or the sense of the winding of the coil causes Φ_{m1} and Φ_{m2} to aid each other, L_{12} (L_{21}) would be positive as in the case here. If they oppose L_{12} (L_{21}) would be negative.

Now, λ_1 may be written

$$\begin{aligned} \lambda_1 &= L_{l1} i_1 + L_{m1} i_1 + L_{12} i_2 \\ &= L_{l1} i_1 + L_{m1} \left(i_1 + \frac{N_2}{N_1} i_2 \right) \end{aligned} \tag{1.66}$$

Here we multiplied L_{12}, given by (1.63), by $\frac{N_1}{N_2}$ whereupon

$$\begin{aligned} \frac{N_1}{N_2} L_{12} &= \left(\frac{N_1}{N_2} \right) \frac{N_1 N_2}{\mathcal{R}_m} \\ &= L_{m1} \end{aligned} \tag{1.67}$$

thus,

$$L_{12} = \frac{N_2}{N_1} L_{m1} \tag{1.68}$$

From (1.66), we see there is a common inductance L_{m1} that carries the currents i_1 and $\frac{N_2}{N_1} i_2$. We have referred $\frac{N_2}{N_1} i_2$ to the winding with N_1 turns; if we would have expressed λ_2 rather than λ_1 we could have referred i_1 to the winding with N_2 turns by $\frac{N_1}{N_2}i_1$. Let's stay with $\frac{N_2}{N_1}i_2$, that is generally written, for compactness, as

$$i_2' = \frac{N_2}{N_1}i_2 \tag{1.69}$$

or

$$N_1 i_2' = N_2 i_2 \tag{1.70}$$

From (1.70), i_2' flowing in N_1 produces the same mmf as i_2 flowing in N_2. To maintain the same power in primed variables

$$v_2' i_2' = v_2 i_2 \tag{1.71}$$

Thus

$$v_2' = \frac{N_1}{N_2}v_2 \tag{1.72}$$

Since λ_2 has units of V s, the same turns ratio used for voltages can be applied to it; thus,

$$\lambda_2' = \frac{N_1}{N_2}\lambda_2 \tag{1.73}$$

There is considerable algebraic manipulation to get to the equivalent circuit. We'll only set forth the steps. First, let us write the voltage equations in terms of 1 and 2 variables.

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt} \tag{1.74}$$

$$v_2 = r_2\, i_2 + \frac{d\lambda_2}{dt} \tag{1.75}$$

where r_1 and r_2 are the resistances of winding 1 and winding 2, respectively. Since we are referring to the two variables to the one winding, (1.74) remains unchanged in form; however, (1.75) becomes

$$v_2' = r_2' i_2' + \frac{d\lambda_2'}{dt} \tag{1.76}$$

The flux linkage equations become

$$\lambda_1 = L_{l1} i_1 + L_{m1}(i_1 + i_2') \tag{1.77}$$

$$\lambda_2' = L_{l2}' i_2' + L_{m1}(i_1 + i_2') \tag{1.78}$$

where

$$L_{l2}' = \left(\frac{N_1}{N_2}\right)^2 L_{l2} \tag{1.79}$$

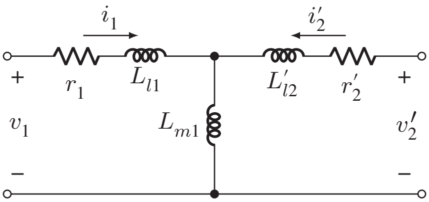


Figure 1.6 Transformer equivalent *T* circuit with winding 1 selected as reference winding.

$$r_2' = \left(\frac{N_1}{N_2}\right)^2 r_2 \tag{1.80}$$

The equivalent circuit is shown in Figure 1.6. This circuit is valid for all modes of operation, transient and steady state; however, it is valid only for a magnetically linear system. Also, we will find that with a slight modification the equivalent circuit shown in Figure 1.6 is valid for the steady-state performance of an induction machine.

1.4.2 Field Energy

Before leaving our work with transformers, let us derive an expression for the energy stored in the field W_f. In the case of a single coil, the field energy was $\frac{1}{2}Li^2$. Here, we have two coils with mutual coupling between them. In this case, the energy from the electrical system is

$$W_e = \int e_1 i_1 dt + e_2 i_2 dt \tag{1.81}$$

Since r_1 and r_2 are external to the respective coils and using unreferred variables

$$e_1 = \frac{d\lambda_1}{dt} \tag{1.82}$$

$$e_2 = \frac{d\lambda_2}{dt} \tag{1.83}$$

Substituting (1.82) and (1.83) into (1.81) yields

$$W_e = \int i_1 d\lambda_1 + i_2 d\lambda_2 \tag{1.84}$$

From (1.64) and (1.65)

$$d\lambda_1 = L_{11} di_1 + L_{12} di_2 \tag{1.85}$$

$$d\lambda_2 = L_{22} di_2 + L_{12} di_1 \tag{1.86}$$

Substituting (1.85) and (1.86) into (1.84) and since $W_f = W_e$, we have

$$W_f = \int [i_1(L_{11}di_1 + L_{12}di_2) + i_2(L_{22}di_2 + L_{12}di_1)] \tag{1.87}$$

For a magnetically linear system these integrals can be calculated in two steps; first let the current in winding 1 be the variable of integration and vary it from zero to i_1 while holding i_2 and di_2 at zero. Next hold the current in winding 1 at i_1 ($di_1 = 0$) and let the current in winding 2 be the variable of integration varying from zero to i_2. Performing the first step, we have

$$W_{f(1)} = \int_0^{i_1} L_{11}\xi d\xi = \frac{1}{2}L_{11}i_1^2 \tag{1.88}$$

For the second step

$$\begin{aligned} W_{f(2)} &= \int_0^{i_2} L_{12}i_1 d\xi + L_{22}\xi d\xi \\ &= L_{12}i_1 i_2 + \frac{1}{2}L_{22}i_2^2 \end{aligned} \tag{1.89}$$

Thus

$$W_f = \frac{1}{2}L_{11}i_1^2 + L_{12}i_1 i_2 + \frac{1}{2}L_{22}i_2^2 \tag{1.90}$$

The energy stored in the coupling field is W_f given by (1.90) less the energy stored in the leakage inductances. In a problem at the end of the chapter, you are asked to express (1.90) in terms of i_1 and i_2' and to identify the energy stored in the coupling field.

The pattern is clear, the field energy may be expressed for multiple electrical inputs (coils) as $\frac{1}{2}$ the self-inductance of each coil times the square of the current flowing in the coil and the mutual inductance between each set of coils times the product of the current flowing in the mutually coupled coils. For three coupled coils

$$W_f = \frac{1}{2}L_{11}i_1^2 + \frac{1}{2}L_{22}i_2^2 + \frac{1}{2}L_{33}i_3^2 + L_{12}i_1 i_2 + L_{13}i_1 i_3 + L_{23}i_2 i_3 \tag{1.91}$$

Now, for multiple electrical inputs (windings), (1.54) becomes

$$\sum_{j=1}^{J} \lambda_j i_j = W_f + W_c \tag{1.92}$$

where J is the number of windings. Thus,

$$W_c = \sum_{j=1}^{J} \lambda_j i_j - W_f \tag{1.93}$$

In SP1.4.3, you are asked to prove that W_c is W_f using (1.91) and (1.93).

Example 1C ***Parameters of the transformer equivalent circuit***

It is instructive to illustrate the method of deriving an equivalent T circuit from open- and short-circuit measurements of the transformer. When winding 2 of the two-winding transformer shown in Figure 1.6 is open circuited and a voltage of 110 V (rms) at 60 Hz is applied to winding 1, the average power supplied to winding 1 is 6.66 W. The measured current in winding 1 is 1.05 A (rms). Next, with winding 2 short-circuited, the current flowing in winding 1 is 2 A when the applied voltage is 30 V at 60 Hz. The average input power is 44 W. If we assume $L_{l1} = L'_{l2}$, an approximate equivalent T circuit can be determined from these measurements with winding 1 selected as the reference winding.

The average power supplied to winding 1 may be expressed from (1.31) as

$$P_1 = |\widetilde{V}_1|\,|\widetilde{I}_1| \cos \varphi_{pf} \tag{1C.1}$$

where

$$\varphi_{pf} = \theta_{ev}(0) - \theta_{ei}(0) \tag{1C.2}$$

Here, $\widetilde{V}_1$ and $\widetilde{I}_1$ are phasors with the positive direction of $\widetilde{I}_1$ taken in the direction of voltage drop, and $\theta_{ev}(0)$ and $\theta_{ei}(0)$ are the phase angles of $\widetilde{V}_1$ and $\widetilde{I}_1$, respectively. Solving for φ_{pf} during the open-circuit test, we have

$$\varphi_{pf} = \cos^{-1} \frac{P_1}{|\widetilde{V}_1|\,|\widetilde{I}_1|} = \cos^{-1} \frac{6.66}{(110)(1.05)} = 86.7° \tag{1C.3}$$

Although $\varphi_{pf} = -86.7°$ is also a legitimate solution of (1C.3), the positive value is taken since $\widetilde{V}_1$ leads $\widetilde{I}_1$ in an inductive circuit. With winding 2 open-circuited, the input impedance of winding 1 is

$$Z = \frac{\widetilde{V}_1}{\widetilde{I}_1} = r_1 + j(X_{l1} + X_{m1}) \tag{1C.4}$$

With $\widetilde{V}_1$ as the reference phasor, $\widetilde{V}_1 = 110\underline{/0°}$, $\widetilde{I}_1 = 1.05\underline{/-86.7°}$. Thus,

$$r_1 + j(X_{l1} + X_{m1}) = \frac{110\underline{/0°}}{1.05\underline{/-86.7°}} = 6 + j104.6\,\Omega \tag{1C.5}$$

From (1C.5), $r_1 = 6\ \Omega$. We also see from (1C.5) that $X_{l1} + X_{m1} = 104.6\ \Omega$.

For the short-circuit test, we will assume that $\widetilde{I}_1 = -\widetilde{I}'_2$ since transformers are designed so that at rated frequency $X_{m1} \gg |r'_2 + jX'_{l2}|$. Hence, using (1C.1) again,

$$\varphi_{pf} = \cos^{-1} \frac{44}{(30)(2)} = 42.8° \tag{1C.6}$$

In this case, the input impedance is $Z = (r_1 + r'_2) + j(X_{l1} + X'_{l2})$. This may be determined as

$$Z = \frac{30\underline{/0°}}{2\underline{/-42.8°}} = 11 + j10.2\,\Omega \tag{1C.7}$$

Hence, $r'_2 = 11 - r_1 = 5\,\Omega$ and, since it is assumed that $X_{l1} = X'_{l2}$, both are $10.2/2 = 5.1\ \Omega$. Therefore, $X_{m1} = 104.6 - 5.1 = 99.5\ \Omega$. In summary, $r_1 = 6\ \Omega$,

$L_{l1} = 13.5$ mH, $L_{m1} = 263.9$ mH, $r'_2 = 5\,\Omega$, $L'_{l2} = 13.5$ mH. It is left to the reader to verify the conversion from X's to L's.

SP1.4.1 Consider the transformer and parameters calculated in Example 1C. Winding 2 is short-circuited and 12 V (dc) is applied to winding 1. Calculate the steady-state values of i_1 and i_2. Repeat with winding 2 open-circuited. [$I_1 = 2$ A and $I_2 = 0$ in both cases]

SP1.4.2 Calculate the no-load current (winding 2 open-circuited) for the transformer given in Example 1C if $V_1 = \sqrt{2}10\cos 100t$. [$\tilde{I}_1 = 0.352\underline{/-77.8^\circ}$ A]

SP1.4.3 Determine W_c from (1.91) and (1.93). Show that $W_c = W_f$.

SP1.4.4 Use the equivalent T circuit given in Figure 1.6 to identify the energy stored in the coupling field. [$\frac{1}{2}L_{m1}(i_1^2 + i'^2_2) + L_{m1}i_1i'_2$]

1.5 Two- and Three-phase Systems

High-voltage transmission, most inverter-supplied electric drives, and the alternator of your car are examples of three-phase systems. Although two-phase systems are not common, a two-phase system is far less involved when it comes to machine analysis than its three-phase counterpart. Fortunately, once the derivations have been set forth for a two-phase machine, the extension to a three-phase machine is straightforward and easily achieved. This section is devoted to the introduction of two- and three-phase systems.

By definition, a two-phase set of variables is balanced if the variables are equal-amplitude sinusoidal quantities in time quadrature (90° out of time phase). A three-phase set of variables is balanced if the sinusoidal variables are equal-amplitude quantities that are 120° out of time phase with each other.

1.5.1 Two-phase Systems

In the broadest sense of the above definition, two-phase balanced sets may be expressed as

$$f_a(t) = \pm f(t)\cos\theta_{ef} \tag{1.94}$$

$$f_b(t) = \pm f(t)\sin\theta_{ef} \tag{1.95}$$

where

$$\theta_{ef}(t) = \int_0^t \omega_e(\xi)d\xi + \theta_{ef}(0) \tag{1.96}$$

In (1.94) and (1.95), $f(t)$ can represent voltage, current, or flux linkage. The amplitude may be any function of time; however, for steady-state balanced conditions $f(t)$ is a constant. In (1.96), ω_e is the electrical angular velocity and ξ is a dummy variable of integration. Equations (1.94) and (1.95) express four balanced two-phase sets. Like signs of (1.94) and (1.95) define balanced sets where $f_a(t)$ leads $f_b(t)$ by 90°: an *ab* sequence; for unlike signs $f_a(t)$ lags $f_b(t)$ by 90°: a *ba* sequence.

For steady-state balanced conditions ω_e is constant and (1.96) becomes

$$\theta_{ef}(t) = \omega_e t + \theta_{ef}(0) \tag{1.97}$$

Whereupon (1.94) and (1.95) are written as

$$F_a(t) = \pm\sqrt{2}F\cos[\omega_e t + \theta_{ef}(0)] \tag{1.98}$$

$$F_b(t) = \pm\sqrt{2}F\sin[\omega_e t + \theta_{ef}(0)] \tag{1.99}$$

For like signs of (1.98) and (1.99) $\tilde{F}_a = j\tilde{F}_b$; for unlike signs $\tilde{F}_a = -j\tilde{F}_b$.

Most large horsepower electric machines are three phase and smaller household machines are single phase. Therefore, it can be argued that there is no need to treat two-phase devices. This is an understandable position; however, we will find that it is convenient to first analyze a two-phase device and then extend this to a three-phase device. This turns out to be much less involved and more instructive than to start the analysis with the three-phase device. Although this is perhaps not evident at this juncture, it will become very evident as we get into the later chapters.

It is helpful to take a brief look at the stator winding arrangement of the two-phase machine, shown in Figure 1.7. The windings are assumed to be identical in parameters and distribution. The displacement around the stator is denoted ϕ_s. The two windings are displaced 90° from each other. The voltage equations may be written (dropping the functional notation)

$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt} \tag{1.100}$$

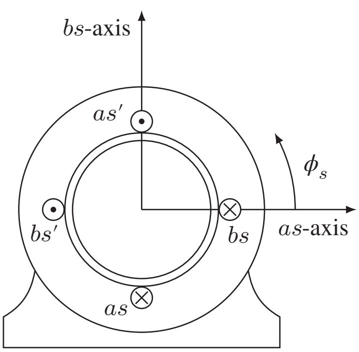


Figure 1.7 Elementary two-pole two-phase concentrated stator windings.

$$v_{bs} = r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \tag{1.101}$$

In matrix form

$$\mathbf{v}_{abs} = \mathbf{r}_s \mathbf{i}_{abs} + p\boldsymbol{\lambda}_{abs} \tag{1.102}$$

The stator windings are identical, and the air gap is uniform. The magnetic axes are orthogonal, (thus $L_{asbs} = 0$) and the flux linkage equations may be written

$$\begin{aligned}\lambda_{as} &= L_{asas} i_{as} \\ &= (L_{ls} + L_{ms}) i_{as}\end{aligned} \tag{1.103}$$

$$\begin{aligned}\lambda_{bs} &= L_{bsbs} i_{bs} \\ &= (L_{ls} + L_{ms}) i_{bs}\end{aligned} \tag{1.104}$$

where L_{ls} is the leakage inductance and L_{ms} is the magnetizing inductance of the stator windings. In matrix form

$$\begin{bmatrix}\lambda_{as} \\ \lambda_{bs}\end{bmatrix} = \begin{bmatrix}L_{ss} & 0 \\ 0 & L_{ss}\end{bmatrix} \begin{bmatrix}i_{as} \\ i_{bs}\end{bmatrix} \tag{1.105}$$

or

$$\boldsymbol{\lambda}_{abs} = \mathbf{L}_s \mathbf{i}_{abs} \tag{1.106}$$

where

$$L_s = \begin{bmatrix}L_{ss} & 0 \\ 0 & L_{ss}\end{bmatrix} \tag{1.107}$$

and

$$L_{ss} = L_{ls} + L_{ms} \tag{1.108}$$

An important feature of multiphase systems is that the instantaneous power is constant for balanced operation. You are asked to show this in SP1.5.2. Recall that in a single-phase system the instantaneous power has an average value and a double frequency component.

1.5.2 Three-phase Systems

A three-phase balanced set may be expressed as

$$f_a(t) = f(t) \cos\theta_{ef} \tag{1.109}$$

$$f_b(t) = f(t) \cos\left(\theta_{ef} - \frac{2}{3}\pi\right) \tag{1.110}$$

$$f_c(t) = f(t) \cos\left(\theta_{ef} + \frac{2}{3}\pi\right) \tag{1.111}$$

This set is referred to as an *abc* sequence, since $f_a(t)$ leads $f_b(t)$ by 120° and $f_b(t)$ leads $f_c(t)$ by 120°. An *acb* sequence is

$$f_a(t) = f(t)\cos\theta_{ef} \tag{1.112}$$

$$f_b(t) = f(t)\cos\left(\theta_{ef} + \frac{2}{3}\pi\right) \tag{1.113}$$

$$f_c(t) = f(t)\cos\left(\theta_{ef} - \frac{2}{3}\pi\right) \tag{1.114}$$

For steady-state balanced conditions, the *abc* sequence may be written

$$F_a(t) = \sqrt{2}F\cos\left[\omega_e t + \theta_{ef}(0)\right] \tag{1.115}$$

$$F_b(t) = \sqrt{2}F\cos\left[\omega_e t - \frac{2}{3}\pi + \theta_{ef}(0)\right] \tag{1.116}$$

$$F_c(t) = \sqrt{2}F\cos\left[\omega_e t + \frac{2}{3}\pi + \theta_{ef}(0)\right] \tag{1.117}$$

With $\tilde{F}_a = F\underline{/\theta_{ef}(0)}$, $\tilde{F}_b = F\underline{/\theta_{ef}(0) - \frac{2}{3}\pi}$, and $\tilde{F}_c = F\underline{/\theta_{ef}(0) + \frac{2}{3}\pi}$. For an *acb* sequence $\tilde{F}_b$ and $\tilde{F}_c$ are interchanged.

The windings of a three-phase stator are shown in Figure 1.8. The magnetic axes of the windings are displaced 120° and the windings are often "wye-connected"

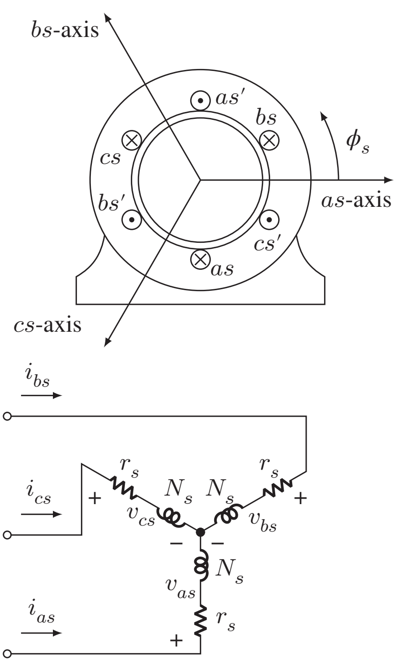


Figure 1.8 Elementary two-pole three-phase concentrated stator windings.

as shown. We'll talk more about three-phase connections later. The voltage equations may be written (again dropping the functional notation)

$$v_{as} = r_s i_{as} + \frac{d\lambda_{as}}{dt} \tag{1.118}$$

$$v_{bs} = r_s i_{bs} + \frac{d\lambda_{bs}}{dt} \tag{1.119}$$

$$v_{cs} = r_s i_{cs} + \frac{d\lambda_{cs}}{dt} \tag{1.120}$$

In matrix form

$$\mathbf{v}_{abcs} = \mathbf{r}_s \mathbf{i}_{abcs} + p\boldsymbol{\lambda}_{abcs} \tag{1.121}$$

Since the windings are displaced 120° from each other, there is a mutual coupling between the stator windings. Let us assume that we can move the *bs* winding clockwise through the iron until it is "on top" of the *as* winding at $\phi_s = 0$. The coupling would be maximum positive. Now assume we can rotate the *bs* winding counterclockwise back to $\phi_s = 120°$ where the mutual inductance between the *as* and *bs* windings can be approximated as

$$\begin{aligned} L_{asbs} &= L_{ms} \cos 120° \\ &= -\frac{1}{2} L_{ms} \end{aligned} \tag{1.122}$$

where L_{ms} is the magnetizing inductance of the stator windings. Following this same approach, we can express the flux-linkage matrix as

$$\begin{bmatrix} \lambda_{as} \\ \lambda_{bs} \\ \lambda_{cs} \end{bmatrix} = \begin{bmatrix} L_{ss} & -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & L_{ss} & -\frac{1}{2}L_{ms} \\ -\frac{1}{2}L_{ms} & -\frac{1}{2}L_{ms} & L_{ss} \end{bmatrix} \begin{bmatrix} i_{as} \\ i_{bs} \\ i_{cs} \end{bmatrix} \tag{1.123}$$

where

$$L_{ss} = L_{ls} + L_{ms} \tag{1.124}$$

Equation (1.123) may also be written as

$$\boldsymbol{\lambda}_{abcs} = \mathbf{L}_s \mathbf{i}_{abcs} \tag{1.125}$$

Example 1D ***Voltage equations for a three-wire system***

A three-phase stator similar to that given in Figure 1.8 is connected to a three-phase source as shown in Figure 1D.1. Assume the stator is symmetrical, that is, the windings have the same resistance and same number of turns and displaced 120°. The stator could be that of induction, synchronous, or brushless dc machines. The source voltages e_{ga}, e_{gb}, and e_{gc} may be of any form. Express v_{as}, v_{bs}, and v_{cs} in terms of e_{ga}, e_{gb}, and e_{gc}.

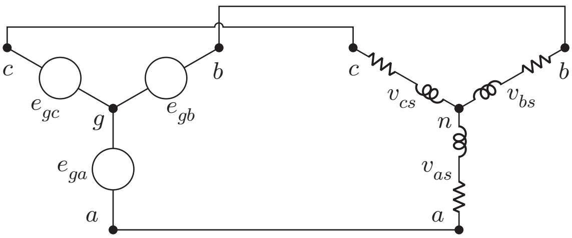


Figure 1D.1 Three-phase source connected to symmetrical stator windings.

From Figure 1D.1 we can write

$$e_{ga} = v_{as} + v_{ng} \tag{1D.1}$$

$$e_{gb} = v_{bs} + v_{ng} \tag{1D.2}$$

$$e_{gc} = v_{cs} + v_{ng} \tag{1D.3}$$

Adding (1D.1)–(1D.3) yields

$$e_{ga} + e_{gb} + e_{gc} = v_{as} + v_{bs} + v_{cs} + 3v_{ng} \tag{1D.4}$$

Let us look at $v_{as} + v_{bs} + v_{cs}$. From (1.118)–(1.120)

$$v_{as} + v_{bs} + v_{cs} = r_s(i_{as} + i_{bs} + i_{cs}) + p(\lambda_{as} + \lambda_{bs} + \lambda_{cs}) \tag{1D.5}$$

In a three-wire, wye-connected stator the sum of $i_{as} + i_{bs} + i_{cs}$ must be zero regardless of the form of the currents. Now from (1.123)

$$\lambda_{as} + \lambda_{bs} + \lambda_{cs} = L_{ss}(i_{as} + i_{bs} + i_{cs}) - L_{ms}(i_{as} + i_{bs} + i_{cs}) = 0 \tag{1D.6}$$

Thus,

$$v_{as} + v_{bs} + v_{cs} = 0 \tag{1D.7}$$

The question is, will this be the case when we bring the rotor into play? We will find that for the electromechanical devices we'll consider, it will be true. Substituting (1D.7) into (1D.4) yields

$$v_{ng} = \frac{1}{3}(e_{ga} + e_{gb} + e_{gc}) \tag{1D.8}$$

Going back to (1D.1)–(1D.3) we can write

$$\begin{aligned} v_{as} &= e_{ga} - v_{ng} \\ &= \frac{2}{3}e_{ga} - \frac{1}{3}(e_{gb} + e_{gc}) \end{aligned} \tag{1D.9}$$

$$\begin{aligned} v_{bs} &= e_{gb} - v_{ng} \\ &= \frac{2}{3}e_{gb} - \frac{1}{3}(e_{gc} + e_{ga}) \end{aligned} \tag{1D.10}$$

$$v_{cs} = e_{gc} - v_{ng}$$
$$= \frac{2}{3}e_{gc} - \frac{1}{3}(e_{ga} + e_{gb}) \quad (1D.11)$$

We will make use of these equations when considering electric drives.

SP1.5.1 In Figure 1D.1, let $e_{ga} = 1$, $e_{gb} = 0$, $e_{gc} = \cos\omega_e t$. Determine v_{as}, v_{bs}, and v_{cs}. [$\frac{2}{3} - \frac{1}{3}\cos\omega_e t$; $-\frac{1}{3} - \frac{1}{3}\cos\omega_e t$; $-\frac{1}{3} + \frac{2}{3}\cos\omega_e t$]. Note that $v_{as} + v_{bs} + v_{cs} = 0$.

SP1.5.2 In a two-phase system, let $V_a = \sqrt{2}V\cos[\omega_e t + \theta_{ev}(0)]$, $I_a = \sqrt{2}I\cos[\omega_e t + \theta_{ei}(0)]$, $V_b = \sqrt{2}V\sin[\omega_e t + \theta_{ev}(0)]$, and $I_b = \sqrt{2}I\sin[\omega_e t + \theta_{ei}(0)]$. Show that the total instantaneous power is $P = 2VI\cos[\theta_{ev}(0) - \theta_{ei}(0)]$.

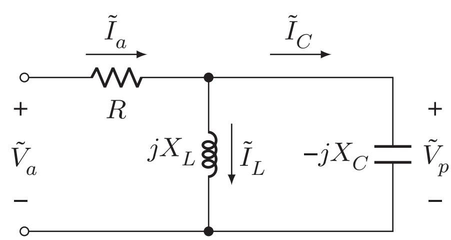
1.6 Problems

1.1 Using the same circuit parameters as given in Example 1A; $R = 6\ \Omega$, $L = 20$ mH, and $C = 1 \times 10^3\ \mu$F, consider the circuit shown in Figure 1.9 Let the 60-Hz source voltage be $\tilde{V}_a = 110\angle 0°$. Determine all phasors and draw the phasor diagram.

1.2 For the magnetic system in Example 1B, let $V(t) = \sqrt{2}\ 10\cos 377t$. Establish $I(t)$ and express $W_f(t)$.

1.3 During the short-circuit test of the transformer in Example 1C, which is a step-down distribution transformer, it was noticed that the I_2 was 6 A. In redoing the open-circuit test the open-circuit current $|\tilde{I}_1|$ was not 1.05 A but 0.6 A. Upon checking the wiring it was discovered that the lower terminal of N_1 was mistakenly connected to the top terminal of N_2 (see Figure 1.5) and the 110-V source was connected between the top terminal of N_1 and the

Figure 1.9 Series-parallel circuit.



bottom terminal of N_2. Justify that $|\widetilde{I}_1| = 0.6$ A. The equivalent T circuit will not yield the correct answer.

1.4 During the open-circuit test performed in Example 1C the rms voltage across the open-circuit 2 winding was 34.8 V. Determine X_{m2} ($\omega_e L_{m2}$).

1.5 Express the field energy W_f stored in the fields of the coils shown in Figure 1.5 in terms of L_{l1}, L'_{l2}, L_{m1}, i_1, and i'_2. Identify the energy stored in the coupling field.

1.6 Consider Example 1D and Figure 1D.1. The load is symmetrical and the source voltages are given in Figure 1.10. (a) Plot v_{ng}, v_{as}, v_{bs}, and v_{cs}. (b) Connect n to g in Figure 1D.1 and repeat part (a).

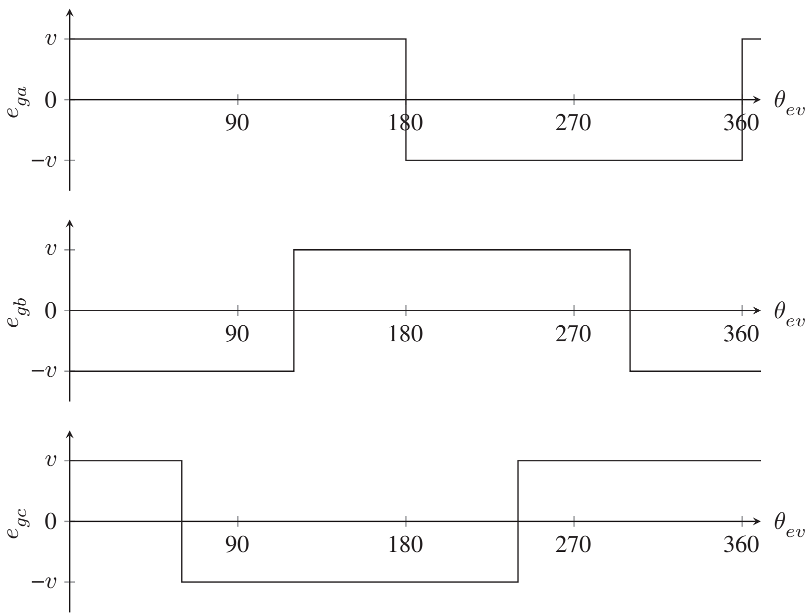


Figure 1.10 Waveforms of the source voltages of Figure 1D.1.