1

Introduction

In this chapter we simply want to present some examples of physical control systems. The intent here is to show why feedback control is necessary and to give the "big picture" of what must be done to control physical systems.

1.1 Aircraft

Figure 1.1 is a drawing of a simple propeller powered airplane. There are four forces on the airplane: *lift* which is primarily provided by the wings, *drag* which is basically wind resistance, *thrust* which is provided by the propeller, and *gravity*.

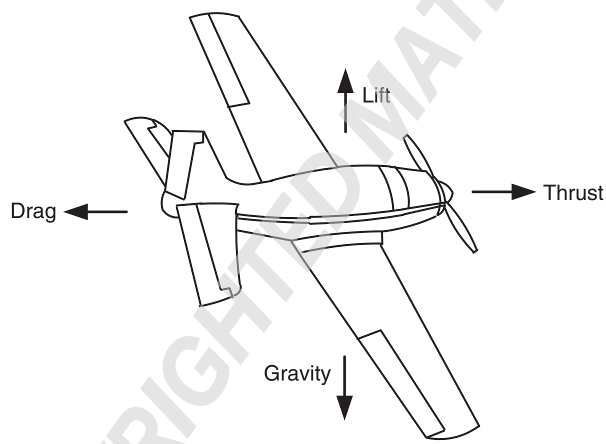

Figure 1.1. The four forces on an aircraft.

Lift is due to the difference in air pressure between the top and bottom of an airfoil. The airfoil is a generic term that here refers to the wings, horizontal tail or the vertical tail of the aircraft. Figure 1.2 on the next page shows streamlines of air moving past an airfoil (wing). The air going over the top travels faster (as it goes a greater distance) than the air on the bottom of the wing. Because of this speed difference, it turns out that the air pressure on the top of the wing is then less than the air pressure on the bottom of the wing. The resulting upward force is what we call *lift*. We call lift an *aerodynamic* force. Figure 1.3 on the next page shows the air flow streamlines across an airfoil in a wind tunnel.

The horizontal cylindrical glass tube (Venturi tube) in Figure 1.4 on the next page is used to experimentally demonstrate that air pressure decreases as the air speed increases. The original flow of air from the right in the Venturi tube is forced to flow through a bottleneck into a reduced diameter tube. Although not obvious, at speeds below supersonic, air is

An Introduction to System Modeling and Control, First Edition. John Chiasson.

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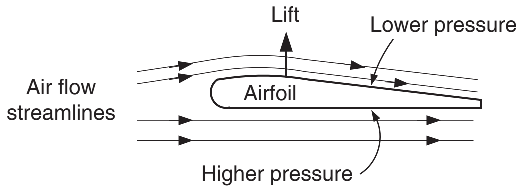

Figure 1.2. A lift force is due to the shape of the airfoil that results in the pressure above the airfoil being less than the pressure below it.

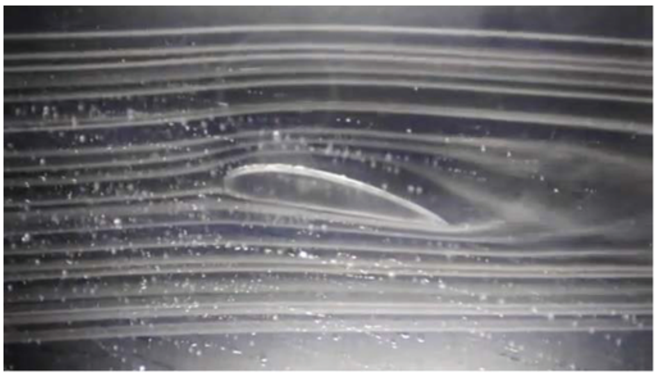

Figure 1.3. Airflow streamlines on a wing in a wind tunnel. Source: Screenshot from http://www.decodedscience.com/how-does-an-airplane-fly-lift-weight-thrust-and-drag-in-action/5200.

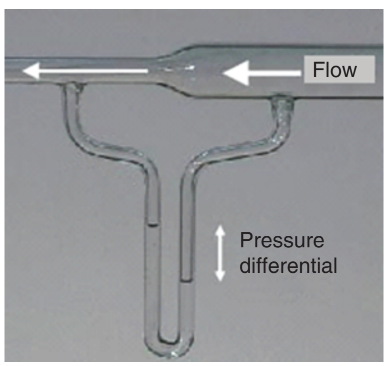

Figure 1.4. Venturi tube showing air flow through a bottleneck reducing the cross-section of the tube. In this situation the air is essentially incompressible. So the air speeds up going through the bottleneck as the mass flow is constant. The ∪-shaped tube below the Venturi tube (called a manometer) is filled with water to show the pressure on the left-side is reduced compared to the pressure on the right-side. Source: ComputerGeezer and Geof [8]. “Venturiflow”, https://commons.wikimedia.org/wiki/File:Venturi-Flow.png, 2010, Licensed under CC BY-SA 3.0.

essentially incompressible so here its density is the same on both sides of the bottleneck.[1] The bottleneck results in increasing the air speed as it goes through it. That is, the same mass of air per unit time is flowing in both tubes (because the air does not compress) so it must speed up on the left side as the tube cross section is smaller there. The ∪-shaped columns below the Venturi tube are filled with water (called a manometer) and show the experimental fact that the air pressure on the left side is *less* than the air pressure on the right side. As air speeds up its pressure drops! This is referred to as Bernoulli's principle. The key point for airfoils is that the shape of the wing causes the air speed on the top of the wing to increase resulting in lower pressure on the top of the wing and therefore producing lift.

The left side of Figure 1.5 shows the centerline of the wing aligned with the air speed v. The air speed is simply the speed of the aircraft with respect to the air. The right side of Figure 1.2 shows the centerline of the wing at an angle α with respect to the air speed. As long as α is not too large (8 to 20 degrees depending on the plane), the lift force increases with α. We refer to α as the *angle of attack*.

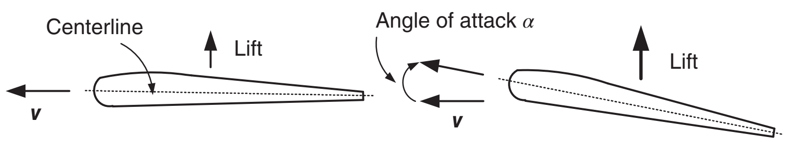

Figure 1.5. Angle of attack.

Let's go back to the airplane as shown in Figure 1.6. The *wings*, *horizontal tail*, and *vertical tail* are generically referred to as *airfoils*. The *ailerons* on the wings, the *elevators* on the horizontal tail, and the *rudder* on the vertical tail are referred to as *control surfaces*.

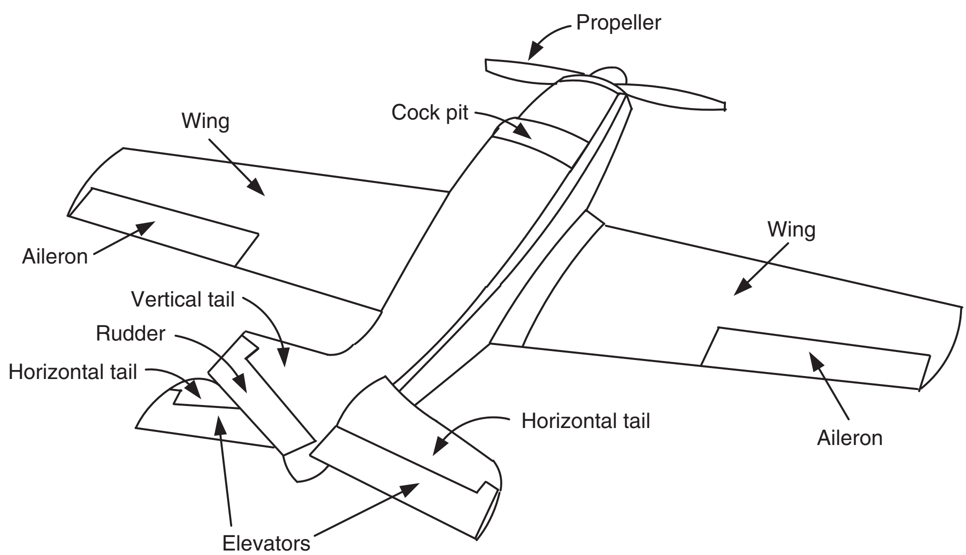

Figure 1.6. The airfoils and the control surfaces on the aircraft.

[1]Another way to say this is that the air is *not* compressed going through the bottleneck.

As indicated in Figure 1.7, the control surface is connected by a hinge to the airfoil. The pilot can rotate the control surface about the hinge. As shown in Figure 1.7, if the control surface is rotated down (relative to the centerline) then the airfoil will pitch down due to the (aerodynamic) force on the control surface. Conversely, if the control surface is rotated up, it will cause the airfoil to pitch up.

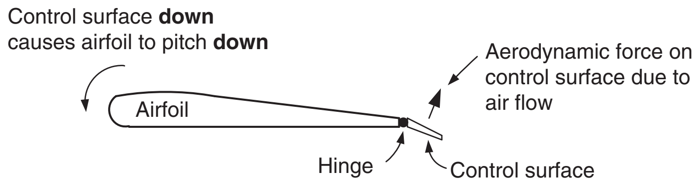

Figure 1.7. Control surface used to pitch the airfoil up or down. With the control surface down, the airfoil pitches down.

Control of Pitch Using the Elevators

As a first example of how control surfaces are used, we consider the elevators on the horizontal tail. Figure 1.8 shows the elevator (control surface) on the horizontal wing rotated up, which results in the aircraft being pitched up about its center of mass. By moving the elevator up or down, the pilot can cause the aircraft to pitch up or down, respectively.

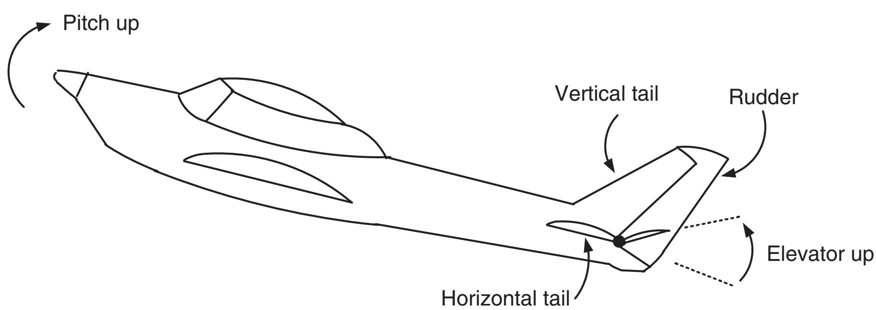

Figure 1.8. Using the elevators to pitch the aircraft up.

Control of Roll Using the Ailerons

Figure 1.9 shows the aircraft with its left aileron down and its right aileron is up. The aerodynamic forces on the two ailerons then cause the airplane to roll to the pilot's right. By adjusting the angle of the ailerons, the pilot can roll right or left.

Control of Yaw Using the Rudder

The final control surface is the rudder on the vertical tail. As indicated in Figure 1.10, if the pilot rotates the rudder to the right, then the plane will yaw (rotate) to the right and vice versa.

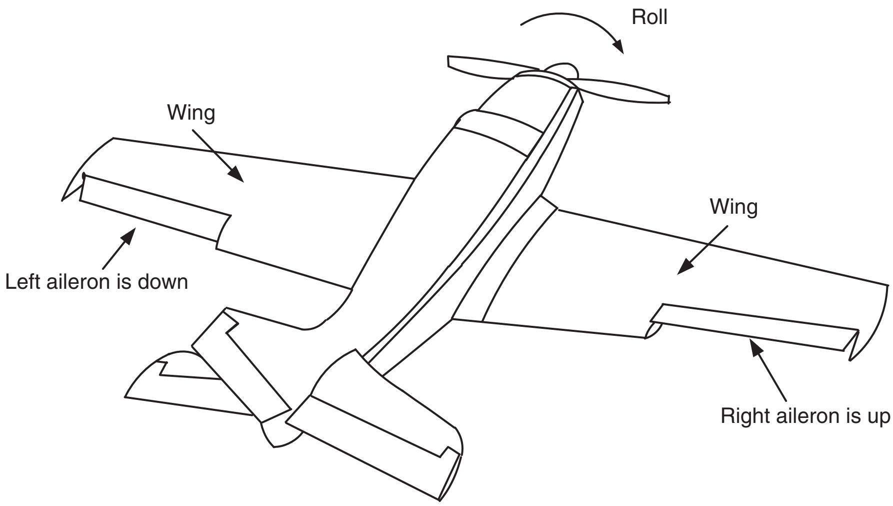

Figure 1.9. Using the ailerons to roll the aircraft.

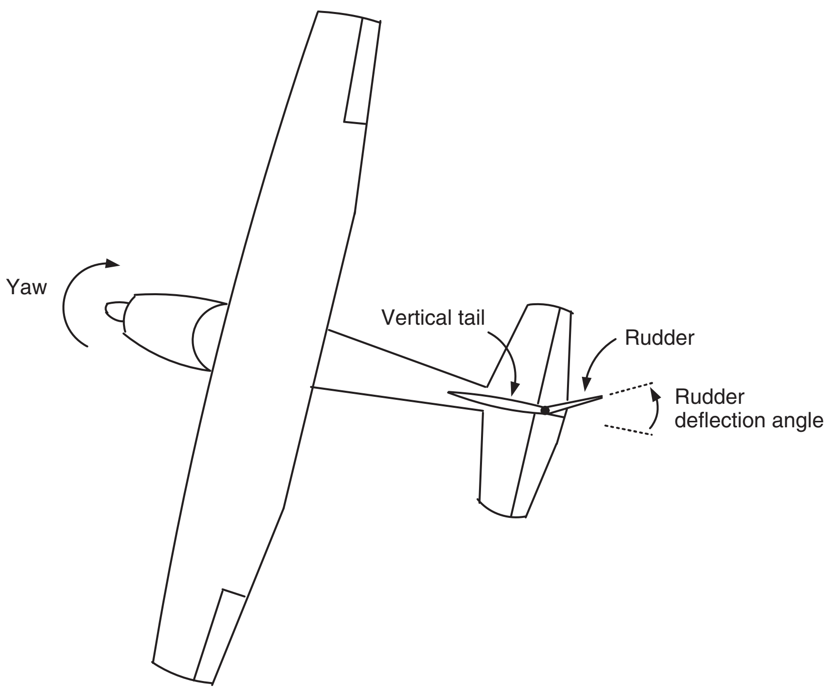

Figure 1.10. Using the rudder to change the heading of the aircraft.

Controlling the Airplane

The pilot has the four input controls:

(1) The elevator to control pitch.

(2) The ailerons to control roll.

(3) The rudder to control yaw.

(4) An engine connected to a propeller to control forward thrust.

Figure 1.11 depicts how a pilot in a small airplane controls the elevator, ailerons, and rudder. In Figure 1.11 the pilot's hands are shown on the *yoke* or *center stick* (looks like a bit like a car's steering wheel). If the pilot pulls the yoke toward himself, the plane will pitch up while alternatively pushing the yoke away from himself the plane will pitch down. See the YouTube animation at [9].

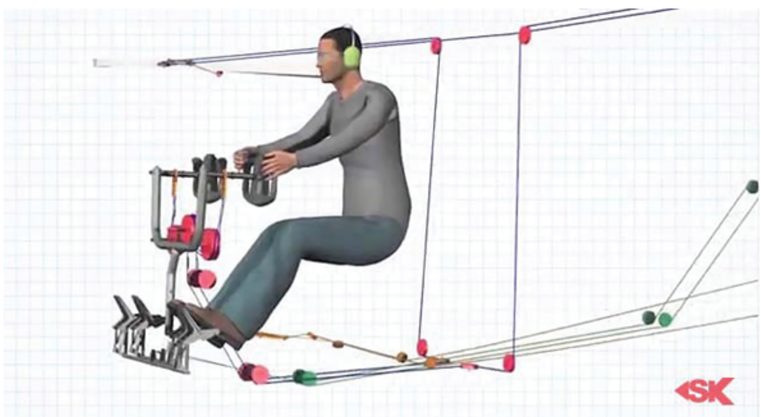

Figure 1.11. Pilot using a yoke, pedals, and throttle to control the aircraft. Source: Courtesy of Steve Karp [9]. "How It Works Flight Controls", October 12, 2013, https://www.youtube.com/watch?v=AiTk5r-4coc.

If the pilot turns the yoke to the right, the plane will roll to the right and conversely if the pilot turns the yoke to the left the plane will rotate to the left.

Figure 1.11 shows the pilot with each foot on a pedal. Pushing on the right pedal the plane will yaw to the right while pushing on the left pedal the plane will yaw to the left.

Not shown in Figure 1.11 is a throttle[2] that the pilot can use to control the engine speed and therefore the propeller thrust.

The angular position of the airplane as specified by its pitch, roll, and yaw is referred to as the *attitude* of the aircraft. The pilot uses the thrust input (propeller) to maintain air flow over the wings. On take off, the propeller must accelerate the airplane to a speed such that the flow of the air across the wings is fast enough to produce the necessary lift. The pilot will then adjust the pitch angle (using the elevators) to obtain an angle of attack that provides the necessary lift to get to the desired altitude. During the climb the pilot may bank (roll) and yaw using the ailerons and rudder, respectively to head off in

[2] A throttle controls the amount of fuel to the engine.

the direction of the destination. At a cruising speed at some fixed altitude the propeller is primarily producing enough thrust to cancel out the drag force so the plane can maintain air speed (and therefore the lift force). The first aircraft required a lot of physical effort on the part of the pilot to control them using mechanical linkages to move the control surfaces. However, using these inputs the pilot can make the airplane take off, cruise, and land safely even though there are cross winds, air gusts, etc. disrupting the plane's motion.

Automatic Control (Autopilot)

The basic idea of *automatic control* is to replace the human pilot with an autopilot. The autopilot is simply a computer with inputs to it being the air speed, the aircraft acceleration, and the aircraft attitude (roll, pitch, and yaw angles). The autopilot obtains the airspeed measurement from a *Pitot tube* mounted on the outside of the plane, the acceleration from a three-axis accelerometer set mounted on the airplane and the angular rates from a three-axis gyroscope set also mounted on the plane. Using these measurements, the autopilot continuously (typically every millisecond) determines what angles the rudder, elevators and ailerons should be at in order to keep the plane at a proper attitude as well as the throttle value so the engine thrust is able to maintain the required air speed. The autopilot sends out the required control surface positions to the control surfaces where motors move these surfaces to these commanded values. The autopilot also sends the required throttle value to the engine where another motor maintains the throttle at this value. Such an autopilot can maintain level flight despite wind gusts acting on the plane.

1.2 Quadrotors

Figure 1.12 is a picture of a quadrotor, which basically consists of four propellers all in the same plane and 90 degrees apart connected to a main body. By adjusting the speed of each propeller one can control the thrust (aerodynamic force) produced by each propeller. The propeller speeds are managed by an on board microprocessor that controls the electric motor of each propeller.

Figure 1.12. Photo of a Parrot quadrotor in hover. Source: Courtesy of Professor Aykut Satici.

First let's look at a single propeller connected to an electric motor as indicated in Figure 1.13. For the purposes of this discussion think of the propeller/motor system as being on a frictionless surface. Thus the bottom of the motor is free to slide or rotate without being stopped by any friction from the table.

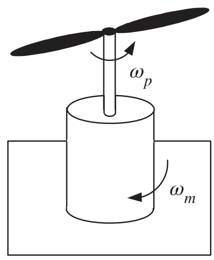

Figure 1.13. Conservation of angular momentum requires $J_p\omega_p = J_m\omega_m$.

Let the propeller have a moment of inertia J_p about the spin axis and the motor have a moment of inertial J_m about same axis. As indicated by the curved arrows in Figure 1.13, if the propeller rotates counterclockwise then $\omega_p > 0$, while if the motor rotates clockwise then $\omega_m > 0$. The angular velocity of the propeller is denoted by ω_p and it has angular momentum $L_p = J_p\omega_p$. The angular velocity of the motor is denoted by ω_m and it has angular momentum $L_m = -J_m\omega_m$. With no external forces acting on the motor–propeller system of Figure 1.13, the principle of *conservation on angular momentum* tells us that its total angular momentum is constant, i.e. $L_p + L_m =$ constant. At startup the propeller is off and the motor is stationary so this constant is zero. After turning on the propeller we must continue to have $L_p + L_m = 0$ or

$$J_p\omega_p - J_m\omega_m = 0. \tag{1.1}$$

Consequently, if the propeller is spinning counterclockwise the motor must spin clockwise. Typically $J_p \ll J_m$ so $\omega_p \gg \omega_m$.

Figure 1.14 is a schematic representation of a quadrotor in flight. The unit orthogonal vectors $\hat{\boldsymbol{x}}_e, \hat{\boldsymbol{y}}_e, \hat{\boldsymbol{z}}_e$ represent earth (inertial) coordinates fixed to the earth.[3] $\hat{\boldsymbol{z}}_e$ points straight up from the earth while $\hat{\boldsymbol{x}}_e, \hat{\boldsymbol{y}}_e$ are tangent to the earth and orthogonal to each other. We have $\hat{\boldsymbol{z}}_e = \hat{\boldsymbol{x}}_e \times \hat{\boldsymbol{y}}_e$ so $\hat{\boldsymbol{x}}_e, \hat{\boldsymbol{y}}_e, \hat{\boldsymbol{z}}_e$ form a right-handed coordinate system. Let $\hat{\boldsymbol{x}}_b, \hat{\boldsymbol{y}}_b, \hat{\boldsymbol{z}}_b$ be unit orthogonal vectors attached to the quadrotor (rigid body) with their origin at the center of mass of the quadrotor. The unit vector $\hat{\boldsymbol{x}}_b$ points to the front of the quadrotor, the unit vector $\hat{\boldsymbol{y}}_b$ points to the left side of the quadrotor, and the unit vector $\hat{\boldsymbol{z}}_b$ points up (from the point of view of the quadrotor). Note that $\hat{\boldsymbol{z}}_b = \hat{\boldsymbol{x}}_b \times \hat{\boldsymbol{y}}_b$ so that they also form a right-handed coordinate system. $\vec{\boldsymbol{r}}_{cm}$ is a vector from the origin of the $\hat{\boldsymbol{x}}_e, \hat{\boldsymbol{y}}_e, \hat{\boldsymbol{z}}_e$ coordinate system to the center of mass of the quadrotor, which is the origin of the $\hat{\boldsymbol{x}}_b, \hat{\boldsymbol{y}}_b, \hat{\boldsymbol{z}}_b$ coordinate system. Both sets of vectors span Euclidean 3-*space*, which we denote as $\mathbf{E}^3$.

Figure 1.15 is a schematic of a quadrotor in hover, i.e., $\hat{\boldsymbol{z}}_b = \hat{\boldsymbol{z}}_e$ with all the propellers rotating at the same angular speed ω_p and thus each producing the same upward thrust (force) f where $f = mg$ with m the mass of the quadrator. Note in Figure 1.15 that the front and rear propellers rotate counterclockwise, while the left and right propellers rotate

[3]The "hat" over the vectors is to denote them as *unit* vectors.

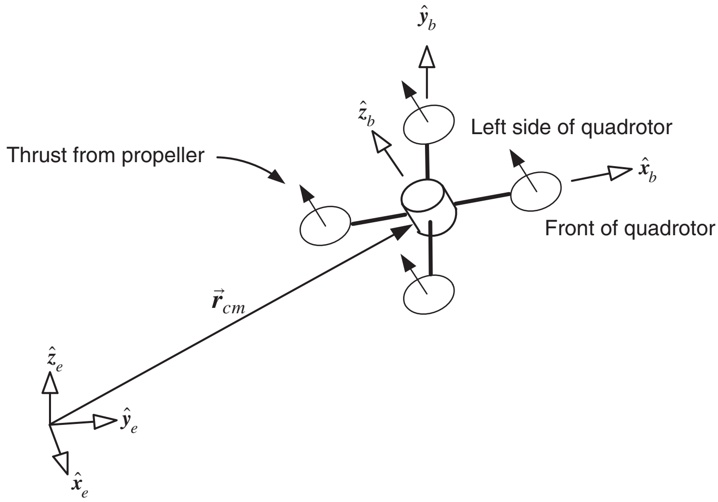

Figure 1.14. Quadrotor with the inertial and body coordinate systems shown.

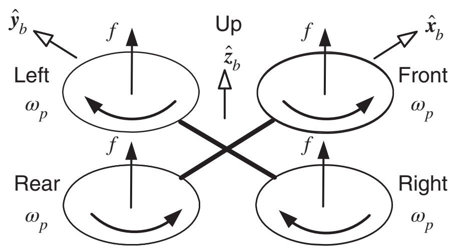

Figure 1.15. A hovering quadrotor. The front and back propellers spin counter clockwise, while the left and right propellers spin clockwise.

clockwise. As a result the total angular momentum L_{z_b} of the four propellers about the $\hat{\boldsymbol{z}}_b$ axis is zero as

$$\begin{aligned} L_{z_b} &= L_{z_b\_left} + L_{z_b\_right} + L_{z_b\_front} + L_{z_b\_rear} \\ &= -J_{z_b}\omega_p - J_{z_b}\omega_p + J_{z_b}\omega_p + J_{z_b}\omega_p \\ &= 0 \end{aligned} \tag{1.2}$$

where J_{z_b} is the moment of inertia of a *propeller*. As $L_{z_b} = 0$ in hover the body of the quadrotor must also have zero angular momentum and thus it does not rotate about the $\hat{\boldsymbol{z}}_b$ axis.

Due to the construction of the quadrotor its thrust is always in the direction of the $\hat{\boldsymbol{z}}_b$ axis. As a consequence, the quadrotor body must be rotated to have a component of its thrust point in the direction of travel. This is done by roll, pitch, and yaw motions.

Figure 1.16 on the next page shows how the quadrotor performs a *roll* motion, that is, rotates about its $\hat{\boldsymbol{x}}_b$ axis. In Figure 1.16 p denotes the angular velocity of the quadrotor body about the $\hat{\boldsymbol{x}}_b$ axis. For the quadrotor to roll to its right the propeller speed on the right side is decreased by $\Delta\omega_p$ with a consequential decrease of Δf in its thrust, while the left propeller has its rotor speed increased by $\Delta\omega_p$ with an increase in its thrust of Δf.

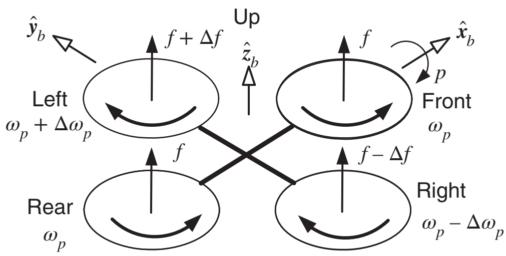

Figure 1.16. The quadrotor executing a roll motion.

With this approach for rolling the quadrotor the total angular momentum L_{z_b} of the four propellers about the $\hat{\boldsymbol{z}}_b$ axis is zero as

$$\begin{aligned} L_{z_b} &= L_{z_b\_left} + L_{z_b\_right} + L_{z_b\_front} + L_{z_b\_rear} \\ &= -J_{z_b}(\omega_p + \Delta\omega_p) - J_{z_b}(\omega_p - \Delta\omega_p) + J_{z_b}\omega_p + J_{z_b}\omega_p \\ &= 0. \end{aligned} \tag{1.3}$$

By conservation of angular momentum the body of the quadrotor does not rotate about the $\hat{\boldsymbol{z}}_b$ axis during this roll motion.

Figure 1.17 shows how the quadrotor performs a *pitch* motion, which is a rotation about its $\hat{\boldsymbol{y}}_b$ axis. In Figure 1.17 q denotes the angular velocity of the quadrotor body about the $\hat{\boldsymbol{y}}_b$ axis. For the quadrotor to pitch down the propeller speed of the rear propeller is increased by $\Delta\omega_p$ with a consequential increase in its thrust of Δf, while the front propeller has its rotor speed decreased by $\Delta\omega_p$ with an decrease in its thrust of Δf. With this way of pitching the quadrotor the total angular momentum of the four propellors about the $\hat{\boldsymbol{z}}_b$ axis is zero as

$$\begin{aligned} L_{z_b} &= L_{z_b\_left} + L_{z_b\_right} + L_{z_b\_front} + L_{z_b\_rear} \\ &= +J_{z_b}\omega_p + J_{z_b}\omega_p - J_{z_b}(\omega_p - \Delta\omega_p) - J_{z_b}(\omega_p + \Delta\omega_p) \\ &= 0. \end{aligned} \tag{1.4}$$

As in the case of a roll motion, during a pitch maneuver the body of the quadrotor does not rotate about the $\hat{\boldsymbol{z}}_b$ axis.

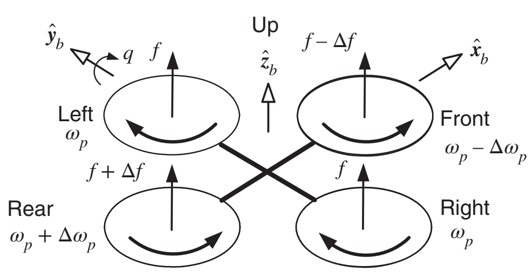

Figure 1.17. The quadrotor executing a pitch motion.

Finally Figure 1.18 shows how to get the quadrotor body to rotate about its $\hat{\boldsymbol{z}}_b$ axis, which is referred to as a *yaw* motion. In Figure 1.18 r denotes the angular velocity of the quadrotor body about the $\hat{\boldsymbol{z}}_b$ axis. For the quadrotor to yaw to the left (counterclockwise

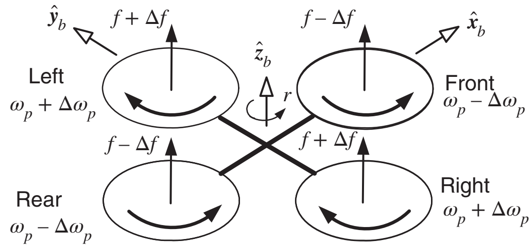

Figure 1.18. The quadrotor executing a yaw motion.

from above), the front and rear propeller speeds are decreased by $\Delta\omega_p$ with a consequential decrease in each of their thrusts by Δf, while the left and right propellers have their speeds increased by $\Delta\omega_p$ with an increase in their thrusts of Δf each. In contrast to the pitch and roll motions, the total angular momentum about the $\hat{\boldsymbol{z}}_b$ axis of the four propellers is not zero. Specifically, their total angular momentum is

$$\begin{aligned} L_{z_b} &= L_{z_b\_left} + L_{z_b\_right} + L_{z_b\_front} + L_{z_b\_rear} \\ &= -J_{z_b}(\omega_p + \Delta\omega_p) - J_{z_b}(\omega_p + \Delta\omega_p) + J_{z_b}(\omega_p - \Delta\omega_p) + J_{z_b}(\omega_p - \Delta\omega_p) \\ &= -4J_{z_b}(\Delta\omega_p). \end{aligned} \tag{1.5}$$

L_{z_b} negative tells us that the net angular momentum of the propellers is in the clockwise direction. As angular momentum is conserved, the body of the quadrotor will rotate counterclockwise (left). With J_b the moment of inertia of the quadrotor *body* about the $\hat{\boldsymbol{z}}_b$ axis its angular momentum must be $L_b = J_b r = 4J_{z_b}(\Delta\omega_p)$ so that $L_b + L_{z_b} = 4J_{z_b}(\Delta\omega_p) - 4J_{z_b}(\Delta\omega_p) = 0$. The angular velocity of the quadrotor body is then

$$r = (4J_{z_b}/J_b)\Delta\omega_p. \tag{1.6}$$

Automatic Control

We can consider the inputs to the quadrotor to be the four propeller speeds.[4] Typically a quadrotor has an inertial measurement unit (IMU), which consists of a three axis accelerometer, a three axis rate gyro, and a three axis magnetometer. The three accelerometers provide the inertial acceleration (i.e., the acceleration with respect to the earth) of the quadrotor in the $\hat{\boldsymbol{x}}_b, \hat{\boldsymbol{y}}_b, \hat{\boldsymbol{z}}_b$ directions. The three axis rate gyro gives angular velocity vector (p, q, r) of the quadrotor in the $\hat{\boldsymbol{x}}_b, \hat{\boldsymbol{y}}_b, \hat{\boldsymbol{z}}_b$ coordinate system. The three axis magnetometer gives the ambient magnetic field[5] in the $\hat{\boldsymbol{x}}_b, \hat{\boldsymbol{y}}_b, \hat{\boldsymbol{z}}_b$ directions. Using an IMU there are methods to estimate the orientation of the quadrotor relative to the earth coordinate system (orientation of $\hat{\boldsymbol{x}}_b, \hat{\boldsymbol{y}}_b, \hat{\boldsymbol{z}}_b$ with respect to $\hat{\boldsymbol{x}}_e, \hat{\boldsymbol{y}}_e, \hat{\boldsymbol{z}}_e$) as well as the velocity vector of the center of mass of quadrotor with respect to earth. Based on the quadrotor's orientation, velocity, and position, a feedback controller (an algorithm on a microprocessor) figures out how fast each propeller should spin to make the quadrotor follow a reference trajectory.

[4] There is an "inner" controller that controls the propeller speeds by controlling the voltage of the motor connected to the propeller.

[5] Same as the earth's magnetic field if there are not power lines close by.

1.3 Inverted Pendulum

A classic control problem in academia is the inverted pendulum on a cart. It consists of a rod of length ℓ which is free to rotate about a pivot as shown in Figure 1.19. The idea is to apply the appropriate force u to the cart to keep it under the pendulum rod, i.e., to the keep the rod vertical. Specifically, one can get a measurement of θ and x (say every millisecond) and wants to determine the value of u (also every millisecond) that keeps the pendulum at a fixed position in x and keeps $\theta = 0$. In Chapter 13 we will derive the nonlinear differential equations of motion of this pendulum. For now we just state them as follows:

$$(J + m\ell^2)\frac{d^2\theta}{dt^2} = mg\ell\sin(\theta) - m\ell\frac{d^2x}{dt^2}\cos(\theta) \tag{1.7}$$

$$(M+m)\frac{d^2x}{dt^2} + m\ell\left(\frac{d^2\theta}{dt^2}\cos(\theta) - \left(\frac{d\theta}{dt}\right)^2\sin(\theta)\right) = u(t). \tag{1.8}$$

The point here is that for any input $u(t)$ the solution to these differential equations gives position of the cart $x(t)$ and the angle $\theta(t)$ of the pendulum rod.

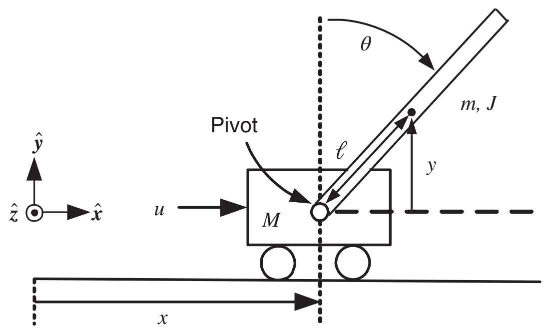

Figure 1.19. Inverted pendulum.

These equations may look complicated, but that is only because they are! To simplify them consider the situation for which $|\theta|$ and $|\dot{\theta}| = |d\theta/dt|$ remain small. Then $\sin(\theta) \approx \theta, \cos(\theta) \approx 1$, and $\left(\dfrac{d\theta}{dt}\right)^2 \sin(\theta) \approx (\dot{\theta})^2\theta \approx 0$ so that Eqs. (1.7) and (1.8) can be approximated by the *linear* differential equations

$$(J + m\ell^2)\frac{d^2\theta}{dt^2} = mg\ell\theta - m\ell\frac{d^2x}{dt^2} \tag{1.9}$$

$$(M+m)\frac{d^2x}{dt^2} + m\ell\frac{d^2\theta}{dt^2} = u(t). \tag{1.10}$$

In your first differential equations course you primarily studied *linear* differential equations. The reason for this is that we know how to solve them! As we will show in later chapters, it is also known how to *control* a physical system with a linear differential equation model. By the control of the inverted pendulum we mean that given the measurements $x(t), \theta(t)$ the input value $u(t)$ can be chosen to keep the pendulum rod vertical. If this controller based on the linear model keeps $|\theta|$ and $|\dot{\theta}|$ small, then the linear model is a valid approximation to the actual pendulum and the controller will also work on the actual pendulum.

We can use Laplace transforms to convert linear differential equations into *algebraic* equations. In Chapters 2 and 3 we will do a comprehensive review of Laplace transforms. For now we just note that the Laplace transfer function model of the inverted pendulum is

$$X(s) = \frac{1}{Mm\ell^2 + J(M+m)} \frac{(J+m\ell^2)s^2 - mg\ell}{s^2(s^2-\alpha^2)} U(s) \tag{1.11}$$

$$\theta(s) = -\frac{1}{Mm\ell^2 + J(M+m)} \frac{m\ell}{s^2-\alpha^2} U(s) \tag{1.12}$$

$$\alpha^2 = \frac{mg\ell(M+m)}{Mm\ell^2 + J(M+m)}. \tag{1.13}$$

Control Problem

The control problem is to use the measured values of $x(t)$ and $\theta(t)$ to determine the force $u(t)$ applied to the cart so that the pendulum rod does not fall. The control algorithm is simply a function that gives the value of $u(t)$ given the values of $x(t)$ and $\theta(t)$. A controller can be designed using the above transfer function model or the above differential equation model. In the first half of the book we base the control design on the transfer function model, while in later chapters of the book a differential equation model is used.

1.4 Magnetic Levitation

In this example we consider a magnetic levitation system. The schematic given in Figure 1.20 shows a wire wrapped around a cylindrical core of iron making an electromagnet.

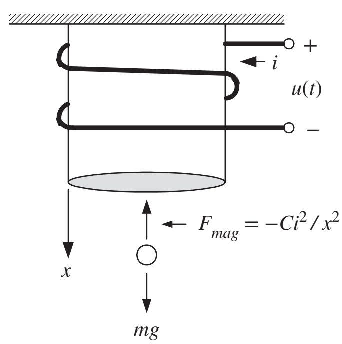

Figure 1.20. A simple magnetic levitation system.

Applying a voltage $u(t)$ to the coil results in a current $i(t)$. This current along with the iron core produces a magnetic field that extends below the electromagnet. This magnetic field then magnetizes the steel ball, i.e., makes it into a magnet. It turns out that the magnetic force of attraction between the steel ball and the electromagnet is upward and proportional to the square of the current and inversely proportional to the square of the distance. Mathematically, we write that $F_{mag} = -Ci^2/x^2$ where C is a constant. Note in the figure that positive x is downward and $x = 0$ corresponds to the bottom of the iron core.

Mathematical Model

There are current-command amplifiers that allow one to have the amplifier put out a desired current. Using such an amplifier we can consider the input to the magnetic levitation system to be the current $i(t)$. The differential equation model of this magnetic levitation system is then (see Chapter 13)

$$\frac{dx}{dt} = v \tag{1.14}$$

$$m\frac{dv}{dt} = -C\frac{i^2}{x^2} + mg. \tag{1.15}$$

Here C is the force constant, m is the mass of the steel ball, and g is the acceleration of gravity. This differential equation model is important because for any given input current $i(t)$ to the coil, the solution of these differential equations then gives the response of the position $x(t)$ and velocity $v(t) = \dot{x}(t)$ of the steel ball. However this model is nonlinear. As in the case of the inverted pendulum, we can obtain an approximate linear model. Specifically, let the desired (constant) position of the steel ball be x_0 and choose the current i_0 to make the ball's acceleration zero, i.e.,

$$0 = -C\frac{i_0^2}{x_0^2} + mg \quad \text{or} \quad i_0 = x_0\sqrt{\frac{mg}{C}}. \tag{1.16}$$

With $\Delta x \triangleq x - x_0, \Delta\dot{x} \triangleq \dot{x} - \dot{x}_0 = \dot{x}, u \triangleq i - i_0$ we will show in Chapter 13 that

$$m\Delta\ddot{x} = \frac{2g}{x_0}\Delta x - \frac{2g}{i_0}u \tag{1.17}$$

is a valid *linear* model for $\Delta x, \Delta\dot{x}$, and u small. After studying Chapter 3 you will be able to show that the corresponding Laplace transform model is

$$\Delta X(s) = -\frac{2g/i_0}{s^2 - 2g/x_0}U(s). \tag{1.18}$$

Figure 1.21 is a magnetic levitation system built by W. Barie [10]. It shows light being shone across the top of the ball to a photodetector sensor. Using this setup the position of the ball beneath the electromagnet can be found.

Control Problem

Based on measurements of $x(t)$ and $v(t)$, one wants to choose the input voltage $u(t)$ so the ball maintains a fixed position x_0 below the electromagnet, i.e., it does not fall nor gets sucked into the magnet. A procedure to choose $u(t)$ based on the values of $x(t)$ and $v(t)$ is called the control algorithm or controller. The control algorithm is found using either the differential equation model or the Laplace transform model.

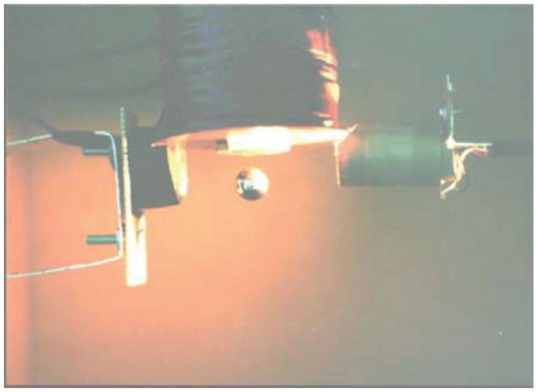

Figure 1.21. A laboratory magnetic levitation system. Source: Barie and Chiasson [10]. "Linear and nonlinear state-space controllers for magnetic levitation", International Journal of Systems Science, vol. 27, no. 11, pp. 1153–1163, November 1996. DOI-https://doi.org/10.1080/00207729608929322.

1.5 General Control Problem

In general there is some physical system that needs to be controlled. The steps involved are:

- Determine a mathematical model of this system (typically a set of differential equations). The model describes how the *output variables* behave for any given *input*.
- Design the control algorithm. The feedback control algorithm processes measurements of the output through a differential equation to produce the inputs to the physical system that force the output to a desired value.
- Simulate the combined controller and mathematical model to check if the system outputs are controlled as desired. If not, modify or (if necessary) redesign the controller. If the controller doesn't work in simulation, then it won't work in practice!
- The final step is to implement the controller on the actual physical system.